Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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Session # 10

October 8, 2024

EE 3101 Digital Signal Processing

Session 10

EE3101
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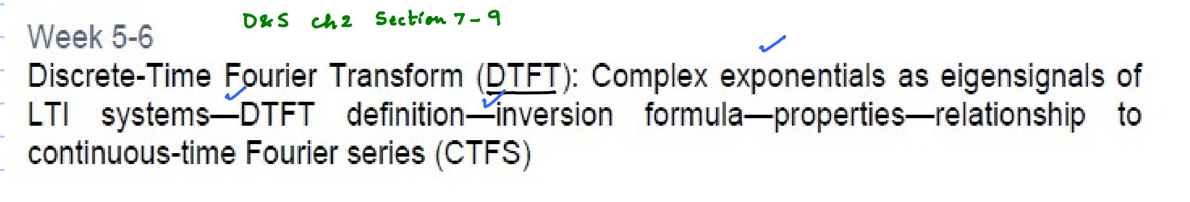
Outline

Last session
- Quantization

O&S ch2 Section 7-9



Week 5-6

Discrete-Time Fourier Transform (DTFT): Complex exponentials as eigensignals of LTI systems—DTFT definition—inversion formula—properties—relationship to continuous-time Fourier series (CTFS)

O&S ch 2 Sec 7 Representation of Sequences by Fourier Transforms
ch 2 Sec 8 Symmetry Properties of Fourier Transform
ch 2 Sec 9 Fourier Transform Theorems

Properties & Applications

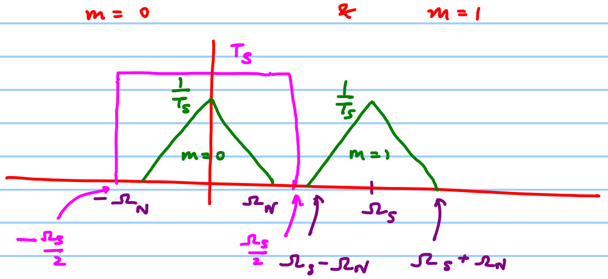
Reading Assignment

O&S ch 2 Sec 7 : Representation of Sequences by Fourier Transforms

$$X_s(j\Omega) = \underbrace{\frac{1}{T_s}}_{\text{scale factor}} \sum_{m=-\infty}^{\infty} X_c(j(\Omega - m\Omega_s))$$

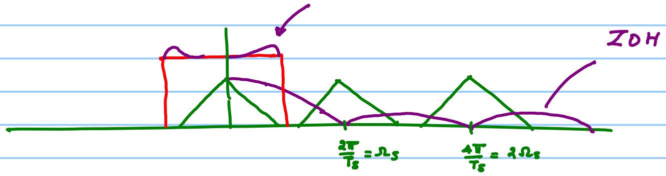
many copies

shifted by multiples of Ω_s



$$h_r(t) = \mathrm{Sinc}\left(\frac{t}{T_s}\right)$$

Compensation in Reconstruction filter



$$H_0(j\Omega) = e^{-j\Omega\frac{T_s}{2}} \frac{2\sin\frac{\Omega T_s}{2}}{\Omega}$$ Zero Order Hold

Modified Reconstruction Filter

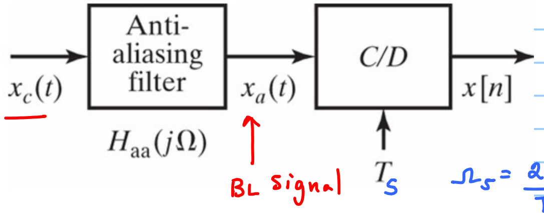
$$H_r'(j\Omega) = \frac{H_r(j\Omega) \rightarrow \text{original}}{H_0(j\Omega) \rightarrow \text{ZOH}}$$

Quantization

$x_c(t) \xrightarrow{\text{Samp.}} x[n] \xrightarrow{\text{Quant}} \hat{x}[n]$

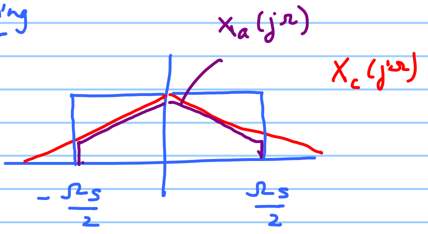
Disc. in time
Cont in amplitude
(∞ bits)

