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A STATISTICAL ANALYSIS OF EXCITED NUCLEAR STATES

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Abstract: The many known excited states of some medium-weight nucleides are analysed in terms of the total number of states as a function of the excitation energy Low-energy nuclear temperatures can be determined and are found to agree well with those obtained from evaporation spectra There is a marked odd-even difference in the number of states of odd and even nuclei, appearing as a displacement of the curves with respect to each other by the pairing energy Information on the spin distribution among the excited states is obtained and is related to a nuclear moment of inertia The theoretical interpretation of the odd-even effect and a constant temperature is discussed in terms of an energy-gap model

1. Introduction

The individual properties of excited nuclear states have been intensively studied for a number of years It has always been recognized that the highly excited states seen in neutron or proton resonances can be analysed in terms of their statistical properties. However, at lower excitation the level spectra are now in many cases so well known and contain so many states that, parallel to the individual level analysis, a statistical analysis may be applied At present, (pp') and (dp) experiments permit the study of 50 to 100 levels in a single nucleus, and it should be possible to raise the number even further. Nuclear temperatures are ordinarily determined from evaporation spectra. We can now determine the temperatures directly and thus obtain a long-needed check on the evaporation temperatures The fact that the data from the low-energy experiments can be directly connected with the neutron resonance data is very important in that it considerably extends the region in which the average spectrum can be studied In this paper we shall present a method of analysing the data of nuclear level densities, with an application to some of the best-studied medium-weight nuclei. As we now study the real level density, it is easy to make a direct comparison between the level densities of different nuclei and investigate the influence of the individuality of the nuclei.

The level densities from neutron resonances refer to only a few spin states Together with the total level density which we find here, the distribution of spins among the nuclear states is easily derived.

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2. Method of Statistical Analysis

In the following we shall confine our statistical study to an analysis of the level density and spin distribution for particular nuclei As we are dealing with the individual nuclear states, we shall obtain the level density $\hat{\rho}(E)$ not as a smooth function of the excitation energy E, but rather as a sum of δ-functions, each corresponding to one state A level density $\rho(E)$ which is a smooth function of energy is obtained by averaging over a suitable energy interval The averaging is most easily performed on the step function $N(E)$, the total number of states with excitation energies less than E. For this function, the averaging is in practice obvious for $N(E) > 10$ (see figures), and in the following we shall assume it to have been carried out.

By plotting the quantity $\log N(E)$ as a function of E we can obtain a kind of nuclear temperature τ which is somewhat different from the ordinary nuclear temperature T defined by the level density †. We have the relations

$$\frac{\mathrm{d}}{\mathrm{d}E}\log N(E) = \frac{1}{\tau} = \frac{\rho(E)}{N(E)} \tag{1}$$

so that

$$N(E) = \tau\rho(E) \tag{2}$$

Therefore, if we know $\rho(E)$ and τ, we also know $N(E)$. This relation is important because it permits us to relate a level density determined from, say, neutron resonances to the known level-number curve $N(E)$ at lower energy, obtained directly from a knowledge of the nuclear states.

The temperature τ is related to the ordinary nuclear temperature T by

$$\frac{\mathrm{d}}{\mathrm{d}E}\log \rho(E) = \frac{1}{T} = \frac{1}{\tau}\left(1 - \frac{\mathrm{d}\tau}{\mathrm{d}E}\right). \tag{3}$$

It is well known that $\rho(E)$ has an essentially exponential behaviour. Therefore we expect the two temperatures to be nearly equal If $N(E)$ depends upon the excitation energy like a pure exponential, they are indeed equal.

