Project 2

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1 Task 1

In this task we will use this equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{1}$$

to show that

$$t = \int_T^{T_0} \frac{s'(T)dT}{s(T)\sqrt{24\pi G(T)}} \tag{2}$$

Rewriting equation 1, we get

$$\left(\frac{1}{a}da\right)^2 = \frac{8\pi G}{3}\rho dt^2 \tag{3}$$

which gives us

$$\int \frac{1}{a}da = \int \sqrt{\frac{8\pi G}{3}}\rho dt \tag{4}$$

$$t = \int \sqrt{\frac{3}{8\pi G\rho}}\frac{1}{a}da \tag{5}$$

Now we need to relate the scale factor to the temperature. This can be found by using the following relation

$$a^3 s(T) = constant \tag{6}$$

We derive this with respect to time and get

$$3a^2\frac{da}{dt}s(T) + a^3\frac{ds(T)}{dt} = 0 \tag{7}$$

sorting the scale factor terms on one side we thus get

$$\frac{1}{a}da = -\frac{ds(T)}{3s(T)} \tag{8}$$

We insert this into equation 5 and get

$$t = -\int \sqrt{\frac{3}{8\pi G\rho}}\frac{ds(T)}{3s(T)}$$

The limits are found by the relation that $t(a_0) = 0$ and $t(T_0) = 0$, and thus we have a relation for the limits of the integral below.

$$t = -\int_{T_0}^{T} \frac{ds(T)}{dT}\frac{dT}{s(T)}\sqrt{\frac{1}{24\pi G\rho}}$$

$$t = \int_T^{T_0} \frac{s'(T)}{s(T)} \frac{dT}{\sqrt{24\pi G\rho}}$$

and we are done.

2 Task 2

It can be shown that the entropy density of a photon gas is given as

$$s(T)_\gamma = \frac{2\pi^2}{45} g_{*s} k_B \left(\frac{k_b T}{\hbar c}\right)^3$$

where g_{*s} is the degrees of freedom for the photons, here equal to 2. We thus have

$$s(T)_\gamma = \frac{4\pi^2}{45} k_B \left(\frac{k_b T}{\hbar c}\right)^3 \tag{9}$$

Now we have to find the entropy density for the electrons and positrons. From the project we have that the entropy density is given as

$$s(T) = \frac{\rho c^2 + P}{T} \tag{10}$$

where

$$\rho c^2 = \int_0^\infty E(p) n(p,T) dp$$

$$P(T) = \int_0^\infty \frac{(pc^2)}{3E(p)} n(p,T) dp$$

$$E(p) = \sqrt{(pc)^2 + (m_e c^2)^2}$$

$$n(p,T) = \frac{4\pi g p^2}{(2\hbar\pi)^2} \frac{1}{\exp\left[E(p)/k_B T\right] + 1}$$

Inserting these four equations into equation 10 we get

$$s(T) = \frac{1}{T} \int_0^\infty \left(E(p) + \frac{pc^2}{3E(p)}\right) n(p,T) dp \tag{11}$$

Here g, the degrees of freedom for the electrons is equal to 2, but we have to consider the positrons as well, so we multiply with another factor of 2. This gives us the following expression for the total entropy density as

$$s(T) = \frac{4\pi^2}{45}k_B\left(\frac{k_bT}{\hbar c}\right)^3 +$$
$$\frac{1}{T}\int_0^\infty \frac{16\pi p^2}{(2\pi\hbar)^3}\left(\sqrt{(pc^2)^2+(m_ec^2)^2} + \frac{(pc^2)}{3\sqrt{(pc^2)^2+(m_ec^2)^2}}\right)\frac{dp}{\exp\left[\sqrt{(pc^2)^2+(m_ec^2)^2}/k_BT\right]+1} \tag{12}$$

