Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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Session # 8

October 1, 2024

EE3101 Digital Signal Processing

Session 7

Outline

Last session

- Interconnection of LTI systems
- Examples

Week 1-2

Introduction to sampling - Review of Signals and Systems: Basic operations on signals - Discrete time complex exponentials and other basic signals – system properties (linearity, time-invariance, memory, causality, BIBO stability) -- impulse response and convolution

Week 3-4

Sampling: Impulse train sampling—relationship between impulse trained sampled continuous-time signal spectrum and the DTFT of its discrete-time counterpart—scaling of the frequency axis—relationship between true frequency and digital frequency—reconstruction through sinc interpolation—aliasing—effect of sampling at a discontinuous point—relationship between analog and digital sinc—effects of oversampling—discrete-time processing of continuous-time signals

$$\underset{\text{Cont. Time}}{CT} \xrightarrow{\text{Sampling}} \underset{\text{Discrete Time}}{DT}$$

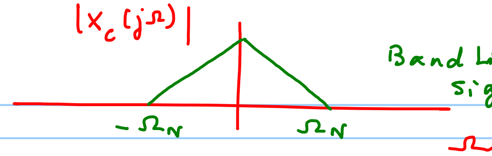
$$CT \xleftarrow[\text{Reconstruction}]{} DT$$

Reading Assignment

O&S Chapter 4: Sampling of CT signals

Rawat Chapter 2: Sampling and Quantization

Nyquist Sampling Theorem



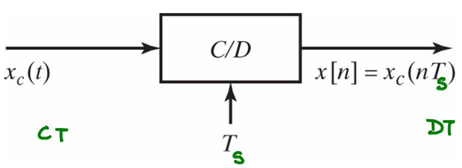
Band Limited signal

Let $x_c(t)$ be a band-limited signal with $X_c(j\Omega) = 0$ $\quad |\Omega| \geq \Omega_N$ rads/sec

$x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT_s)$ $\quad n = 0, \pm 1, \pm 2, \dots$

if $\Omega_s = \dfrac{2\pi}{T_s} \geq 2\Omega_N$

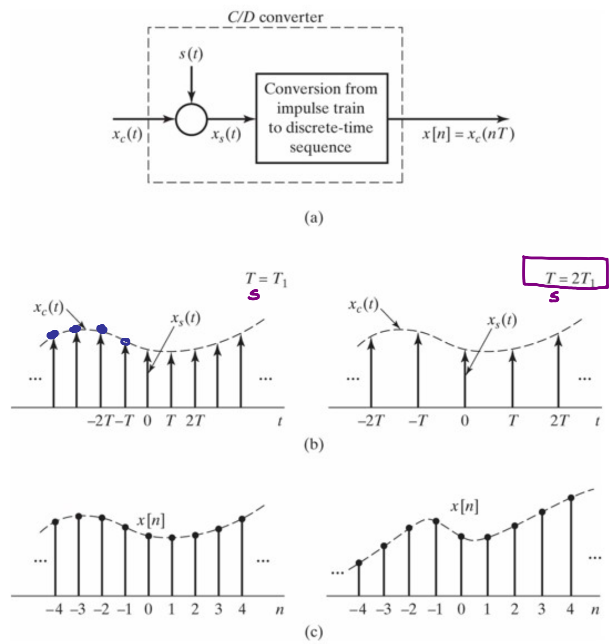
Nyquist rate $= 2\Omega_N$ $\quad \Omega_s$ = rads/sec



