Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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Session # 22

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2/12/24

EE 3101 Digital Signal Processing

EE3101
Session 22

Session 22

Outline

Last session

- FIR Linear Phase
  Type I, II, III, IV
- Parks-McClellan Filter design
  FIR Lin ϕ equiripple filters

Today

- DT Fourier Series

Week 9-10 O&S chapter 5

Frequency Domain Analysis of LTI Systems: Frequency response of systems with rational transfer function—definitions of magnitude and phase response—frequency response of single complex zero/pole—frequency response of simple configurations (second order resonator, notch filter, averaging filter, comb filter, allpass systems)—

phase response—definition of principal phase—zero-phase response—conditions that have to be met for a filter to have generalized linear phase—four types of linear phase FIR filters—on the zero locations of a linear phase FIR filter—constrained zeros at z = 1 and at z = -1 and their implications on choice of filters Type I through Type IV (with focus on Type I)

O&S Chapter 7 Filter Design Techniques. Parks-McClellan

Week 11-12 (DFT & FFT)

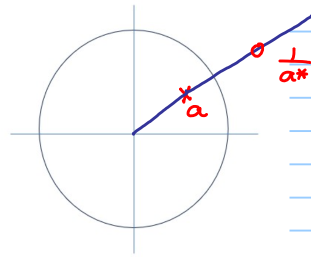
Discrete Fourier Transform (DFT): Definition of the DFT and inverse DFT—relationship to discrete-time Fourier series—matrix representation—DFT as the samples of the DTFT and the implied periodicity of the time-domain signal—circular convolution and its relationship with linear convolution—effect of zero padding—introduction to the Fast Fourier Transform (FFT) algorithm—decimation-in-time

Reading Assignment

O&S ch 8 Discrete Fourier Transform

Allpass Systems

$$\boxed{H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}} = \frac{z^{-1}(1 - a^* z)}{(1 - az^{-1})}$$

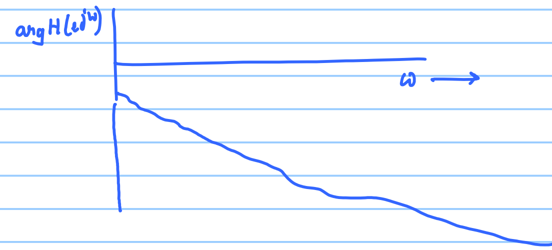


$$H_{ap}(e^{jw}) = e^{-jw}\frac{(1 - a^* e^{jw})}{(1 - a e^{-jw})} \implies |H_{ap}(e^{jw})| = 1 \quad \forall w$$

Causal, stable allpass $\Rightarrow$ all poles inside the unit circle $r < 1$

① $\tau(w) > 0 \; \forall w$ Group Delay is positive $\forall w$

② $\arg H_{ap}(e^{jw})$ is a monotone decreasing function



Magnitude Squared Response

$$|H(e^{jw})|^2 = H(e^{jw})\, H^*(e^{jw})$$

$$\underbrace{h[n] * h^*[-n]}_{\text{autocorrelation}} \longleftrightarrow |H(e^{jw})|^2$$

$$h[n] \longleftrightarrow H(e^{jw})$$

$$h^*[-n] \longleftrightarrow H^*(e^{jw})$$

$$h[n] * h^*[-n] \longleftrightarrow H(z)\; H^*\left(\frac{1}{z^*}\right)$$

Observations about Magnitude Squared function

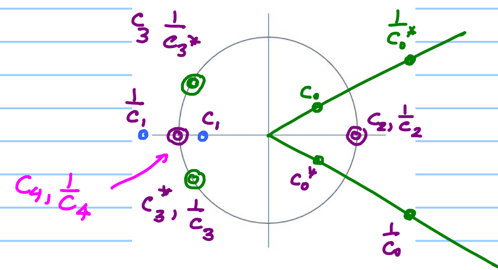
If c_0 is a zero/pole, $\frac{1}{c_0^*}$ is also a zero/pole

If $h[n]$ is real valued, then $c_0, c_0^*, \frac{1}{c_0^*}, \frac{1}{c_0}$

If zero/pole on real axis, $c_1, \frac{1}{c_1}$ are pole/zero

If zero/pole on unit circle, but not on real axis $c_3, \frac{1}{c_3^*}$

If zero/pole @ $z = \pm 1$, double zero/pole $c_4, 1/c_4$



Properties Min ϕ systems

(Min ϕ lag systems)

* Causal, stable $\longrightarrow$ all poles & zeros inside unit circle
* Min ϕ lag compared to any other TF with same mag response
* Min GD "
* Max partial energy "

