Physics Programming - Basics

Bodies

Hello physics world

A typical physics simulation is just a collection of bodies that collide. And before we can simulate collisions, we need to discuss how to represent the bodies. Every body has a position in space. This can be represented with a vector in cartesian coordinates. We will use the Vec3 already present in the code. Also, bodies have orientations. We're going to use quaternions to represent the orientation of our bodies. The final property that a body must have to be represented in our simulation is a shape. The shape defines the actual geometry of the body. There's all kinds of shapes that we may want to use to represent a body, but in this course, we're only going to bother with spheres. However, since we will simulate more general shapes later, we're going to create a base shape class as well as a sphere shape class that inherits from it.

Let's implement the Body ans the Sphere shape.

```
#pragma once
#include "code/Math/Vector.h"
#include "code/Renderer/model.h"
#include "code/Math/Quat.h"

class Body
{
public:
    Vec3 position;
    Quat orientation;
    Shape* shape;
};
```

```
#pragma once
class Shape {
public:
    enum class ShapeType
    {
        SHAPE_SPHERE,
    };
    virtual ShapeType GetType() const = 0;

};

class ShapeSphere: public Shape {
public:
    ShapeSphere(float radiusP) : radius(radiusP) {}
    ShapeType GetType() const override { return ShapeType::SHAPE_SPHERE; }
    float radius;
};
```

We will create our first render with:

```
void Scene::Initialize() {
    Body body;
    body.position = Vec3( 0, 0, 0 );
    body.orientation = Quat( 0, 0, 0, 1 );
    body.shape = new ShapeSphere( 1.0f );
    bodies.push_back( body );
}
```

This will display a small sphere. The simulation runs at 60 FPS.

Body space and world space.

We should probably quickly cover the difference between world space and body space. When things collide, we're going to have information about where in world space that collision occured. And sometimes, we're going to want to know where specifically on the body that hit is located. In order to do this, we'll need a function that can transform a world space point to local space. And we'll also want a function that can take us from local space to world space.

We also need to consider center of mass. We'll get deeper into this concept later in this course. This is important to note because while body space is similar to model space, they are different. Model space is the space centered about the shape geometry's origin. However, body space is centered about the center of mass. Fortunately for our sphere shapes, their center of mass coincides with their geometric center; which makes things easier for this course.

So, let's go ahead and add the center of mass to the shape class.

```
class Shape {
public:
    enum class ShapeType
    {
        SHAPE_SPHERE,
    };

    virtual ShapeType GetType() const = 0;
    virtual Vec3 GetCenterOfMass() const { return centerOfMass; }

protected:
    Vec3 centerOfMass;
};

class ShapeSphere : public Shape {
public:
    ShapeSphere(float radiusP) : radius(radiusP)
    {
        centerOfMass.Zero();
    }

    ShapeType GetType() const override { return ShapeType::SHAPE_SPHERE; }
    float radius;
};
```

And now the world space and body space functions to our body class.

```
class Body
{
public:
    Vec3 position;
    Quat orientation;
    Shape* shape;

    Vec3 GetCenterOfMassWorldSpace() const;
    Vec3 GetCenterOfMassBodySpace() const;

    Vec3 WorldSpaceToBodySpace(const Vec3& worldPoint);
    Vec3 BodySpaceToWorldSpace(const Vec3& bodyPoint);
};
```

```
#include "Body.h"
#include "Shape.h"

Vec3 Body::GetCenterOfMassWorldSpace() const
{
 const Vec3 centerOfMass = shape->GetCenterOfMass();
 const Vec3 pos = position + orientation.RotatePoint(centerOfMass);
 return pos;
}

Vec3 Body::GetCenterOfMassBodySpace() const
{
 return shape->GetCenterOfMass();
}

Vec3 Body::WorldSpaceToBodySpace(const Vec3& worldPoint)
{
 const Vec3 tmp = worldPoint - GetCenterOfMassWorldSpace();
 const Quat invertOrient = orientation.Inverse();
 Vec3 bodySpace = invertOrient.RotatePoint(tmp);
 return bodySpace;
}

Vec3 Body::BodySpaceToWorldSpace(const Vec3& bodyPoint)
{
 Vec3 worldSpace = GetCenterOfMassWorldSpace()
                  + orientation.RotatePoint(bodyPoint);
 return worldSpace;
}
```

Gravity

Now that we have a body added to the scene and drawn to screen, it would be nice if it also moved. You may have noticed that objects move when an external force is applied to them. And you may have also noticed that all objects are affected by gravity. Therefore, a simple way to get objects moving in our little physics experiment is to apply gravity.

Positions change when they have velocity and the equation relating the change of position with the velocity is: dx = v ∗ dt

Velocities change when there is an acceleration. The equation relating the change of velocity with the acceleration is: dv = a ∗ dt

The acceleration due to the gravity applies to all objects equally, and near the surface of the Earth is approximately 9.8m/s 2. We, however are going to approximate this further to a nice round 10m/s 2 .

Since an object's velocity is typically unique to itself, this is a new property that we should add to the body class:

```
class Body
{
public:
    Vec3 position;
    Quat orientation;
    Vec3 linearVelocity;
    ...
```

We call this linear velocity because we are currently only concerned with translational velocity. We will get to rotational velocity later. Now, we can add gravity to the velocity of each body and then update the positions of the bodies by modifying the update function of the scene class like so:

```
void Scene::Update( const float dt_sec ) {
    for (int i = 0; i < bodies.size(); ++i) {
        bodies[i].linearVelocity += Vec3(0, 0, -10) * dt_sec;
    }
    for (int i = 0; i < bodies.size(); ++i) {
        bodies[i].position += bodies[i].linearVelocity * dt_sec;
    }
}
```

To test, do not forget to unpause the simulation by hitting T.

Gravity as impulse

While directly adding the acceleration, due to gravity, to the velocity of a body works, it's not exactly best for us in the long run. Gravity, is a force, and forces act on bodies that have mass. Force is defined as:

`F = m ∗ a`

And the force of gravity (near the surface of the earth) is defined as:

`F = m ∗ g`

where g is 9.8m/s 2 (or for us 10m/s 2). By setting these two equations equal to each other and solving for acceleration, we quickly find that the acceleration due to gravity is g.

Now, it’s perfectly valid to deal with gravity this way on paper, but we are going to try and make a robust physics simulation. And for the sake of the simulation it would be better if we applied all forces in a uniform way. In this sense, we would like to be agnostic about gravity and just treat it as we would any other force. The physics “engine” that we’re going to build will handle forces indirectly by applying impulses. This means we need to know what impulses are and how they affect momentum. Momentum is defined as:

`p = m * v` (1)

You can think of momentum as the amount a moving object would like to keep moving in a particular direction. So, the more momentum an object has, the more force is required to change its momentum. The change in the momentum is related to the applied force by:

`dp = F * dt`

Now the definition of impulse, J, is the change of momentum, so this means:

`J ≡ dp = F * dt` (triple equal is "equal by definition")

And from the relation (1), then:

`J = m * dv` (because `dp = m * dv`)

Which, solving for dv yields:

`dv = J/m`

This means that we can easily calculate the impulse due to gravity by multiplying the force of gravity by the time delta between frames, `dt`. And from that we can find the change in velocity by simply dividing by the mass of the body. Now we can go ahead and modify the body class to apply impulses.

```
class Body
{
public:
    ...
    Vec3 linearVelocity;
    float inverseMass;
    ...
    void ApplyImpulseLinear(const Vec3& impulse);
};


void Body::ApplyImpulseLinear(const Vec3& impulse)
{
 if (inverseMass == 0.0f) return;
 // dv = J / m
 linearVelocity += impulse * inverseMass;
}
```

It’s important to note that we’re actually storing the inverse mass, not the mass itself. There’s a couple of reasons for this. The first is that we need a way to describe objects with infinite mass. Nothing in the real world actually has infinite mass, but for the kinds of forces that we’re going to simulate, we can approximate certain objects as having infinite mass. For instance, a tennis ball thrown against the sidewalk. The tennis ball imparts a force upon the ground and the ground in turn imparts a force upon the ball, and the Earth itself will move a little. But the amount the Earth moves is so small that we can easily approximate this collision as the Earth having an infinite mass.

And now we can adjust our update loop to apply gravity as an impulse, instead of directly using it as an acceleration.

```
void Scene::Update( const float dt_sec ) {
    for (int i = 0; i < bodies.size(); ++i) {
        Body& body = bodies[i];
        float mass = 1.0f / body.inverseMass;
        // Gravity needs to be an impulse I
        // I == dp, so F == dp/dt <=> dp = F * dt
        // <=> I = F * dt <=> I = m * g * dt
        Vec3 impulseGravity = Vec3(0, 0, -10) * mass * dt_sec;
        body.ApplyImpulseLinear(impulseGravity);
    }

    // Position update
    for (int i = 0; i < bodies.size(); ++i) {
        bodies[i].position += bodies[i].linearVelocity * dt_sec;
    }
}
```

Finally, we can also add a ground body, that won't fall under the influence of gravity either:

```
void Scene::Initialize() {
    Body body;
    body.position = Vec3(0, 0, 10);
    body.orientation = Quat(0, 0, 0, 1);
    body.shape = new ShapeSphere(1.0f);
    body.inverseMass = 1.0f;
    bodies.push_back(body);

    Body earth;
    earth.position = Vec3(0, 0, -1000);
    earth.orientation = Quat(0, 0, 0, 1);
    earth.shape = new ShapeSphere(1000.0f);
    earth.inverseMass = 0.0f;
    bodies.push_back(earth);
}
```

Collisions

Sphere collisions

Sadly, we still don't have any means of colliding two bodies. Which means running our code so far, will cause the small sphere to pass through the large sphere. This clearly won't work for us. Fortunately, sphere sphere collisions are the simplest to test. And we can easily add this to our code.

