Note Title 02-09-2024

Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

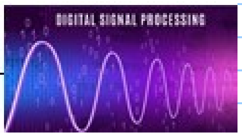
BS Electronic Systems programme

R. David Koilpillai

September – December 2024

Session # 26

December 10, 2024



Session 26

Outline

Last session

- DFT properties
- Examples
- Linear convolution using DFT

Week 11-12 (DFT & FFT)

Discrete Fourier Transform (DFT): Definition of the DFT and inverse DFT—relationship to discrete-time Fourier series—matrix representation—DFT as the samples of the DTFT and the implied periodicity of the time-domain signal—circular convolution and its relationship with linear convolution—effect of zero padding—introduction to the Fast Fourier Transform (FFT) algorithm—decimation-in-time

Today

- FFT
- Linear convolution using FFT

Reading Assignment

O&S ch 9 Fast Fourier Transform (FFT)

O&S ch 8

✓ 8.1 Discrete Fourier Series (DFS) or (DTFS)

✓ 8.2 Properties of DFS

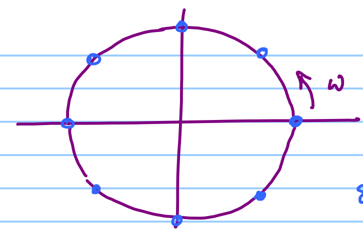
✓ 8.3 Fourier Transform of periodic signals

✓ 8.4 Sampling the FT

✓ 8.5 Discrete Fourier Transform (DFT) ⇒ Fourier Representation of Finite Duration Sequences

✓ 8.6 Properties of DFT

✓ 8.7 Computing Linear Convolution using the DFT



8 samples of the DTFT

O&S ch 9 Computation of the DFT

✓ 9.1 Direct Computation

✓ 9.2 Decimation-in-Time FFT

$N = 2^{\gamma}$ (Radix-2)

Not covered:

9.3 Decimation-in-Freq FFT → Read & verify

9.5 More General FFT Algorithms } Read.

9.7 Effects of Quantization

→ Different values of N; Different Radix

DFT

Inverse — Synthesis Equation

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn} \quad \text{(Inverse)}$$

Forward — Analysis Equation

$$X[\ell] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}\ell n} \quad \text{(Forward)}$$

Viewing the DFS as an orthogonal transform

$$\underline{X} = \underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}}_{N\times 1 \text{ vector}} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^{1} & W_N^{2} \cdots & W_N^{N-1} \\ 1 & W_N^{2} & W_N^{4} \cdots & W_N^{2(N-1)} \\ & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{\substack{N\times N \\ \underline{D_N}}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{N\times 1 \text{ vector } \underline{x}}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\boxed{\begin{array}{ll} \text{Analysis eqn.} & \underline{X} = \underline{D_N} \cdot \underline{x} \\ \text{Synthesis eqn} & \underline{x} = \underline{D_N}^{-1}\, \underline{X} \end{array}}$$

$$\underline{D_N}^{-1} = \frac{1}{N}\underline{D_N}^{*}$$

Relationship between DTFT, DFS, DFT

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

Choose N such that there is no overlap between time-shifted copies of $x[n]$

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \xrightarrow{\text{Sampling}} \tilde{X}[k] \xleftrightarrow{\text{Inv DFS}} \tilde{x}[n] \longleftrightarrow x[n]$$

$$\tilde{X}[k] \updownarrow X[k], \qquad X[k] \xleftrightarrow{\text{DFT}} x[n]$$

$$\left[\begin{aligned} \tilde{X}[k] &= X[((k))_N] \\ \tilde{x}[n] &= x[((n))_N] \end{aligned}\right.$$

$$X[k] = \begin{cases} \tilde{X}[k] & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \tilde{x}[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

If $x[n]$ is real valued

$$|X[k]| = \Big[\underbrace{Re^2[X[k]]}_{\text{even}} + \underbrace{Im^2[X[k]]}_{\text{odd} \times \text{odd} = \text{even}}\Big]^{\frac{1}{2}} = \Big|X\big[((-k))_N\big]\Big|$$

Magnitude of DFT coefficients $|X[k]|$ $k = 0, \ldots N-1$ is even function of k

$$\underbrace{\arg X[k]}_{\text{odd function}} = \tan^{-1} \frac{Im\, X[k] \ (\text{odd})}{Re\, X[k] \ (\text{even})}$$

Arg of DFT coefficients is odd function of k

$$|X[0]| \quad |X[1]| \quad |X[2]| \quad \cdots \quad |X[N-2]| \quad |X[N-1]|$$

even symmetry $|X[k]|$

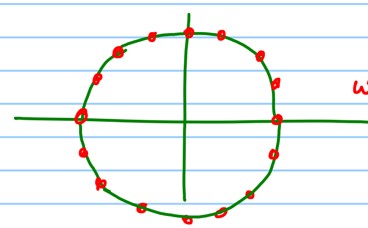
$$|X[k]| = |X[N-k]| \qquad k = 1, \ldots N-1$$