$x[n] = x_c(t)\Big|_{t = nT_s}$

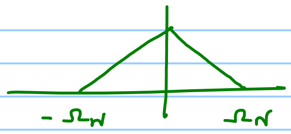
BL Signal

Bw = Ω_N



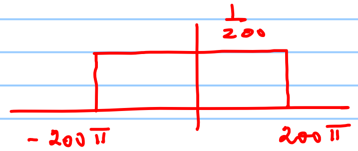
Nyquist sampling rate

* Fourier Transform $\quad x_c(t) \longleftrightarrow X_c(j\Omega)$



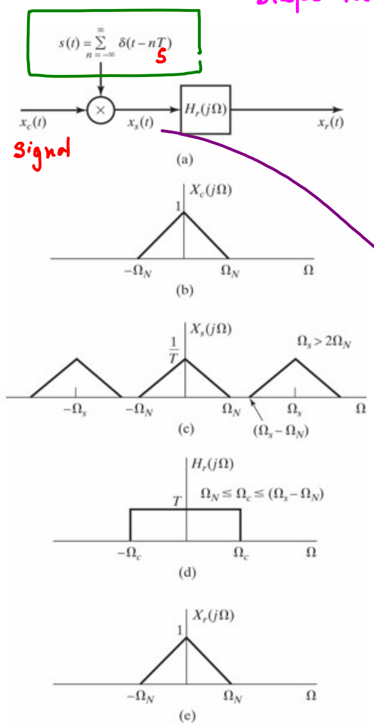
example

$$x_c(t) = \text{sinc}(200t) = \frac{\sin 200\pi t}{200\pi t} \longleftrightarrow$$



Mathematical Framework for Sampling

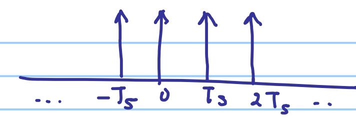
Steps in sampling

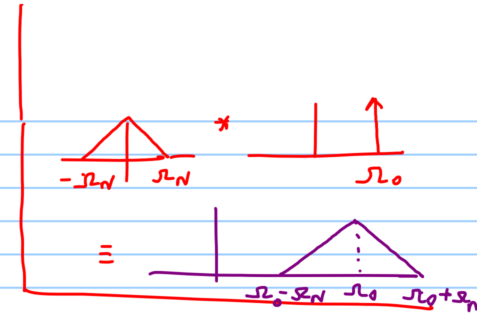


$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

train of Dirac Delta

1





$$x_c(t)\cdot s(t) = x_c(t)\cdot \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT_s)\,\delta(t-nT_s)$$

$x_c(t)\,\delta(t-mT_s) = x_c(mT_s)\,\delta(t-mT_s)$

$$x_s(t) = x_c(t)\cdot s(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

evaluate

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

periodic $\rightarrow$ Yes

period $\rightarrow$ T_s

$$s(t) = \sum_{m=-\infty}^{\infty} a_m e^{j\frac{2\pi}{T_s}mt}$$

$$a_m = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)\, e^{-j\frac{2\pi}{T_s}nt}\,dt = \frac{1}{T_s} \quad \forall m$$

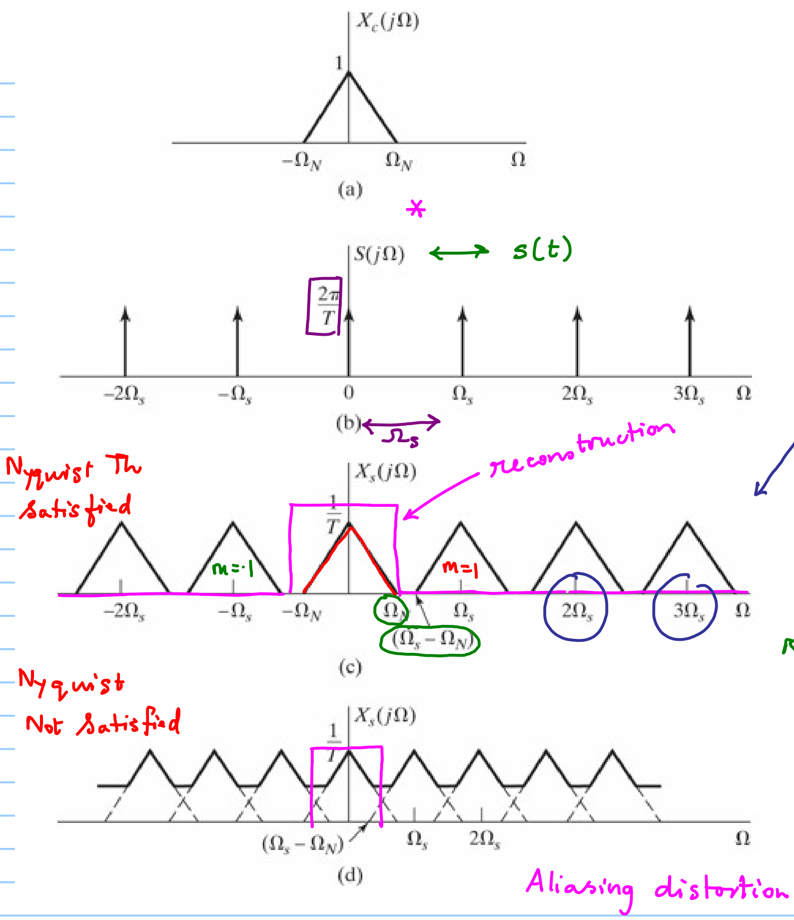
$$s(t) = \frac{1}{T_s}\sum_{m=-\infty}^{\infty} e^{j\frac{2\pi}{T_s}mt}$$

