Project 3

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Here are all the tasks from the Gravitational waves paper. The python scripts used will be named with the page number they're linked to. Some of the scripts have multiple tasks in them, and will be named will all the pages relevant.

1 Page 8

1.1 Task 1

We have that

$$\eta = \frac{m_a m_b}{(m_a + m_b)^2}$$

and $M = m_a + m_b$ so we get

$$\eta M = \frac{m_a m_b}{(m_a + m_b)^2}(m_a + m_b) = \frac{m_a m_b}{m_a + m_b} \tag{1}$$

For two equal masses $m_a = m_b$ we have

$$\eta = \frac{m_a^2}{(2m_a)^2} = \frac{m_a^2}{4m_a^2} = \frac{1}{4} \tag{2}$$

For masses $m_a >> m_b$ we have

$$\eta = \frac{m_a m_b}{(m_a + m_b)^2} \approx \frac{m_a m_b}{m_a^2} = \frac{m_b}{m_a} \tag{3}$$

1.2 Task 2

Here we will show that

$$\dot{r} = \frac{-2N(GM)^3\eta}{c^5 r^3} \tag{4}$$

We have that

$$U_{tot} = \frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2 - G\frac{m_a m_b}{r} = -G\frac{m_a m_b}{2r} \tag{5}$$

and

$$\frac{dE_g}{dt} = \frac{NG(\eta M)^2 r^4 \omega^6}{c^5} \tag{6}$$

where

$$\omega^2 = \frac{GM}{r^3}$$

Here the change in gravitational energy in equation 6 is equal to the change in the total energy from equation 5, so we get that

$$\frac{dU_{tot}}{dt} = \frac{dE_g}{dt}$$

$$N\frac{G(\eta M)^2 r^4 \omega^6}{c^5} = \frac{d}{dt}(-G\frac{m_a m_b}{2r}) \tag{7}$$

$$\frac{NG\eta^2 M^2 r^4 (GM)^3}{c^5 r^9} = \frac{G m_a m_b}{2r^2}\dot{r}$$

$$\dot{r} = -\frac{N\eta 2(GM)^3}{c^5 r^3} \tag{8}$$

Note here that all constants are strictly positive, so we have to assign a negative sign in order for the distance to decrease.

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2.1 Task 1

Here I will show that we can rewrite equation 8 as

$$\dot{r} = -\frac{\eta N c}{4}\left[\frac{r_s}{r}\right]^3 \tag{9}$$

where $r_s = \frac{2GM}{c^2}$. We have

$$\dot{r} = -\frac{N\eta 2(GM)^3}{c^5 r^3} \tag{10}$$

$$= -\frac{2^3}{4}\frac{Nc\eta(GM)^3}{c^6 r^3} \tag{11}$$

$$= -\frac{\eta N c}{4}\left[\frac{r_s}{r}\right]^3 \tag{12}$$

2.2 Task 2

Here I will show that

$$r^4(t) = N\eta r_s^3 c(t_c - t) \tag{13}$$

is a solution to equation 9. We have

$$\frac{d}{dt}r^4(t) = 4r^3\dot{r} = \frac{d}{dt}(t_c - t)N\eta r_s^3 c \tag{14}$$

$$\dot{r} = -\frac{\eta N c}{4}\left[\frac{r_s}{r}\right]^3 \tag{15}$$

2.3 Task 3

Here we will show that

$$\begin{align} r^4(t) &= r_i^4 - N\eta r_s^3 c(t - t_i) \tag{16} \\ &= N\eta r_s^3 c(t_c - t) \tag{17} \end{align}$$

$$\begin{align} r^4(t) &= r_i^4 - N\eta r_s^3 c(t - t_i) \tag{18} \\ &= r_i^4 - N\eta r_s^3 ct + N\eta r_s^3 ct_i \tag{19} \end{align}$$

When the two objects collide, at $t = t_c$, we get that r is equal to zero, so we have that

$$r_i^4 - N\eta r_s^3 ct_c + N\eta r_s^3 ct_i = 0$$

$$-r_i^4 + N\eta r_s^3 ct_c = N\eta r_s^3 ct_i$$

We insert this into equation 16 and get

$$\begin{align} r^4(t) &= r_i^4 - N\eta r_s^3 ct + N\eta r_s^3 ct_i \tag{20} \\ &= r_i^4 - N\eta r_s^3 ct - r_i^4 + N\eta r_s^3 ct_c \tag{21} \\ &= N\eta r_s^3 c(t_c - t) \tag{22} \end{align}$$

2.4 Task 4

We see that as $r \to r_s$ we get $\mid \dot{r} \mid \to \eta\frac{8c}{5}$ in GR and $\mid \dot{r} \mid \to \eta\frac{c}{10}$ for EM. The best case for maximum rate of approach is when the masses are about the same, thus $\eta = 1/4$, so we get that $\mid \dot{r} \mid \to \eta\frac{2c}{5}$ in GR and $\mid \dot{r} \mid \to \eta\frac{c}{40}$ in EM, where both are pretty close to the speed of light compared to most moving objects in space.

2.5 Task 5

Keppler's third law tells us that $\omega^2 = GM/r^3$. Using the definition of the Schwartzchild radii, we get that

$$\omega(t) = \sqrt{\frac{GM}{r(t)^3}} = \sqrt{\frac{c^2 r_s}{2r(t)^3}} = \frac{c}{\sqrt{2}}\sqrt{\frac{r_s}{r(t)^3}} \tag{23}$$

2.6 Task 6

The velocity of an orbiting body in a binary is given as $v = \omega r/2$. Inserting from equation 23 we get

$$v = \frac{\omega r}{2} = \frac{c}{2\sqrt{2}}\sqrt{\frac{r_s}{r^3}}r \tag{24}$$

Now, the distance from the the center axis of the orbit will be $r/2$, since r is the separation distance between them, so we thus get

$$v = \frac{c}{2\sqrt{2}}\sqrt{\frac{r_s}{(r/2)^3}}r = \frac{c\sqrt{8}}{\sqrt{8}}\sqrt{\frac{r_s}{r}} \tag{25}$$

As $r \to r_s$ we then get

$$v = c \tag{26}$$

which in non-relativistic mechanics is allowed.

3 Page 14

3.1 Task 1

We start with the absolute value of the Coloumb force

$$\mid F_c \mid = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2} \tag{27}$$

Using the substitutions of $-G = \frac{1}{4\pi\epsilon}$ and $m_a m_b = Q_1 Q_2$ we thus get

$$\mid F_G \mid = -G\frac{m_a m_b}{r^2} \tag{28}$$

which is Newton's law of gravitation. Note the minus sign, as gravity is an attractive force, and since as far as we know there is no negative mass[1] we need the negative sign. The Coloumb force however takes this into account as one uses charges with either positive or negative charge, leading to equal charges repelling and opposites attracting.