Each sphere has a position, and this position is the center of the sphere in world space. A simple way to check for overlap is to compute the distance between these two points and then compare them with the sum of the two radii. If the distance is less than the two radii, then there’s an intersection.

Create a new Intersections class.

```
#pragma once
#include "Body.h"
#include "Shape.h"

class Intersections
{
public:
    bool Intersect(Body& a, Body& b);
};
```

```
#include "Intersections.h"

bool Intersections::Intersect(Body& a, Body& b)
{
    const Vec3 ab = b.position - a.position;
    if (a.shape->GetType() == Shape::ShapeType::SHAPE_SPHERE
    && b.shape->GetType() == Shape::ShapeType::SHAPE_SPHERE) {
        ShapeSphere* sphereA = reinterpret_cast<ShapeSphere*>(a.shape);
        ShapeSphere* sphereB = reinterpret_cast<ShapeSphere*>(b.shape);
        const float radiusAB = sphereA->radius + sphereB->radius;
        // We compare squares
        if (ab.GetLengthSqr() < radiusAB * radiusAB) {
            return true;
        }
    }

    return false;
}
```

We’re going to compare every body against every other body for an intersection. And since we need to do something when things intersect, we’re going to just set their velocity to zero for now.

```
void Scene::Update(const float dt_sec)
{
    ...
    // Collision checks
    for (int i = 0; i < bodies.size(); ++i)
    {
        for (int j = i+1; j < bodies.size(); ++j)
        {
            Body& bodyA = bodies[i];
            Body& bodyB = bodies[j];
            if (bodyA.inverseMass == 0.0f && bodyB.inverseMass == 0.0f)
```

```
                continue;
            if (Intersections::Intersect(bodyA, bodyB))
            {
                bodyA.linearVelocity.Zero();
                bodyB.linearVelocity.Zero();
            }
        }
    }

    // Position update
    ...
}
```

A couple of notes here. There’s no need to bother testing for intersections among pairs of bodies that both have infinite mass, since those bodies are never going to move anyway. So, we just skip those.

Running this new code, the body in the air falls until it hits the ground, then it stops. We now have our first glimpse into collision detection and we’re headed in the right direction. Now, it’s great that it stops. But if you look at it closely, you’ll notice that the two spheres are actually interpenetrating. What can we do about that?

Contact and projection methods

We would like to work towards a physically based collision response for pairs of bodies, but first we need to solve this interpenetration. One way of solving it, is to use what’s known as the projection method. This uses the contacts points on the surface of each body and their masses to separate them appropriately.

Let’s start by defining a data structure for the contacts in a new Contact class:

```
#pragma once
#include "code/Math/Vector.h"
#include "Body.h"
class Contact
{
public:
    Vec3 ptOnAWorldSpace;
    Vec3 ptOnALocalSpace;
    Vec3 ptOnBWorldSpace;
    Vec3 ptOnBLocalSpace;

    Vec3 normal;
    float separationDistance;
    float timeOfImpact;

    Body* a{ nullptr };
    Body* b{ nullptr };

    static void ResolveContact(Contact& contact);
};
```

```
#include "Contact.h"

void Contact::ResolveContact(Contact& contact)
{
    Body* a = contact.a;
    Body* b = contact.b;

    a->linearVelocity.Zero();
    b->linearVelocity.Zero();
}
```

We update Intersection to take a contact parameter and use the Contact class in the Scene:

```
bool Intersections::Intersect(Body& a, Body& b, Contact& contact)
{
    contact.a = &a;
    contact.b = &b;
    const Vec3 ab = b.position - a.position;
    contact.normal = ab;
    contact.normal.Normalize();

    if (a.shape->GetType() == Shape::ShapeType::SHAPE_SPHERE
    && b.shape->GetType() == Shape::ShapeType::SHAPE_SPHERE) {
        ShapeSphere* sphereA = static_cast<ShapeSphere*>(a.shape);
        ShapeSphere* sphereB = static_cast<ShapeSphere*>(b.shape);

        contact.ptOnAWorldSpace = a.position
                                + contact.normal * sphereA->radius;
        contact.ptOnBWorldSpace = b.position
                                - contact.normal * sphereB->radius;

        const float radiusAB = sphereA->radius + sphereB->radius;
        // We compare squares
        if (ab.GetLengthSqr() < radiusAB * radiusAB) {
            return true;
        }
    }

    return false;
}
```

(Be careful: the contact point on b is position MINUS normal * radius.)

```
void Scene::Update(const float dt_sec)
{
    ...
    // Collision checks
    for (int i = 0; i < bodies.size(); ++i)
    {
        for (int j = i+1; j < bodies.size(); ++j)
        {
            Body& bodyA = bodies[i];
```

```
                Body& bodyB = bodies[j];
                if (bodyA.inverseMass == 0.0f && bodyB.inverseMass == 0.0f)
                    continue;

                Contact contact;
                if (Intersections::Intersect(bodyA, bodyB, contact))
                {
                    Contact::ResolveContact(contact);
                }
            }
        }
        ...
    }
```

This does not change the behaviour. So now we return to the problem of interpenetration and how best to separate two bodies.

For our little program, we have one object with infinite mass and another of finite mass. Since the infinite mass shouldn't move, we could just move the finite mass along the direction of the contact normal by the penetration distance. While this would work, in this case, it wouldn't be a good idea to simply move the body with the lower mass. After all, if we had two bodies with equal mass, then both bodies should move equally.

A useful concept that we can exploit is the center of mass of the system. The center of mass of N massive particles is defined as:

$$x_{cm} = \frac{\sum_i x_i * m_i}{\sum_i m_i}$$

What we're going to want to do is to separate the colliding bodies such that the center of mass of the two bodies doesn't change. So, for two masses the equation for the center of mass is:

$$x_{cm} = \frac{x_1 * m_1 + x_2 * m_2}{m_1 + m_2}$$

Or using inverse mass:

$$x_{cm} = \frac{x_1 * m_2^{-1} + x_2 * m_1^{-1}}{m_1^{-1} + m_2^{-1}}$$

The current separation distance, d, is defined as: $d = x_1 - x_2$

Let's define x_1' and x_2' as the values of x_1 and x_2 when d is equal to 0. We want to find such vlues. We also know that x_{cm} is constant.

In the case where $d = 0$:

$$x_{cm} = \frac{x_1 * m_1 + x_2 * m_2}{m_1 + m_2} = \frac{x_1' * m_2^{-1} + x_2' * m_1^{-1}}{m_1^{-1} + m_2^{-1}}$$

Which is equavalent to:

$$x_1 * m_1 + x_2 * m_2 = x_1' * m_2^{-1} + x_2' * m_1^{-1}$$

So we can express x_1' and x_2' in function of x_1 and x_2 :

$$x'_1 = x_1 + \frac{d * m_1^{-1}}{m_1^{-1} + m_2^{-1}}$$

$$x'_2 = x_2 - \frac{d * m_2^{-1}}{m_1^{-1} + m_2^{-1}}$$

We can use those values in code:

```
void Contact::ResolveContact(Contact& contact)
{
    Body* a = contact.a;
    Body* b = contact.b;

    a->linearVelocity.Zero();
    b->linearVelocity.Zero();

    // If object are interpenetrating, use this to set them on contact
    const float tA = a->inverseMass / (a->inverseMass + b->inverseMass);
    const float tB = b->inverseMass / (a->inverseMass + b->inverseMass);
    const Vec3 d = contact.ptOnBWorldSpace - contact.ptOnAWorldSpace;

    a->position += d * tA;
    b->position -= d * tB;
}
```

Velocity and elasticity

Conservation of momentum

Now that the penetration problem is solved, we need to solve the velocity problem. We can't just set the velocities of our bodies to zero anytime they collide, since that's not very realistic and also boring to watch.

Fortunately, we have the physical laws of conservation of momentum and conservation of energy to exploit. Recall that momentum is defined as:

$$p = m * v$$

And kinetic energy T is defined as :

$$T = 1/2 * m * v^2$$

Because those quantities are conserved, we can write

$$m_1 * v_1 + m_2 * v_2 = m_1 * v'_1 + m_2 * v'_2$$

$$m_1 * v_1^2 + m_2 * v_2^2 = m_1 * v_1'^2 + m_2 * v_2'^2$$

If we reorganize to put v_1 on one side and v_2 on the other side:

$$m_1 * (v_1 - v'_1) = -m_2 * (v_2 - v'_2)$$

$$m_1 * (v_1^2 - v_1'^2) = -m_2 * (v_2^2 - v_2'^2)$$

Or :

$$m_1 * (v_1 - v'_1) = -m_2 * (v_2 - v'_2)$$

$$m_1 * (v_1 - v_1') * (v_1 + v_1') = -m_2 * (v_2 - v_2') * (v_2 + v_2')$$

We can use the one before last equation, in the last equation, so it simplifies in :

$$(v_1 + v_1') = (v_2 + v_2')$$

$$\Leftrightarrow v_2' = v_1 + v_1' - v_2$$

We can now write that:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$\Leftrightarrow m_1 v_1' = m_1 v_1 + m_2 v_2 - m_2 v_2'$$

$$\Leftrightarrow m_1 v_1' = m_1 v_1 + m_2 v_2 - m_2 v_1 - m_2 v_1' + m_2 v_2$$

$$\Leftrightarrow (m_1 + m_2) v_1' = (m_1 - m_2) v_1 + 2 m_2 v_2$$

If we multiply everything in the v_2' equation by $(m_1 + m_2)$ and use the previous result:

