On the possible expansion histories of the universe

Id 15131
(Dated: March 16, 2021)

When fitting the density parameters and w to find the combination with highest probability of being true, I found that $\Omega_{m0} = 0.24$, $\Omega_{w0} = 0.64$ and $w = -1.05$. I can with high confidence claim this as the model used to find them had a relative error proportional of at most 10^{-14}. From the calculations we found two possible scenarios, one expansion with no matter in the universe and one with matter, both resulting "the Big Freeze". In the time up to today the universe with matter expanded far more rapidly than the universe with no matter inside.

CONTENTS

I. INTRODUCTION

The universe is wast and almost impossible for the human intellect to conceive. Many great minds have however tried, from Anaxagoras, the greek philosopher, who claimed that all things existed from the beginning in the most simple form, to John Philoponus, who claimed that the universe followed temporal finitism. However, the idea of a "Big Bang" was first posed by Belgian priest and professor Georges Lemaitre in 1931, giving birth to yet another ensamble of questions as to how everything works.

This report aims to explore possible expansion histories of the universe, by developing a model for the luminosity distance, to fit the density parameters. Using an energy plot, we will explore other possible expansions.

II. THEORY

A. Equations and constants

In order to explore possible expansion histories, we have to vary the amount of all that is inside the universe. This has a special notation, denoted as Ω_{i0}, where the 0 subscript indicates that it is a measurement today in time, and the i indicates what sort of component we are looking at. Matter for instance has the subscript "m", radiation has "r", and the cosmological constant has Λ. These densities are in units of the critical density, given as

$$\Omega_{i0} = \frac{\rho_0}{\rho_{c0}} \tag{1}$$

where $\rho_{c0} = 1.879 * 10^{-29} h^2 \text{ g } cm^{-3}$, where h is the dimensionless Hubble constant equal to 0.7. We also have the curvature density parameter denoted as

$$\Omega_{k0} = \frac{-k^2 c^2}{a_0^2 H_0^2} \tag{2}$$

where k is the spatial curvature parameter with possible values being $k = 0, -1$ or 1, c is the speed of light, H_0 is the Hubble constant at this point in time, approximately equal to $70 kms^{-1} Mpc^{-1}$, and a_0 is the scale factor today. It can also be shown that the scale factor is related to the red shift, and the relation is given as

$$a(z) = \frac{a_0}{1+z} \tag{3}$$

In this report we will use the First Friedman equation to model the universe. The first Friedman equation can be written as

$$\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \Omega_{mo}\left(\frac{a_0}{a}\right)^3 + \Omega_{ko}\left(\frac{a_0}{a}\right)^2 + \Omega_{wo}\left(\frac{a_0}{a}\right)^{3(1+w)} \tag{4}$$

where we denote $\dot{a} = \frac{da}{dt}$.

One way to measure distance in the universe is by using the luminosity distance. This allows us to consider the expansion of the universe, when measuring distances. The luminosity distance is given as

$$\begin{aligned} d_L(z) &= a_0(1+z)r \\ &= \frac{c(1+z)}{H_0\sqrt{\mid \Omega_{k0} \mid}} S_k\left[\sqrt{\mid \Omega_{k0} \mid}\int_0^z \frac{dz'}{H(z')/H_0}\right] \end{aligned} \tag{5}$$

where S_k is a piece-wise function given as

$$S_k(x) = \begin{cases} \sin(x) & \text{if } k = 1 \\ x & \text{if } k = 0 \\ \sinh(x) & \text{if } k = -1 \end{cases} \tag{6}$$

The relative error is given as

$$err_{rel} = \frac{\mid sample_{model} - sample_{analytic} \mid}{sample_{analytic}} \tag{7}$$

B. Probability and statistics

In order to find the best fit of our given parameters, we wish to maximize the likelihood as a function of said parameters. This is equivalent to minimize the quantity

$$\chi^2(\Omega_{w0}, \Omega_{m0}) = \sum_{i=0}^{N} \frac{(d_L(z_i; \Omega_{w0}, \Omega_{m0}) - d_L^i)^2}{\sigma_i^2} \tag{8}$$

This however does not guaranty that the most probable set of parameters is the actual true set of values. We only know from the relation above the range for which the true set of parameters lie, and it can be shown that there is a 95% probability that the given set of values is the true values if they satisfy the following relation

$$\chi^2 - \chi^2_{min} < 6.17 \tag{9}$$

where χ^2_{min} is the minimum value of χ^2.