[1] Anti-matter would have opposite charge but same sign in mass

3.2 Task 2

In a static system, the change in motion is zero, and so the relativistic effects do not appear. Thus, the retarded time correction is not necessary.

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4.1 Task 1

When you go swimming in the ocean, you might find waves crushing in to shore. Most of us as children then loved to jump into them, being pushed back. Here the water in and of it self does not move in the plane, they follow curve of the wave, moving up and down, but moving the energy forward, eventually hitting you or me or any other child, pushing us backwards.

Another example of waves carrying energy is the fact that sunlight, being electromagnetic waves, tan or if one is not cautious, burns the skin. The waves hits the skin with a given energy, and if it is high enough, it will tan or even burn the skin. This can in some cases ionize the skin cells, thus allowing for tumors or cancer cells to grow. That should be the on every sunscreen bottle, be aware of energy-filled waves from the sky.

4.2 Task 2

4.2.1 A

Let's assume that

$$S = \frac{1}{2}\vec{E}^2 + \frac{\vec{B}^2}{\mu_0} \tag{29}$$

is the energy density. To find the energy contained in the cylinder at some given time, we multiply S by the crossection of the cylinder

$$E_{em} = \left[\frac{1}{2}\vec{E}^2 + \frac{\vec{B}^2}{\mu_0}\right]\pi r^2 h \tag{30}$$

4.2.2 B

If we assume that the waves move along the center axis of the cylinder, at the speed of light, then the time it takes to move from one end to another is given as

$$t = \frac{h}{c} \tag{31}$$

where h is the length of the cylinder, and c is the speed of light.

4.2.3 C

Power is energy per unit time, so if we use the relation from before we get

$$\left[\frac{1}{2}\vec{E}^2 + \frac{\vec{B}^2}{\mu_0}\right]\frac{\pi r^2 h}{h/c} = \left[\frac{c}{2}\vec{E}^2 + \frac{\vec{B}^2}{\mu_0}\right]\pi r^2 = S_{EM}\pi r^2 \tag{32}$$

Dividing by πr^2 we thus get that S_{EM} is power per unit area

4.3 Task 3

Here we will verify that

$$< E^2 \sin^2(\omega t) >= \frac{1}{2}E^2 \tag{33}$$

We have that

$$< E^2 \sin^2(\omega t) >= E^2 < \sin^2(\omega t) > \tag{34}$$

The average of any linear sine or cosine term over a period of 2π is zero so we get

$$\begin{aligned} < E^2 \sin^2(\omega t) > &= E^2 < 1/2 - \frac{\cos(2\omega t)}{2} > & (35)\\ &= E^2 \left[< 1/2 > - < \frac{\cos(2\omega t)}{2} >\right] & (36)\\ &= \frac{E^2}{2} & (37)\end{aligned}$$

4.4 Task 4

Magnetic field strength has the units $B = kgs^{-2}A^{-1}$. Electric field strength divided by c is given as $E/c = \frac{kgms^{-3}A^{-1}}{m/s} = kgs^{-2}A^{-1}$, which is the same as for magnetic field strength.

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We have that the distance R_a is given as

$$R_a = R\sqrt{1 + (\frac{r_a}{R})^2 - 2\frac{\vec{R}\cdot\vec{r_a}}{R^2}} \tag{38}$$

First we envoke the fact that $R >> r_a, r_b$, which gives us

$$R_a = R\sqrt{1 - 2\frac{\vec{R}\cdot\vec{r_a}}{R^2}}$$

Second, we introduce the unit vector $\hat{R} = \vec{R}/R$

$$R_a = R\sqrt{1 - 2\frac{\hat{R}\cdot\vec{r_a}}{R}}$$

Lastly we will use the binomial relation that $(1+x)^n = 1 + nx + ...$ to get

$$R_a = R\sqrt{1 - 2\frac{\hat{R}\cdot\vec{r_a}}{R}} = R\left(1 - 2\frac{\hat{R}\cdot\vec{r_a}}{R}\right)^{1/2} \approx R - \frac{1}{2}2\hat{R}\cdot\vec{r_a} = R - \hat{R}\cdot\vec{r_a}$$

This approximation will yield the same result for r_b.

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We have the gravitational vector potential

$$\vec{A_g} = -\frac{G}{c^2}\frac{m_a\vec{v_a}(t_{Ra})}{R_a} - \frac{G}{c^2}\frac{m_b\vec{v_b}(t_{Rb})}{R_b} \tag{39}$$

where t_{Ra}, t_{Rb} is the retarded time for the respective objects a, b. Using the approximation that

$$v_a(t_{Ra}) = v(t_R) + \frac{dv_a(t_R)}{dt}\Delta t_a \tag{40}$$

and the fact that $R \approx R_a \approx R_b$ we get

$$\begin{aligned}
\vec{A_g} &= -\frac{G}{c^2R}\left[mv_a(t_R) + m_a\frac{dv_a(t_R)}{dt}\Delta t_a\right] - \frac{G}{c^2R}\left[m_bv_b(t_R) + m_b\frac{dv_b(t_R)}{dt}\Delta t_b\right] \\
&= -\frac{G}{c^2R}\left[m_av_a(t_R) + m_bv_b(t_R)\right] + \frac{G}{c^2R}\left[m_a\frac{dv_a(t_R)}{dt}\Delta t_a + m_b\frac{dv_b(t_R)}{dt}\Delta t_b\right]
\end{aligned}$$

Now we use the two relations

$$\frac{dv_a}{dt} = -\omega^2\vec{r_a}$$

and

$$\Delta t_a = \frac{\hat{R}\cdot\vec{r_a}}{c}$$

we get

$$\vec{A_g} = -\frac{G}{c^2R}\left[m_av_a(t_R) + m_bv_b(t_R)\right] + \frac{G\omega^2}{c^3R}\left[m_a\vec{r_a}(\hat{R}\cdot\vec{r_a}) + m_b\vec{r_b}(\hat{R}\cdot\vec{r_b})\right] \tag{41}$$

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Let's expand on the gravitational vector potential.

$$\vec{A}_g = -\frac{G}{c^2R}\left[m_a v_a(t_R) + m_b v_b(t_R)\right] + \frac{G\omega^2}{c^3R}\left[m_a\vec{r}_a(\hat{R}\cdot\vec{r}_a) + m_b\vec{r}_b(\hat{R}\cdot\vec{r}_b)\right] \tag{42}$$

The first term can be ignored as the net sum of momentum for a binary is a constant, and so when we find the time derivative, this term disappears, which gives us

$$\vec{A}_g = \frac{G\omega^2}{c^3R}\left[m_a\vec{r}_a(\hat{R}\cdot\vec{r}_a) + m_b\vec{r}_b(\hat{R}\cdot\vec{r}_b)\right] \tag{43}$$

