Introduction to dynamics

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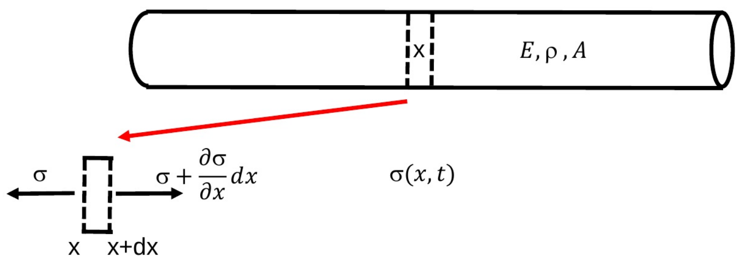
Outline

Context: continuum mechanics, small deformation, linear elasticity

Objectives: quickly brush up wave dynamics concepts to later interpret physics of dynamic fracture

- 1D wave propagation; c-t diagram
- Dispersion, wave reflection and transmission
- 3D dynamics; fundamental waves
- Rayleigh wave

1D wave propagation: In an infinite elastic bar



Newton's second law: $$\left(\sigma + \frac{\partial \sigma}{\partial x} dx\right) A - \sigma A = \rho A dx \frac{\partial^2 u }{\partial^2 x}$$

With constitutive law: $$\sigma = E \epsilon = E \frac{\partial u}{\partial x}$$

Gives wave equation: $$\frac{\partial^2 u }{\partial^2 t} = \frac{E}{\rho} \frac{\partial^2 u }{\partial^2 x} = c^2 \frac{\partial^2 u }{\partial^2 x}$$

with $c = \sqrt{\frac{E}{\rho}}$

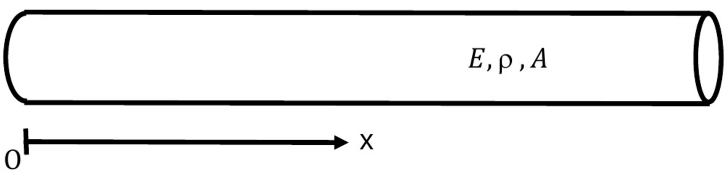
Solution: left and right propagating waves $$u(x,t) = f(x+ct)+ g(x-ct)$$

Boundary conditions set f and g

Particle velocity **is not** wave velocity ($v = \dot{u} \ll c$)

