Welcome!

#pod-031

Week #3, Day 2

(Reviewed by: Deepak)

neuromatch academy

facebook Reality Labs

Penn University of Pennsylvania
mindCORE Center for Outreach, Research, and Education
UC Irvine
Queen's University
PennState
University of Minnesota
CIFAR
IEEE brain
Simons Foundation
Templeton World Charity Foundation
The Kavli Foundation
think : theory
Tianqiao & Chrissy Chen Institute
wellcome
Gatsby
Bernstein Network Computational Neuroscience
NBDT
hhmi janelia Research Campus

Agenda

- Tutorial 1 (Neural Rate models)
    - 3 exercises + 2 bonus
- Tutorial 2 (Wilson-Cowan Model)
    - 4 exercises + 4 bonus

Tutorial #1
Explanations

Objective

The brain is a complex system, not because it is composed of a large number of diverse types of neurons, but mainly because of how neurons are connected to each other. The brain is indeed a network of highly specialized neuronal networks.

The activity of a neural network constantly evolves in time. For this reason, neurons can be modeled as dynamical systems. The dynamical system approach is only one of the many modeling approaches that computational neuroscientists have developed (other points of view include information processing, statistical models, etc.). How the dynamics of neuronal networks (network activity dynamics) affect the representation and processing of information in the brain is an open question. However, signatures of altered brain dynamics present in many brain diseases (e.g., in epilepsy or Parkinson's disease).

- simulation and study one of the simplest models of biological neuronal networks treating them as a single homogeneous population and approximate their dynamics using a single one-dimensional equation describing the evolution of their average spiking rate in time (build a firing rate model of a single population of excitatory neurons).
- Visualization of the response of the population as a function of parameters such as threshold level and gain, using the frequency-current (F-I) curve.
- Numerically simulate the dynamics of the excitatory population and find the fixed points of the system. Further investigation of the stability of the fixed points by linearizing the dynamics around them.

Dynamics of a single excitatory population

Individual neurons respond by spiking. When we average the spikes of neurons in a population, we can define the average firing activity of the population further used to study how the population-averaged firing varies as a function of time and network parameters.

W3D2_pod31

firing rate dynamic -

$$\tau \frac{dr}{dt} = -r + F(w \cdot r + I_{ext})$$

Synaptic strength of recurrent input to the population

controls time scale of evolution of avg firing rate

$r(t)$ avg firing rate of excitatory population at time t

external input

transfer function (f-I curve of individual neurons)

represents population activation function in response to all received inputs

Transfer function

The transfer function $F(\cdot)$ represents the gain of the population as a function of the total input. The gain is often modeled as a sigmoidal function, i.e., more input drive leads to a nonlinear increase in the population firing rate. The output firing rate will eventually saturate for high input values.

F-I curve

In electrophysiology, a neuron is often characterized by its spike rate output in response to input currents

F-I curve

Sigmoidal F

$$F(x;a,\theta) = \frac{1}{1+e^{-a(x-\theta)}} - \frac{1}{1+e^{a\theta}}$$

output spike frequency in response to diff. injected current

transfer function

input to population

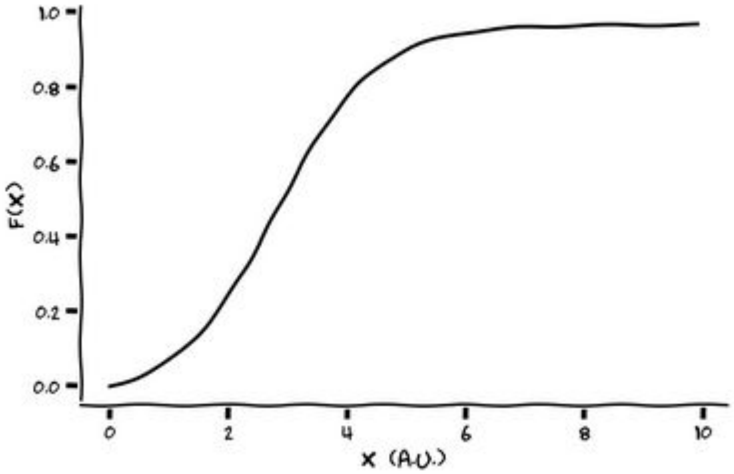
threshold

gain

$F(0;a,\theta)=0$

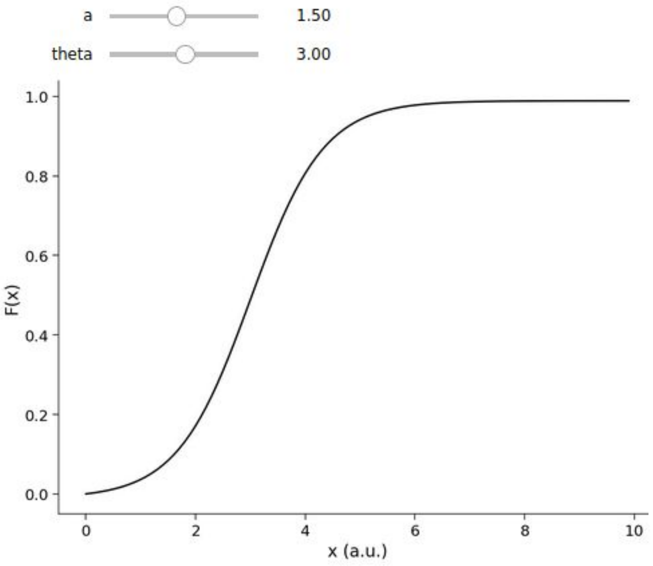
Other monotonic transfer functions

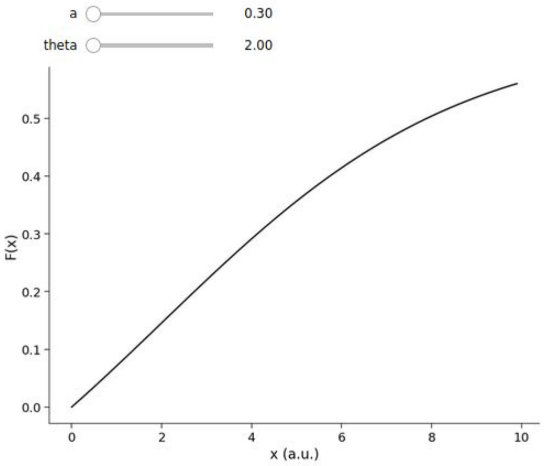
Many other transfer functions (generally monotonic) can be also used. Examples are the rectified linear function ReLU(x) or the hyperbolic tangent tanh(x).

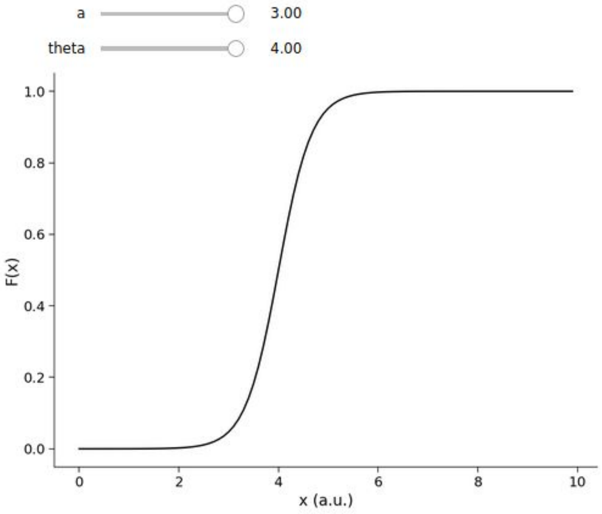


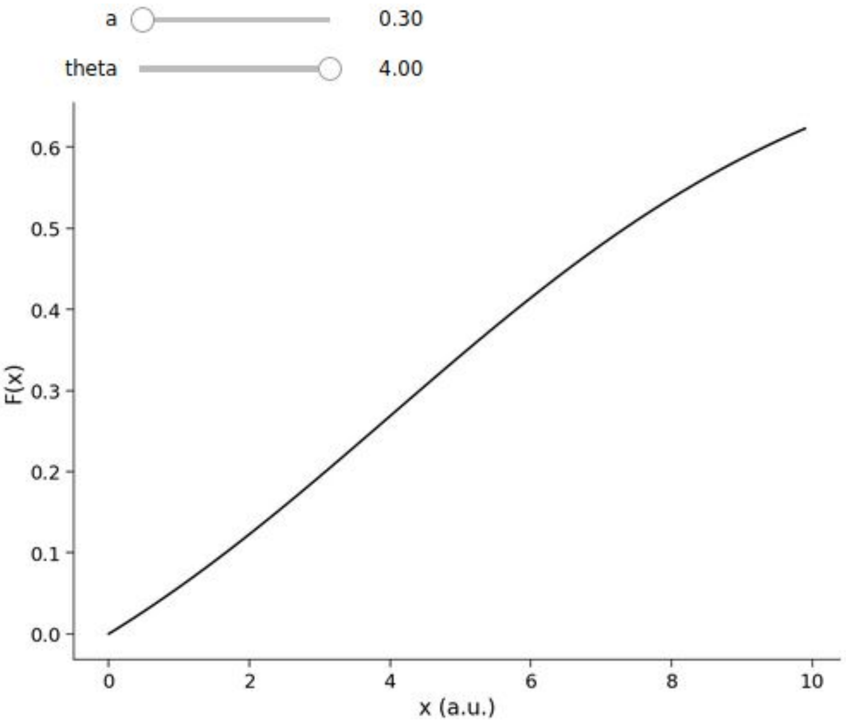
Parameter exploration of F-I curve

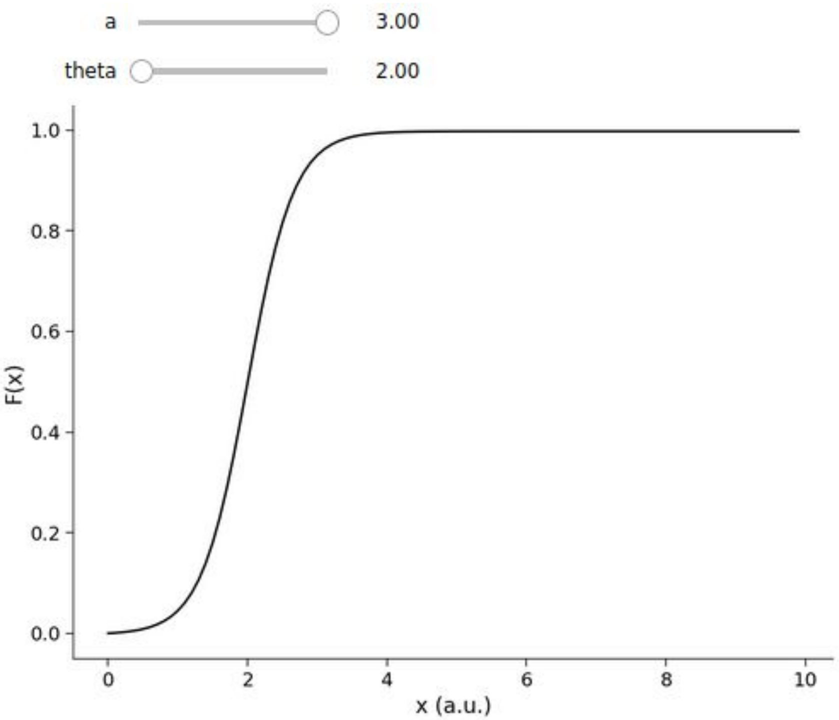
How do the gain and threshold parameters affect the curve?








a
0.30
theta
4.00
F(x)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0
2
4
6
8
10
x (a.u.)


a
3.00
theta
2.00
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
x (a.u.)

Observation

For the function we have chosen to model the F-I curve,

- a determines the slope (gain) of the rising phase of the F-I curve aka controls the gain of the neuron population
- theta determines the input at which the function F(x) reaches its mid-value (0.5) aka controls the threshold at which the neuron population starts to respond

That is, theta shifts the F-I curve along the horizontal axis.

Simulation scheme of E dynamics

Because $F(\cdot)$ is a nonlinear function, the exact solution can not be determined via analytical methods. Therefore, numerical methods must be used to find the solution.

approximate using Euler method on a time grid of stepsize Δt

$r[k] = r[k\Delta t]$

$$\frac{dr}{dt} \sim \frac{r[k+1] - r[k]}{\Delta t}$$

$$\Delta r[k] = \frac{\Delta t}{\tau}\left[-r[k] + F\left(w \cdot r[k] + I_{ext}[k]; a, \theta\right)\right]$$

updating @ each time step by -

$$r[k+1] = r[k] + \Delta r[k]$$

Dynamics

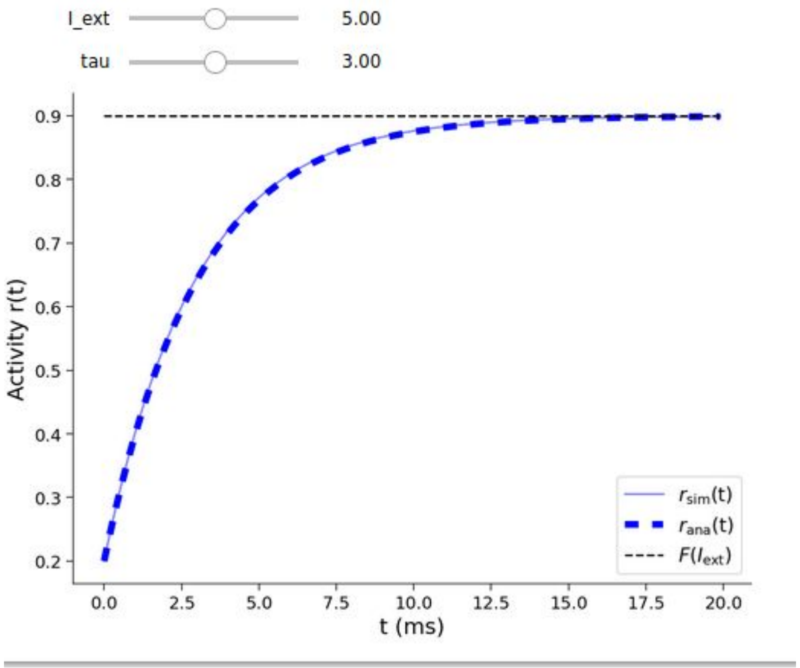
Note that w=0, as in the default setting, means no recurrent input to the neuron population. Hence, the dynamics are entirely determined by the external input Iext.

