Note Title 02-09-2024

Electrical Engineering
IIT Madras

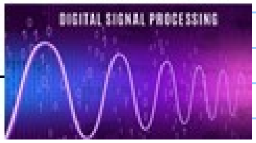
EE 3101
Digital Signal Processing

BS Electronic Systems programme

R. David Koilpillai

September – December 2024

September 10, 2024



EE 3101 Digital Signal Processing

Outline

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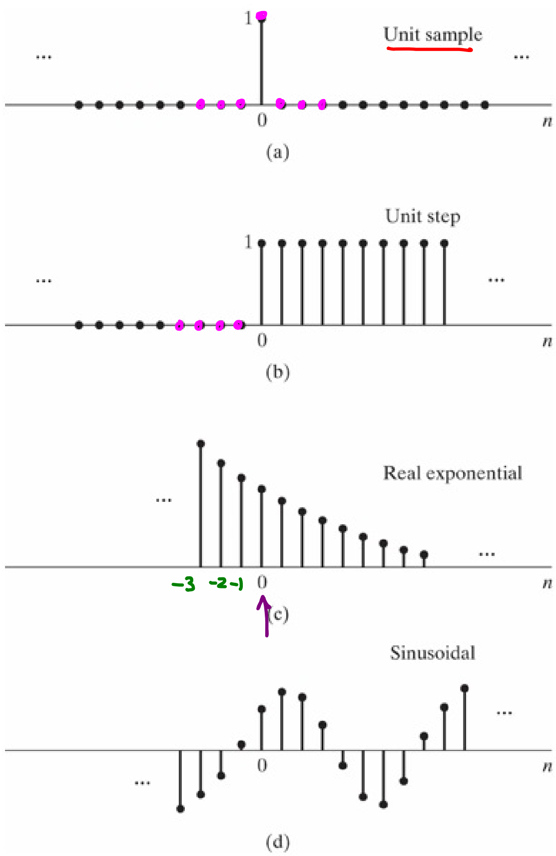
Last session
- Basic DT signals
- DT sinusoids
- Basic operations on DT signals

Week 1-2
✓ Introduction to sampling - Review of Signals and Systems: Basic operations on signals - Discrete time complex exponentials and other basic signals – system properties (linearity, time-invariance, memory, causality, BIBO stability) -- impulse response and convolution

