Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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EE3101 Digital Signal Processing

Session 4

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Outline

Last session

Week 1-2

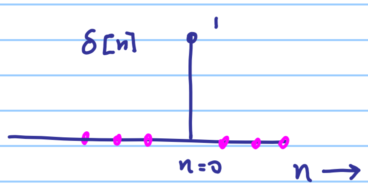
Introduction to sampling - Review of Signals and Systems: Basic operations on signals - Discrete time complex exponentials and other basic signals – system properties (linearity, time-invariance, memory, causality, BIBO stability) -- impulse response and convolution

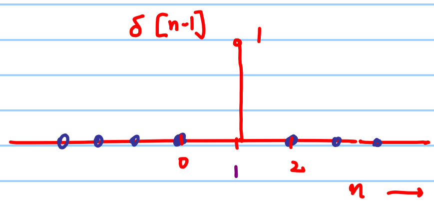
Reading Assignment

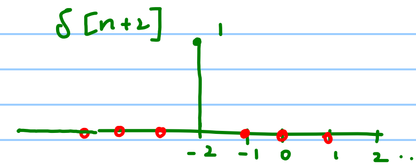
Oppenheim & Schafer : Sec 2.0 – 2.3

Rawat : Chapter 1

unit sample







$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$

Gen representation of any DT sequence $x[n]$

Sequences

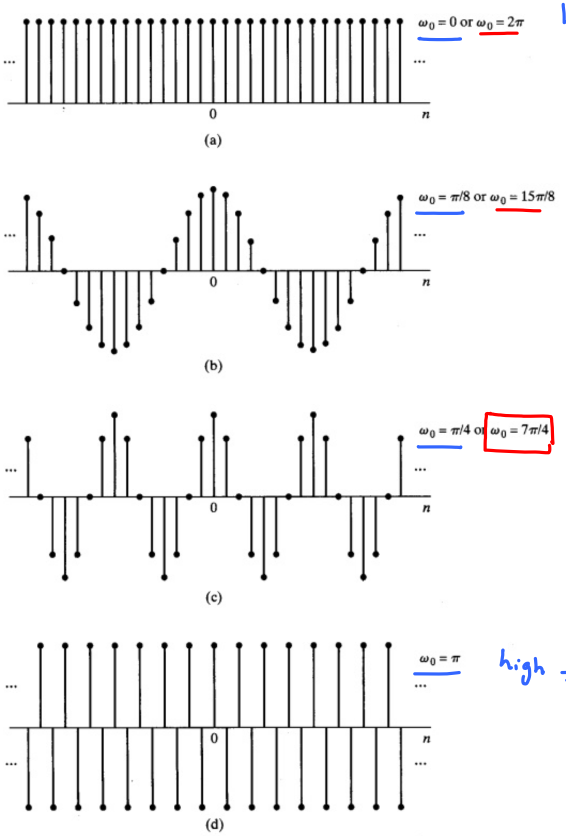
* Finite length, Infinite Length
* causal, right-sided
* anticausal, Left-sided
* Non-causal
* Length of finite length sequence $N = N_2 - N_1 + 1$ $x[n]$ non-zero in the range $[N_1, N_2]$

DT Sinusoids

$A\cos(\omega_1 n + \phi) = A\cos(\omega_0 n + 2\pi k n + \phi) = A\cos(\omega_0 n + \phi)$