C. Models of the universe

1. *The Einstein - de Sitter Model*

In the case of a spatially flat ($k = 0$) dust filled universe, we have the Einstein - de Sitter model. Here the equation of state $w = 0$, and with $\Omega_{m0} = 1$ and $\Omega_{w0} = 0$.

2. *The Milne Model*

In the case of a negative spatial curve ($k = -1$), without any energy density, we have the Milne model. Here we have that $\Omega_{k0} = 1$ and all other $\Omega_{i0} = 0$. This model is not consistent with observations, but it has a simple analytical solution to both equation 4 and equation 5.

3. *The ΛCDM model*

In the case of a spatially flat ($k = 0$) dust filled universe with a cosmological constant, we have the ΛCDM model. Here we have that $\Omega_{m0} = 0.3$. This model contains dark matter (the cosmological constant), cold dark matter (CDM) and ordinary matter.

D. Treating the Friedman equation as an energy function

In the appendix, we have rewritten the first Friedman equation, given in section II A. We see here that the left term always is equal to or larger than zero, much like a kinetic energy. Thus we can treat the right term as a potential energy, and use this analogy to explore possible expansion histories for the universe.

III. METHOD

The Python3 code written for this project requires several packages in how ever order to run. The following versions where used by me, but slightly older ones could also be used and should not necessarily be problematic.

- numpy 1.19.4
- matplotlib 3.3.3
- scipy 1.5.4
- progress 1.5

The progress package is not necessary for running the scripts, and is by default commented out. But since some of the calculations takes a while, its gives an indication as to how far you are along.

To find the ideal combination of Ω_{m0} and Ω_{wo} we use the following algorithm:

Result: Return $\chi^2(\Omega_{w0}, \Omega_{m0})$
Initialize arrays;
i = 0;
while $\Omega_{w0,i} \leq \Omega_{w0,max}$ **do**
  j = 0;
  while $\Omega_{m0,j} \leq \Omega_{m0,max}$ **do**
    if $H^2 \geq 0$ **then**
      Calculate $\chi^2(\Omega_{w0,i}, \Omega_{m0,j})$;
    else
      Set $\chi^2 = NaN$;
    end
    increase j;
  end
  increase i;
end

Algorithm 1: Calculating χ^2 for a given combination of Ω_{m0} and Ω_{w0}

The same algorithm was also used to find the ideal combination of w and Ω_{w0} when $\Omega_{m0} = 0.3$.

IV. RESULTS

A. Testing the model against analytical solutions

I tested the numerical luminosity model for the luminosity distance against the Einstein - de Sitter model and against the Milne model, as shown below:

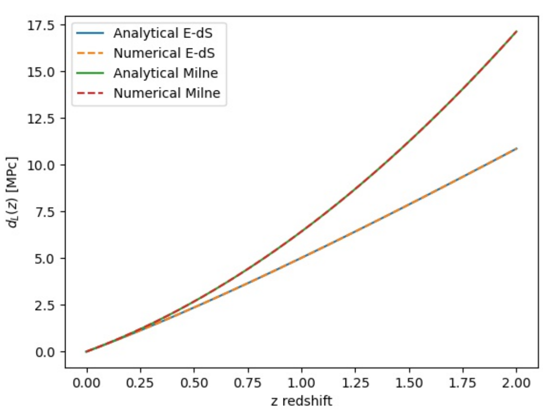

Figure 1: $dL(z)$ as a function of the red shift, in unit Megaparsec.

