Introduction to dynamics

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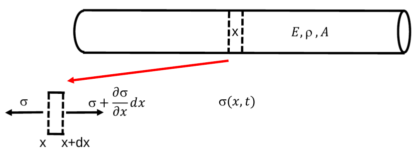
Outline

Context: continuum mechanics, small deformation, linear elasticity

Objectives: quickly brush up wave dynamics concepts to later interpret physics of dynamic fracture

- 1D (fragment) wave propagation; c-t diagram
- Wave reflection and transmission
- 3D dynamics; fundamental waves
- (Crack) Surface wave; Rayleigh

1D wave propagation: In an infinite elastic bar



Newton's second law:

$$\left(\sigma + \frac{\partial \sigma}{\partial x} dx\right) A - \sigma A = \rho A dx \frac{\partial^2 u}{\partial^2 x}$$

With constitutive law:

$$\sigma = E\epsilon = E\frac{\partial u}{\partial x}$$

Gives wave equation:

$$\frac{\partial^2 u}{\partial^2 t} = \frac{E}{\rho}\frac{\partial^2 u}{\partial^2 x} = c^2 \frac{\partial^2 u}{\partial^2 x}$$

with $c = \sqrt{\frac{E}{\rho}}$

Solution: left and right propagating waves

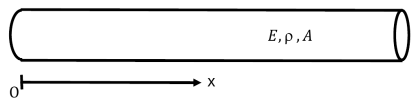
$$u(x,t) = f(x+ct) + g(x-ct)$$

Boundary conditions set f and g

Rq: Particle velocity **is not** wave velocity ($v = \dot{u} \ll c$)

1D wave propagation

Space-Time (ct) diagram



Initial conditions:

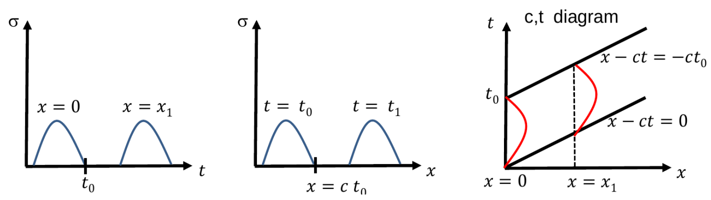
$$u(x,0) = 0 \qquad v(x,0) = 0$$

Boundary conditions:

$$\sigma(0,t) = \sigma_0 h(t) \qquad \forall t < t_0$$

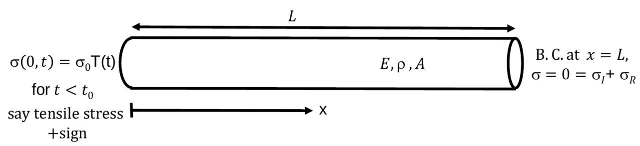
Right propagation:

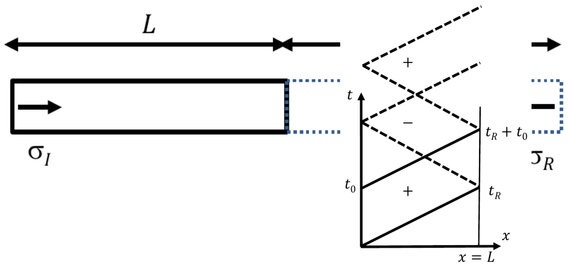
$$\sigma(x,t) = f(x-ct) = \sigma_0 h(t - x/c)$$



Finite size bar; reflection

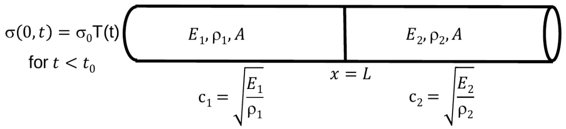
Incident and reflected wave

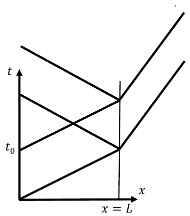




Interaction with interfaces

bimaterials: Incident, reflected, and transmitted waves





$$\sigma_I = f_I(x - c_1 t)$$
$$\sigma_R = f_R(x - L + c_1(t - t_R))$$
$$\sigma_T = f_T(x - L - c_2(t - t_R))$$