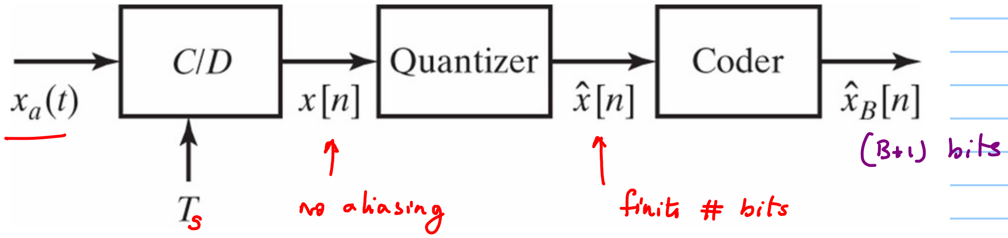
Disc. in time
Disc in amplitude (finite # bits)



$\Omega_S = \frac{2\pi}{T_S}$

No aliasing



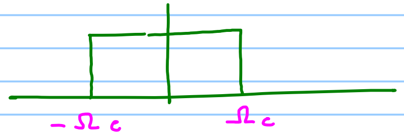


B+1 bit representation
fractional
sign

2's complement representation

$f_s = 10000\,Hz$ $\quad T_s = \frac{1}{10000}$

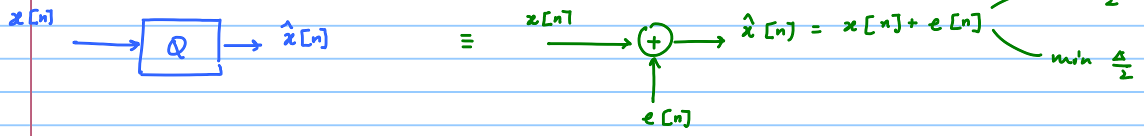
Antialiasing filter $\Omega_c = 2\pi 5000 = 10000\pi$



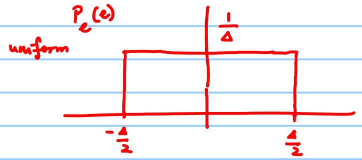
$\Omega_c = 10000\pi$

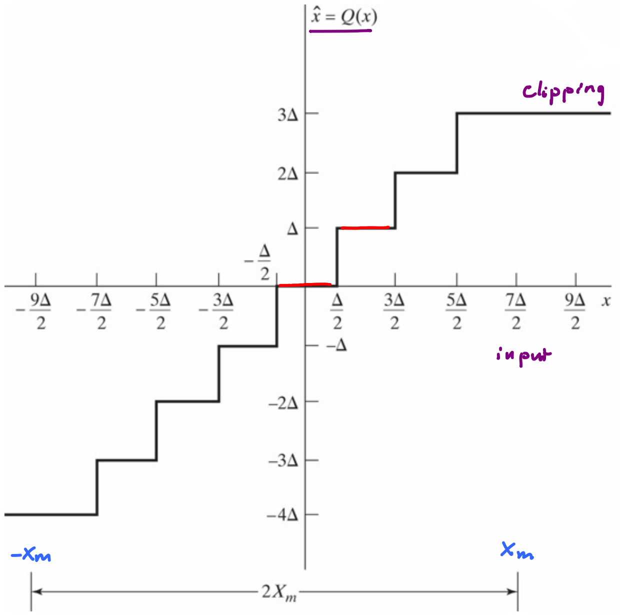
Uniform Quantizer via rounding

(B+1) quantizer



$\hat{x}[n] = x[n] + e[n]$ max $\frac{\Delta}{2}$, min $\frac{\Delta}{2}$





Characteristics of e[n]

① e[n] is Wide Sense Stationary (WSS) Random Process

② e[n] error seq is uncorrelated with x[n]

③ e[n] is a white noise random noise

④ PDF of e[n] $P_e(e)$ is uniform

$\Delta = \frac{X_m}{2^B} = 2^{-B} X_m \qquad \sigma_e^2 = 2^{-2B} \frac{X_m^2}{12}$