Now, we will introduce two new vectors

$$\vec{r}_a = \frac{m_b\vec{r}}{m_a + m_b} \tag{44}$$

and

$$\vec{r}_b = -\frac{m_a\vec{r}}{m_a + m_b} \tag{45}$$

Inserting these into the equation we get above we get

$$\begin{aligned}
\vec{A}_g &= \frac{G\omega^2}{c^3R}\left[m_a\vec{r}_a(\hat{R}\cdot\vec{r}_a) + m_b\vec{r}_b(\hat{R}\cdot\vec{r}_b)\right] && (46)\\
&= \frac{G\omega^2}{c^3R}\left[\frac{m_a m_b}{ma+mb}\frac{\vec{r}(\hat{R}\cdot m_b\vec{r})}{m_a+m_b} - \frac{m_a m_b}{ma+mb}\frac{(-1)\vec{r}(\hat{R}\cdot m_a\vec{r})}{m_a+m_b}\right] && (47)\\
&= \frac{G\omega^2}{c^3R}\frac{m_a m_b}{(m_a+m_b)^2}(m_a+m_b)\vec{r}(\hat{R}\cdot\vec{r}) && (48)\\
&= \frac{G\omega^2}{c^3R}\eta M\vec{r}(\hat{R}\cdot\vec{r}) && (49)
\end{aligned}$$

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We now introduce r as a vector with angular components

$$\vec{r} = r[\hat{x}\cos(\omega t_R) + \hat{y}\sin(\omega t_R)] \tag{50}$$

which gives us our gravitational vector potential as

$$\vec{A}_g = \frac{G\eta M\omega^2 r^2}{c^3R}[[\hat{x}\cos(\omega t_R) + \hat{y}\sin(\omega t_R)]\hat{R}\cdot\hat{x}\cos(\omega t_R)] \tag{51}$$

We can rewrite this using the relation $n_x = \hat{R}\cdot\hat{x}$ to get

$$\vec{A}_g = \frac{G\eta M\omega^2 r^2}{2c^3R}[\hat{x}n_x(1+\cos(2\omega t_R)) + \hat{y}n_x\sin(2\omega t_R)] \tag{52}$$

Now, by using the fact that $\hat{x} = \sin(\theta_i)\hat{R} + \cos(\theta_i)\hat{e_1}$, $\hat{y} = \hat{e_2}$ and $n_x = \sin(\theta_i)$ we get

$$\vec{A}_g = \frac{G\eta M\omega^2 r^2}{2c^3 R}\sin(\theta_i)[\sin(\theta_i)\hat{R}(1+\cos(2\omega t_R)) + (1+\cos(2\omega t_R))\cos(\theta_i)\hat{e_1} + \sin(2\omega t_R)\hat{e_2}] \tag{53}$$

If we now only focus on the tranverse terms we get

$$\vec{A}_{g,trans}(\vec{R},t) = \frac{G\eta M\omega^2 r^2}{2c^3 R}\sin(\theta_i)[\cos(\theta_i)(1+\cos(2\omega t_R))\hat{e_1} + \sin(2\omega t_R)\hat{e_2}] \tag{54}$$

To find the gravitational vector field $\vec{g}_{rad}$ we take the time derivative of the equation above

$$\begin{aligned}
\vec{g}_{rad} &= -\frac{\partial \vec{A}_{g,trans}}{\partial t} && (55)\\
&= -\frac{G\eta M\omega^2 r^2}{2c^3 R}\sin(\theta_i)[-2\omega\cos(\theta_i)\sin(2\omega t_R)\hat{e_1} + 2\omega\cos(2\omega t_R)\hat{e_2}] && (56)\\
&= \frac{G\eta M\omega^3 r^2}{2c^3 R}[2\sin(\theta_i)\cos(\theta_i)\sin(2\omega t_R)\hat{e_1} - 2\sin(\theta_i)\cos(2\omega t_R)\hat{e_2}] && (57)\\
&= \frac{G\eta M\omega^3 r^2}{2c^3 R}[\sin(2\theta_i)\sin(2\omega t_R)\hat{e_1} - 2\sin(\theta_i)\cos(2\omega t_R)\hat{e_2}] && (58)\\
& && (59)
\end{aligned}$$

Which is what we were supposed to get.

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9.0.1 A

We have that

$$\omega(t) = \sqrt{\frac{GM}{r^3(t)}}$$

Here we want to find the angular frequency of the binary stars, with total mass of 20 solar masses, and estimated that the separation of the stars is approximately the sum of their radii. Now, if we assume that mass $m = d * V$, where d is density and V is volume, we then know that $m \propto r^3$, so the radii could be approximated to be about a third of the sun radii. This gives us

$$\omega(t) = \sqrt{\frac{GM}{r^3(t)}} = \sqrt{\frac{G20M_\odot}{(2*(10)^{1/3}_{r_\odot})^3}} = 3.139 * 10^{-4} rad/s$$

9.0.2 B

To find the amplitude of the gravitational vector field, we calculate the constants outside the angular terms. We know the binary is about $R = 1000$ light years away, $\eta = 1/4$, and using the information found in the last task, we insert into equation 55 and we get

$$\frac{G\eta M\omega^3 r^2}{2c^3 R} = 3.617 * 10^{-16} m/s^2 \approx 10^{-16} m/s^2 \tag{60}$$

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$$< \vec{S_G} >= \frac{c}{8\pi G} < \vec{g_{rad}}^2 + c^2 \vec{B_{G\,rad}}^2 > \hat{z} \tag{61}$$

Using the relation that $\vec{g_{rad}}^2 = c^2 \vec{B_{G\,rad}}^2$, we get

$$\begin{aligned}
< \vec{S_G} > &= \frac{c}{8\pi G} < \vec{g_{rad}}^2 + \vec{g_{rad}}^2 > \hat{z} &(62)\\
&= \frac{c}{4\pi G} < \vec{g_{rad}}^2 > &(63)\\
&= \frac{c}{4\pi G} < \vec{g_{rad}} \cdot \vec{g_{rad}} > &(64)\\
&= \frac{c}{4\pi G} < \frac{G^2(\eta M)^2 \omega^6 r^4}{4c^6 R^2}[(\sin(2\theta_i)\sin(2\omega t_R))^2 + (2\sin(\theta_i)\cos(2\omega t_R)^2] > &(65)\\
&= \frac{G(\eta M)^2 \omega^6 r^4}{16 c^5 R^2} < \frac{1}{2}\sin(2\theta_i) + \frac{4}{2}\sin^2(\theta_i) > &(66)\\
&= \frac{G(\eta M)^2 \omega^6 r^4}{32 c^5 R^2}[\sin(2\theta_i) + 4\sin^2(\theta_i)] &(67)
\end{aligned}$$

