Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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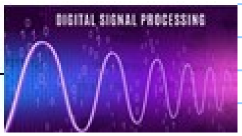
September – December 2024

Session # 17

November 12, 2024



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Note Title

EE3101

Session 17

03-01-2018

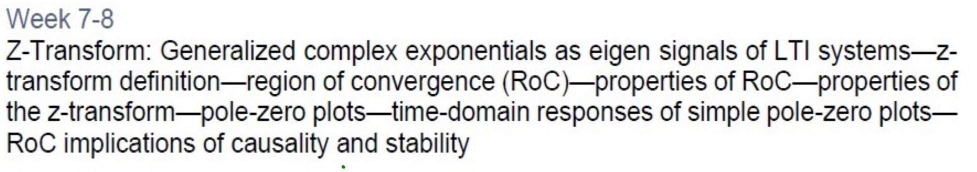
EE 3101 Digital Signal Processing

Session 17

Outline

Last session
- ZT pairs
- ZT properties
- Pole - Zero plots
- RoC, Causality, Stability

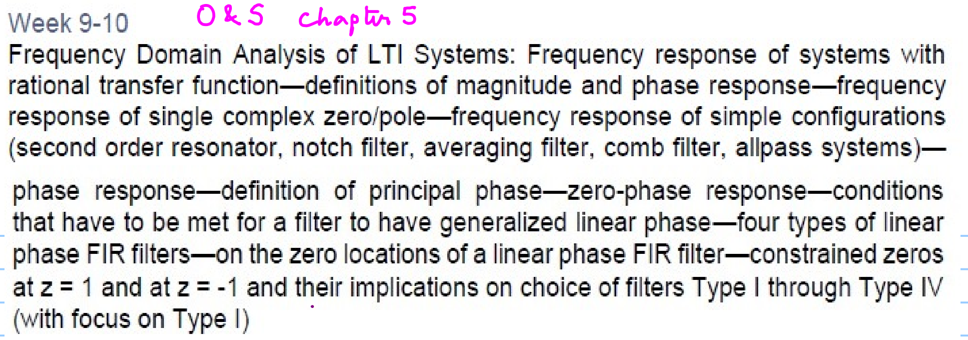
Today
- Inverse Z Transform
- Intro to Ch 5



Week 7-8

Z-Transform: Generalized complex exponentials as eigen signals of LTI systems—z-transform definition—region of convergence (RoC)—properties of RoC—properties of the z-transform—pole-zero plots—time-domain responses of simple pole-zero plots—RoC implications of causality and stability

O & S chapter 5



Week 9-10

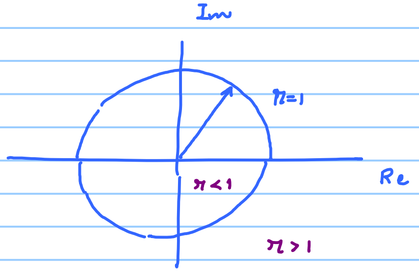
Frequency Domain Analysis of LTI Systems: Frequency response of systems with rational transfer function—definitions of magnitude and phase response—frequency response of single complex zero/pole—frequency response of simple configurations (second order resonator, notch filter, averaging filter, comb filter, allpass systems)—phase response—definition of principal phase—zero-phase response—conditions that have to be met for a filter to have generalized linear phase—four types of linear phase FIR filters—on the zero locations of a linear phase FIR filter—constrained zeros at z = 1 and at z = -1 and their implications on choice of filters Type I through Type IV (with focus on Type I)

Reading Assignment

O & S ch 5 Transform Analysis of LTI systems

Z Transform is a generalization of the DTFT

Defn $X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left[x[n] r^{-n}\right] e^{-j\omega n}$



$$x[n] \underset{\text{Inv. ZT}}{\overset{Z}{\rightleftarrows}} X(z) \quad \text{ROC}$$

ZT can exist even if DTFT does not exist

General form of ROC

$$0 \le \overbrace{\overbrace{r_R < |z|}^{\text{right-seq}} \underbrace{< r_L}_{\text{Left-sided}}}^{\text{Two-sided}} \le \infty$$

$H(z) = \frac{1}{1+3z^{-1}} = \frac{z}{z+3}$ pole @ $z=-3$

zero @ $z=0$

Pole-zero plots

poles = # zeros in finite Z plane

$$H(z) = \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{m=1}^{N}(1-d_m z^{-1})}$$