$$(m_1 + m_2) v_2' = (m_1 + m_2) v_1 + (m_1 + m_2) v_1' - (m_1 + m_2) v_2$$

$$\Leftrightarrow (m_1 + m_2) v_2' = (m_1 + m_2) v_1 + (m_1 - m_2) v_1 + 2 m_2 v_2 - (m_1 + m_2) v_2$$

$$\Leftrightarrow (m_1 + m_2) v_2' = 2 m_1 v_1 - m_1 v_2 + m_2 v_2$$

Now, recall the definition of impulse is the change in momentum:

$$J_1 = m_1 * (v_1' - v_1)$$

$$J_2 = m_2 * (v_2' - v_2)$$

So:

$$J_1 = m_1 * v_1' - m_1 * v_1 = \frac{m_1}{m_1 + m_2}((m_1 - m_2) v_1 + 2 m_2 v_2) - m_1 * v_1$$

$$\Leftrightarrow (m_1 + m_2) J_1 = m_1 (m_1 - m_2) v_1 + 2 m_1 m_2 v_2 - m_1 (m_1 + m_2) v_1$$

$$\Leftrightarrow (m_1 + m_2) J_1 = -m_1 m_2 v_1 + 2 m_1 m_2 v_2 - m_1 m_2 v_1 = 2 m_1 m_2 v_2 - 2 m_1 m_2 v_1$$

$$\Leftrightarrow J_1 = \frac{2 m_1 m_2 (v_2 - v_1)}{m_1 + m_2}$$

If we do the same process for J_2 :

$$J_2 = \frac{2 m_1 m_2 (v_1 - v_2)}{m_1 + m_2} = -J_1$$

Since we're storing the inverse masses of our bodies, it would be nice if the equation was written in terms of the inverse masses. This is pretty simple too, we only need to multiply the numerator and denominator by the inverse masses, which gives:

$$J_1 = \frac{2(v_2 - v_1)}{m_1^{-1} + m_2^{-1}}$$

$$J_2 = \frac{2(v_1 - v_2)}{m_1^{-1} + m_2^{-1}}$$

Now that was for the one dimensional case. Fortunately translating this into 3D is pretty easy. However, I'm just going to use an informal argument to formulate it. Basically, we only care about the velocities that are normal to the collision. Anything tangential, can be ignored, and the impulse response of the collision should only be along the normal of the contact.

So, to get the component of the relative velocity to the normal, we only need to perform the dot product with the normal, which gives the projection of the velocity on the normal.

We can now update the code:

```
void Contact::ResolveContact(Contact& contact)
{
    Body* a = contact.a;
    Body* b = contact.b;

    const float invMassA = a->inverseMass;
    const float invMassB = b->inverseMass;

    // Collision impulse
    const Vec3& n = contact.normal;
    const Vec3& velAb = a->linearVelocity - b->linearVelocity;
    const float impulseValueJ = -2.0f * velAb.Dot(n)
                                / (invMassA + invMassB);
    const Vec3 impulse = n * impulseValueJ;

    a->ApplyImpulseLinear(impulse);
    b->ApplyImpulseLinear(impulse * -1.0f);

    // If object are interpenetrating, use this to set them on contact
    const float tA = invMassA / (invMassA + invMassB);
    const float tB = invMassB / (invMassA + invMassB);
    const Vec3 d = contact.ptOnBWorldSpace - contact.ptOnAWorldSpace;

    a->position += d * tA;
    b->position -= d * tB;
}
```

Note that we removed the linear velocity `Zero()` call.

Elasticity

The collision response in the last paragraph is physically accurate. However, it only simulates a collision that perfectly conserves kinetic energy. Which is very closely how billiard balls bounce off each other, but what if we wanted to simulate something a little softer? Suppose instead of having two very hard billiard balls, we have two balls made of clay. Such a collision would actually lose a lot of kinetic energy. Of course, it should go without saying, that in the real world such a collision doesn't lose energy, it's simply converted from kinetic energy to thermal energy. If you ever have the opportunity, you may notice that a large chunk of metal that's been twisted from a high impact collision is very hot.

Let's quickly define some terms. A collision that perfectly conserves kinetic energy is called elastic. And a collision that loses some or all of its kinetic energy is called inelastic. Since our simulation already simulates an elastic collision, the question becomes "how do we simulate an inelastic collision?"

Well, we can introduce a new variable, e, that represents elasticity. This is also known as the constitution of restitution. And sometimes it's also referred to as "bounciness". e is a number in the range [0, 1] and it relates to the initial and final velocties by the equation:

$$e = \frac{v_2' - v_1'}{v_2 - v_1}$$

Anf if we recall from the last paragraph, the definition of impulse and conservation of momentum gave us these equations:

$$v_1' = v_1 + \frac{J}{m_1}$$

$$v_2' = v_2 - \frac{J}{m_2}$$

Therefore:

$$v_2' - v_1' = v_2 - v_1 - \frac{J}{m_2} - \frac{J}{m_1}$$

So, using the definition of e:

$$-e(v_2 - v_1) = v_2 - v_1 - \frac{J}{m_2} - \frac{J}{m_1}$$

$$\Leftrightarrow e(v_2 - v_1) = -(v_2 - v_1) + \frac{J}{m_2} + \frac{J}{m_1}$$

$$\Leftrightarrow e(v_2 - v_1) + (v_2 - v_1) = \frac{J}{m_2} + \frac{J}{m_1}$$

$$\Leftrightarrow (e+1)(v_2 - v_1) = \frac{J}{m_2} + \frac{J}{m_1}$$

$$\Leftrightarrow J = (1+e)\frac{v_2 - v_1}{m_1^{-1} + m_2^{-1}}$$

If e is set to 1, we have the same equation we had in the previous paragraph.

Let's add the elasticity variable in the body class:

```
class Body
{
public:
    Vec3 position;
    Quat orientation;
    Vec3 linearVelocity;
    float inverseMass;
    float elasticity;
    Shape* shape;
    ...
```

And in the ResolveContact function, we need to determine which body's elasticity to use. In a professional engine for games, you may have some more parameters exposed to allow designers to control how the elasticity is combined for two bodies. However, I like to use the simple pattern of multiplying the two elasticities together. This way, the elasticity that's used reflects the physical properties of both, while remaining in the [0, 1] range. The argument I use for this is if a hard billiard ball hits another ball that's made of clay, then they'll tend to stick. But if you had two balls made of clay, then they should stick even more. So, this should be a pretty good approximation of what you'd expect from the real world.

```
void Contact::ResolveContact(Contact& contact)
{
    Body* a = contact.a;
    Body* b = contact.b;

    const float invMassA = a->inverseMass;
    const float invMassB = b->inverseMass;

    const float elasticityA = a->elasticity;
    const float elasticityB = b->elasticity;
    const float elasticity = elasticityA * elasticityB;

    // Collision impulse
    const Vec3& n = contact.normal;
    const Vec3& velAb = a->linearVelocity - b->linearVelocity;
    const float impulseValueJ = -(1.0f + elasticity) * velAb.Dot(n)
                                / (invMassA + invMassB);
    const Vec3 impulse = n * impulseValueJ;
    ...
```

Now, test the code with an object elasticity of 0.5 and a grond elasticity of 1.

```
void Scene::Initialize() {
    Body body;
    body.position = Vec3(0, 0, 10);
    body.orientation = Quat(0, 0, 0, 1);
    body.shape = new ShapeSphere(1.0f);
    body.inverseMass = 1.0f;
    body.elasticity = 0.5f;
    bodies.push_back(body);

    Body earth;
    earth.position = Vec3(0, 0, -1000);
    earth.orientation = Quat(0, 0, 0, 1);
    earth.shape = new ShapeSphere(1000.0f);
    earth.inverseMass = 0.0f;
    earth.elasticity = 1.0f;
    bodies.push_back(earth);
}
```

Rotations

Angular velocity

You may have noticed that we're still missing rotations. Fortunately, dealing with rotation is almost as simple as handling translation. To do this we must introduce the concepts of **torque** τ (tau) and **angular momentum** L. These quantities are defined as:

$$\tau = I.\alpha = r \times F$$

$$L = I.\omega = r \times p$$

With α the angular acceleration, ω angular velocity, r the position vector (a vector from the point about which the torque is being measured to the point where the force is applied) and p the linear moment (mass times velocity in a line).