Result

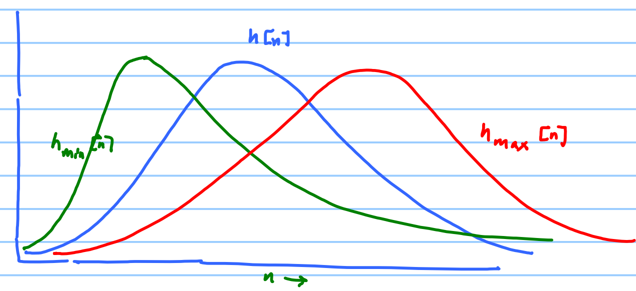
Any rational TF $H(z)$ (causal & stable) can be expressed as

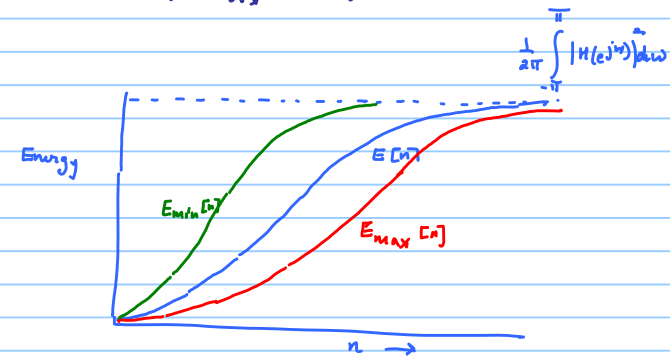
$$H(z) = H_{min}(z)\, H_{ap}(z)$$

LTI system causal & stable

$H(z)$	$H_{min}(z)$	$H_{max}(z)$
mixed	all zeros inside unit circle	all zeros outside

$$|H(e^{jw})| = |H_{min}(e^{jw})| = |H_{max}(e^{jw})| \quad \forall\, w$$





Partial Energy

$$E_{min}[n] \geq E[n] \geq E_{max}[n]$$

$\downarrow$

Min GD

Min ϕ lag

Linear Phase

Zero phase → example Ideal LPF $h[n]$

$h[n]$ real, causal

$$\left\{\begin{array}{l} h[n] \longleftrightarrow H(e^{jw}) \\ h[-n] \longleftrightarrow H^*(e^{jw}) \end{array}\right.$$

$h_e[n] \longleftrightarrow \underbrace{H_R(e^{jw})}$ Real valued

$h_o[n] \longleftrightarrow j\,\underbrace{H_I(e^{jw})}$ Real-valued

$j = e^{j\frac{\pi}{2}}$

M = even

$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4}$

M = 4
# coeff. = M+1 = 5

M = odd

$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$

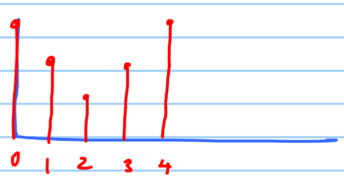
M = 3
# coeff. = M+1 = 4

$\arg H_e(e^{jw}) = -wN_0$ Linear phase

$$\boxed{h_e[n-N_0] \longleftrightarrow e^{-jwN_0} H_R(e^{jw})}$$

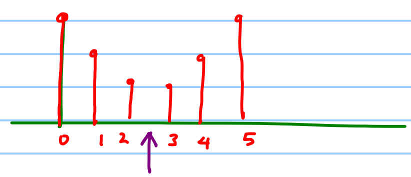
To make even seq $h_e[n]$ Causal

Type I Lin ϕ FIR



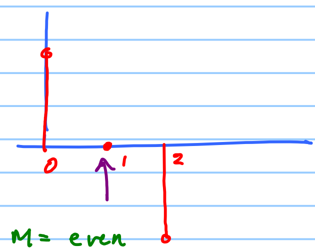
order = M = even
even symmetry

Type II Lin ϕ FIR



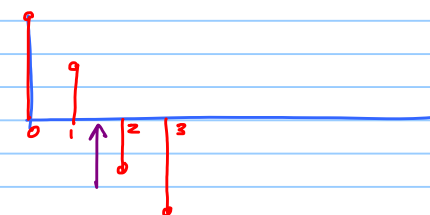
M = odd
even symmetry

Type III Lin ϕ FIR



M = even
odd symmetry

Type IV Lin ϕ FIR



M = odd
odd symmetry

nc = no constraint

Table

Symm	M	Type	$H(e^{j\omega})$	@ $\omega = 0$	@ $\omega = \pi$	Applic.
even	even	I	$e^{-j\omega \frac{M}{2}} \sum_{k=0}^{\frac{M}{2}} a[k] \cos \omega k$	nc	nc	Any filter
even	odd	II	$e^{-j\omega \frac{M}{2}} \sum_{k=1}^{\frac{M+1}{2}} b[k] \cos \omega (k-\frac{1}{2})$	nc	0	except HPF
odd	even	III	$je^{-j\omega \frac{M}{2}} \sum_{k=1}^{\frac{M}{2}} c[k] \sin \omega k$	0	0	HPF x, LPF x; Differentiator, Hilbert Transformer
odd	odd	IV	$je^{-j\omega \frac{M}{2}} \sum_{k=1}^{\frac{M+1}{2}} d[k] \sin \omega (k-\frac{1}{2})$	0	nc	HPF ✓, LPF x; "