Reading Assignment

Oppenheim & Schafer : Sec 2.0, 2.1
Rawat : Sec 1.1 – 1.3

Basic DT signals



unit sample

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

unit step

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Exponential

$x[n] = A\alpha^n$ A, α are real numbers

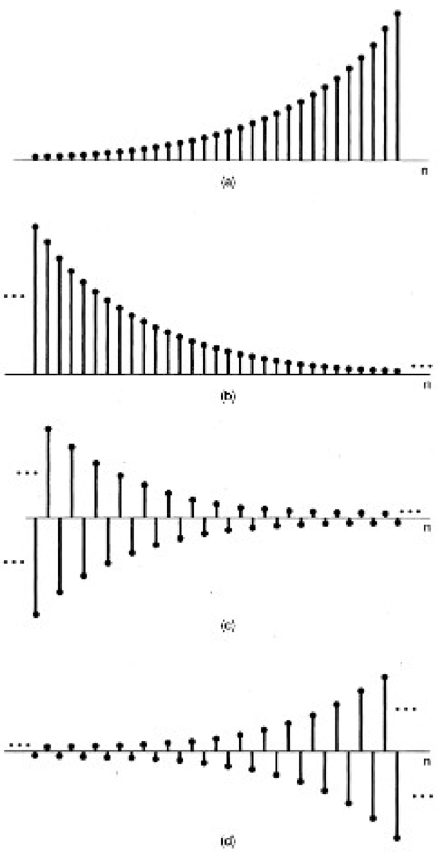
(Decaying) A positive

$0 < \alpha < 1$

Sinusoidal

$x[n] = A\cos(\omega_0 n + \phi)$ A, ω_0, ϕ are real numbers

Family of Exponential Signals



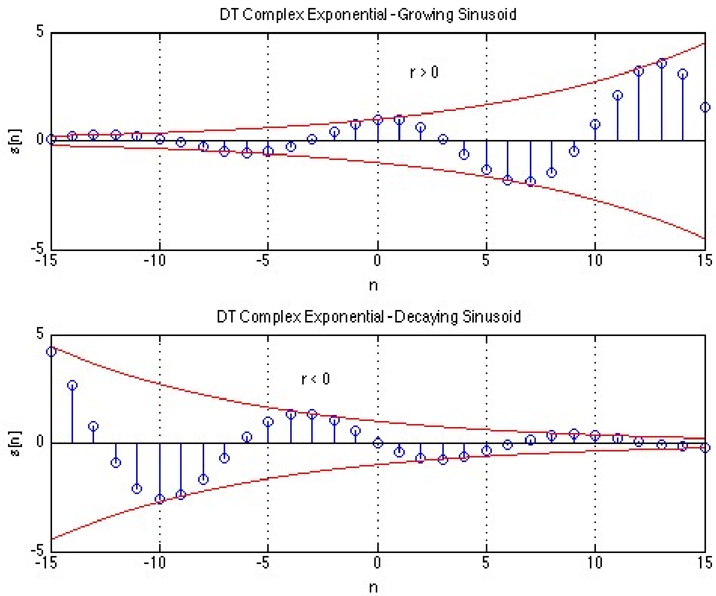
growing $\alpha > 1$

decaying $0 < \alpha < 1$

decaying & oscillating

growing & oscillating

Sinusoidal signals



exp. growing sinusoid

$A\alpha^n [\cos(\omega_0 n + \phi)]$

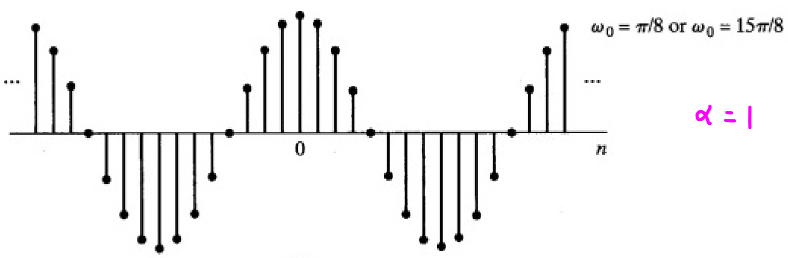
$\alpha > 1$

exp. decaying sinusoid

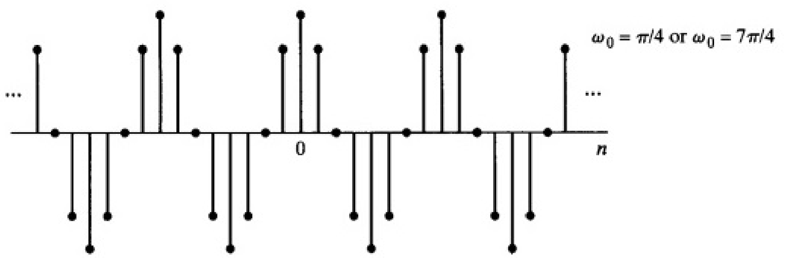
$A\alpha^n [\cos(\omega_0 n + \phi)]$

