UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	IN5270/IN9270 — Numerical methods for partial differential equations
Day of examination:	December 6th-7th, 2021
Examination hours:	Dec. 6, 09.00 – Dec. 7, 09.00

This problem set consists of 4 pages.

Appendices:	None
Permitted aids:	All (but without *any* help from other people)

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Advice to the students

- For each task, please first describe *in your own words* the main idea and/or the overall derivation/computation process, before diving into the specific details. The details should be provided in such a way that allows the examiner to precisely evaluate your actual level of comprehension. If relevant, please also provide a discussion in your own words afterwards.
- The PDF version of this set of exam questions can also be downloaded from the semester webpage of IN5270, within the duration of the home exam.
- On Dec. 6, between 12:30 and 13:30, there will be a lecturer's round ("trøsterunde") via Zoom. The meeting ID is 696 5801 8886, with passcode 425343. (*Note*: Each student will have to wait for her/his turn in the digital waiting room.)
- You should submit a *single* PDF containing all your answers. You can freely choose between typewriting your answers or scanning/photographing (parts of) them as handwritten texts. For more info, please read the frontpage of this home exam in the Inspera system.

Weighting of the problems

Problem 1: 50%
Problem 2: 50%

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Problem 1 Solving a 1D stationary linear PDE with finite elements (weight 50%)

The following 1D PDE

$$-u_{xx} + au = f(x), \quad 0 < x < L \tag{1}$$

is to be solved, where $f(x)$ is a given function and $a > 0$ is a scalar constant. Neumann boundary conditions apply on both the boundary points, more specifically, we have $u_x = C$ at $x = 0$ and $u_x = D$ at $x = L$.

1a (weight 6%)

Please derive in detail the weak variational form of (1) for the purpose of using the Galerkin method.

1b (weight 10%)

We will use a *non-uniform* mesh with in total N_e elements, where all elements are of the P1 type. Suppose element number e is of length h_e, such that $\sum_{e=0}^{N_e-1} h_e = L$. Explain in detail how to compute, with help of a standardized reference P1 element, the element matrix associated with an interior element, that is, with element index $0 < e < N_e - 1$. (There is no need to compute the element vector.)

1c (weight 10%)

Explain in detail how the element matrix will be, respectively, for the leftmost element (with index $e = 0$) and the rightmost element (with index $e = N_e - 1$). Please also present the formulas for computing the element vectors for these two elements. (But you don't need to compute the actual values in these element vectors.)

1d (weight 12%)

How many rows will there be in the global matrix that arises from assembling all the N_e element matrices? What is the total number of nonzero values in the global matrix? Explain in detail what the actual values will be on row number i in the global matrix.

1e (weight 12%)

If the boundary condition on the left point $x = 0$ is changed to a so-called Robin condition:

$$u_x = r(u - B), \tag{2}$$

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where r and B are scalar constants. (The boundary condition on the right boundary point remains unchanged.)

Please explain in detail how the weak variational form will change due to the above change of boundary condition. Please also explain the exact changes that will happen to the resulting global matrix and global vector.

Problem 2 Solving a 2D nonlinear diffusion equation with finite differences (weight 50%)

We consider the following 2D nonlinear diffusion equation:

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+f(u), \quad (x,y)\in\Omega=(0,1)^2,\ t\in(0,T], \quad (3)$$

where $\kappa > 0$ is a scalar constant and $f(u)$ is a given nonlinear function in u. The initial condition is of the form

$$u(x,y,0)=I(x,y), \quad (x,y)\in\Omega,$$

and the homogeneous Neumann boundary condition, $\frac{\partial u}{\partial n}=0$, is applicable on the entire boundary (that is, the four sides $x=0$, $y=0$, $x=1$ and $y=1$).

2a (weight 5%)

Use the backward Euler scheme in time and derive the resulting time discrete problem for each time step, with Δt as the time step size.

2b (weight 12%)

Suppose we use a 2D uniform mesh in space with N_x small intervals in the x direction and N_y small intervals in the y direction. The values of N_x and N_y may be different. If central finite differences are used to discretize the time discrete problem from question **2a**, what will be the resulting system of nonlinear algebraic equations? (How does each nonlinear algebraic equation look like? How many nonlinear algebraic equations are there in total?)

2c (weight 8%)

Explain in detail how Picard iterations can be used to solve the system of nonlinear algebraic equations from question **2b**.

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2d (weight 15%)

Explain in detail how Newton's method can be used to solve the system of nonlinear algebraic equations from question **2b**.

2e (weight 10%)

Suppose you have implemented a computer program, based on the backward Euler scheme in time, central finite differencing in space, and Picard iterations or Newton's method as above, to solve the 2D nonlinear diffusion equation (3). This code can freely change the mesh resolution $\Delta x, \Delta y, \Delta t$.

If the code is to be run by your friend on a powerful computer and there is a 1-hour time limit, how would you recommend the choice of Δx, Δy and Δt with the purpose of obtaining a highest possible accuracy of the numerical solution?

Remarks: Your friend can tell you beforehand the chosen value of T, but not the actual choices of $I(x, y)$ and $f(u)$. You can experiment with the code beforehand as much as possible on your own PC, but you don't have access to your friend's powerful computer. The powerful computer has S times the speed of your own PC, and your friend has just "one go" of a 1-hour execution on the powerful computer.