Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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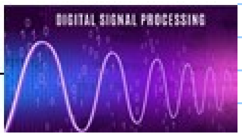
Session # 25

December 9, 2024



Session 24

Outline

Last session

- DFT
- Properties
- Examples

Today

- DFT properties
- Examples
- Linear convolution using DFT
- FFT - Introduction

Week 11-12 (DFT & FFT)

Discrete Fourier Transform (DFT): Definition of the DFT and inverse DFT—relationship to discrete-time Fourier series—matrix representation—DFT as the samples of the DTFT and the implied periodicity of the time-domain signal—circular convolution and its relationship with linear convolution—effect of zero padding—introduction to the Fast Fourier Transform (FFT) algorithm—decimation-in-time

Reading Assignment

O&S ch 8 Discrete Fourier Transform (DFT)

O&S ch 9 Fast Fourier Transform (FFT)

O&S ch 8

✓ 8.1 Discrete Fourier Series (DFS) or (DTFS)

✓ 8.2 Properties of DFS

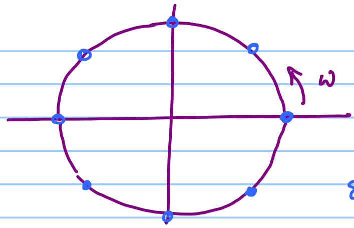
✓ 8.3 Fourier Transform of periodic signals

✓ 8.4 Sampling the FT

✓ 8.5 Discrete Fourier Transform (DFT) ⇒ Fourier Representation of Finite Duration Sequences

✓ 8.6 Properties of DFT

✓ 8.7 Computing Linear Convolution using the DFT



8 samples of the DTFT

DFT

Inverse Synthesis Equation $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$ (Inverse)

Forward Analysis Equation $X[\ell] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}\ell n}$ (Forward)

Viewing the DFS as an orthogonal transform

$$\underline{X} = \underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}}_{N\times 1 \text{ vector}} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 \cdots & W_N^{2(N-1)} \\ & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{\substack{N\times N \\ \underline{D_N}}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{N\times 1 \text{ vector } \underline{x}}$$

$W_N = e^{-j\frac{2\pi}{N}}$

$$\boxed{\begin{array}{ll} \text{Analysis eqn.} & \underline{X} = \underline{D_N} \cdot \underline{x} \\ \text{Synthesis eqn} & \underline{x} = \underline{D_N}^{-1} \underline{X} \end{array}}$$

$\underline{D_N}^{-1} = \frac{1}{N} \underline{D_N}^*$

Relationship between DTFT, DFS, DFT

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

$$\begin{tikzcd}
X(e^{j\omega}) \arrow[r] & \tilde{X}[k] \arrow[r, leftrightarrow, "DFS"] \arrow[d, leftrightarrow] & \tilde{x}[n] \arrow[r, leftrightarrow] & x[n] \arrow[dll, leftrightarrow, "DFT"] \\
 & X[k] & &
\end{tikzcd}$$

Properties of DFS

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$$

Shift in time $\quad \tilde{x}[n-m] \longleftrightarrow e^{-j\frac{2\pi}{N}km}\,\tilde{X}[k] = W_N^{mk}\,\tilde{X}[k]$

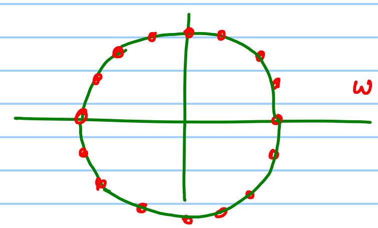
Duality $\quad \tilde{X}[n] \longleftrightarrow N\,\tilde{x}[-n]$

Shift in freq $\quad W_N^{-\ell n}\,\tilde{x}[n] \longleftrightarrow \tilde{X}[k-\ell]$

Periodic conv in time $\quad \tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\,\tilde{x}_2[n-m] \longleftrightarrow \tilde{X}_1[k]\,\tilde{X}_2[k]$

Periodic conv in freq $\quad \tilde{x}_3[n] = \tilde{x}_1[n]\,\tilde{x}_2[n] \longleftrightarrow \tilde{X}_3[k] = \frac{1}{N}\sum_{m=0}^{N-1} \tilde{X}_1[m]\,\tilde{X}_2[k-m]$

$$\boxed{\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}}$$



DFT

Linear Convolution

$x_1[n] = \{1, 2, 0, 1\}$

$x_2[n] = \{2, 2, 1, 1\}$

$$\begin{array}{c} 1\ 1\ 2 \end{array}\underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}\ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

Linear convolution

$$\underset{N=4}{\{1, 2, 0, 1\}} * \underset{N=4}{\{2, 2, 1, 1\}} \longrightarrow \underset{N=7}{y[n]} = \{2, 6, 5, 5, 4, 1, 1\}$$