$\boxed{\text{Periodicity in frequency for DT Sinusoids with period} = 2\pi}$

Illustration DT sinusoids

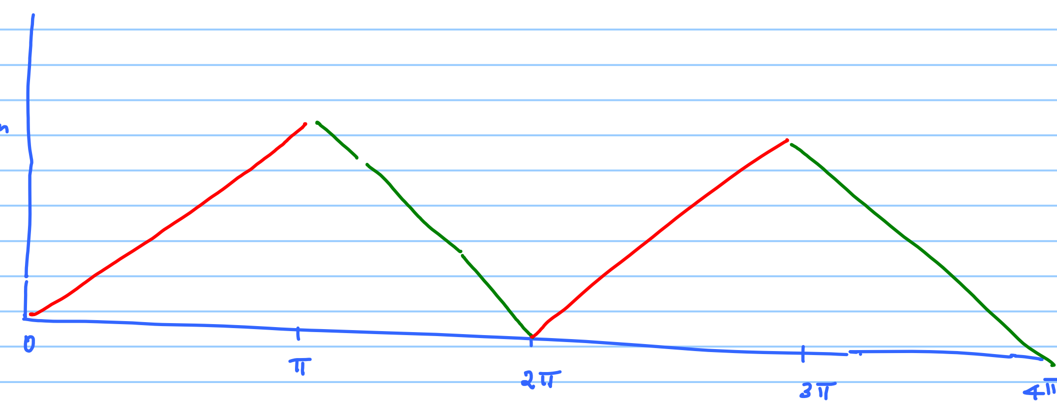


Low freq

$> \omega = \pi$

high freq

$A\cos(\omega_0 n + \phi)$



$A\cos(\omega_0 n + \phi)$

$\omega_0 = \pi$

Observation #1 ***

DT sinusoids are periodic in frequency

Period = 2π or any multiple of 2π

Time behaviour

CT → all sinusoids are period $T = \frac{2\pi}{\Omega_0}$

DT $A \cos(\omega_0 n + \phi)$

If Periodic with period $= N$ $\Rightarrow$ $\underbrace{A \cos(\omega_0 (n + kN) + \phi)}_{x[n+kN]} = A \cos(\omega_0 n + \phi)$

$x[n + kN] = x[n]$ $\forall n$

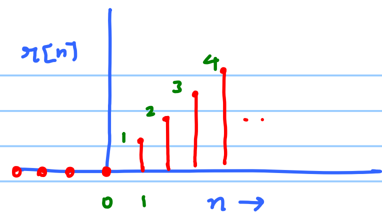
period

A DT sinusoid is periodic iff $\boxed{\frac{2\pi}{\omega_0} = \frac{N}{m}}$ Rational function

N, m are integers

$N =$ period of periodic seq

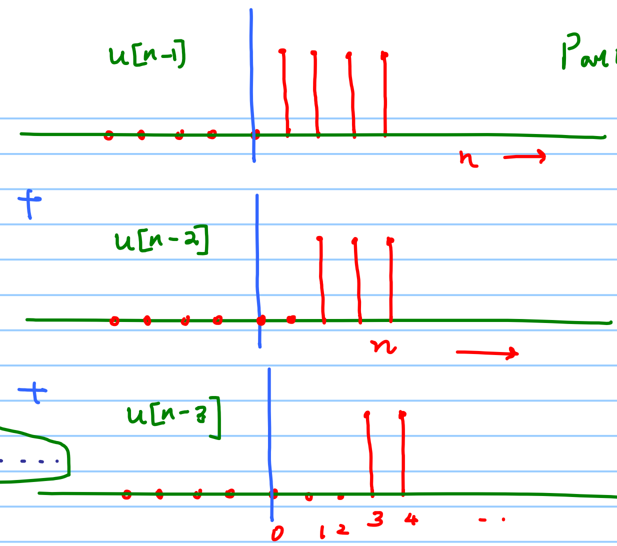
Unit ramp



$r[n] = n \cdot u[n]$

$= \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$

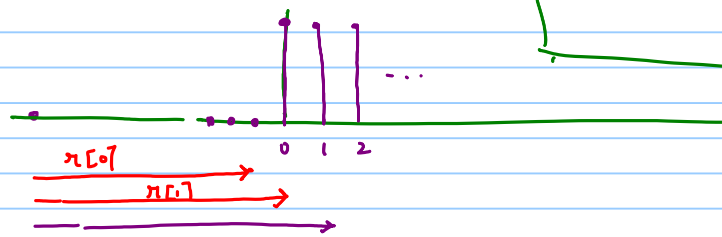
Part ③



Show That

① $r[n] = \sum_{k=0}^{\infty} k\,\delta[n-k] = 0\cdot\delta[n] + 1\cdot\delta[n-1] + 2\,\delta[n-2] + \cdots$

② $r[n] = \sum_{k=-\infty}^{n-1} u[k]$



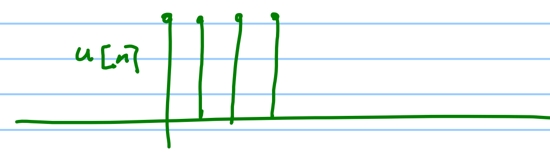
$r[0] = 0$

$r[1] = 1$

$r[2] = 2$

$\vdots$

③ $r[n] = \sum_{m=1}^{\infty} u[n-m] = u[n-1] + u[n-2] + u[n-3] + \cdots$



DT Sequence $x[n]$

Real valued

If $x[n] = x[-n]$ even sequence

$x[n] = -x[-n]$ odd sequence

$x[n] = x_e[n] + x_o[n]$

$x_e[n] = \frac{1}{2}[x[n] + x[-n]]$

$x_o[n] = \frac{1}{2}[x[n] - x[-n]]$

Complex valued sequences

$x[n] = x^*[-n]$ Conjugate symm

$x[n] = -x^*[-n]$ Conjugate antisymm

$x_{cs}[n] = \frac{1}{2}[x[n] + x^*[-n]]$

$x_{ca}[n] = \frac{1}{2}[x[n] - x^*[-n]]$

$x[n] = x_{cs}[n] + x_{ca}[n]$

Periodic signal

$x[n] = x[n+N] \quad \forall n$ (1)