Though in principle we can determine $N(E)$ from a known level density $\rho(E)$ with the aid of eq (2), the situation is usually complicated because only level densities for certain spin values and parities are known. We have therefore to correct for such effects before applying eq (2). Theoretically, we expect on very general grounds that the density of states of definite parity and spin j can to a good approximation be expressed by

$$\rho_j(E) = (2j+1)\mathrm{e}^{-\frac{j(j+1)}{2\sigma^2}}\rho_0(E). \tag{4}$$

† Here and in the following we shall call $\{(\mathrm{d}/\mathrm{d}E)\log\rho(E)\}^{-1}$ the nuclear temperature T, as this is the quantity ordinarily determined by experiment

This form of the spin distribution assumes only that the spins associated with the excited degrees of freedom are coupled at random For further details we refer the reader to appendix A of ref [1]) Thus, from the density of states of spin 0, $\rho_0(E)$, we obtain for the total level density $\rho(E)$ (with the assumption of an equal probability of either parity)

$$\begin{aligned}\rho(E) &= 2\rho_0(E)\times\sum_j (2j+1)\mathrm{e}^{-\frac{j(j+1)}{2\sigma^2}} \\ &\approx 4\rho_0(E)\times\int_0^\infty x\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\,\mathrm{d}x = 4\sigma^2\rho_0(E)\end{aligned} \tag{5}$$

The dispersion σ is directly connected with the number n of spin-carrying elementary excitations through the relation

$$\sigma^2 = n\sigma_0{}^2, \tag{6}$$

where σ_0 is the spin dispersion for a single elementary excitation. A knowledge of the variation of σ^2 with energy is therefore equivalent to a knowledge of the variation of the number of elementary excitations with energy We can always express σ in terms of a nuclear moment of inertia $\mathscr{I}$ by the relation

$$\sigma^2 = \frac{\mathscr{I}T}{\hbar^2}, \tag{7}$$

which formally indicates that a part $\hbar^2 j(j+1)/2\mathscr{I}$ of the excitation energy is used to rotate the nucleus If the system has a classical analogue, we have, at sufficiently high excitation,

$$\mathscr{I} = \mathscr{I}_{\text{rigid}}. \tag{8}$$

This relation has been shown explicitly for certain models [2,3])

Little is known about the actual value of σ. It is theoretically expected that the moment of inertia will be smaller than its rigid-body value as it approaches this limit at high excitation [4]) At low excitation, deviations in either direction can be expected Angular distributions in compound nucleus processes in the iron region indicate a reduced moment of inertia [1,5]) (cf also ref [6]))

3. Examples

In the following we shall analyse experimental data from (pp') and (dp) reactions for the low-energy region and data from neutron resonances for the high-energy region according to the method outlined in the preceding section Limitations of the analysis may be brought about by

1) selection rules which lead to such a weak population of some states that they are not observed,

2) finite resolution, which may lead to underestimation of $N(E)$ when

adjoining states are not resolved This effect becomes increasingly important with increasing excitation

A calculation of the l-values of the ingoing particles shows that levels with a wide range of spin values of both parities should be excited For target nuclei of spin 0 or $\frac{1}{2}$ it is possible, however, that some high-spin states are not populated. That selection rules are unimportant is further confirmed by the fact that the same states are populated in both (pp') and (dp) reactions up to quite high excitations [7,8])

The second limitation has a considerably greater influence The number of unresolved levels is a function of the spacing between the levels At present, up to 40—70 levels, with very few unresolved levels among them, can be assigned to a nucleide in the iron region

The number of unresolved levels has been estimated from the resolution given in the experimental papers and is indicated in the figures by arrows. The values thus obtained are only approximate and must not be taken too literally

The following analysis is chiefly based on the high-precision (dp)-measurements of the ONR-Generator group at MIT, level-densities from neutron resonances being also included when available.

