Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

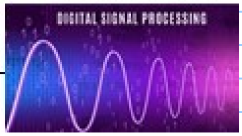
BS Electronic Systems programme

R. David Koilpillai

September – December 2024

Session #6

September 24, 2024



EE3101 Digital Signal Processing

Session 6

Outline

Last session

- Convolution
- Energy signals, Power signals
- BIBO stability

Week 1-2

Introduction to sampling - Review of Signals and Systems: Basic operations on signals - Discrete time complex exponentials and other basic signals – system properties (linearity, time-invariance, memory, causality, BIBO stability) -- impulse response and convolution

Week 3-4

Sampling: Impulse train sampling—relationship between impulse trained sampled continuous-time signal spectrum and the DTFT of its discrete-time counterpart—scaling of the frequency axis—relationship between true frequency and digital frequency—reconstruction through sinc interpolation—aliasing—effect of sampling at a discontinuous point—relationship between analog and digital sinc—effects of oversampling—discrete-time processing of continuous-time signals

- Examples
- Interconnection of LTI systems

$\boxed{\text{Reading Assignment}}$

Oppenheim & Schafer : Sec 2.0 – 2.5 completed

Rawat : Chapter 1

O & S Chapter 4

CT $\xrightarrow{\text{Sampling}}$ DT

Cont. Time — Discrete Time

Session 5

Exercises

Check the following DT for Linearity

* 1. $y[n] = n\,x[n]$ L
2. $y[n] = x^2[n]$ NL
* 3. $y[n] = Re\{x[n]\}$ NL
4. $y[n] = x[n]\,x[n-1]$ NL

Linearity < scaling, superposition

① $y[n] = n\,x[n]$ LINEAR ✓

$x_1[n] \longrightarrow y_1[n] = n\,x_1[n]$

$x_2[n] \longrightarrow y_2[n] = n\,x_2[n]$

$a\,x_1[n] + b\,x_2[n] \longrightarrow y[n] = n\left(a\,x_1[n] + b\,x_2[n]\right) = a\,y_1[n] + b\,y_2[n]$

③ $y[n] = Re\{x[n]\}$ $\quad x[n] = x_R[n] + j\,x_I[n]$

a is real const

$x[n] \longrightarrow y[n] = x_R[n]$

$x_1[n] = a\,x[n] \longrightarrow y_1[n] = a\,x_R[n] = a\,y[n]$

$= a\,x_R[n] + j\,a\,x_I[n]$

Scaling ✗

scaling valid for all constants "a" – real ✓
– complex

$a = j = \sqrt{-1}$

$x_2[n] = a\,x[n] = j\,x[n] \longrightarrow Re\{x_2[n]\}$

$= -x_I[n] + j\,x_R[n]$ $\quad = -x_I[n] \neq j\,x_R[n]$

Scaling not Satisfied
Not Linear

Check if the following DT Systems are Time/shift invariant.

Exercises

* ① $y[n] = x[-n]$
② $y[n] = a\,x[n]$
③ $y[n] = n\,x[n]$
* ④ $y[n] = x[2n]$

SV = Shift Variant
SI = Shift Invariant

Check for shift invariance

$x[n] \longrightarrow y[n]$

$x[n-n_0] \longrightarrow y[n-n_0]$ then DT system is shift invariant

Exercise

* ① $y[n] = x[-n]$ SV
② $y[n] = a\,x[n]$ SI
③ $y[n] = n\,x[n]$ SV
* ④ $y[n] = x[2n]$ SV

① $y[n] = x[-n]$

$x_1[n] \longrightarrow y_1[n] = x_1[-n]$

$x_2[n] = x_1[n-n_0] \longrightarrow y_2[n] = x_2[-n] = x_1[-n-n_0]$

$y_1[n-n_0] = x_1[-(n-n_0)] = x_1[-n+n_0]$

$y_2[n] \neq y_1[n-n_0]$

④ $y[n] = x[2n]$

$x_1[n] = x[n-n_0] \longrightarrow y_1[n] = x_1[2n] = x[2n-n_0]$

$y[n-n_0] = x[2(n-n_0)] = x[2n-2n_0]$

$y_1[n] \neq y[n-n_0]$

Review

$x[n] * \delta[n] = x[n]$

$x[n] * \delta[n-1] = x[n-1]$

$x[n-1] * \delta[n] = x[n-1]$

$x[n-1] * \delta[n-1] = x[n-2]$

$x[n-n_0] * \delta[n-n_1] = x[n-n_0-n_1] = x[n-(n_0+n_1)]$

Convolution - Analytical Method

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

Homework

LTI system $\qquad h[n] = a^n u[n-1] \qquad 0 < a < 1$

$x[n] = u[n-1]$

$$= \begin{cases} 1 & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$h[0] = 0$