Let $N > M$

zeros @ $z = c_k$ $k=1,\cdots M$, $(N-M)$ zeros @ $z=0$

poles @ $z = d_m$ $m=1,\cdots N$

Table 3.1 SOME COMMON z-TRANSFORM PAIRS

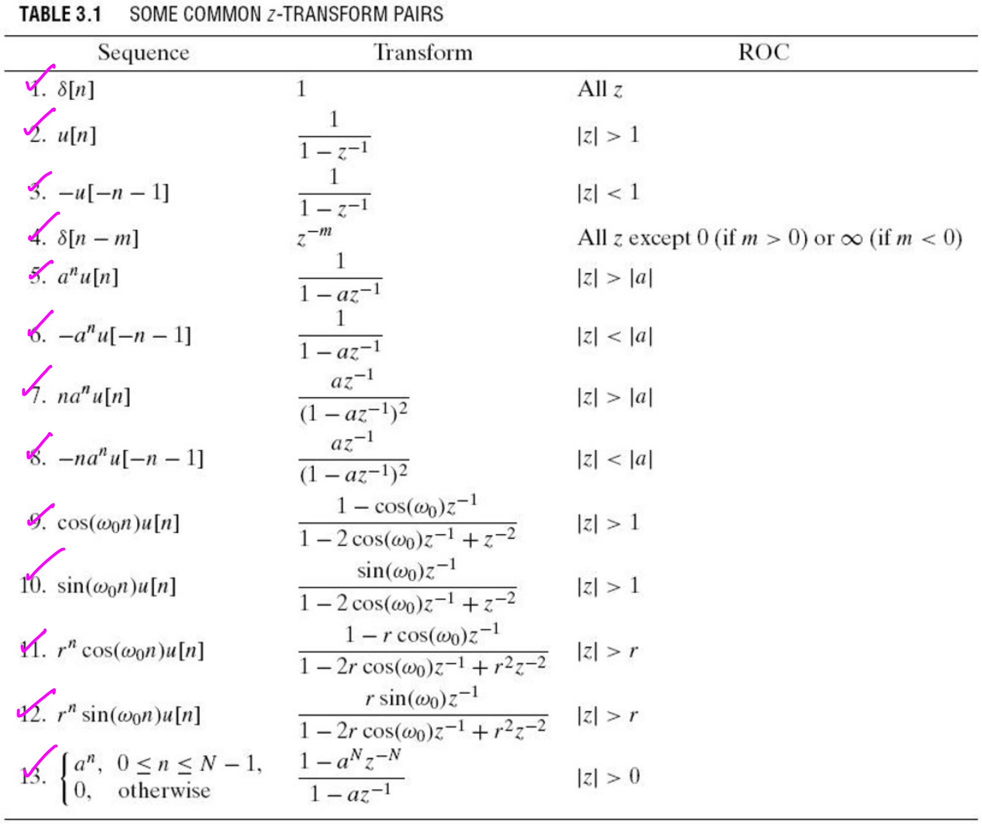


TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
✓ 1. $\delta[n]$	1	All z
✓ 2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
✓ 3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
✓ 4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
✓ 5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
✓ 6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
✓ 7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
✓ 8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
✓ 9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
✓ 10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
✓ 11. $r^n \cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
✓ 12. $r^n \sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
✓ 13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