Signal to Quantiz. Noise Ratio

$SQNR = 10 \log_{10} \frac{\sigma_x^2}{\sigma_e^2}$

With $\sigma_x = \frac{X_m}{4}$ $SQNR = [6.02B - 1.25] dB$

Condition $\sigma_x = \frac{X_m}{4}$

Case 1 $\sigma_x = \frac{X_m}{3}$ $SQNR = (6.02B + 1.26) dB$

Discrete Time Fourier Transform

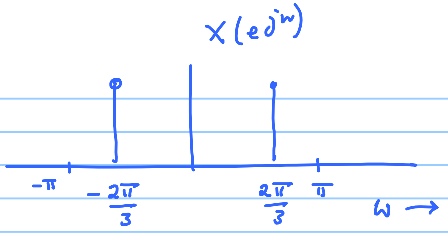
DT seq $x[n] \xrightarrow[\text{DTFT}]{\mathcal{F}} X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$

$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw \xleftarrow[\text{Inv DTFT}]{\mathcal{F}^{-1}} X(e^{jw})$ (PROVED)

$$X(e^{jw}) = X_R(e^{jw}) + j\, X_I(e^{jw})$$

$$= \underbrace{\left|X(e^{jw})\right|}_{\text{Magnitude Spectrum}} e^{j\, \underbrace{\arg\{X(e^{jw})\}}_{\text{phase spectrum}}}$$

Ex 1. $x_c(t) = \cos 4000\pi t$ $\quad f_0 = 2000$ Hz



Case 1 $\quad T_s = \frac{1}{6000}$ $\quad \Omega_s = 12000\pi$ ✓ satisfying Nyquist

$$x[n] = \cos 4000\pi n T_s = \cos \frac{4000\pi n}{6000} = \cos \frac{2\pi}{3} n$$

Case 2 $\quad T_s = \frac{1}{1500}$ X Nyquist not satisfied

$$x[n] = \cos \frac{4000\pi n}{1500} = \cos\left(\frac{8\pi n}{3}\right) = \cos\left(2\pi + \frac{2\pi}{3}\right) n = \cos \frac{2\pi}{3} n$$

Case 3 $\quad x_c(t) = \cos 16000\pi t$ $\quad f_0 = 8000$ Hz

$T_s = \frac{1}{6000}$ X Nyquist not satisfied

$$x[n] = \cos \frac{16000\pi n}{6000} = \cos\left(\frac{8\pi n}{3}\right) = \cos \frac{2\pi}{3} n$$

How many CT sinusoids sampled @ $T_s = \frac{1}{6000}$ will produce $x[n] = \cos \frac{2\pi}{3} n$

$$\cos 4000\pi t \Big|_{t = nT_s} = \cos \frac{2\pi}{3} n \quad \text{(Case 1)}$$

m is integer $\quad \cos\left((4000\pi + m\Omega_s) t\right)\Big|_{t = nT_s} = \cos\left(4000\pi + m \cdot 12000\pi\right)\frac{n}{6000} = \cos\left(\frac{2\pi}{3} + m \cdot 2\pi\right) n = \cos \frac{2\pi}{3} n$

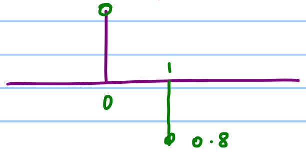
$m = 1, 2, \cdots$

$m=1 \quad 16000\pi \qquad m=2 \quad 28000\pi \qquad m=3 \quad 40000\pi \quad \ldots$ $\quad \infty$ numbers of CT sinusoids $\longrightarrow \cos \frac{2\pi}{3} n$

DTFT Example

① $h[n] = \delta[n] - \alpha\,\delta[n-1]$

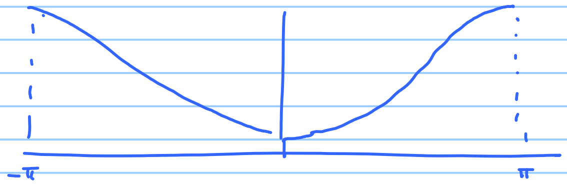
example $\alpha = 0.8$



$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-jwn}$$

$$H(e^{jw}) = 1 - \alpha e^{-jw} = \underbrace{1 - \alpha\cos w}_{H_R(e^{jw})} + j\,\underbrace{\alpha \sin w}_{H_I(e^{jw})}$$

$$|H(e^{jw})|^2 = (1 - \alpha\cos w)^2 + (\alpha\sin w)^2 = 1 + \alpha^2 - 2\alpha\cos w$$