$0 < \alpha < 1$

DT Sinusoidal signals



(b)



$\alpha = 1$

Continuous time sinusoid $A\cos(\Omega_0 t + \theta)$

$x[n] = A\cos(\omega_0 n + \phi)$

ω_0 DT frequency

dimensionless

$\boxed{\omega_0 n = \text{rads}}$

$\boxed{\Omega_0 T_s = \omega_0}$

Ω_0 rads/sec

T_s sec

$\boxed{\omega_0 \text{ rads}}$

$$x[n] = A\alpha^n e^{j(\omega_0 n + \phi)}$$

$$= A\alpha^n \left[\underbrace{\cos(\omega_0 n + \phi)} + j \underbrace{\sin(\omega_0 n + \phi)} \right]$$

$j = \sqrt{-1}$

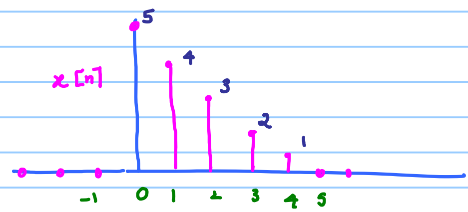
Basic Operations – Delay / Advance

$x[n]$

$y[n] = x[n-n_0]$ $\quad$ n_0 integer

$n_0 > 0$ delay $\quad$ $-\infty < n < \infty$

$n_0 < 0$ advance

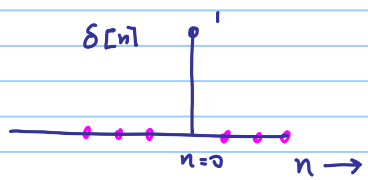


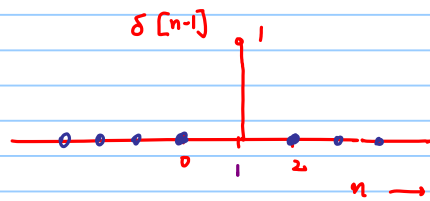
$x[n] = \{0,0,0,\underset{\uparrow}{5},4,3,2,1,0,0,0\}$