Now, we can rewrite this equation as a constant times a function we will call $S(x)$. We start by defining x as $x = \frac{mc^2}{k_BT}$, and then define

$$\sqrt{y^2+x^2} = \frac{E}{k_BT} = \frac{\sqrt{(pc)^2+(mc^2)^2}}{k_BT}$$

Solving for y we find that y is

$$y^2 = \frac{(pc)^2+(mc^2)^2}{(k_BT)^2} - \frac{m^2c^4}{(k_BT)^2}$$
$$y = \frac{pc}{k_BT} \tag{13}$$

Now we can substitute p for y by solving for p and deriving

$$p = \frac{k_BT}{c}y$$
$$dp = \frac{k_BT}{c}dy \tag{14}$$

As mentioned earlier, the rewritten equation is on the form of a constant times another function $S(x)$. The constant we choose is thus the entropy term for the photon gas. Thus, the equation $S(x)$ is on the form $1 + k\int \ldots dy$, where k is a constant. Now lets look at the entropy density term for the electrons and positrons, with some k_BT multiplied in

$$S(T)_e = \frac{1}{T}\int_0^\infty \frac{16\pi p^2 k_BT}{(2\pi\hbar)^3}\left(E/k_BT + \frac{(pc/k_BT)^2}{3E/k_T}\right)\frac{dp}{\exp\left[E/k_BT\right]+1} \tag{15}$$

this can now be rewritten as

$$S(T)_e = \frac{1}{T}\int_0^\infty \frac{16\pi(k_BT)^4}{(2\pi\hbar c)^3}\left(\sqrt{x^2+y^2} + \frac{y^2}{3\sqrt{x^2+y^2}}\right)\frac{dy}{\exp\left[\sqrt{x^2+y^2}\right]+1} \tag{16}$$

$$S(T)_e = \frac{4\pi^2}{45}k_B\frac{(k_BT)^3}{(\hbar c)^3}\frac{45}{2\pi^4}\int_0^\infty \left(\sqrt{x^2+y^2} + \frac{y^2}{3\sqrt{x^2+y^2}}\right)\frac{dy}{\exp\left[\sqrt{x^2+y^2}\right]+1} \tag{17}$$

Now we have the common constant for both the entropy density terms, and we can thus rewrite equation 13 as

$$s(T) = \frac{4\pi^2}{45} k_B \frac{(k_B T)^3}{(\hbar c)^3} \left[1 + \frac{45}{2\pi^4} \int_0^\infty \left(\sqrt{x^2+y^2} + \frac{y^2}{3\sqrt{x^2+y^2}} \right) \frac{dy}{\exp\left[\sqrt{x^2+y^2}\right] + 1} \right] \tag{18}$$

$$= \frac{4\pi^2}{45} k_B \frac{(k_B T)^3}{(\hbar c)^3} S(x = \frac{mc^2}{k_B T}) \tag{19}$$

3 Task 3

In this task we will show the two following relations

$$a \propto \frac{1}{T S^{1/3}(x)} \tag{20}$$

$$T_\nu = A T S^{1/3}(x) \tag{21}$$

We start with equation 6. Here we see that the scale factor is proportional to the entropy density to the one third power. And from the equation above we see that the entropy density is proportional to $T^3 S(x)$, and thus the first relation is proved. It can be shown that the neutrino temperature, $T_\nu \propto \frac{1}{a}$. If we insert equation 20 into this relation, we get that

$$T_\nu \propto T S^{1/3}(x)$$

If we multiply the right side by a constant A, we can thus allow to equate the equation, proving equation 21.