Here is a real world torque explanation:

https://www.youtube.com/watch?v=ekpSQHxXFBM

Angular momentum is a little more tricky:

https://www.youtube.com/watch?v=MULe4xv3lVk

These quantities are related to orientation by :

$$d\theta = \omega * dt$$

$$d\omega = \alpha * dt$$

Angular momentum and torque are related to each other in a similar way that force and momentum are related:

$$dL = \tau.dt = J$$

$$dL = I.d\omega \Rightarrow d\omega = I^{-1}.J$$

Now something that does make this a little more complicated is I. I is called the inertia tensor. It is similar to mass, where it is a measure of how much an object "resists" change to its angular momentum. We will go into more detail about this in the next lesson, but for this lesson it is only important to note that it is a 3x3 matrix. The inertia tensor is calculated from the mass distribution of the body. Since our bodies have a uniform density and the shapes of the bodies are spheres, the inertia tensor is defined as:

$$I = \begin{pmatrix} 2/5MR^2 & 0 & 0 \\ 0 & 2/5MR^2 & 0 \\ 0 & 0 & 2/5MR^2 \end{pmatrix}$$

Now, typically in a physics simulation, you only need the inverse inertia tensor; similar to how you really only need to store the inverse mass, as opposed to the mass of a body. However, I thought it would be a little easier to digest this concept if we stored the proper inertia tensor instead of its inverse. In order to access this in code, we're going to add a function to get it from the Shape class:

```
class Shape {
public:
    ...
    virtual Mat3 InertiaTensor() const = 0;
    ...
};

class ShapeSphere : public Shape {
...
    Mat3 InertiaTensor() const override;
...
};
```

```
#include "Shape.h"
#include "code/Math/Matrix.h"

Mat3 ShapeSphere::InertiaTensor() const
{
    Mat3 tensor;
    tensor.Zero();
    tensor.rows[0][0] = 2.0f * radius * radius / 5.0f;
    tensor.rows[1][1] = 2.0f * radius * radius / 5.0f;
    tensor.rows[2][2] = 2.0f * radius * radius / 5.0f;
}
```

As you've probably noticed, the shapes don't have mass. So to get the full inertia tensor, we're going to also add code to the body class. And we'll want to be able to access the inertia tensor in both world space and local body space.

```
class Body
{
public:
    ...
    Mat3 GetInverseInertiaTensorBodySpace() const;
    Mat3 GetInverseInertiaTensorWorldSpace() const;
    ...
```

```
Mat3 Body::GetInverseInertiaTensorBodySpace() const
{
    Mat3 inertiaTensor = shape->InertiaTensor();
    Mat3 inverseInertiaTensor = inertiaTensor.Inverse() * inverseMass;
    return inverseInertiaTensor;
}

Mat3 Body::GetInverseInertiaTensorWorldSpace() const
{
    Mat3 inertiaTensor = shape->InertiaTensor();
    Mat3 inverseInertiaTensor = inertiaTensor.Inverse() * inverseMass;
    Mat3 orient = orientation.ToMat3();
    inverseInertiaTensor = orient * inverseInertiaTensor
                                  * orient.Transpose();
    return inverseInertiaTensor;
}
```

Now that we've got the inertia tensor sorted out, we can get back to the angular impulse. Recall that we've already defined it to be:

$$dL = \tau.dt = J$$

$$dL = I.d\omega \Rightarrow d\omega = I^{-1}.J$$

It's important to note that these vector quantities are normal to the plane of rotation. For instance, a rotating bicycle wheel. Its angular momentum is a vector quantity that is parallel to the axis of the wheel. And when you apply a torque to the wheel by peddling harder or applying the break, the torque is a vector quantity that is also parallel to the axis of the wheel. With this, we have enough knowledge to implement the angular impulse function:

```
class Body
{
public:
    Vec3 position;
    Quat orientation;
    Vec3 linearVelocity;
    Vec3 angularVelocity;
    ...
    void ApplyImpulseAngular(const Vec3& impulse);
};


void Body::ApplyImpulseAngular(const Vec3& impulse)
{
    if (inverseMass == 0.0f) return;

    // L = I w = r x p
    // dL = I dw = r x J
    // dw = I^-1 * ( r x J )
    angularVelocity += GetInverseInertiaTensorWorldSpace() * impulse;

    // Clamp angular velocity
    // -- 30 rad per seconds, sufficient for now
    const float maxAngularSpeed = 30.0f;
    if (angularVelocity.GetLengthSqr() > maxAngularSpeed * maxAngularSpeed)
    {
        angularVelocity.Normalize();
        angularVelocity *= maxAngularSpeed;
    }
}
```

Let's take a break for a moment to discuss two curious chunks of code.

The first is the angular velocity clamp in the ApplyImpulseAngular function. It is common practice to clamp both the linear and angular velocity of bodies in game simulations. This is mainly for performance. It'll become more obvious why this is an issue when we get to the broadphase, and continuous collision detection for non-spherical bodies.

Now, we haven't bothered with clamping the linear velocity in this lesson, mainly because we don't need it. None of the setups that we'll explore in this course will have objects that move fast enough to cause performance issues. There will be a teleportation bug that we discuss later, but this will get fixed with continous collision detection. The next line of code I'd like to discuss is from the GetInverseInertiaTensorWorldSpace function:

```
inverseInertiaTensor = orient * inverseInertiaTensor
                              * orient.Transpose();
```

This is the world space inverse inertia tensor. You see, the inertia tensor that we had calculated before is in the local or model space of the body. But, the impulse that we're applying is in world space. So, we need to transform the local space inverse inertia tensor into world space. And one way to do that, is by multiplying by the inverse orientation matrix and the orientation matrix.

One way to think of this is an order of operations. We have a world space angular impulse that we'll dub J_ω. And the inverse inertia tensor is in local space, so we need to transform J_ω into local space, and we can do that with:

$$J_{local} = R^{-1}.J_{world}$$

R is the orientation matrix.

Now we can apply the angular impulse in local space to get the local space change in angular velocity:

$$d\omega_{local} = I.J_{local}$$

And then in order to get the change in angular velocity back into world space, we simply use the orientation matrix:

$$d\omega_{world} = R.d\omega_{local}$$

So, there's two ways to think of this. Either we're transforming the inverse inertia tensor from local space to world space:

$$I_{world} = R.I_{local}.R^{-1}$$

Or we're transforming the impulse into local space, applying it, then transforming the change of velocity from local space to world space.

One last note. The rotation matrix has a determinant of one, because it doesn't scale any vertices, it simply rotates them. So that means the inverse of the matrix and the transpose of the matrix are the same thing. A matrix with this special property is known as an orthogonal matrix. So we can also write:

$$I_{world} = R.I_{local}.R^{T}$$

General impulses

We have now covered both linear impulses and angular impulses. One can think of linear impulses as a general impulse that's applied through the center of mass of the body. And the angular impulses we're applying are also through the center of mass of the body.

However, in practice, very few impulses that are applied to a body will ever be exactly applied through the center of mass. And this means that we'll need to account for that. Which means we need to start with the center of mass.

We've already touched on center of mass when we discussed the projection method for separating bodies. If we recall the equation it is defined by:

$$x_{cm} = \frac{\sum_i x_i * m_i}{\sum_i m_i}$$

Now, since we are only going to simulate bodies with uniform mass densities, the center of mass of each body will be based solely upon the body's shape.

Fortunately for us, a sphere's center of mass is its geometric center. So we've been able to get away with using the body's position as equivalent to the center of mass. But it's probably worthwhile for us now to address it, especially since we won't be restricted to spheres in the next lesson.

If we have a shape where we can't assume its center of mass is also its geometric center, then how do we use the body's position and orientation to get the center of mass in world space?

Well, to do that, it is the same thing we do to transform the vertices of a model from model space to world space. The first thing we need to do is rotate the center of mass by the body's orientation, and then follow that up with a translation by the body's position.

So, since most impulses will be applied to a point on the surface of the body, we can presume we have both the position as well as the impulse itself. Now, how do we figure out what the linear impulse and angular impulses are from the position and impulse itself?

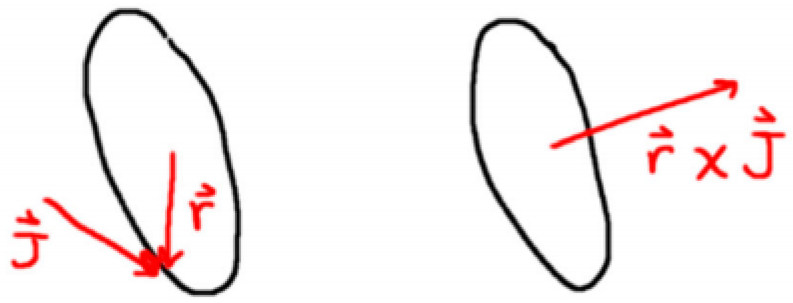
Well, as it turns out the linear impulse will just be the impulse itself. But we need the position of the impulse to figure out the angular impulse. Fortunately, we can just use the definition of the angular impulse to do that.

Recall that:

$$L = I.\omega = r \times p$$

$$\Rightarrow dL = I.d\omega = r \times J_{linear}$$

$$\Rightarrow J_{angular} = r \times J_{linear}$$



And translating this into code:

```
class Body
{
public:
    ...
    /// <summary>
    /// Apply impulse on a specific world space
    /// </summary>
    /// <param name="impulsePoint">
    /// The world space location of the application of the impulse
    /// </param>
    /// <param name="impulse">
    /// The world space direction and magnitude of the impulse
    ///</param>
    void ApplyImpulse(const Vec3& impulsePoint, const Vec3& impulse);
};
```

```
void Body::ApplyImpulse(const Vec3& impulsePoint, const Vec3& impulse)
{
    if (inverseMass == 0.0f) return;
    ApplyImpulseLinear(impulse);

    // Applying impulse must produce torques through the center of mass
    Vec3 position = GetCenterOfMassWorldSpace();
    Vec3 r = impulsePoint - position;
    Vec3 dL = r.Cross(impulse); // World space
    ApplyImpulseAngular(dL);
}
```

Then we need to change the way we update the body position. For now we directly change the position in the Scene::Update function. Let's rather create a Body::Update:

```
class Body
{
public:
    ...
    void Update(const float dt_sec);
    ...
```

Now we just need to figure out how to update the orientation from the angular velocity and then implement it. Updating the orientation from the angular velocity is slightly more complicated than updating position. If the body’s shape is asymmetric, then the object will precess and cause an internal torque on itself. This may be counter intuitive, but you can search for videos on the internet of astronauts demonstrating this with a T-handle in orbit. Some names of this effect are ”The Tennis Racket Theorem”, ”Dzhanibekov effect”, and ”Intermediate Axis Theorem”.

