Solving the Poisson equation with finite elements

Obligatory project No. 2 in IN5270/IN9270

Fall 2021

General information

- Collaboration/discussion among the students is encouraged, but each student should submit her/his own Python code and a short project report.
- Please organize your submission as a directory named `fem_project` to hold all your files of this project, inside the individually assigned GitHub respository.
- The *short* project report is ment to contain the needed mathematical/numerical derivations and important explanations, and summarize the main results. LaTeX is the preferred format, but there are several other options[1] too. Regardless of format, the report must be in an easy-to-read format like PDF or HTML (or Jupyter notebook).

Solving 1D Poisson equation with finite elements

The following 1D Poisson equation

$$-u_{xx} = f(x), \quad 0 < x < 1 \tag{1}$$

is to be solved, where $f(x)$ is a given function. On the left boundary point, $x = 0$, the Neumann boundary condition $u_x = C$ is valid, where C is a nonzero scalar constant. On the right boundary point, $x = 1$, the Dirichlet boundary condition $u = D$ is valid, where D is a nonzero scalar constant.

We will use in total N_e equal-sized elements, each of size $h = 1/N_e$, to discretize the solution domain $\Omega = [0, 1]$. (You can assume that $N_e \geq 2$ is an even number.) The first $N_e/2$ elements are of the P2 type and cover the left half of Ω, whereas the remaining $N_e/2$ elements are of the P1 type and cover the right half of Ω.

Hint: Each P2 element involves three nodes and each P1 element involves two nodes. The rightmost P2 element shares one node with the leftmost P1 element, at $x = \frac{1}{2}$.

[1] `http://hplgit.github.io/teamods/writing_reports/index.html`

Task 1

Derive the weak variational formulation of the above 1D Poisson equation.

Task 2

Calculate the actual values of the element matrix for the leftmost P2 element (which borders the left end-point $x = 0$). Please carry out the calculation by making use of a standardized P2 reference element. Please also explain how the Neumann boundary condition will be included in the element vector for that P2 element. (*Note*: without exactly knowing $f(x)$ the actual values of the element vector cannot be calculated.)

Task 3

Calculate the actual values of the element matrix for the rightmost P1 element (which borders the right end-point $x = 1$). Please carry out the calculation by making use of a standardized P1 reference element. Please also explain how the Dirichlet boundary condition will be included in the element vector for that P1 element. (*Note*: without exactly knowing $f(x)$ the actual values of the element vector cannot be calculated.)

Task 4

Find an analytical solution $u_{\mathrm{e}}(x)$ of the 1D Poisson equation by choosing a matching $f(x)$ and the corresponding values of C and D. (The analytical solution should not be too trivial, such as a constant solution or a linear function.)

Task 5

Write a Python program to solve the 1D Poisson equation that includes

- Cellwise computation (numerical integration can be used to calculate the element vectors);
- Assembly of the element matrices and vectors into a linear system;
- Solution of the resulting linear system (using existing functionality);
- Plot of the numerical solution produced by the finite element method against the exact solution.

Solving 2D Poisson equation with finite elements (optional)

This optional part encourages you to use FEniCS to solve a 2D Poisson equation on the unit square:

$$-\nabla \cdot \nabla u(x, y) = f(x, y), \tag{2}$$

where $0 < x, y < 1$. On the bottom boundary, $y = 0$, a Neumann boundary condition $-u_y = C(x)$ is prescribed, whereas on the remaining boundary the boundary condition is of the Dirichlet type $u = D(x)$ or $u = D(y)$.

Task 6 (optional)

Find an analytical solution $u_{\mathrm{e}}(x, y)$ of the 2D Poisson equation by choosing a matching $f(x, y)$ and the corresponding Neumann and Dirichlet boundary conditions.

Task 7 (optional)

Derive a weak varaitional formulation of the 2D Poisson equation.

Task 8 (optional)

Implement a FEniCS program to solve the 2D Poisson equation. Carry out a sequence of numerical experiments with refined meshes and estimate the convergence rate (with respect to the L_2 norm of the error).