

Introduction to dynamics

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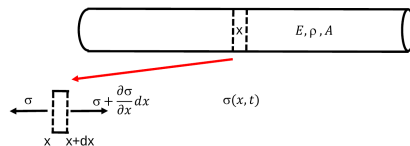
Outline

Context: continuum mechanics, small deformation, linear elasticity

Objectives: quickly brush up wave dynamics concepts to later interpret physics of dynamic fracture

- 1D (fragment) wave propagation; c-t diagram
- Wave reflection and transmission
- 3D dynamics; fundamental waves
- (Crack) Surface wave; Rayleigh

1D wave propagation: In an infinite elastic bar



Newton's second law:

$$\left(\sigma + \frac{\partial \sigma}{\partial x} dx\right) A - \sigma A = \rho A dx \frac{\partial^2 u}{\partial t^2}$$

With constitutive law:

$$\sigma = E\epsilon = E \frac{\partial u}{\partial x}$$

Gives wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\text{with } c = \sqrt{\frac{E}{\rho}}$$

Solution: left and right propagating waves

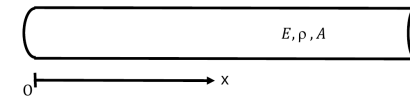
$$u(x, t) = f(x + ct) + g(x - ct)$$

Boundary conditions set f and g

Rq: Particle velocity **is not** wave velocity ($v = \dot{u} \ll c$)

1D wave propagation

Space-Time (ct) diagram



Initial conditions:

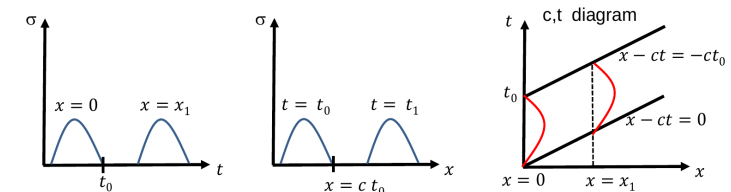
$$u(x, 0) = 0 \quad v(x, 0) = 0$$

Boundary conditions:

$$\sigma(0, t) = \sigma_0 h(t) \quad \forall t < t_0$$

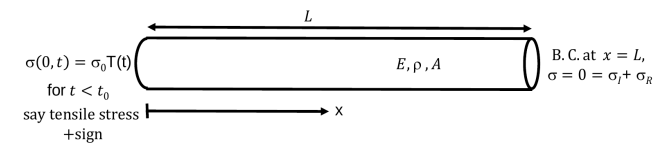
Right propagation:

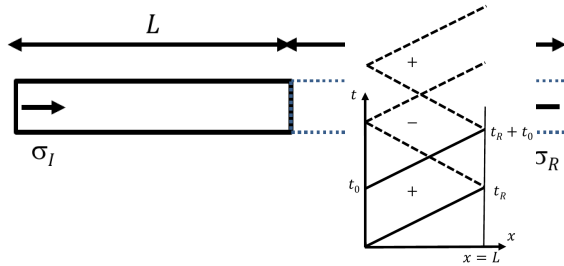
$$\sigma(x, t) = f(x - ct) = \sigma_0 h(t - x/c)$$



Finite size bar; reflection

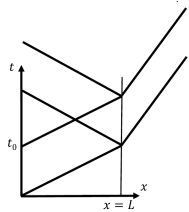
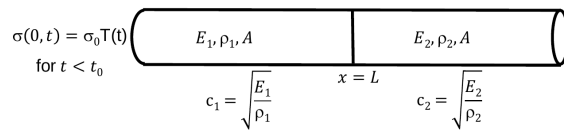
Incident and reflected wave





Interaction with interfaces

bimaterials: Incident, reflected, and transmitted waves



$$\begin{aligned}\sigma_I &= f_I(x - c_1 t) \\ \sigma_R &= f_R(x - L + c_1(t - t_R)) \\ \sigma_T &= f_T(x - L - c_2(t - t_R))\end{aligned}$$

Continuity: $f_T = \frac{2R}{1+R}f_I$ and $f_R = \frac{R-1}{1+R}f_I$

$$\begin{aligned}\forall t, \text{ at } x = L \\ \sigma_I + \sigma_R &= \sigma_T \\ u_I + u_R &= u_T\end{aligned}$$

or

$$v_I + v_R = v_T$$

with R the impedance ratio

$$R = \frac{\rho_2 c_2}{\rho_1 c_1}$$

3D wave propagation: In elastic isotropic medium

Equations of motion:

$$\text{div}(\sigma) = \rho \ddot{\mathbf{u}}$$

With constitutive law:

$$\sigma = \lambda \text{tr}(\epsilon)I + 2\mu\epsilon$$

Kinematics:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Planar waves solutions:

$$u(\mathbf{x}, t) = f(\mathbf{p} \cdot \mathbf{x} - ct)\mathbf{d}$$

with

- \mathbf{d} : direction of displacement
- \mathbf{p} : direction of propagation
- c : wave velocity
- $\mathbf{p} \cdot \mathbf{p} = 1$

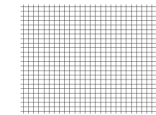
Equation of motion yields an eigenvalue/eigenvector problem of accoustic tensor:

$$[(\lambda + \mu)\mathbf{p} \otimes \mathbf{p} + \mu(\mathbf{p} \cdot \mathbf{p})I]\mathbf{d} = \rho c^2 \mathbf{d}$$

Solutions of acoustic tensor: Two solutions

$$\text{Solution 1: } \mathbf{d} = d\mathbf{p} \Rightarrow c_L = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

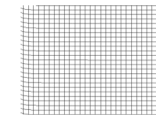
Longitudinal waves, P-waves,
or Push waves,



Pressure waves, L waves,
Primary waves (earthquake warning)

$$\text{Solution 2: } \mathbf{d} \cdot \mathbf{p} = 0 \Rightarrow c_S = \sqrt{\frac{\mu}{\rho}}$$

Shear waves \equiv S-wave



Two possible directions for \mathbf{p}

animations from Wikipedia

Fundamental waves: Helmholtz decomposition

One can show (with Helmholtz decomposition) that all waves can be decomposed as a sum of c_L and c_T waves.

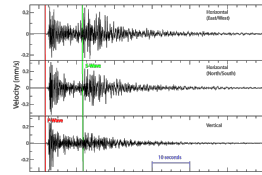
Rayleigh waves: surface waves

- *Rayleigh* waves (often called «ground roll») are of particular interest to seismologists
 - Decay slower than bulk waves: carry energy over long distances

Ratio of celerities:

$$\frac{c_L}{c_S} = \sqrt{\frac{2(1-\nu)}{1-2\nu}}$$

- If $\nu = 0$, $\frac{c_L}{c_S} = \sqrt{2}$
- In all cases: $c_S < \sqrt{2}c_L$



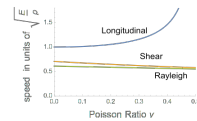
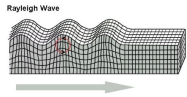
- Useful to localize earthquakes
- Emergence of «peculiar» waves: such as Love, Lamb, Stoneley and Rayleigh waves

- Rayleigh wave speed c_R

$$c_R < c_S < c_L$$

- Freund:

$$\frac{c_R}{c_S} = \sqrt{\frac{0.862 + 114\nu}{1 + \nu}}$$



images from Wikipedia

- Waves propagating along cracks are concerned

Conclusions

- Existence of P-Waves and S-Waves, propagating at c_s c_T
- Surface Rayleigh waves propagate at $c_R < c_s$