1D wave propagation

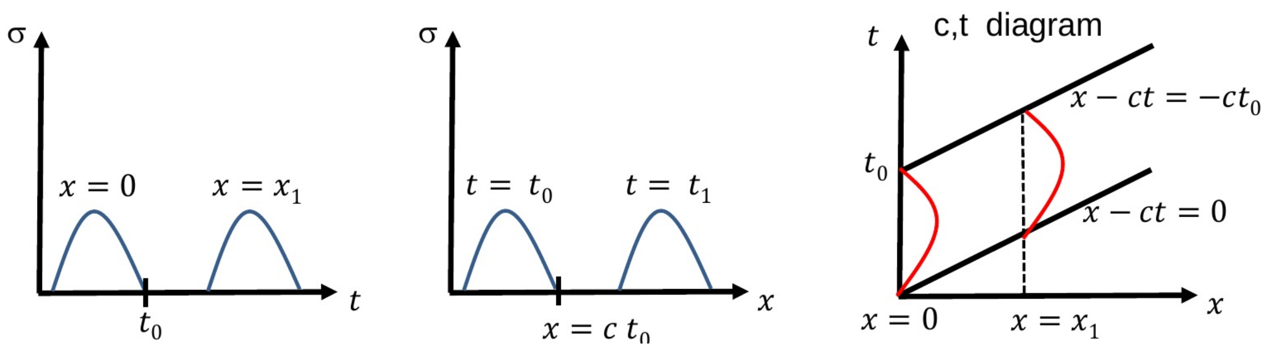
Space-Time (ct) diagram



Initial conditions: $$u(x, 0) = 0 \qquad v(x, 0) = 0$$

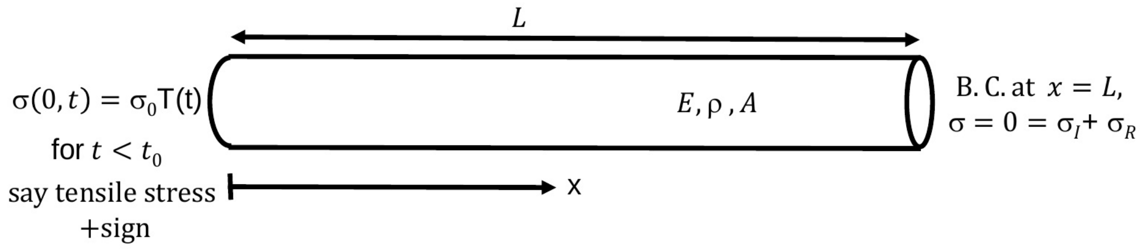
Boundary conditions: $$\sigma(0, t) = \sigma_0 h(t) \qquad \forall t < t_0$$

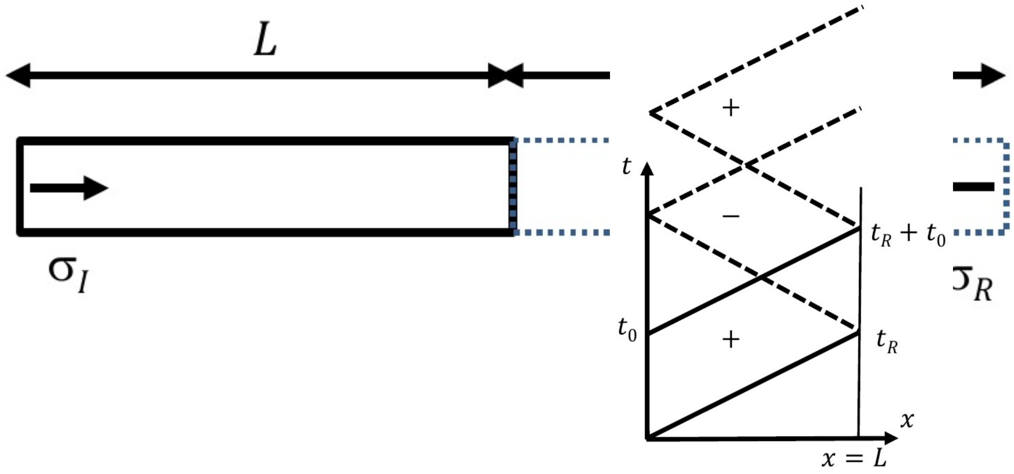
Right propagation: $$\sigma(x, t) = f(x-ct) = \sigma_0 h(t-x/c)$$



Finite size bar; reflection

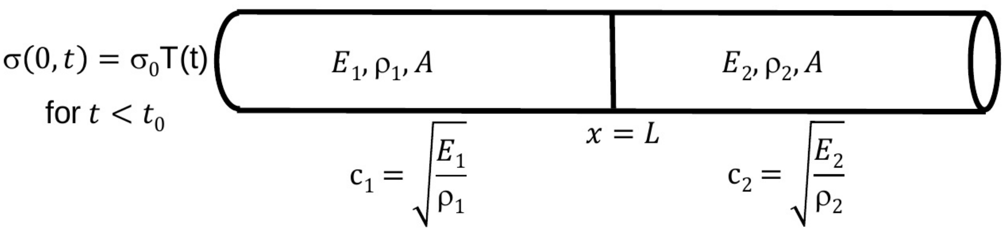
Incident and reflected wave

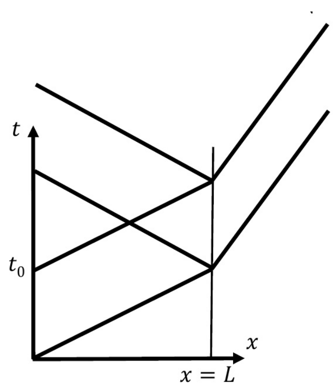




Interaction with interfaces

bimaterials: Incident, reflected, and transmitted waves





$$\sigma_I = f_I(x-c_1t)$$ $$\sigma_R = f_R(x-L+c_1(t-t_R))$$ $$\sigma_T = f_T(x-L-c_2(t-t_R))$$