* $x[n] = x[n+2N]$

* Period of a periodic seq is the smallest integer satisfying (1)

Time Scaling

$x[n] = \{0, 0, \ldots 0, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0 \cdot 0 \ldots\}$

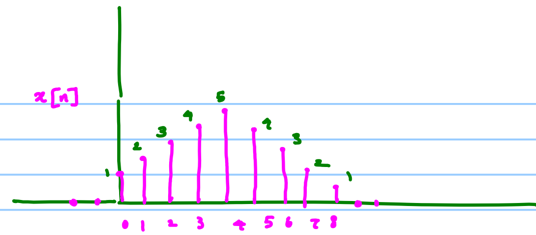
$y[n] = x[Mn]$

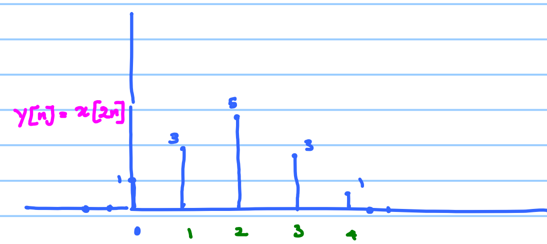
$x[n] \rightarrow \boxed{\downarrow M} \rightarrow y[n] = x[Mn]$

M integer

Compression

Down - Sampling





Expansion or Upsampling

$$\rightarrow \boxed{\uparrow 2} \rightarrow$$

$$y[n] = \begin{cases} x\left[\frac{n}{2}\right] & \text{if } n = 0, \pm 2, \pm 4, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$y[0] = x[0]$

$y[1] = 0$

$y[2] = x[1]$

$y[3] = 0$

$y[4] = x[2]$

$\vdots$

$x[n] = \{ 0, 0, 0, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0 \ldots \}$

$y[n] = \{ \ldots 0, 0, 0, 0\ 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 4, 0, 3, 0, 2, 0, 1, 0 \ldots \}$

$$y_1[n] = \begin{cases} x\left[\frac{n}{3}\right] & \text{if } n = 0, \pm 3, \pm 6, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = \begin{cases} x\left[\frac{n}{N}\right] & \text{if } n = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$y_1[n]\ \{ \ldots\ 0\ 0\ 1\ 0\ 0\ 2\ 0\ 0\ 3 \ldots \}$

insertion of $\underline{\underline{(N-1)}}$ zeros between every pair of samples

shift
time scaling
time reversal

$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x[Mn]$

$$x[n] \longrightarrow \boxed{\uparrow N} \longrightarrow \begin{cases} x\left[\frac{n}{N}\right] & \text{if } n = 0, \pm N, \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = x\left[\frac{M}{N} n\right]$$

1. Upsampling
2. Downsampling

HW

$$y[n] = x\left[-\frac{M}{N} n - n_0\right]$$

1. Shift
2. upsampling
3. downsamp
4. Time reversal

$$x_1[n] = x[n - n_0]$$

$$x_2[n] = \begin{cases} x_1\left[\frac{n}{N}\right] & \text{if } n = 0, \pm N, \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$x_3[n] = x_2[Mn] = \begin{cases} x_1\left[\frac{Mn}{N}\right] & \text{if } n = 0, \pm N, \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = x_3[-n] = x_2[-Mn] = \begin{cases} x_1\left[\frac{-Mn}{N}\right] & \text{if } n = 0, \pm N, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} x\left[-\frac{Mn}{N} - n_0\right] & \text{if } n = 0, \pm N, \dots \\ 0 & \text{otherwise} \end{cases}$$

Example

1. $x[n] = \{1, 3, 4, -2, 5\}$

$$x[n] \longrightarrow \boxed{\downarrow 2} \xrightarrow{x_1[n]} \boxed{\uparrow 3} \longrightarrow y_1[n]$$

$$x[n] \longrightarrow \boxed{\uparrow 3} \xrightarrow{x_2[n]} \boxed{\downarrow 2} \longrightarrow y_2[n]$$