Type I

$$H(e^{j\omega}) = \underbrace{e^{-j\omega \frac{M}{2}}}_{\text{phase}} \underbrace{\sum_{k=0}^{\frac{M}{2}} a[k] \cos \omega k}_{\text{Amplitude}}$$

real, real

$A(e^{j\omega})$ < real valued, + or -ve

$$\arg H(e^{j\omega}) = -\omega \frac{M}{2}$$

$$GD \quad \tau(\omega) = +\frac{M}{2}$$

FIR lin ϕ with real coefficient

$h[n] = \pm h[M-n]$

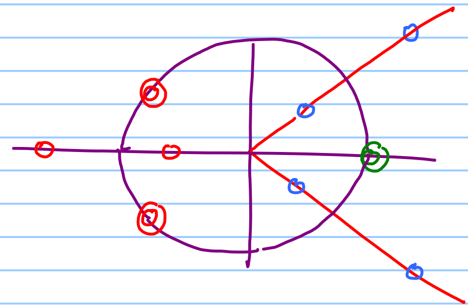
$H(z) = h[0] + h[1]z^{-1} + \dots + h[1]z^{-(M-1)} + h[0]z^{-M}$

$$\boxed{H(z) = z^{-M} H(z^{-1})}$$

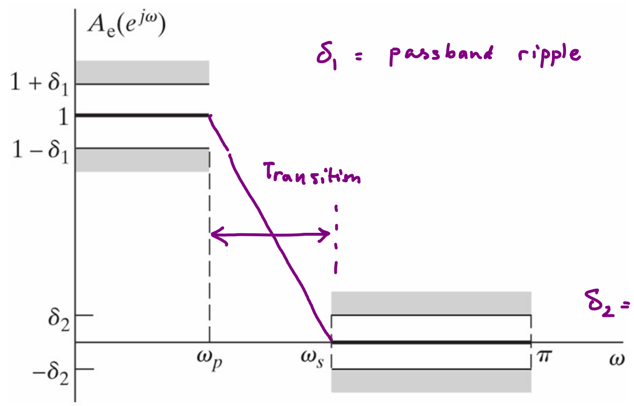
If z_0 is a zero of $H(z)$ FIR lin ϕ TF, then $\frac{1}{z_0}$ is also a zero

h[n] is real-valued

If z_0 is a zero, z_0^* is a zero $\Bigg\}$ $\frac{1}{z_0}$ is a zero, $\frac{1}{z_0^*}$ is a zero

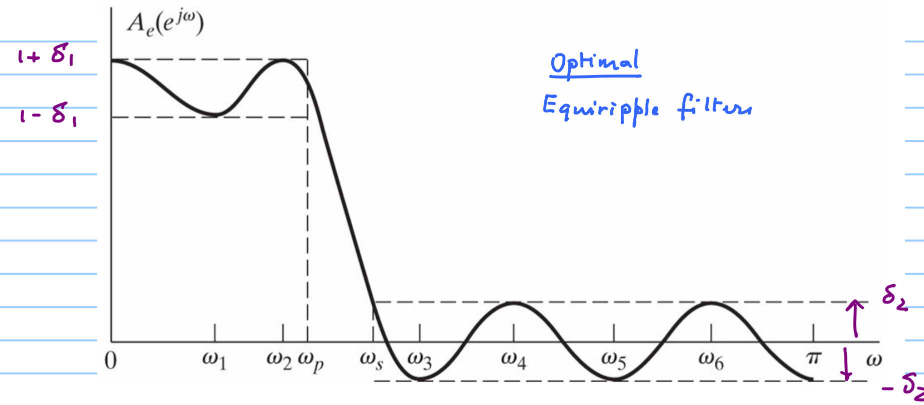


Filter Specifications (LPF)



δ_1 = passband ripple

δ_2 = stopband Attenuation



Optimal

Equiripple filters

Parks - McClellan Algorithm

$\left.\begin{array}{l}\omega_p \\ \omega_s\end{array}\right\}$ $\Delta\omega = \omega_s - \omega_p$ Transition band

δ_1

δ_2

M

$\left.\begin{array}{ll}\text{Passband} & [0, \omega_p] \\ \text{Stopband} & [\omega_s, \pi]\end{array}\right\}$ Transition band $[\omega_p, \omega_s]$

LPF

ω_p passband edge (radians)

ω_s stopband edge (radians)