@ $w = 0$ $(1-\alpha)^2 = 0.04$

@ $w = \pi$ $(1+\alpha)^2 = (1.8)^2$

High Pass Filter

Phase spectrum
" response

$$\phi_h(w) = \arg H(e^{jw}) = \tan^{-1}\left(\frac{H_I(e^{jw})}{H_R(e^{jw})}\right)$$

$$= \tan^{-1}\left(\frac{\alpha\sin w}{1-\alpha\cos w}\right)$$

Group delay of $H(e^{jw})$ $= \tau_h(w) = -\frac{d}{dw}\phi_h(w)$

$$= \frac{\alpha^2 - \alpha\cos w}{1+\alpha^2 - 2\alpha\cos w}$$

VERIFY

$$\boxed{\frac{d}{dx}\tan^{-1}(f(x)) = \frac{1}{1+f^2(x)}\frac{d}{dx}f(x)}$$

Ex 2

$x[n] = a^n u[n]$ $|a|<1$ (decaying exponential Causal signal)

$$X(e^{jw}) = \sum_{n=0}^{\infty} a^n e^{-jwn} = \sum_{n=0}^{\infty} \left(ae^{-jw}\right)^n = \frac{1}{1-ae^{-jw}} \checkmark$$

Convergence condition of Geom. Series

$|ae^{-jw}|<1 \Rightarrow |a|<1$ (given)

True

$$x[n] = a^n u[n] \longleftrightarrow X(e^{jw}) = \frac{1}{1-ae^{-jw}}$$

(Verify Lowpass filter)

@ $w=0$ $X(e^{jw}) = \frac{1}{1-a}$

@ $w=\pi$ $X(e^{jw}) = \frac{1}{1+a}$

Ex 2b $x[n] = a^n u[-n-1]$ $|a|>1$

$$x[n] = \{ \cdots \; a^2 \; a \; 1 \; \underset{\uparrow}{0} \; 0 \; 0 \}$$

$$X(e^{jw}) = \sum_{n=-1}^{-\infty} a^n e^{-jwn}$$

Substitution $\boxed{m=-n}$

$$= \sum_{n=-\infty}^{-1} a^n e^{-jwn}$$

$$= \sum_{m=1}^{\infty} a^{-m} e^{jwm} = \sum_{m=1}^{\infty} \left(\frac{1}{a}e^{jw}\right)^m = \frac{\frac{1}{a}e^{jw}}{1-\frac{1}{a}e^{jw}}$$

$$= \frac{-1}{1-ae^{-jw}}$$

$\left|\frac{1}{a}e^{jw}\right|<1$

$\Rightarrow \frac{1}{|a|}<1$

$\Rightarrow |a|>1$ True

Ex3

$x[n] = a^{-n} \; u[-n-1]$ $\qquad |a| < 1$

$$X(e^{j\omega}) = \sum_{n=-1}^{-\infty} a^{-n} e^{-j\omega n} = \sum_{m=1}^{\infty} (ae^{j\omega})^m = \frac{ae^{j\omega}}{1-ae^{j\omega}} \qquad |ae^{j\omega}| < 1$$

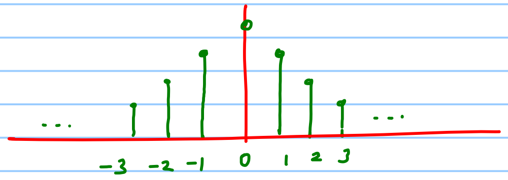
$m = -n$ $\qquad |a| < 1$

Ex 4

$x[n] = a^{|n|}$ $\qquad |a| < 1$

$$\boxed{\text{Low pass filter}}$$

$$= \underbrace{a^{n} u[n]}_{\text{causal}} + \underbrace{a^{-n} u[-n-1]}_{\text{anticausal}}$$