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The integral is given as

$$I = R^2 \int_0^{2\pi} d\phi \int_0^{\pi} [(\sin^2(2\theta) + 4\sin^2(\theta)] \sin(\theta)\, d\theta \tag{68}$$

Solving we get

$$
\begin{aligned}
I &= R^2 2\pi \int_0^{\pi} [(\sin^2(2\theta) + 4\sin^2(\theta)] \sin(\theta)\, d\theta && (69) \\
&= R^2 2\pi \left[\int_0^{\pi} \sin^2(2\theta)\sin(\theta)\, d\theta + \int_0^{\pi} 4\sin^3(\theta)\, d\theta \right] && (70) \\
&= R^2 2\pi \left[\int_0^{\pi} 4\sin^2(\theta)\cos^2(\theta)\sin(\theta)\, d\theta + \int_0^{\pi} 4\sin^3(\theta)\, d\theta \right] && (71) \\
& && (72)
\end{aligned}
$$

We solve the two integrals separate.

$$
\begin{aligned}
I_1 &= \int_0^{\pi} 4\sin^2(\theta)\cos^2(\theta)\sin(\theta)\, d\theta && (73) \\
&= -4\int_0^{\pi} \sin(\theta)(\cos^2(\theta) - 1)\cos^2(\theta)\, d\theta && (74) \\
& && (75)
\end{aligned}
$$

Using the substitution $u = -\cos(\theta)$ and $du = -sin(\theta)d\theta$ we get

$$
\begin{aligned}
I_1 &= 4\int u^2(u^2 - 1)\, du \\
&= \int (u^4 - u^2)\, du \\
&= \left[\frac{u^5}{5} - \frac{u^3}{3} \right] \\
&= \frac{(-\cos(\theta))^5}{5} - \frac{(-\cos(\theta))^3}{3} \Bigg|_0^{\pi} \\
&= -\frac{\cos^5(\theta)}{5} - \frac{\cos^3(\theta)}{3} \Bigg|_0^{\pi} \\
&= \frac{16}{15}
\end{aligned}
$$

The second integral is

$$
\begin{aligned}
I_2 &= \int_0^{\pi} 4\sin^3(\theta)\, d\theta \\
&= \int_0^{\pi} (1 - \cos^2(\theta))\sin(\theta)\, d\theta
\end{aligned}
$$

using the substitution $u = \cos(\theta)$ and $du = -\sin(\theta)$ we get

$$
\begin{aligned}
I_2 &= -4 \int (1 - u^2)\, du \\
&= 4 \left[\frac{u^3}{3} - u \right] \Bigg|_0^{\pi} \\
&= 4 \left[\frac{\cos(\theta)^3}{3} - \cos(\theta) \right] \Bigg|_0^{\pi} \\
&= \frac{16}{3}
\end{aligned}
$$

Our integral is thus

$$I = 2\pi R^2 (I_1 + I_2) = 2\pi R^2 \frac{96}{15} = \frac{64}{5}\pi R^2 \tag{76}$$

which is what we needed to prove.

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12.1 Task 1

We have ηM which is the reduced mass and $\omega^2 r^2$ which is velocity squared. We see thus that the term is proportional to kinetic energy of the binary system.

12.2 Task 2

We have that

$$\frac{c^5}{G} = \frac{m^5}{s^5} \frac{kgs^2}{m^3} = \frac{kgm^2}{s^3} = \frac{J}{s} \tag{77}$$

Inserting the numerical value for c and G, we get that

$$L_0 = \frac{c^5}{G} = 3.628 * 10^{52} J/s \tag{78}$$

12.3 Task 3

Rewriting the equation from the task, we get

$$\frac{dE_g}{dt} \approx \left(\frac{\eta M}{r} \right)^2 v^6 \frac{G}{c^5}$$

Here we see that the gravitational wave energy radiated away goes as $1/r^2$, which is proportional to a gravitational force.

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We have the following relation

$$\frac{r(t)}{r_s} = \left[\left(\frac{r_i}{r_s} \right)^4 - \frac{Nc\eta(t - t_i)}{r_s} \right]^{1/4} \tag{79}$$

we know that when $t = t_* \; r(t_*) = r_s$ so we get

$$\frac{r(t_*)}{r_s} = 1 = \left[\left(\frac{r_i}{r_s} \right)^4 - \frac{Nc\eta(t_* - t_i)}{r_s} \right]^{1/4} \tag{80}$$

$$1 = \left(\frac{r_i}{r_s} \right)^4 - \frac{Nc\eta(t_* - t_i)}{r_s}$$

$$\left(\frac{r_i}{r_s} \right)^4 - 1 = \frac{Nc\eta(t_* - t_i)}{r_s}$$

$$t_* - t_i = \frac{1}{N\eta c} \left[\left(\frac{r_i}{r_s} \right)^4 r_s - r_s \right]$$

$$t_* = t_i + \frac{r_s}{N\eta c} \left[\left(\frac{r_i}{r_s} \right)^4 - 1 \right] \tag{81}$$

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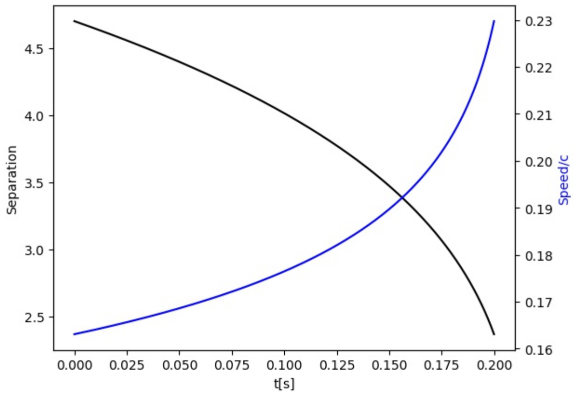
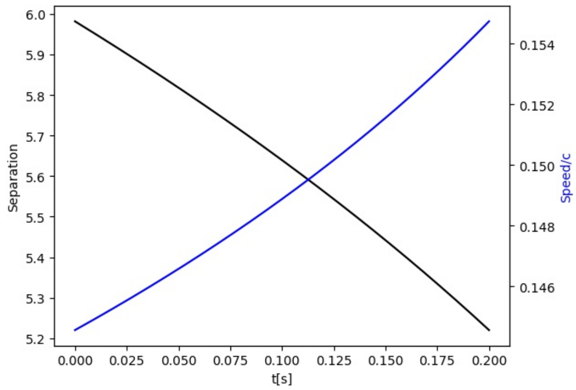
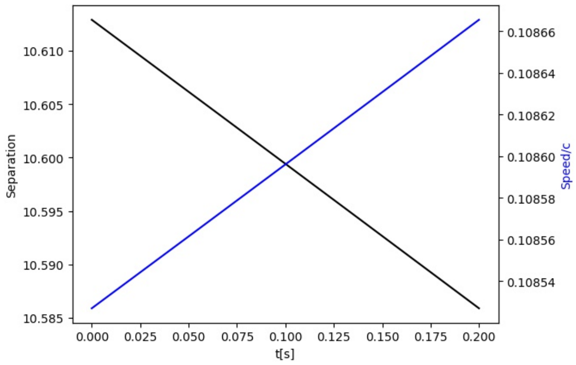
When η changes, the schwartzchild radii changes, and thus the velocity and separation changes. In the short time span it hard to know how fast they will go, but we expect that as they get closer and closer, the velocity increases significantly, and in the first image.