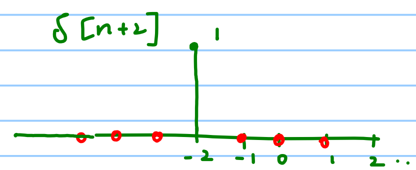
$y_1[n] = x[n-1] = \{0,0,\underset{\uparrow}{0},5,4,3,2,1,0,0,0\}$

$y_2[n] = x[n+2] = \{0,0,0,5,4,\underset{\uparrow}{3},2,1,0,0,0\}$

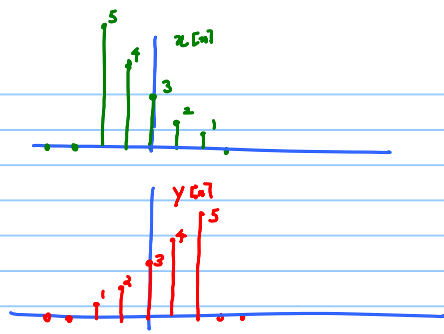
unit sample







Time Reversal



$y[n] = x[-n]$

$x[n] = \{0, 0, 0, 5, 4, 3, 2, 1, 0, 0, 0 \dots\}$

$y[n] = \{0, 0, 0 \; 0 \; 1 \; 2 \; 3, 4, 5, 0, 0 \dots\}$

Basic Operations on DT sequences

DT sequences $x_1[n]$, and $x_2[n]$

Sequence / signal addition $\quad y_1[n] = x_1[n] + x_2[n] \quad \forall n$

$y_1[0] = x_1[0] + x_2[0]$

$y_1[1] = x_1[1] + x_2[1]$

$\vdots$

Scalar addition $\quad y_2[n] = \alpha + x_2[n] \quad \forall n$

$y_2[0] = \alpha + x_2[0]$

$y_2[1] = \alpha + x_2[1]$

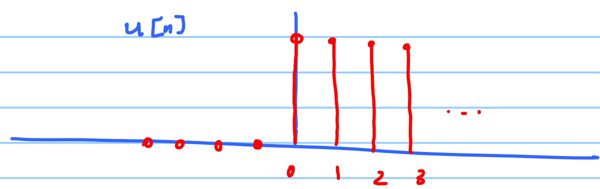
$\vdots$

Sequence multiplication $\quad y_3[n] = x_1[n] \cdot x_2[n] \quad \forall n$

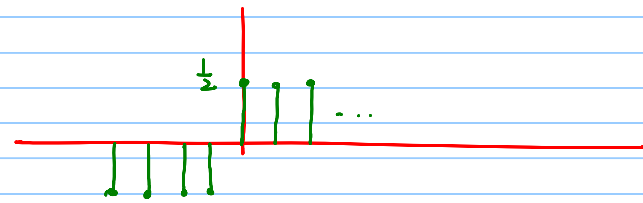
$y_3[0] = x_1[0] \cdot x_2[0]$

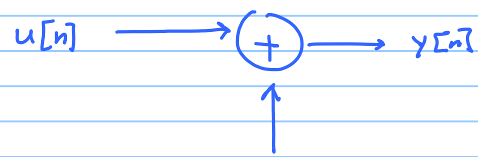
$\vdots$

Scalar multiplication $\quad y_4[n] = \alpha \, x_1[n] \quad \forall n.$



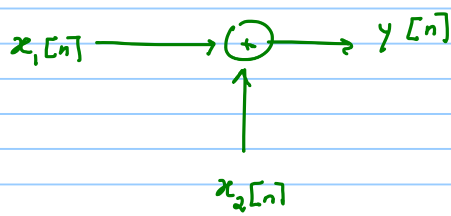
$y[n] = u[n] - \frac{1}{2}$





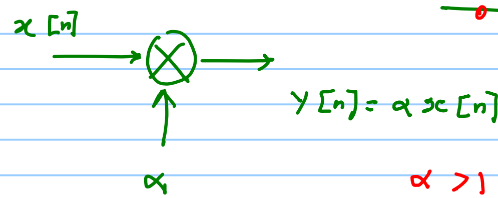
$\alpha = -\frac{1}{2}$

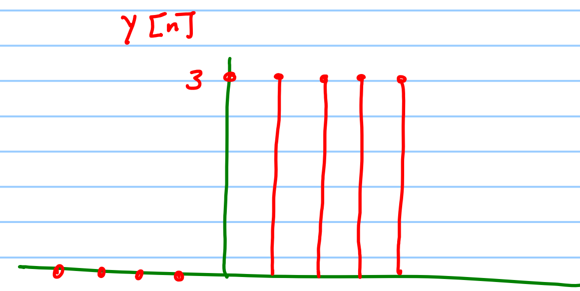
Addition



Example

$y[n] = \alpha\, u[n] \quad \alpha = 3$





$\alpha > 1$ amplification

$\alpha < 1$ attenuation

Example

$$x_1[n] = \{1.5,\ \underset{\uparrow}{2},\ 3.4\ \ -5\ \ 10\}$$

$$x_2[n] = \{2.2\ \ \underset{\uparrow}{3}\ \ 2\ \ 4.2\ \ 8\}$$

(1) $y_1[n] = x_1[n] + x_2[n] \quad = \{3.7,\ \underset{\uparrow}{5},\ 5.4,\ -0.8,\ 18\}$

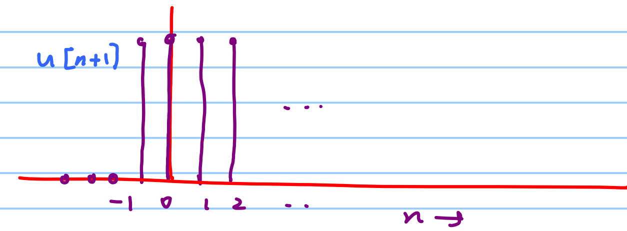
(2) $y_2[n] = x_1[n]\, x_2[n] \quad = \{3.3,\ \underset{\uparrow}{6},\ 6.8,\ -21,\ 80\}$

(3) $y_3[n] = \frac{3}{2} x_2[n] \quad = \{3.3,\ \underset{\uparrow}{4.5},\ 3,\ 6.3,\ 12\}$

(4) $\left.\begin{array}{l} y_4[n] = \alpha + x_1[n] \\ \alpha = 2 \end{array}\right\} = \{4.2,\ \underset{\uparrow}{5},\ 4,\ 6.2,\ 10\}$

Sequences

* Finite length, Infinite Length
* causal, right-sided
* anticausal, Left-sided
* Non-causal
* Length of finite length sequence $N = N_2 - N_1 + 1$
  $x[n]$ non-zero in the range $[N_1, N_2]$



Examples

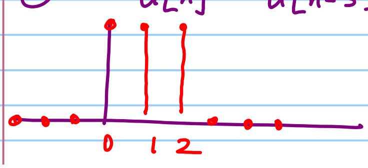
(1) $u[n]$ – infinite Length, causal, right-sided