The difference between the model and the analytical is very small, and so below is the relative error as a function of the red shift:

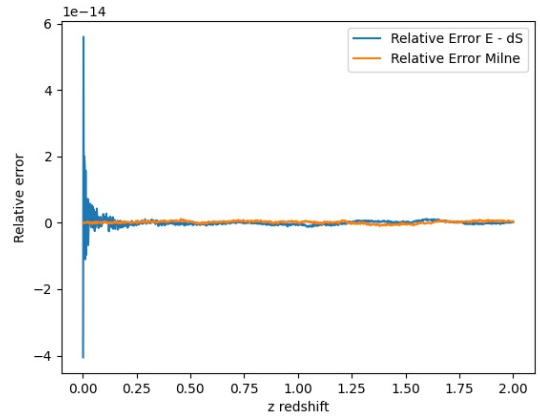

Figure 2: Relative error as a function of the redshift

B. Fitting of parameters

When testing for different combinations of Ω_{wo} and Ω_{mo} I found in the plots below values for χ^2. Note that in figure 3, figure 4, figure 5 and figure 6, I have used the base 10 logarithm of χ^2 to avoid very large scales, thus the true difference in the plots are quite larger:

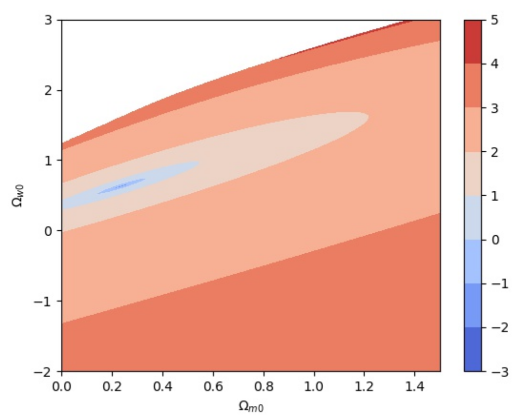

Figure 3: χ^2 as a function of Ω_{wo} and Ω_{mo}.

Below we also have χ^2 in the 95% confidence region.

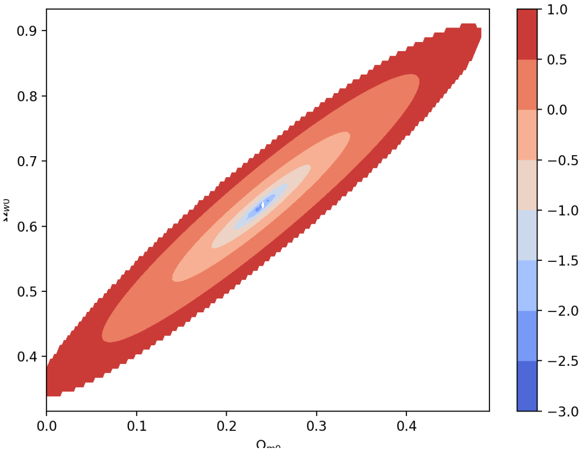

Figure 4: χ^2 as a function of Ω_{wo} and Ω_{mo} in the 95% confidence region.

From the plots above we can see that the combination with the lowest χ^2 value is approximately $\Omega_{wo} = 0.64$ and $\Omega_{mo} = 0.24$. Now, assuming that $\Omega_{mo} = 0.3$, we can check the best combination for w, the equation of state parameter, and Ω_{wo}. Here we found the following values when calculating χ^2:

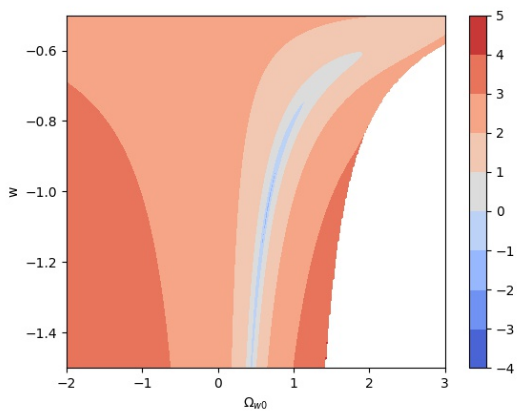

Figure 5: χ^2 as a function of Ω_{wo} and w.

Below we also have χ^2 in the 95% confidence region.

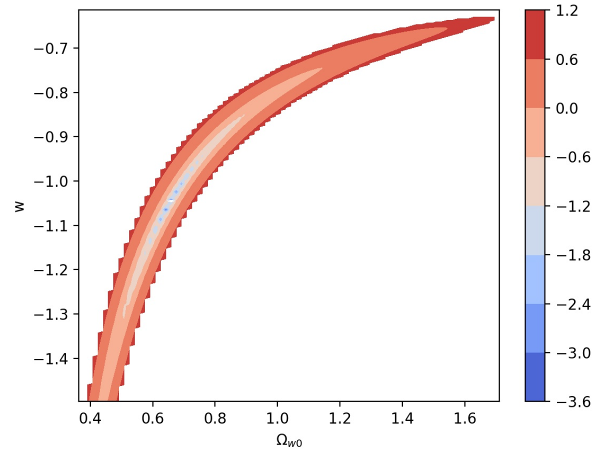

Figure 6: χ^2 as a function of Ω_{wo} and w in the 95% confidence region.