sufficient to evaluate $k = 0, 1, \ldots \frac{N}{2}$ or $\omega = \pi$

Properties of DFS

			DFT
	$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$		
Shift in time	$\tilde{x}[n-m] \longleftrightarrow e^{-j\frac{2\pi}{N}km}\,\tilde{X}[k] = W_N^{mk}\,\tilde{X}[k]$	$\longleftrightarrow$	$x[((n-m))_N] \longleftrightarrow W_N^{km}\,X[k]$
Duality	$\tilde{X}[n] \longleftrightarrow N\,\tilde{x}[-k]$	$\longleftrightarrow$	$X[n] \longleftrightarrow N\,x[((-k))_N]$
Shift in freq	$W_N^{-\ell n}\,\tilde{x}[n] \longleftrightarrow \tilde{X}[k-\ell]$	$\longleftrightarrow$	$W_N^{-\ell n}\,x[n] \longleftrightarrow X[((k-\ell))_N]$
Periodic conv in time	$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\,\tilde{x}_2[n-m] \longleftrightarrow \tilde{X}_1[k]\,\tilde{X}_2[k]$	$\longleftrightarrow$	$\sum_{m=0}^{N-1} x_1[m]\,x_2[((n-m))_N] \longleftrightarrow X_1[k]\,X_2[k]$
Periodic conv in freq	$\tilde{x}_3[n] = \tilde{x}_1[n]\,\tilde{x}_2[n] \longleftrightarrow \tilde{X}_3[k] = \frac{1}{N}\sum_{m=0}^{N-1} \tilde{X}_1[m]\,\tilde{X}_2[k-m]$	$\longleftrightarrow$	

$$\boxed{\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}}$$



$\omega = \frac{2\pi}{N}k$

$N = 16$

Linear Convolution

$x_1[n] = \{1, 2, 0, 1\}$

$x_2[n] = \{2, 2, 1, 1\}$

$$
\begin{array}{cc}
 & \overset{0}{\downarrow} \\
1\ 1\ 2 & \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}
\end{array}
\underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \\ 4 \\ 1 \\ 1 \end{bmatrix}
$$

Linear convolution

$\{1, 2, 0, 1\} * \{2, 2, 1, 1\} \longrightarrow y[n] = \{2, 6, 5, 5, 4, 1, 1\}$

$\uparrow N = 4 \qquad \uparrow N = 4 \qquad N = 7 \uparrow$

$x_1[n] = \{1, 2, 0, 1, 0, 0, 0\}$ $\quad N = 7$

$x_2[n] = \{2, 2, 1, 1, 0, 0, 0\}$ $\quad N = 7$

$$
\begin{array}{cc}
000\ 11\ 2 & \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \end{bmatrix}
\end{array}
\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \\ 4 \\ 1 \\ 1 \end{bmatrix}
$$

$x_1[n]$ ⑦ $x_2[n]$

Circular / Periodic convolution

$x_1[n]$ ④ $x_2[n] = x_3[n]$

$$
\begin{array}{cc}
\boxed{1\ 1\ 2} & \begin{bmatrix} 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}
\end{array}
\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}
$$

$4 \times 4 \qquad 4 \times 1 \qquad 4 \times 1$

$y[n] = x[n] * h[n]$ Linear Convolution

$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$

$\tilde{h}[n] \longleftrightarrow \tilde{H}[k]$

$\tilde{y}[n] \longleftrightarrow \underbrace{\tilde{X}[k]\,\tilde{H}[k]}_{\tilde{Y}[k]}$ With zero padding

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 \cdots & W_N^{2(N-1)} \\ & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$\underline{D}_N$

$W_N = e^{-j\frac{2\pi}{N}kn}$

$\underline{X} = \underline{D}_N \, \underline{x}$

Properties of DFT

① $X[0] = \sum_{n=0}^{N-1} x[n]$

② $x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]$

③ $X\left[\frac{N}{2}\right] = \sum_{n=0}^{N-1} x[n]\, W_N^{\left(\frac{N}{2}\right)n}$

@ $k = \frac{N}{2}$ $\quad = \sum_{n=0}^{N-1} (-1)^n x[n] = X\left[\frac{N}{2}\right]$

$W_N^{\frac{N}{2}n} = e^{-j\frac{2\pi}{\cancel{N}}\cdot\left(\frac{\cancel{N}}{\cancel{2}}\right)n}$

$= e^{-jn\pi} = (-1)^n$

④ $x\left[\frac{N}{2}\right] = \frac{1}{N}\sum_{k=0}^{N-1} (-1)^k X[k]$

⑤ $W_N^{\frac{N}{2}} = -1$

⑥ $W_N^{\frac{N}{4}} = -j$

⑦ $W_N^{\frac{3N}{4}} = j$

⑧ $W_N^{k+N} = W_N^k$

⑨ $W_N^{k+\frac{N}{2}} = -W_N^k$

⑩ $W_N^{k2} = W_{\frac{N}{2}}^k$

⑪ $W_N^* = W_N^{N-1} = W_N^{-1}$

⑫ $\left(W_N^m\right)^* = W_N^{-m} = W_N^{N-m}$

⑬ $\left(W_N^{mk}\right)^* = W_N^{-mk} = W_N^{(N-m)k}$

⑭ $W_N^{\left(k+\frac{N}{2}\right)n} = (-1)^n W_N^{kn}$

Zero padding

$x[n]$ is an N-point seq

$$y[n] = \begin{cases} x[n] & 0 \le n \le N-1 \\ 0 & N \le n \le rN-1 \end{cases}$$