δ_1 passband ripple

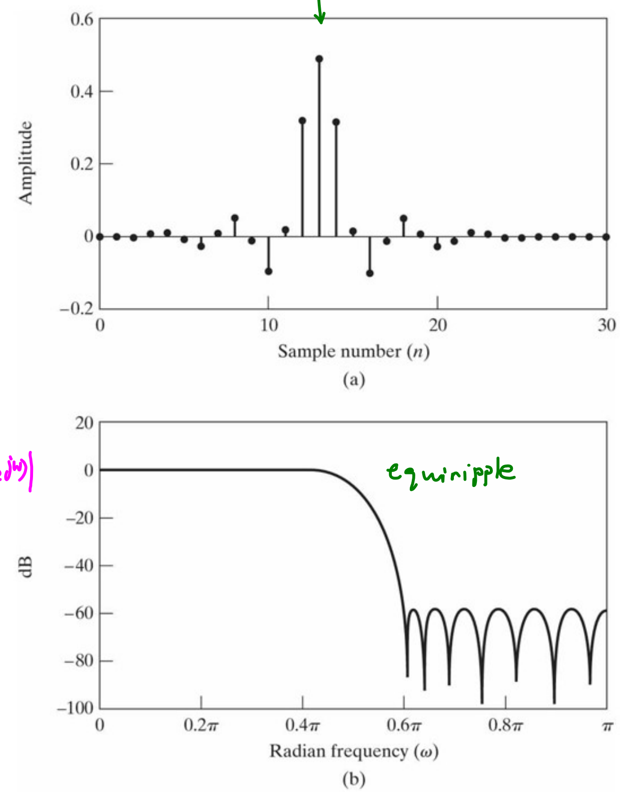
δ_2 stopband ripple   Attenuation $= -20\log_{10}\delta_2$ (dB)

Estimation of Filter Order M = filter order $= \dfrac{-10\log_{10}\delta_1\delta_2 - 13}{2.324\,\Delta\omega}$

PM Filter ≡ Equiripple

point of symmetry

$20 \log_{10} |H(e^{j\omega})|$



Parks - McClellan (PM) Design Example

$\omega_p = 0.4\pi$

$\omega_s = 0.6\pi$

$K = 10$

$M = 26$ (Type I FIR filter)

$\boxed{\delta_1 = K\delta_2}$

$\delta_1 = 0.009$

Filter Order estimation

$$M = \frac{-10 \log_{10} \delta_1 \delta_2 - 13}{2.324\, \Delta\omega}$$

$\delta_1 = 0.009$

$\delta_2 = 0.0009$

$\Delta\omega = 0.2\pi$

$M \sim 25.96 = 26.$

Verify via design that filter satisfies

1. the specifications for δ_1 and δ_2
2. Equiripple response

DSP
ch8

Discrete Fourier Transform

* Representation of DT seq $x[n]$ → DTFT $X(e^{jw})$; ZT $X(z)$

DTFT

$$X(e^{jw}) = \sum_{n=\cancel{-\infty}\,0}^{\cancel{\infty}\,N-1} x[n]\, e^{-jwn}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw$$

Time	Freq
discrete (in n)	continuous (in w)
aperiodic	periodic in w
	period = 2π

Discrete Time Fourier Series

If $\tilde{x}[n]$ is periodic ⇒ period = N

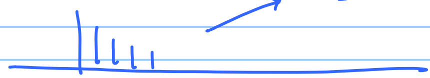
$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} C_k\, e^{j\frac{2\pi}{N}kn}$$

Fourier Coefficients

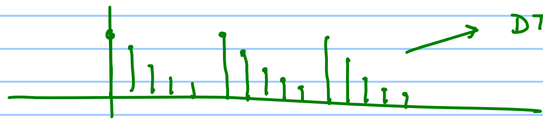
$$C_k = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\frac{2\pi}{N}nk}$$

Time	Freq
discrete	$\boxed{\text{discrete}}$
periodic	periodic

N distinct values



DTFT $X(e^{j\omega})$ continuous



DTFS discrete

Discrete Fourier Transform (DFT)

efficient implem

Fast Fourier Transform (FFT)

DTFT

$\updownarrow$

DFS $\longrightarrow$ DFT $\longrightarrow$ FFT

O&S ch 8

8.1 Discrete Fourier Series (DFS) or (DTFS)

8.2 Properties of DFS

8.3 Fourier Transform of periodic signals

8.4 Sampling the FT

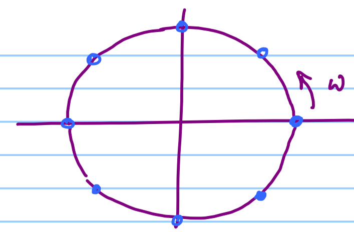
8.5 Discrete Fourier Transform (DFT) ⇒ Fourier Representation of Finite Duration Sequences

8.6 Properties of DFT

Application of DFT ⟶ Convolution

Using DFT we obtain periodic convolution

8.7 Computing Linear Convolution using the DFT



8 samples of the DTFT

Discrete Fourier Series

* Consider a DT periodic seq $\tilde{x}[n]$ with period N

$$\Rightarrow \quad \tilde{x}[n] = \tilde{x}[n + rN] \quad \text{for any integers } r, N$$

