Reconstruction Algorithms in Computerized Tomography

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Tomography: Slice Imaging

Computerized Tomography (CT)

A technique for imaging the cross sections of an object using a series of x-ray measurements taken from different angles around the object.

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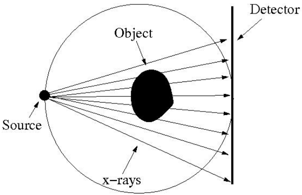
- They shared the 1979 Nobel Prize in Medicine.
- Since then researchers have tried to make
  - Scanning faster,
  - Dosage used optimal,
  - Reconstruction algorithms more efficient.

Reconstruction Algorithms

- Direct, e.g. Filtered back-projection (FBP).
- Iterative
  - Algebraic, e.g. algebraic reconstruction techniques (ART).
  - Statistical image reconstruction techniques (SIRT): (Weighted) Least squares or Likelihood (Poisson), e.g. maximum log-likelihood.



FBP- Modeling Assumptions

- Straight lines (no refraction/diffraction)
- Monochromatic source (X-ray photons of the same wave-length/energy)
- Beer's Law: X-ray intensity attenuates linearly in matter

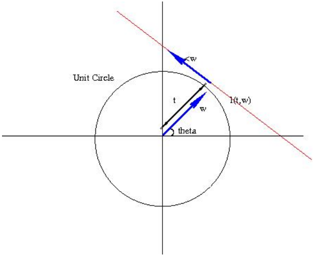
$$\frac{dI}{ds} = -\mu I \quad \Rightarrow \quad \boxed{\int_l \mu \, ds = -\ln\left(\frac{I}{I_0}\right)}$$

I x-ray intensity
μ attenuation coefficient
s arc length

In medical imaging, the *Radon transform* of a function provides a mathematical model for the measured attenuation.

$$[\mathcal{R}\mu(x,y)](t,\omega) := \int_{l_{(t,\omega)}} \mu(x,y)\, ds \,,$$

- $l_{(t,\omega)} = \{ t\omega + s\widehat{\omega} \ : \ s \in \mathbb{R} \}$
- $\widehat{\omega}$ unit vector perpendicular to unit vector ω .
  $\omega(\theta) = (\cos\theta, \sin\theta)$, with orientation $\det(\omega\widehat{\omega}) > 0$.



- t affine parameter.

Fourier Transform

The *Fourier Transform* of μ defined on $\mathbb{R}$ is

$$\widehat{\mu}(t) = \int_{-\infty}^{+\infty} \mu(x) e^{-ixt} \, dx \, .$$

Fourier Inversion Formula

$$\mu(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{\mu}(t) e^{ixt} \, dt \, .$$

These formulas can be extended to higher dimensions.

Central Slice Theorem

Key to efficient tomographic imaging

Relates the measured projection data (Radon transform) to the two-dimensional Fourier transform of the object cross sections.

Theorem

Let μ be an absolutely integrable function in the natural domain of Radon transform. For any real number r and unit vector ω, we have the identity

$$\int_{-\infty}^{+\infty} [\mathcal{R}\mu(x,y)](t,\omega)e^{-itr}\,dt = \widehat{\mu}(r\omega)\,.$$

Filtered Back-Projection

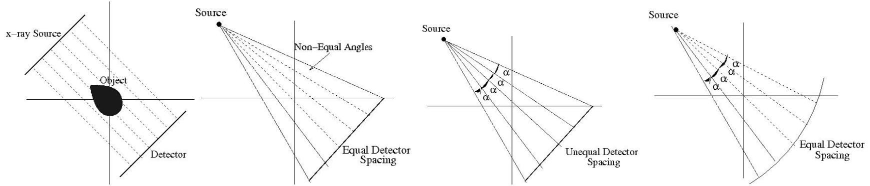
Central slice theorem and fourier inversion formula give an inversion formula for the Radon transform, a.k.a. filtered back-projection (FBP) formula.

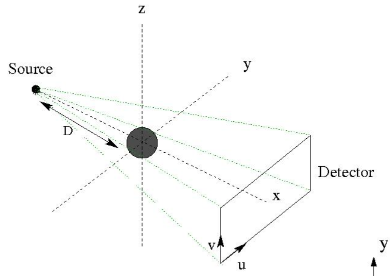
FBP in two steps

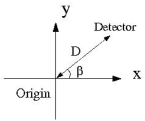
- Radial integral: Filtering
- Angular integral: Back-Projection (BP)

FBP is a good starting point for the development of practical algorithms.