$x_1[n] = \{0, 0, 1, 4, 5, 0 \ldots\}$

$y_1[n] = \{0, 0, 0\ 1, 0, 0\ 4, 0, 0, 5, 0, 0 \ldots\}$

$x_2[n] = \{0\ 0\ 1, 0, 0, 3, 0, 0, 4, 0, 0, -2, 0, 0, 5, 0\ 0 \ldots\}$

$y_2[n] = \{0\ 1\ 0\ 0\ 4, 0, 0, 5, 0 \ldots\}$

$y_2[n] = y_1[n]$

$x[n] \to \boxed{\downarrow 2} \to \boxed{\uparrow 2} \to y_3[n]$

$$Y_3[n] = \{0,\ 1,\ 0,\ 4,\ 0,\ 5,\ 0\ \ldots\}$$

$x[n] \to \boxed{\uparrow 2} \to \boxed{\downarrow 2} \to Y_4[n]$

$$Y_4[n] = \{0 \cdot\cdot\ 1,\ 3,\ 4,\ -2,\ 5,\ 0 \ldots\}$$

$$Y_4[n] = x[n]$$

$$Y_3[n] \neq Y_4[n]$$

Important Results

$x[n] \to \boxed{\uparrow N} \to \boxed{\downarrow N} \to y[n] = x[n]$

$x[n] \to \boxed{\uparrow N} \to \boxed{\downarrow M} \to y_1[n]$ $\qquad x[n] \to \boxed{\downarrow M} \to \boxed{\uparrow N} \to y_2[n]$

$Y_1[n] = Y_2[n]$ if and only if N & M are relatively prime

$x[n] \to \boxed{\uparrow N} \to \boxed{\downarrow N} \to y_3[n] \neq x[n] \to \boxed{\downarrow N} \to \boxed{\uparrow N} \to y_4[n]$

$$Y_3[n] \neq Y_4[n]$$

Example 2

$x_1[n]$ is periodic with period N_1
$x_2[n]$ " period N_2

Is $x_1[n]\, x_2[n]$ periodic ?

If periodic with period $= N$

$$x[n] = x_1[n]\, x_2[n]$$

$$x[n+N] = \underbrace{x_1[n+N]}_{x_1[n]}\ \underbrace{x_2[n+N]}_{x_2[n]} = x[n]$$

$N = p N_1 \qquad N = q N_2 \longrightarrow N = p N_1 = q N_2$

$$\frac{N_1}{N_2} = \frac{q}{p}$$

$$x[n+N] = x_1[n+pN_1]\ x_2[n+qN_2]$$

Period $= N = p N_1 = q N_2$

Application

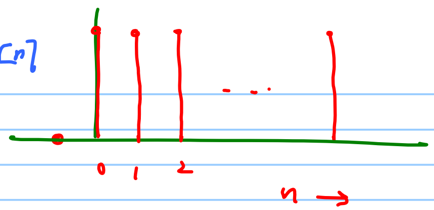
$N_1 = 90$
$N_2 = 54$

$x_1[n]\, x_2[n]$ will be periodic with period $N = 270$

$$\frac{N_1}{N_2} = \frac{90}{54} = \frac{5}{3} \quad \frac{q}{p}$$

$N = p N_1 = 270$
$q N_2 = 270$

① Is $u[n]$ periodic ?