$h[1] = a$

$h[2] = a^2$

$\vdots$

$$\left.\begin{matrix} h[0] = 0 \\ h[1] = a \\ h[2] = a^2 \\ \vdots \end{matrix}\right\} \begin{matrix} a^n & n \geq 1 \\ 0 & \text{otherwise} \end{matrix}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} u[k-1]\, a^{n-k}\, \underbrace{u[n-k-1]}_{u[n-1-k]} = \sum_{k=1}^{n-1} a^{n-k} = a^n \sum_{k=1}^{n-1} a^{-k} = a^n \left[\frac{1}{a} + \frac{1}{a^2} + \cdots + \frac{1}{a^{n-1}} \right]$$

$$= a^{n-1} \left[1 + \frac{1}{a} + \cdots + \frac{1}{a^{n-2}} \right]$$

$$= \left(a^{n-1}\right) \frac{\frac{1}{a^{n-1}} - 1}{\frac{1}{a} - 1}$$

$$y[n] = \frac{a - a^n}{1 - a} u[n]$$

Any LTI system $\longrightarrow$ fully characterized by its impulse response $h[n]$

$$x[n] * h[n] \equiv h[n] * x[n] \quad \text{commutative}$$

$$h[n] = \sum_{k=-\infty}^{\infty} h_1[k] \cdot h_2[n-k] \quad (1)$$

$$x[n] \longrightarrow \boxed{h_1[n]} \xrightarrow{w[n]} \boxed{h_2[n]} \longrightarrow y_1[n]$$

compare with

$$x[n] \longrightarrow \boxed{\underbrace{h_1[n] * h_2[n]}_{h[n]}} \longrightarrow y_2[n]$$

$$y_2[n] = \sum_{m=-\infty}^{\infty} x[m]\, \underline{h[n-m]} \quad (2)$$

Substitute (1) in (2)

$$= \sum_{m=-\infty}^{\infty} x[m] \sum_{k=-\infty}^{\infty} h_1[k]\, \underbrace{h_2[n-m-k]}_{h_2[n-(m+k)]}$$

$$r = m+k$$

$$= \sum_{m=-\infty}^{\infty} x[m] \sum_{r=-\infty}^{\infty} h_1[r-m]\, h_2[n-r]$$

$$= \sum_{r=-\infty}^{\infty} \underbrace{\left(\sum_{m=-\infty}^{\infty} x[m]\, h_1[r-m] \right)}_{w[r]} h_2[n-r]$$

$$y_2[n] = \sum_{r=-\infty}^{\infty} w[r]\, h_2[n-r]$$

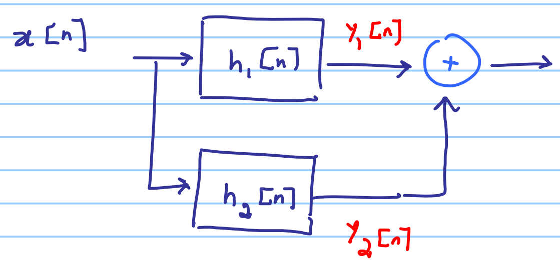
$$\boxed{y_2[n] = y_1[n]}$$

$$x[n] \to \boxed{h_1[n]} \to \boxed{h_2[n]} \to y[n] \equiv x[n] \to \boxed{h_1[n] * h_2[n]} \to y[n] \equiv x[n] \to \boxed{h_2[n] * h_1[n]} \to y[n]$$

Result :

$$x[n] \to \boxed{h_1[n]} \to \boxed{h_2[n]} \to y[n] \equiv x[n] \to \boxed{h_2[n]} \to \boxed{h_1[n]} \to y[n]$$

$$\equiv x[n] \to \boxed{h_1[n] * h_2[n]} \to y[n]$$



$$y[n] = y_1[n] + y_2[n]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h_1[k] + \sum_{k=-\infty}^{\infty} x[n-k]\, h_2[k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k] \underbrace{\left(h_1[k] + h_2[k] \right)}_{h[n]}$$