For the calculation of the value of $N(E)$ corresponding to the neutron-resonance level density we have used the temperature τ from the $N(E)$-curve at lower energy This extrapolation is justifiable because, whenever it has been possible to compare, τ has been found to agree closely with the temperatures from evaporation spectra, and furthermore, minor uncertainties in the temperature introduce only small changes in the calculated value for $N(E)$

Because the quantity σ (which determines the cut-off for the distribution of spins) is practically unknown, we have marked in the figures the calculated number of levels corresponding to different values of σ

3 1 Al^{28} AND S^{33}

Al^{28} (fig 1) This nucleus has been studied by (dp)-reactions up to the neutron binding energy [9]) Up to 4 5 MeV, practically all levels should be resolved, but even at neutron binding energy about one half of the states seem to be observed according to fig 1 In this nucleus, the neutron resonances have been very intensively studied by Hibdon [6]), who has resolved no less than 66 resonances in the first half MeV. We have determined the level density in the energy range 175 keV $< E_n <$ 375 keV for the incident neutron, where s, p and d waves give important contributions to the cross section The spin $\frac{5}{2}$ of Al^{27} together with these l-values assures that most resonances in this energy range are excited. This is also clearly indicated by the value of $N(E)$ calculated from the level density with the use of the value

of the temperature determined at lower excitation The level density has been determined over an appreciable energy region, and the local fluctu-

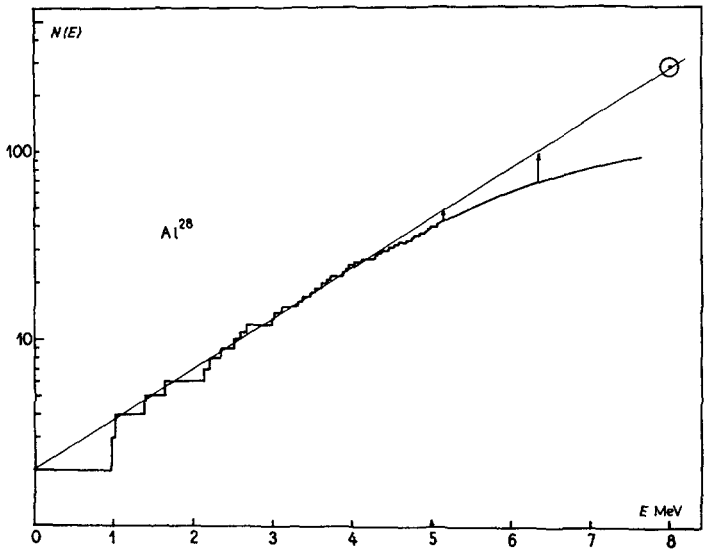

Fig 1 The total number of states up to the excitation energy E in Al^{28} vs E as obtained by (dp)-reaction [9]) The arrows indicate the number of unresolved levels estimated from the resolution The point from the neutron-resonance level density [6]) has been calculated by means of eq (2) The curve corresponds to a nuclear temperature τ of 1 6 MeV

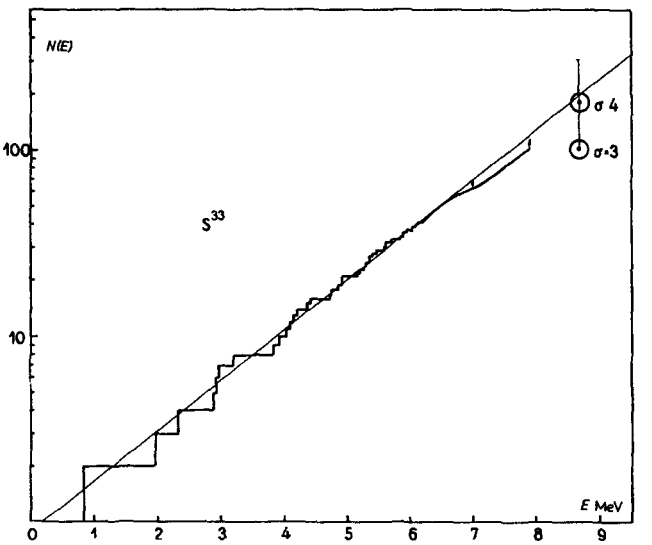

Fig 2 The total number of states up to the excitation energy E in S^{33} vs E as obtained by (dp)-reaction [10]) The arrows indicate the estimated number of unresolved levels The point from the neutron-resonance level density [11]) has been calculated by means of eq (2) with the spin distribution of eq (4) Its value for two different σ and its estimated uncertainty are indicated The curve corresponds to a nuclear temperature τ of 1 6 MeV and to σ 4