How does $r$sim(t) change with different Iext values? How does it change with different τ values? Investigate the relationship between

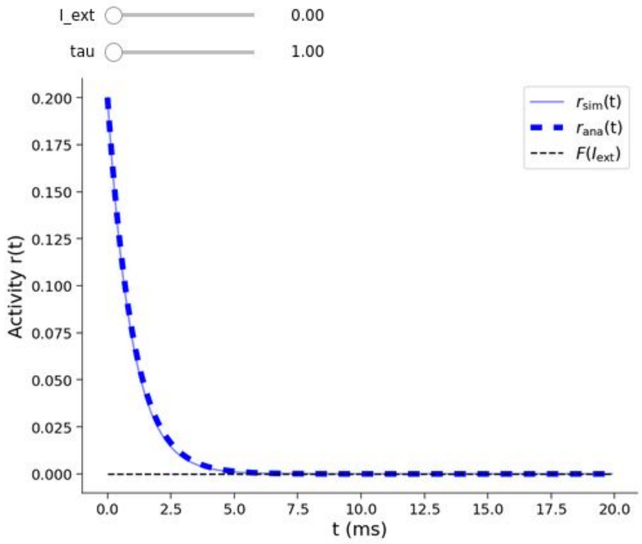
F(Iext;a,θ) and the steady value of r(t).

Note that, $r$ana(t) denotes the analytical solution.

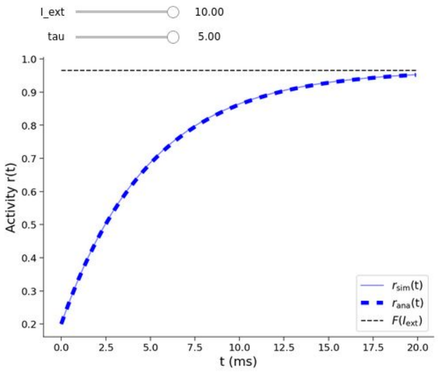
W3D2_pod31



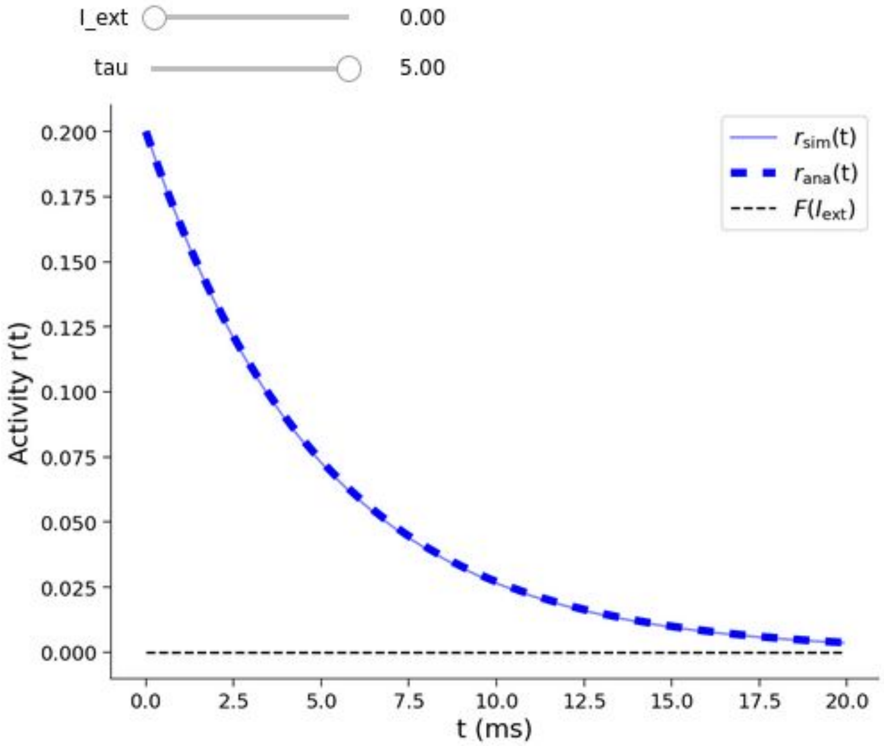
I_ext
5.00
tau
3.00
Activity r(t)
t (ms)
$r_{sim}(t)$
$r_{ana}(t)$
$F(I_{ext})$

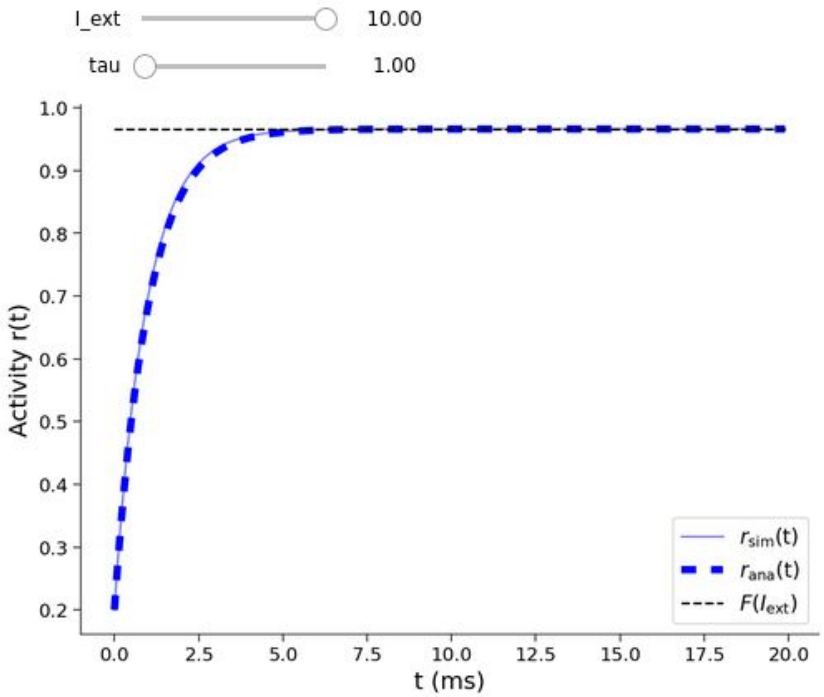


I_ext
0.00
tau
1.00
Activity r(t)
t (ms)
$r_{sim}(t)$
$r_{ana}(t)$
$F(I_{ext})$



I_ext
10.00
tau
5.00
Activity r(t)
t (ms)
$r_{sim}(t)$
$r_{ana}(t)$
$F(I_{ext})$


I_ext 0.00
tau 5.00
Activity r(t)
t (ms)
$r_{sim}(t)$
$r_{ana}(t)$
$F(I_{ext})$


I_ext 10.00
tau 1.00
Activity r(t)
t (ms)
$r_{sim}(t)$
$r_{ana}(t)$
$F(I_{ext})$

Observations

Given the choice of F-I curve and dynamics of the neuron population the neurons have two fixed points or steady-state responses irrespective of the input.

- Weak inputs to the neurons eventually result in the activity converging to zero
- Strong inputs to the neurons eventually result in the activity converging to max value

The time constant tau, does not affect the steady-state response but it determines the time the neurons take to reach to their fixed point.

Food for thought

$r_E(t)$ either decays to zero or reaches a fixed non-zero value.

Why doesn't the solution of the system "explode" in a finite time? In other words, what guarantees that $r_E(t)$ stays finite? As the F-I curve is bounded between zero and one, the system doesn't explode.

The f-curve guarantees this property

Which parameter would you change in order to increase the maximum value of the response?

One way to increase the maximum response is to change the f-I curve. For example, the ReLU is an unbounded function, and thus will increase the overall maximal response of the network.

Fixed points of the single population system

At first the system output quickly changes, with time, it reaches its maximum/ minimum value and does not change anymore. The value eventually reached by the system is called the **steady state** of the system, or the **fixed point**.

in steady states $\frac{dr}{dt} = 0$

activity

F : non-linear

Solving r: $-r_{steady} + F(w \cdot r_{steady} + I_{ext}; a, \theta) = 0$

when it exists, solution defines fixed point of dynamical system

$w = 0 \Rightarrow$ analytically computing solution

deduce role of τ in determining convergence to fixed point

$$r(t) = \left[F(I_{ext}; a, \theta) - r(t=0)\right]\left(1 - e^{-t/\tau}\right) + r(t=0)$$

Visualization of the fixed points

When it is not possible to find the solution analytically, a graphical approach can be taken. To that end, it is useful to plot dr/dt as a function of r. The values of r for which the plotted function crosses zero on the y axis correspond to fixed points. Value of **T** influences how quickly the activity will converge to the steady state from its initial value.

$w = 5.0$

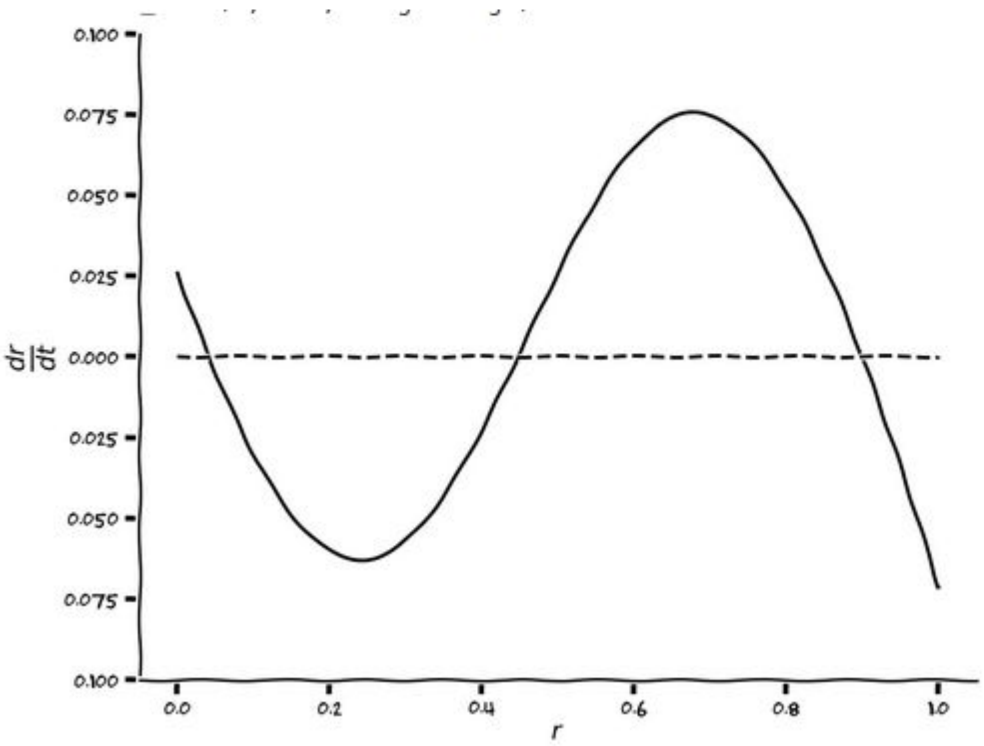
$I^{ext} = 0.5$

$$\frac{dr}{dt} = \frac{\left[-r + F\left(w \cdot r + I^{ext}\right)\right]}{\tau}$$

#TASK : plot $\frac{dr}{dt}$ as a function of r

Check for presence of fixed points.

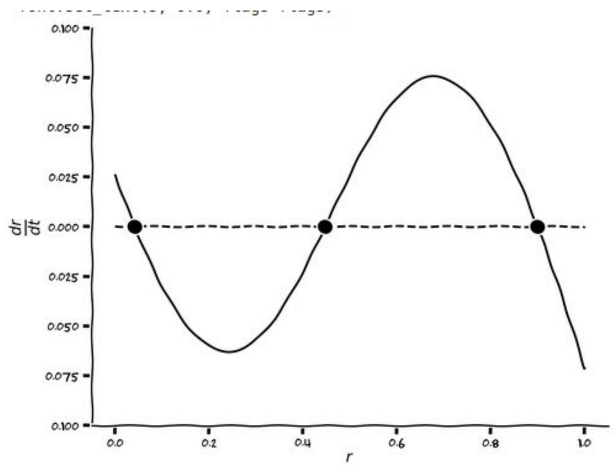
W3D2_pod31



0.100
0.075
0.050
0.025
0.000
0.025
0.050
0.075
0.100
dr/dt
0.0
0.2
0.4
0.6
0.8
1.0
r

Fixed point calculation

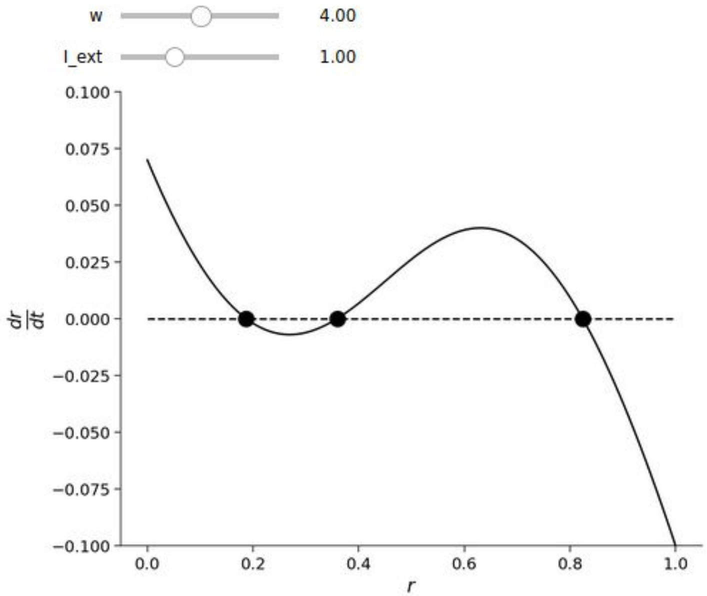
We will now find the fixed points numerically. To do so, we need to specify initial values (rguess) for the root-finding algorithm to start from. From the line dr/dt, initial values can be chosen as a set of values close to where the line crosses zero on the y axis (real fixed point).