$$\tau = \omega \times I.\omega$$

$$\tau = I.\alpha$$

$$\alpha = I^{-1}.(\omega \times I.\omega)$$

$$d\omega = \alpha.dt$$

$$d\theta = \omega.dt$$

And now, in order to update the orientation, instead of using addition, we update it with multiplication:

$$q' = dq * q$$

```
void Body::Update(const float dt_sec)
{
    position += linearVelocity * dt_sec;

    // We have an angular velocity around the center of mass,
    // this needs to be converted to relative to model position.
    // This way we can properly update the orientation
    // of the model
    Vec3 positionCM = GetCenterOfMassWorldSpace();
    Vec3 CMToPositon = position - positionCM;

    // Total torques is equal to external applied
    // torques + internal torque (precession)
    // T = Texternal + w x I * w
    // Texternal = 0 because it was applied in the collision
    // response function
    // T = Ia = w x I * w
    // a = I^-1 (w x I * w)
    Mat3 orientationMat = orientation.ToMat3();
    Mat3 inertiaTensor = orientationMat
                     * shape->InertiaTensor() * orientationMat.Transpose
    Vec3 alpha = inertiaTensor.Inverse()
                 * (angularVelocity.Cross(inertiaTensor * angularVelocity));
    angularVelocity += alpha * dt_sec;

    // Update orientation
    Vec3 dAngle = angularVelocity * dt_sec;
    Quat dq = Quat(dAngle, dAngle.GetMagnitude());
    orientation = dq * orientation;
    orientation.Normalize();

    // Get the new model position
    position = positionCM + dq.RotatePoint(CMToPositon);
}
```

Now let's call this function in Scene:

```
void Scene::Update(const float dt_sec)
{
    ...
    // Position update
    for (int i = 0; i < bodies.size(); ++i) {
        bodies[i].Update(dt_sec);
    }
}
```

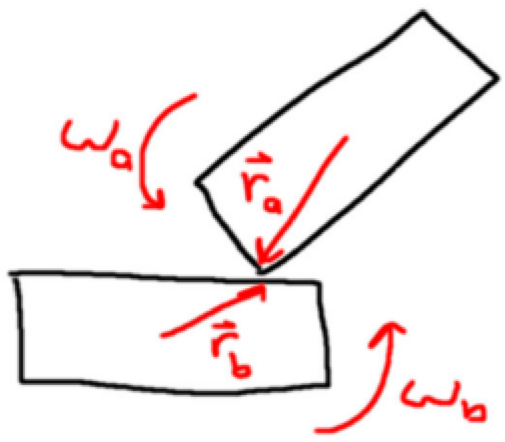
For now it changes nothing.

Add angular collision impulse

You might've thought that we had finished the contact resolution function. But unfortunately, the impulse calculated for the impact did not take into account angular velocity. And that is the goal of this lesson.

Since angular velocity doesn't really matter for the impact impulse of spherical shapes, we won't be fully testing it yet. It'll matter much more when we investigate other shapes in the next lesson. But, I want to introduce it for two reasons. One reason is for completeness. Since this lesson is all about resolving ballistic contacts, it is important to include the angular velocity in that contact response. The other reason is so that when we begin the next lesson, we no longer have to worry about the ResolveContact function. It'll be finished and we can just focus on the new tasks.

Assume we have two rotating bodies that collide with each other like in next figure:



Now, if you recall from before we had the following equations from the linear conservation of momentum:

$$v_1' = v_1 + \frac{J}{m_1}$$

$$v_2' = v_2 - \frac{J}{m_2}$$

But now we also need to consider the conservation of angular momentum as well:

$$\omega_1' = \omega_1 + I^{-1}.(r_1 \times n).J$$

$$\omega_2' = \omega_2 + I^{-1}.(r_2 \times n).J$$

And we also had from elasticity that:

$$v_{12}' = -e * v_{12}$$

Only now the total linear velocity at the point of impact is:

$$v_{1_{total}} = v_1 + r_1 \times \omega_1$$

$$v_{2_{total}} = v_2 + r_1 \times \omega_2$$

Using these equations to solve the impulse gives:

$$J = \frac{(1+e)*(v_2 - v_1)}{m_1^{-1} + m_2^{-1} + (I_1^{-1}(r_1 \times n) \times r_1 + I_2^{-1}(r_2 \times n) \times r_2).n}$$

In code, we can write:

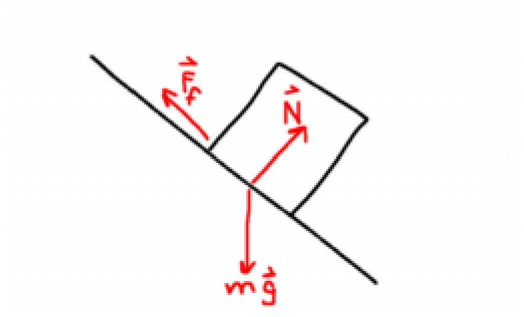
```
void Contact::ResolveContact(Contact& contact)
{
    Body* a = contact.a;
    Body* b = contact.b;

    const float invMassA = a->inverseMass;
    const float invMassB = b->inverseMass;

    const float elasticityA = a->elasticity;
    const float elasticityB = b->elasticity;
    const float elasticity = elasticityA * elasticityB;

    const Vec3 ptOnA = contact.ptOnAWorldSpace;
    const Vec3 ptOnB = contact.ptOnBWorldSpace;

    const Mat3 inverseWorldInertiaA = a->GetInverseInertiaTensorWorldSpace()
    const Mat3 inverseWorldInertiaB = b->GetInverseInertiaTensorWorldSpace()
    const Vec3 n = contact.normal;
    const Vec3 rA = ptOnA - a->GetCenterOfMassWorldSpace();
    const Vec3 rB = ptOnB - b->GetCenterOfMassWorldSpace();

    const Vec3 angularJA =
        (inverseWorldInertiaA * rA.Cross(n)).Cross(rA);
    const Vec3 angularJB =
        (inverseWorldInertiaB * rB.Cross(n)).Cross(rB);
    const float angularFactor = (angularJA + angularJB).Dot(n);

    // Get world space velocity of the motion and rotation
    const Vec3 velA = a->linearVelocity + a->angularVelocity.Cross(rA);
    const Vec3 velB = b->linearVelocity + b->angularVelocity.Cross(rB);

    // Collision impulse
    const Vec3& velAb = velA - velB;
    // -- Sign is changed here
    const float impulseValueJ = (1.0f + elasticity) * velAb.Dot(n)
                            / (invMassA + invMassB + angularFactor);
    const Vec3 impulse = n * impulseValueJ;

    a->ApplyImpulse(ptOnA, impulse * -1.0f); // ...And here
    b->ApplyImpulse(ptOnB, impulse * 1.0f); // ...And here

    // If object are interpenetrating, use this to set them on contact
    const float tA = invMassA / (invMassA + invMassB);
    const float tB = invMassB / (invMassA + invMassB);
    const Vec3 d = contact.ptOnBWorldSpace - contact.ptOnAWorldSpace;

    a->position += d * tA;
    b->position -= d * tB;
}
```

Frictions

Now that we've included rotation into our simulation. We can introduce friction. Sliding friction (also known as Coulomb friction) is defined as:

$$F_f = \mu * N$$

With F_f the force of friction, N the normal force and μ the coefficient of friction.



In the real world, we tend to make a distinction between static coefficients of friction and kinetic coefficients of friction. However, we're not going to bother with this, and we will only have a singular value for µ. Another important note is that the force of friction is always in the opposite direction of the velocity. And it always removes energy from the system, it never makes a sliding object slide faster. So it would be more appropriate to write:

$$F_f \leqslant \mu * N$$

So how do we calculate the normal force? Well, since our impulse response is already along the direction of the normal, we can treat that as our normal force. And then if we calculate the tangential velocity of bodyA and bodyB at the contact point, then we can use that as the direction that the force of friction is applied.

However, a surprisingly effective approach to collision is to treat it as a tangential collision that removes a portion of the energy each frame. As long as the values for friction that we use are between [0, 1) then we can use all the previous mathematics we've already derived. Only instead of using the normal to calculate the impulses, we use the tangential velocity for the direction. This yields:

$$J = \frac{\mu * v_t}{m_1^{-1} + m_2^{-1} + (I_1^{-1}(r_1 \times \hat{v_t}) \times r_1 + I_2^{-1}(r_2 \times \hat{v_t}) \times r_2).\hat{v_t}}$$

(The hat means the vector is normalized.)