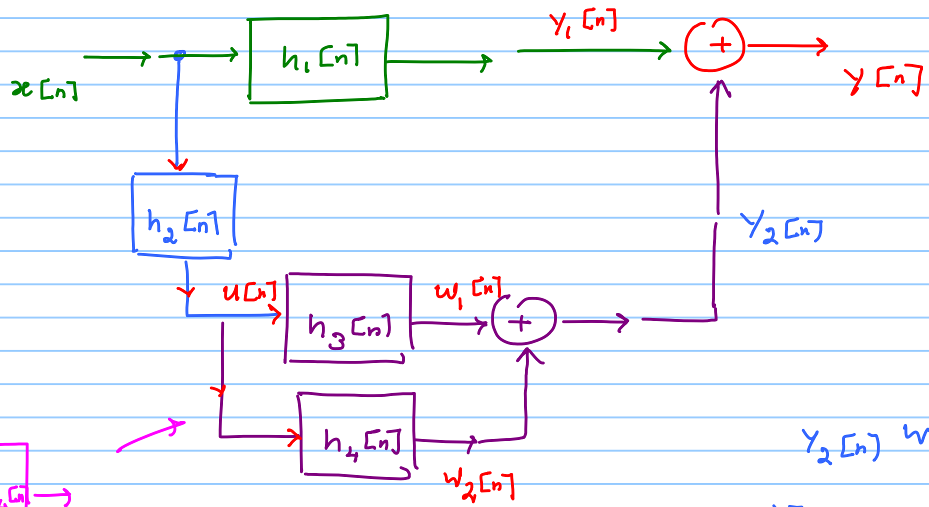
$$h[n] = h_1[n] + h_2[n] * [h_3[n] + h_4[n]]$$ — final expression

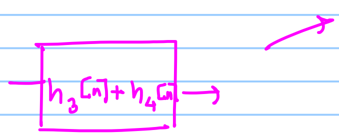
Property 2

$$x[n] * (h_1[n] + h_2[n]) \equiv x[n] * h_1[n] + x[n] * h_2[n]$$

Distribution Property

Example





$$y_1[n] = x[n] * h_1[n]$$

$$u[n] = x[n] * h_2[n]$$

$$w_1[n] = u[n] * h_3[n]$$

$$= x[n] * h_2[n] * h_3[n]$$

$$w_2[n] = x[n] * h_2[n] * h_4[n]$$

$$y_2[n] = w_1[n] + w_2[n] = x[n] * (h_2[n] * h_3[n] + h_2[n] * h_4[n])$$

$$y[n] = x[n] * \left[\overbrace{h_1[n] + h_2[n] * h_3[n] + h_2[n] * h_4[n]}^{h[n]}\right]$$

Session 5

Causality

A DT system $y[n] = T\{x[n]\}$ is causal if for every choice of n_0 the output seq $y[n_0]$ depends only on the input seq values $x[n_0]$, $x[n_0-1]$, $x[n_0-2]$ ...

$y[n_0]$ depends on $x[n]$ $\quad n \le n_0$

LTI systems $\rightarrow$ characterized by unit sample response $h[n]$

$\rightarrow$ Causal if $\boxed{h[n] = 0 \qquad n < 0}$

Energy of DT Seq $x[n]$

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Power of DT seq $x[n]$

① $$P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$$

② Periodic signal Period $= N$

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

Periodic Signals $\mathcal{E}_x = \infty$, P_x finite Power signals

Finite duration signal $\mathcal{E}_x =$ finite $P_x \to 0$ Energy signals

Ex. Given $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$

Obtain energy of $x_1[n]$, $\mathcal{E}_{x_1}$, where $x_1[n] = x[-n]$

$$\mathcal{E}_{x_1} = \sum_{n=-\infty}^{\infty} |x_1[n]|^2 = \sum_{n=-\infty}^{\infty} |x[-n]|^2$$