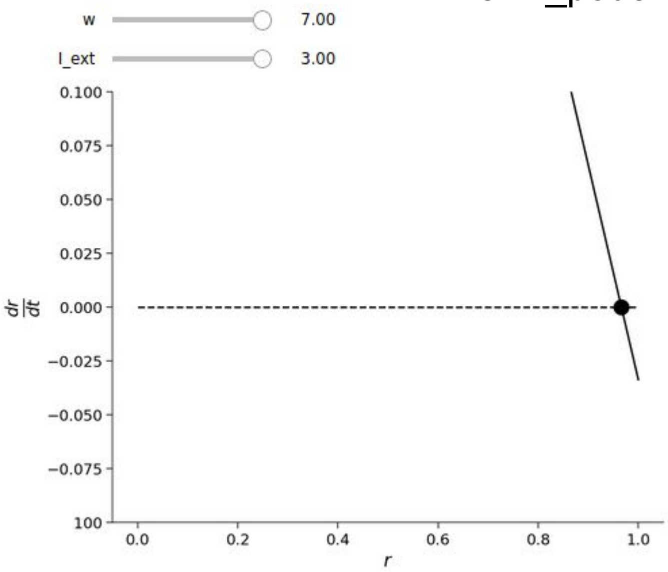


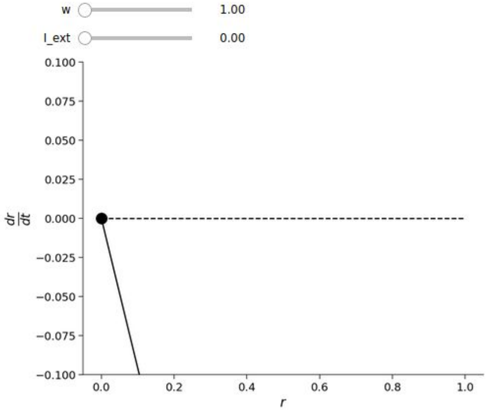
fixed points as a function of recurrent and external inputs.

Study changes when the recurrent coupling w and the external input Iext take different values.

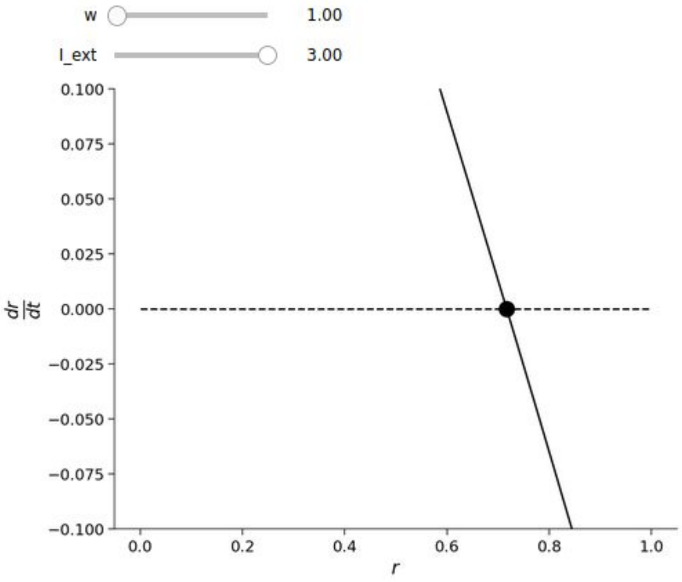
W3D2_pod31

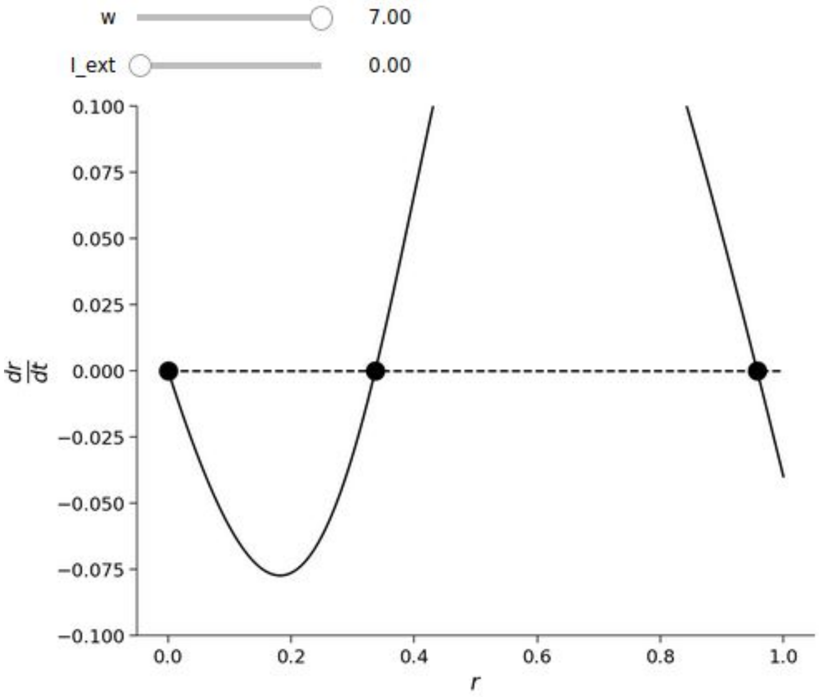

w
4.00
I_ext
1.00
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
$\frac{dr}{dt}$
0.0
0.2
0.4
0.6
0.8
1.0
r


w
7.00
I_ext
3.00
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
100
$\frac{dr}{dt}$
0.0
0.2
0.4
0.6
0.8
1.0
r


w
1.00
I_ext
0.00
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
$\frac{dr}{dt}$
0.0
0.2
0.4
0.6
0.8
1.0
r

W3D2_pod31


w
1.00
I_ext
3.00
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
0.0
0.2
0.4
0.6
0.8
1.0
r
dr/dt


w
7.00
I_ext
0.00
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
0.0
0.2
0.4
0.6
0.8
1.0
r
dr/dt

Observations

The fixed points of the single excitatory neuron population are determined by both recurrent connections w and external input I_ext.

system shows two different steady-states when w = 0. But when w doe not equal 0, for some range of w the system shows three fixed points (the middle one being unstable) and the steady state depends on the initial conditions (i.e. r at time zero.).

W3D2_pod31

Summary

investigate the dynamics of a rate-based single population of neurons.

effect of the input parameters and the time constant of the network on the dynamics of the population.

find the fixed point(s) of the system.

Tutorial #1 Bonus Explanations

W3D2_pod31

Objective

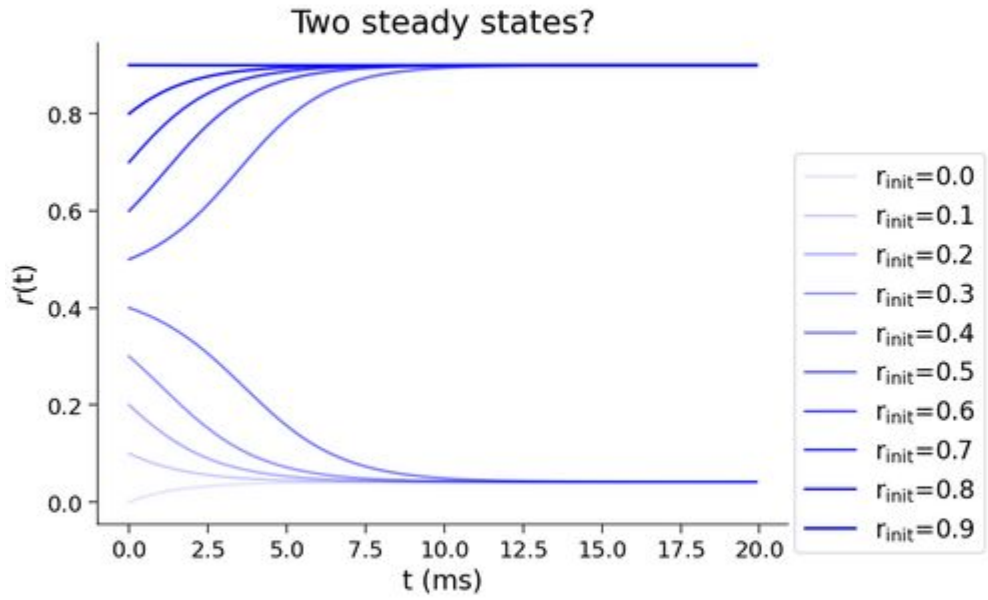
- determine the stability of a fixed point by linearizing the system.
- add realistic inputs to our model.

Stability of a fixed point

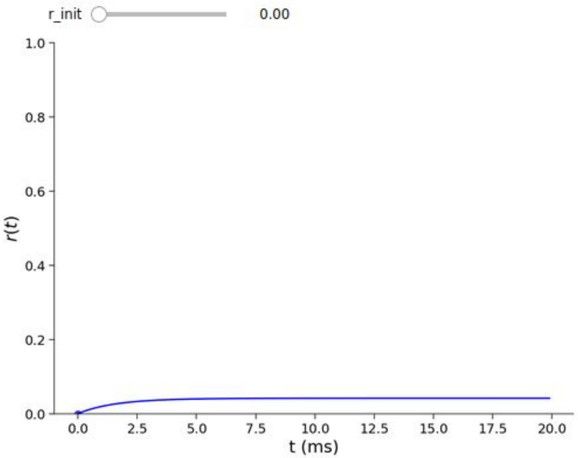
Let $w=5.0$ and $I_{ext}=0.5$, and investigate the dynamics of $r(t)$ starting with different initial values $r(0)\equiv r_{init}$.

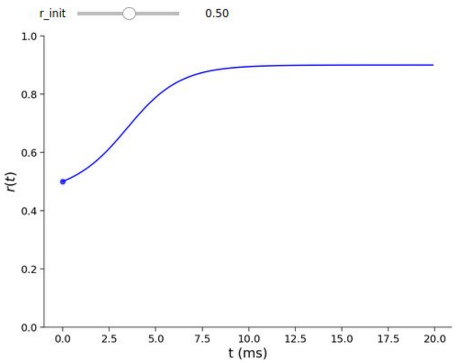
Plot the trajectories of $r(t)$ with $r_{init}=0.0, 0.1, 0.2, \ldots, 0.9$

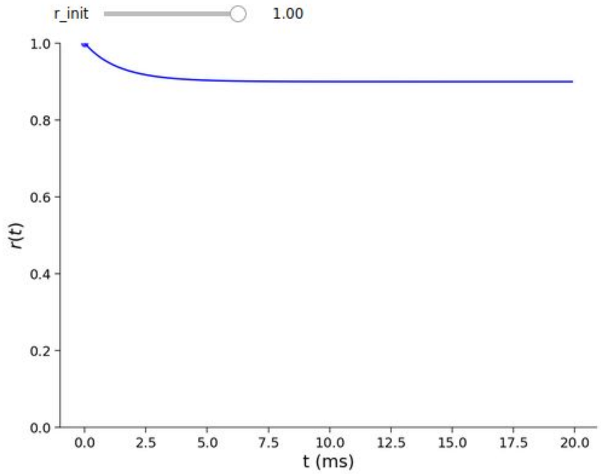
.



DYNAMICS AS A FUNCTION OF THE INITIAL VALUE







Observations

choose the initial value in this demo and see in which direction

the system output moves.

When r_init is in the vicinity of the leftmost fixed points

it moves towards the left most fixed point.

When r_init is in the vicinity of the

rightmost fixed points it moves towards the rightmost fixed point.

Stability analysis via linearization of the dynamics

Generic linear system : $\frac{dx}{dt} = \lambda(x-b)$

fixed point $x=b$

Analytical solution : $x(t) = b + (x(0)-b)e^{\lambda t}$

Small perturbation of activity $x(0) = b+\varepsilon$ $|\varepsilon| \ll 1$.

Evolution of perturbation with time (using analytical solution for $x(t)$)

$$\varepsilon(t) = x(t) - b = \varepsilon e^{\lambda t}$$

$\lambda < 0$, $\varepsilon(t)$ decays to 0, $x(t)$ converges to b (stable fixed point)

$\lambda > 0$, $\varepsilon(t)$ grows with time, $x(t)$ leaves fixed pt b exponentially (unstable fixed point)

W3D2_pod31

Stability of fixed point r^* of excitatory population dynamics

$r = r^* + \varepsilon$

determining time evolution of perturbation $\varepsilon(t)$

$$\tau \frac{d\varepsilon}{dt} \sim -\varepsilon + wF'(w \cdot r^* + I_{ext}; a, \theta]\,\varepsilon$$

derivative of transfer function

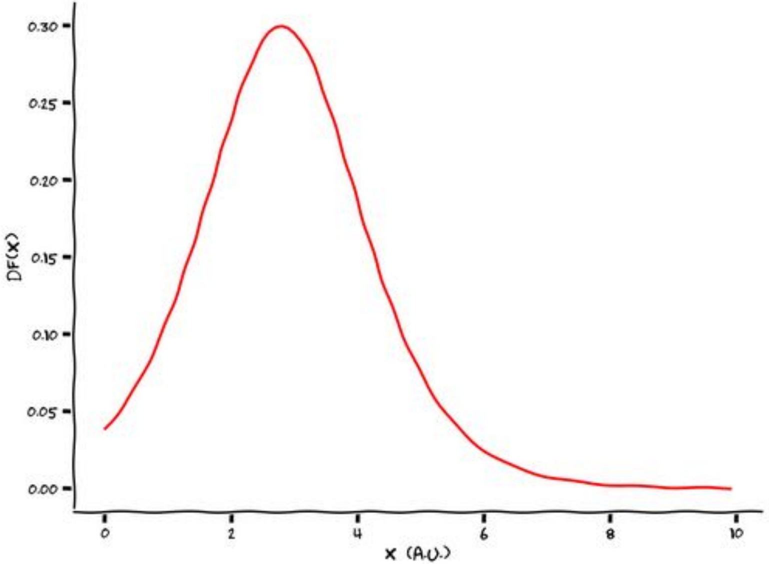
$$\frac{d\varepsilon}{dt} \sim \frac{\varepsilon}{\tau}\left[-1 + wF'(w \cdot r^* + I_{ext}; a, \theta)\right]$$

using the linear system

$$\lambda = \frac{\left[-1 + wF'(w \cdot r^* + I_{ext}; a, \theta)\right]}{\tau}$$

Eigenvalue of dynamic system

determines whether the perturbation will grow or decay to zero, i.e., λ defines the stability of the fixed point. This value is called the **eigenvalue** of the dynamical system.



derivative of sigmoid transfer function

$$\frac{dF}{dx} = \frac{d}{dx}\left(1 + e^{-a(x-\theta))}\right)^{-1}$$

$$= a e^{-a(x-\theta)} \cdot \left(1 + e^{-a(x-\theta)}\right)^{-2}$$

Steady states and stability

However, when we simulated the dynamics and varied the initial conditions rinit, we could only obtain **two** steady states.

check the stability of each of the three fixed points by calculating the corresponding eigenvalues with the function eig_single.