$x[n]$ N-point seq

$y[n] = \{ (\ x[n]\ )(\ 0\ )(\ 0\ ) \ldots \}$

rN point seq

$y[n]$ is obtained by $x[n]$ padded with $(r-1)N$ zeros

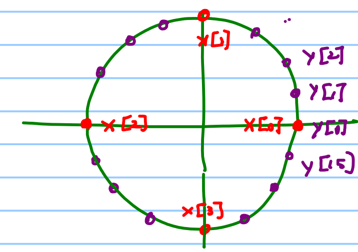
$$y[n] \leftrightarrow Y[k] = \underbrace{\sum_{n=0}^{rN-1} y[n]\, W_{rN}^{kn}}_{rN\text{-point DFT}} = \sum_{n=0}^{N-1} x[n]\, W_{rN}^{kn} \qquad k = 0, 1, \ldots rN-1$$

$$\boxed{X[m] = \sum_{n=0}^{N-1} x[n]\, W_N^{mn}} \quad \text{N-point DFT}$$

N = 4
r = 4

$Y[0]$ $Y[1]$ $Y[2]$ $Y[3]$ $Y[4]$ $Y[5]$ $Y[6]$ $Y[7]$ $Y[8]$ $Y[9]$ $Y[10]$ $Y[11]$ $Y[12]$ $Y[13]$ $Y[14]$ $Y[15]$

$Y[0] \updownarrow X[0]$, $Y[4] \updownarrow X[1]$, $Y[8] \updownarrow X[2]$, $Y[12] \updownarrow X[3]$



N = 4
r = 4

Zero padding in time-domain $\Rightarrow$ Higher resolution in freq

Examples

① Find the 10-pt IDFT of $X[k] = 1 + 2\,\delta[k]$

$$X[k] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\underline{x} = \frac{1}{10} D_{10}^{*} \underline{X}$$

$$= \frac{1}{10} \underline{D_{10}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{2}{10} \underline{D_{10}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{x} = \frac{1}{10} \begin{bmatrix} 10 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{2}{10} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 0.2 \\ \vdots \\ 0.2 \end{bmatrix}$$

(2) $x[n] \longleftrightarrow X[k]$ Length N

$y[n] \longleftrightarrow Y[k]$

$x[n] \, (N) \, y[n] \longleftrightarrow X[k]\,Y[k]$

$$x[n] \, (N) \, y[n] = \sum_{m=0}^{N-1} x[m]\, y[((n-m))_N]$$

$\underbrace{x[n] \, (N) \, y^*[((-n))_N]}_{y_1[n]} \longleftrightarrow X[k]\,Y^*[k]$

$$\underbrace{x[n] \, (N) \, y^*[((-n))_N]}_{y_1} = \sum_{m=0}^{N-1} x[m]\, y_1[((n-m))_N]$$

$$= \sum_{m=0}^{N-1} x[m]\, y^*[((m-n))_N]$$

$r_{xy}[n] \longleftrightarrow X[k]\,Y^*[k]$

Circular cross correlation $x[n]$ & $y[n]$

$r_{xy}[n]$

If $y[n] = x[n]$ $\quad r_{xx}[n] = x[n] \, (N) \, x^*[((-n))_N] \longleftrightarrow X[k]\,X^*[k]$

$$r_{xx}[0] = \sum_{m=0}^{N-1} |x[m]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

Parseval relationship
$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

Parseval relation via Matrix representation

$$\underline{X} = \underline{D}_N \underline{x} \quad (1)$$

$$\underline{x} = \frac{1}{N} \underline{D}_N^* \underline{X}$$

$$\frac{1}{N} \underline{X}^H \underline{X} = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{N} \underbrace{\underline{x}^H \underline{D}_N^H}_{X^H} \underbrace{\underline{D}_N \underline{x}}_{X} = \frac{\cancel{N}}{\cancel{N}} \underline{x}^H \underline{x} = \underline{x}^H \underline{x}$$

$$= \sum_{n=0}^{N-1} |x[n]|^2$$

Verify $\underline{D}_N^H \underline{D}_N = N \underline{I}$

Parseval relation is verified ✓

④ The even samples of a 11-point DFT $\hat{X[k]}$ of a length real valued seq. $x[n]$ is given

Determine the missing samples of $X[k]$

$$\begin{bmatrix} X[0] \\ X[2] \\ X[4] \\ X[6] \\ X[8] \\ X[10] \end{bmatrix} = \begin{bmatrix} 4 \\ -1+j3 \\ 2+j5 \\ 9-j6 \\ -5-j8 \\ \sqrt{3}-j2 \end{bmatrix}$$