From above we see that the ideal combination of w and Ω_{wo} is $w = -0.97$ and $\Omega_{wo} = 0.65$.

C. Energy diagrams for expansion history

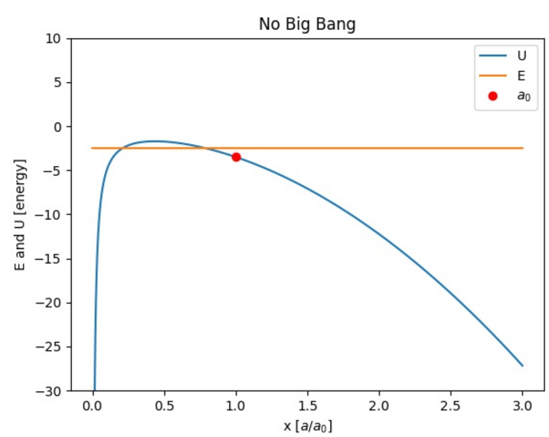

Figure 7: No Big Bang solution, $w = -1$, $\Omega_{m0} = 0.5$, and $\Omega_{w0} = 3$

Above we see a solution with no big bang, where $a(z)$ has a minimum value.

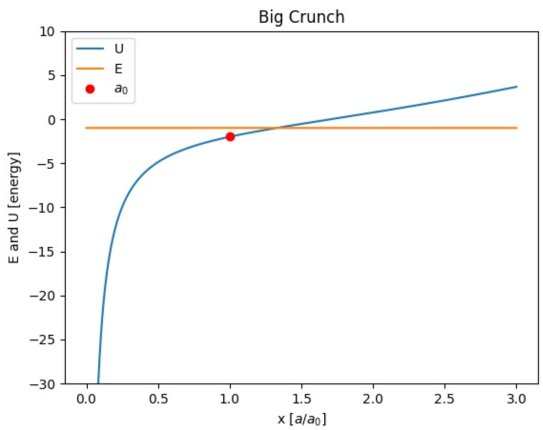

Figure 8: Big Crunch solution, $w = -1$, $\Omega_{m0} = 2.5$, and $\Omega_{w0} = -0.5$

Above we see a solution with the big crunch, where $a(z)$ has a maximum value. Here the universe has a rapid expansion which decreases and then suddenly turns.

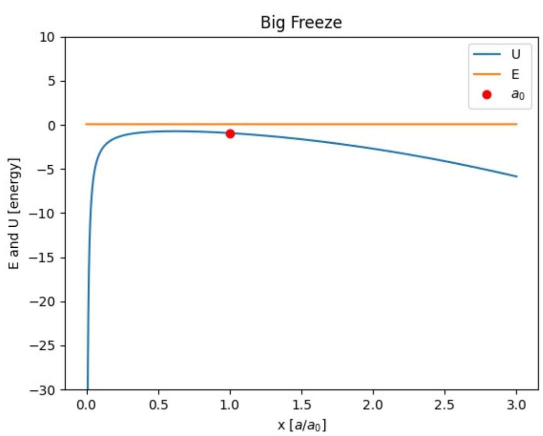

Figure 9: Big freeze solution, $w = -1.05$, $\Omega_{m0} = 0.24$, and $\Omega_{w0} = 0.64$

Above we see a solution with the big freeze, where $a(z)$ has no maximum value. Here the universe will expand for ever.