Continuity: $f_T = \frac{2R}{1+R} f_I$ and $f_R = \frac{R-1}{1+R} f_I$

$$\forall t, \text{ at } x = L$$
$$\sigma_I + \sigma_R = \sigma_T$$
$$u_I + u_R = u_T$$

or

$$v_I + v_R = v_T$$

with R the impedance ratio

$$R = \frac{\rho_2 c_2}{\rho_1 c_1}$$

3D wave propagation: In elastic isotropic medium

Equations of motion:

$$div(\sigma) = \rho \ddot{\mathbf{u}}$$

With constitutive law:

$$\sigma = \lambda tr(\epsilon) I + 2\mu\epsilon$$

Kinematics:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Planar waves solutions:

$$u(\mathbf{x}, t) = f(\mathbf{p} \cdot \mathbf{x} - ct)\mathbf{d}$$

with

- $\mathbf{d}$: direction of displacement
- $\mathbf{p}$: direction of propagation
- c: wave velocity
- $\mathbf{p} \cdot \mathbf{p} = 1$

Equation of motion yields an eigenvalue/eigenvector problem of accoustic tensor:

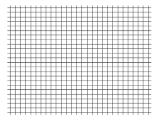
$$[(\lambda + \mu)\mathbf{p} \otimes \mathbf{p} + \mu(\mathbf{p} \cdot \mathbf{p})\mathbf{I}]\, \mathbf{d} = \rho c^2 \mathbf{d}$$

Solutions of acoustic tensor: Two solutions

Solution 1: $\mathbf{d} = d\mathbf{p} \Rightarrow c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}$

Longitudinal waves, P-waves,
or Push waves,

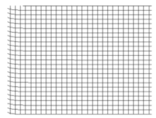
Pressure waves, L waves,
Primary waves (earthquake warning)



Solution 2: $\mathbf{d} \cdot \mathbf{p} = 0 \Rightarrow c_S = \sqrt{\frac{\mu}{\rho}}$

Shear waves ≡ S-wave

Two possible directions for $\mathbf{p}$



animations from Wikipedia

Fundamental waves: Helmoltz decomposition

One can show (with Helmoltz decomposition) that all waves can be decomposed as a sum of c_L and c_T waves.

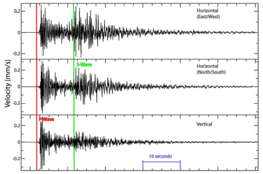
Rayleigh waves: surface waves

- *Rayleigh* waves (often called «ground roll») are of particular interest to seismologists
  - Decay slower than bulk waves: carry energy over long distances

Ratio of celerities:

$$\frac{c_L}{c_S} = \sqrt{\frac{2(1-\nu)}{1-2\nu}}$$

- If $\nu = 0$, $\frac{c_L}{c_S} = \sqrt{2}$
- In all cases: $c_S < \sqrt{2} c_L$



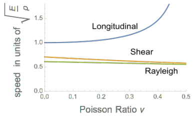
- Useful to localize earthquakes
- Emergence of «peculiar» waves: such as Love, Lamb, Stoneley and Rayleigh waves

- Rayleight wave speed c_R

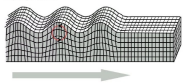
$$c_R < c_S < c_L$$

- Freund:

$$\frac{c_R}{c_S} = \sqrt{\frac{0.862 + 114\nu}{1+\nu}}$$



images from Wikipedia



Conclusions

- Existence of P-Waves and S-Waves, propagating at c_s c_T
- Surface Rayleigh waves propagate at $c_R < c_s$

- Waves propagating along cracks are concerned