(a) Separation and velocity for two $35M_\odot$ black holes, with $\eta = 1/4$



(b) Separation and velocity for a $20M_\odot$ and $35M_\odot$ black hole, with $\eta = 0.23$



(c) Separation and velocity for a $1M_\odot$ and $30M_\odot$ black hole, with $\eta = 0.03$

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15.0.1 Task 1

The term for the gravitational field radiated is given as

$$\vec{g}_{rad} = \frac{G\eta M\omega^3 r^2}{2c^3 R}[\sin(2\theta_i)\sin(2\omega t_R)\hat{e_1} - 2\sin(\theta_i)\cos(2\omega t_R \hat{e_2}] \tag{82}$$

Dotting with $\hat{x}$ gives us contribution along $\vec{R}$, which gives us

$$\vec{g}_{rad} \cdot \hat{x} = \frac{G\eta M\omega^3 r^2}{2c^3 R}\sin(\beta)\sin(2\theta_i)\sin(2\omega t_R) \tag{83}$$

The units are

$$\frac{m^3 kg m^2 s^3}{s^2 kg s^3 m m^3} = \frac{m}{s^2}$$

which are units of acceleration.

15.1 Task 3

The differential equation for a sinusoidally-driven damped oscillator is given as

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega t) \tag{84}$$

We know that the driving frequency 2ω is much larger that the resonance frequency. For a damped oscillator this means that the position of the moving object barely changes. One can think of this as the force switching direction so fast that the system does not have time to react to it. Thus, the position can be cancelled out as the net sum og position will be about zero. When this happens the derivative term also goes to zero, because the net change is so small. we thus get

$$\frac{d^2x}{dt^2} = \frac{F_0}{m}\cos(\omega t) \tag{85}$$

which is on the same form as our expression in equation 83.

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16.1 Task 1

We have that

$$\sin(2\omega t) - \sin(2\omega t - 2\omega\Delta t) = \sin(2\omega t) - \sin(2\omega t)\cos(2\omega\Delta t) + \cos(2\omega t)\sin(2\omega\Delta t) \tag{86}$$

$$= \cos(2\omega t)\sin(2\omega\Delta t) \tag{87}$$

$$\approx 2\omega\Delta t\cos(2\omega t) \tag{88}$$

Here we used the small angle approximation for cosine. Using the LIGO observatory, we get that

$$2\omega\Delta t = 2\omega\frac{\Delta R}{c} \tag{89}$$

$$= 2\omega\frac{4km}{c} \tag{90}$$

$$= \frac{2\pi 200 * 4km}{c} \tag{91}$$

$$= 0.03353 << 1 \tag{92}$$

16.2 Task 2

Using the calculation from the last task we get

$$\begin{aligned}
\Delta L &= \Delta x_2(t) - \Delta x_1(t) && (93)\\
&= -\frac{G\eta M\omega r^2}{8c^3R}\sin(\beta)\sin(2\theta_i)[\sin(2\omega t) - \sin(2\omega(t-\Delta t))] && (94)\\
&= -\frac{G\eta M\omega r^2}{8c^3R}\sin(\beta)\sin(2\theta_i)2\omega\Delta t\cos(2\omega t) && (95)\\
&= -\frac{G\eta M\omega^2 r^2}{4c^3R}\sin(\beta)\sin(2\theta_i)\Delta t\cos(2\omega t) && (96)
\end{aligned}$$

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We have that

$$h(t) = \frac{\Delta L_x(t) - \Delta L_y(t)}{L} = \frac{\eta(GM)^{5/3}\omega^{2/3}}{2c^4R}\sin(\beta)\cos(\beta)\sin(2\theta_i)\cos(2\omega t) \quad (97)$$

We also have the definition of the Schwartzchild radii defined in section 2. We thus get that the amplitude of the strain signal is given as

$$\mid h(t) \mid = \frac{(GM)^{-1/3}\eta\omega^{2/3}r_s^2}{8R} \quad (98)$$

Using Kepler's third law, we get $(\omega^2)^{1/3} = \left(\frac{GM}{r^3}\right) = \left(\frac{(GM)^{1/3}}{r}\right)$, which gives us

$$\begin{aligned}
\mid h(t) \mid &= \frac{(GM)^{-1/3}\eta\omega^{2/3}r_s^2}{8R} && (99)\\
&= \frac{(GM)^{-1/3}\eta(GM)^{1/3}r_s^2}{8Rr} && (100)\\
&= \frac{\eta r_s^2}{8Rr} && (101)
\end{aligned}$$

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18.1 Task 1

We will show here we can derive

$$\omega(t) = \frac{1}{\sqrt{2}\tau_{chirp}}\left[N\frac{t_c - t}{\tau_{chirp}}\right]^{-3/8} \quad (102)$$

from

$$\dot{\omega} = \frac{3N\eta(GM^{5/3})\omega^{11/3}}{c^5} \tag{103}$$

We have

$$\int_{\omega(t)}^{\infty} \omega^{-11/3}\, d\omega = \int_{t_c}^{t} \frac{3N\eta(GM)^{5/3}}{c^5}\, d\tau \tag{104}$$

$$\omega^{-8/3} = \frac{8\eta(GM)^{5/3}}{c^5}(t_c - t)N$$

$$\omega = \frac{8^{-3/8}\eta^{-3/8}(GM)^{-5/8}}{c^{-5*3/8}}[(t_c - t)N]^{-3/8}$$

Now, we have that $\tau_{chirp} = 2GM_{chirp}/c^3$, and that $M_{chirp} = \eta^3 M^5$. We then get

$$\begin{aligned}\left[\frac{2^4 2^5 (GM)^5 \eta^3}{c^{15}}\right]^{-1/8} &= \frac{1}{\sqrt{2}}\left[\frac{2^5 M_{chirp}^5 G^5}{c^{15}}\right]^{-1/8} \\ &= \frac{1}{\sqrt{2}}\tau_{chirp}^{-5/8} \\ &= \frac{1}{\sqrt{2}}\frac{1}{\tau_{chirp}}\left[\frac{1}{\tau_{chirp}}\right]^{-3/8}\end{aligned}$$

