AST3220, spring 2021: Project 2

In addition to the solution of the analytical part, the deliverables are the following:

- Your code in a separate file, ready to compile and run
- All graphs and other output asked for in the following. Note that the plots must be included in your report. It is not enough that your code generates them.
- Your report can be either a straightforward set of answers to the questions, or in the form of a paper. Either way is fine.
- The maximum score on the project is 100 points.

In the lectures we looked at the relationship between the neutrino temperature T_ν and the photon temperature T after the neutrinos decoupled. Among the assumptions we made, was that the electrons and positrons become non-relativistic as soon as the temperature drops below $k_B T = m_e c^2$, where m_e is the electron (and positron) mass. This is not strictly correct, and you will in the following look at the relation between T_ν and T when the electrons and positrons are treated more accurately. The following expressions for the energy density and pressure of a gas of fermions will be useful:

$$
\begin{aligned}
\rho c^2(T) &= \int_0^\infty E(p) n(p,T) dp \\
P(T) &= \int_0^\infty \frac{(pc)^2}{3E(p)} n(p,T) dp \\
n(p,T) &= \frac{4\pi g p^2}{(2\pi\hbar)^3} \frac{1}{\exp[E(p)/k_B T] + 1} \\
E(p) &= \sqrt{p^2c^2 + m^2c^4},
\end{aligned}
$$

where g is the number of internal degrees of freedom and m is the rest mass of the particle in question. In the lectures we showed that the entropy density is given by

$$
s(T) = \frac{\rho c^2 + P}{T},
$$

and that entropy conservation implies $a^3 s(T) = \text{constant}$.

1) (10 points) Show that entropy conservation together with the first Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

gives

$$t = \int_T^{T_0} \frac{s'(T)dT}{s(T)\sqrt{24\pi G\rho(T)}},$$

where $s'(T) = \frac{ds}{dT}$ and I have defined $t(T_0) = 0$.

2) (10 points) Show that the total entropy density for photons, electrons and positrons can be written as

$$\begin{aligned} s(T) &= \frac{4\pi^2}{45}k_B\left(\frac{k_BT}{\hbar c}\right)^3 \\ &+ \frac{1}{T}\int_0^\infty \frac{16\pi p^2 dp}{(2\pi\hbar)^3}\frac{1}{\exp[E_e(p)/k_BT]+1}\left[E_e(p) + \frac{(pc)^2}{3E_e(p)}\right] \\ &= \frac{4\pi^2}{45}k_B\left(\frac{k_BT}{\hbar c}\right)^3 \mathcal{S}\left(x = \frac{m_ec^2}{k_BT}\right), \end{aligned}$$

where $E_e = \sqrt{p^2c^2 + m_e^2c^4}$ and

$$\mathcal{S}(x) = 1 + \frac{45}{2\pi^4}\int_0^\infty y^2\left(\sqrt{y^2+x^2} + \frac{y^2}{3\sqrt{y^2+x^2}}\right)\frac{1}{\exp(\sqrt{y^2+x^2})+1}dy.$$

(Note that $\mathcal{S}(x)$ here has nothing to do with the function we called $\mathcal{S}$ when we discussed distances.)

3) (10 points) I showed in the lectures that after decoupling the neutrinos still follow the Fermi-Dirac distribution with $T_\nu \propto 1/a$. Show that entropy conservation implies

$$a \propto \frac{1}{T\mathcal{S}^{1/3}(x = m_ec^2/k_BT)},$$

and that we therefore have

$$T_\nu = AT\mathcal{S}^{1/3}(x = m_ec^2/k_BT),$$

where A is a constant.

4) (10 points) What is the relation between T and T_ν at very high temperatures ($T \to \infty$)? Use this to find A and show that

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T\mathcal{S}^{1/3}\left(x = \frac{m_e c^2}{k_B T}\right).$$

The following integral may be useful:

$$\int_0^\infty \frac{y^3 dy}{\exp(y) + 1} = \frac{7\pi^4}{120}.$$

5) (10 points) Show that the total energy density can be written as

$$\begin{aligned}
\rho c^2 &= \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3} + \frac{7}{8}\frac{\pi^2}{30}6\frac{(k_B T_\nu)^4}{(\hbar c)^3} \\
&+ \int_0^\infty \frac{16\pi p^2 dp}{(2\pi\hbar)^3}\frac{\sqrt{p^2c^2 + m_e^2c^4}}{\exp(\sqrt{p^2c^2 + m_e^2c^4}/k_B T) + 1} \\
&= \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}\mathcal{E}(x = m_e c^2/k_B T),
\end{aligned}$$

where

$$\mathcal{E}(x) = 1 + \frac{21}{8}\left(\frac{4}{11}\right)^{4/3}\mathcal{S}^{4/3}(x) + \frac{30}{\pi^4}\int_0^\infty \frac{y^2\sqrt{y^2 + x^2}dy}{\exp(\sqrt{y^2 + x^2}) + 1}.$$

6) (10 points) Show that

$$t = \sqrt{\frac{15\hbar^3}{24\pi^3 G m_e^4 c^3}}\int_{m_e c^2/k_B T_0}^{m_e c^2/k_B T}\left(3 - \frac{x\mathcal{S}'(x)}{\mathcal{S}(x)}\right)\mathcal{E}^{-1/2}(x) x dx$$

7) (10 points) Write a code to evaluate $\mathcal{S}(x)$ and plot it. Use physical arguments to find the value of $\mathcal{S}$ in the limits $x = 0$ and $x \gg 1$. Check that your code agrees with these limits.

8) (30 points) Take $T_0 = 10^{11}$ K. Write a code to evaluate $t(T)$ and $T_\nu(T)$, and complete the table on the next page.

T (K)	T_ν/T	t (s)
10^{11}	1.000	0
$6 \cdot 10^{10}$		
$2 \cdot 10^{10}$		
10^{10}		
$6 \cdot 10^{9}$		
$3 \cdot 10^{9}$		
$2 \cdot 10^{9}$		
10^{9}		
$3 \cdot 10^{8}$		
10^{8}		
10^{7}		
10^{6}		