$u[n+1]$ – infinite Length, non-causal, right-sided

$u[-n]$ – infinite Length, non-causal, left-sided

$u[-n-1]$ – infinite Length, anti-causal, left-sided

(2) $u[n] - u[n-3]$

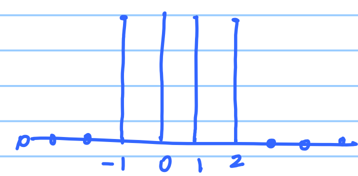


finite length, causal

Length $= N = 3$

(3) $u[n+1] - u[n-3]$ finite length, non-causal

Length $N = 4$



DT Sinusoids Properties

$x[n] = A\sin(\omega_0 n + \phi)$ or $A\cos(\omega_0 n + \phi)$

$x(t) = A\sin(\Omega_0 t + \theta)$ Ω_0 $0 \longrightarrow \infty$

low freq freq

Freq $\omega_1 = \omega_0 + 2\pi k$ shifted by a multiple of 2π

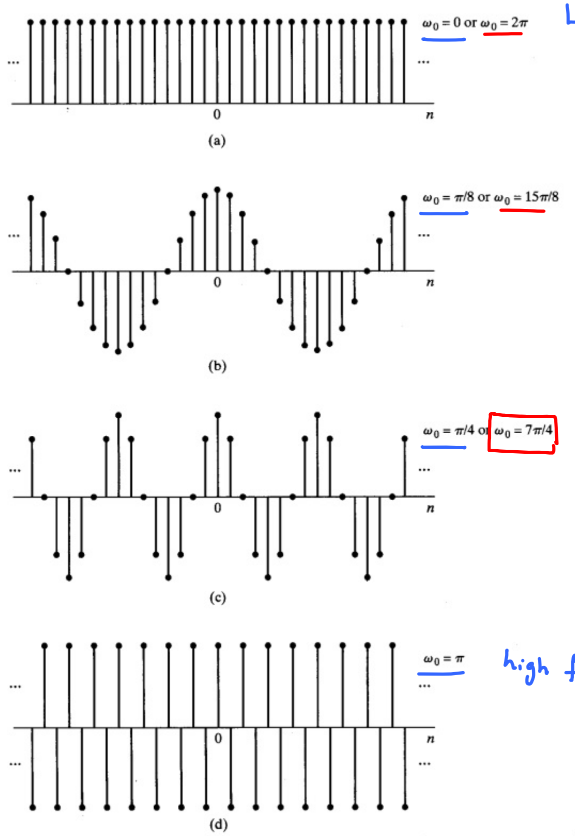
$A\cos(\omega_1 n + \phi) = A\cos((\omega_0 + 2\pi k)n + \phi)$

$= A\cos(\omega_0 n + 2\pi k n + \phi) = A\cos(\omega_0 n + \phi)$

$\boxed{\text{Periodicity in frequency for DT Sinusoids}}$

Illustration

DT sinusoids



Low freq

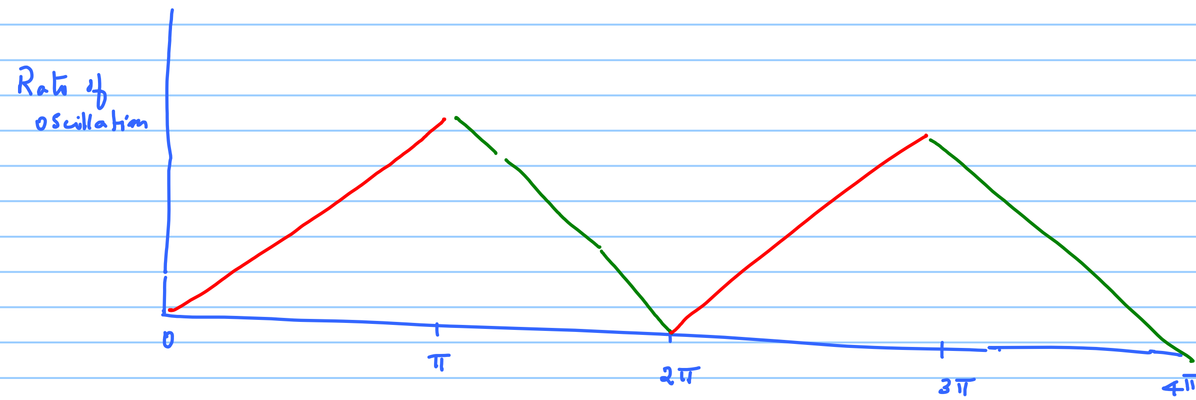
$A\cos(\omega_0 n + \phi)$

$> \omega = \pi$

high freq

$A\cos(\omega_0 n + \phi)$

$\omega_0 = \pi$



Observation #1 ***

DT sinusoids are periodic in frequency

Period = 2π or any multiple of 2π

Time behavior

CT → all sinusoids are period $T = \frac{2\pi}{\Omega_0}$

DT $A\cos(\omega_0 n + \phi)$

If Periodic with period $= N$ $\Rightarrow$ $\underbrace{A\cos(\omega_0(n + kN) + \phi)}_{x[n+kN]} = A\cos(\omega_0 n + \phi)$

$x[n+kN] = x[n]$

$\omega_0 kN$ = multiple of 2π

$k=1$ $\omega_0 N$ = multiple of 2π

$\omega_0 N = 2\pi m$ m is integer

A DT sinusoid is periodic iff $\boxed{\frac{2\pi}{\omega_0} = \frac{N}{m}}$ Rational function