We thus get

$$\omega(t) = \frac{8^{-3/8}\eta^{-3/8}(GM)^{-5/8}}{c^{-5*3/8}}[(t_c - t)N]^{-3/8} \tag{105}$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\tau_{chirp}}\left[\frac{(t_c - t)N}{\tau_{chirp}}\right]^{-3/8} \tag{106}$$

Now, we can use this equation to find $\phi(t)$ given as

$$\phi(t) = -\frac{1}{5}\left[\frac{16}{N^{3/5}}\right]^{5/8}\left[\frac{t_c - t}{\tau_{chirp}}\right]^{5/8} + \phi_c$$

We start with

$$\phi(t) = \int_{t_0}^{t} \omega(\tau)\, d\tau \tag{107}$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{\tau_{chirp}}\right]^{5/8} N^{-3/8} \int_{t_0}^{t} (t_c - t)^{-3/8}\, d\tau \tag{108}$$

$$= -\frac{1}{5} \left[\frac{1}{\tau_{chirp}}\right]^{5/8} N^{-3/8} 2^{5/2} \left[(t_c - t)^{5/8} - (t_c - t_0)^{5/8}\right] \tag{109}$$

$$= -\frac{1}{5} \left[\frac{1}{\tau_{chirp}}\right]^{5/8} \left[\frac{16}{N}\right]^{-3/8} \left[(t_c - t)^{5/8} - (t_c - t_0)^{5/8}\right] \tag{110}$$

$$= -\frac{1}{5} \left[\frac{1}{\tau_{chirp}}\right]^{5/8} \left[\frac{16}{N}\right]^{-3/8} (t_c - t)^{5/8} - \phi_c \tag{111}$$

18.2 Task 2

Using that $N = 32/5$ we get that the numerical factors is

$$\begin{aligned}
\frac{1}{5}\left[\frac{16}{(32/5)^{3/5}}\right]^{5/8} &= \frac{1}{5}[2 * 5^{3/5}]^{5/8} \\
&= \frac{1}{5}2^{5/8}5^{3/8} \\
&= 2^{5/8}5^{-8/8+3/8} \\
&= 2^{5/8}5^{-5/8} \\
&= \left[\frac{2}{5}\right]^{5/8}
\end{aligned}$$

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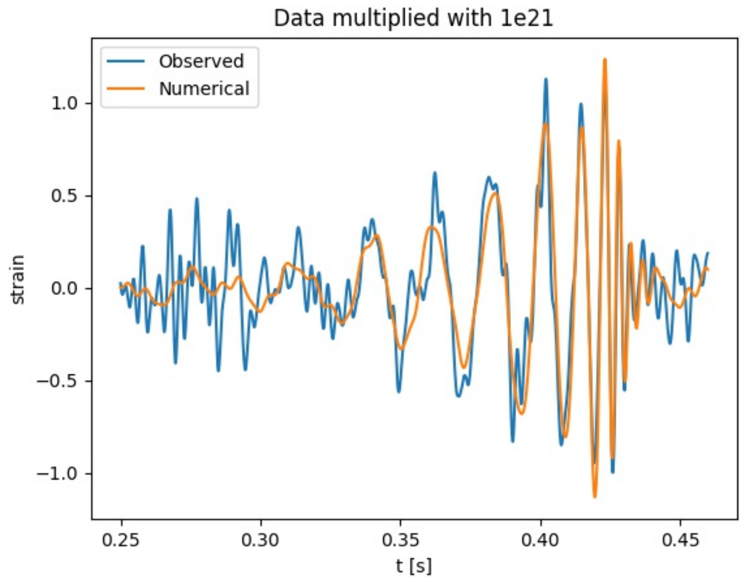
Using the observed and numerically found data from Hanford, Washington, we get the following image

Figure 2: Strain data from



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In this task we will use the orbital frequency to determine the mass of the two objects. We used the following five images to determine the average frequency for the first two periods. We found that $\omega \approx 263 rad/s$. Using this data we found that the total chirp mass is $M_{chirp} = 23.13 M_\odot$, which with an $\eta = 1/4$ gives us the the chirp mass of each black hole is half the total mass, $M_{chirp,a} = M_{chirp,b} = 11.5 M_\odot$. This is a bit lower than what was used by the simulation, but there are multiple reasons why we get this estimate. It is very hard to estimate the mass of black holes far away, but it is remarkable that a model like this can get relatively close to the actual mass.

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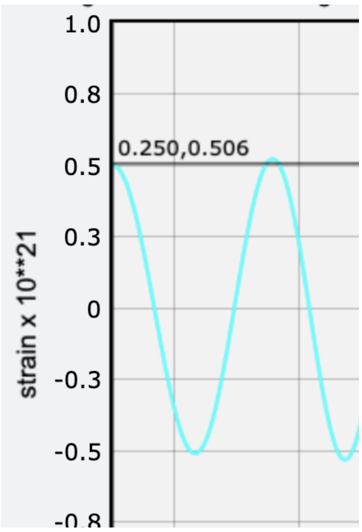
The separation of the binary system is measured to be

- EM: $r = 1.719 * 10^7 m$
- GR: $r = 1.015 * 19^6 m$

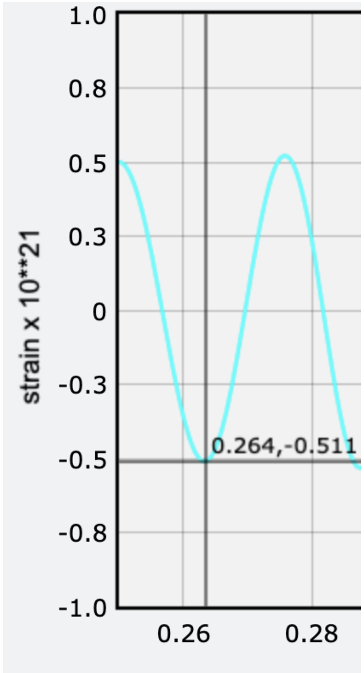
This separation is, by comparison to the sun's diameter

- EM: $r = 1.23 * 10^{-3} sun\, diameter$
- GR: $r = 7.29 * 10^{-4} sun\, diameter$

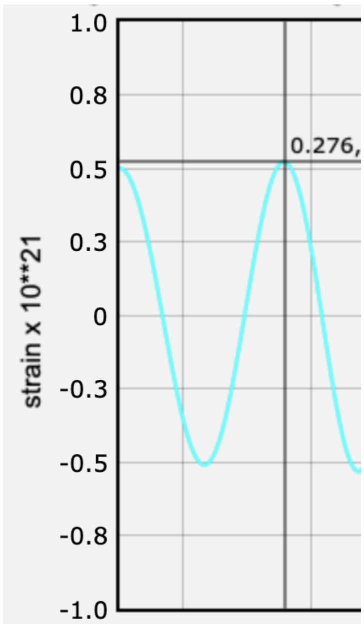
which is significantly less than the diameter of the sun.