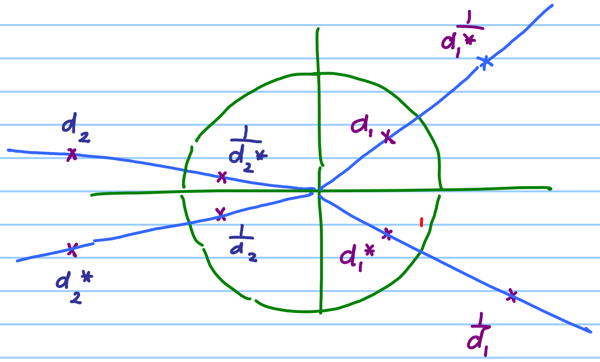
TABLE 3.2 SOME z-TRANSFORM PROPERTIES

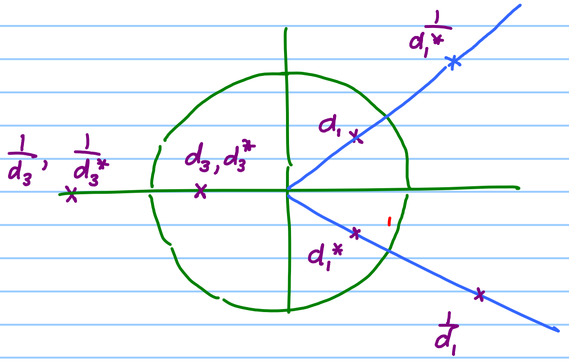
	Property Number	Section Reference	Sequence		Transform	ROC
			$x[n]$	$\longleftrightarrow$	$X(z)$	R_x
			$x_1[n]$	$\longleftrightarrow$	$X_1(z)$	R_{x_1}
			$x_2[n]$	$\longleftrightarrow$	$X_2(z)$	R_{x_2}
Linearity	1 ✓	3.4.1	$ax_1[n] + bx_2[n]$		$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
Shift	2 ✓	3.4.2	$x[n - n_0]$		$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
Multip. by expon. seq	3 ✓	3.4.3	$z_0^n x[n]$		$X(z/z_0)$	$\|z_0\|R_x$ $\|z_0\|r_R < \|z\| < \|z_0\|r_L$
Differentiation	4 ✓	3.4.4	$nx[n]$		$-z\dfrac{dX(z)}{dz}$	R_x
	5 ✓	3.4.5	$x^*[n]$		$X^*(z^*)$	R_x
	6 ✓		$\mathcal{R}e\{x[n]\}$		$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
	7 ✓		$\mathcal{I}m\{x[n]\}$		$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
	8 ✓	3.4.6	$x^*[-n]$		$X^*(1/z^*)$	$1/R_x$
	9 ✓	3.4.7	$x_1[n] * x_2[n]$		$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
			$R_{xx}[n]$		$X\left(\frac{1}{z}\right) X(z)$	$\max\left(r_R, \frac{1}{r_L}\right) < \|z\| < \min\left(r_L, \frac{1}{r_R}\right)$

Conjugation

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

			Pole/zero location
$x[n] \longleftrightarrow X(z)$	$r_R < \lvert z\rvert < r_L$		$z = d_k$
$x^*[n] \longleftrightarrow X^*(z^*)$	$r_R < \lvert z\rvert < r_L$		$z = d_k^*$
$x[-n] \longleftrightarrow X\left(\frac{1}{z}\right)$	$\frac{1}{r_L} < \lvert z\rvert < \frac{1}{r_R}$		$z = \frac{1}{d_k}$
$x^*[-n] \longleftrightarrow X^*\left(\frac{1}{z^*}\right)$	$\frac{1}{r_L} < \lvert z\rvert < \frac{1}{r_R}$		$z = \frac{1}{d_k^*}$





Exercises

① $x[n] = \left(\frac{1}{2}\right)^{n+1} u[n+1]$ ST ROC $\frac{1}{2} < |z| < \infty$

② Given $X(z) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{-n+1} z^{n}$ Obtain ROC $|z| < \frac{1}{2}$

③ $x[n] = -\left(\frac{1}{2}\right)^{n} u[-n-1] + 2^{n} u[-n-1]$ ST ROC $|z| < \frac{1}{2}$

#9 Convolution Property of ZT

$x_1[n] \longleftrightarrow X_1(z) \quad R_{x_1}$

$x_2[n] \longleftrightarrow X_2(z) \quad R_{x_2}$

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] \longleftrightarrow X_1(z)\, X_2(z) \quad \text{ROC} \quad R_{x_1} \cap R_{x_2}$$

(take into account pole-zero cancellation)

Autocorrelation $x[n]$

$x[n]$ is real-valued

$$r_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]\, x[k+n] \longleftrightarrow X\left(\frac{1}{z}\right) X(z) \qquad \max\left(r_R, \frac{1}{r_L}\right) < |z| < \min\left(r_L, \frac{1}{r_R}\right)$$

Verify $r_{xx}[n] = r_{xx}[-n]$

$\frac{1}{r_L} < |z| < \frac{1}{r_R}$ $\qquad$ $r_R < |z| < r_L$

Initial Value Theorem

If $x[n]$ is causal $\left(\begin{array}{l} x[n] = 0 \text{ for } n < 0 \\ x[0] \neq 0 \end{array} \right)$