$$X[1] = \sqrt{3}+j2 = X^*[10]$$

$$X[3] = -5+j8 = X^*[8]$$

$$\vdots \qquad \vdots$$

$$X[9] = -1-j3 = X^*[2]$$

$$x[n] = x^*[n]$$

$$\downarrow \qquad \downarrow$$

$$\underbrace{X[k] = X^*[((-k))_N]}$$

$$X[k] = X^*[N-k] = X^*[11-k]$$

$$X[1] = X^*[11-1] = X^*[10]$$

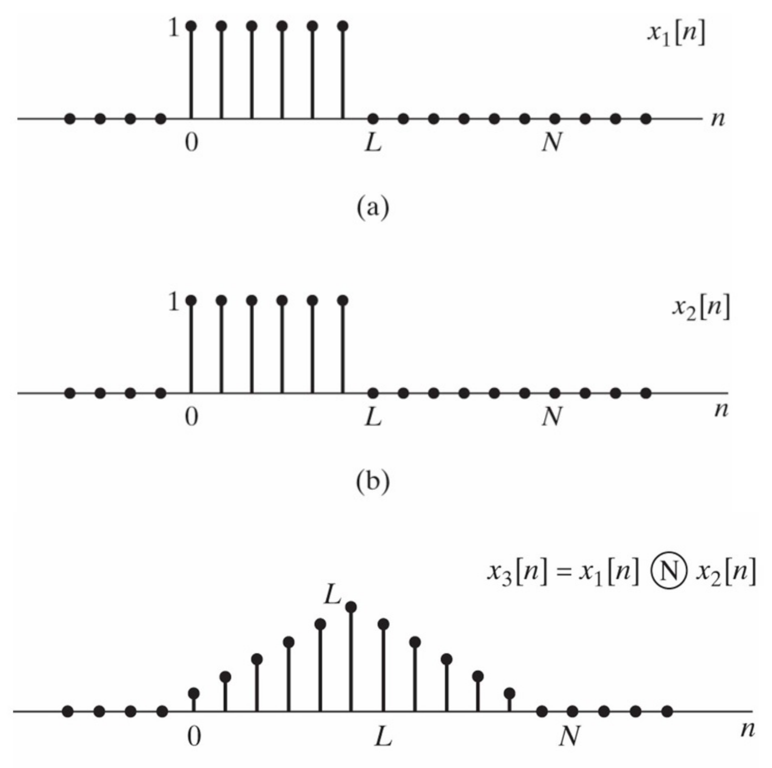
5 $X[k] = DFT\{x[n]\}$ $\quad x[n] \overset{DFT}{\longleftrightarrow} X[k]$

a $DFT\{\underbrace{DFT\{x[n]\}}_{X[k]}\} = N\, x[((-k))_N]$

b $DFT\{DFT\{\underbrace{DFT\{DFT\{x[n]\}\}}_{N\, x[((-n))_N] = x_1[n]}\}\}$

$DFT\{DFT\{x_1[n]\}\} = N\, x_1[((-n))_N] = N \cdot N \cdot x[n] = N^2 x[n]$

$L = 6$ $\quad N = 11$



$$\underbrace{[1\ 1\ 1\ 1\ 1\ 1]}_{\text{Length} = 6} \; (6) \; \underbrace{[1,1,1,1,1,1]}_{\text{Length}} \rightarrow [6,6,6,6,6,6]$$

Linear conv. $x_1[n] * x_2[n]$

Length = 11

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \vdots & & & & & & & & & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Circular Convolution $\qquad x_1[n] \; ⑥ \; x_2[n]$

$$\begin{bmatrix} 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \end{bmatrix} \begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix} = \begin{bmatrix} 6\\6\\6\\6\\6\\6 \end{bmatrix}$$

$x_1[n]$ Length L
$x_2[n]$ Length P $\Big\} \; x_1[n] * x_2[n]$ Length $L+P-1$

Zeropadding $(P-1) \rightarrow$ Seq. length $L+P-1$

Zeropadding $(L-1) \longrightarrow$ Seq. length $L+P-1$

$x_1[n] \; ⑩ \; x_2[n]$

$$\underbrace{\begin{bmatrix} 1&0&0&0&0&1&1&1&1&1 \\ 1&1&0&0&0&0&1&1&1&1 \\ 1&1&1&0&0&0&0&1&1&1 \\ 1&1&1&1&0&0&0&0&1&1 \\ 1&1&1&1&1&0&0&0&0&1 \\ 1&1&1&1&1&1&0&0&0&0 \\ 0&1&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&1&0&0 \\ 0&0&0&1&1&1&1&1&1&0 \\ 0&0&0&0&1&1&1&1&1&1 \end{bmatrix}}_{10 \times 10} \underbrace{\begin{bmatrix} 1\\1\\1\\1\\1\\1\\0\\0\\0\\0 \end{bmatrix}}_{10 \times 1} = \begin{bmatrix} 2\\2\\3\\4\\5\\6\\5\\4\\3\\2 \end{bmatrix} \begin{matrix} \times \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \end{matrix}$$

$x_1[n] \; ⑪ \; x_2[n]$
$= x_1[n] * x_2[n]$

$$\begin{bmatrix} 1\\2\\3\\4\\5\\6\\5\\4\\3\\2\\1 \end{bmatrix}$$

If # zeros to be added is not correct

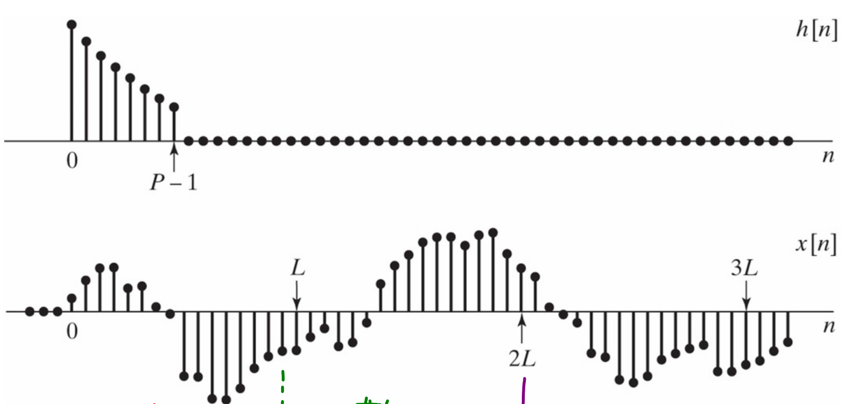
$\Rightarrow N \neq L+P-1$

Some samples in error

$$N = \underset{11}{N_{Lc}} - 1 = 10$$

$$\# \text{ Samples in error} = 11 - 10 = 1$$

Convolution using DFT



LTI system

filter

Length = P

Data Seq

Length of Data Seq >> Length of the filter

$y[n] = x[n] * h[n]$

Linear convolution

#0 #1 #2

Segments of Length L

$$x[n] = \sum_{r=0}^{\infty} x_r[n - rL]$$

Block #r $\quad x_r[n] = x[n + rL] \quad 0 \le n \le L-1$

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} x_r[n - rL] * h[n] = \sum_{r=0}^{\infty} y_r[n - rL]$$