* $\tilde{x}[n]$ can be represented by a sum of harmonically related complex exponentials

* Complex exponential

$e^{j\omega n}$

periodic with period $= N$

with period N

k^{th} harm.

$$\boxed{e_k[n] = e^{j\frac{2\pi}{N}kn}}$$

fundamental freq $= \frac{2\pi}{N}$

harmonics $\frac{2\pi}{N}k \quad k = 0, 1, \dots N-1$

distinct sinusoids $= N$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\frac{2\pi}{N}kn}$$

* In CT, FS representation involves ∞ harmonically related complex exponentials

In DT, FS " " finite # complex exponentials

= Period N

$$e_k[n] = e^{j\frac{2\pi}{N}kn} \qquad k = 0, 1, \ldots N-1$$

Notation 1

$$W_N = e^{-j\frac{2\pi}{N}} \qquad e_k[n] = W_N^{-kn}$$

Notation 2

basis functions $\quad \psi[k,n] = e^{j\frac{2\pi}{N}kn} = e_k[n]$

Prop 1

$$e_{k+lN}[n] = e^{j\frac{2\pi}{N}(k+lN)n}$$

$$= e^{j\frac{2\pi}{N}kn} \underbrace{e^{j\frac{2\pi}{\cancel{N}}l\cancel{N}n}}_{=1} = e^{j\frac{2\pi}{N}kn}$$

$$\boxed{e_{k+lN}[n] = e_k[n]}$$

Prop 2

$$\frac{1}{N}\sum_{n=0}^{N-1} \psi[k,n]\,\psi^*[l,n] = \begin{cases} 1 & l = k \\ 0 & l \neq k \end{cases} \qquad \text{Orthogonality}$$

Verify

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} \cdot e^{-j\frac{2\pi}{N}ln} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n} \;\ldots\ldots\; = \begin{cases} 1 & (k-l) = mN \\ 0 & \text{otherwise} \end{cases}$$

$k = 0, \ldots N-1$

$l = 0, \ldots N-1$

$\Rightarrow (k-l) = 0$

$\Rightarrow k = l$

Synthesis Equation

$$\boxed{\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\, e^{j\frac{2\pi}{N}kn}} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\, W_N^{-kn}$$

Multiply both sides by $e^{-j\frac{2\pi}{N}\ell n}$ & take Summation

$$\sum_{n=0}^{N-1}\tilde{x}[n]\, e^{-j\frac{2\pi}{N}\ell n} = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\, e^{j\frac{2\pi}{N}kn}\right) e^{-j\frac{2\pi}{N}\ell n}$$

Interchange order of Summation

$$= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-\ell)n}}_{\text{Orthogonality}}$$

$$\boxed{\sum_{n=0}^{N-1}\tilde{x}[n]\, e^{-j\frac{2\pi}{N}\ell n} = \tilde{X}[\ell]}$$

Analysis Equation

$$\boxed{\begin{array}{lll} \text{Inverse} & \text{Synthesis Equation} & \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\frac{2\pi}{N}kn} \\ \text{Forward} & \text{Analysis Equation} & \tilde{X}[\ell] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\frac{2\pi}{N}\ell n} \end{array}}$$

DFS pair

Viewing the DFS as an orthogonal transform

$$\underline{\tilde{X}} = \underbrace{\begin{bmatrix} \tilde{X}[0] \\ \tilde{X}[1] \\ \vdots \\ \tilde{X}[N-1] \end{bmatrix}}_{N\times 1 \text{ vector}} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 \cdots & W_N^{2(N-1)} \\ & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{\substack{N\times N \\ \underline{D_N}}} \underbrace{\begin{bmatrix} \tilde{x}[0] \\ \tilde{x}[1] \\ \vdots \\ \tilde{x}[N-1] \end{bmatrix}}_{N\times 1 \text{ vector}}$$

$\underline{\tilde{x}}$

$$W_N = e^{-j\frac{2\pi}{N}}$$

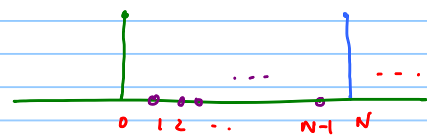
Analysis eqn. $\underline{\tilde{X}} = \underline{D_N} \cdot \underline{\tilde{x}}$ ← orthogonal Transformation