$m = -n$ $$\sum_{m=-\infty}^{\infty} |x[m]|^2 = \mathcal{E}_x$$

$$\boxed{\mathcal{E}_{x_1} = \mathcal{E}_x}$$

$P_x = 0$

Ex. Compute P_x and $\mathcal{E}_x$

(1) $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$ $\qquad \mathcal{E}_{x_1} = \sum_{n=-\infty}^{\infty} |x_1[n]|^2 = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

$\mathcal{E}_{x_1} = \frac{4}{3}$

$P_{x_1} = 0$

(2) $x_2[n] = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{8}\right)}$ $\qquad |x_2[n]| = \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{8}\right)} \right| = 1$

$\mathcal{E}_{x_2} = \sum_{n=-\infty}^{\infty} |x_2[n]|^2 = \infty$

$P_{x_2} = \frac{1}{2K+1} \sum_{n=-K}^{K} |x_2[n]|^2 = \frac{2K+1}{2K+1} = 1$

$\mathcal{E}_{x_2} = \infty$

$P_{x_2} = 1$

(3) $x_3[n] = \cos \frac{\pi}{4} n$

$$\mathcal{E}_{x_3} = \sum_{n=-\infty}^{\infty} \left| \cos^2 \frac{\pi}{4} n \right| \qquad \cos^2 \frac{\pi}{4} n = \frac{1}{2}\left(1 + \cos \frac{\pi}{2} n\right)$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right.}\left.+ \cancel{\frac{1}{2} \cos \frac{\pi}{2} n} \right)$$

$\mathcal{E}_{x_3} = \infty$

$P_{x_3} = \frac{1}{2}$

Boundedness

A seq $\{x[n]\}$ is said to be bounded if $|x[n]| \leq B_x < \infty$

eg $A\cos(\omega_0 n + \phi) = x[n]$ $|x[n]| \leq A$ $\Rightarrow$ $x[n]$ is bounded

$x_2[n] = A\alpha^n u[n]$ $\alpha > 1$ $x[n]$ is not bounded

Absolutely Summable $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

Square Summable $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$ $\Rightarrow$ Energy signal

---

LTI system $\longrightarrow$ impulse response $h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\cdot x[n-k] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Given $x[n]$ is bounded $|x[n]| \leq B_x < \infty$ $\forall n$

If $\sum_{k=-\infty}^{\infty} |h[k]| \leq A < \infty$ Absolute summability

$\Rightarrow$ $y[n]$ is also bounded

BIBO Stability

① $y[n] = n\,x[n]$

Bounded input

$|x[n]| \le B_x < \infty$

$|y[n]| = |n|\,|x[n]|$

$\le |n|\,B_x$

$\lim_{n \to \infty} |n|\,B_x \to \infty$

not bounded $\Rightarrow$ DT not BIBO stable

② $y[n] = e^{x[n]}$

$\underbrace{|x[n]| \le B_x}$ $< \infty$

$-B_x \le x[n] \le B_x$

$e^{-B_x} \le y[n] = e^{x[n]} \le e^{B_x}$

$y[n] \le e^{B_x}$ $\Rightarrow$ $y[n]$ is bounded DT system is BIBO stable

③ $y[n] = \underbrace{\sum_{k=0}^{\infty} \rho^k x[n-k]}_{\text{convolution}}$

$\left.\begin{array}{l} h[n] * x[n] \\ h[n] = \rho^n u[n] \end{array}\right\}$ $|\rho| < 1$ decaying exponential

$|\rho| > 1$ growing exponential

$|x[n]| \leq B_x < \infty$

$$|y[n]| = \left| \sum_{k=0}^{\infty} \rho^k x[n-k] \right| \leq \sum_{k=0}^{\infty} |\rho|^k \underbrace{|x[n-k]|}_{\leq B_x}$$