$y_r[n] = x_r[n] * h[n] \leftarrow$ Linear conv-

$y_r[n] = \underbrace{x_r[n] * h[n]}_{\text{Length } L \quad \text{Length } P}$ linear convolution

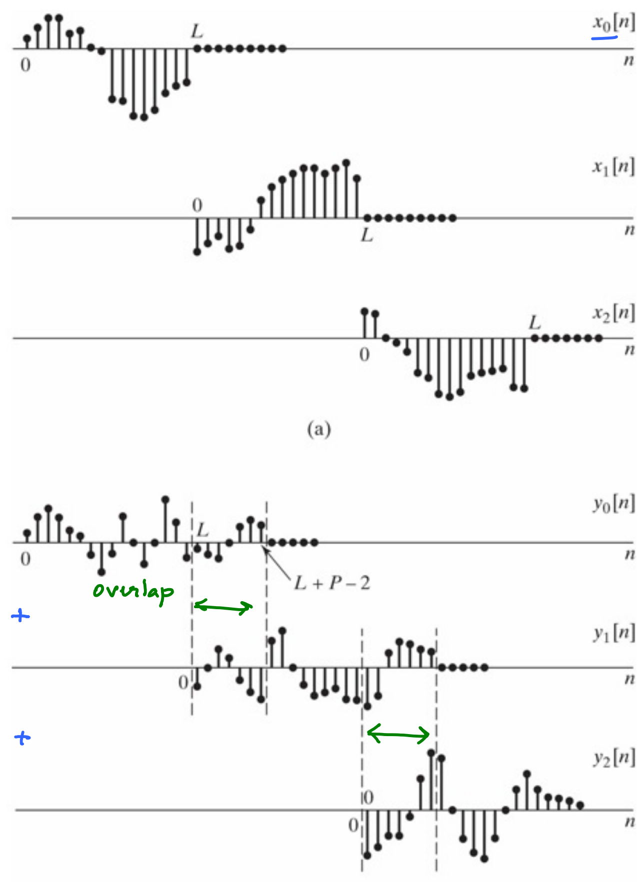
$y_r[n]$ → Length $L+P-1$

$x_r[n]$ Length L, Append $(P-1)$ zeros → $(L+P-1)$ pt DFT $X_r[k]$

$h[n]$ Length P, Append $(L-1)$ zeros → $(L+P-1)$ pt DFT $H[k]$

$$\underbrace{x_r[n] * h[n]}_{L+P-1 \text{ point seq}} \longleftrightarrow \underbrace{X_r[k]\,H[k]}_{Y_r[k]}$$

OVERLAP - ADD METHOD



(a)

$y_0[n] = x_0[n] * h[n]$

Add +

$y_1[n] = x_1[n] * h[n]$

+

Convolution using DFT

Overlap Add Method.

1. Add $(L-1)$ zeros to $h[n]$ & compute $(L+P-1)$ pt DFT $H[k]$ (Store in memory)

2. Each data segment $x_r[n]$ $\Big\}$ $(L+P-1)$ pt DFT $X_r[k]$
   Add $(P-1)$ zeros

3. $X_r[k]\,H[k] = Y_r[k]$

   $x_r'[n]$ $(L+P-1)$ $h'[n]$

   $\equiv \quad x_r[n] * h[n]$

4. $y_r[n] = \text{IDFT}\{X_r[k]\,H[k]\} = x_r[n] * h[n]$

5. $$y[n] = \sum_{r=0}^{\infty} y_r[n - rL]$$ ← overlap & add

Ch 9 Efficient Computation of DFT $\equiv$ Fast Fourier Transform (FFT)

Discrete Time Domain
- periodic $\longleftrightarrow$ DFS
- - - - - DFT $\longleftrightarrow$ FFT
- finite length aperiodic $\longleftrightarrow$ DTFT

Computational Complexity

1 Complex Multiplication = 1 CM = $(a+jb)(c+jd) = (ac-bd) + j(ad+bc)$
$= 4M + 2A$

1 CM = 4M + 2A

1 complex Addition = 1 CA = $(a+c) + j(b+d)$ = 2A

$$\boxed{\begin{aligned} 1\ CM &= 4M + 2A \\ 1\ CA &= 2A \end{aligned}}$$

Focus of FFT $\Rightarrow$ reducing # computation $\Rightarrow$ reducing # CMs

Direct Computation of DFT

$\underline{X} = \underline{D}_N \, \underline{x}$

Each DFT coeff $X[k] \sim (N)\,CM + (N-1)\,CA$

Entire DFT $\sim (N^2)\,CM + [N(N-1)]\,CA \quad \sim O(N^2)$

Complexity of Direct Comp. of DFT grows exponentially as $\sim N^2$

1965 Cooley & Tukey Algorithm for efficient computation of DFT

Basic Idea

N- point DFT $\longrightarrow$ 2 $\left(\frac{N}{2}\right)$-pt DFTs

$O(N^2)$

$2\times O\left(\frac{N}{2}\right)^2 \sim \frac{N^2}{2}$

Symmetries in DFT Matrix D_N

$W_N^{(k+\frac{N}{2})n} = (-1)^n W_N^{kn}$

Using all symmetry properties

$N = 2^\gamma$ Total comp. complexity of FFT $\left(\frac{N}{2}\log_2 N\right)$ CM + $\left(N\log_2 N\right)$ CA