$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

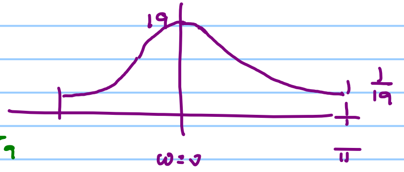
$|-1| = 1 = -(-1)$

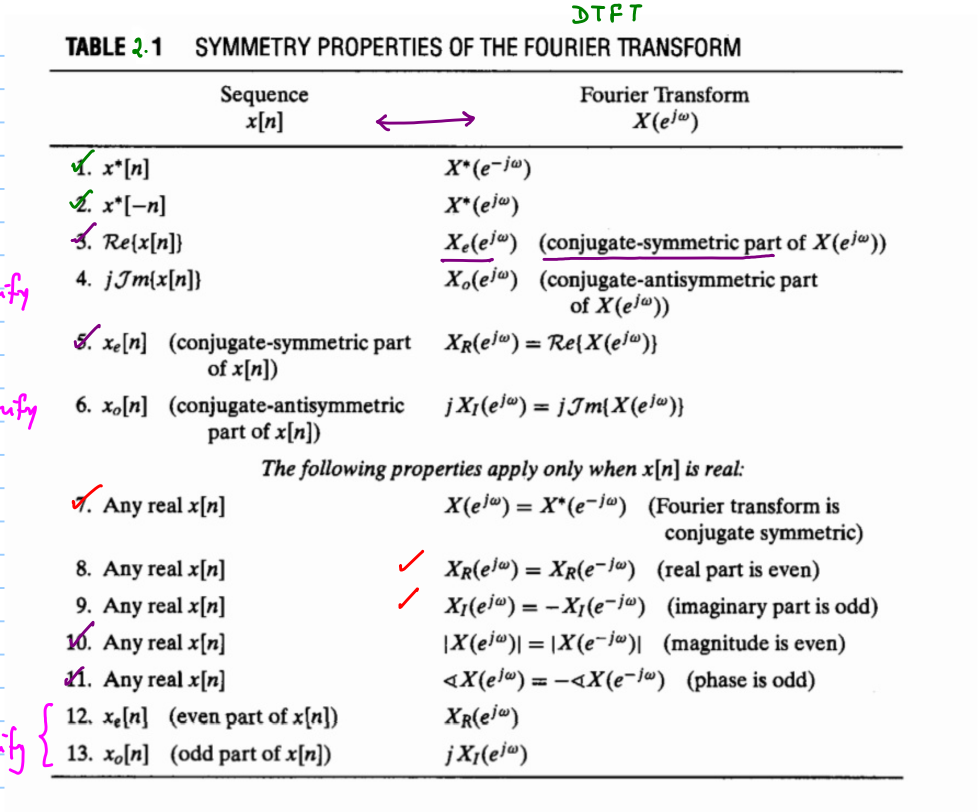
$$X(e^{j\omega}) = \frac{1-\cancel{ae^{j\omega}}+\cancel{ae^{j\omega}}-a^2}{1+a^2-2a\cos\omega} = \frac{1-a^2}{1+a^2-2a\cos\omega}$$

Ex $a : 0.9$

$$X(e^{j0}) = \frac{1+a}{1-a} = 19$$

$$X(e^{j\pi}) = \frac{1-a}{1+a} = \frac{0.1}{1.9} = \frac{1}{19}$$





DTFT

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

	Sequence $x[n]$	$\longleftrightarrow$	Fourier Transform $X(e^{j\omega})$
	✓ 1. $x^*[n]$		$X^*(e^{-j\omega})$
	✓ 2. $x^*[-n]$		$X^*(e^{j\omega})$
	✓ 3. $\mathcal{R}e\{x[n]\}$		$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
Verify	4. $j\mathcal{I}m\{x[n]\}$		$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
	✓ 5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)		$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
Verify	6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)		$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
	The following properties apply only when $x[n]$ is real:		
	✓ 7. Any real $x[n]$		$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
	8. Any real $x[n]$	✓	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
	9. Any real $x[n]$	✓	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
	✓ 10. Any real $x[n]$		$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
	✓ 11. Any real $x[n]$		$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
Verify	12. $x_e[n]$ (even part of $x[n]$)		$X_R(e^{j\omega})$
	13. $x_o[n]$ (odd part of $x[n]$)		$jX_I(e^{j\omega})$