Synthesis eqn $\underline{\tilde{x}} = \underline{D_N}^{-1} \underline{\tilde{X}} = \frac{1}{N} \underline{D_N}^{*} \underline{\tilde{X}}$

$$\boxed{\underline{D_N}^{-1} = \frac{1}{N} \underline{D_N}^{*}}$$

Ex 1

$$\tilde{x}[n] = \begin{cases} 1 & n = 0 \\ 0 & 1 \le n \le N-1 \end{cases}$$



$$\tilde{X}[k] = 1 \quad \forall k.$$

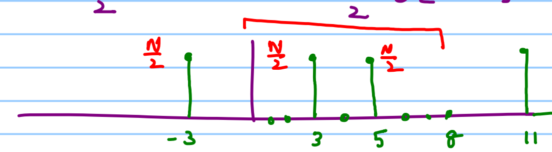
Ex2

$$\tilde{x}[n] = \cos \frac{2\pi}{N} r n \quad 0 \le n \le N-1 \quad = \frac{1}{2}\left[e^{j\frac{2\pi}{N} r n} + e^{-j\frac{2\pi}{N} r n}\right] = \frac{1}{2}\left[W_N^{-rn} + W_N^{rn}\right]$$

$r=3$
$N=8$

$$\tilde{X}[k] = \underbrace{\frac{1}{2}\sum_{n=0}^{N-1} W_N^{-rn} W_N^{kn}}_{= \begin{cases} N & \text{if } k = r \\ 0 & \text{otherwise} \end{cases}} + \underbrace{\frac{1}{2}\sum_{n=0}^{N-1} W_N^{rn} W_N^{kn}}_{= \begin{cases} N & \text{if } k = -r = N-r \\ 0 & \text{otherwise} \end{cases}}$$

$$= \frac{N}{2} \delta[k-r] + \frac{N}{2} \delta[k+r]$$



Properties of DFS

1. Linearity

Same period N
$$\left\{\begin{array}{l} \tilde{x}_1[n] \longleftrightarrow \tilde{X}_1[k] \\ \tilde{x}_2[n] \longleftrightarrow \tilde{X}_2[k] \end{array}\right\}$$

$$\alpha\,\tilde{x}_1[n] + \beta\,\tilde{x}_2[n] \longleftrightarrow \alpha\,\tilde{X}_1[k] + \beta\,\tilde{X}_2[k]$$

5 Shift of a sequence

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$$

$$\tilde{x}[n-m] \longleftrightarrow e^{-j\frac{2\pi}{N}km}\,\tilde{X}[k] = W_N^{mk}\,\tilde{X}[k]$$

Let $\tilde{x}_1[n] = \tilde{x}[n-m]$

$$\tilde{X}_1[k] = \sum_{n=0}^{N-1} \tilde{x}_1[n]\,W_N^{kn} = \sum_{n=0}^{N-1} \tilde{x}[n-m]\,W_N^{kn}$$

Let $n-m = \ell$

$$= \sum_{\ell=-m}^{N-1-m} \tilde{x}[\ell]\,W_N^{(\ell+m)k} = W_N^{mk}\left(\sum_{\ell=-m}^{N-1-m} \tilde{x}[\ell]\,W_N^{\ell k}\right) = W_N^{mk}\left[\sum_{\ell=0}^{N-1-m} \tilde{x}[\ell]\,W_N^{\ell k} + \sum_{\ell=-m}^{-1} \tilde{x}[\ell]\,W_N^{\ell k}\right]$$

1

$$\sum_{l=-m}^{-1} \tilde{x}[l]\, W_N^{lk}$$

$\tilde{x}[n]$ is periodic $\Rightarrow$ $\tilde{x}[l] = \tilde{x}[N+l]$

$$W_N^{lk} = W_N^{(N+l)k}$$

$$= \sum_{l=-m}^{-1} \tilde{x}[N+l]\, W_N^{(N+l)k}$$

Let $j = N + l$

$$= \sum_{j=N-m}^{N-1} \tilde{x}[j]\, W_N^{jk} \qquad (2)$$

Substituting (2) in (1)

$$\tilde{X}_1[k] = W_N^{mk} \underbrace{\left[\sum_{n=0}^{N-m-1} \tilde{x}[n]\, W_N^{nk} + \sum_{n=N-m}^{N-1} \tilde{x}[n]\, W_N^{nk} \right]}_{\tilde{X}[k]}$$