In code:

```
void Contact::ResolveContact(Contact& contact)
{
    ...
    a->ApplyImpulse(ptOnA, impulse * -1.0f); // ...And here
    b->ApplyImpulse(ptOnB, impulse * 1.0f); // ...And here

    // Friction-caused impulse
    const float frictionA = a->friction;
    const float frictionB = b->friction;
    const float friction = frictionA * frictionB;

    // -- Find the normal direction of the velocity
    // -- with respect to the normal of the collision
    const Vec3 velNormal = n * n.Dot(velAb);
    // -- Find the tengent direction of the velocity
    // -- with respect to the normal of the collision
    const Vec3 velTengent = velAb - velNormal;
    // -- Get the tengential velocities relative to the other body
    Vec3 relativVelTengent = velTengent;
    relativVelTengent.Normalize();
    const Vec3 inertiaA =
        (inverseWorldInertiaA * rA.Cross(relativVelTengent)).Cross(rA);
    const Vec3 inertiaB =
        (inverseWorldInertiaB * rB.Cross(relativVelTengent)).Cross(rB);
    const float inverseInertia = (inertiaA + inertiaB).Dot(relativVelTengent

    // -- Tengential impulse for friction
    const float reducedMass =
        1.0f / (a->inverseMass + b->inverseMass + inverseInertia);
    const Vec3 impulseFriction = velTengent * reducedMass * friction;
    // -- Apply kinetic friction
    a->ApplyImpulse(ptOnA, impulseFriction * -1.0f);
    b->ApplyImpulse(ptOnB, impulseFriction * 1.0f);


    // If object are interpenetrating, use this to set them on contact
    ...
}
```

We need to update the body class and to add the firction and update the scene.

```
class Body
{
public:
    Vec3 position;
    Quat orientation;
    Vec3 linearVelocity;
    Vec3 angularVelocity;

    float inverseMass;
    float elasticity;
    float friction;
    ...



void Scene::Initialize() {
    Body body;
    body.position = Vec3(0, 0, 10);
    body.orientation = Quat(0, 0, 0, 1);
    body.shape = new ShapeSphere(1.0f);
```

```
        body.inverseMass = 1.0f;
        body.elasticity = 0.5f;
        body.friction = 0.5f;
        body.linearVelocity = Vec3(1, 0, 0);
        bodies.push_back(body);

        Body earth;
        earth.position = Vec3(0, 0, -1000);
        earth.orientation = Quat(0, 0, 0, 1);
        earth.shape = new ShapeSphere(1000.0f);
        earth.inverseMass = 0.0f;
        earth.elasticity = 0.99f;
        earth.friction = 0.5f;
        bodies.push_back(earth);
    }
```

As we can see, running this code with friction, the dynamic body slides and then starts rotating. And everything is finally behaving the way we would expect if we dropped a bunch of balls or marbles.

Solving collision bugs and optimizing

Continuous collision detection

Now, that we've fully formed our contact resolution. It's time to discuss the limitations of checking for collisions at finite time steps. The most glaring problem that we have right now, is that we don't consider the velocities of objects when we check for collision. This means, that if we have some very thin geometry and an object that moves very fast (a classic example is a bullet vs a paper thin wall or a fast moving sphere) then the object may end up passing through the geometry. This is known as teleportation.

We can fix this problem with what's known as continuous collision detection. All we really need to do is take the velocities of the bodies, and the time step of the frame, into account when we check for intersections. Then, if the two bodies collide, we also need to return the time of impact.

To illustrate the basics of this, let's look at a sphere-sphere collision. We can imagine that in the frame's time step, the two spheres move a certain distance, and their trajectories sweep out a capsule in space (the "trail" of the sphere).

We could also simplify the situation by using the relative velocities of the two spheres. In this case, one body will appear to be at rest and the other body will move more. Now we only have to collide between a capsule and a sphere. This in turn can be calculated as a line segment versus a sphere with the combined radius of both sphere A and sphere B : we check if a line which represents the relative velocity would enter in the max collision zone of both spheres.

Solving for a ray trace vs a sphere is pretty common in computer graphics. So it should be pretty simple to solve the equations for intersection.

Direction:

$$d = b - a$$

Ray equation in function of time t:

$$r(t) = a + d * t$$

We define s being the vector between the center of the sphere and the point on the ray at time t:

$$s(t) = c_{sphere} - r(t)$$

We want to solve for t such as $s.s = R^2$, which means the point on the ray is exactly on the surface of the sphere.

$$R^2 - s.s = 0$$

$$\Leftrightarrow R^2 - c^2 - r(t)^2 + 2c.r = 0$$

$$\Leftrightarrow R^2 - c^2 - a^2 - a.d * t - d^2 * t^2 = 0$$

This is a second degree polynomial. Whe can find 0, 1 or 2 solutions. We create a Ray-Sphere collision function in the Intersection class:

```
bool Intersections::RaySphere(const Vec3& rayStart, const Vec3& rayDir,
    const Vec3& sphereCenter, const float sphereRadius, float& t0, float& t1
{
    const Vec3& s = sphereCenter - rayStart;
    const float a = rayDir.Dot(rayDir);
    const float b = s.Dot(rayDir);
    const float c = s.Dot(s) - sphereRadius * sphereRadius;

    const float delta = b * b - a * c;
    const float inverseA = 1.0f / a;

    if (delta < 0) {
        // No solution
        return false;
    }

    const float deltaRoot = sqrtf(delta);
    t0 = (b - deltaRoot) * inverseA;
    t1 = (b + deltaRoot) * inverseA;


    return true;
}
```

With this helper function, we can create a sphere-sphere collision function:

```
bool Intersections::SphereSphereDynamic(
    const ShapeSphere& shapeA, const ShapeSphere& shapeB,
    const Vec3& posA, const Vec3& posB, const Vec3& velA, const Vec3& velB,
    const float dt, Vec3& ptOnA, Vec3& ptOnB, float& timeOfImpact)
{
    const Vec3 relativeVelocity = velA - velB;

    const Vec3 startPtA = posA;
    const Vec3 endPtA = startPtA + relativeVelocity * dt;
    const Vec3 rayDir = endPtA - startPtA;

    float t0 = 0;
    float t1 = 0;
    if (rayDir.GetLengthSqr() < 0.001f * 0.001f)
    {
        // Ray is too short, just check if already intersecting
        Vec3 ab = posB - posA;
        float radius = shapeA.radius + shapeB.radius + 0.001f;
        if (ab.GetLengthSqr() > radius * radius)
        {
            return false;
        }
    }
    else if (!RaySphere(startPtA, rayDir, posB,
        shapeA.radius + shapeB.radius, t0, t1))
    {
        return false;
    }

    // Change from [0, 1] to [0, dt];
    t0 *= dt;
    t1 *= dt;

    // If the collision in only in the past, there will be
    // no future collision for this frame
    if (t1 < 0) return false;

    // Get earliest positive time of impact
    timeOfImpact = t0 < 0.0f ? 0.0f : t0;

    // If the earliest collision is too far in the future,
    // then there's no collision this frame
    if (timeOfImpact > dt) {
        return false;
    }

    // Get the points on the respective points of collision
    // and return true
    Vec3 newPosA = posA + velA * timeOfImpact;
    Vec3 newPosB = posB + velB * timeOfImpact;
    Vec3 ab = newPosB - newPosA;
    ab.Normalize();

    ptOnA = newPosA + ab * shapeA.radius;
    ptOnB = newPosB - ab * shapeB.radius;
    return true;
}
```

We can now update the Intersect function to our our new collision detection. We add a dt argument:

```
bool Intersections::Intersect(Body& a, Body& b,
    const float dt, Contact& contact)
{
    contact.a = &a;
    contact.b = &b;
    const Vec3 ab = b.position - a.position;
    contact.normal = ab;
    contact.normal.Normalize();

    if (a.shape->GetType() == Shape::ShapeType::SHAPE_SPHERE
    && b.shape->GetType() == Shape::ShapeType::SHAPE_SPHERE) {
        ShapeSphere* sphereA = static_cast<ShapeSphere*>(a.shape);
        ShapeSphere* sphereB = static_cast<ShapeSphere*>(b.shape);

        Vec3 posA = a.position;
        Vec3 posB = b.position;
        Vec3 valA = a.linearVelocity;
        Vec3 velB = b.linearVelocity;

        if (Intersections::SphereSphereDynamic(*sphereA, *sphereB,
            posA, posB, valA, velB, dt,
            contact.ptOnAWorldSpace, contact.ptOnBWorldSpace,
            contact.timeOfImpact))
        {
            // Step bodies forward to get local space collision points
            a.Update(contact.timeOfImpact);
            b.Update(contact.timeOfImpact);

            // Convert world space contacts to local space
            contact.ptOnALocalSpace =
                a.WorldSpaceToBodySpace(contact.ptOnAWorldSpace);
            contact.ptOnBLocalSpace =
                b.WorldSpaceToBodySpace(contact.ptOnBWorldSpace);

            Vec3 ab = a.position - b.position;
            contact.normal = ab;
            contact.normal.Normalize();

            // Unwind time step
            a.Update(-contact.timeOfImpact);
            b.Update(-contact.timeOfImpact);

            // Calculate separation distance
            float r = ab.GetMagnitude()
                      - (sphereA->radius + sphereB->radius);
            contact.separationDistance = r;
            return true;
        }
    }

    return false;
}
```

We should be able to detect collisions that happen between the frames. However, what's this time of impact parameter and how do we use it for properly resolving the collision?

Time of impact

Let's imagine a situation where there's three objects and they might all potentially collide. But two collide before the other, and then the change in velocities from that collision might prevent the third object from colliding with either. So, not only do we need to check for collisions, but we need to sort our collisions by TOI and perform our contact resolution in proper temporal order.

Obviously, when bodies collide, their velocities change. So, if a body has two potential collisions, but the first collision changes its velocity enough, to where the second collision's toi changes or doesn't happen at all... then we need to account for that. Something we could do is update all bodies by the TOI of the first collision, resolve that contact, then brute force recalculate all possible collisions for the simulation. If we worked in aerospace and were designing a jet engine, then sure, we could dedicate 100+hours to simulate 5 minutes of an eagle being sucked into the engine. And we'd be right to do that since we want to make sure the engine doesn't explode and kill everyone on the plane. However, we're making a simulation intended for games. We don't need super ultra accuracy as much as we need efficiency. For a high performance game we only have 16ms for each frame, and probably less than 2ms to service physics. So it's okay for us to have have some errors in our simulation. As long as they are not so blatently obvious as to offend any would be users. Now what are we to do?