$x_1[n] = \{1, 2, 0, 1, 0, 0, 0\}$ $\quad N = 7$

$x_2[n] = \{2, 2, 1, 1, 0, 0, 0\}$ $\quad N = 7$

$$\begin{array}{c} 0\,0\,0\ 1\,1\ 2 \end{array}\begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$x_1[n]$ (7) $x_2[n]$

Circular / Periodic convolution

$x_1[n]$ (4) $x_2[n] = x_3[n]$

$$\boxed{1\ 1\ 2}\ \underset{4\times4}{\begin{bmatrix} 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}}\ \underset{4\times1}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} = \underset{4\times1}{\begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}}$$

$y[n] = x[n] * h[n]$ Linear convolution

$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$

$\tilde{h}[n] \longleftrightarrow \tilde{H}[k]$

$\tilde{y}[n] \longleftrightarrow \underbrace{\tilde{X}[k]\,\tilde{H}[k]}_{\tilde{Y}[k]}$ With zero padding

Discrete Fourier Transform (DFT)

* Finite length seq. $x[n]$ $\qquad$ $x[n]$ is non-zero in the range $0 \le n \le N-1$

  length $= N$

* Can append zeros to increase the length to M

  Append $(M-N)$ zeros

* Associated periodic signal $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$

$$x[n] = \begin{cases} \tilde{x}[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{x}[n] = x[n \bmod N] = x[((n))_N]$$

$((n))_N \in [0, 1, \ldots N-1]$

modulo operation

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$$

* Choose the Fourier Coefficients from one period of $\tilde{X}[k]$

$$X[k] = \begin{cases} \tilde{X}[k] & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{X}[k] = X[k \bmod N] = X[((k))_N]$$

DFS

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, W_N^{kn}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, W_N^{-kn}$$

DFT

Analysis (Forward)

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n]\, W_N^{kn} & k = 0, \ldots N-1 \\ 0 & \text{otherwise} \end{cases}$$

Synthesis (Inverse)

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn} & n = 0, \ldots N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}} X[k]$$

* Focus on DFT $\quad x[n] \longleftrightarrow X[k]$

* Underlying periodicity

$$x[n] \longleftrightarrow \boxed{\tilde{x}[n] \longleftrightarrow \tilde{X}[k]} \longleftrightarrow X[k]$$

DFT

$x_1[n] \longleftrightarrow X_1[k]$ (N-point)

$x_2[n] \longleftrightarrow X_2[k]$

TABLE 8.2 SUMMARY OF PROPERTIES OF THE DFT

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km} X[k]$
6.	$W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7. Circular Convolution	$\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8. multiplic.	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1[\ell]X_2[((k-\ell))_N]$
9.	$x^*[n]$	$X^*[((-k))_N]$
10.	$x^*[((-n))_N]$	$X^*[k]$

5 circular shift in time

$x[n] \xleftrightarrow{DFT} X[k]$

$x[((n-m))_N] \longrightarrow e^{-j\frac{2\pi}{N}mk} X[k]$

$\tilde{x}[n-m] \xleftrightarrow{DFS} e^{-j\frac{2\pi}{N}km} \tilde{X}[k]$

$W_N = e^{-j\frac{2\pi}{N}}$

$x[((n-m))_N] \longleftrightarrow W_N^{km} X[k]$

4 Duality

$\tilde{X}[n] \longrightarrow N\tilde{x}[-k]$

$X[n] \longrightarrow Nx[((-k))_N]$

Verify

6 circular shift in freq.

8 Multiplication in time

#7

Periodic Convolution $\sum_{m=0}^{N-1} x_1[m]\, x_2[((n-m))_N] \longleftrightarrow X_1[k]\, X_2[k]$

$x_1[n] \circledN x_2[n]$

For $\tilde{x}_1[n] \longleftrightarrow \tilde{X}_1[k]$

$\tilde{x}_2[n] \longleftrightarrow \tilde{X}_2[k]$

$$\sum_{m=0}^{N-1} \tilde{x}_1[m]\, \tilde{x}_2[n-m] \longleftrightarrow \tilde{X}_1[k]\, \tilde{X}_2[k]$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\sum_{m=0}^{N-1} x_1[m]\, x_2[((n-m))_N] \longleftrightarrow X_1[k]\, X_2[k]$$

$x_1[n] \circledN x_2[n]$

periodic convolution in time $\longleftrightarrow$ Product of DFT coeffts

$$\underline{D_N} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 \cdots & W_N^{2(N-1)} \\ & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$W_N = e^{-j\frac{2\pi}{N}kn}$$

$$\underline{X} = \underline{D_N}\,\underline{x}$$

Properties of DFT

① $X[0] = \sum_{n=0}^{N-1} x[n]$

② $x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]$

③ $X\left[\frac{N}{2}\right] = \sum_{n=0}^{N-1} x[n]\, W_N^{\left(\frac{N}{2}\right)n}$