Fourier Series

$$s(t) = \frac{1}{T_s}\sum_{m=-\infty}^{\infty} e^{j\frac{2\pi}{T_s}mt} \quad (2)$$

$$e^{j\Omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\Omega-\Omega_0) \quad (1)$$

$$s(t) \longleftrightarrow S(j\Omega) = \frac{2\pi}{T_s}\sum_{m=-\infty}^{\infty}\delta(\Omega - m\Omega_s)$$



$X_c(j\Omega) * S(j\Omega)$

$$\frac{1}{\cancel{2\pi}} X_c(j\Omega) * \frac{\cancel{2\pi}}{T_s} \sum_{m=-\infty}^{\infty} \delta(\Omega - m\Omega_s)$$

$$\boxed{\frac{1}{T_s} \sum_{m=-\infty}^{\infty} X_c(j(\Omega - m\Omega_s)) = X_s(j\Omega)}$$

scale factor

repetition of spectrum

shifts multiples of Ω_s

Result #1

Original signal @ $\Omega = 0$ $[-\Omega_N, \Omega_N]$

copy $m=1$ @ $\Omega = \Omega_s$ $[\Omega_s - \Omega_N, \Omega_s + \Omega_N]$

Copy @ $m=-1$ @ $\Omega = -\Omega_s$

Do not want copies to overlap @ $m=0$, $m=1$

$\Omega_N \leq \Omega_s - \Omega_N$ $\quad 2\Omega_N \leq \Omega_s$

Nyquist Theorem

$\boxed{\Omega_s \geq 2\Omega_N}$

$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \xleftrightarrow{\mathcal{F}} S(j\Omega) = \boxed{\frac{2\pi}{T_s}} \sum_{m=-\infty}^{\infty} \delta(\Omega - m\Omega_s)$

$$\boxed{\Omega_s = 2\pi F_s = \frac{2\pi}{T_s}}$$

$x_s(t) = x_c(t) \cdot s(t) \longleftrightarrow X_s(j\Omega) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} X_c(j(\Omega - m\Omega_s))$

Do not want copies to overlap @ $m=0$, $m=1$

$\Omega_N \leq \Omega_s - \Omega_N$ $\quad \boxed{2\Omega_N \leq \Omega_s}$

Nyquist Theorem

$\boxed{\Omega_s \geq 2\Omega_N}$

Ideal Reconstruction Filter

$\equiv$ Lowpass filter with cutoff $\Omega_c = \frac{\Omega_s}{2}$

$X_s(j\Omega)$ Spectrum of the sampled signal

Reconstructed signal $x_r(t) \longleftarrow \qquad \longleftarrow$

$x_c(t) \longrightarrow x_s(t) \longrightarrow x[n]$

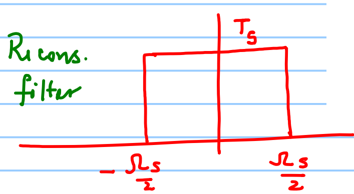
① Remove unwanted copies $m = \pm 1, \pm 2$