$N = 1024$

$N^2 = 10,48,576$

$N\log_2 N = 10,240$

$N = 2^{\nu}$ $\quad \nu$ integer (always use ZP if needed)

$x[n]$ $\quad x[0]\quad x[2]\quad x[4]\ \ldots$ even samples $x[2n]$

$\quad x[1]\quad x[3]\quad x[5]\ \ldots$ odd samples $x[2n+1]$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_N^{k2r} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_N^{k(2r+1)}$$

$$W_N^{k2r} = W_{\frac{N}{2}}^{kr}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_{\frac{N}{2}}^{kr} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_N^{k}\, W_{\frac{N}{2}}^{kr}$$

$$X[k] = \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_{\frac{N}{2}}^{kr}}_{G[k]} + \underbrace{W_N^{k}}_{\text{Scale Factor}} \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{\frac{N}{2}}^{kr}}_{H[k]}$$

$k = 0, 1 \ldots N-1$

$\left.\begin{array}{l} G[k] \\ H[k] \end{array}\right\} \quad k = 0, 1 \ldots \frac{N}{2}-1$

$G[k]$ $\quad \frac{N}{2}$ point DFT of even samples of $x[n]$

$H[k]$ $\quad \frac{N}{2}$ point DFT of odd samples of $x[n]$

$$\boxed{X[k] = G[k] + W_N^k H[k]}$$

$\frac{N}{2}$ point DFT $\sim \left[\left(\frac{N}{2}\right)^2 CM + \frac{N}{2}\left(\frac{N}{2}-1\right) CA \right] \times 2$