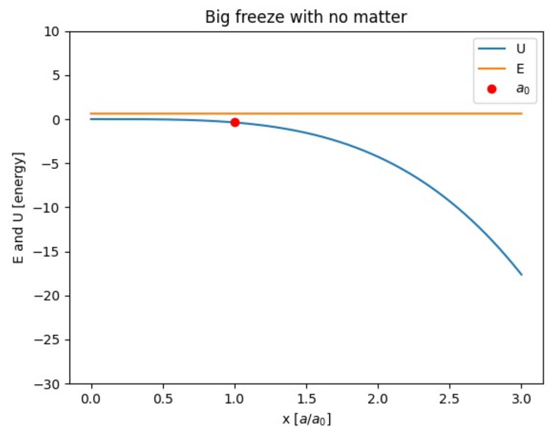

Figure 10: Big freeze solution with no matter, $w = -1.5$, $\Omega_{m0} = 0$, and $\Omega_{w0} = 0.377$

Above we have a solution with the big freeze, but without matter. This universe increases very slowly up to about 1.5 times the age of the universe.

Note here that the y axis is not really energy, but analogous to energy. The x axis is here given as a/a_0, and the red dot indicates the scale factor today.

V. DISCUSSION AND ANALYSIS

A. Accuracy of model

Before we can continue with the fitting of the density parameters, we should check that the model actually works. Figure 1 shows the numerical solution versus the analytical solution for both the Einstein - de Sitter model and the Milne model. All though these are specific cases, and not the current model for the universe, they have easily derivable analytical solutions, shown in the appendix. The method used here is in fact so accurate that the relative error is proportional to 10^{-14}.

B. Fitting of parameters and other possible expansion histories

To find the combination of the densities with highest probability of being the true value, we chose ranges from zero to one for both Ω_{m0} and Ω_{w0}. In figure 3 we see a white area. This area is set to NaN, and shows number combinations that results in "no Big Bang" solutions when used in the first Friedman equation. The aforementioned image has a resolution of 700×700, which on my MacBook Pro M1 computer took about 26 minutes and 10 seconds to compute. This proved to be quite long, and perhaps not necessary, but I wanted a good value for the optimal solution. It would be interesting to write the code in C++ and do the same calculations there, as they probably would be much faster. From figure 4 and figure 6 we get that the combination of values with highest probability of being true is

- $\Omega_{m0} = 0.24$
- $\Omega_{w0} = 0.64$
- $w = -1.05$

All though we cannot claim that they are the actual true values, the minimized error is so low that we with high confidence can be very certain that they are.

Figure 7, 8, 9 and 10 are energy diagrams. They display four different scenarios for expansion, with carefully selected parameters. Note the red dot, that indicates the scale factor today, shows us what happens to the universe. The orange line is the total energy, and so the potential, i.e the blue line, cannot cross this section because it would give us a negative kinetic energy, which is not possible. Now, the potential can actually cross the line, it is just that a dot moving a long the line cannot. And so depending on which of the side of the cross the dot is, it tells us what happens with the universe. For example, we know that since the red dot in figure 7 is on the right side of the cross, it will never reach zero, and so we have found a minimum value for the scale factor. Thus the universe has no start, and thus no big bang. In figure 8 we see that the red dot can move up til the cross side from the left, which tells us that the universe expands and then shrinks again into a big crunch. The last figure tells us that the universe rapidly expands, slows down, and then slowly increases its expansion speed again.

The last two figures were calculated with values from the 95% confidence area shown in section IV C. After testing for different combinations inside the confidence area, I found that figure 9 and figure 10 where the two most different cases for the expansion. Although the parameters are quite close for these diagrams compared to the two before, they tell two completely different stories. Figure 9 shows us the case with the best probability of being the true case, as shown in the list above. Here the universe has a rapid expansion before slowing down, and then slowly increasing the expansion rate. If we let time go far into the future, this evolution would result into what is called the Big Freeze, where galaxies slowly move away from another until they are so far away that one cannot see the others. The smallest stars would eventually burn out, and the universe would be a dark, cold place.

Figure 10 has no matter, which is drastically different from what we know about our universe, and this is also visible in the figure. We see here that the universe slowly increases until the scale factor is what it is today, and then speed up. This universe would expand much quicker here than in the case above, and be cold and dark quite quickly.

C. No Big Bang case

Now, one could ask the question, "What if the universe had no Big Bang?", and how can we know this not to be true? Well let us assume that the universe had no Big Bang. A consequence of this is that the scale factor never can be zero. This leaves us with three options.