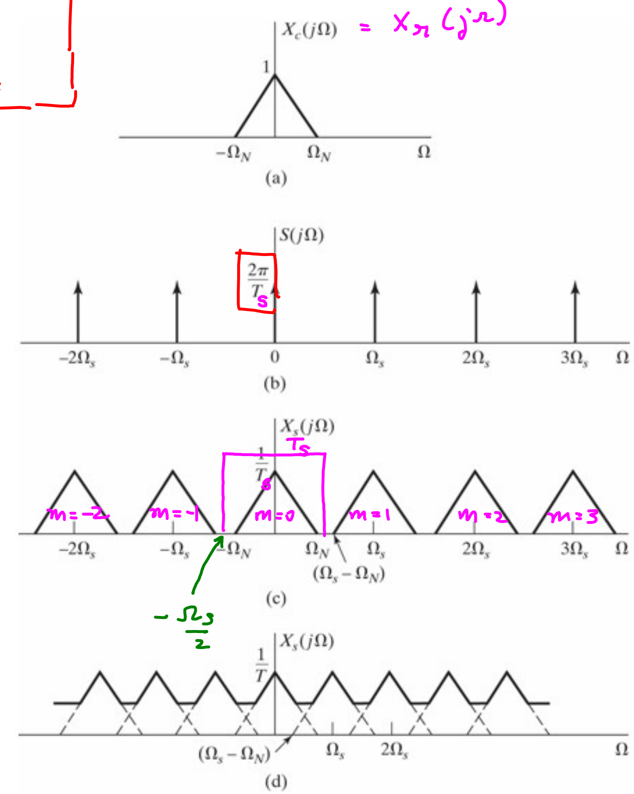
② Scale factor T_s

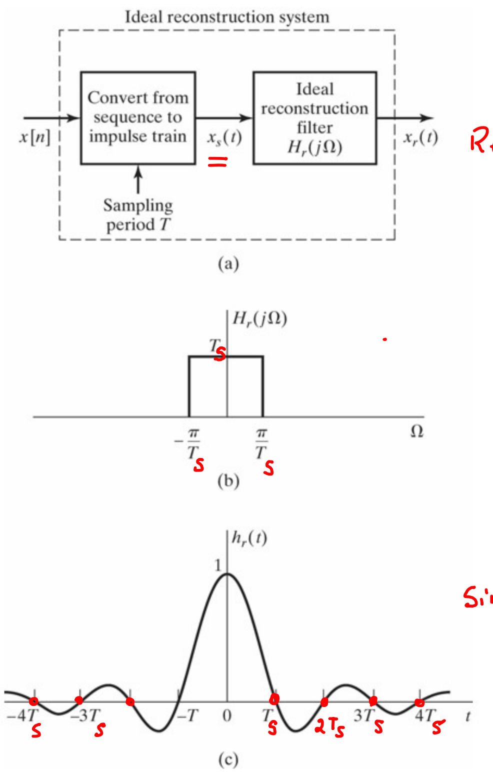
CT signal non-zero for all t

CT signal non-zero only @ nT_s

Recons. filter







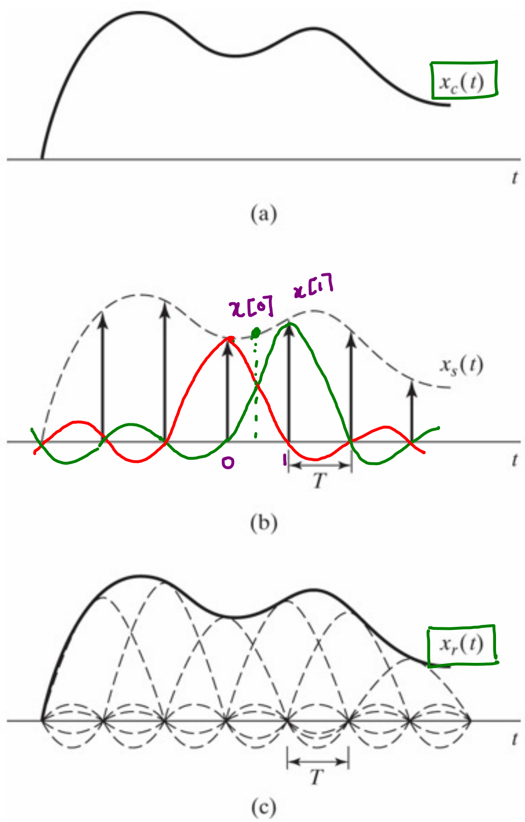
Reconstructed signal

$\mathrm{Sinc}\left(\frac{t}{T_s}\right)$

impulse response of reconstruction filter

$$h_r(t) = \frac{1}{2\pi}\int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} T_s\, e^{j\Omega t}\, d\Omega = \frac{\sin \frac{\pi t}{T_s}}{\frac{\pi t}{T_s}}$$

$$\boxed{x_r(t) = x_s(t) * h_r(t)} \longleftrightarrow X_s(j\Omega)\, H_r(j\Omega)$$



Dirac Delta functions

Reconstruction

impulse response $h_r(t)$

$$x_s(t) \longrightarrow \boxed{\text{LTI Reconstruct.}} \longrightarrow x_r(t) = x_s(t) * h_r(t)$$

$$h_r(t) = \frac{\sin \frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} = \text{sinc}\left(\frac{t}{T_s}\right)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t - nT_s)$$

Scaled Dirac Delta func

$$x_r(t) = x_s(t) * h_r(t)$$