N, m are integers

Fourier representation

Discrete Time signal with period $= N$

$\omega_0 N = 2\pi m$

$\boxed{\frac{2\pi}{\omega_0} = \frac{N}{m}} \Rightarrow \omega_0 = \frac{2\pi}{N} m$

m is integer

$m = 0, 1, \ldots N-1$

$\omega_0 = \frac{2\pi}{N} k$ $\quad \underbrace{k = 0, 1, 2, 3, \ldots N-1}$

# of periodic sequences with period $= N$
(sinusoidal)

$\Rightarrow N$ sequences

Observation $\delta[n]$ $u[n]$

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$ True

① $u[n] - u[n-1] = \delta[n]$ T/F True

② $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$ True

$= \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$

③ $$\boxed{u[n] = \sum_{k=-\infty}^{n} \delta[k]} = \underset{0}{\delta[-\infty]} + \cdots \underset{0}{\delta[-3]} + \underset{0}{\delta[-2]} + \underset{0}{\delta[-1]} + \underset{1}{\delta[0]} + \underset{0}{\delta[1]}$$

TRUE

$= u[n]$

④ $$\sum_{n=-\infty}^{\infty} \delta[n] = \underbrace{\sum_{n=-\infty}^{-1} \delta[n]}_{=0} + \underbrace{\delta[0]}_{=1} + \underbrace{\sum_{n=1}^{\infty} \delta[n]}_{=0} = 1$$

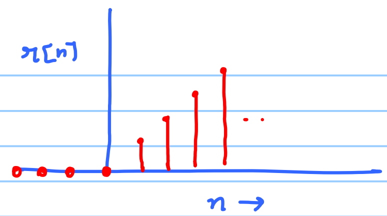
⑤ $$\boxed{\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-k] = x[k]}$$

$$\underbrace{x[-1]\,\delta[n+1]}_{=0} + \underbrace{x[0]\,\delta[n]}_{=0} + \cdots + \underbrace{x[k]\,\delta[0]}_{=x[k]} + \underbrace{x[k+1]\,\delta[1]}_{=0} + \cdots$$

Unit ramp

$r[n] = n \cdot u[n]$

$= \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$



<u>Show That</u>

① $r[n] = \sum_{k=0}^{\infty} k\,\delta[n-k]$

② $r[n] = \sum_{k=-\infty}^{n-1} u[k]$

③ $r[n] = \sum_{m=1}^{\infty} u[n-m]$

DT Sequence $x[n]$

Real valued

If $x[n] = x[-n]$ even sequence

$x[n] = -x[-n]$ odd sequence

$x[n] = x_e[n] + x_o[n]$

$x_e[n] = \frac{1}{2}\left[x[n] + x[-n]\right]$

$x_o[n] = \frac{1}{2}\left[x[n] - x[-n]\right]$

Complex valued sequences

$x[n] = x^*[-n]$ Conjugate symm

$x[n] = -x^*[-n]$ Conjugate antisymm

$x_{cs}[n] = \frac{1}{2}\left[x[n] + x^*[-n]\right]$

$x_{ca}[n] = \frac{1}{2}\left[x[n] - x^*[-n]\right]$

$x[n] = x_{cs}[n] + x_{ca}[n]$

Periodic signal

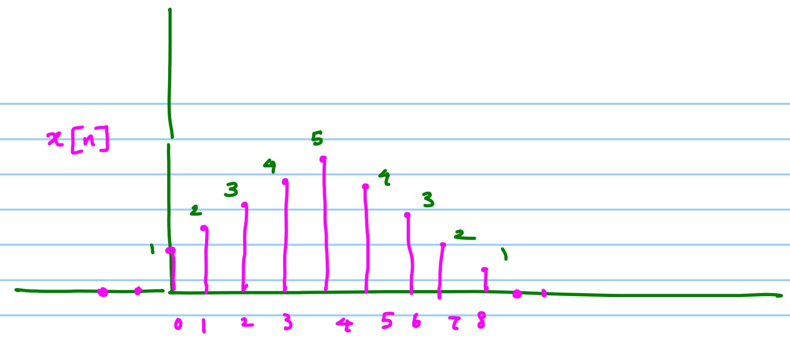
$x[n] = x[n+N] \quad \forall n$ (1)