4 Task 4

In this task we will find the constant A from the previous task. We see that T and T_ν are proportional at very high temperatures since they both are inversely proportional to the scale factor. So when T goes to infinity, a goes to zero, and this is the same for the neutrino temperature as well. Now, as T goes to infinity, we see that equation 21 is rerwritten as

$$A = \lim_{T \to \infty} \frac{1}{S^{1/3}(x)} \tag{22}$$

Note that x is inversely proportional to T, so when T goes to infinity, x goes to zero. We then get

$$A = \frac{1}{\left(1 + \frac{45}{2\pi^4} \int_0^\infty (y + \frac{y^2}{3y}) \frac{y^2}{\exp(y)+1} dy\right)^{1/3}} \tag{23}$$

$$A = \frac{1}{\left(1 + \frac{45}{2\pi^4}\int_0^\infty \frac{4y^3}{3\exp{(y)}+1}dy\right)^{1/3}} \tag{24}$$

$$A = \frac{1}{(1 + \frac{45*4*7*\pi^4}{2*\pi^4*3*120})^{1/3}} = \left(\frac{4}{11}\right)^{1/3} \tag{25}$$

So equation 21 can no be written as

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} TS^{1/3}(x) \tag{26}$$

5 Task 5

In this task we will show that the total energy density can be written as

$$\rho c^2_{tot} = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3} + \frac{7}{8}\frac{6\pi^2}{30}\frac{(k_B T_\nu)^4}{(\hbar c)^3} + \int_0^\infty \frac{16\pi p^2}{(2\hbar c)^3}\frac{\sqrt{p^2c^2 + m_e^2c^4}}{\exp{[\sqrt{p^2c^2+m_e^2c^4}]}+1}dp \tag{27}$$

Now we have to consider bosons, fermions and neutrinos, as they all contribute to the energy density. For ultrarelativistic particles, which neutrinos and bosons are during decoupling, the following relation holds.

$$\rho_i c^2 = \frac{\pi^2}{30}g_i\frac{(k_B T)^4}{(\hbar c)^3}\left(\frac{T_i}{T}\right)^4 \tag{28}$$

Here, the notation i describes what kind of particle we are calculating for, for example bosons.

$$\rho c^2_{bosons} = \frac{2\pi^2}{30}\frac{(k_B T)^4}{(\hbar c)^3}$$

$$\rho c^2_{neutrinos} = \frac{7}{8}\frac{6\pi^2}{30}\frac{(k_B T)^4}{(\hbar c)^3}$$

Neutrinos have 6 degrees of freedom, and a factor of 7/8 because they are fermions. For non ultra-relativistic fermions the energy density is given as

$$\rho c^2_{fermions} = \int_0^\infty \frac{2*4\pi p^2 *2}{(2\hbar\pi)^3}\frac{\sqrt{p^2c^2+m_e^2c^4}}{\exp{[\sqrt{p^2c^2+m_e^2c^4}/k_BT]}+1}dp$$

The total energy density is thus

$$\rho c^2_{tot} = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3} + \frac{7}{8}\frac{6\pi^2}{30}\frac{(k_B T)^4}{(\hbar c)^3} + \int_0^\infty \frac{16\pi p^2}{(2\hbar\pi)^3}\frac{\sqrt{p^2c^2+m_e^2c^4}}{\exp{[\sqrt{p^2c^2+m_e^2c^4}/k_BT]}+1}dp \tag{29}$$

Now, we can rewrite this relation as a constant times a function $\epsilon(x)$. We set the constant to be the energy density for the bosons. Now, the remaining two terms have to be altered so that we can pull this constant out. We substitute the neutrino temperature using equation 26, and get

$$\rho c^2_{neutrino} = \frac{7}{8}\frac{6\pi^2}{30}\frac{(k_B T)^4}{(\hbar c)^3}\left(\frac{4}{11}\right)^{4/3} S^{4/3}(x)$$

which can be written as

$$\rho c^2_{neutrino} = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}\frac{21}{8}\left(\frac{4}{11}\right)^{4/3} S^{4/3}(x)$$

The fermion energy density can also be rewritten, using some of the substitutions from task 2.