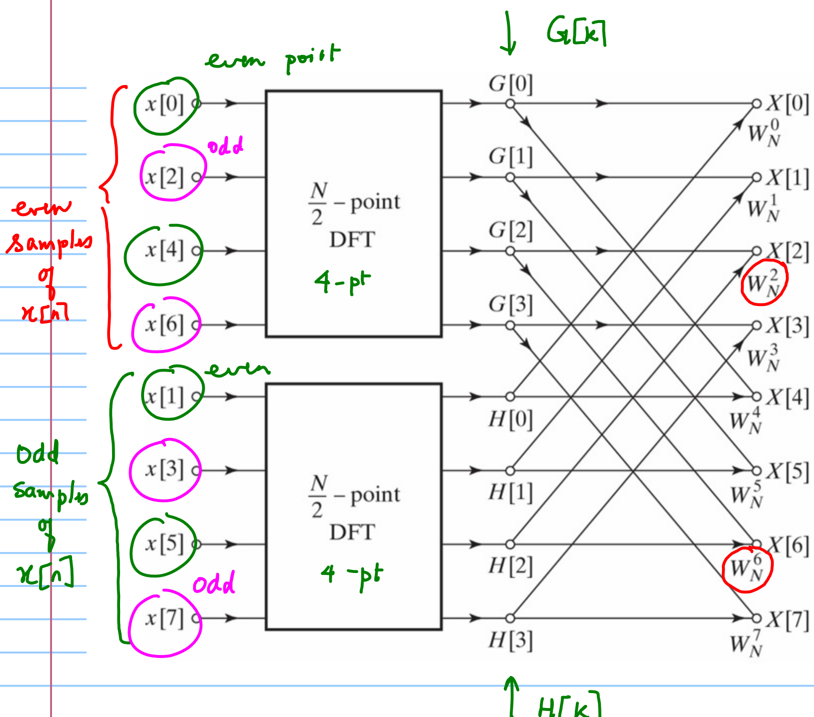
for $G[k]$ & $H[k]$

Combing $G[k]$ & $H[k]$ $\quad N\ CM + N\ CA$

Total complexity $\sim \left(\frac{N^2}{2} + N\right) CM + \frac{N^2}{2} CA$

Lower than Direct Computation of N-point DFT any $N > 2$

8-point DFT



$$\boxed{X[k] = G[k] + W_N^k H[k]}$$

$$X[2] = G[2] + W_N^2 H[2]$$

$$X[6] = G[2] + W_N^6 H[2]$$

$$G[6] = G[2]$$

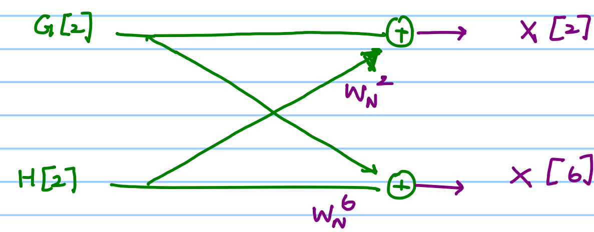
$$H[6] = H[2]$$

$$X[2] = G[2] + W_8^2 H[2]$$

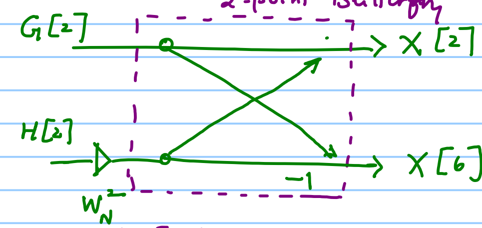
$$X[6] = G[2] + W_8^6 H[2]$$

$$W_8^6 = W_8^4 \cdot W^2 = -W_8^2$$

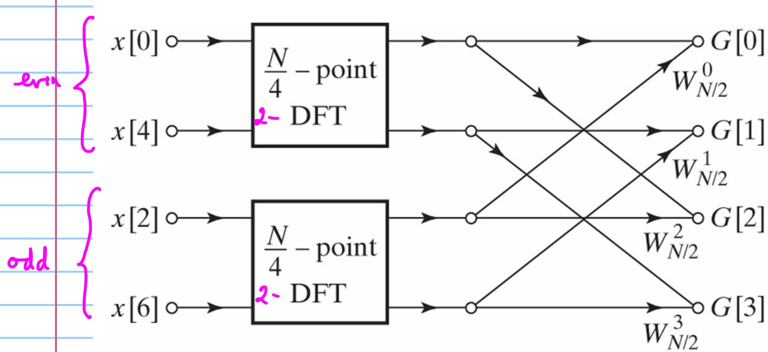
$$W_8^4 = -1$$

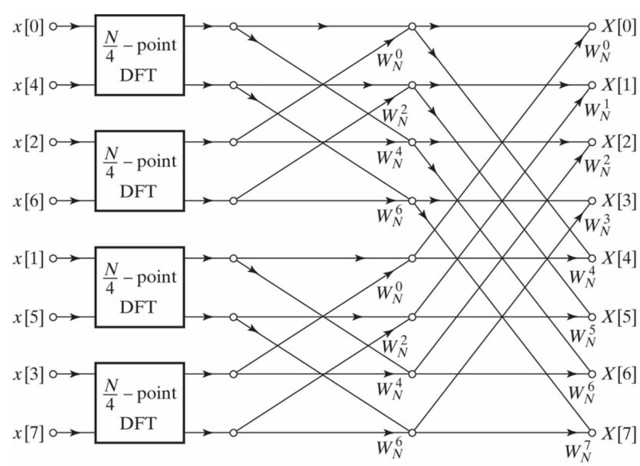


2-point Butterfly



Twiddle Factor



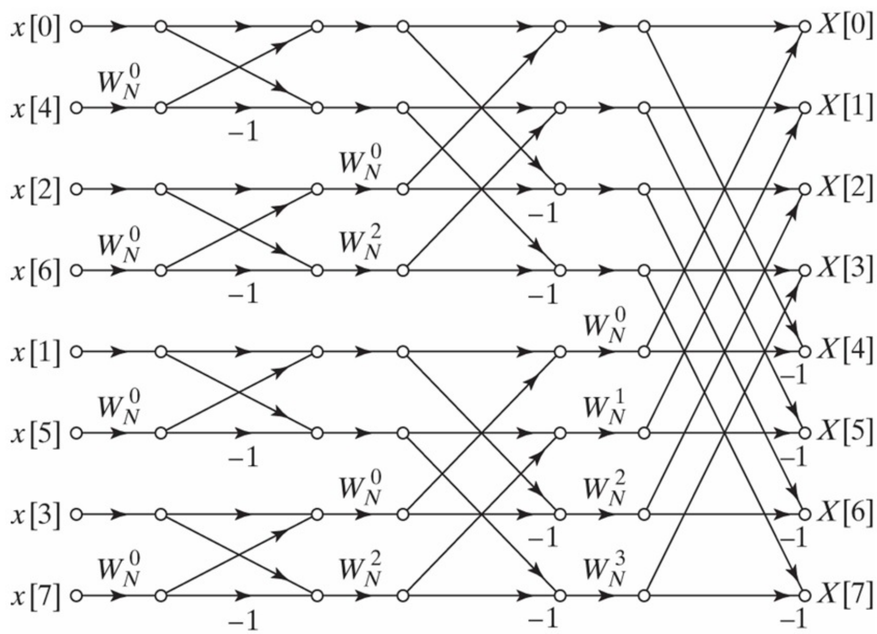


2-pt DFT $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Overall $N = 2^{\nu} \rightarrow 2^{\nu-1}$ pt DFT $\rightarrow 2^{\nu-2} \cdots \rightarrow 2^1$ DFT

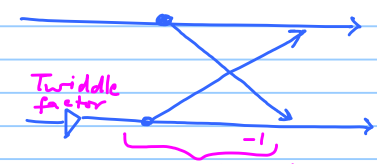
stages $= \log_2 N = \nu$

Implement 8-point DFT ≡ 8-point FFT



2-pt DFT 4-pt 8-pt

Basic Building Block



$$\begin{bmatrix} X_m[p] \\ X_m[q] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{2-pt Butterfly}} \underbrace{\begin{bmatrix} 1 & \\ & W_N^r \end{bmatrix}}_{\text{Twiddle factor}} \underbrace{\begin{bmatrix} X_{m-1}[p] \\ X_{m-1}[q] \end{bmatrix}}_{\text{input}}$$