Check the sign of each eigenvalue (i.e., stability of each fixed point).

Food for thought

assumption: $w>0$, i.e., we considered a single population of **excitatory** neurons. What do you think will be the behavior of a population of inhibitory neurons, i.e., where $w>0$ is replaced by $w<0$?

We set the weight to $w<0$.

System has only one fixed point and that

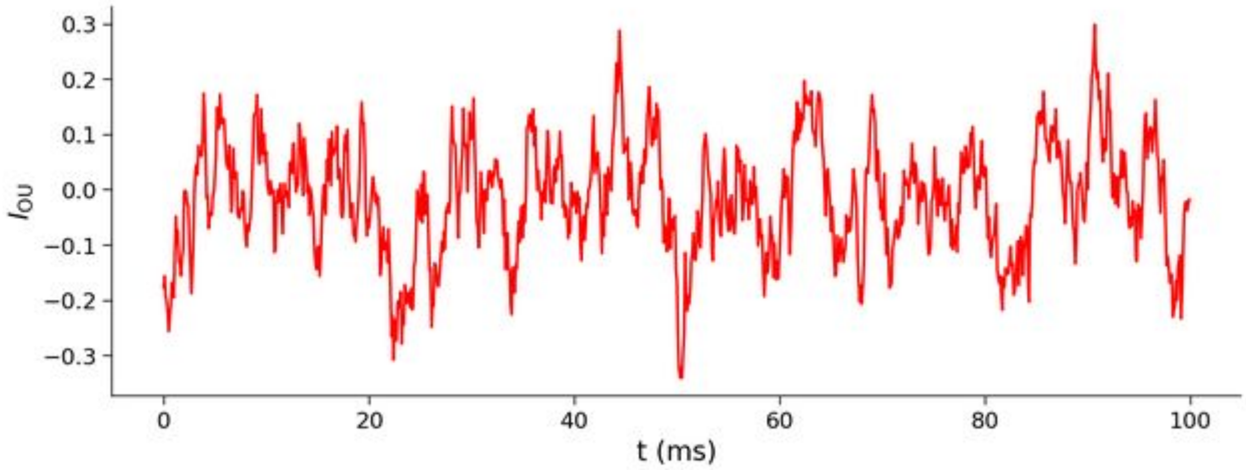
is at zero value. For this particular dynamics, the system will eventually converge

to zero.

W3D2_pod31

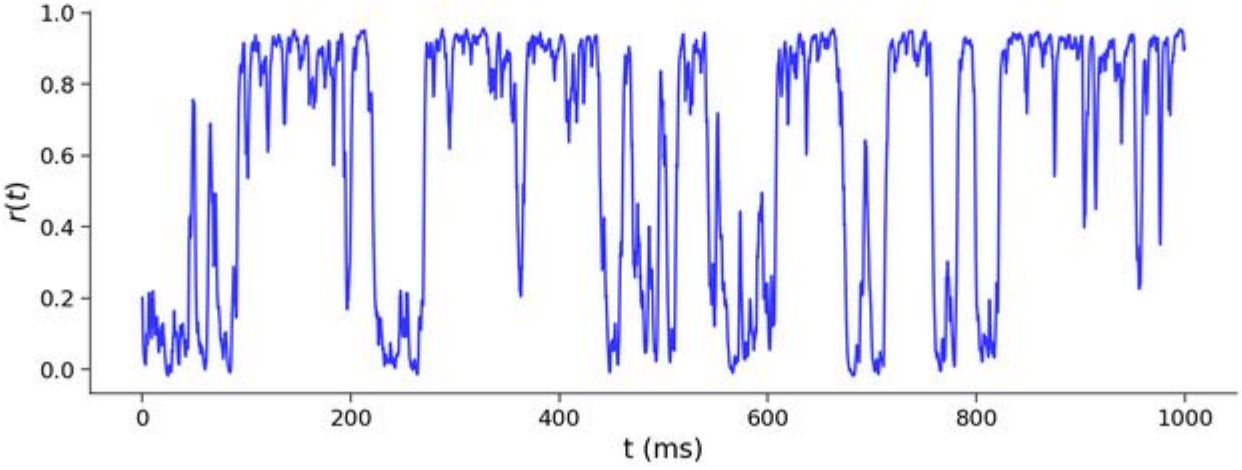
OU process to generate noisy input to the neuron

$$\tau_n \frac{d}{dt}\eta(t) = -\eta(t) + \sigma_n\sqrt{2\tau_n}\,\xi(t)$$



Up-Down transition

In the presence of two or more fixed points, noisy inputs can drive a transition between the fixed points! Stimulate an E population for 1,000 ms applying OU inputs.



Tutorial #2
Explanations

Agenda

Extend approach to include both excitatory and inhibitory neuronal populations in our network as opposed to excitatory only. A simple, yet powerful model to study the dynamics of two interacting populations of excitatory and inhibitory neurons, is the so-called **Wilson-Cowan** rate model. We calculate equations for the firing rate dynamics of a 2D system composed of an excitatory (E) and an inhibitory (I) population of neurons, simulate the dynamics of the system, plot the frequency-current (F-I) curves for both populations (i.e., E and I) and visualize/inspect the behavior of the system using ***phase plane analysis***, ***vector fields***, and ***nullclines***.

Further, find and plot the ***fixed points*** of the Wilson-Cowan model, investigate the stability by linearizing its dynamics and examining the ***Jacobian matrix***, and learn how the Wilson-Cowan model can reach an oscillatory state. Also, visualize the behavior of an Inhibition-stabilized network, simulate working memory.

Mathematical description of the WC model

MANY OF THE RICH DYNAMICS RECORDED IN THE BRAIN ARE GENERATED BY THE INTERACTION OF EXCITATORY AND INHIBITORY SUBTYPE NEURONS.

Coupled differential equations

excitatory population

avg activation/firing rate of population @ time t

$$\tau_E \frac{dr_E}{dt} = -r_E + F_E(w_{EE} r_E - w_{EI} r_I + I_E^{ext}; a_E, \theta_E)$$

inhibitory population

$$\tau_I \frac{dr_I}{dt} = -r_I + F_I(w_{IE} r_E - w_{II} r_I + I_I^{ext}; a_I, \theta_I)$$

Control timescales of dynamics of each population

Connection strengths:

w_{EE} $(E \to E)$

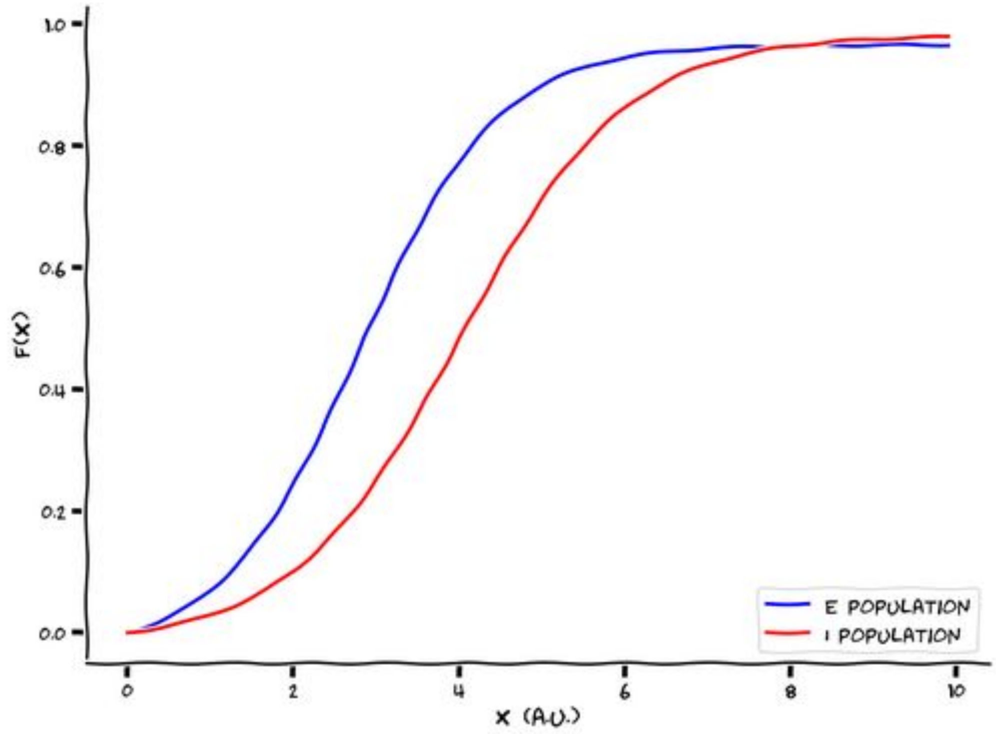
w_{EI} $(I \to E)$

w_{IE} $(E \to I)$

w_{II} $(I \to I)$

connections from inhibitory to excitatory

W3D2_pod31



Numerical integration with euler method
(dynamics simulated on time grid of stepsize Δt)

Updates for activity of populations:

$$r_E[k+1] = r_E[k] + \Delta r_E[k]$$

$$r_I[k+1] = r_I[k] + \Delta r_I[k]$$

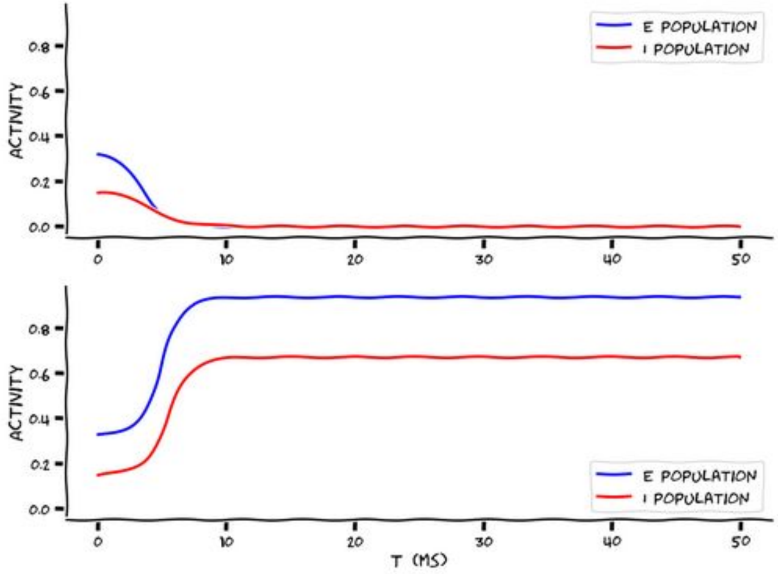
increments :

$$\Delta r_E[k] = \frac{\Delta t}{\tau_E}\Big[-r_E[k] + F_E\big(w_{EE} r_E[k] - w_{EI} r_I[k] + I_E^{ext}[k];\, a_E, \theta_E\big)\Big]$$

$$\Delta r_I[k] = \frac{\Delta t}{\tau_I}\Big[-r_I[k] + F_I\big(w_{IE} r_E[k] - w_{II} r_E[k] + I_I^{ext}[k];\, a_I, \theta_I\big)\Big]$$

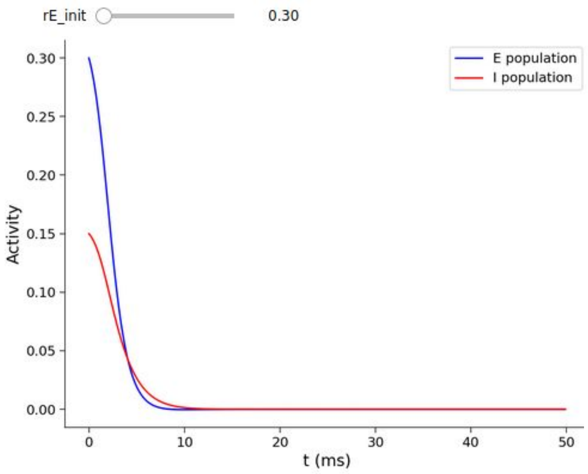
Temporal evolution

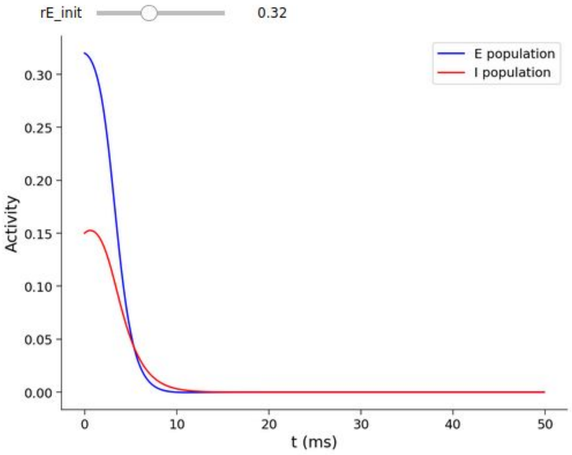
Show the temporal evolution of excitatory (r_E, blue) and inhibitory (r_I, red) activity for two different sets of initial conditions.