$$\begin{aligned}\rho c^2_{fermion} &= \int_0^\infty \frac{16\pi p^2}{(2\pi\hbar)^3 c}\frac{k_b T\sqrt{x^2+y^2}}{\exp\left[\sqrt{x^2+y^2}\right]+1}dp \\ &= \int_0^\infty \frac{16\pi(k_B T)^3}{(2\hbar c\pi)^3}\frac{k_b T y^2\sqrt{x^2+y^2}}{\exp\left[\sqrt{x^2+y^2}\right]+1}dy \\ &= \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}\frac{30}{\pi^4}\int_0^\infty \frac{y^2\sqrt{x^2+y^2}}{\exp\left[\sqrt{x^2+y^2}\right]+1}dy\end{aligned}$$

Now we have a common constant in all three energy densities, and we thus get

$$\rho c^2_{tot} = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}\left[1+\frac{21}{8}\left(\frac{4}{11}\right)^{4/3} S^{4/3}(x)+\frac{30}{\pi^4}\int_0^\infty \frac{y^2\sqrt{x^2+y^2}}{\exp\left[\sqrt{x^2+y^2}\right]+1}dy\right] \quad (30)$$

where we define

$$\epsilon(x) = 1+\frac{21}{8}\left(\frac{4}{11}\right)^{4/3} S^{4/3}(x)+\frac{30}{\pi^4}\int_0^\infty \frac{y^2\sqrt{x^2+y^2}}{\exp\left[\sqrt{x^2+y^2}\right]+1}dy \quad (31)$$

6 Task 6

In this task we will show the following relation

$$t = \sqrt{\frac{15\hbar^3}{24\pi^3 G m_e^4 c^3}}\int_{m_e c^2/k_B T_0}^{m_e c^2/k_B T}\left[3-\frac{xS'(x)}{S(x)}\right]\epsilon^{-1/2}(x)x\,dx \quad (32)$$

We showed in task 1 that the time is given as

$$t = \int_T^{T_0} \frac{s'(T)}{s(T)} \frac{dT}{\sqrt{24\pi G\rho}} \tag{33}$$

Here we multiply c so that we get the energy density in the denominator

$$t = \int_T^{T_0} \frac{s'(T)}{s(T)} \frac{dTc}{\sqrt{24\pi G\rho c^2}} \tag{34}$$

Inserting the total energy density from task 5, we thus get

$$t = \frac{1}{\sqrt{24\pi G}} \int_T^{T_0} \frac{s'(T)}{s(T)} \sqrt{\frac{15\hbar c^3}{\pi^2 (k_B T)^4}} \frac{cdT}{\sqrt{\epsilon(x)}} \tag{35}$$

Now we have to find $s'(T)$. This is given as

$$\begin{aligned} s'(T) &= \frac{4\pi^2}{45} k_B 3 \left(\frac{k_B}{\hbar c}\right)^3 T^2 S(x) + \frac{4\pi^2}{45} k_B \left(\frac{k_B T}{\hbar c}\right)^3 S'(x) x'(T) \\ &= \frac{4\pi^2}{45} k_B 3 \left(\frac{k_B}{\hbar c}\right)^3 T^2 S(x) + \frac{4\pi^2}{45} k_B \left(\frac{k_B T}{\hbar c}\right)^3 S'(x) \left(-\frac{m_e c^2}{k_B T^2}\right) \end{aligned}$$

inserting this into the integral we get

$$t = \sqrt{\frac{15\hbar}{24\pi G}} \int_T^{T_0} \left(\frac{3}{T} - \frac{m_e c^2}{k_B T^2} \frac{S'(x)}{S(x)}\right) \sqrt{\frac{c^3}{\pi^2 (k_B T)^4}} \frac{cdT}{\sqrt{\epsilon(x)}} \tag{36}$$

$$t = \sqrt{\frac{15\hbar}{24\pi^3 G}} \int_T^{T_0} \left(3 - x\frac{S'(x)}{S(x)}\right) \frac{c\sqrt{c}}{(k_B T)^2} \frac{cdT}{\sqrt{\epsilon(x)} T} \tag{37}$$