@ $k = \frac{N}{2}$ $\quad = \sum_{n=0}^{N-1} (-1)^n x[n] = X\left[\frac{N}{2}\right]$

$$W_N^{\frac{N}{2}n} = e^{-j\frac{2\pi}{\cancel{N}}\cdot\left(\frac{\cancel{N}}{\cancel{2}}\right)n}$$

$$= e^{-jn\pi} = (-1)^n$$

④ $x\left[\frac{N}{2}\right] = \frac{1}{N}\sum_{k=0}^{N-1}(-1)^k X[k]$

⑤ $W_N^{\frac{N}{2}} = -1$

⑥ $W_N^{\frac{N}{4}} = -j$

⑦ $W_N^{\frac{3N}{4}} = j$

⑧ $W_N^{k+N} = W_N^k$

⑨ $W_N^{k+\frac{N}{2}} = -W_N^k$

⑩ $W_N^{k2} = W_{\frac{N}{2}}^k$

⑪ $W_N^* = W_N^{N-1} = W_N^{-1}$

⑫ $\left(W_N^m\right)^* = W_N^{-m} = W_N^{N-m}$

⑬ $\left(W_N^{mk}\right)^* = W_N^{-mk} = W_N^{(N-m)k}$

⑭ $W_N^{\left(k+\frac{N}{2}\right)n} = (-1)^n W_N^{kn}$

Props # 9, 10

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

$$x[n] \longleftrightarrow X[k]$$

$$x^*[n] \longleftrightarrow X^*[((-k))_N]$$

$$x^*[((-n))_N] \longleftrightarrow X^*[k]$$

$$\tilde{X}^*[k] = \sum_{n=0}^{N-1} \tilde{x}^*[n]\, W_N^{-kn}$$

$$\tilde{X}^*[-k] = \sum_{n=0}^{N-1} \tilde{x}^*[n]\, W_N^{kn}$$

$$\tilde{x}^*[n] \longleftrightarrow \tilde{X}^*[-k]$$

$$\downarrow$$

$$x^*[n] \longleftrightarrow X^*[((-k))_N]$$

$$\tilde{x}^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}^*[k]\, W_N^{+kn}$$

$$\tilde{x}^*[-n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}^*[k]\, W_N^{-kn}$$

$$\tilde{x}^*[-n] \longleftrightarrow \tilde{X}^*[k]$$

$$\downarrow \qquad\qquad \downarrow$$

$$x^*[((-n))_N] \longleftrightarrow X^*[k]$$

11. $\mathcal{R}e\{x[n]\}$ $\quad X_{ep}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$

Verify 12. $j\mathcal{I}m\{x[n]\}$ $\quad X_{op}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$

13. $x_{ep}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$ $\quad \mathcal{R}e\{X[k]\}$

Verify 14. $x_{op}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$ $\quad j\mathcal{I}m\{X[k]\}$

Properties 15–17 apply only when $x[n]$ is real.

15. Symmetry properties

$$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{I}m\{X[k]\} = -\mathcal{I}m\{X[((-k))_N]\} \\ |X[k]| = |X[((-k))_N]| \\ \angle\{X[k]\} = -\angle\{X[((-k))_N]\} \end{cases}$$

16. $x_{ep}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$ $\quad \mathcal{R}e\{X[k]\}$

17. $x_{op}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$ $\quad j\mathcal{I}m\{X[k]\}$

#11

$$Re\{x[n]\} = \frac{1}{2}[x[n] + x^*[n]]$$

$$x[n] \updownarrow X[k] \qquad x^*[n] \updownarrow X^*[((-k))_N]$$

$$Re\{x[n]\} \longleftrightarrow \underbrace{\frac{1}{2}\left[X[k] + X^*[((-k))_N]\right]}_{X_{ep}[k]}$$