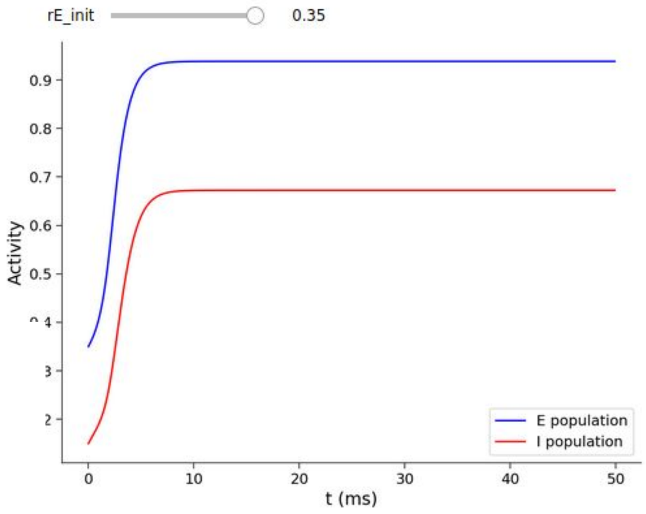


population trajectories with different initial values

It is evident that the steady states of the neuronal response can be different when different initial states are chosen.







Phase plane analysis

Graphical approach called **phase plane analysis** to study the dynamics of a 2-D system like the Wilson-Cowan model.

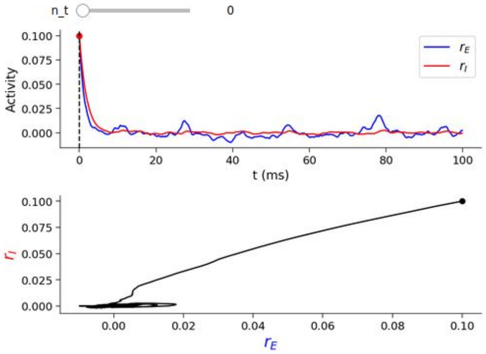
So far, we have plotted the activities of the two populations as a function of time, i.e., in the Activity-t plane, either the (t,rE(t)) plane or the (t,rI(t)) one.

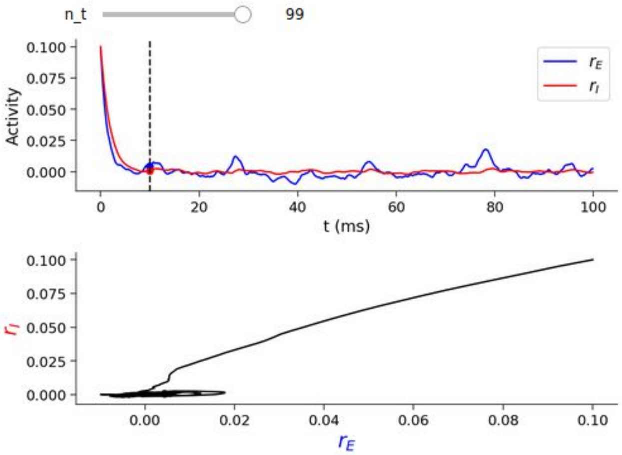
Instead, we can plot the two activities rE(t) and rI(t) against each other at any time point t. This characterization in the rI-rE plane

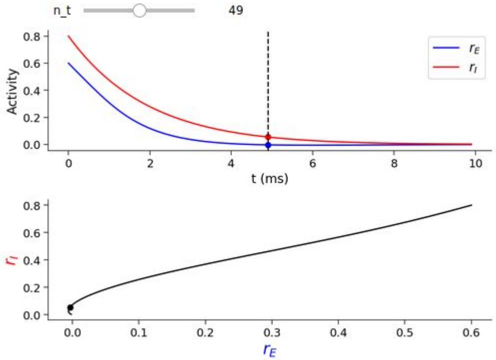
(rI(t), rE(t)) is called the **phase plane**. Each line in the phase plane indicates how both rE and rI evolve with time.

From the Activity - time plane to the r_I-r_E phase plane

Visualize the system dynamics using both the Activity-time and the (rE, rI) phase plane. The circles indicate the activities at a given time t, while the lines represent the evolution of the system for the entire duration of the simulation.







Observations

- Phase plane portraits allows us to visualize out of all possible states which states a system can take.
- among all possible pairs of values for r1 and r2, this system can take only a limited number of states (those that lie on the black line).
- There are other things we can infer from the phase portraits e.g. fixed points, trajectory to the fixed points etc.
- Explicit information about time is not visible in the phase portraits

Nullclines of the Wilson-Cowan Equations

An important concept in the phase plane analysis is the "nullcline" which is defined as the set of points in the phase plane where the activity of one population (but not necessarily the other) does not change.

Excitatory Nullclines. $\frac{dr_E}{dt} = 0$

inhibitory Nullclines $\frac{dr_I}{dt} = 0$

$$-r_E + F_E\left(w_{EE} r_E - w_{EI} r_I + I_E^{ext}; a_E, \theta_E\right) = 0$$

$$-r_I + F_I\left(w_{IE} r_E - w_{II} r_I + I_I^{ext}; a_I, \theta_I\right) = 0$$

Computing nullclines

along nullcline of inhibitory population, calculate excitatory activity

$$r_E = \frac{1}{w_{IE}}\left[w_{II} r_I + F_I^{-1}(r_I; a_I \theta_I) - I_I^{ext}\right].$$

inverse of inhibitory transfer function

W3D2_pod31

Computing Nullclines

along nullcline of excitatory population, calculate inhibitory activity:

$$r_I = \frac{1}{w_{EI}}\left[w_{EE} r_E - F_E^{-1}(r_E; a_E, \theta_E) + I_E^{ext}\right]$$

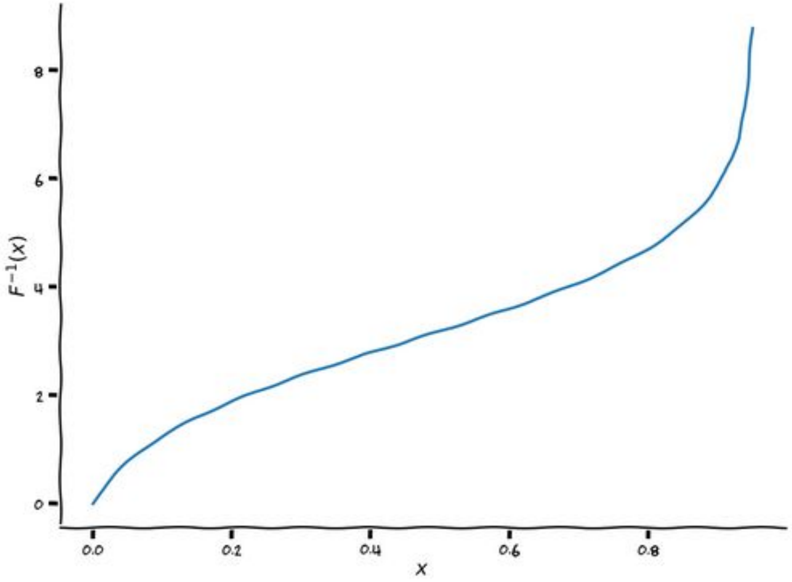
inverse of excitatory transfer function

Calculating inverse of transfer functions.

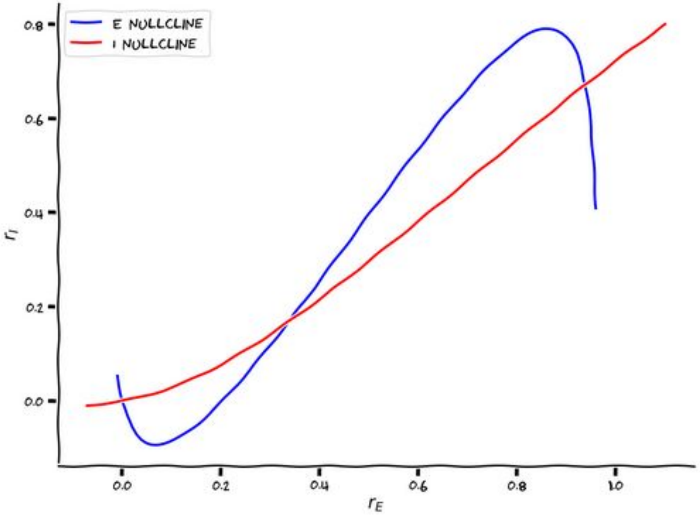
inverse of sigmoid shaped f/ function –

$$F^{-1}(x;a,\theta) = \frac{-1}{a} \ln\left[\frac{1}{x + \frac{1}{1+e^{a\theta}}} - 1\right] + \theta$$

W3D2_pod31



8
6
4
2
0
$F^{-1}(x)$
0.0
0.2
0.4
0.6
0.8
x



E NULLCLINE
I NULLCLINE
0.8
0.6
0.4
0.2
0.0
r_I
0.0
0.2
0.4
0.6
0.8
1.0
r_E

Observations

Note that by definition along the blue line in the phase-plane spanned by r_E, r_I, $\frac{dr_E(t)}{dt} = 0$, therefore, it is called a nullcline.

That is, the blue nullcline divides the phase-plane spanned by r_E, r_I into two regions: on one side of the nullcline $\frac{dr_E(t)}{dt} > 0$ and on the other side $\frac{dr_E(t)}{dt} < 0$.

The same is true for the red line along which $\frac{dr_I(t)}{dt} = 0$. That is, the red nullcline divides the phase-plane spanned by r_E, r_I into two regions: on one side of the nullcline $\frac{dr_I(t)}{dt} > 0$ and on the other side $\frac{dr_I(t)}{dt} < 0$.

Vector field

- phase plane and the nullcline curves help us understand the behavior of the Wilson-Cowan model
- The activities of the *E* and *I* populations *rE*(*t*) and *rI*(*t*) at each time point *t* correspond to a single point in the phase plane, with coordinates (*rE*(*t*), *rI*(*t*))
- Therefore, the time-dependent trajectory of the system can be described as a continuous curve in the phase plane, and the tangent vector to the trajectory,
- The map of tangent vectors in the phase plane is called **vector field.**

The behavior of any trajectory in the phase plane is determined by

initial conditions

$$\left(r_E(0), r_I(0)\right)$$

vector field

$$\left(\frac{dr_E(t)}{dt}, \frac{dr_I(t)}{dt}\right)$$

W3D2_pod31

time dependent trajectory.

$$\left(\frac{dr_E(t)}{dt}, \frac{dr_I(t)}{dt} \right)$$

indicates ① direction towards which activity is evolving.

② how fast activity is changing along each axis.

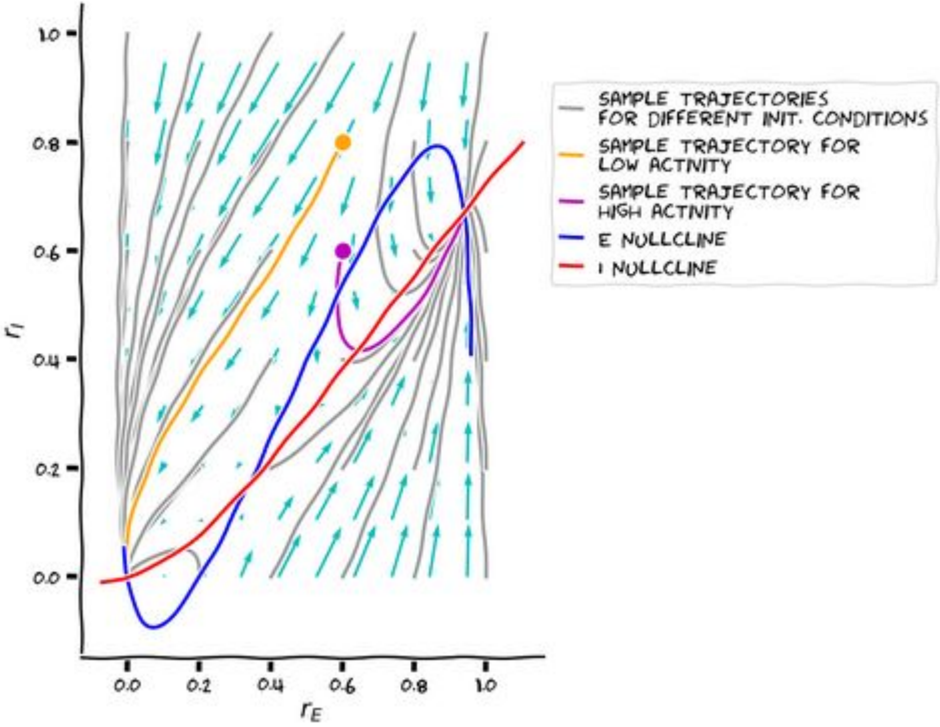
for each point (E, I) in phase plane, compute tangent vector

$\left(\frac{dr_E}{dt}, \frac{dr_I}{dt} \right)$ indicates behavior of system when it traverses that point

W3D2_pod31

Value of vector field

In general, the value of the vector field at a particular point in the phase plane is represented by an arrow. The orientation and the size of the arrow reflect the direction and the norm of the vector, respectively.



W3D2_pod31

Compute/plot vector field $\left(\frac{dr_E}{dt}, \frac{dr_I}{dt}\right)$

$$\frac{dr_E}{dt} = \left[-r_E + F_E\left(w_{EE}r_E - w_{EI}r_I + I_E^{ext}; a_E, \theta_E\right)\right]\frac{1}{\tau_E}$$

$$\frac{dr_I}{dt} = \left[-r_I + F_I\left(w_{IE}r_E - w_{II}r_I + I_I^{ext}; a_I, \theta_I\right)\right]\frac{1}{\tau_I}$$

Observations

- Trajectories seem to follow the direction of the vector field
- Different trajectories eventually always reach one of two points depending on the initial conditions.
- The two points where the trajectories converge are the intersection of the two nullcline curves.

Food for thought

There are, in total, three intersection points, meaning that the system has three fixed points.

- One of the fixed points (the one in the middle) is never the final state of a trajectory. Why is that?
  Because the middle fixed point is unstable. Trajectories only point to stable fixed points.
- Why the arrows tend to get smaller as they approach the fixed points?

  The slope of dr/dt determines the speed at which the system states will evolve.At the nullclines dr_e/dt = 0 and/or dr_i/dt = 0. That means as we move close to the nullclines the dr/dt becomes smaller and smaller. Therefore the system state evolves slower and slower as we approach the nullcline.