Continuity: $f_T = \frac{2R}{1+R}f_I$ and $f_R = \frac{R-1}{1+R}f_I$

$$\forall t, \text{at } x=L$$ $$\sigma_I + \sigma_R = \sigma_T$$ $$u_I + u_R = u_T$$

or $$v_I + v_R = v_T$$

with R the impedance ratio $$R=\frac{\rho_2 c_2}{\rho_1 c_1}$$

3D wave propagation: In elastic isotropic medium

Equations of motion: $$div(\mathbf \sigma) = \rho \ddot{\mathbf u}$$

With constitutive law: $$\mathbf \sigma = \lambda tr(\mathbf \epsilon) I + 2\mu \mathbf \epsilon$$

Kinematics: $$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{i, i})$$

Shear planar waves solutions:

$$u(\mathbf x, t) = f(\mathbf p\cdot \mathbf x - ct) d$$

with

- $\mathbf d$: direction of displacement
- $\mathbf p$: direction of propagation
- c: wave velocity
- $\mathbf p \cdot \mathbf p = 1$

Yields an eigenvalue/eigenvector problem of accoustic tensor:

$$\left[(\lambda + \mu) \mathbf p \otimes \mathbf p + \mu (\mathbf p\cdot \mathbf p) \mathbf I \right] \mathbf d = \rho c^2 \mathbf d$$

Solutions of acoustic tensor: Two solutions

Solution 1: $\mathbf d = d \mathbf p \Rightarrow c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}$

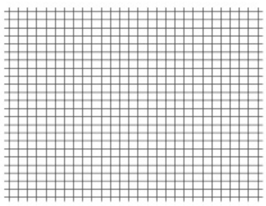
> Longitudinal waves, P-waves, or Push waves,

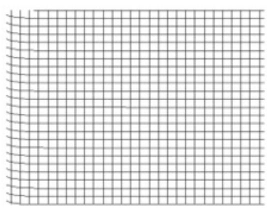
Pressure waves, L waves,
Primary waves (earthquake warning)

Solution 2: $\mathbf d \cdot \mathbf p= 0 \Rightarrow c_S = \sqrt{\frac{\mu}{\rho}}$

> Shear waves $\equiv$ S-wave

Two directions for $\mathbf p$





animations from Wikipedia

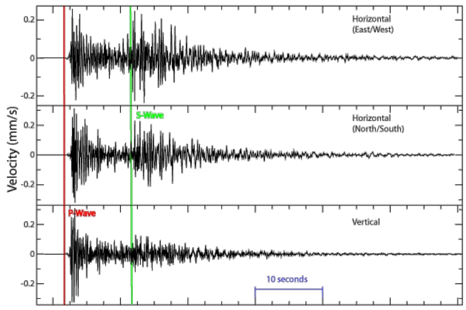
Fundamental waves: Helmoltz decomposition

One can show (with Helmoltz decomposition) that all waves can be decomposed as a sum of c_L and c_T waves; these are called fundamental waves.

Ratio of celerities:

$$\frac{cL}{c_S} = \sqrt{\frac{2(1 - \nu)}{1-2\nu}}$$

- If $ν = 0$, $\frac{c_L}{c_S} = \sqrt{2}$
- In all cases: $c_S < \sqrt{2} c_L$



- Useful to localize earthquakes
- Emergence of «peculiar» waves: such as Love, Lamb, Stoneley and Rayleigh waves

Rayleigh waves: surface waves

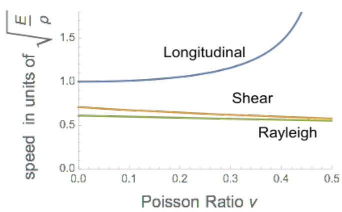
- *Rayleigh* waves (often called «ground roll») are of particular interest to seismologists
  - Decay slower than bulk waves: carry energy over long distances

- Rayleight wave speed c_R

$$c_R < c_S < c_L$$

- Freund:

$$\frac{c_R}{c_S} = \sqrt{\frac{0.862 + 114\nu}{1+\nu}}$$



images from Wikipedia