#13

$$x_{ep}[n] = \frac{1}{2}[x[n] + x^*[((-n))_N]]$$

$$x[n] \updownarrow X[k] \qquad x^*[((-n))_N] \updownarrow X^*[k]$$

$$x_{ep}[n] \longleftrightarrow \frac{1}{2}[X[k] + X^*[k]] = Re\{X[k]\}$$

$x[n]$ real valued $\Rightarrow$ $x[n] = x^*[n]$

$$X[k] = X^*[((-k))_N]$$

$Re\{X[k]\} = Re\{X^*[((-k))_N]\}$ even

$Im\{X[k]\} = -Im\{X^*[((-k))_N]\}$ odd

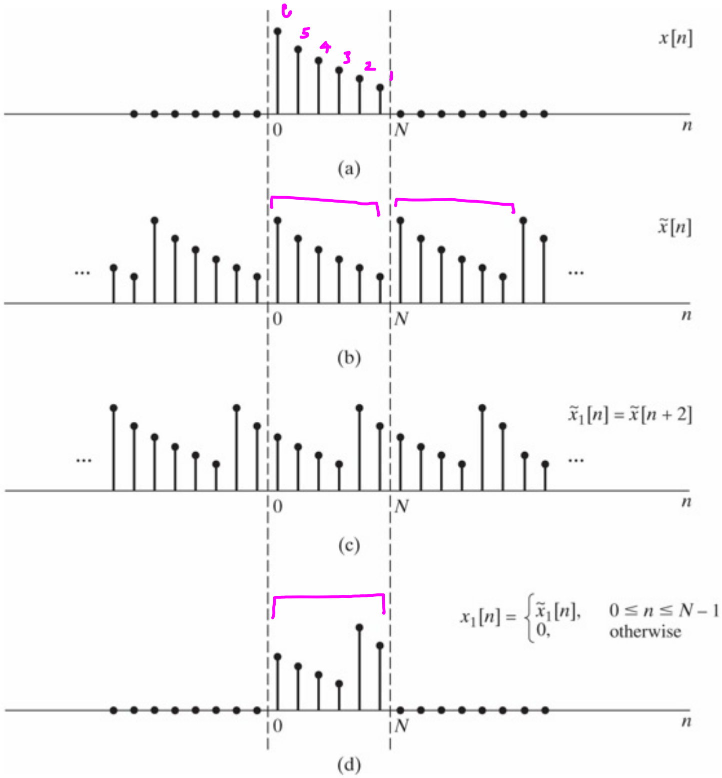
$$|X[k]| = \left[\underbrace{Re^2[X[k]]}_{\text{even}} + \underbrace{Im^2[X[k]]}_{\text{odd} \times \text{odd} = \text{even}} \right]^{\frac{1}{2}} = \left| X[((-k))_N] \right|$$

Magnitude of DFT coefficients $|X[k]|$ $k = 0, \dots N-1$ is even funcn of k

$$\underbrace{\arg X[k]}_{\text{odd function}} = \tan^{-1} \frac{Im\, X[k] \ (\text{odd})}{Re\, X[k] \ (\text{even})}$$

Arg of DFT coefficients is odd function of k

Ex 1



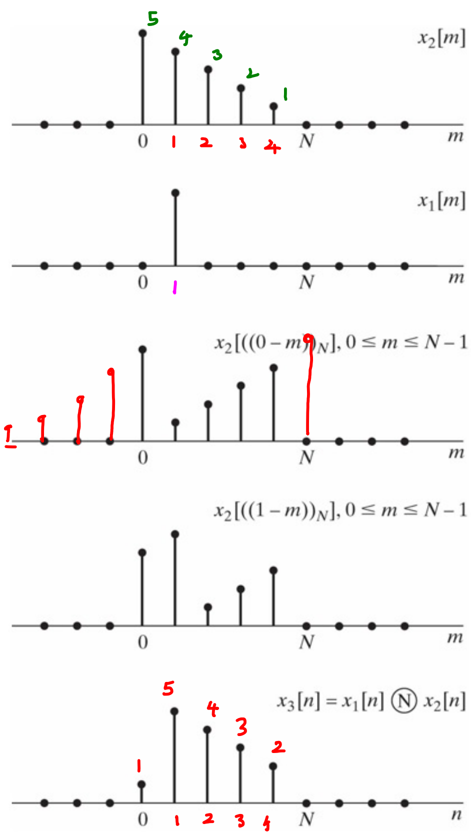
$x[n] = \{6, 5, 4, 3, 2, 1\}$

$\tilde{x}[n] = \{\cdots\ 6, 5, 4, 3, 2, 1,\ 6, 5, 4, 3, 2, 1,\ 6, 5, 4, 3, 2, 1,\ \cdots\}$

$\tilde{x}[n+2]$

$x[n+2] = \{4, 3, 2, 1, 6, 5\}$

Ex 2



$x_2[n] * x_1[n]$

$x_1[n] = \delta[n-1]$

$\tilde{x}_2[n] * \delta[n-1] = \tilde{x}_2[n-1]$

$$\sum_{m=0}^{N-1} x_1[m]\, x_2[((n-m))_N] = x_3[n]$$

$x_2[n] = \{5, 4, 3, 2, 1\}$

$x_2[((-n))_N] = \{5, 1, 2, 3, 4\}$

$N = 5$

1 2 3 4

$$\underbrace{\begin{bmatrix} 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}}_{x_2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{x_1[n]}$$