- The universe is eternally increasing
- The universe is static
- The scale factor fluctuates in size

In the first case the scale factor goes towards zero as $t \to -\infty$, in which case we would expect an infinite red shift. This we know is not true, because we can measure red shift.
In the second case we have a static universe, and with infinite time we would get thermal equilibrium, which we know is not the case.
In the third case we know that the scale factor has at least one minimum. This is problematic too because it allows for blue shifting of light, which we do not observe, and probably could not observe anyway since we know that the universe is now expanding. This would also lead to a max value for the red shift, as shown in equation 3, and we would then at some point see that the light would not get more red shifted.

VI. CONCLUSION

The expansion history of the universe can be modelled using the first Friedman equation, and by fitting parameters we found the combination with the highest probability of being the true one. We tested against experimental values with a model for the luminosity distance, and can with high confidence assert that the model gave the true values, as the model showed a relative error of at the most $err_{rel} \propto 10^{-14}$. From this model I found that the combination with highest probability of being true was $\Omega_{m0} = 0.24$, $\Omega_{w0} = 0.64$ and $w = -1.05$, as shown in figure 4 and 6. The values have however few decimal places, as one have to zoom into the plot quite a lot to find the remaining ones. With high resolution, in other words lots of data points, this showed to be troublesome. However, as the plots shows, we are quite close using only two decimals. I also found that for the two universes with value combination inside the 95% confidence area, the universe with matter expanded far more rapidly in the time up to today than the one without matter. This equals a bit more out when the time increases a lot further.

Appendix A: Calculations

1. Rewriting the first Friedman equation

We have that the first Friedman equation is given as

$$\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \Omega_{mo}\left(\frac{a_0}{a}\right)^3 + \Omega_{ko}\left(\frac{a_0}{a}\right)^2 + \Omega_{wo}\left(\frac{a_0}{a}\right)^{3(1+w)}$$

We then do the following variable changes $x = a/a_0$ and $\tau = H_0 t$. From this we find that their respective derivatives are given as

$$\frac{\dot{a}}{a} = \frac{\dot{x}}{x} = \frac{1}{x}\frac{dx}{dt}$$

and

$$d\tau = H_0 dt$$

We then substitute for x and multiply equation 4 with x^2 and get

$$\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 x^2 = \frac{\Omega_{mo}}{x} + \Omega_{ko} + \frac{\Omega_{wo}}{x^{3(1+w)}}$$

We then solve for Ω_{ko} and get

$$\frac{1}{H_0^2}\left(\frac{dx}{dt}\right)^2 + \left[-\left(\frac{\Omega_{mo}}{x} + \frac{\Omega_{wo}}{x^{3(1+w)}}\right)\right] = \Omega_{ko}$$

which can be rewritten as

$$\left(\frac{dx}{d\tau}\right)^2 + \left[-\left(\frac{\Omega_{mo}}{x} + \frac{\Omega_{wo}}{x^{3(1+w)}}\right)\right] = \Omega_{ko} = constant \tag{A1}$$

2. Analytical solutions to the first Friedman equation

a. *The Einstein-de Sitter model*

In the Einstein - de Sitter model we have a spatially flat universe, i.e $k = 0$ and $\Omega_{m0} = 1$, with $w = 0$. Thus we get that equation 4 is reduced to

$$H(z) = H_0\sqrt{(1+z)^3}$$

If we insert this into equation 5 we get that

$$d_L(z) = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{(1+z)^{3/2}}$$

which gives us

$$d_L(z) = \frac{2c}{H_0}(1 + z - \sqrt{1+z})$$

b. *The Milne model*

In the Milne model we have a negative curve on space, i.e $k = -1$ with $\Omega_{k0} = 1$ and $w = -1/3$. Thus we get that equation 4 is reduced to

$$H(z) = H_0\sqrt{\Omega_{k0}(1+z)^2} = 1 + z$$

If we insert this into equation 5 we get that

$$\begin{aligned} d_L(z) &= \frac{c(1+z)}{H_0} S_{-1}\left[\int_0^z \frac{dz'}{(1+z)}\right] \\ &= \frac{c(1+z)}{H_0}\sinh\left[\int_0^z \frac{dz'}{(1+z)}\right] \end{aligned}$$

which gives us

$$d_L(z) = \frac{cz}{2H_0}(z+2)$$