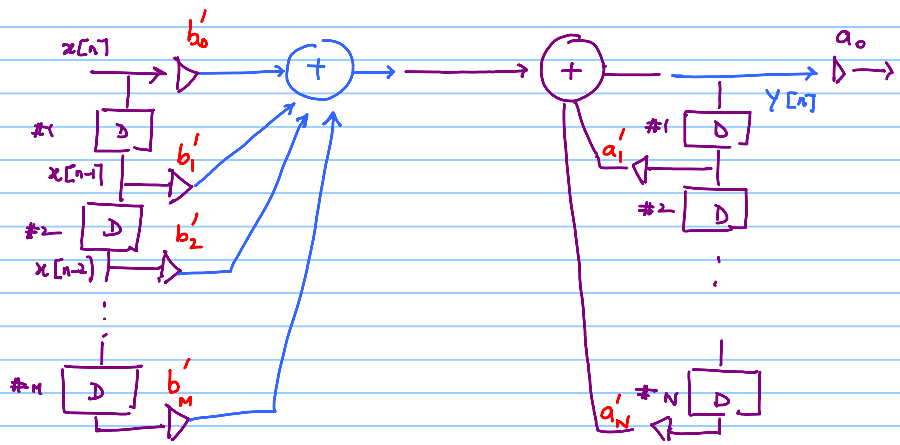
$$\leq B_x \sum_{k=0}^{\infty} |\rho|^k = \underbrace{\frac{B_x}{1-|\rho|}}$$ true only if $|\rho| < 1$

$$B_y = \frac{B_x}{1-|\rho|}$$

An important class of LTI are those for whom the input/out relationship is given

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{m=0}^{M} b_m\, x[n-m] \quad a_k\text{'s} \ \& \ b_k\text{'s constants}$$

$\vdots$



$$\boxed{\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{m=0}^{M} b_m\, x[n-m]}$$

$$a_0 y[n] + a_1 y[n-1] + \cdots a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

$$y[n] = \underbrace{-\left(\frac{a_1}{a_0}\right)}_{a_1'} y[n-1] - \cdots \underbrace{-\left(\frac{a_N}{a_0}\right)}_{a_N'} y[n-N]$$

$$+ \underbrace{\left(\frac{b_0}{a_0}\right)}_{b_0'} x[n] + \underbrace{\left(\frac{b_1}{a_0}\right)}_{b_1'} x[n-1] + \cdots + \underbrace{\left(\frac{b_M}{a_0}\right)}_{b_M'} x[n-M]$$

$$a_1' = -\left(\frac{a_1}{a_0}\right)$$

$\vdots$

Time Domain $\longrightarrow$ Freq Domain

Freq. Domain Representation of DT signals & System

$\boxed{x[n] = e^{j\omega_0 n}}$ $\quad -\infty < n < \infty$ eigenfunctions of all LTI systems

Discrete Time Fourier Transform

FS coefficients

$$x[n] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

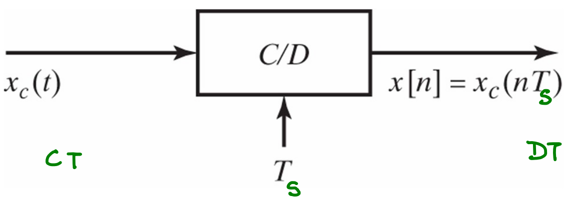
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$X(e^{j\omega})$ is a periodic signal with period $= 2\pi$