$$x[n] \xleftrightarrow{DTFT} X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$$

$$X^*(e^{jw}) = \sum_{m=-\infty}^{\infty} x^*[n]\, e^{jwn}$$

Replace $w \leftarrow -w$

$$X^*(e^{-jw}) = \sum_{m=-\infty}^{\infty} x^*[n]\, e^{-jwn}$$

$$x^*[n] \xleftrightarrow{DTFT} X^*(e^{-jw})$$

$$X^*(e^{jw}) = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{jwn}$$

$m = -n$

$$X^*(e^{jw}) = \sum_{m=-\infty}^{\infty} x^*[-m]\, e^{-jwm}$$

$$x^*[-m] \xleftrightarrow{DTFT} X^*(e^{jw})$$

$x[n]$

$x_{cs}[n] = \frac{1}{2}\left[x[n] + x^*[-n]\right]$

$x_{ca}[n] = \frac{1}{2}\left[x[n] - x^*[-n]\right]$

#3 $\mathrm{Re}\{x[n]\} = \frac{1}{2}\left[x[n] + x^*[n]\right] \longleftrightarrow \frac{1}{2}\left[X(e^{jw}) + X^*(e^{-jw})\right]$

conjug. symm part $X_e(e^{jw}) = \frac{1}{2}\left[X(e^{jw}) + X^*(e^{-jw})\right]$

$\mathrm{Re}\{x[n]\} \longleftrightarrow X_e(e^{jw})$

#4 $j\,\mathrm{Im}\{x[n]\} \longleftrightarrow X_o(e^{jw}) = \frac{1}{2}\left[X(e^{jw}) - X^*(e^{-jw})\right]$ = Conjugate Antisymmetric part of $X(e^{jw})$

#5 $x_e[n] = \frac{1}{2}\left[x[n] + x^*[-n]\right] \longleftrightarrow \frac{1}{2}\left[X(e^{jw}) + X^*(e^{jw})\right] = X_R(e^{jw})$

Verify #6 $x_o[n] = \frac{1}{2}\left[x[n] - x^*[-n]\right] \longleftrightarrow jX_I(e^{jw})$

Consider $x[n]$ is real-valued

$$x[n] = x^*[n] \Longrightarrow X(e^{j\omega}) = X^*(e^{-j\omega}) \Rightarrow \text{Conjugate symmetric}$$

$$X_R(e^{j\omega}) + j X_I(e^{j\omega}) = X_R(e^{-j\omega}) - j X_I(e^{-j\omega})$$

Comparing LHS & RHS Real & imaginary parts

$$X_R(e^{j\omega}) = X_R(e^{-j\omega})$$

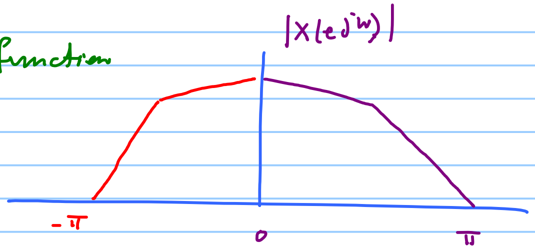
Real part of $X(e^{j\omega}) = X_R(e^{j\omega})$ is an even function

$$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$$

$X_I(e^{j\omega})$ is an odd function of ω

#10 $$\underset{\text{even}}{\left|X(e^{j\omega})\right|} = \sqrt{\underset{\underbrace{\text{even} \times \text{even}}_{\text{even}}}{X_R^2(e^{j\omega})} + \underset{\underbrace{\text{odd} \times \text{odd}}_{\text{even}}}{X_I^2(e^{j\omega})}}$$

$|X(e^{j\omega})|$ is an even function



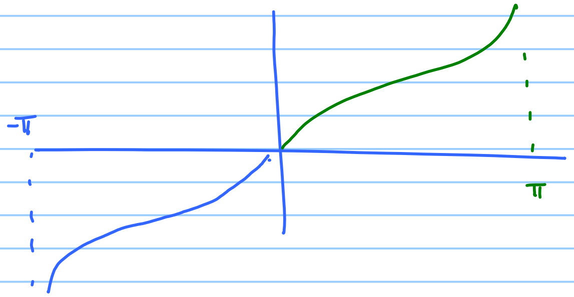
$$\operatorname{Arg} X(e^{j\omega}) = \tan^{-1} \frac{X_I(e^{j\omega})}{X_R(e^{j\omega})} = \phi_x(\omega)$$