$$\tilde{x}[n-m] \longleftrightarrow W_N^{mk}\, \tilde{X}[k]$$

#4 Duality

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, \underbrace{W_N^{nk}}$$

(1) Summation
(2) over N terms

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, W_N^{-kn}$$

(1) Summation
(2) over N terms

Same set of Sinusoids

$$W_N^{-kn} = W_N^{(N-k)n}$$

Synthesis eqn

$$N\,\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{X}[k]\, W_N^{-nk}$$

$$N\,\tilde{x}[-n] = \sum_{k=0}^{N-1} \tilde{X}[k]\, W_N^{nk}$$

Compare with Analysis Equation

Computed using the Forward Transformation

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$$

$$\tilde{X}[n] \longleftrightarrow N\,\tilde{x}[-n]$$

scaling

indexing

Principle of Duality

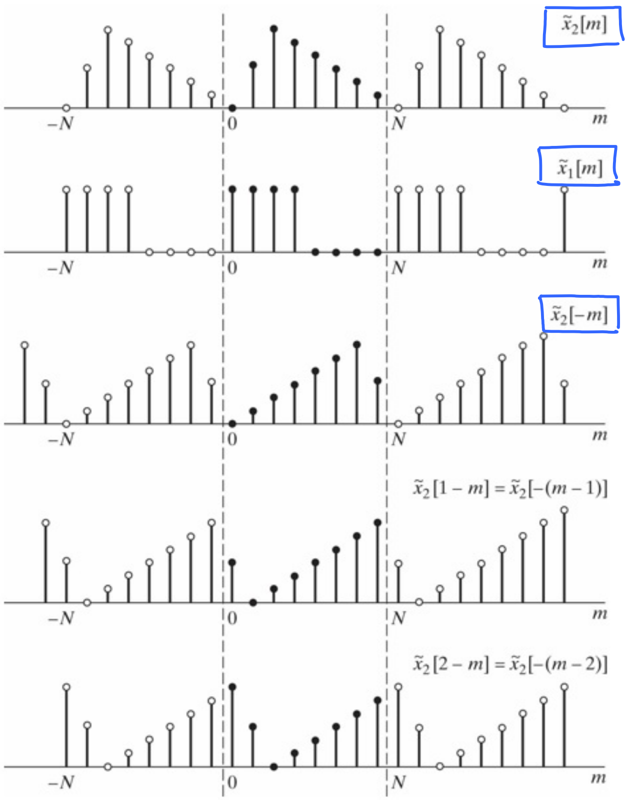
TABLE 8.1 SUMMARY OF PROPERTIES OF THE DFS

	Periodic Sequence (Period N)	DFS Coefficients (Period N)
	1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
Linearity	2. $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
	3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
Duality	4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
Shift in time	5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
	6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$ shift in freq
	7. $\underbrace{\sum_{m=0}^{N-1} \overbrace{\tilde{x}_1[m]}^{N}\overbrace{\tilde{x}_2[n-m]}^{N}}_{N}$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
	8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
	9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
	10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$

Verify

Prove property #6 Shift in freq using the principle of duality

Periodic Convolution



$N = 8$

Time - reversed

Symmetry Properties (Verify)

11.	$\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12.	$j\mathcal{I}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13.	$\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14.	$\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$
	Properties 15–17 apply only when $x[n]$ is real.	
15.	Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$
16.	$\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17.	$\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$$

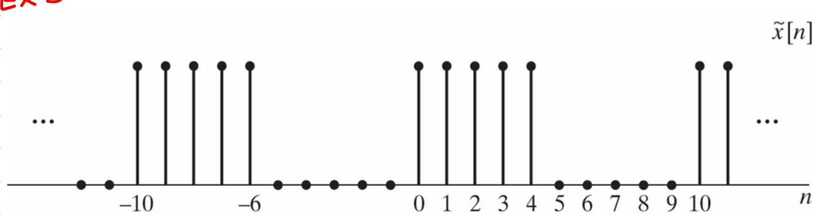
$$\tilde{x}^*[n] \longleftrightarrow \tilde{X}^*[-k]$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk}$$

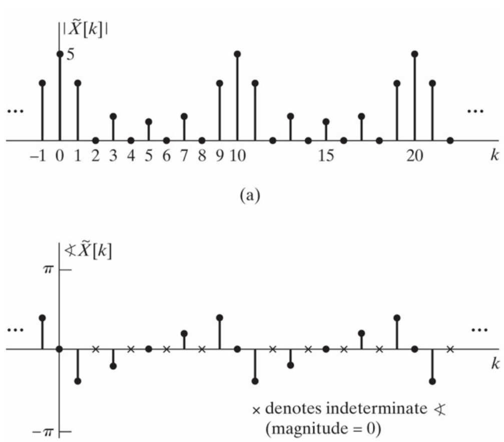
$$\tilde{X}^*[k] = \sum_{n=0}^{N-1} \tilde{x}^*[n] W_N^{-nk}$$

$$\tilde{X}^*[-k] = \sum_{n=0}^{N-1} \tilde{x}^*[n] W_N^{nk}$$

O&S
Ex 3



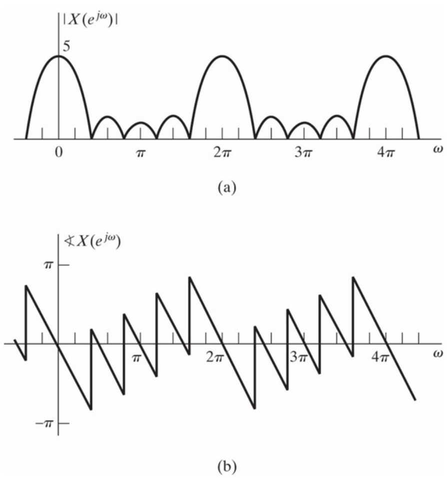
$\tilde{x}[n]$
–10
–6
0 1 2 3 4 5 6 7 8 9 10
n



