# Simulating dynamics with finite-elements

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## Outline

- fundamental solid mechanics concepts
- Strong and weak forms of elasto-dynamics equations
- Semi-discrete equations of motion
- Temporal discretization
- Newmark algorithm
    - Explicit dynamics; mass lumping
    - Newmark implicit

**Further reading:**

**T. Belytschko, W. Liu, B. Moran**. *Nonlinear Finite Elements for Continua and Structures*. (John Wiley \& Sons, 2000).

**T. Hughes**. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. (Dover Publications Inc., 2003).

## Elasto-dynamics

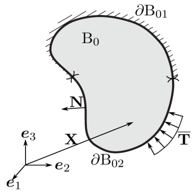
**Initial boundary value problem (IBVP):**

$$\nabla_0 \cdot \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \ddot{\mathbf{x}} \quad \text{in} \quad B_0$$

with:

- $P$ first Piola-Kirchhoff stress tensor
- $\ddot{\mathbf{x}}(\mathbf{X}, t)$ material acceleration

- $\mathbf{F} = \nabla_0 \mathbf{x}$ deformation gradient
- $P \cdot N = T(\mathbf{X}, t)$ in $\partial B_{02}$
- $\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t)$ in $\partial B_{01}$
- $B$ body forces




**Initial conditions:**

- $\mathbf{x}(\mathbf{X}, 0) = \mathbf{x}_0(\mathbf{X})$
- $\dot{\mathbf{x}}(\mathbf{X}, 0) = \mathbf{v}_0(\mathbf{X})$

**Constitutive equations for hyperelastic materials:**

$$W(\mathbf{F}) \equiv \text{strain-energy density/undeformed volume}$$
$$P_{iJ} = \frac{\partial W}{\partial F_{iJ}}(\nabla_0 \mathbf{x}) \quad in \quad B_0$$

**Formulate weak form of equations:**

$$\int_{B_0} [\rho_0(\ddot{\mathbf{x}} - \mathbf{B}) - \nabla_0 \cdot \mathbf{P}] \cdot \eta \, dV_0 = \mathbf{0}$$
$$\forall \eta \text{ admissible} \quad i.e. \quad \eta | \partial B_{01} = 0$$

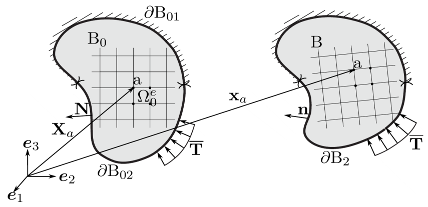
**Integration by part**

$$\int_{B_0} [\rho_0(\ddot{\mathbf{x}} - \mathbf{B}) \cdot \eta + \mathbf{P} : \nabla_0 \eta] \, dV_0 - \int_{\partial B_0} \underbrace{(\mathbf{P} \cdot \mathbf{N})}_{\bar{\mathbf{T}}} \cdot \eta \, dS_0 = 0$$

# Semi-discrete equations of motion:

- Discretize in space (finite dimensional space $\mathcal{V}_h$ )
- Keep time as a continuous variable (for now)

$$\mathbf{x}_h(X,t) = \sum_{a=1}^{N} \mathbf{x}_a(t) N_a(X)$$




**Derived fields:**

$$\dot{\mathbf{x}}_h(\mathbf{X},t) = \sum_a \dot{\mathbf{x}}_a(t) N_a(\mathbf{X})$$
$$\ddot{\mathbf{x}}_h(\mathbf{X},t) = \sum_a \ddot{\mathbf{x}}_a(t) N_a(\mathbf{X})$$
$$\mathbf{F}_h(\mathbf{X},t) = \sum_a \mathbf{x}_a(t) \nabla_0 N_a(\mathbf{X})$$

**Insert fields into weak form**

$$\eta_h = \sum_a^N \eta_a N_a \qquad \text{and} \qquad \mathbf{x}(X,t) = \sum_a \mathbf{x}_a(t) N_a$$

Leads to:

