AST3220, spring 2021: Project 3

The compulsory annoying sermon

This project consists of a set of tasks, some analytical, some numerical. You can structure your answers as a report with an introduction, methods, results, discussion and conclusion, but you can also choose to just answer the questions, one by one. Both approaches are fine. It is important that you explain how you think, just writing down a bunch of equations with no explanations will not give you a maximum score. You should write your report/answers using LaTeX. Posting handwritten lecture notes and solutions to problems is a privilege that belongs to the lecturer alone. Your figures should have a clear layout with proper axis labels and units, and with a caption explaining what the figure shows. The figures should be referenced in the main text. You are also required to hand in your source code in a form that can be easily compiled. Please don't write your name anywhere. If you do, we will deduct 5 points from your total score.

For this project, you can choose between two alternatives. The first option is a numerical study of inflation. The second is strictly outside the scope of this course, but it will give you insight into gravitational waves from binary black holes.

Alternative 1: Inflation without approximation

In the lectures and in the problems we have studied inflation analytically with the slow-roll approximation. If we forego nice, closed expressions we can however, solve the full equations numerically, and this is what you will do in this project.

I remind you that the Planck energy, Planck mass, and Planck length are defined by, respectively

$$E_{\mathrm{P}}^2 = \frac{\hbar c^5}{G}, \quad m_{\mathrm{P}}^2 = \frac{\hbar c}{G}, \quad l_{\mathrm{P}}^2 = \frac{\hbar G}{c^3}. \tag{1}$$

Assuming spatial flatness and that the scalar field dominates the energy density, the equations governing the evolution of the scalar field and the scale factor are

$$\ddot{\phi} + 3H\dot{\phi} + \hbar c^3 V'(\phi) = 0 \tag{2}$$

$$H^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi)\right]. \tag{3}$$

Before solving these equations numerically it is useful to rewrite them in terms of dimensionless quantities. First, define

$$H_i^2 \equiv \frac{8\pi G}{3c^2} V(\phi_i), \tag{4}$$

where ϕ_i is the initial value of the field, and then introduce the variables

$$\begin{aligned} \tau &= H_i t && (5) \\ h &= \frac{H}{H_i} && (6) \\ \psi &= \frac{\phi}{E_{\mathrm{P}}} && (7) \\ v &= \frac{\hbar c^3}{H_i^2 E_{\mathrm{P}}^2} V && (8) \end{aligned}$$

a) Check that these variables are dimensionless.

b) Show that equations (2) and (3) can be rewritten as

$$\begin{aligned} \frac{d^2\psi}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{dv}{d\psi} &= 0 && (9) \\ h^2 &= \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] && (10) \end{aligned}$$

We need to think about the initial conditions. It is convenient to shift the origin of the time coordinate so we can start at $\tau = 0$. The condition for h is trivial: $h(0) = H(0)/H_i = 1$. For $\psi = \phi/E_{\mathrm{P}}$ we should choose a value that makes sure that we get inflation, and that means that the slow-roll conditions should be fulfilled. But since the equation for the scalar field is a second-order equation, we also need an initial value for $d\psi/d\tau$.

c) Give an argument for why

$$\left(\frac{d\psi}{d\tau}\right)_{\tau=0} = -\frac{1}{3}\left(\frac{dv}{d\psi}\right)_{\psi=\psi_i}, \tag{11}$$

where ψ_i is the inital value of ψ.

We are now ready to look at specific models. Let's try

$$V(\phi) = \frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi^2, \tag{12}$$

from the example starting on page 115 in the lecture notes. Let $mc^2 = 0.01E_\mathrm{P}$.

d) Use the slow-roll conditions to choose an appropriate initial value for the field.

e) Solve equations (9) and (10) numerically and plot the results. Based on the lectures, how would you expect ψ to behave? Does the numerical solution conform with your expectation?

f) In the same plot, plot the slow-roll solution from the lecture notes. When does it start to deviate significantly from the exact, numerical solution?

g) Using the result for h, plot $\ln[a(\tau)/a_i]$, where $a_i = a(\tau = 0)$. (Hint: Start with the definition $H = \dot{a}/a$ and integrate.) Estimate how many e-foldings we get and compare with the slow-roll result.

h) Show that in terms of the dimensionless variables

$$\frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v} \tag{13}$$

i) What would you expect the ratio in equation (13) to be in the slow-roll regime? And in the oscillating phase? Plot the numerical result and compare.

j) Assume $a(t) \propto t^p$ at late times, after the slow-roll conditions break down, and use your numerical results to find the value of p by trial and error (or in a smarter way, if you can think of one). Which value would you expect? Why?

k) Repeat d)-g) with i) $V(\phi) = \lambda\phi^4/(\hbar c)^3$ and ii) $V(\phi) = V_0 e^{-\lambda\phi}$. Compare with results from the lectures and the weekly problems.

Alternative 2: Gravitational waves

If you choose this option, you will work your way through the article ”Gravitational waves from orbiting binaries without general relativity: a tutorial” by R. C. Hilborn (link on the course webpage). The article includes a number of exercises, of which you should do the following:

- Both on page 8
- All on page 10
- The one on page 14
- Both on page 17
- The exercises on page 19, 20, 21, 22
- The first exercise on page 23
- All exercises on page 24, 25, 26, 28, and 29
- The first and the third on page 32
- All on page 33, 34, 37, 38, 39, 40, 42, 43, 48, 50, and 51
- Extra exercise: Reproduce figure 6 in the paper
- Another extra exercise: Explain at least two important differences between the gravitational waves from inflation and the gravitational waves considered in this article