ations in the temperature should therefore be negligible Corrections to the calculated point for unresolved levels and for a temperature increasing with energy tend to increase its value but are probably negligible. This must therefore be considered independent confirmation that Hibdon observes most neutron resonances in this energy range for the incident neutrons We can furthermore conclude that the nuclear temperature is practically constant up to about 8 MeV for this nucleus This is also indicated by its high value, $\tau = 1\ 6$ MeV, in the low-energy part of the curve

$\underline{S^{33}}$ (fig 2) Also in the case of sulphur the levels have been studied by (dp)-reaction up to the neighbourhood of the neutron binding energy [10]) The neutron-resonance level density has been estimated from the three known s-wave resonances [11]), but the few resonances lead to a rather large uncertainty The data are consistent with the curve obtained from the (dp)-reaction and give an approximately constant nuclear temperature of 1 6 MeV (equal to that for Al^{28}) and a value for σ of about 4 As in the case of Al^{28}, it would be of considerable interest to have also a study by (dp)-reaction of the neutron resonant states

3 2 Mn AND Fe ISOTOPES

For the heavier nucleides now to be discussed, the present experimental resolution is not sufficient to study all the nuclear levels up to the neutron binding energy Consequently, the directly determined part of the $N(E)$-curve and the point calculated from neutron resonances are separated by several MeV, and we must extrapolate over this region. It is fortunate that the nuclear temperatures from the evaporation spectra of α-particles have been very carefully studied in this region (Ni—Zn) [5, 12])

The temperatures τ obtained below for Mn and Fe are 1 15—1 2 MeV, while the evaporation temperatures range from 0 94 to 1 16 MeV The differences in the temperatures are of the order to be expected from the fact that different nuclei are studied and from the influence on the evaporation spectrum of the spin distribution among the excited states We can therefore conclude that the temperature is nearly constant up to 10 MeV, and can also rather safely carry out the extrapolations

$\underline{Mn^{56}}$ (fig 3) The estimates of the number of unresolved levels are based on this nucleus At 2 4 MeV excitation, there was an indication of a break in the $N(E)$-curve and an appreciable increase of the temperature [13]) to a value above that determined from α-particle evaporation [12]) This effect appeared just in the region where resolution effects should start to become important A special investigation showed that this was indeed a resolution effect and that the true curve is in very good agreement with the expected curve [14]) † The temperature is 1 15 MeV The point at neutron binding energy is based on

† I am indebted to P R Christensen, O Elbek and M C Olesen for these unpublished data

6 resonances [15]) and indicates a value for σ of 4 5 This value is somewhat uncertain owing to a rather large uncertainty in the level density

Fe^{57} (fig. 4) The $N(E)$-curve is obtained from (pp') and (dp) data [7,8]). The level density is based on 8 neutron resonances [16]) The temperature is 1.2 MeV, σ is approximately 3.7