* $x[n] = x[n+2N]$

* Period of a periodic seq is the smallest integer satisfying (1)

Time Scaling

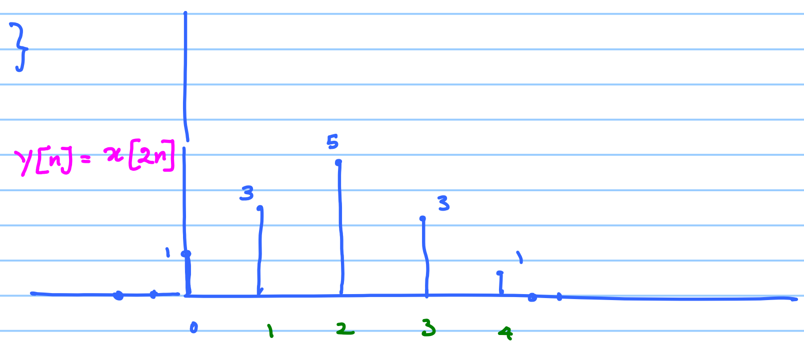
$x[n] = \{0, 0, \ldots 0, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0 \cdot 0 \cdots\}$



$y[n] = x[2n]$

$y[0] = x[0]; \quad y[1] = x[2]; \quad y[2] = x[4] \ldots$

$y[n] = \{0, 0, \ldots 0, 1, \quad 3, \quad 5, \quad 3, \quad 1, 0 \cdot 0 \cdots\}$



$y[n] = x[Mn]$ M integer

$x[n] \rightarrow \boxed{\downarrow M} \rightarrow y[n] = x[Mn]$

Compression

Down - sampling

Example

$y_1[n] = x[3n]$

$= \{0, 0 \ldots 0 \; 0 \; 1, 4, 3, 0, 0\}$

Expansion or Upsampling $\rightarrow \boxed{\uparrow 2} \rightarrow$

$$y[n] = \begin{cases} x\left[\frac{n}{2}\right] & \text{if } n = 0, \pm 2, \pm 4, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$y[0] = x[0]$

$y[1] = 0$

$y[2] = x[1]$

$y[3] = 0$

$y[4] = x[2]$

$\vdots$

$$x[n] = \{0, 0, 0, \underset{\uparrow}{1}, 2, 3, 4, 5, 4, 3, 2, 1, 0 \ldots\}$$

$$y[n] = \{\ldots 0, 0, 0, 0\ \underset{\uparrow}{1}, 0, 2, 0, 3, 0, 4, 0, 5, 0, 4, 0, 3, 0, 2, 0, 1, 0 \ldots\}$$

$$y_1[n] = \begin{cases} x\left[\frac{n}{3}\right] & \text{if } n = 0, \pm 3, \pm 6, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = \begin{cases} x\left[\frac{n}{N}\right] & \text{if } n = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$y_1[n]\ \{\ldots\ 0\ 0\ \underset{\uparrow}{1}\ 0\ 0\ 2\ 0\ 0\ 3 \ldots\}$$

insertion of $(N-1)$ zeros between every pair of samples

① Shift
② Time reversal

Example

$x[n] = \{1, 2, 3, 4, 5, 6\}$

① $y_1[n] = x[n-3]$ $\{0, 1, 2, 3, 4, 5, 6\}$

② $y_2[n] = x[-n]$ $\{6, 5, 4, 3, 2, 1\}$

③ $y_3[n] = x[-n+1]$ $y'[n] = x[n+1]$ $\{1, 2, 3, 4, 5, 6\}$

$y_3[n] = y'[-n]$

$y_3[n] = x[-n+1]$ $\{6, 5, 4, 3, 2, 1\}$ ✓

④ $y_4[n] = x[-n-2]$ $y'[n] = x[n-2]$ $\{1, 2, 3, 4, 5, 6\}$

$y_4[n] = y'[-n]$

$= x[-n-2]$ $\{6, 5, 4, 3, 2, 1\}$

shift
× time reversal
time scaling

shift
time scaling
time reversal

$y_5[n] = x[3n-1]$

$y'[n] = x[n-1]$

$y_5[n] = y'[3n]$

$= x[3n-1]$

$\{1, 2, 3, 4, 5, 6\}$

$\{0, 2, 5, 0\}$

$x[n] \rightarrow \boxed{\downarrow M} \rightarrow x[Mn]$

$x[n] \rightarrow \boxed{\uparrow N} \rightarrow \begin{cases} x\left[\frac{n}{N}\right] & \text{if } n = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$