$x[n+N] \overset{?}{=} u[n]$

$N=1$

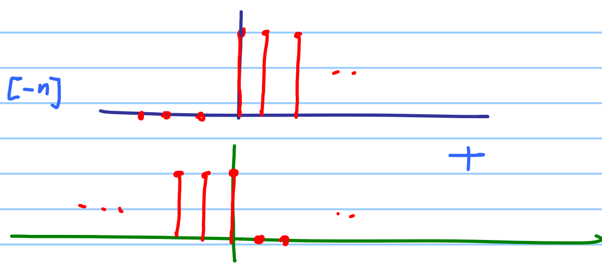
$n=0 \quad x[0+1] = x[0]$ ✓

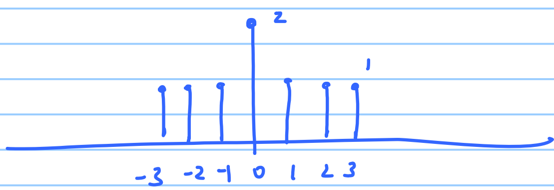
$n=1 \quad x[1+1] = x[1]$ ✓

$n=-2 \quad x[-1] = x[-2]$ ✓

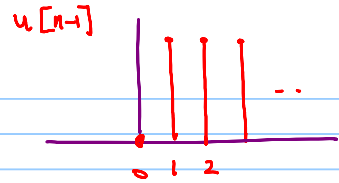
$n=-1 \quad x[0] = x[-1]$ ✗

② $x_1[n] = u[n] + u[-n]$

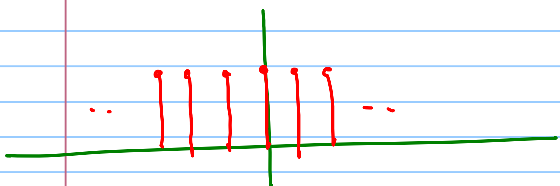


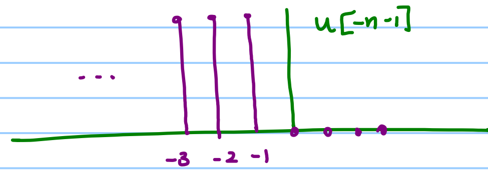


Not periodic



$x_2[n] = u[n] + u[-n-1]$

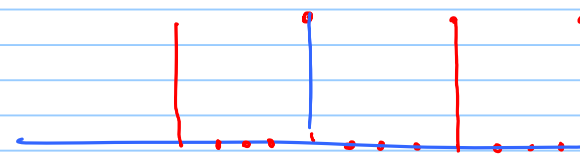




periodic seq

period $N = 1$

$x_2[n] = x_2[n+1] \quad \forall n$



3 $\quad x_3[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k] = \delta[n] + \delta[n-4] + \delta[n-8] + \dots$

$+ \delta[n+4] + \delta[n+8] + \dots$

Periodic = Yes.

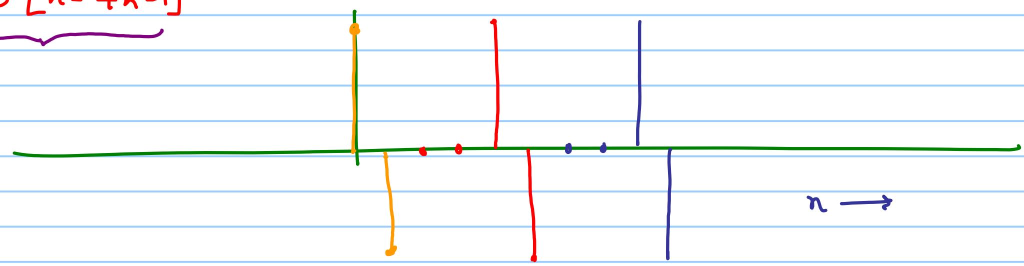
period $N = 4$.

$$x_4[n] = \sum_{k=-\infty}^{\infty} \overset{\text{periodic, period}=4}{\delta[n-4k]} - \underbrace{\overset{\text{periodic, period}=4}{\delta[n-4k-1]}}$$

$N = pN_1 = qN_2$

$N = \text{LCM}(N_1, N_2)$

periodic, period $= N = 4$



Even & Odd signals

$x_1[n] = x_1[-n]$ even

$x_1[n] = -x_1[-n]$ odd

Given

$x_1[n]$ odd signal

$x_2[n]$ even signal

$x[n] = x_1[n]\, x_2[n]$

$x[-n] = \underbrace{x_1[-n]}_{-x_1[n]}\, \underbrace{x_2[-n]}_{x_2[n]}$

$x[-n] = -x_1[n]\, x_2[n]$

$\boxed{x[-n] = -x[n]}$ $x[n]$ is odd

$x_1[n]$ even, $x_2[n]$ even $\Rightarrow$ $x_1[n]\, x_2[n]$ even

$x_1[n]$ odd, $x_2[n]$ odd $\Rightarrow$ $x_1[n]\, x_2[n]$ even

If $x[n]$ is odd

$x[n] = -x[-n]$

$n=0$ $\quad x[0] = -x[0] \qquad 2x[0] = 0 \Rightarrow x[0] = 0$

$$\sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-\infty}^{-1} x[n] + \underbrace{\cancel{x[0]}}_{=0} + \sum_{n=1}^{\infty} x[n]$$