$x_3[0] = 1$

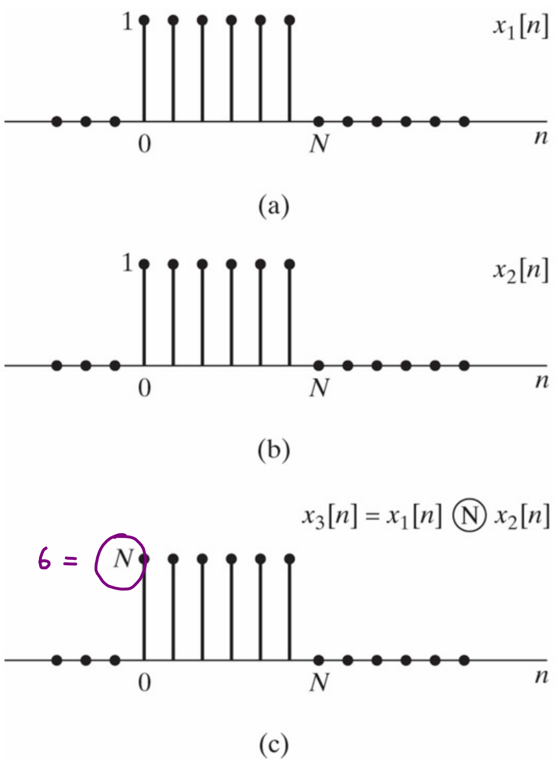
$x_3[1] = 5$

$x_3[2] = 4$

$x_3[3] = 3$

$x_3[4] = 2$

Ex 3



N = 6

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}$$

$$x_1[n] \textcircled{N} x_2[n] \longleftrightarrow X_1[k]\, X_2[k]$$

$$\underline{D_6} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{x}_1 = \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{X}_1 = \underline{X}_2 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{X}_3 = \underline{X}_1 \cdot \underline{X}_2 = \begin{bmatrix} 36 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3[n] = \frac{1}{6} \sum_{k=0}^{k-1} X_3[k]\, W_N^{-kn}$$

$$x_3[n] = 6 \quad \forall\, n$$

Multiplication in Time

$$x_1[n]\, x_2[n] \longleftrightarrow \frac{1}{N} X_1[k] \circledN X_2[k] = \frac{1}{N} \sum_{m=0}^{N-1} X_1[m]\, X_2[((k-m))_N]$$

Multiplication in Freq

$$x_1[n] \circledN x_2[n] \longleftrightarrow X_1[k]\, X_2[k]$$

Ex $\quad X[k] = \{1, 0, 1, 0\}$

$x[n] = ?$ $\qquad \underline{x} = \underline{D}_4^{-1} X = \frac{1}{N} \underline{D}_4^{*} X$

$$\underline{x} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ 1 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ 1 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$\left\{\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right\} \longleftrightarrow \{1, 0, 1, 0\}$$

Ex. Zero padding

$x[n]$ is an N-point seq

$$y[n] = \begin{cases} x[n] & 0 \le n \le N-1 \\ 0 & N \le n \le rN-1 \end{cases}$$

$x[n]$ N-point seq

$y[n] = \{ (\; x[n] \;) (\; 0 \;) (\; 0 \;) \; - - . \; \}$

rN point seq

$y[n]$ is obtained by $x[n]$ padded with $(r-1)N$ zeros

$$y[n] \longleftrightarrow Y[k] = \underbrace{\sum_{n=0}^{rN-1} y[n] \, W_{rN}^{kn}}_{rN\text{-point DFT}} = \sum_{n=0}^{N-1} x[n] \, W_{rN}^{kn} \qquad k = 0, 1, \ldots \; rN-1$$

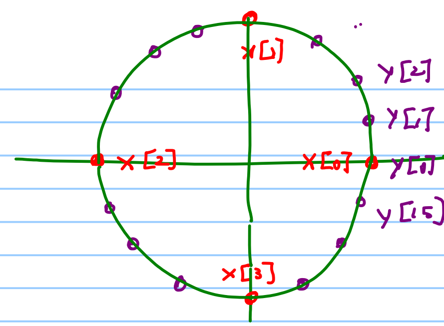
$$Y[rm] = \sum_{n=0}^{N-1} x[n] \, W_{rN}^{rmn} = \sum_{n=0}^{N-1} x[n] \, W_{N}^{mn} = X[k]$$

$$W_{rN} = e^{-j\frac{2\pi}{rN}}$$

$$W_{rN}^{rmn} = e^{-j\frac{2\pi}{\cancel{r}N}\cancel{r}mn} = e^{-j\frac{2\pi}{N}mn} = W_N^{mn}$$