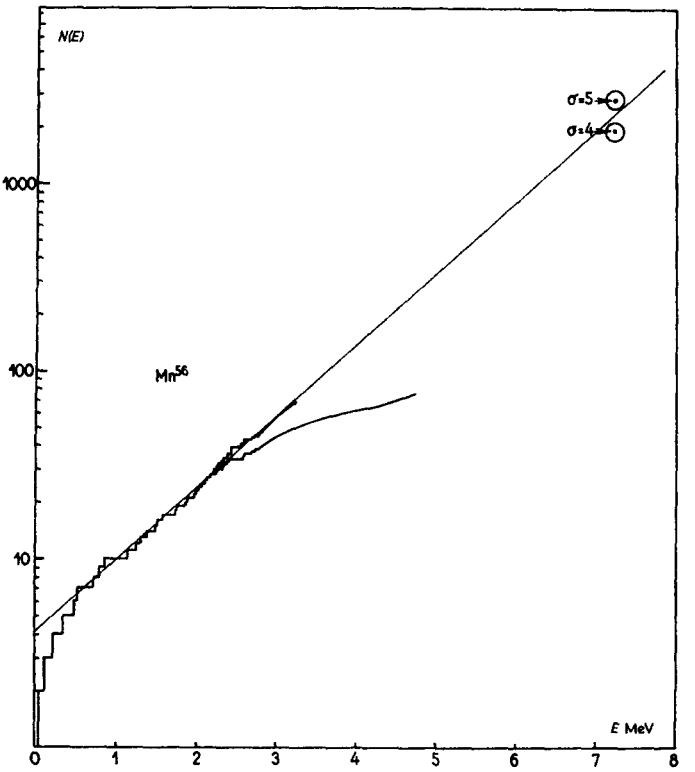

Fig 3 The total number of states up to the excitation energy E in Mn^{56} vs E as obtained from (dp)-reaction [13]) The higher curve for $2 < E < 3$ MeV has been obtained from a special investigation of the influence of the resolution [14]) The point from the neutron-resonance level density [15]) has been calculated by means of eq (2) with the spin distribution of eq (4) The curve corresponds to a nuclear temperature τ of 1 15 MeV and to σ 4 5

Mn^{56}, Fe^{55}, Fe^{57}, Fe^{58} (fig. 5) We have plotted the $N(E)$-curves for these nuclei in the same figure in order to compare even, odd, and odd-mass nuclei. The difference is striking For $N(E) > 10$ we obtain three groups of rather parallel curves, displaced with respect to each other by roughly 1 MeV. The energy difference between the extreme curves, 2 2 MeV, is quite close to the pairing energy for these nuclei.

4. Discussion

The analysis of the preceding section brings up several interesting properties of the nuclear level density in the low-energy region

The outstanding feature is the displacement of the curves for odd and

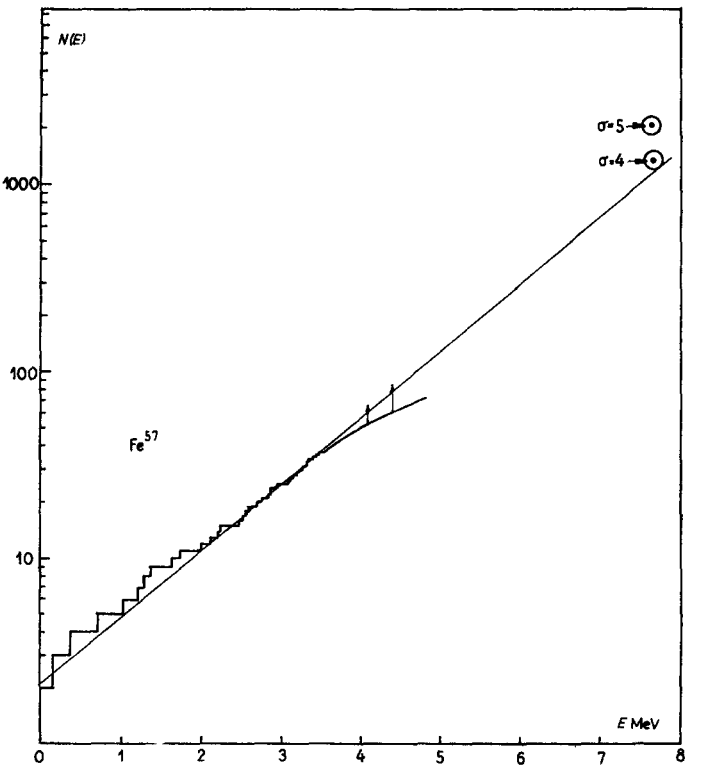

Fig 4 The total number of states up to the excitation energy E in Fe^{57} vs E as obtained from (pp′) and (dp)-reactions [7,8]) The arrows correspond to the number of unresolved levels estimated from the resolution The point from the neutron-resonance level density [18]) has been calculated by means of eq (2) with the spin distribution of eq (4) The curve corresponds to a nuclear temperature τ of 1 2 MeV and to σ 3 7