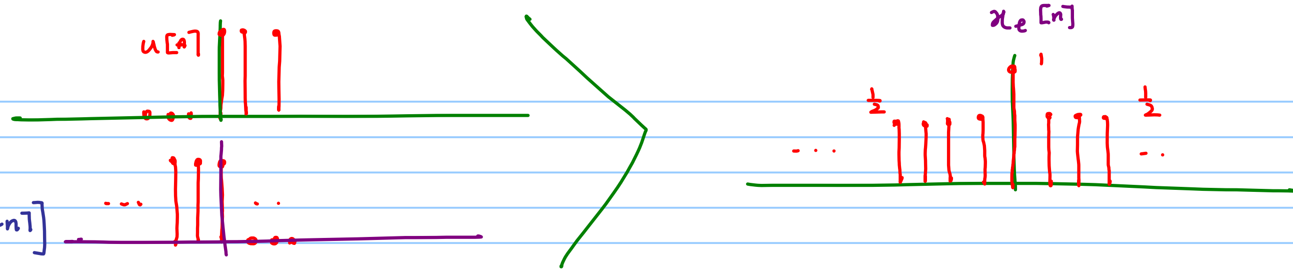
$\ldots + \cancel{x[-3]} + \cancel{x[-2]} + \cancel{x[-1]} \quad x[1] + x[2] + x[3] + \ldots$

$= 0$

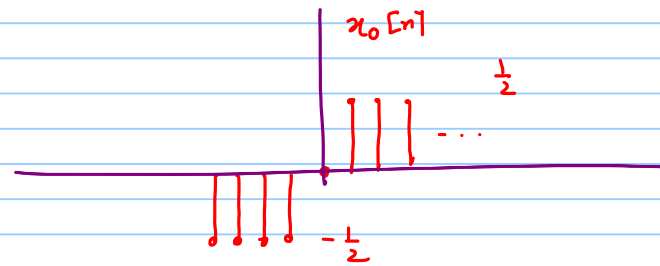
Example

$x[n] = u[n]$

$x_e[n] = \frac{1}{2}\left[u[n] + u[-n]\right]$



$x_o[n] = \frac{1}{2}\left[u[n] - u[-n]\right]$



✓ $x_e[n] + x_o[n] = x[n] = u[n]$

Example

$x[n] = \{4, -2, 4, -6\}$

$x_e[n] = \{2, -4, 4, -4, 2\}$

$x_o[n] = \{2, 2, 0, -2, -2\}$

Ex $x[n] = [\,0,\ 1+j4,\ -2+j3,\ 2-j4,\ -6-j5,\ 7,\ -j3\,]$

Verify

$x_{cs}[n] = \frac{1}{2}[x[n] + x^*[-n]] = [\,j1.5 \quad 4+j2 \quad -4+j4 \quad 2 \quad -4-j4 \quad 4-j2 \quad -j1.5\,]$

$x_{ca}[n] = \frac{1}{2}[x[n] - x^*[-n]] = [\,-j1.5 \quad -3+j2 \quad 2-j \quad -j4 \quad -2-j \quad 3+j2 \quad -j1.5\,]$

$x[n] = x_{cs}[n] + x_{ca}[n]$

DT System

$$x[n] \longrightarrow \boxed{T\{\quad\}} \longrightarrow y[n]$$

Properties of DT Systems

1. Memoryless property

Output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n

Linearity

Systems that satisfy principle of superposition → scaling, additivity

$$T\{x_1[n]\} = y_1[n] \qquad x_1[n] \xrightarrow{T} y_1[n]$$

$$T\{x_2[n]\} = y_2[n]$$

Scaling $\quad T\{a\,x_1[n]\} = a\,y_1[n]$ homogeneity

Additivity $\quad T\{x_1[n] + x_2[n]\} = y_1[n] + y_2[n]$

DT System is $\boxed{\text{linear}}$ if

$$T\{a\,x_1[n] + b\,x_2[n]\} = \underbrace{a\,y_1[n] + b\,y_2[n]}_{\text{additivity}}$$

scaling

$$\boxed{\begin{array}{c} x_1[n] \xrightarrow{T\{\ \}} y_1[n] \\ x_2[n] \xrightarrow{T\{\ \}} y_2[n] \end{array}}$$