We can't recalculate every single ricochet because fast moving objects might cause so many secondary and tertiary collisions that our performance tanks and the simulation is no longer "interactive". One thing we could do is add an "isDynamic" member to the body class. Only update a body's position when it's set to dynamic. And a body is dynamic until it has a collision. This would solve the teleportation bug, and not tank performance.

But it's a solution that has its own problems. If from frame to frame an object has a collision very early in the frame, then its update will appear to stutter. How do we solve that? Well, instead of setting "isDynamic" to false when it has a single collision, we could wait until its second or third collision to set it to false. That would make a happy medium of avoiding stutter and keeping performance up.

However, it won't be what we do here. Instead, we won't bother recalculating the secondary collision for bodies. This means that it's possible for two bodies to still teleport through each other, but it should now be hidden from the user. This way if there's a cluster of bodies, and a fast moving projectile, then it will still hit the proper first collision. But it might teleport through some other bodies or even have some phantom collisions (collisions that shouldn't have occured). However, the player probably won't notice this error, since the projectile still hit the first proper collision.

We first need a static helper function to sort contacts in function of their time of impact:

```
int Contact::CompareContact(const void* p1, const void* p2)
{
    const Contact& a = *(Contact*)p1;
    const Contact& b = *(Contact*)p1;
    if (a.timeOfImpact < b.timeOfImpact) {
        return -1;
    }
    else if (a.timeOfImpact == b.timeOfImpact) {
        return -0;
    }
    return 1;
}
```

Now we can change our scene update:

```
    void Scene::Update(const float dt_sec)
    {
        // Gravity
        for (int i = 0; i < bodies.size(); ++i)
        {
            Body& body = bodies[i];
            float mass = 1.0f / body.inverseMass;
            // Gravity needs to be an impulse I
            // I == dp, so F == dp/dt <=> dp = F * dt
            // <=> I = F * dt <=> I = m * g * dt
            Vec3 impulseGravity = Vec3(0, 0, -10) * mass * dt_sec;
            body.ApplyImpulseLinear(impulseGravity);
        }

        // Collision checks
        int numContacts = 0;
        const int maxContacts = bodies.size() * bodies.size();
        Contact* contacts = (Contact*)alloca(sizeof(Contact) * maxContacts);
        for (int i = 0; i < bodies.size(); ++i)
        {
            for (int j = i+1; j < bodies.size(); ++j)
            {
                Body& bodyA = bodies[i];
                Body& bodyB = bodies[j];
                if (bodyA.inverseMass == 0.0f && bodyB.inverseMass == 0.0f)
                    continue;

                Contact contact;
                if (Intersections::Intersect(bodyA, bodyB, dt_sec, contact))
                {
                    contacts[numContacts] = contact;
                    ++numContacts;
                }
            }
        }

        // Sort times of impact
        if (numContacts > 1) {
            qsort(contacts, numContacts, sizeof(Contact),
                Contact::CompareContact);
        }

        // Contact resolve in order
        float accumulatedTime = 0.0f;
        for (int i = 0; i < numContacts; ++i)
        {
            Contact& contact = contacts[i];
            const float dt = contact.timeOfImpact - accumulatedTime;
            Body* bodyA = contact.a;
            Body* bodyB = contact.b;

            // Skip body par with infinite mass
            if (bodyA->inverseMass == 0.0f && bodyB->inverseMass == 0.0f)
                continue;

            // Position update
            for (int j = 0; j < bodies.size(); ++j) {
                bodies[j].Update(dt);
            }

            Contact::ResolveContact(contact);
            accumulatedTime += dt;
        }

        // Other physics behavirous, outside collisions.
        // Update the positions for the rest of this frame's time.
        const float timeRemaining = dt_sec - accumulatedTime;
        if (timeRemaining > 0.0f)
        {
            for (int i = 0; i < bodies.size(); ++i) {
                bodies[i].Update(timeRemaining);
            }
        }
    }
```

As you can see, we now have the collision loop broken up a little. We first collect all the contacts, and then sort them based upon their timeOfImpact. Then we resolve each contact in temporal order. As we do that, we then update the bodies. It's almost as if we've broken the entire update loop into smaller slices of time.

Something else that I think is important is a modification to the ResolveContact function. In theory, if we resolve a collision at the time the collision happens, then we don't need to manually push the two objects apart. So, now we can change the ResolveContact function to only perform the projection method when the time of impact is zero:

```
void Contact::ResolveContact(Contact& contact)
{
    ...
    // -- Apply kinetic friction
    a->ApplyImpulse(ptOnA, impulseFriction * -1.0f);
    b->ApplyImpulse(ptOnB, impulseFriction * 1.0f);


    // If object are interpenetrating, use this to set them on contact
    if (contact.timeOfImpact == 0.0f)
    {
        const float tA = invMassA / (invMassA + invMassB);
        const float tB = invMassB / (invMassA + invMassB);
        const Vec3 d = ptOnB - ptOnA;

        a->position += d * tA;
        b->position -= d * tB;
    }
}
```

Let’s go ahead and setup a scene to test this. We’ll add a floating body with infinite mass, and another body with high velocity.

```
void Scene::Initialize() {
    Body fast;
    fast.position = Vec3(-3, 0, 3);
    fast.orientation = Quat(0, 0, 0, 1);
    fast.shape = new ShapeSphere(1.0f);
    fast.inverseMass = 1.0f;
    fast.elasticity = 0.5f;
    fast.friction = 0.5f;
    fast.linearVelocity = Vec3(500, 0, 0);
    bodies.push_back(fast);

    Body immobile;
    immobile.position = Vec3(0, 0, 3);
    immobile.orientation = Quat(0, 0, 0, 1);
    immobile.shape = new ShapeSphere(1.0f);
    immobile.inverseMass = 1.0f;
    immobile.elasticity = 0.5f;
    immobile.friction = 0.5f;
    immobile.linearVelocity = Vec3(0, 0, 0);
    bodies.push_back(immobile);

    Body earth;
    earth.position = Vec3(0, 0, -1000);
    earth.orientation = Quat(0, 0, 0, 1);
    earth.shape = new ShapeSphere(1000.0f);
    earth.inverseMass = 0.0f;
    earth.elasticity = 0.99f;
    earth.friction = 0.5f;
    bodies.push_back(earth);
}
```

Optimizing collision detection with bounds

Checking the bounds of two shapes is significantly faster than performing an actual collision check. So an easy way to speed up these collision checks is by determining if their bounds overlap. Odds are good the bounds will not overlap and then we’ve determined there’s no collision, without the hard work of actually checking the collision. So let’s have a look at a typical implementation of a bounds class (already in the Start code):

```
class Bounds {
public:
    Bounds() { Clear(); }
    Bounds( const Bounds & rhs ) : mins( rhs.mins ), maxs( rhs.maxs ) {}
    const Bounds & operator = ( const Bounds & rhs );
    ~Bounds() {}

    void Clear() { mins = Vec3( 1e6 ); maxs = Vec3( -1e6 ); }
    bool DoesIntersect( const Bounds & rhs ) const;
    void Expand( const Vec3 * pts, const int num );
    void Expand( const Vec3 & rhs );
    void Expand( const Bounds & rhs );

    float WidthX() const { return maxs.x - mins.x; }
    float WidthY() const { return maxs.y - mins.y; }
    float WidthZ() const { return maxs.z - mins.z; }

public:
    Vec3 mins;
    Vec3 maxs;
};
```

```
#include "Bounds.h"
#include "../../Body.h"

const Bounds& Bounds::operator = (const Bounds& rhs) {
    mins = rhs.mins;
    maxs = rhs.maxs;
    return *this;
}


bool Bounds::DoesIntersect(const Bounds& rhs) const {
    if (maxs.x < rhs.mins.x || maxs.y < rhs.mins.y || maxs.z < rhs.mins.z) {
        return false;
    }
    if (rhs.maxs.x < mins.x || rhs.maxs.y < mins.y || rhs.maxs.z < mins.z) {
        return false;
    }
    return true;
}

void Bounds::Expand(const Vec3* pts, const int num) {
    for (int i = 0; i < num; i++) {
        Expand(pts[i]);
    }
}

void Bounds::Expand(const Vec3& rhs) {
    if (rhs.x < mins.x) {
        mins.x = rhs.x;
    }
    if (rhs.y < mins.y) {
        mins.y = rhs.y;
    }
    if (rhs.z < mins.z) {
        mins.z = rhs.z;
    }

    if (rhs.x > maxs.x) {
        maxs.x = rhs.x;
    }
    if (rhs.y > maxs.y) {
        maxs.y = rhs.y;
    }
    if (rhs.z > maxs.z) {
        maxs.z = rhs.z;
    }
}

void Bounds::Expand(const Bounds& rhs) {
    Expand(rhs.mins);
    Expand(rhs.maxs);
}
```

As you can see the bounds class only stores the minimum and maximum points. And looking at the DoesIntersect function, you can see why this check is so fast. It takes almost no work to check for overlap between bounds. Now, we need to modify our shape classes too:

```
class ShapeSphere : public Shape
{
public:
    ...
    Bounds GetBounds(const Vec3& pos, const Quat& orient) const override;
    Bounds GetBounds() const override;

    float radius;
};



Bounds ShapeSphere::GetBounds(const Vec3& pos, const Quat& orient) const
{
    Bounds tmp;
    tmp.mins = Vec3(-radius) + pos;
    tmp.maxs = Vec3(radius) + pos;
    return tmp;
}

Bounds ShapeSphere::GetBounds() const
{
    Bounds tmp;
    tmp.mins = Vec3(-radius);
    tmp.maxs = Vec3(radius);
    return tmp;
}
```

Notice that the GetBounds function takes in a position and orientation. This is mostly for our added convenience. When doing bounds checks on bodies, we need to take into account the position and orientation (for non-spherical bodies).