Tutorial #2 Bonus Explanations

Objective

More advanced concepts on dynamical systems:

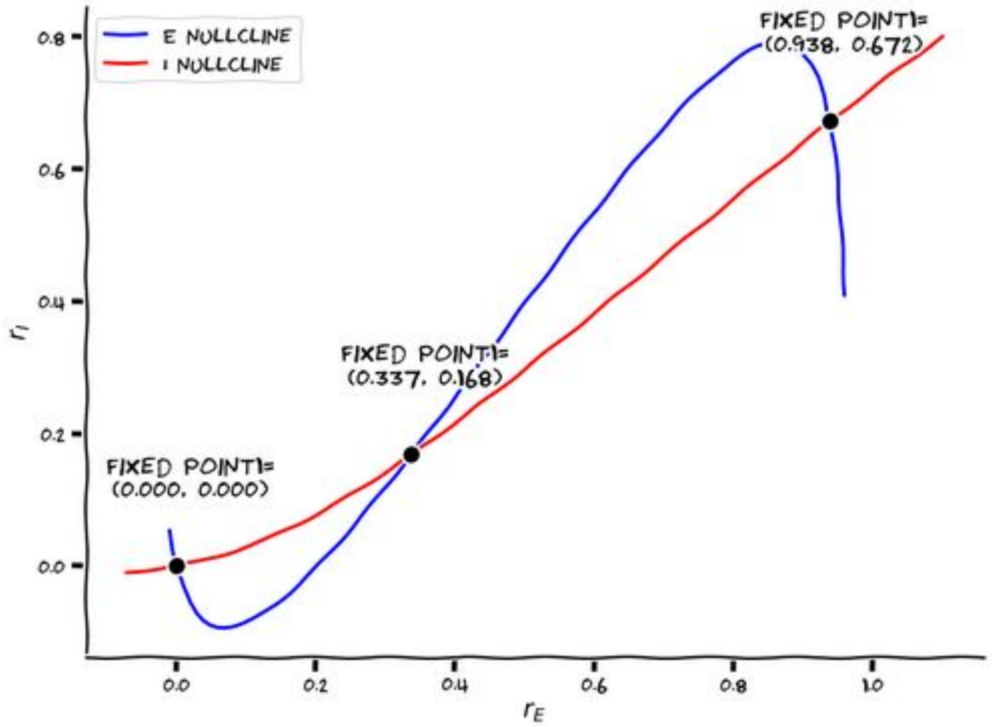
- find the fixed points on such a system, and to investigate its stability by linearizing its dynamics and examining the **Jacobian matrix**.
- identify conditions under which the Wilson-Cowan model can exhibit oscillations.

Two applications of the Wilson-Cowan model:

- Visualization of an Inhibition-stabilized network
- Simulation of working memory

Fixed Points of the E/I system

Intersection points of the two nullcline curves are the fixed points of the Wilson-Cowan model.
Find the coordinate of all fixed points for a given set of parameters.



Find the fixed points of the Wilson-Cowan model

The system features three fixed points with the parameters we used. To find their coordinates, we need to choose proper initial value since the algorithm can only find fixed points in the vicinity of the initial value.

Note that you can choose the values near the intersections of the nullclines as the initial values to calculate the fixed points.

Stability of a fixed point and eigenvalues of the Jacobian Matrix

$$\frac{dr_E}{dt} = G_E(r_E, r_I) = \frac{1}{\tau_E}\left[-r_E + F_E\left(w_{EE} r_E - w_{EI} r_I + I_E^{ext}; a, \theta\right)\right]$$

$$\frac{dr_I}{dt} = G_I(r_E, r_I) = \frac{1}{\tau_I}\left[-r_I + F_I\left(w_{IE} r_E - w_{II} r_I + I_I^{ext}; a, \theta\right)\right]$$

at fixed points: $\frac{dr_E}{dt} = \frac{dr_I}{dt} = 0$

Jacobian matrix

Therefore, if the initial state is exactly at the fixed point, the state of the system will not change as time evolves.

However, if the initial state deviates slightly from the fixed point, there are two possibilities the trajectory will be attracted back to the

1. The trajectory will be attracted back to the fixed point
2. The trajectory will diverge from the fixed point.

These two possibilities define the type of fixed point, i.e., stable or unstable.

The stability of a fixed point(r*E,r*I) can be determined by linearizing the dynamics of the system. The linearization will yield a matrix of first-order derivatives called the Jacobian matrix:

W3D2_pod31

The eigenvalues of the Jacobian matrix calculated at the fixed point will determine whether it is a stable or unstable fixed point.

$$J = \begin{pmatrix} \frac{\partial}{\partial r_E} G_E(r_E^*, r_I^*) & \frac{\partial}{\partial r_I} G_E(r_E^*, r_I^*) \\ \frac{\partial}{\partial r_E} G_I(r_E^*, r_I^*) & \frac{\partial}{\partial r_I} G_I(r_E^*, r_I^*) \end{pmatrix}$$

compute the derivatives needed to build the Jacobian matrix using the chain and product rules!

for excitatory neurons.

$$\frac{\partial}{\partial r_E} G_E(r_E^*, r_I^*) = \frac{1}{\tau_E}\left[-1 + w_{EE} F_E'\left(w_{EE} r_E^* - w_{EI} r_I^* + I_E^{ext}; a_E, \theta_E\right)\right]$$

$$\frac{\partial}{\partial r_I} G_E(r_E^*, r_I^*) = \frac{1}{\tau_E}\left[-w_{EI} F_E'\left(w_{EE} r_E^* - w_{EI} r_I^* + I_E^{ext}; a_E, \theta_E\right)\right]$$

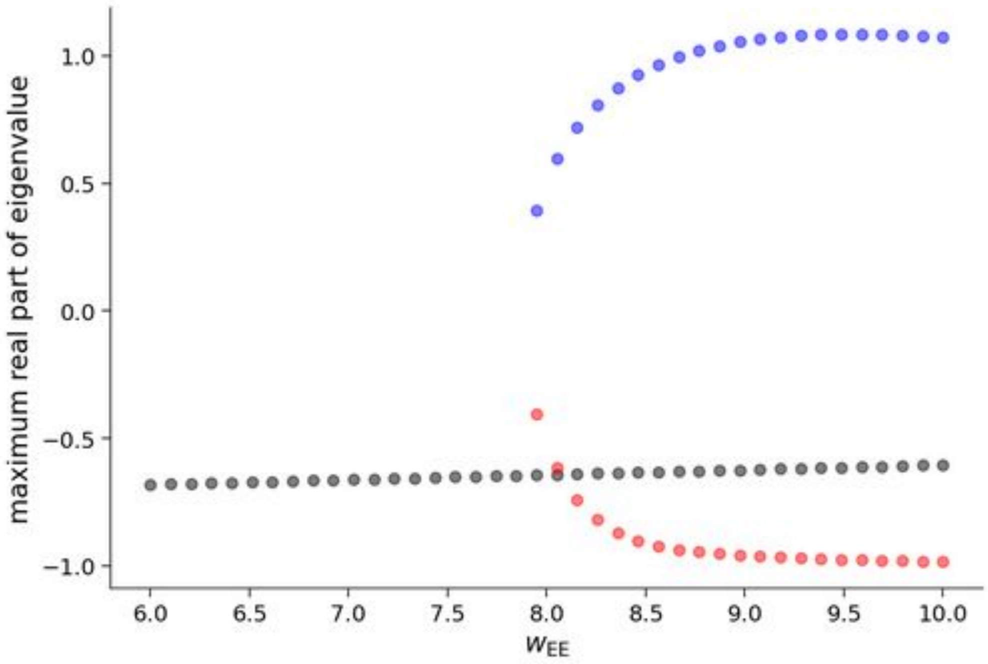
Pitchfork bifurcation

As is evident, the stable fixed points correspond to the negative eigenvalues, while unstable point corresponds to at least one positive eigenvalue.

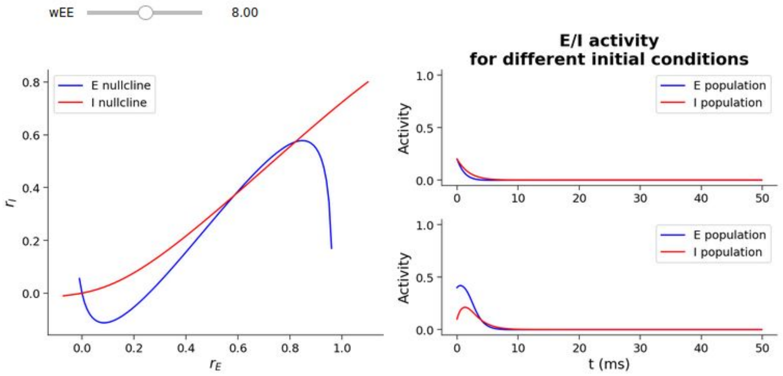
The sign of the eigenvalues is determined by the connectivity (interaction) between excitatory and inhibitory populations.

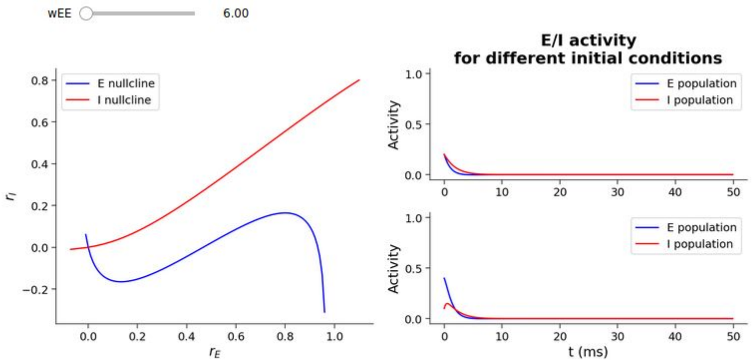
* Critical change is referred to as **pitchfork bifurcation.**

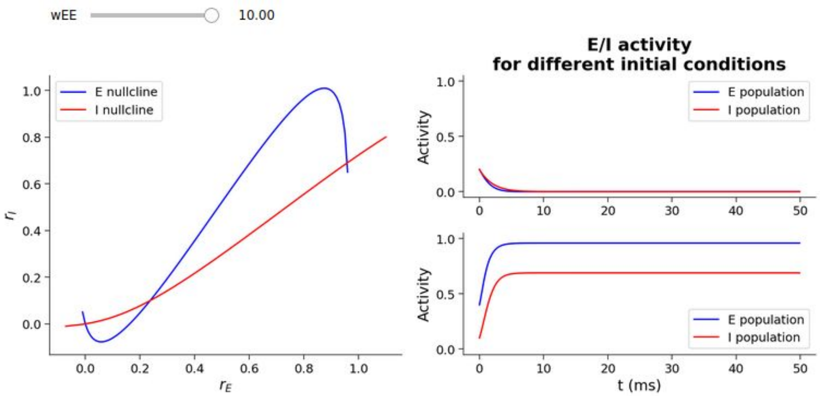
effect of wEE on the nullclines and the eigenvalues



Nullclines position in the phase plane changes with parameter values







Observations

- For low values of wEE there is only one fixed point and it is stable so initial conditions do not matter and the system always converge to the only fixed point

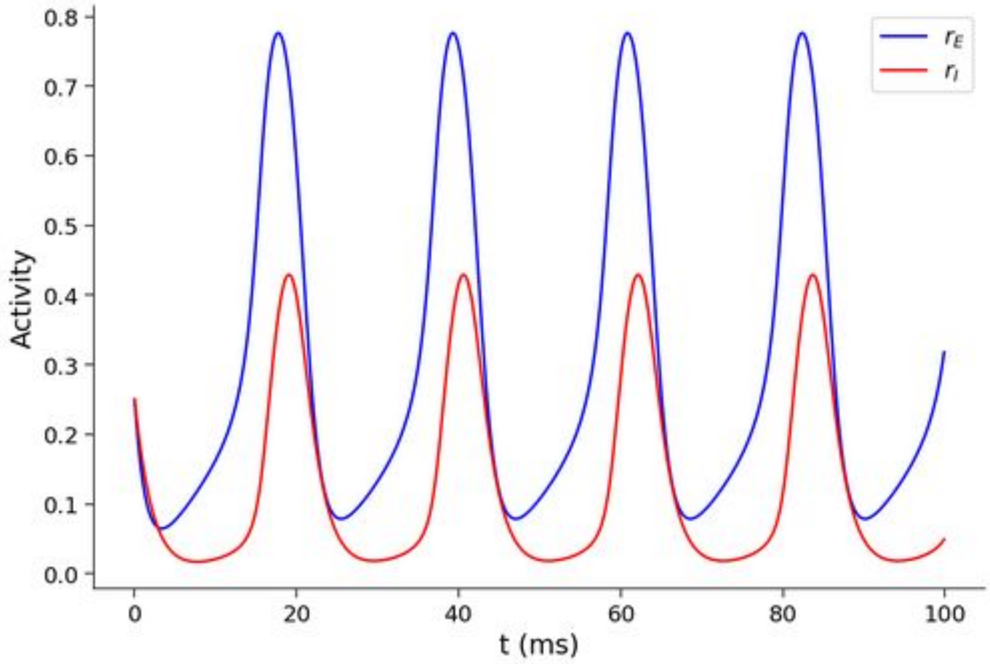
- For high values of wEE we have three fixed points of which two are stable and one is unstable (or saddle). Now it matters where the initial conditions are. If the initial conditions are in the attractor region os the high activity fixed point then the system will converge to that (the bottom example).

We can also investigate the effect of different wEI, wIE, wII, τE, τI, and $IextE$ on the stability of fixed points (complex eigen values). In addition, we can also consider the perturbation of the parameters of the gain curve $F(\cdot)$.

Limit cycle - Oscillations

When at least one pair of eigenvalues is complex, oscillations arise. The stability of oscillations is determined by the real part of the eigenvalues (+ve real part oscillations will grow, -ve real part oscillations will die out). The size of the complex part determines the frequency of oscillations. For instance, if we use a different set of parameters, then we shall observe that the E and I population activity start to oscillate!