Example

$$\boxed{y[n] = 2x[n]}$$

$x_1[n] \longrightarrow y_1[n] = 2x_1[n]$

$x_2[n] \longrightarrow y_2[n] = 2x_2[n]$

$\alpha x_1[n] \longrightarrow y[n] = 2\alpha x_1[n] = \alpha y_1[n]$ Scaling ✓

$x_1[n] + x_2[n] \longrightarrow y[n] = 2(x_1[n] + x_2[n])$

$= 2x_1[n] + 2x_2[n]$

$= y_1[n] + y_2[n]$

Verify

$$\alpha x_1[n] + \beta x_2[n] \xrightarrow{T\{\ \}} \alpha y_1[n] + \beta y_2[n]$$

$\boxed{\text{Yes}}$

Linear

$y[n] = 2x[n]$ straight line, slope 2
passing through the origin

Input is 0, then output must be zero

$y[n] = 2x[n] - 3$ Linear

If $x[n]=0$, $y[n] = -3$

$x_1[n] \longrightarrow y_1[n] = 2x_1[n] - 3$

$x_2[n] \longrightarrow y_2[n] = 2x_2[n] - 3$

$\alpha x_1[n] \longrightarrow y[n] = 2\alpha x_1[n] - 3$

$\alpha y_1[n] = 2\alpha x_1[n] - 3\alpha$

Scaling not satisfied $\Rightarrow$ not linear

$x_1[n] + x_2[n] \longrightarrow y[n] = 2(x_1[n] + x_2[n]) - 3 = 2x_1[n] + 2x_2[n] - 3$

$y_1[n] + y_2[n] = (2x_1[n] - 3) + (2x_2[n] - 3) = 2x_1[n] + 2x_2[n] - 6$

Additivity not satisfied

$\Rightarrow$ Not linear

Examples Check the following DT for linearity

1. $y[n] = n\,x[n]$

2. $y[n] = x^2[n]$

3. $y[n] = Re\{x[n]\}$

4. $y[n] = x[n]\,x[n-1]$

Time Invariance

$T\{x_1[n]\} \longrightarrow y_1[n]$

$T\{x_1[n-n_0]\} \longrightarrow y[n-n_0]$

System is Time Invariant

Example

$y[n] = \sin(x[n])$ Linear ✗

$x[n-n_0] \longrightarrow y_1[n] = \sin(x[n-n_0])$

$y[n-n_0] = \sin(x[n-n_0]) = y_1[n]$

Time/shift invariant

Exercise

1. $y[n] = x[-n]$
2. $y[n] = a\,x[n]$
3. $y[n] = n\,x[n]$
4. $y[n] = x[2n]$

LTI

$\boxed{\text{Linearity \& Time Invariance}}$ $\equiv$ Impulse Response

$$T\{a\,x_1[n] + b\,x_2[n]\} = a\,y_1[n] + b\,y_2[n]$$

$$T\{x_1[n-n_0]\} = y_1[n-n_0]$$

I $$x[n] = \sum_{k=-\infty}^{\infty} \underset{\uparrow \text{ scale}}{x[k]}\, \underbrace{\delta[n-k]}_{\text{unit impulses shifted in time}} \quad (1)$$

II $$y[n] = T\{x[n]\}$$

From (1)

$$y[n] = T\left\{ \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \right\}$$

superposition

$$= \sum_{k=-\infty}^{\infty} T\{\underbrace{x[k]}_{\text{scale factor}}\,\delta[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,\underbrace{T\{\delta[n-k]\}}_{h[n-k]} \quad \text{Shift invariance}$$

$$\boxed{T[\delta[n]] = h[n]}$$

unit impulse   impulse response

$$= \sum_{k=-\infty}^{\infty} x[k]\,\underbrace{h[n-k]}_{h[-k+n]}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{h[n-k]}_{\substack{\text{shift} \\ \text{Time reversal}}}$$

$h[n] = T\{\delta[n]\}$ unit sample response

Computing the Output

1. Obtain $h[n]$ unit sample response
2. Time reversal $h[n]$
3. Shift the time-reversed seq
   → at each shift, obtain one output point

DT Convolution

$$y[n] = x[n] * h[n]$$
$$= h[n] * x[n]$$

CT LTI characterized $\boxed{\text{impulse response } h(t)}$

$$y(t) = x(t) * h(t)$$

computed via integral

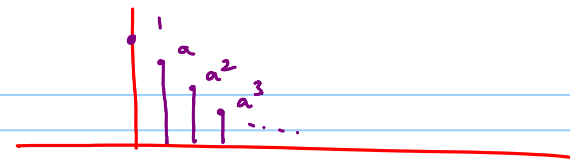
$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