$$\boxed{X[m] = \sum_{n=0}^{N-1} x[n] \, W_N^{mn}} \quad \text{N-point DFT}$$

$$Y[k] = \sum_{n=0}^{N-1} x[n] \cdot W_{rN}^{k}$$



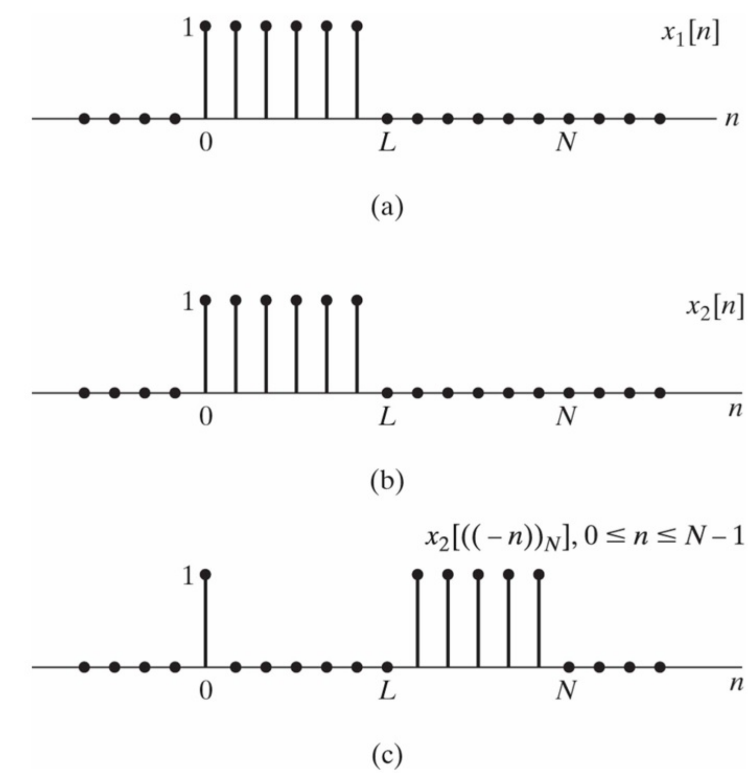
$N = 4$
$r = 4$

$Y[0]$ $Y[1]$ $Y[2]$ $Y[3]$ $Y[4]$ $Y[5]$ $Y[6]$ $Y[7]$ $Y[8]$ $Y[9]$ $Y[10]$ $Y[11]$ $Y[12]$ $Y[13]$ $Y[14]$ $Y[15]$

$Y[0] \leftrightarrow X[0]$ $\qquad Y[4] \leftrightarrow X[1]$ $\qquad Y[8] \leftrightarrow X[2]$ $\qquad Y[12] \leftrightarrow X[3]$

Zero padding in time-domain $\Rightarrow$ Higher resolution in freq

$L = 6$ $N = 11$

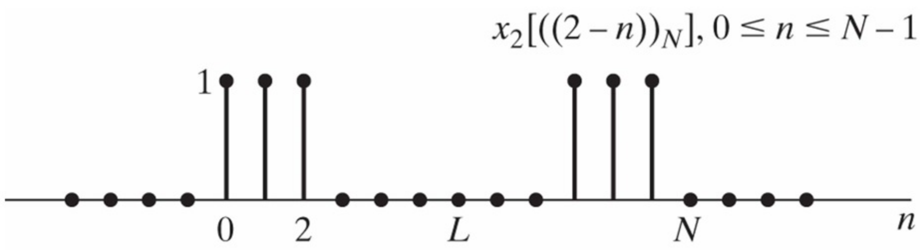


$$\underbrace{[1\ 1\ 1\ 1\ 1\ 1]}_{\text{Length} = 6} \ ⑥ \ \underbrace{[1, 1, 1, 1, 1, 1]}_{\text{Length}}$$

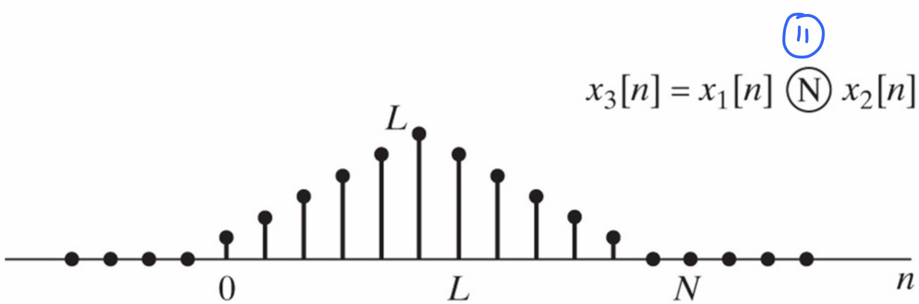
Linear conv. $x_1[n] * x_2[n]$

Length = 11

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \vdots & & & & & & & & & & \\ & & & & & & & & & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