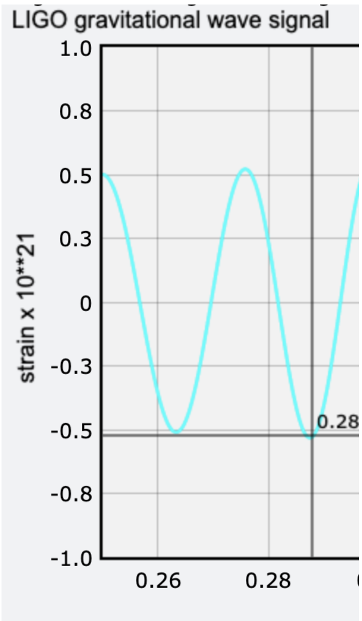
(a) Start point for first period



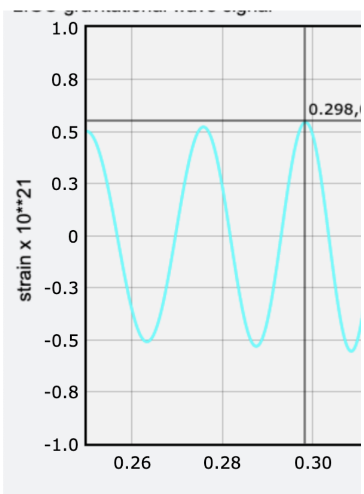
(b) Start time for change in angular frequency



(c) End point for first period, and start point for second period



(d) End time for change in angular frequency



(e) End point for second period

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22.1 Task 1

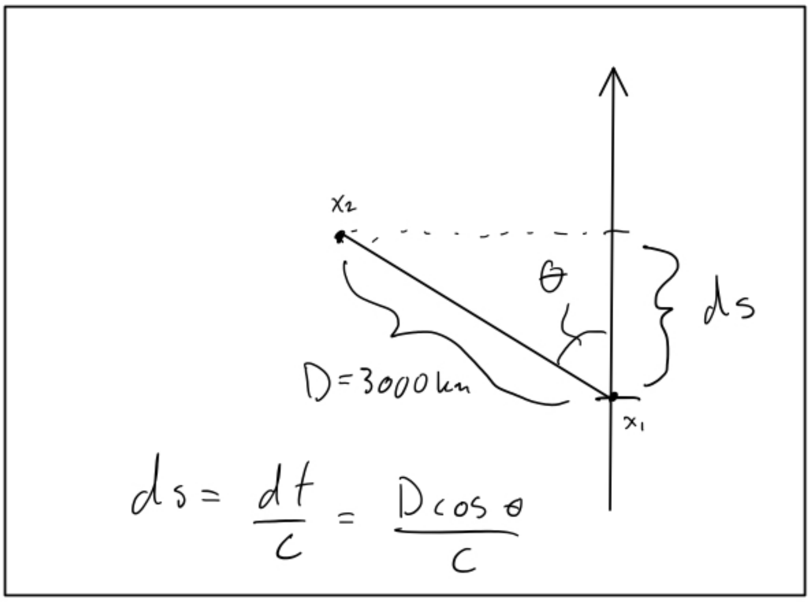
The estimated distance of the binary is about 1.3 billion light years away. This corresponds to about 400 Mpc. The distance is also about 7000 Milky ways away, outside the Supercluster that the local group is a member of. To scale that even further, the Milky Way is one of many members of the local group.

22.2 Task 2

The luminosity of the sun and Sagitarius A, the supermassive black hole are both large. They are however very different systems than the black hole binary. The sun radiates energy almost exclusively through electromagnetic radiation, and given that the sun is a very average star, and in the middle of its life span, its limunosity is not extreme, yet. The supermassive black hole at the center of the Milky Way is indeed massive, but stable. There is mass moving into it from the surroundings, but this is not enough to create extreme luminosity either. The binary black holes however, are massive in size, moving very fast in declining orbit. Because they are so massive and move at relativistic speed, they loose a lot of mass radiated away through the gravitational waves.

22.3 Task 3

Figure 4: x_1 and x_2 are the LIGO observatories in the United States and the arrow shows the direction of the waves propagating through space.



If the signal takes 6.9 ms, and the distance between the two observatories is 3000 km we get that the angle θ is given as

$$\theta = \arccos(\frac{c}{D} 6.9ms) \approx 46^\circ \tag{112}$$

22.4 Task 4

The longest time the signal can take is if the cosine term is maximized, which is when the angle is zero degrees. This gives us a time delay of

$$dt = \frac{D}{c}\cos(\theta = 0) = \frac{D}{c} = \frac{1}{100}s \tag{113}$$

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We have that the magnitude of the gravitational vector field is given as

$$\vec{g}_{rad} = \frac{G\eta M\omega^2 r^2}{2c^3 R}[\sin(2\omega t_R)\hat{e_1} - \sqrt{2}\cos(2\omega t_R)\hat{e_2}] \tag{114}$$

solving for the distance R we get

$$\mid R \mid = \frac{G\eta M\omega^2 r^2}{2c^3 \mid \vec{g}_{rad} \mid} \tag{115}$$

Using the data from the GW150914 binary we get that the distance for which the gravitational magnitude is the same as earth is 0.25 AU. At this distance the gravitational acceleration switches back and forth with earth acceleration, which would be quite unpleasant. From equation 114 we see that the change in gravitational vector field goes as ω^3, so it will change quite rapidly as the two black holes gets closer and closer. It is quite safe to say that even in a stable orbit around the the binary, one would not be alive for long, or at least one would be super dizzy.

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24.1 Task 1

We start by defining the magnitude of the strain from the EM model

$$\mid h(t)_{EM} \mid = \frac{\eta (GM)^{5/3}\omega^{2/3}}{2c^4 R} \tag{116}$$

we use Kepler's third law, and the definition of η to get

$$| h(t) | = \frac{m_a m_b}{(m_a + m_b)^2} \frac{G^{5/3}(m_a + m_b)^{5/3} G^{1/3}(m_a + m_b)^{1/3}}{2c^4 Rr} \tag{117}$$

$$= \frac{1}{R} \frac{m_a m_b G^2}{2c^4 r} \tag{118}$$

This is proportional to the strain strength from GR, all though it is 4 and 8 times smaller than the GR strains.