Now, in order to substitute T into x, we differentiate x with respect to T:

$$x = \frac{m_e c^2}{k_B T}$$

$$dx = -\frac{m_e c^2}{k_B T^2} dT = -\frac{x}{T} dT$$

We thus get

$$t = \sqrt{\frac{15\hbar}{24\pi^3 G}} \int_{m_e c^2/k_B T_0}^{m_e c^2/k_B T} \left(3 - x\frac{S'(x)}{S(x)}\right) \epsilon^{-1/2}(x) \frac{m_e^2 c^4 \sqrt{c}}{m_e^2 c^2 (k_B T)^2} \frac{dx}{x} \tag{38}$$

$$t = \sqrt{\frac{15\hbar}{24 m_e^4 \pi^3 G c^3}} \int_{m_e c^2/k_B T_0}^{m_e c^2/k_B T} \left(3 - x\frac{S'(x)}{S(x)}\right) \epsilon^{-1/2}(x) x dx \tag{39}$$

and we are done.

7 Task 7

In the appendix below the code is listed. To ensure that the numerical integration, we can test for two cases. The first case is when x is zero. In task 2 we defined it as being inverse proportional to the temperature, so when x is zero, the temperature goes to infinity. This is not a realistic value, but the integral in $S(x)$ is easy enough to solve. We expect the analytical value to be

$$S(0) = 1 + \frac{45}{2\pi^4}\frac{4}{3}\frac{7\pi^4}{120} = 2.75$$

and we got the same result for the numerical solution, which can be viewed in the terminal when running the code. Testing for the array of temperatures we get the following scatter plot for $S(x)$

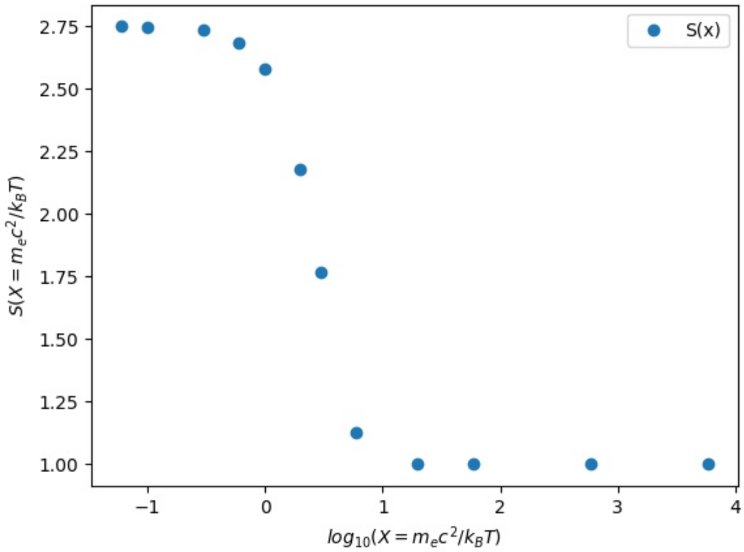

Figure 1: $S(x)$ as a function of X, set in log 10 scale along the x-axis.

8 Task 8

In this task we will calculate the neutrino temperature and the time from task 6. The table of values are below

T(K)	$\mathbf{T}_\nu/\mathbf{T}(K)$	**t(s)**
10^{11}	0.9999	0
$6 * 10^{10}$	0.9997	0.01768
$2 * 10^{10}$	0.9979	0.2394
10^{10}	0.9920	0.9971
$6 * 10^{9}$	0.9786	2.8512
$3 * 10^{9}$	0.9255	12.5832
$2 * 10^{9}$	0.8630	32.0871
10^{9}	0.7432	168.4805
$3 * 10^{8}$	0.7137	1981.6999
10^{8}	0.7137	17780.5375
10^{7}	0.7137	$1.774 * 10^{6}$
10^{6}	0.7137	$1.774 * 10^{8}$