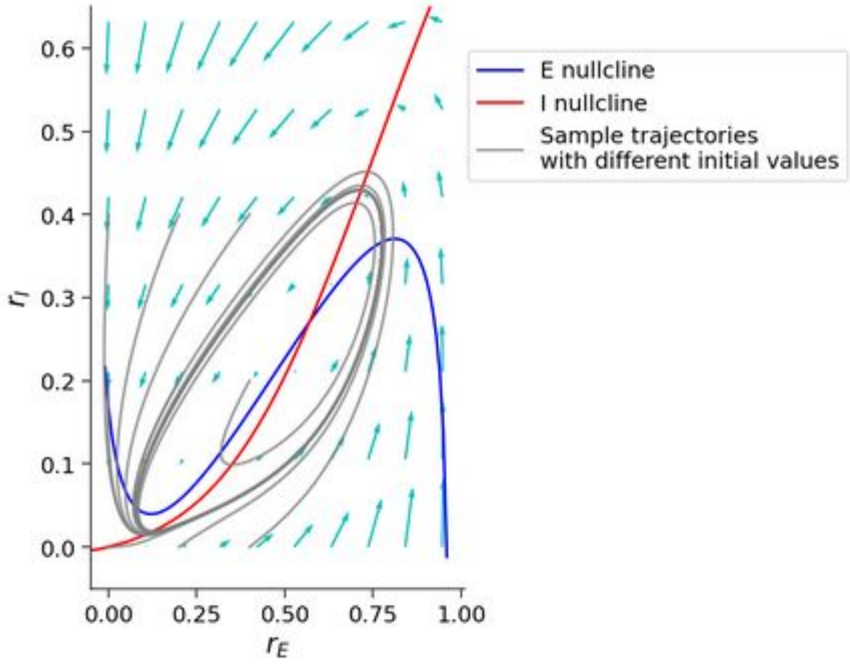
W3D2_pod31



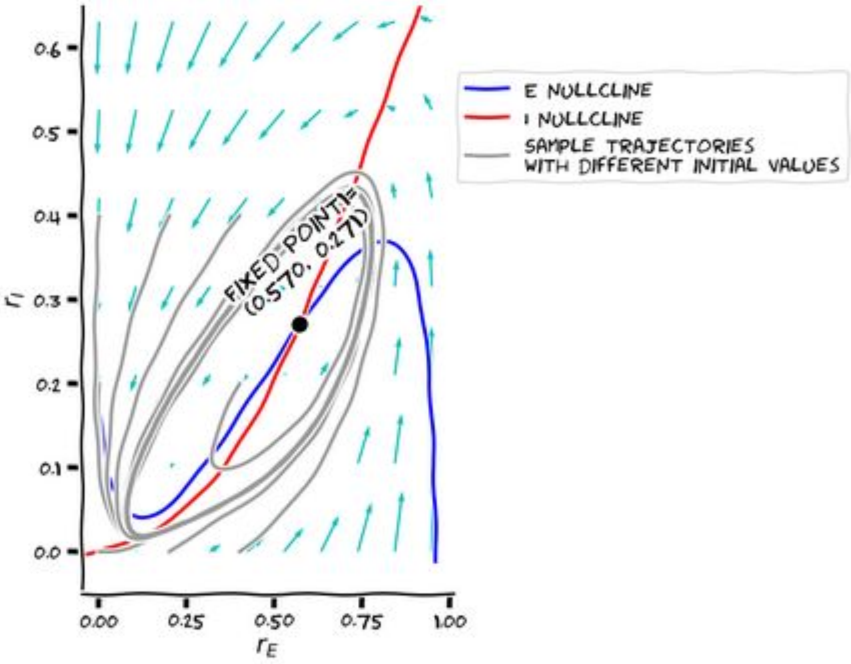
Plot the phase plane

Understand the oscillations of the population behavior using the phase plane. By plotting a set of trajectories with different initial states, we can see that these trajectories will move in a circle instead of converging to a fixed point. This circle is called "limit cycle" and shows the periodic oscillations of the E and I population behavior under some conditions.

W3D2_pod31



E nullcline
I nullcline
Sample trajectories
with different initial values
r_I
r_E
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.00
0.25
0.50
0.75
1.00



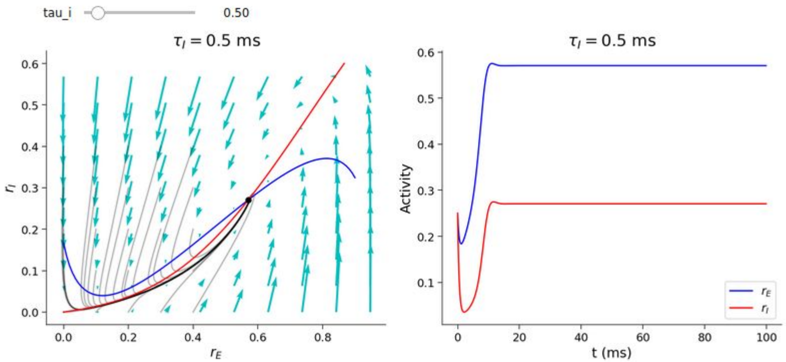
E NULLCLINE
I NULLCLINE
SAMPLE TRAJECTORIES
WITH DIFFERENT INITIAL VALUES
FIXED POINT1=
(0.570, 0.271)
r_I
r_E
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.00
0.25
0.50
0.75
1.00

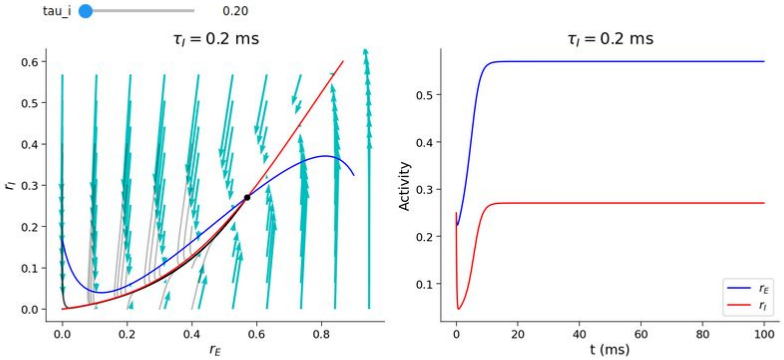
Limit cycle and oscillations.

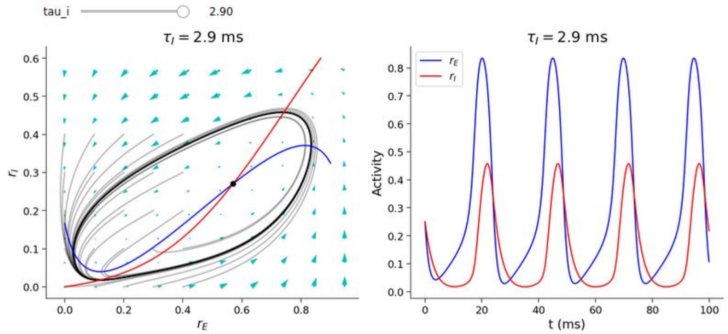
Change of model parameters changes the shape of the nullclines and, accordingly, the behavior of the E and I populations from steady fixed points to oscillations. However, the shape of the nullclines is unable to fully determine the behavior of the network. The vector field also matters.

Investigating the effect of time constants on the population behavior. By changing the inhibitory time constant τ_I, the nullclines do not change, but the network behavior changes substantially from steady state to oscillations with different frequencies.Such a dramatic change in the system behavior is referred to as a **bifurcation**.

W3D2_pod31


tau_i
0.50
$\tau_I = 0.5$ ms
r_I
r_E
$\tau_I = 0.5$ ms
Activity
t (ms)
r_E
r_I


tau_i
0.20
$\tau_I = 0.2$ ms
r_I
r_E
$\tau_I = 0.2$ ms
Activity
t (ms)
r_E
r_I


tau_i
2.90
$\tau_I = 2.9$ ms
r_I
r_E
$\tau_I = 2.9$ ms
Activity
t (ms)
r_E
r_I

W3D2_pod31

Explanation

Both τ_E and τ_I feature in the Jacobian of the two population network. So here is seems that the by increasing $\tau_I$the eigenvalues corresponding to the stable fixed point are becoming complex.

Intuitively, when τ_I is smaller, inhibitory activity changes faster than excitatory activity. As inhibition exceeds above a certain value, high inhibition inhibits excitatory population but that in turns means that inhibitory population gets smaller input (from the exc. connection). So inhibition decreases rapidly. But this means that excitation recovers -- and so on ...

Inhibition Stabilised Network

$$\frac{d}{dt}\vec{R} = \begin{bmatrix} \frac{\partial G_E}{\partial r_E} & \frac{\partial G_E}{\partial r_I} \\ \frac{\partial G_I}{\partial r_E} & \frac{\partial G_I}{\partial r_I} \end{bmatrix} \vec{R}$$

$[r_E, r_I]^T$

vector of E/I activity

Excitatory Subpopulation

$$\frac{\partial r_E}{dt} = \frac{\partial G_E}{\partial r_E} \cdot r_E + \frac{\partial G_E}{\partial r_I} \cdot r_I$$

W3D2_pod31

around fixed point (r_E^*, r_I^*)

$$\frac{\partial}{\partial r_E} G_E(r_E^*, r_I^*) = \frac{1}{\tau_E}\left[-1 + w_{EE} F_E'(w_{EE} r_E^* - w_{EI} r_I^* + I_E^{ext}; a_E, \theta_E)\right]$$

$$\frac{\partial}{\partial r_I} G_E(r_E^*, r_I^*) = \frac{1}{\tau_E}\left[-w_{EI} F_E'(w_{EE} r_E^* - w_{EI} r_I^* + I_E^{ext}; a_E, \theta_E)\right]$$

$$\frac{\partial}{\partial r_E} G_I(r_E^*, r_I^*) = \frac{1}{\tau_I}\left[w_{IE} F_I'(w_{IE} r_E^* - w_{II} r_I^* + I_I^{ext}; a_I, \theta_I) \right]$$

$$\frac{\partial}{\partial r_I} G_I(r_E^*, r_I^*) = \frac{1}{\tau_I}\left[-1 - w_{II} F_I'(w_{IE} r_E^* - w_{II} r_I^* + I_I^{ext}; a_I, \theta_I) \right]$$

Leak effect

it is clear that $\partial GE / \partial rI$ is negative since the dF/dx is always positive.

Recurrent inhibition from the inhibitory activity (I) can reduce the excitatory (E) activity. However,

$\partial GE/\partial rE$ has negative terms related to the "leak" effect, and positive term related to the recurrent excitation. Therefore, it leads to two different regimes:

$$\frac{\partial}{\partial r_E} G_E(r_E^*, r_I^*) < 0$$

Non inhibition stabilised network (non ISN) regime

$$\frac{\partial}{\partial r_E} G_E(r_E^*, r_I^*) > 0$$

Inhibition stabilised network (ISN) regime.

Null cline analysis : ISN.

$$r_E = F_E(w_{EE} r_E - w_{EI} r_I + I_E^{ext} ; a_E, \theta_E).$$

firing rate r_E
function of r_I

$$\frac{dr_E}{dr_I} = F_E' \left(w_{EE} \frac{dr_E}{dr_I} - w_{EI} \right)$$

$$\Longleftrightarrow \left(1 - F_E' w_{EE}\right) \frac{dr_E}{dr_I} = -F_E' w_{EI}$$

$$\Longleftrightarrow \frac{dr_E}{dr_I} = \frac{F_E' \cdot w_{EI}}{F_E' w_{EE} - 1}$$

W3D2_pod31

In phase plane $r_I - r_E$ plane,

obtain slope along E nullcline:

$$\frac{dr_I}{dr_E} = \frac{F_E' w_{EE} - 1}{F_E' w_{EI}}$$

Obtain slope along I nullcline:

$$\frac{dr_I}{dr_E} = \frac{F_I' w_{IE}}{F_I' w_{II} + 1}$$

find that $\left(\frac{dr_I}{dr_E}\right)_{I\text{-nullcline}} > 0$

Sign of $\left(\frac{dr_I}{dr_E}\right)_{E\text{-nullcline}}$ depends on sign of $\left(F_E' w_{EE} - 1\right)$

$\Rightarrow \left(\frac{dr_I}{dr_E}\right)_{E\text{-nullcline}}$ < 0 non ISN regime

> 0 ISN regime

W3D2_pod31

Conclusion #1

The stability of a fixed point can determine the relationship between the slopes

the fixed point is stable when the Jacobian matrix (J)) has two eigenvalues with a negative real part, which indicates a positive determinant of J , i.e det(J)>0

.

W3D2_pod31

$$J = \begin{pmatrix} \frac{1}{\tau_E}\left(w_{EE}F'_E - 1\right) & -\frac{1}{\tau_E} w_{EI} F'_E \\ \frac{1}{\tau_I} w_{IE} F'_I & \frac{1}{\tau_I}\left(-w_{II} F'_I - 1\right) \end{pmatrix}$$

let $T = \begin{bmatrix} T_E & 0 \\ 0 & T_I \end{bmatrix}$, $F = \begin{bmatrix} F_E' & 0 \\ 0 & F_I' \end{bmatrix}$, $W = \begin{bmatrix} W_{EE} & -W_{EI} \\ W_{IE} & -W_{II} \end{bmatrix}$

identity matrix

$$J = T^{-1}(FW - I)$$

$$\det(J) = \det\left(T^{-1}(FW - I)\right)$$

$$= \left(\underbrace{\det(T^{-1})}_{>0}\right)\left(\det(FW - I)\right)$$

$\Rightarrow$ sign of $\det(J)$ is the same sign as $\det(FW - I)$

$$\det(FW - 1) = (F_E' w_{EI})(F_I' w_{IE}) - (F_I' w_{II} + 1)(F_E' w_{EE} - 1) > 0$$

$$\frac{\left(\dfrac{de_I}{de_E}\right)_{I\text{-nullcline}}}{\left(\dfrac{de_I}{de_E}\right)_{E\text{-nullcline}}} > 1$$

Stable fixed point: I nullcline has a steeper slope than E nullcline.

Conclusion 2: Effect of adding input to the inhibitory population.