Oppenheim & Schafer Section 2.6 & 2.7

Will be covered in Week 5,6

Sampling O&S chapter 4



BL Signal

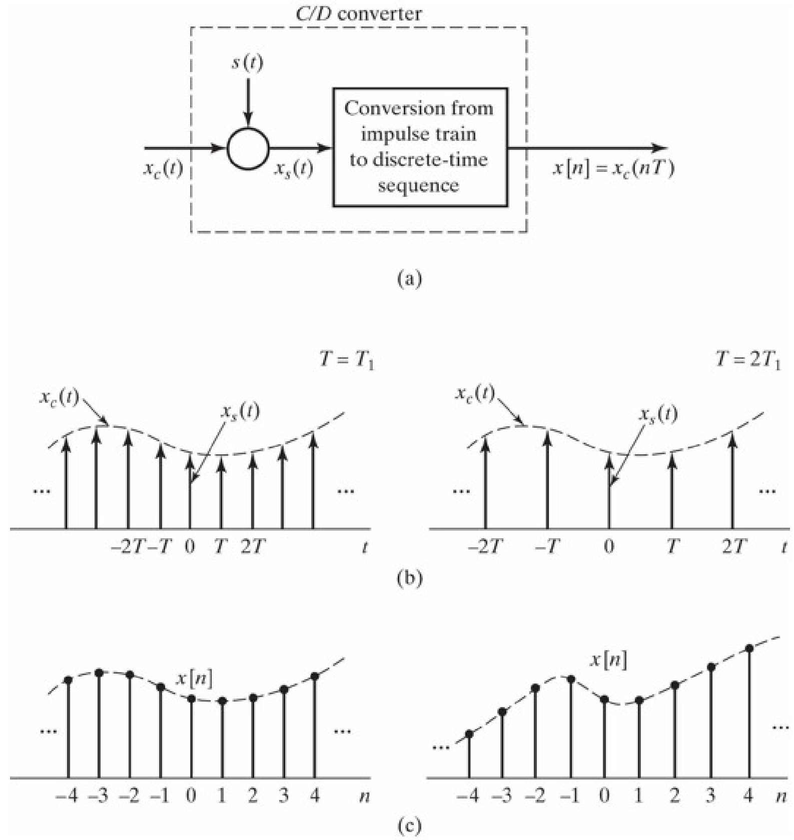
$BW = \Omega_N$

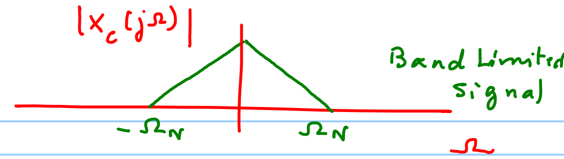
$$x[n] = x_c(t)\Big|_{t = nT_s}$$

Satisfies Nyquist Theorem

$$\Omega_s \geq 2\Omega_N$$

$$\frac{2\pi}{T_s} \geq 2\Omega_N$$





Nyquist Sampling Theorem

Let $x_c(t)$ be a band-limited signal with $X_c(j\Omega) = 0$ $|\Omega| \geq \Omega_N$ rads/sec

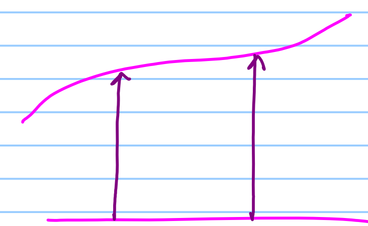
$x_c(t)$ is uniquely determined by its samples

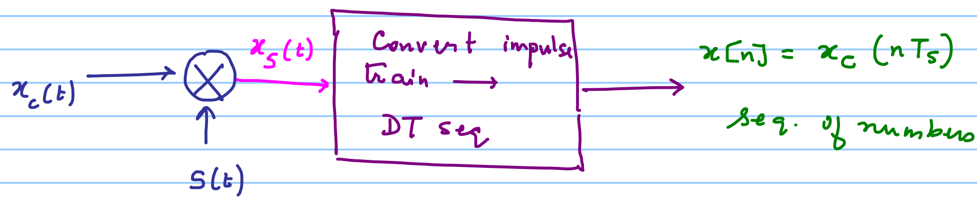
$$x[n] = x_c(nT_s) \qquad n = 0, \pm 1, \pm 2, \dots$$

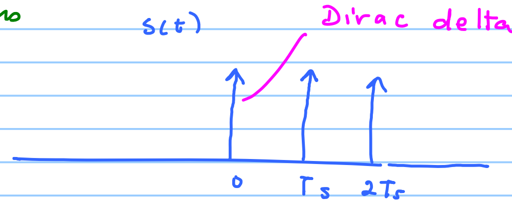
if $\Omega_s = \dfrac{2\pi}{T_s} \geq 2\Omega_N$

$$\boxed{\text{Nyquist rate} = 2\Omega_N}$$

Ω_s = rads/sec

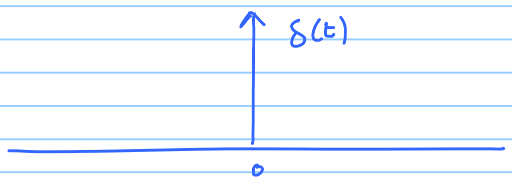




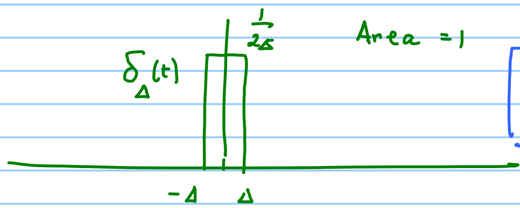


$$s(t) = \text{impulse train} = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Properties of Dirac Delta function



