

Introduction to dynamics

G. Anciaux

Civil Engineering, Materials Science, EPFL



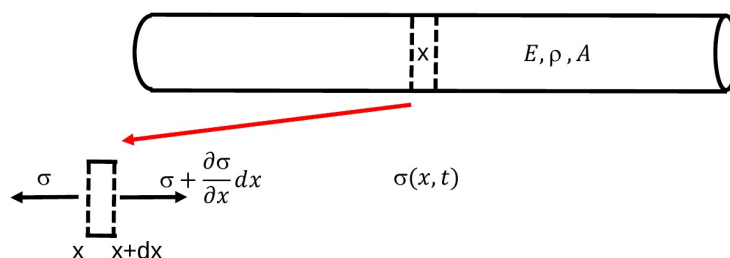
Outline

Context: continuum mechanics, small deformation, linear elasticity

Objectives: quickly brush up wave dynamics concepts to later interpret physics of dynamic fracture

- 1D wave propagation; c-t diagram
- Dispersion, wave reflection and transmission
- 3D dynamics; fundamental waves
- Rayleigh wave

1D wave propagation: In an infinite elastic bar



Newton's second law:
$$\left(\sigma + \frac{\partial \sigma}{\partial x} dx \right) A - \sigma A = \rho A dx \frac{\partial^2 u}{\partial x^2}$$

With constitutive law: $\sigma = E \epsilon = E \frac{\partial u}{\partial x}$

Gives wave equation: $\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

with $c = \sqrt{\frac{E}{\rho}}$

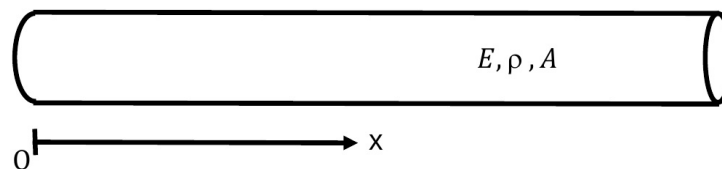
Solution: left and right propagating waves $u(x,t) = f(x+ct) + g(x-ct)$

Boundary conditions set f and g

Particle velocity **is not** wave velocity ($v = \dot{u} \ll c$)