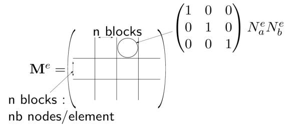
$$\int_{B_0} \left[ \rho_0 \left( \sum_a \ddot{\mathbf{x}}_a(t) N_a - \mathbf{B} \right) \cdot \sum_b \eta_b N_b + \mathbf{P} : \left( \sum_b \eta_b \nabla_0 N_b \right) \right] dV_0 - \int_{\partial B_{02}} \bar{\mathbf{T}} \cdot \left( \sum_b \eta_b N_b \right) dS_0 = 0$$

**Reordering the terms:**

$$\sum_a \sum_b \eta_b \ddot{\mathbf{x}}_a(t) \underbrace{\left( \int_{B_0} \rho_0 N_a N_b dV_0 \right)}_{M} + \sum_b \eta_b \underbrace{\left( \int_{B_0} \mathbf{P} : (\nabla_0 N_b)\, dV_0 \right)}_{\mathbf{f}^{int}} =$$
$$\sum_b \eta_b \underbrace{\left( \int_{B_0} \rho_0 \mathbf{B} N_b + \int_{\partial B_{02}} \bar{\mathbf{T}} \cdot N_b dS_0 \right)}_{\mathbf{f}^{ext}} = 0$$

**Indicial notations:**

$$M_{iakb} = \int_{B_0} \rho_0 \delta_{ik} N_a N_b dV_0 \qquad M_{iakb} = \sum_{e=1}^{E} M^e_{iakb} = \sum_{e=1}^{E} \int_{\Omega^e_0} \rho_0 \delta_{ik} N^e_a N^e_b dV_0$$




$$M_{iakb} = \int_{B_0} \rho_0 \delta_{ik} N_a N_b dV_0$$
$$f^{int}_{ia}(x,\dot{x}) = \int_{B_0} P_{iJ}(\dot{F}_h, F_h) N_{a,J} dV_0$$
$$f^{ext}_{ia}(t) = \int_{B_0} \rho_0 B_i N_a dV_0 + \int_{B_{02}} \bar{T}_i N_a dS_0$$

(continuous in time)

**Semi-discrete equations of motion (matricial)**

$$M\ddot{x} + f^{int}(x,\dot{x}) = f^{ext}(t)$$

*Rq: all degrees of freedom are placed contiguously*

**Initial conditions:**

- $x(0) = x_0$
- $\dot{x}(0) = v_0$

**Objective: compute trajectory** $x(t), t \in [0,T]$

Definition:

- Tangent stiffness matrix: $K(x,\dot{x}) \equiv \frac{\partial f^{int}(x,\dot{x})}{\partial x}$
- Tangent damping matrix: $C(x,\dot{x}) \equiv \frac{\partial f^{int}(x,\dot{x})}{\partial \dot{x}}$

Linear behavior $\Rightarrow f^{int}(x,\dot{x}) = Kx + C\dot{x}$
$\Rightarrow M\ddot{x} + Kx + C\dot{x} = f^{ext}(t)$

**Temporal discretization** = **time stepping algorithm**

Sample the solution (approximately)

$$t_0, t_1 = t_0 + \Delta t, \ldots, t_n = t_0 + n\Delta t$$

$x_0, x_1, \ldots . x_n$
$v_0, v_1, \ldots, v_n$
$a_0, a_1, \ldots, a_n$

A set of rules is needed to compute $(x_{n+1}, v_{n+1}, a_{n+1})$ from $(x_n, v_n, a_n)$ which is consistent with equations of motion.

**Newmark algorithm**

$$x_{n+1} = x_n + \Delta t v_n + \Delta t^2 \left[ \left( \frac{1}{2} - \beta \right) a_n + \beta a_{n+1} \right]$$
$$v_{n+1} = v_n + \Delta t[(1-\gamma) a_n + \gamma a_{n+1}]$$