$\rightarrow$ odd function of ω

$$\phi_x(-\omega) = \tan^{-1} \frac{X_I(e^{-j\omega})}{X_R(e^{-j\omega})} = -\tan^{-1} \frac{X_I(e^{j\omega})}{X_R(e^{-j\omega})} = -\phi_x(\omega)$$

Arg $X(e^{jw})$

odd funcn of ω



#12 $\quad x_e[n] = \dfrac{x[n] + x[-n]}{2} \longleftrightarrow \dfrac{1}{2}\left[X(e^{jw}) + X^*(e^{jw})\right] = X_R(e^{jw})$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

Verify $\left\{ \begin{array}{l} x_e[n] \xleftrightarrow{\text{DTFT}} X_R(e^{jw}) \\ x_o[n] \longleftrightarrow jX_I(e^{jw}) \end{array} \right.$

x[n] is real

$$X^*(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{jwn}$$

$m \leftarrow -n$

$$X^*(e^{jw}) = \sum_{m=-\infty}^{\infty} x[-m]\, e^{-jwm}$$

$x[-m] \longleftrightarrow X^*(e^{jw})$ ①

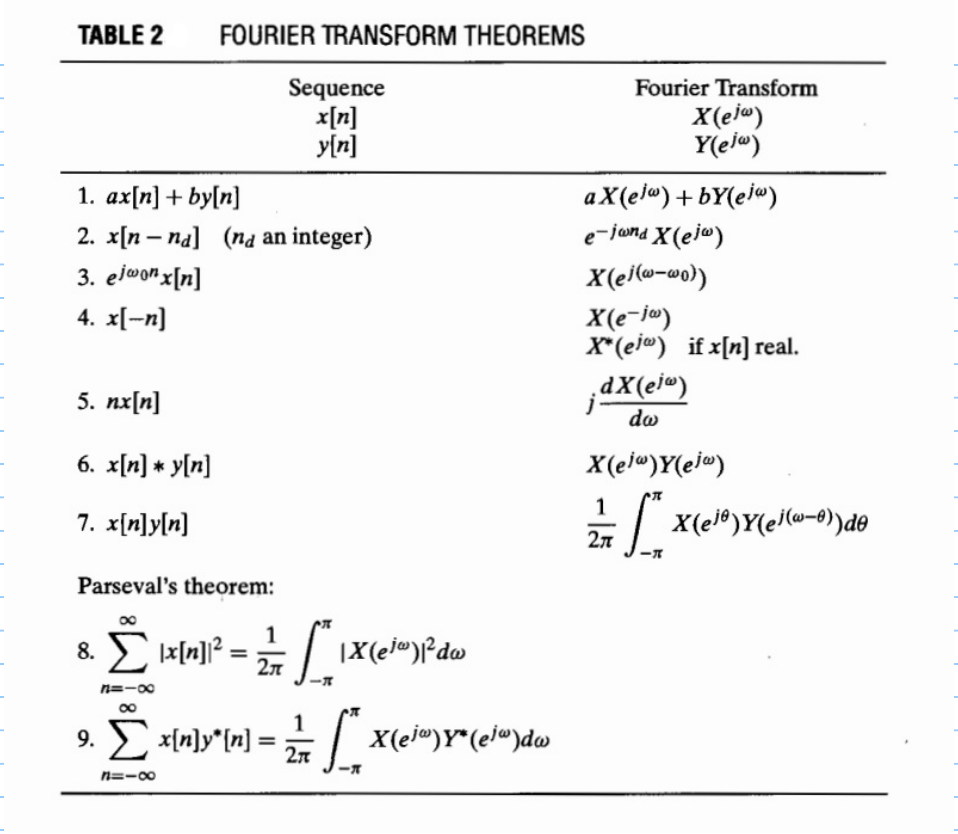


TABLE 2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\displaystyle\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\displaystyle\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	