1D wave propagation

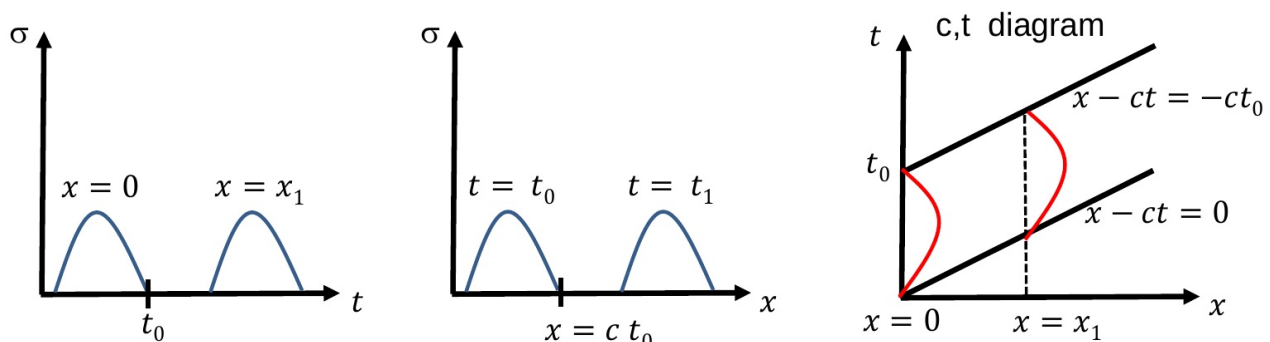
Space-Time (ct) diagram



Initial conditions: $u(x, 0) = 0 \quad v(x, 0) = 0$

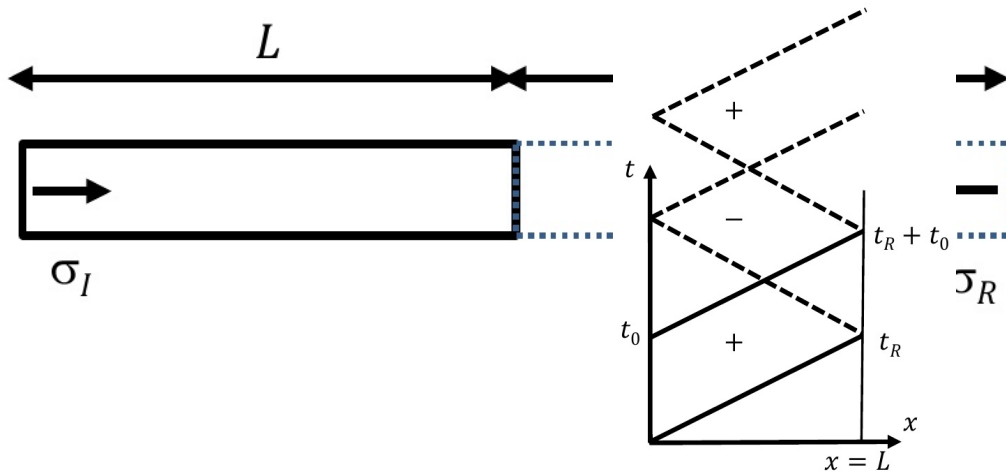
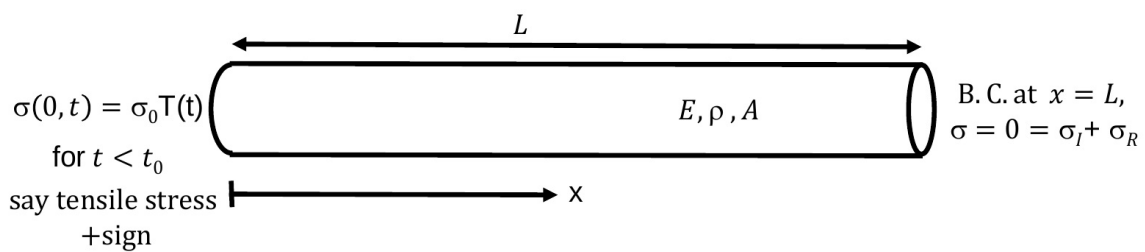
Boundary conditions: $\sigma(0, t) = \sigma_0 h(t) \quad \text{for all } t < t_0$

Right propagation: $\sigma(x, t) = f(x-ct) = \sigma_0 h(t-x/c)$



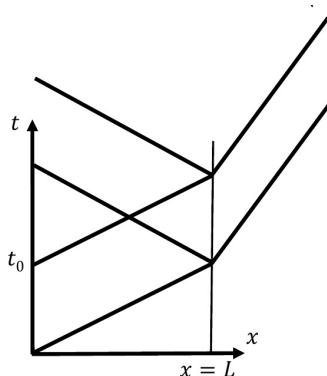
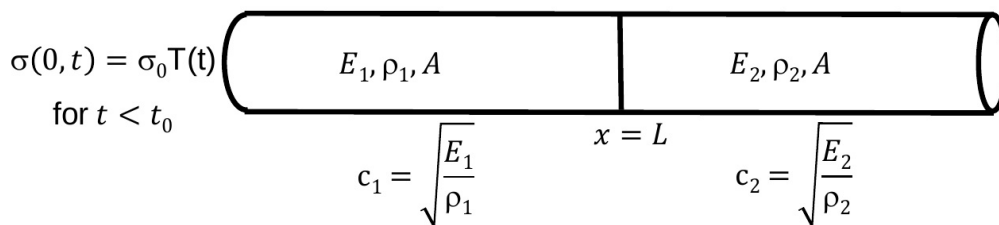
Finite size bar; reflection

Incident and reflected wave



Interaction with interfaces

bimaterials: Incident, reflected, and transmitted waves



$$\sigma_I = f_I(x - c_1 t) \quad \sigma_R = f_R(x - L + c_1(t - t_R)) \quad \sigma_T = f_T(x - L - c_2(t - t_R))$$