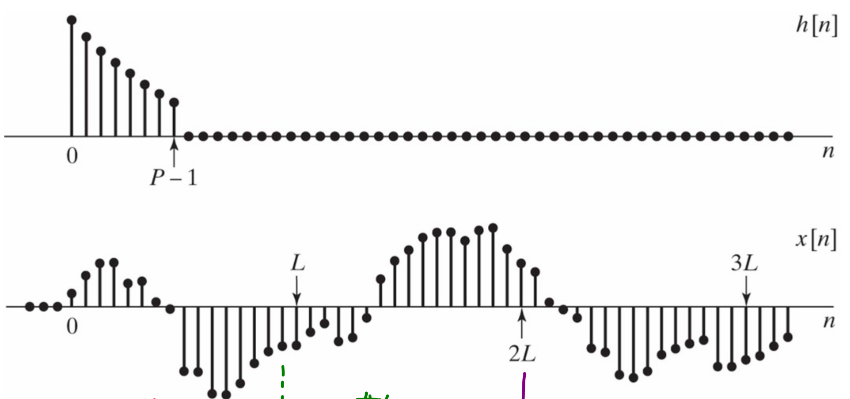


(d)



(e)

$\left\{ \begin{array}{l} \text{6 point seq} \\ \text{5 zeros} \end{array} \right\}$ 11 $\left\{ \begin{array}{l} \text{6 seq} \\ \text{5 zeros} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Linear conv} \\ \text{with } N=11 \end{array} \right\}$



#0 #1 #2

Segments of Length L

LTI system

filter

Length = P

Data Seq

Length of Data Seq >> Length of the filter

$y[n] = x[n] * h[n]$

Linear convolution

$$x[n] = \sum_{r=0}^{\infty} x_r[n - rL]$$

Block # r $\quad x_r[n] = x[n + rL] \quad 0 \le n \le L-1$

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} x_r[n - rL] * h[n] = \sum_{r=0}^{\infty} y_r[n - rL]$$

$y_r[n] = x_r[n] * h[n]$ ← Linear conv-

$$\underbrace{y_r[n] = x_r[n] * h[n]}_{\text{Length } L+P-1}$$

$\underbrace{x_r[n]}_{\text{Length } L} * \underbrace{h[n]}_{\text{Length } P}$ Linear convolution

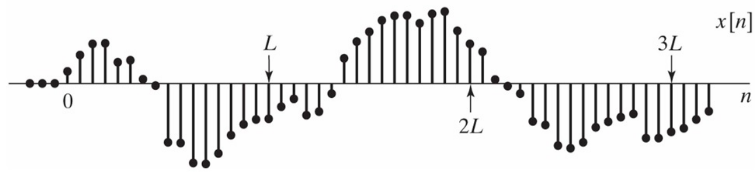
$x_r[n]$ Length L
Append $(P-1)$ zeros

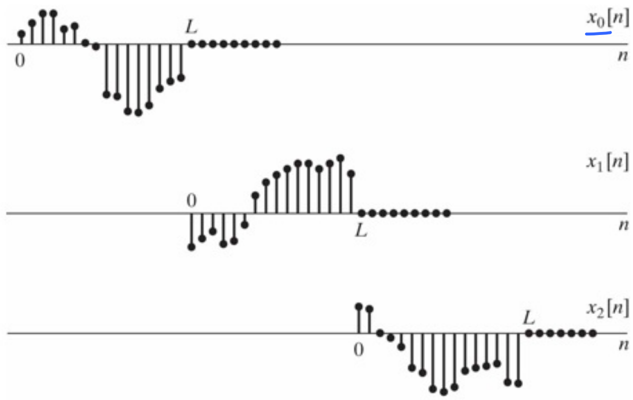
$h[n]$ Length P
Append $(L-1)$ zeros

$(L+P-1)$ pt DFT $X_r[k]$

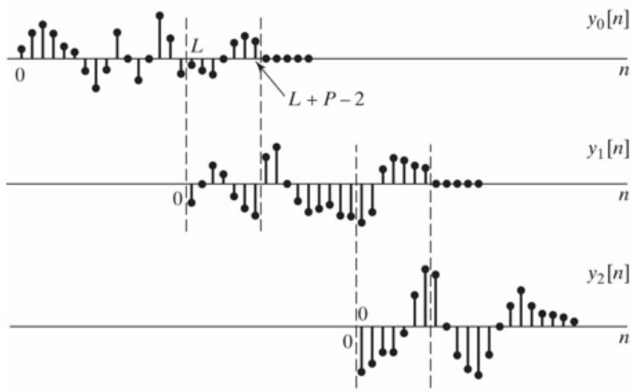
$(L+P-1)$ pt DFT $H[k]$

$$\underbrace{x_r[n] * h[n]}_{L+P-1 \text{ point seq}} \longleftrightarrow \underbrace{X_r[k]\,H[k]}_{Y_r[k]}$$





(a)