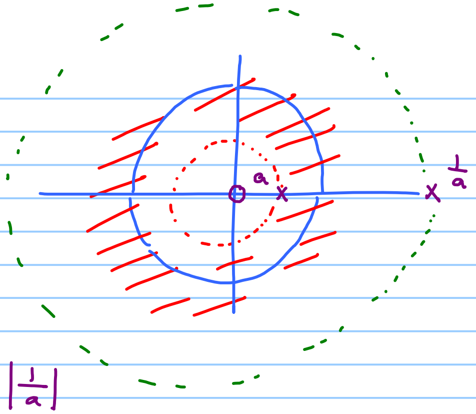
$$\lim_{z \to \infty} X(z) = x[0]$$

Example

$x[n] = a^{n} u[n] \longleftrightarrow X_1(z) = \frac{1}{1-az^{-1}}$ $|z| > |a|$ assume $|a| < 1$

pole @ $z = a$

zero @ $z = 0$



$r_{xx}[n] \longleftrightarrow R_{xx}(z) = X(z)\, X\left(\frac{1}{z}\right)$

$$= \underbrace{\left(\frac{1}{1-az^{-1}}\right)}_{|z| > |a|} \underbrace{\left(\frac{1}{1-az}\right)}_{|z| < \left|\frac{1}{a}\right|} \qquad |a| < |z| < \left|\frac{1}{a}\right|$$

ⓐ Another seq $x_1[n]$ that has the same autocorrelation seq

$r_{x_1x_1}[n] = r_{xx}[n] \longleftrightarrow R_{x_1x_1}(z) = R_{xx}(z)$

$x_1[n] = x[n-n_0]$ $\quad X_1(z) = z^{-n_0} X(z)$

$\quad X_1\left(\frac{1}{z}\right) = z^{n_0} X\left(\frac{1}{z}\right)$ $\Big\}$ $X_1(z) X_1\left(\frac{1}{z}\right) = X(z)\, X\left(\frac{1}{z}\right)$

ⓑ Another seq $x_2[n]$ that has the same autocorrelation as $x[n]$

$x_2[n] = x[-n]$ $\quad X_2(z) = X\left(\frac{1}{z}\right)$

$R_{x_2x_2}(z) = X_2(z) X_2\left(\frac{1}{z}\right) = X\left(\frac{1}{z}\right) X(z)$

Verify

$r_{xx}[-n] = r_{xx}[n]$

Results

c_k must real-valued, or c_k & c_k^* must be roots of $X(z)$

Let $x[n]$ is a real-valued, finite length seq, causal

$$x[n] \longleftrightarrow X(z) = \prod_{K=1}^{N} (1 - c_K z^{-1})$$

$$(1-c_k z^{-1})(1-c_k^* z^{-1}) = \underbrace{1}_{real} - \underbrace{2\,Re\{c_k\}}_{real} z^{-1} + \underbrace{|c_k|^2}_{real} z^{-2}$$

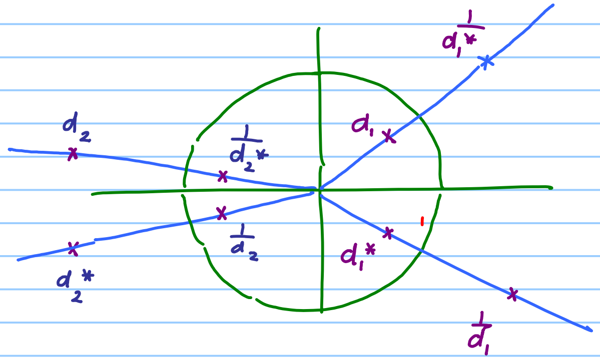
$$R_{xx}(z) = X(z)\, X\left(\frac{1}{z}\right)$$

c_k, c_k^*

$\frac{1}{c_k}, \frac{1}{c_k^*}$

$$\left\{ c_k, c_k^*, \frac{1}{c_k}, \frac{1}{c_k^*} \right\}$$

$$\left\{ c_j, \frac{1}{c_j} \right\}$$



Inverse Z Transform

1. Inspection Method
2. Partial Fraction Expansion (PFE)
3. Power Series method
4. Via Contour Integration

Inspection Method (ZT pairs, ZT properties)

$$\left.\begin{array}{l} H_1(z) = \dfrac{1}{1-az^{-1}} \quad |z| > a \\ \qquad\qquad\qquad 0 < a < 1 \end{array}\right\rangle \quad h_1[n] = a^n u[n]$$

$$H_2(z) = \frac{0.5z}{z^2 - z - 0.25} \quad |z| > 0.5$$

$$= \frac{0.5z^{-1}}{1 - z^{-1} - 0.25z^{-2}} = \frac{0.5z^{-1}}{(1 - 0.5z^{-1})^2}$$

$$na^n u[n] \longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} \quad |z| > a$$

$$h_2[n] = n\left(\frac{1}{2}\right)^n u[n]$$

Ex $X(z) = \frac{1}{z - \frac{1}{4}}$ $|z| > \frac{1}{4}$ Obtain $x[n]$

$= \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}}$

Method 1

$\frac{1}{1-\frac{1}{4}z^{-1}} \longleftrightarrow \left(\frac{1}{4}\right)^n u[n]$