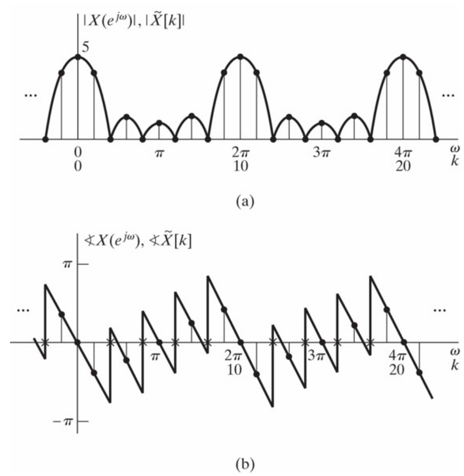
$|\tilde{X}[k]|$
5
–1 0 1 2 3 4 5 6 7 8 9 10
15
20
k
(a)
$\sphericalangle\tilde{X}[k]$
π
$-\pi$
k
× denotes indeterminate $\sphericalangle$
(magnitude = 0)

O&S

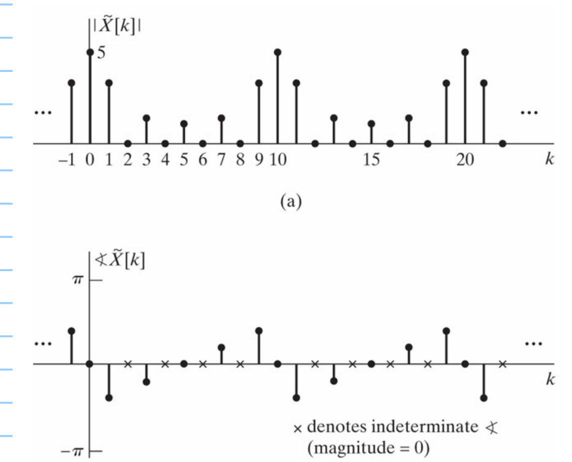
Ex 6



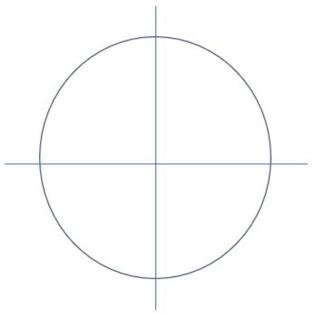
$|X(e^{j\omega})|$
5
0 π 2π 3π 4π ω
(a)
$\sphericalangle X(e^{j\omega})$
π
$-\pi$
π 2π 3π 4π ω
(b)



$|X(e^{j\omega})|, |\tilde{X}[k]|$
5
0 π 2π 3π 4π ω
0 10 20 k
(a)
$\sphericalangle X(e^{j\omega}), \sphericalangle \tilde{X}[k]$
π
$-\pi$
π 2π 3π 4π ω
10 20 k
(b)



$|\tilde{X}[k]|$
5
–1 0 1 2 3 4 5 6 7 8 9 10 15 20 k
(a)
$\sphericalangle \tilde{X}[k]$
π
$-\pi$
k
× denotes indeterminate $\sphericalangle$
(magnitude = 0)



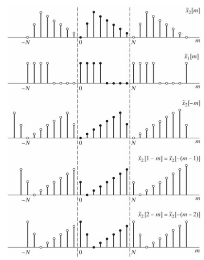
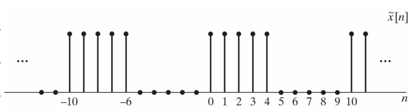
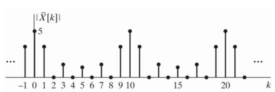


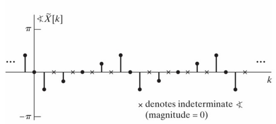
TABLE 8.1 SUMMARY OF PROPERTIES OF THE DFS

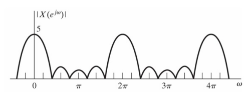
Periodic Sequence (Period N)	DFS Coefficients (Period N)
1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2. $\tilde{x}_1[n]$, $\tilde{x}_2[n]$	$\tilde{X}_1[k]$, $\tilde{X}_2[k]$ periodic with period N
3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7. $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{I}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$



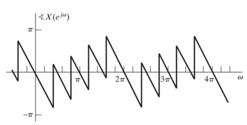


(a)

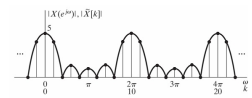




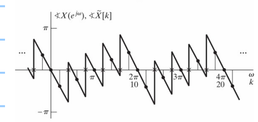
(a)



(b)



(a)



(b)