**Closing with governing equation**

$$Ma_{n+1} + f^{int}_{n+1}(x_{n+1}, v_{n+1}) = f^{ext}_{n+1}$$

**Newmark parameters**

- $\beta \in [0, 12]$
- $\gamma \in [0, 1]$

**Solutions**

- Explicit dynamics
- Implicit dynamics

**Explicit dynamics (no equation solving)**

Explicit member of Newmark's family of algorithm: $\beta = 0, \gamma \neq 0, M$ diagonal, no damping: $f^{int}(x)$

$$\begin{aligned} x_{n+1} &= x_n + \Delta t v_n + \frac{\Delta t^2}{2} a_n \\ a_{n+1} &= M^{-1} \left[ f^{ext}_{n+1} - f^{int}_{n+1}(x_{n+1}) \right] \\ v_{n+1} &= v_n + \Delta t [(1-\gamma) a_n + \gamma a_{n+1}] \end{aligned}$$

- $\Delta t$ restricted by stability condition
- No equation solving
- Impact applications, shock waves,...

**Mass lumping**

Consistent mass:

$$M_{iakb} = \sum_{e=1}^{E} \int_{\Omega_0^e} \rho_0 \delta_{ik} N_a N_b dV_0$$

Leads to optimal error estimates, but speed can be a motivation for breaking the basic rules.

**Diagonal or lumped mass matrix:**

- Economy
- Leads to explicit methods (no equation solving)

$$a_{n+1} = M^{-1} \left[ f^{ext}_{n+1} - f^{int}_{n+1} \right]$$

**Computation of lumped mass matrix?**

One possibility:

- Row/column sum technique: put in diagonal entry the sum of all components in the corresponding row/column.

The diagonal entries of lumped mass matrix are:

$$M^{lumped}_{iaia} = \sum_k \sum_b M_{iakb} = \sum_k \sum_b \sum_e \int_{\Omega_0^e} \rho_0 \delta_{ik} N^e_a N^e_b dV_0$$

Can lump at the element level, then assemble:

$$M^{lumped}_{iaia} = \sum_e \sum_k \sum_b \int_{\Omega_0^e} \rho_0 \delta_{ik} N^e_a N^e_b dV_0$$

**Lumped mass at element level**

$$\begin{aligned} M^{lumped}_{iaia} &= \sum_e \sum_k \sum_b \int_{\Omega_0^e} \rho_0 \delta_{ik} N^e_a N^e_b dV_0 \\ &= \sum_e \sum_b \int_{\Omega_0^e} \rho_0 N^e_a N^e_b dV_0 \\ &= \sum_e \int_{\Omega_0^e} \rho_0 N^e_a \left( \sum_b N^e_b \right) dV_0 \\ &= \sum_e \int_{\Omega_0^e} \rho_0 N^e_a dV_0 \end{aligned}$$

**Mass preservation**

$$\sum_a M^{lumped}_{iaia} = \sum_e \int_{\Omega_0^e} \rho_0 \sum_a N^e_a dV_0 = \int_{B_0} \rho_0 dV_0$$

# Newmark-Implicit

$$\begin{aligned} x_{n+1} &= x_n + \Delta t v_n + \Delta t^2 \left[ \left( \frac{1}{2} - \beta \right) a_n + \beta a_{n+1} \right] \\ v_{n+1} &= v_n + \Delta t [(1-\gamma) a_n + \gamma a_{n+1}] \\ M a_{n+1} &+ f^{int}_{n+1}(x_{n+1}, v_{n+1}) = f^{ext}_{n+1} \end{aligned}$$

**Then, a predictor-corrector Newton-Raphson is required.**

# Stability ($\xi = 0$)

| **Method** | **Type** | $\beta$ | $\gamma$ | **Stability cond.** | **Order** |
|---|---|---|---|---|---|
| Average acc. | Implicit | 1/4 | 1/2 | Unconditional | 2 |
| Linear acc. | Implicit | 1/6 | 1/2 | $\Omega_{crit} = 2\sqrt{3}$ | 2 |
| Fox-Goodwin | Implicit | 1/12 | 1/2 | $\Omega_{crit} = \sqrt{6}$ | 2 |
| Central difference | Implicit | 0 | 1/2 | $\Omega_{crit} = 2$ | 2 |

# Bibliography


**T. Belytschko, W. Liu, B. Moran**. *Nonlinear Finite Elements for Continua and Structures*. (John Wiley \& Sons, 2000).

**T. Hughes**. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. (Dover Publications Inc., 2003).