even nuclei with respect to each other (fig 5) A similar odd-even difference is suggested by the level densities determined from neutron resonances [17–19]). This effect indicates the existence of an energy gap in the nuclear spectrum[20]) similarly as for a system of interacting electrons in a superconductor [21,22]) That such displacements of the level-density curves occur even at low excitation energy has been predicted for a system with an energy gap for the elementary excitations [19]) The energy gap is expected to be of the order of half the pairing energy The displacement is for the Mn—Fe group 2 2 MeV,

while the pairing energy of Fe^{58} is 2 3 MeV, which shows the close agreement between these quantities

It is also seen from the $N(E)$-curves that the nuclear temperature is nearly constant as a function of energy for the nuclei and energy intervals considered Attempts are often made to fit nuclear level densities, using a

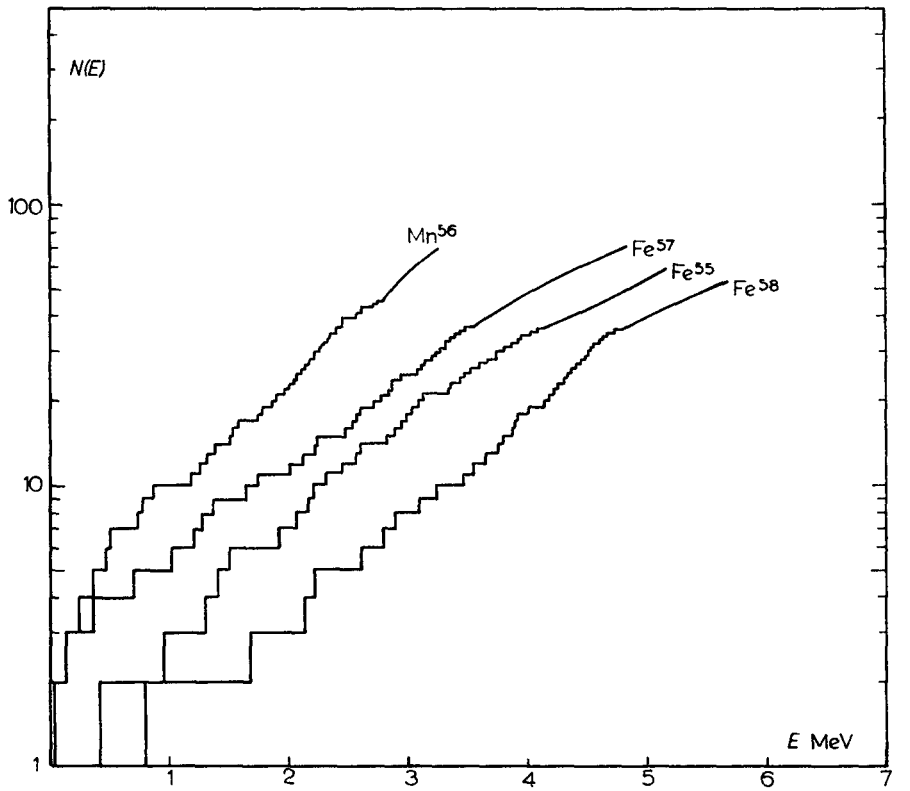

Fig 5 The total number of states up to the excitation energy E for Mn^{56}, Fe^{55}, Fe^{57}, and Fe^{58} vs E The resolution is somewhat higher in the case of Mn^{56} than for the rest The figure compares the total numbers of states for even, odd, and odd-mass nuclei