$y[n] = x\left[\frac{M}{N} n\right]$

1. Upsampling
2. Downsampling

HW $y[n] = x\left[-\frac{M}{N} n - n_0\right]$

1. Shift
2. Upsampling
3. downsamp
4. Time reversal

$$x[n] \longrightarrow \boxed{\uparrow 2} \xrightarrow{x'[n]} \boxed{\downarrow 2} \longrightarrow y[n] = x[n] \ ?$$

$$y[n] = x[n]$$

$$x[n] \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{\uparrow 2} \longrightarrow y[n] \overset{?}{=} x[n]$$

$$x[n] = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$$

$$x'[n] = \{1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0 \ldots\}$$

$$y[n] = \{0\ 0\ 0, 1, 2, 3, 4, 5, 6 \ldots\}$$

DT System

$$x[n] \longrightarrow \boxed{T\{\quad\}} \longrightarrow y[n]$$

Properties of DT Systems

1. Memoryless property

Output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n

$$x[n] \longrightarrow \bigotimes \longrightarrow y[n] = 2x[n]$$

$$\uparrow 2$$

Memoryless system

- Scaling
- Quantizer
- Rectifier

⋮

$\left.\begin{matrix}\text{Scaling}\\ \text{Quantizer}\\ \text{Rectifier}\\ \vdots\end{matrix}\right\}$ Memoryless

$$x[n] \longrightarrow \boxed{Q[x[n]]} \longrightarrow \hat{x}[n]$$

$$x[n] \longrightarrow \boxed{|\quad|} \longrightarrow |x[n]|$$

DTSP operation

Averaging $y[n] = \frac{1}{2}\left[x[n] + x[n-1]\right]$ — Causal, Memory

$y[n] = \frac{1}{3}\left[x[n+1] + x[n] + x[n-1]\right]$ — Non causal, Memory

Linear

Systems that satisfy principle of superposition — scaling, additivity

$T\{x_1[n]\} = y_1[n]$

$x_1[n] \xrightarrow{T} y_1[n]$

$T\{x_2[n]\} = y_2[n]$

Scaling $T\{a x_1[n]\} = a y_1[n]$ homogeneity

Additivity $T\{x_1[n] + x_2[n]\} = y_1[n] + y_2[n]$

DT System is $\boxed{\text{linear}}$ if

$$T\{a x_1[n] + b x_2[n]\} = \underbrace{\overset{\text{Scaling}}{a} y_1[n] + b y_2[n]}_{\text{additivity}}$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$T\{x[n]\} = y[n]$$

Time Invariance

$$\left.\begin{array}{l} T\{x_1[n]\} \longrightarrow y_1[n] \\ T\{x_1[n-n_0]\} \longrightarrow y[n-n_0] \end{array}\right\} \text{System is Time Invariant}$$

LTI

$\boxed{\text{Linearity \& Time Invariance}} \equiv$ Impulse Response

$$T\{a\, x_1[n] + b\, x_2[n]\} = a\, y_1[n] + b\, y_2[n]$$

$$T\{x_1[n-n_0]\} = y_1[n-n_0]$$

$$x[n] = \sum_{k=-\infty}^{\infty} \underset{\text{scale}}{x[k]}\, \underbrace{\delta[n-k]}_{\text{unit impulses shifted in time}}$$

$$y[n] = T\{x[n]\}$$

$$= T\left\{ \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k] \right\} \overset{\text{superposition}}{=} \sum_{k=-\infty}^{\infty} T\{ \underbrace{x[k]}_{\text{scale factor}}\, \delta[n-k] \}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{T\{\delta[n-k]\}}_{h[n-k]}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$\boxed{T[\underset{\text{unit impulse}}{\delta[n]}] = \underset{\text{impulse response}}{h[n]}}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{h[n-k]}_{\text{shift}}$$

Time reversal

Computing the Output

1. Obtain $h[n]$
2. Time reversal $h[n]$
3. Shift the time-reversed seq
   → at each shift, obtain one output point

DT Convolution

$$y[n] = x[n] * h[n]$$
$$= h[n] * x[n]$$

CT LTI characterized impulse response $h(t)$

$$y(t) = x(t) * h(t)$$

computed via integral

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

DT LTI system DT impulse resp $h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$