Different Geometries







Feldkamp-Davis-Kress Algorithm (1984)

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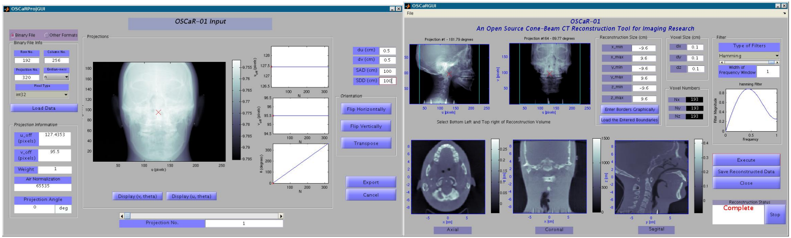
- OSCaR- Open Source Cone-beam CT Reconstruction Tool for Imaging Research



http://www.cs.toronto.edu/~nrezvani/OSCaR.html

FBP Deficiencies

- Difficulty in motion correction.
- Lack of flexibility.
- Not being able to model the noise and therefore more radiation.
- FBP assumes monochromatic source, but in reality we have polychromatic X-rays.
  - When polychromatic X-ray beam passes through matter, low energy photons are absorbed.
  - The beam gradually becomes harder, i.e. X-rays in ranges that are more penetrating are referred to as *hard*, opposed to *soft* X-rays that are more easily attenuated.
  - If not corrected, this may cause artifacts in CT images, as a result of inaccurate measurements.

This phenomenon is called Beam Hardening.

Algebraic Reconstruction Technique (ART)

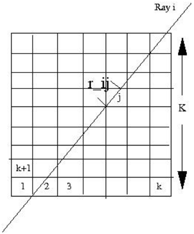
A different approach for CT reconstruction, first introduced in 1970.

In ART,

- it is assumed the cross-sections consist of arrays of unknowns.
- reconstruction problem can be formulated as a system of linear equations.
- choosing a finite collection of basis functions and a method to solve the systems are the next steps.

Pixel Basis

In ART localized basis, $b_{j}{}_{j=1}^{J}$, such as pixel basis are typically used.



- $b_j^K(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in j\text{th-square}, \\ 0 & \text{otherwise.} \end{cases}$
- An approximation to μ is $\bar{\mu}^K = \sum_{j=1}^{J} x_j b_j^K$.
- Radon transform is linear: $\mathcal{R}\bar{\mu}^K = \sum_{j=1}^{J} x_j \mathcal{R} b_j^K$.
- This stays true for many other bases.

Reconstruction Model

- Assume $\{b_j\}$ basis and $\mathcal{R}\mu$ sampled at

$$\{(t_1, \omega_1), \ldots, (t_I, \omega_I)\}\ .$$

- For $i = 1, \ldots, I$ and $j = 1, \ldots, J$, usually $I \neq J$, define measurement matrix as the line integrals: $r_{ij} = \mathcal{R}b_j(t_i, \omega_i)$.
- Vector of measurements will be $p_i = \mathcal{R}\mu(t_i, \omega_i)$.
- Reconstruction problem:

$$\sum_{j=1}^{J} r_{ij} x_j = p_i\ .$$

Iterative methods are used for solving $\mathbf{rx} = \mathbf{p}$.

The size of these systems of equations is usually large making ART methods computationally expensive.

An Example

(Introduction to Mathematics of Medical Imaging– Epstein, 2003)

> *If the square is divided into* $J = 128 \times 128 \simeq 16,000$ *subsquares, then, using the pixel basis, there are 16,000 unknowns. A reasonable number of measurements is 150 samples of the Radon transform at each of 128 equally spaced angles, so that* $I \simeq 19,000$*. That gives a* $19,000 \times 16,000$ *system of equations.*

Kaczmarz Method (Stefan Kaczmarz, 1937)

The principle method used in ART.

- If $\mathbf{rx} = \mathbf{p}$ has a solution, method converges.

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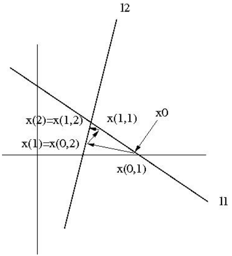
- Many other results by Censor, Elfving, Herman, Gordon, Byrne, and others.