square-root law for the temperature as a function of energy [2, 23]) However, this law is the high-energy asymptotic form, and there is no reason for it to be valid at low excitation A constant temperature might be qualitatively understood as follows [19]) An energy gap introduces a certain ordering in the nucleus. With increasing excitation this ordering is gradually overcome as in a melting system [24]). The temperature therefore tends to be constant When the ordering disappears, we might possibly expect a second-order phase transition [25]) when the system goes into a state that can be described as an ordinary Fermi gas. The energy at which such a phase transition might occur is unknown, and there is so far no experimental evidence of it

Nuclear temperatures have not previously been obtained directly from the excited nuclear states, and thus it was hitherto impossible to verify that the nuclear temperatures from evaporation spectra are the same as those from level densities It is therefore gratifying to observe the agreement of the nuclear temperatures from α-particle evaporation spectra in the Ni—Cu region with those determined here [12]) This is strongly in support of the evaporation approximation giving a good description of a certain class of nuclear reactions

The value of σ obtained from the extrapolated $N(E)$-curves for Mn^{56} and Fe^{57} agrees roughly with the value obtained from the angular distribution of α-particles evaporated from nickel ($\sigma \approx 3\ 6$) [1]) It is still impossible, however, to draw any definite conclusion about a reduction, due to pairing interactions, of the rigid moment of inertia of these nuclei The values so far obtained for σ correspond to moments of inertia reduced by 0—30 %, but the uncertainty is still of the same order as the expected effect [4])

For Al^{28} the value of σ has been determined from the spin assignments to the neutron resonances to be $1\ 7$ [6]), corresponding to a reduction of the moment of inertia to about one third of its rigid-body value, or in a shell-model picture corresponding to an average number of about 2 excited particles This value is rather surprising when compared with the large value in S^{33}, $\sigma = 4$, which we have obtained The latter value seems rather well established because of the obvious need for a large number of additional states with spins higher than $\frac{1}{2}$. The error in σ for sulphur is hardly more than 33 % The sulphur value corresponds to a moment of inertia 50 % higher than the rigid-body value, which, however, falls within the limits of uncertainty The origin of the discrepancy between the Al^{28} and the S^{33} value must be further investigated It cannot be explained as a shell effect, but it might partly be due to the difficulty of assigning spins and higher l-values to neutron resonances

It is seen in fig 5 that the $N(E)$-curve for Fe^{57} lies higher than that for Fe^{55} We associate this with a shell effect, as Fe^{55} has only one neutron outside the $f\frac{7}{2}$ shell, while Fe^{57} has three There is therefore a greater number of recombination possibilities in Fe^{57} than in Fe^{55} at the same energy [26, 27]). The effect is smaller than expected on the basis of an independent particle estimate, presumably owing to pairing effects

It is interesting to compare our results with the very detailed semi-phenomenological calculation of level densities made by Cameron [28]) His calculation is based on an independent-particle shell model The interparticle interactions which give rise to the odd-even shift are only taken into account phenomenologically, by introduction of a "true" ground state for the non-interacting system corresponding to that of an odd nucleus In fitting his data to the experimental level densities he has to adjust his parameters in

such a way that they implicitly correspond to a moment of inertia two orders of magnitude larger than the rigid-body value According to our eq (6) this would imply a very large number of excited particles, and as Cameron's formula has been adjusted to the low-spin neutron resonances, it also implies a total level density about two orders of magnitude larger than the one we obtain (cf our eq (5)) While Cameron's expression should therefore still provide an excellent practical guide for estimates of neutron level densities, care must be taken when spin distributions are calculated from it His large value of the moment of inertia is not surprising The pairing interactions should reduce the level density predicted by the independent-particle model, and this reduction can phenomenologically be achieved also be assuming a large moment of inertia In the present case the independent-particle result has thus been decreased by several orders of magnitude

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