$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undef} & t = 0 \end{cases}$$

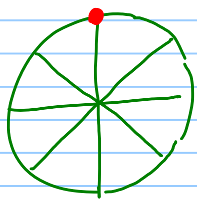


$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)}$$

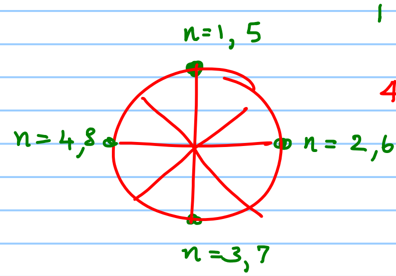
$$\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$$

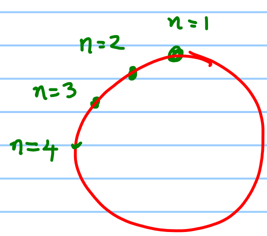
$$\int_{-\infty}^{\infty} x(\tau)\, \delta(\tau)\, d\tau = x(0)$$

Interpretation

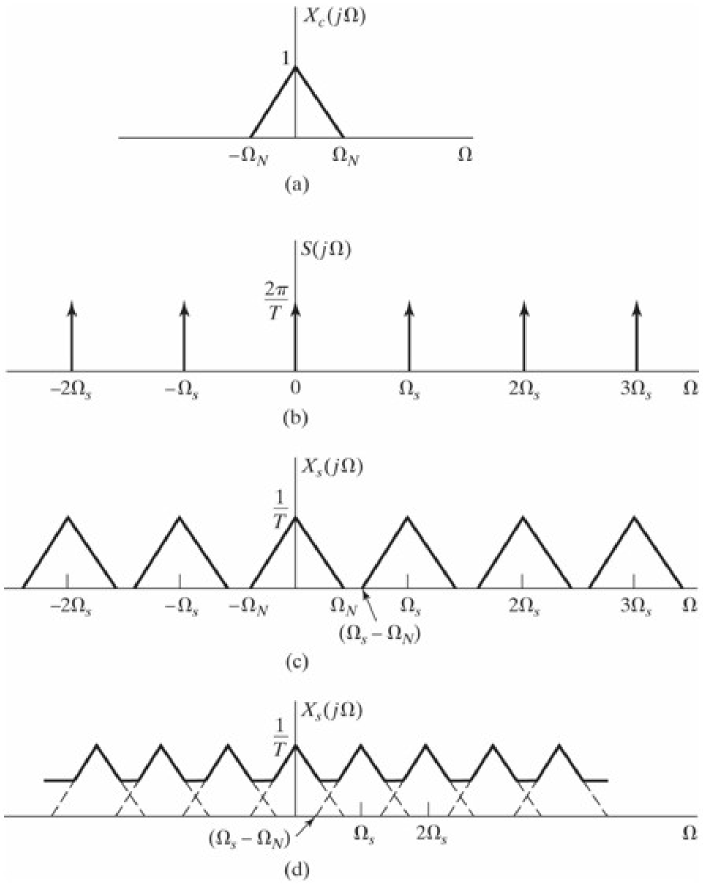


1 rev/sec	Camera Shutter speed	Observation
	1 snap/sec	Stationary
	4 snaps/sec	rotating clockwise ←
	1 snap/0.9 sec	rotating counter clockwise





If Nyquist Theorem not satisfied we cannot recover $x_c(t)$ from $x[n]$



$X_c(j\Omega)$
1
$-\Omega_N$
Ω_N
Ω
(a)
$S(j\Omega)$
$\frac{2\pi}{T}$
$-2\Omega_s$
$-\Omega_s$
0
Ω_s
$2\Omega_s$
$3\Omega_s$
Ω
(b)
$X_s(j\Omega)$
$\frac{1}{T}$
$-2\Omega_s$
$-\Omega_s$
$-\Omega_N$
Ω_N
Ω_s
$2\Omega_s$
$3\Omega_s$
Ω
$(\Omega_s - \Omega_N)$
(c)
$X_s(j\Omega)$
$\frac{1}{T}$
$(\Omega_s - \Omega_N)$
Ω_s
$2\Omega_s$
Ω
(d)