N-point DFT $N = 2^\nu$

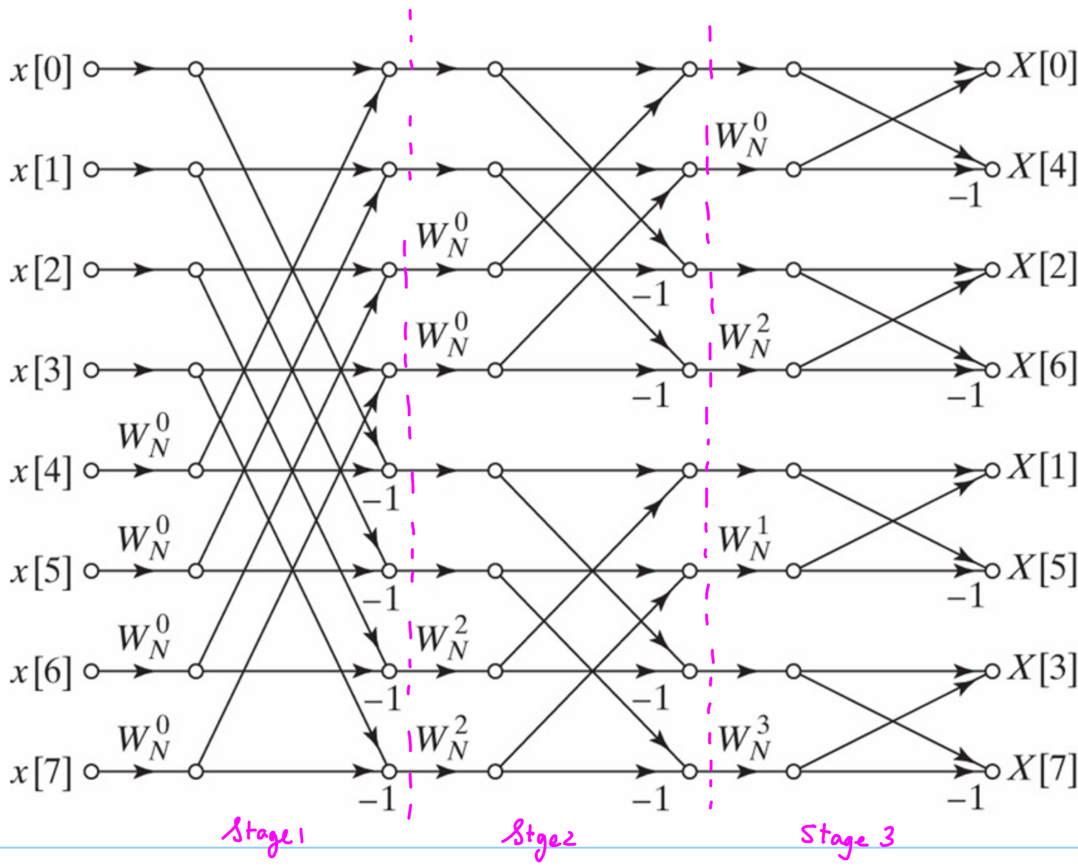
N complex inputs $x[n]$

N complex outputs $X[k]$

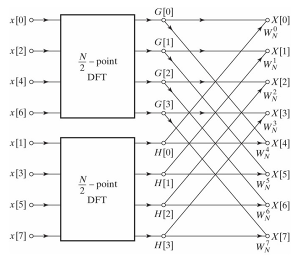
stages = ν

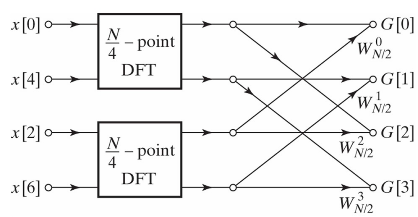
@ each stage $\frac{N}{2}$ Butterflies, $\frac{N}{2}$ Twiddle factors

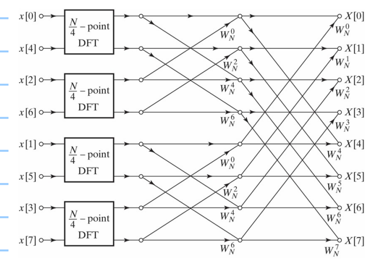
$$\boxed{\text{Total} \quad \frac{N}{2}\log_2 N \ \text{CM} + N\log_2 N \ \text{CA}}$$

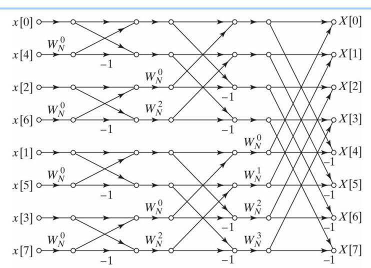


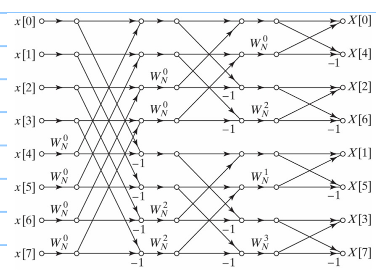
x[0]
x[1]
x[2]
x[3]
x[4]
x[5]
x[6]
x[7]
W_N^0
W_N^0
W_N^0
W_N^0
W_N^0
W_N^0
W_N^2
W_N^2
W_N^0
W_N^2
W_N^1
W_N^3
−1
X[0]
X[4]
X[2]
X[6]
X[1]
X[5]
X[3]
X[7]
Stage 1
Stge 2
Stage 3


x[0]
x[2]
x[4]
x[6]
x[1]
x[3]
x[5]
x[7]
N/2 - point DFT
N/2 - point DFT
G[0]
G[1]
G[2]
G[3]
H[0]
H[1]
H[2]
H[3]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]


x[0]
x[4]
x[2]
x[6]
N/4 - point DFT
N/4 - point DFT
G[0]
G[1]
G[2]
G[3]
$W_{N/2}^0$
$W_{N/2}^1$
$W_{N/2}^2$
$W_{N/2}^3$


x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
N/4 - point DFT
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]


x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]


x[0]
x[1]
x[2]
x[3]
x[4]
x[5]
x[6]
x[7]
X[0]
X[4]
X[2]
X[6]
X[1]
X[5]
X[3]
X[7]