Continuity: $f_T = \frac{2R}{1+R} f_I$
and $f_R = \frac{R-1}{1+R} f_I$

$$\forall t, \text{at } x=L \quad \sigma_I + \sigma_R = \sigma_T \quad u_I + u_R = u_T$$

$$\text{or } v_I + v_R = v_T$$

with R the impedance ratio

$$R = \frac{\rho_2 c_2}{\rho_1 c_1}$$

3D wave propagation: In elastic isotropic medium

Equations of motion: $\text{div}(\mathbf{\sigma}) = \rho \ddot{\mathbf{u}}$

With constitutive law: $\mathbf{\sigma} = \lambda \text{tr}(\mathbf{\epsilon}) \mathbf{I} + 2\mu \mathbf{\epsilon}$

Kinematics: $\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$

Shear planar waves solutions:

$\mathbf{u}(\mathbf{x}, t) = \mathbf{f}(\mathbf{p} \cdot \mathbf{x} - ct) \mathbf{d}$

with

- \mathbf{d} : direction of displacement
- \mathbf{p} : direction of propagation
- c : wave velocity
- $\mathbf{p} \cdot \mathbf{p} = 1$

Yields an eigenvalue/eigenvector problem of acoustic tensor:

$\left[(\lambda + \mu) \mathbf{p} \otimes \mathbf{p} + \mu (\mathbf{p} \cdot \mathbf{p}) \mathbf{I} \right] \mathbf{d} = \rho c^2 \mathbf{d}$

Solutions of acoustic tensor: Two solutions

Solution 1: $\mathbf{d} = \mathbf{p} \Rightarrow c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}$

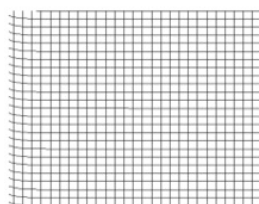
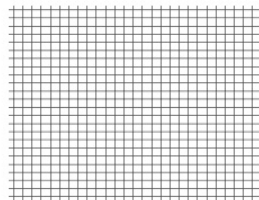
Longitudinal waves, P-waves,
or Push waves,

Pressure waves, L waves,
Primary waves (earthquake warning)

Solution 2: $\mathbf{d} \cdot \mathbf{p} = 0 \Rightarrow c_S = \sqrt{\frac{\mu}{\rho}}$

Shear waves \equiv S-wave

Two directions for \mathbf{p}



animations from Wikipedia

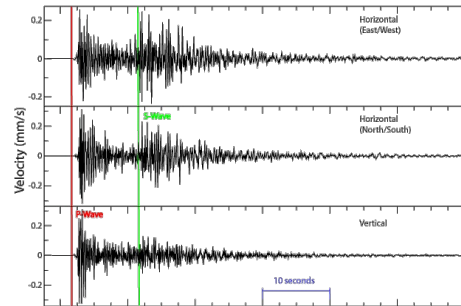
Fundamental waves: Helmholtz decomposition

One can show (with Helmholtz decomposition) that all waves can be decomposed as a sum of c_L and c_T waves.

Ratio of celerities:

$$\frac{c_L}{c_S} = \sqrt{\frac{2(1-\nu)}{1-2\nu}}$$

- If $\nu = 0$, $\frac{c_L}{c_S} = \sqrt{2}$
- In all cases: $c_S < \sqrt{2} c_L$



- Useful to localize earthquakes
- Emergence of «peculiar» waves: such as Love, Lamb, Stoneley and Rayleigh waves

Rayleigh waves: surface waves

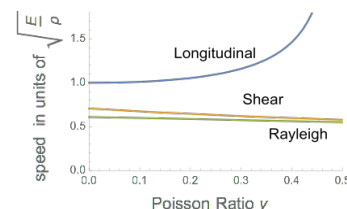
- *Rayleigh* waves (often called «ground roll») are of particular interest to seismologists
 - Decay slower than bulk waves: carry energy over long distances

- Rayleigh wave speed c_R

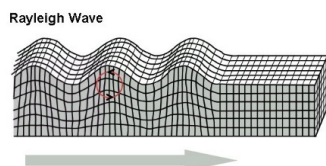
$$c_R < c_S < c_L$$

- Freund:

$$\frac{c_R}{c_S} = \sqrt{\frac{0.862 + 1.14\nu}{1+\nu}}$$



images from Wikipedia





Conclusions

- Existence of P-Waves and S-Waves, propagating at c_s c_T
- Surface Rayleigh waves propagate at c_R
- Waves propagating along cracks are concerned