$|z| > \frac{1}{4}$

$\frac{z^{-1}}{1-\frac{1}{4}z^{-1}} \longleftrightarrow \left(\frac{1}{4}\right)^{n-1} u[n-1]$ Using Time-Shift property

Method 2

$H(z) = \frac{N(z)}{D(z)}$

$N(z)$ is Numerator polynomial order $= M$

$D(z)$ is Denominator polynomial order $= N$

$M < N$ proper rational form

$M \geqslant N$ improper rational form

$\frac{z^{-1}}{1-\frac{1}{4}z^{-1}}$ improper rational form

$$H(z) = \frac{1 + 3z^{-1} + \frac{11}{6}z^{-2} + \frac{1}{3}z^{-3}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

$$\frac{1}{6}z^{-2} + \frac{5}{6}z^{-1} + 1 \;\Big)\; \overset{2z^{-1} + 1}{\overline{\frac{1}{3}z^{-3} + \frac{11}{6}z^{-2} + 3z^{-1} + 1}}$$

Remainder $\frac{1}{6}z^{-1}$

$$= 2z^{-1} + 1 + \underbrace{\frac{\frac{1}{6}z^{-1}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}}_{\text{Proper rational form}}$$

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}} = \frac{-4\left(-\frac{1}{4}z^{-1}\right)}{1 - \frac{1}{4}z^{-1}} = \frac{-4\left(\overbrace{-\frac{1}{4}z^{-1} + 1}\, - 1\right)}{1 - \frac{1}{4}z^{-1}} = -4\frac{\cancel{\left(1 - \frac{1}{4}z^{-1}\right)}}{\cancel{\left(1 - \frac{1}{4}z^{-1}\right)}} + \frac{4}{1 - \frac{1}{4}z^{-1}}$$

$$X(z) = -4 + \frac{4}{1 - \frac{1}{4}z^{-1}} \quad \longleftrightarrow$$

$$\updownarrow$$

$$x[n] = -4\,\delta[n] + 4\left(\frac{1}{4}\right)^{n} u[n]$$

Verify $x[n]$ via Method 1 $\equiv$ $x[n]$ via Method 2

II Partial Fraction Expansion (PFE)

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 \sum_{k=0}^{M} \frac{b_k}{b_0} z^{-k}}{a_0 \sum_{k=0}^{N} \frac{a_k}{a_0} z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1-c_k z^{-1})}{\prod_{k=1}^{N} (1-d_k z^{-1})}$$

Proper Rational Form $M < N$

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{(1-d_k z^{-1})} \longleftrightarrow h[n] = \sum_{k=1}^{N} A_k (d_k)^n u[n]$$

Valid if all d_k's are distinct

ROC $|z| > \max_k \{d_k\}$

Compute A_k's.

Improper Rational Form $M \geq N$

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \text{Proper Rational Form}$$

$$\downarrow$$

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] +$$

PFE

If there are repeated roots

$$H(z) = \frac{C_0 + C_1 z^{-1} + C_2 z^{-2}}{(1-d_0 z^{-1})^3} = \frac{A_1}{1-d_0 z^{-1}} + \frac{A_2}{(1-d_0 z^{-1})^2} + \frac{A_3}{(1-d_0 z^{-1})^3}$$

Compute A_1, A_2, A_3 (Method in O&S Sec 3.3.2)

Ex

$$H(z) = \frac{z^3}{(z-\frac{1}{2})(z+\frac{1}{3})^2} \qquad |z| > \frac{1}{2}$$

$$= \frac{1}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})^2} = \frac{A_1}{(1-\frac{1}{2}z^{-1})} + \frac{C_1}{(1+\frac{1}{3}z^{-1})} + \frac{C_2}{(1+\frac{1}{3}z^{-1})^2}$$