While adding the input δ extI into the inhibitory population, we can find that the E nullcline stays the same, while the I nullcline has a pure left shift:

W3D2_pod31

Original I nullcline equation —

$$r_I = F_I\left(w_{IE} r_E - w_{II} r_I + I_I^{ext} ; a_I, \theta_I\right)$$

remains True if

$$I_I^{ext} \rightarrow I_I^{ext} + \delta I_I^{ext}$$

$$r_E \rightarrow r_E' = r_E - \frac{\delta I_I^{ext}}{w_{IE}}$$

$$r_I = F_I\left(w_{IE} r_E' - w_{II} r_I + I_I^{ext} + \delta I_I^{ext} ; a_I, \theta_I\right)$$

Non-ISN

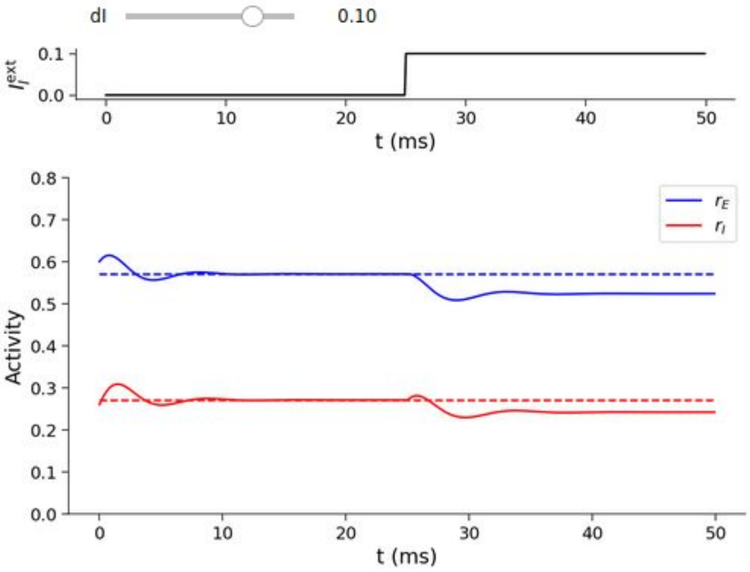
After adding input to the inhibitory population, it can be seen in the trajectories above and the phase plane below that, in an ISN, r_I will increase first but then decay to the new fixed point in which both r_I and r_E are decreased compared to the original fixed point. However, by adding $\delta I_{ext,I}$ into a non-ISN, r_I will increase while r_E will decrease.

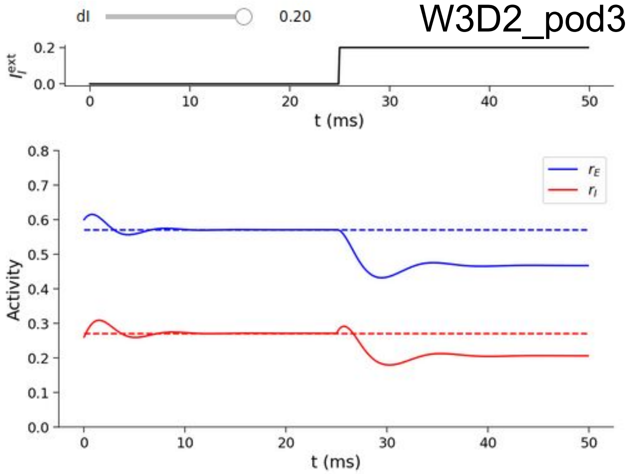
Nullclines of Example ISN and non-ISN

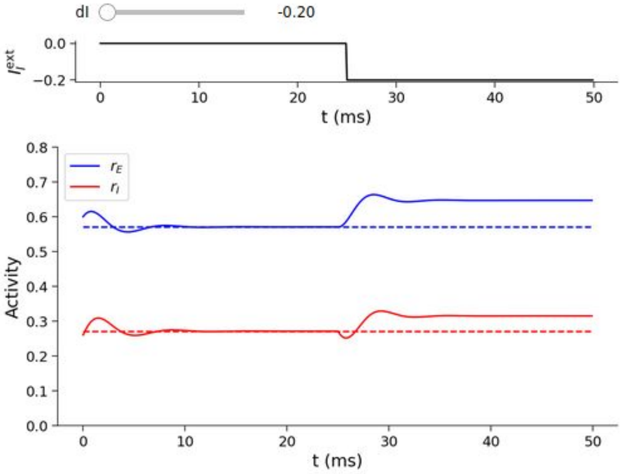
we inject excitatory ($I_{ext,I}>0$) or inhibitory ($I_{ext,I}<0$) drive into the inhibitory population when the system is at its equilibrium (with some parameters)

How does the firing rate of the I population changes with excitatory vs inhibitory drive into the inhibitory population?

W3D2_pod31


dI
0.10
I_I^{ext}
t (ms)
Activity
r_E
r_I


dI
0.20
I_I^{ext}
t (ms)
Activity
r_E
r_I

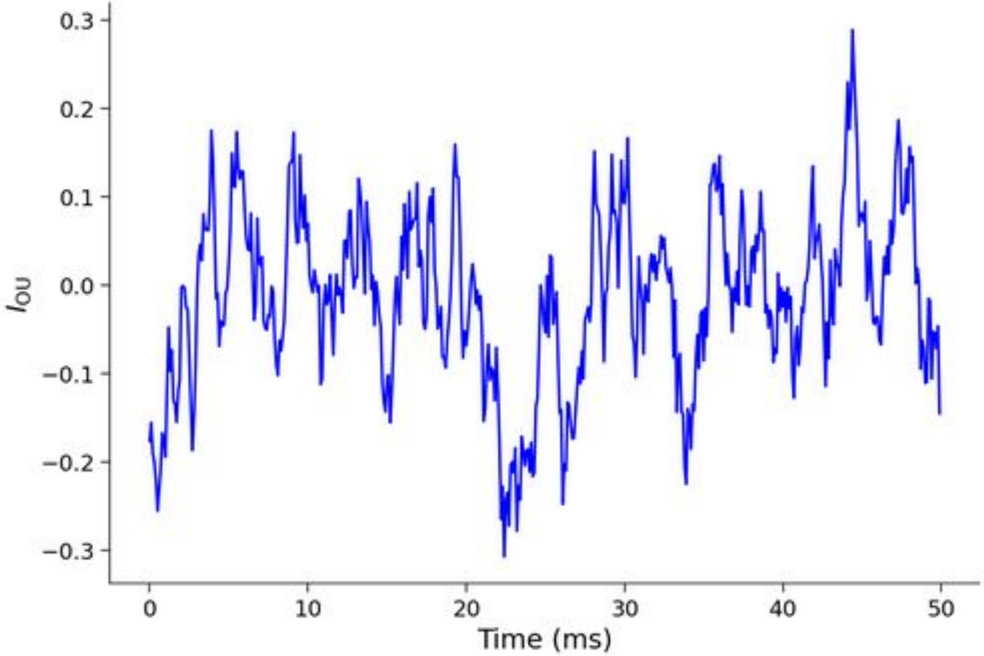

dI
-0.20
I_I^{ext}
t (ms)
Activity
r_E
r_I

Observations

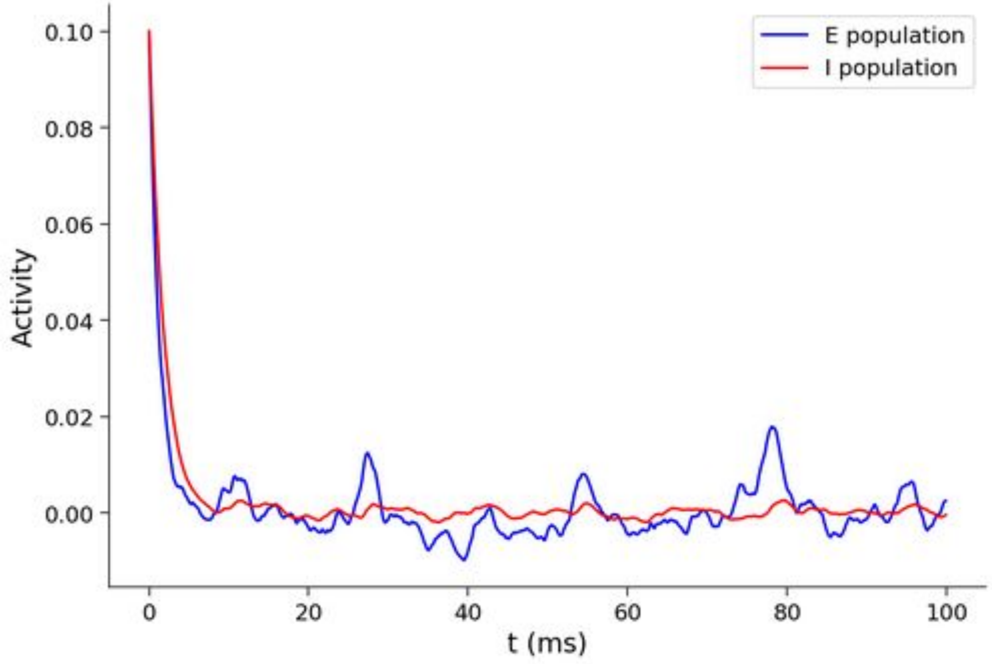
Here we observe a paradoxical effect; if we inject excitatory current to the I population, the r_I goes down, whereas when we inject inhibitory current, the r_I increases. Recall that we inject a constant excitatory current to the E population, which also drives, indirectly, the I population. When Iext>0, the r_I increases but this drives E to a low state, which in turn leads to rI decrease. Whereas, when Iext<0, the effect is negative on I population for a short amount of time, which is sufficient to drive the E population to a high steady state, and then due to E to I connections, the I population activity is increased.

Fixed point and working memory

The input into the neurons measured in the experiment is often very noisy (links). Here, the noisy synaptic input current is modeled as an Ornstein-Uhlenbeck (OU)process.With the default parameters, the system fluctuates around a resting state with the noisy input.



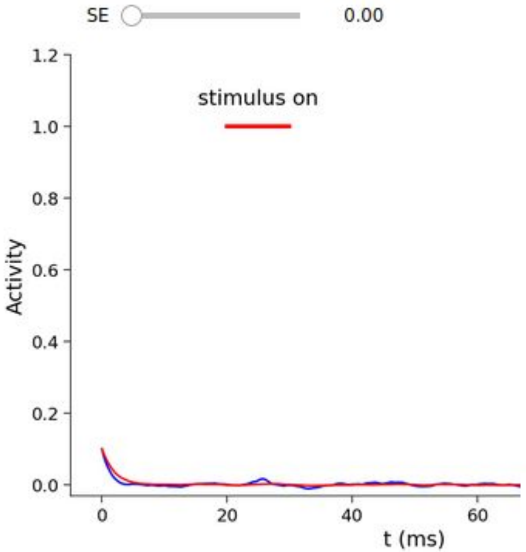
W3D2_pod31

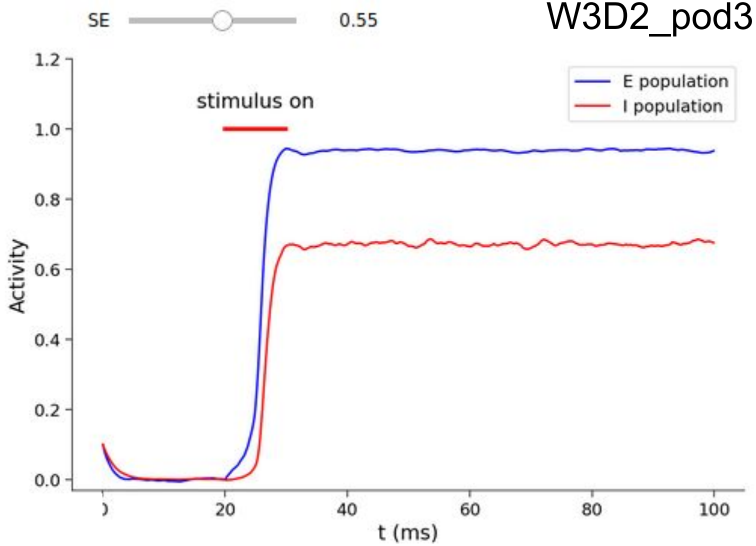

E population
I population
Activity
0.10
0.08
0.06
0.04
0.02
0.00
0
20
40
60
80
100
t (ms)

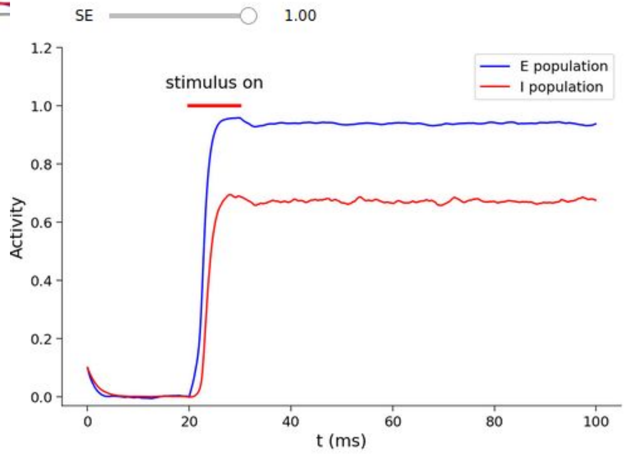
Short pulse induced persistent activity

Use a brief 10-ms positive current to the E population when the system is at its equilibrium. When this amplitude (SE) is sufficiently large, a persistent activity is produced that outlasts the transient input. What is the firing rate of the persistent activity, and what is the critical input strength?

W3D2_pod31


SE
0.00
stimulus on
E population
I population
Activity
t (ms)


SE
0.55
stimulus on
E population
I population
Activity
t (ms)


SE
1.00
stimulus on
E population
I population
Activity
t (ms)

Observations

When a system has more than one fixed points, depending on the input strength, the network will settle in one of the fixed points. In this case, we have two fixed points, one of the fixed points corresponds to high activity. So when input drives the network to the high activity fixed points, the network activity will remain there -- it is a stable fixed point. Because the network retains its activity (persistent activity) even after the input has been removed, we can take the persistent activity as working memory.