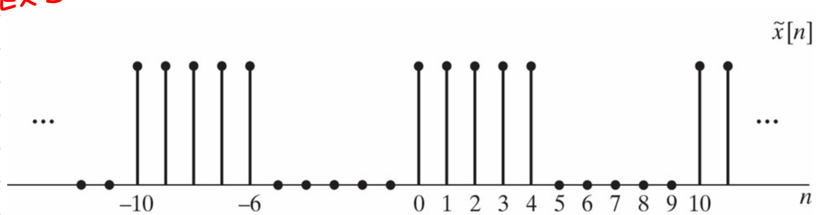
$y_0[n] = x_0[n] * h[n]$

$y_1[n] = x_1[n] * h[n]$

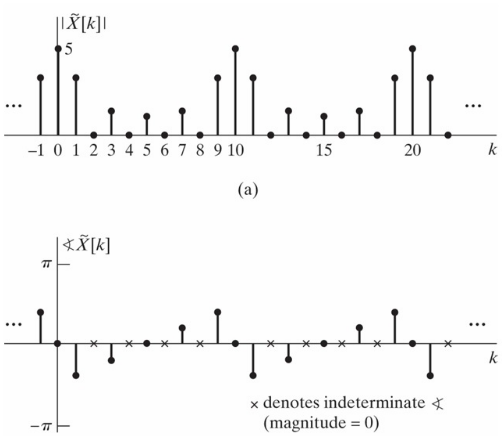
Overlap Add Method

O&S

Ex 3

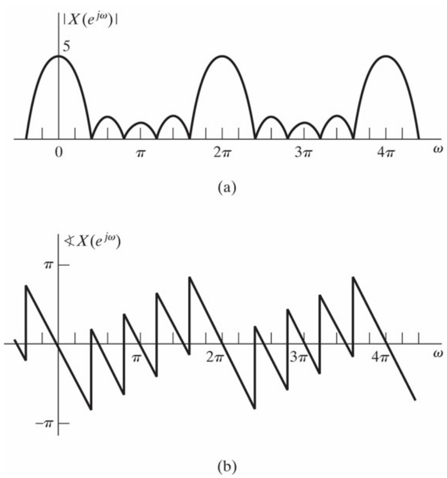


$\tilde{x}[n]$
...
–10
–6
0 1 2 3 4 5 6 7 8 9 10
n

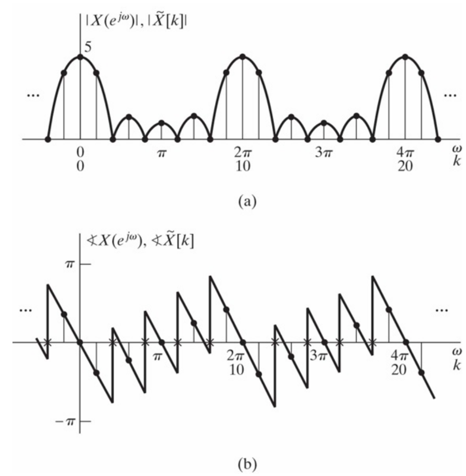


$|\tilde{X}[k]|$
5
–1 0 1 2 3 4 5 6 7 8 9 10
15
20
k
(a)
$\sphericalangle\tilde{X}[k]$
π
$-\pi$
k
× denotes indeterminate $\sphericalangle$
(magnitude = 0)

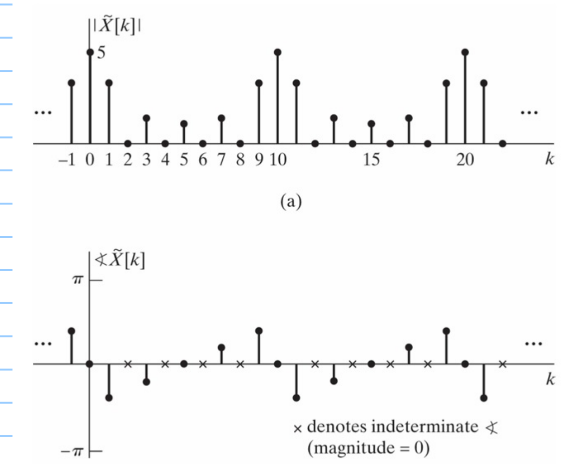
O&S
Ex 6



$|X(e^{j\omega})|$
5
0
π
2π
3π
4π
ω
(a)
$\sphericalangle X(e^{j\omega})$
π
$-\pi$
π
2π
3π
4π
ω
(b)



$|X(e^{j\omega})|, |\tilde{X}[k]|$
5
...
0
0
π
2π
10
3π
4π
20
ω
k
(a)
$\sphericalangle X(e^{j\omega}), \sphericalangle \tilde{X}[k]$
π
$-\pi$
π
2π
10
3π
4π
20
ω
k
(b)



$|\tilde{X}[k]|$
5
...
–1 0 1 2 3 4 5 6 7 8 9 10
15
20
k
(a)
$\sphericalangle \tilde{X}[k]$
π
$-\pi$
k
× denotes indeterminate $\sphericalangle$
(magnitude = 0)