The k-th step of the algorithm



1:

$$x^{(k)} \longrightarrow x^{(k)} - \frac{x^{(k)} \cdot \mathbf{r}_1 - p_1}{\mathbf{r}_1 \mathbf{r}_1} \cdot \mathbf{r}_1 = \mathbf{x}^{(k,1)} \ ,$$

2:

$$x^{(k,1)} \longrightarrow x^{(k,1)} - \frac{x^{(k,1)} \cdot \mathbf{r}_2 - p_2}{\mathbf{r}_2 \mathbf{r}_2} \cdot \mathbf{r}_2 = \mathbf{x}^{(k,2)} \ ,$$

3:

$$x^{(k,l-1)} \longrightarrow x^{(k,l-1)} - \frac{x^{(k,l-1)} \cdot \mathbf{r}_l - p_l}{\mathbf{r}_l \mathbf{r}_l} \cdot \mathbf{r}_l = \mathbf{x}^{(k+1)} \ .$$

Reminder- Beam hardening is a phenomenon that a polychromatic X-ray beam becomes more penetrating, or harder, as it traverses through matter.

Solutions for Beam Hardening

- Pre-filtering: Limiting beam hardening by physically pre-filtering X-ray beams.
- Post-reconstruction: Correct measurements based on material assumptions, e.g. Joseph and Spital method in (1978).
- By incorporating a polychromatic acquisition model.

Statistical image reconstruction techniques (SIRT) for X-ray CT can be developed based on a physical model that accounts for polyenergetic X-ray source.

SIRT for Polyenergetic X-ray CT, *(Elbakri and Fessler, 2002-2003)*

SIR method for X-ray CT based on a physical model that accounts for the polyenergetic X-ray source spectrum and the measurement nonlinearities caused by energy-dependent attenuation.

- Object comprised of known non-overlapping tissue types.
- Attenuation coefficient in jth voxel is modeled:

$$\mu_j(\varepsilon) = \sum_{k=1}^{K} m_k(\varepsilon)\rho_j f_j^k \ ,$$

  - $m(\varepsilon)$: energy-dependent mass attenuation, known for several tissue types.
  - ρ: unknown energy-independent density of the tissue
  - $f_j^k = 1$ if jth voxel belongs to kth material, otherwise 0.

Basic Idea: Given measurements $\{y_i\}_1^I$, find the distribution of linear attenuation coefficients $\{\mu_j\}_1^J$.

- The Poisson model for the measurements y_i's is assumed.
- For estimating the attenuation coefficient, maximum likelihood (ML) for the polyenergetic model, $L(\rho)$, is developed.
- Data is noisy and ML gives noisy reconstruction, hence we need regularization by adding a penalty term, $R(\rho)$ to the likelihood.

- Now our goal would be optimizing

$$\Phi(\rho) = -L(\rho) + \beta R(\rho) ,$$

  where β is a scalar that controls the tradeoff between the data-fit and the penalty terms.

- An iterative method for estimating unknown densities in each voxel is developed,
  - e.g. penalized weighted least squares with ordered subsets (PWLS-OS).

Other Approaches

- IMPACT (De Man et al, 2001)
  - Models object as a linear combination of the spectral properties of two base substances, usually water and bone.
  - Knowledge of the X-ray spectrum is necessary, but a pre-segmented image is not required.

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- Simplified SIR for Polyenergetic X-ray CT (Srivastava and Fessler, 2005)
  - Hybrid method between FBP-JS method and SIRT.
  - SIRT require more knowledge of the X-ray spectrum than is needed in post-reconstruction methods.
  - Uses the same calibration data and tuning parameters in JS, thereby facilitates its practical use.

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- SR Algorithm for Polyenergetic X-ray CT (Chueh et al, 2008)
  - Assumes energy spectrum comprised of several sub-energy spectrums.
  - Needs less a priori information in terms of tissue type and mass attenuation coefficients.

Comparison of Different Reconstruction Algorithms

Compared to FBP,

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- SIRT
  - incorporates the system geometry, detector response, object constraints and any prior knowledge more easily.
  - handles/models the noise more effectively, and therefore dosage used is more optimal.
  - uses a polychromatic acquisition model, since it is nonlinear.
  - has longer computation times, that hinder their use.

Open Problems and Possible Future Work

- How to handle attenuation coefficient and energy levels in SIRT, e.g. decomposition of the energy-dependent attenuation coefficient.
- Reducing the number of coefficients, better optimization methods, more efficient and faster iterative techniques.
- Better estimation methods in SIRT.
- Hybrid method between ART and SIRT.
- Investigating more sophisticated variants to generate the measurement matrix in ART.
- Possibly incorporating compressed sensing into SIRT.
- Using ART and SIRT in "4DCT"– fourth dimension is time.

Thank you for your attention!