$A_1 = 0.36$

$C_1 = 0.24$

$C_2 = 0.4$

$$= \frac{0.36}{1-\frac{1}{2}z^{-1}} + \frac{0.24}{1+\frac{1}{3}z^{-1}} + \frac{0.4}{(1+\frac{1}{3}z^{-1})^2}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$0.36\left(\frac{1}{2}\right)^n u[n] \qquad 0.24\left(-\frac{1}{3}\right)^n u[n] \qquad 0.4(n+1)\left(-\frac{1}{3}\right)^n u[n+1]$$

$$\frac{0\cdot4}{\left(1+\frac{1}{3}z^{-1}\right)^2}$$

$$= \frac{0\cdot4}{\left(-\frac{1}{3}z^{-1}\right)} \underbrace{\frac{\left(-\frac{1}{3}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)^2}}_{n\left(-\frac{1}{3}\right)^n u[n]}$$

Property

$$n a^n u[n] \longleftrightarrow \frac{a z^{-1}}{\left(1-a z^{-1}\right)^2}$$

$$= \frac{0\cdot4}{\left(-\frac{1}{3}\right)} z \cdot \underbrace{\frac{-\frac{1}{3}z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)^2}}_{\downarrow}$$

$$(n+1)\left(-\frac{1}{3}\right)^{n+1} u[n+1]$$

$$\frac{0.4}{\left(-\frac{1}{3}\right)} Z$$

$$\frac{0\cdot4}{\left(-\frac{1}{3}\right)} z \cdot \frac{-\frac{1}{3}z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)^2} \longleftrightarrow \frac{0\cdot4}{\cancel{\left(-\frac{1}{3}\right)}} (n+1)\left(-\frac{1}{3}\right)^{n+\cancel{1}} u[n+1]$$

$$\longleftrightarrow 0\cdot4\,(n+1)\left(-\frac{1}{3}\right)^n u[n+1]$$

IV Power Series Method

Z - transform → Laurent Series → polynomial in z

$$\frac{1}{(1-z^{-1})^2}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$$

$$\frac{1}{(1-z^{-1})^2} = 1 + 2z^{-1} + 3z^{-2} + \cdots \longleftrightarrow \{\underset{\uparrow}{1}, 2, 3, 4 \cdots\}$$

$$\frac{z^{-1}}{(1-z^{-1})^2} = z^{-1} + 2z^{-2} + 3z^{-3} \cdots$$

$$u[n] \longleftrightarrow \frac{1}{1-z^{-1}}$$

$$\underbrace{u[n] * u[n]}_{\text{ramp}} \longleftrightarrow \frac{1}{(1-z^{-1})^2}$$

$n\,u[n]$

Ex $X(z) = \frac{z^{-1}}{(1-z^{-1})^2} \longleftrightarrow \{\underset{\uparrow}{0}, 1, 2 \cdots\}$

$(n-1)\,u[n-1]$ ✓

Ex $H(z) = \ln(1+az^{-1})$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

$$\ln(1+az^{-1}) = az^{-1} - \frac{a^2}{2} z^{-2} + \frac{a^3}{3} z^{-3} \cdots$$

$$\left\{ a, -\frac{a^2}{2}, \frac{a^3}{3}, -\frac{a^4}{4} \cdots \right\}$$

$$x[n] = (-1)^{n-1} \frac{a^n}{n} u[n-1]$$

Power Series via Long Division

$X(z) = \frac{1}{1-az^{-1}}$ $|z| > a$ causal seq.

$$\begin{array}{rl} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \cline{2-2} 1-az^{-1} \big) & \cancel{1} \\ & \cancel{1} - az^{-1} \\ \cline{2-2} & +\cancel{az^{-1}} \\ & +\cancel{az^{-1}} - a^2z^{-2} \\ \cline{2-2} & +a^2z^{-2} \end{array}$$

$$1 + az^{-1} + a^2z^{-2} + \cdots$$

$$\{1, a, a^2, \cdots\}$$

$$a^n u[n]$$

$X(z) = \frac{1}{1-az^{-1}} \qquad |z| < a$

$$\begin{array}{r|l} & -\frac{1}{a}z \;-\; \frac{1}{a^2}z^2 \\ \hline -az^{-1}+1 & \cancel{1} \\ & \cancel{1} - \frac{1}{a}z \\ \hline & +\cancel{\frac{1}{a}z} \\ & +\cancel{\frac{1}{a}z} \;-\; \frac{1}{a^2}z^2 \\ \hline \end{array}$$

$\left\{ -\frac{1}{a},\ -\frac{1}{a^2},\ \ldots \right.$

$$\boxed{x[n] = -a^n u[-n-1]}$$

$$\left\{ \quad \ldots \quad -\frac{1}{a^3} \quad -\frac{1}{a^2}, \quad -\frac{1}{a} \quad \underset{\uparrow}{0} \quad 0 \quad 0 \right\}$$