DT LTI system DT impulse resp $h[n]$

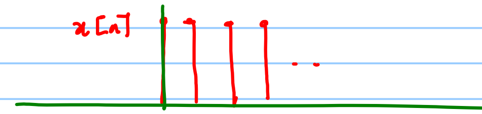
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Ex LTI System $\Rightarrow$ impulse response

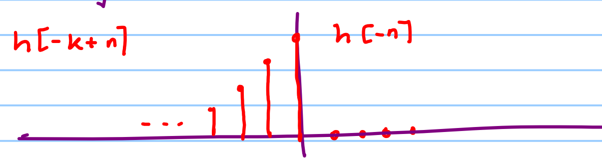
$$h[n] = a^n u[n] \qquad 0 < a < 1$$



$$x[n] = u[n]$$



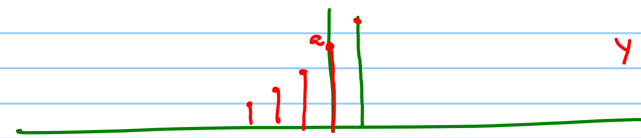
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k]$$

$$= 1$$

$y[0] = 1$



$y[1] = 1 + a$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k+1]$$

$y[2] = 1 + a + a^2$

$$y[n] = 1 + a + a^2 + \cdots a^n \quad u[n]$$

$y[-1] = 0$

$$\boxed{y[n] = \frac{1 - a^{n+1}}{1 - a} u[n]}$$

$y[-2] = 0$

$\vdots$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$n - k = l$$

$$y[n] = \sum_{l=-\infty}^{\infty} x[n-l]\, h[l]$$

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

Flip $x[n]$

Keep $h[n]$ without flipping

$y[n]$ same as before

$y[0] = 1$

$y[1] = 1 + a$

$y[2] = 1 + a + a^2$

$\vdots$

Causality

A DT system $y[n] = T\{x[n]\}$ is causal if for every choice of n_0 the output seq $y[n_0]$ depends only on the input seq values

$$x[n_0],\ x[n_0-1],\ x[n_0-2]\ \ldots$$

$y[n_0]$ depends on $x[n]$ $\quad n \le n_0$

Ex

Averaging Filter

$$y[n] = \frac{1}{3}\left[x[n-1] + x[n] + x[n+1]\right] \rightarrow \text{Not Causal}$$

Linear ? — scaling ✓, additivity ✓

Time invariant ✓

LTI $\Rightarrow$ impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = h[0]\, x[n] + h[1]\, x[n-1] + h[-1]\, x[n+1]$$

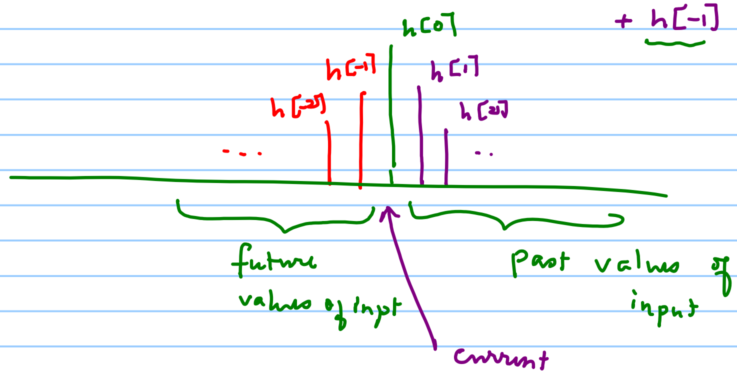
$$h[n] = \left\{ \frac{1}{3},\ \underset{\uparrow}{\frac{1}{3}},\ \frac{1}{3} \right\}$$

Gen Averaging Filter

(Moving Average Filter) $= y[n] = \frac{1}{N} \sum_{k=-M_1}^{M_2} x[n-k]$

$$\sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = h[0]\, x[n] + h[1]\, x[n-1] + \cdots \quad h[M_2]\, x[n-M_2]$$

$$+ h[-1]\, x[n+1] + \cdots \quad \cdots \quad h[-M_1]\, x[n+M_1]$$



$\boxed{h[n] = 0 \qquad n < 0}$ Causal seq

Non Causal

Convert to a Causal filter

$h[n] = \frac{1}{N} \{ 0\ 0 \ldots 0\ 0\ \cancel{1 \ldots 1\ 1}\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \cdots 1\ 0\ 0\ 0 \}$

$-M_1 \qquad\qquad M_2$

$y[n] = \frac{1}{M_2+1} \sum_{k=0}^{M_2} x[n-k]$ Causal version