24.2 Task 2

The full strains are given as

$$h_{em}(t) = \frac{1}{2R} \frac{G^2 m_a m_b}{rc^4} \sin(\beta) \cos(\beta) \sin(2\theta) \cos(2\omega t_R) \tag{119}$$

$$h_+(t) = \frac{1}{R} \frac{2G^2 m_a m_b}{rc^4} (1 + \cos^2(\theta)) \cos(2\omega t_R) \tag{120}$$

$$h_\times(t) = \frac{1}{R} \frac{4G^2 m_a m_b}{rc^4} \cos(\theta) \cos(2\omega t_R) \tag{121}$$

When the angle is $\theta = \pi/2$ we get that the em model strain is zero, the cross polarized strain is also zero, but the + polarized strain is non zero. This means that in GR, we will get a signal at that angle, but we will not with the EM model.

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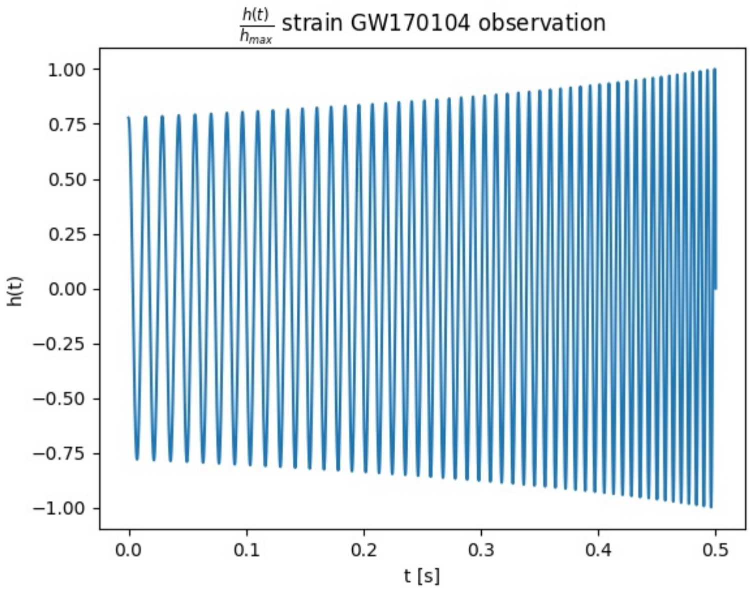
The initial distance is 3.5 times the Schwartzchild radii, and the binary is about 990 Mpc away from us.

Figure 5: GW170104 distortion strain, calculated using the EM model



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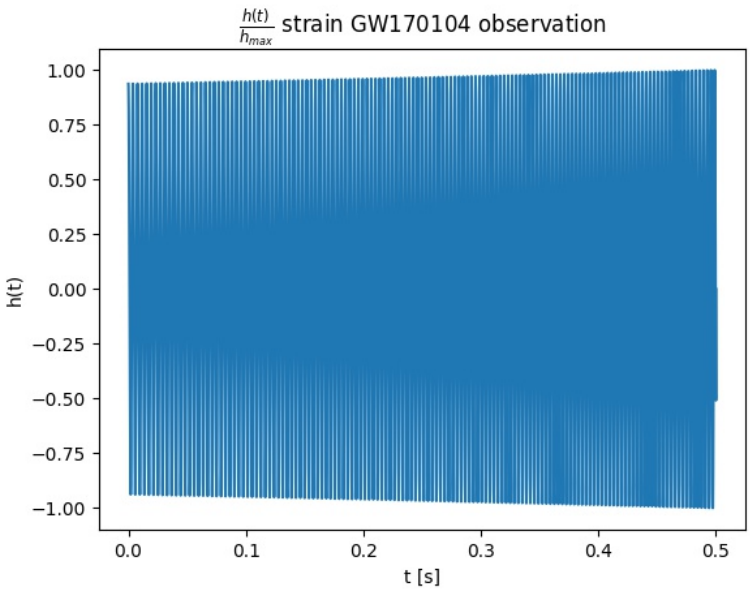
We chose two neutron stars, of masses $m_1 = 1.4M_\odot$ and $m_2 = 1.9M_\odot$. They are about 200 light years away from us, separated by about 9 Schwartzchild radii's. Below we see the signal strain from the neutron star binary.

Figure 6: Neutron binary system distortion strain, calculated using the EM model



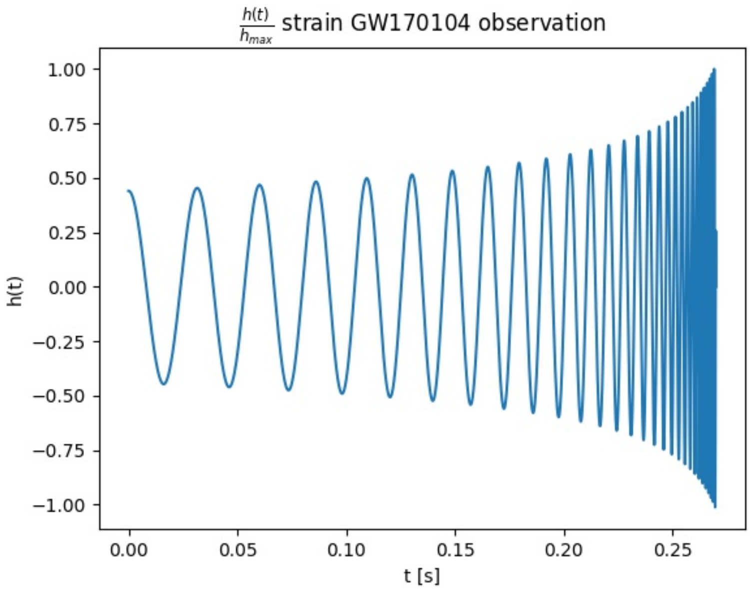
From our calculations, we get that the maximum strain strength is of order of 10^{-18}, which is well below the threshold. Given our system, we also calculated that the furthest away the binary neutron star system could be is about 4 Mpc. This is just around one megaparsec outside the local group, of which the Milky Way is a member.

27 Extra

27.1 Task 1

We tried to the best of our ability to recreate the signal from figure 6, but with the initial information from the figure text, we got calculation error, so we had to decrease the timescale for the system to end around 0.27s. We then get the following signal

Figure 7: Remake of figure 6 signal with total mass $M = 340 M_{\odot}$, $\eta = 1/4$, and $r_i = 1.7 r_s$



27.2 Task 2

This article described gravitational waves from binary black holes, and they differ from inflationary gravitational waves by way of creation. First, the gravitational waves in the article are a result of the binary system orbiting one another. This behavior follow what general relativity needs in order to produce waves. More specifically, GR requires that the angular momentum is two, as with a quadrupole moment. This leads us to the inflationary gravitational waves. The scalar field associated with inflation has angular momentum equal to zero, and so it cannot in and by it self produce the waves. Rather, the field has quantum fluctuations, some of which have a quadrupole moment, and thus will create gravitational waves.

Another difference is that the waves from the binary only depend on the mass it self and the movement and velocity with respect to its center axis. In inflation, the shape and distribution of mass density dictate much more than relative position and velocity.