Optimization: Broadphase and narrowphase

Doing things brute force is pretty effective for getting started. But it’s not very efficient when the number of bodies starts increasing. This means that we’ll need to come up with a clever means of efficiently culling out potential collision pairs.

In order to do this we will introduce the concept of a broadphase and a narrowphase. The narrowphase is pretty much what we’ve been doing this whole time. It’s where we calculate contacts between bodies and then resolve those contacts. The broadphase is where we calculate potential collisions. Basically, we sort the bodies into a data structure that gives us possible collisions pairs. It’s a divide and conquer algorithm where we do a little extra work, at the beginning, to avoid doing wasteful work later.

The data structures that could potentially be used in a broadphase is numerous enough to fill its own book (octree, kd-tree, bvh, etc). So, we’ll just be using the basic, but effective, sort and sweep algorithm (aka sweep and prune). In the one dimensional case, we take the bounds of each body, and then sort them into an array of ascending order. Then we simply create collision pairs from the overlapping bodies.

You might be asking ”which axis should we sort on?” Well, sorting on the z-axis can be problematic when all the bodies have fallen onto the same floor height. Then the algorithm ends up in the worst case scenario and we’re back to poor performance. Should we then prefer the x or y axis? I’d say we shouldn’t take preference over any of them. And instead we should project the bounds onto the (1, 1, 1) axis. That should lead to decent enough performance for any scenario that we’ll encounter in those lessons.

And another question you may have is ”do we need to account for the velocities of the objects during the broadphase?” The answer to that is, yes. We will need to expand the bounds of each body by the distance they would travel in a single timestep. Otherwise all of our work put into continuous collision detection will be for nothing. The code for this is actually pretty straightforward. Create a Broadphase header and cpp file and code this:

```
#pragma once
#include <vector>
#include "Body.h"

struct CollisionPair
{
    int a;
    int b;

    bool operator==(const CollisionPair& rhs) const {
        return (((a == rhs.a) && (b == rhs.b))
            || ((a == rhs.b) && (b == rhs.a)));
    }
    bool operator!=(const CollisionPair& rhs) const {
        return !(*this == rhs);
    }
};

struct PseudoBody
{
    int id;
    float value;
    bool ismin;
};

void BroadPhase(const Body* bodies, const int num,
    std::vector<CollisionPair>& finalPairs, const float dt_sec);
```

```
#include "Broadphase.h"
#include "code/Math/Bounds.h"
#include "Shape.h"

int CompareSAP(const void* a, const void* b) {
    const PseudoBody* ea = (const PseudoBody*)a;
    const PseudoBody* eb = (const PseudoBody*)b;

    if (ea->value < eb->value) {
        return -1;
    }
    return 1;
}

void SortBodiesBounds(const Body* bodies, const size_t num,
                    PseudoBody* sortedArray, const float dt_sec)
{
    Vec3 axis = Vec3(1, 1, 1);
    axis.Normalize();

    for (int i = 0; i < num; i++)
    {
        const Body& body = bodies[i];
        Bounds bounds =
            body.shape->GetBounds(body.position, body.orientation);

        // Expand the bounds by the linear velocity
        bounds.Expand(bounds.mins + body.linearVelocity * dt_sec);
        bounds.Expand(bounds.maxs + body.linearVelocity * dt_sec);


        const float epsilon = 0.01f;
        bounds.Expand(bounds.mins + Vec3(-1, -1, -1) * epsilon);
        bounds.Expand(bounds.maxs + Vec3(1, 1, 1) * epsilon);

        sortedArray[i * 2 + 0].id = i;
        sortedArray[i * 2 + 0].value = axis.Dot(bounds.mins);
        sortedArray[i * 2 + 0].ismin = true;

        sortedArray[i * 2 + 1].id = i;
        sortedArray[i * 2 + 1].value = axis.Dot(bounds.maxs);
        sortedArray[i * 2 + 1].ismin = false;
    }

    qsort(sortedArray, num * 2, sizeof(PseudoBody), CompareSAP);
}

void BuildPairs(std::vector< CollisionPair >& collisionPairs, const PseudoBc
{
    collisionPairs.clear();

    // Now that the bodies are sorted, build the collision pairs
    for (int i = 0; i < num * 2; i++) {
        const PseudoBody& a = sortedBodies[i];
        if (!a.ismin) {
            continue;
        }

        CollisionPair pair;
        pair.a = a.id;

        for (int j = i + 1; j < num * 2; j++) {
            const PseudoBody& b = sortedBodies[j];
            // if we've hit the end of the a element,
            // then we're done creating pairs with a
            if (b.id == a.id) {
                break;
            }

            if (!b.ismin) {
                continue;
            }

            pair.b = b.id;
            collisionPairs.push_back(pair);
        }
    }
}

void SweepAndPrune1D(const Body* bodies, const size_t num,
    std::vector< CollisionPair >& finalPairs, const float dt_sec)
{
    PseudoBody* sortedBodies =
        (PseudoBody*)alloca(sizeof(PseudoBody) * num * 2);

    SortBodiesBounds(bodies, num, sortedBodies, dt_sec);
    BuildPairs(finalPairs, sortedBodies, num);
}

void BroadPhase(const Body* bodies, const int num,
    std::vector< CollisionPair >& finalPairs, const float dt_sec)
{
    finalPairs.clear();

    SweepAndPrune1D(bodies, num, finalPairs, dt_sec);
}
```

This straightforward little algorithm may not seem like much. But this does in fact provide a large performance improvement, especially when we get into more complicated shapes in the next lesson.

We will now modify our core physics loop:

```
void Scene::Update(const float dt_sec)
{
    // Gravity
    for (int i = 0; i < bodies.size(); ++i)
    {
        Body& body = bodies[i];
        float mass = 1.0f / body.inverseMass;
        // Gravity needs to be an impulse I
        // I == dp, so F == dp/dt <=> dp = F * dt
        // <=> I = F * dt <=> I = m * g * dt
        Vec3 impulseGravity = Vec3(0, 0, -10) * mass * dt_sec;
        body.ApplyImpulseLinear(impulseGravity);
    }

    // Broadphase
    std::vector<CollisionPair> collisionPairs;
    BroadPhase(bodies.data(), bodies.size(), collisionPairs, dt_sec);

    // Collision checks (Narrow phase)
    int numContacts = 0;
    const int maxContacts = bodies.size() * bodies.size();
    Contact* contacts = (Contact*)alloca(sizeof(Contact) * maxContacts);
    for (int i = 0; i < collisionPairs.size(); ++i)
    {
        const CollisionPair& pair = collisionPairs[i];
        Body& bodyA = bodies[pair.a];
        Body& bodyB = bodies[pair.b];

        if (bodyA.inverseMass == 0.0f && bodyB.inverseMass == 0.0f)
            continue;

        Contact contact;
        if (Intersections::Intersect(bodyA, bodyB, dt_sec, contact))
        {
            contacts[numContacts] = contact;
            ++numContacts;
        }
    }

    // Sort times of impact
    ...
}
```

Finally, let's test our optimization by modifying the number of bodies we are generating.

```
void Scene::Initialize() {
    Body body;
    for (int i = 0; i < 6; ++i)
    {
        for (int j = 0; j < 6; ++j)
        {
            float radius = 0.5f;
            float x = (i - 1) * radius * 1.5f;
            float y = (j - 1) * radius * 1.5f;
            body.position = Vec3(x, y, 10);
            body.orientation = Quat(0, 0, 0, 1);
            body.shape = new ShapeSphere(radius);
            body.inverseMass = 1.0f;
            body.elasticity = 0.5f;
            body.friction = 0.5f;
            body.linearVelocity.Zero();
            bodies.push_back(body);
        }
    }

    for (int i = 0; i < 3; ++i)
    {
        for (int j = 0; j < 3; ++j)
        {
            float radius = 80.0f;
            float x = (i - 1) * radius * 0.25f;
            float y = (j - 1) * radius * 0.25f;
            body.position = Vec3(x, y, -radius);
            body.orientation = Quat(0, 0, 0, 1);
            body.shape = new ShapeSphere(radius);
            body.inverseMass = 0.0f;
            body.elasticity = 0.99f;
            body.friction = 0.5f;
            bodies.push_back(body);
        }
    }
}
```

Exercice

Expand the terrain to make a bigger and quite flat environment. Make walls around it using other spheres. Now use this physics engine to create a "Petanque" game.

The petanque game is a South of France classical. It plays like this:

- At the beginning of the game, throw a small wooden ball called the "cochonnet" (little piggy). It will serves as an objective for the rest of the game.
- Each of the 2 players has 3 metal balls (elasticity near to zero). Those balls have a radius 4 times bigger than the cochonnet radius.
- One player is randomly selected as the beginner. He or she throws a ball to get the closer possible to the cochonnet (you can augment the friction of the balls).
- Then the second player shoot and tries to get nearer to the cochonnet.

- From then, the player who is further away from the cochonnet plays until he or she has no ball left.
- It is accepted to push other players ball with your own ball. It is called "tirer" (to shoot).
- Trying to aim close to the cochonnet is called "pointer" (to aim).
- The set finishes when all players have thrown their balls. The winner scores one point for each ball closer to the cochonnet. For instance, if the two balls closer to the cochonnet belongs to the first player, the first player scores 2.
- The loser of a set do not score any point.
- The game is won when a player, at the end of a set, has cumulated 13 or more points.
- When a player score 13 vs 0, the losing player has to "kiss Fanny".