$$= \underbrace{\left[\sum_{n=-\infty}^{\infty} x[n]\,\delta(t - nT_s)\right]}_{x_s(t)} * h_r(t)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,h_r(t - nT_s)$$

$$h_r(t) = \frac{\sin \frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} \quad (1)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\, h_r(t - nT_s) \quad (2)$$

Combining (1) & (2)

In eqn (1)

substitute $t \leftarrow t - nT_s$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \pi\left(\frac{t-nT_s}{T_s}\right)}{\pi\left(\frac{t-nT_s}{T_s}\right)}}$$

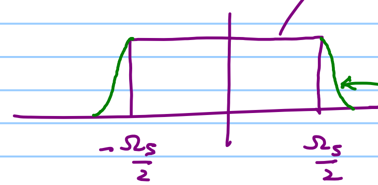
Scale factor

Shifted versions of $h_r(t)$

Oversampling $\Omega_s > \text{Nyquist Rate}$

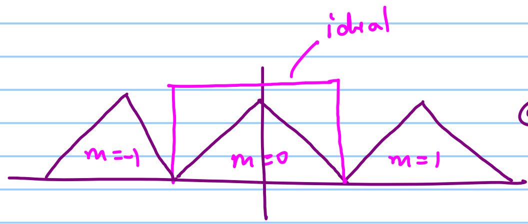
$\Omega_s = \text{Nyquist Rate}$

ideal filter



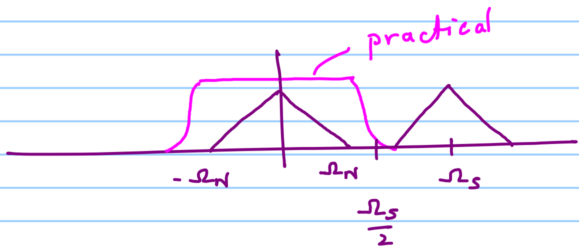
ideal filter
(Not realizeable)

realizeable
practical filter



@ Nyquist rate

need ideal reconstruction filter



> Nyquist rate

$\frac{\Omega_s}{2} > \Omega_N \Rightarrow \Omega_s > 2\Omega_N$

$\Omega_s > \text{Nyquist Rate}$

Compact Disc (CD)

Audio signal 0 – 20 KHz

Nyquist rate 40 KHz

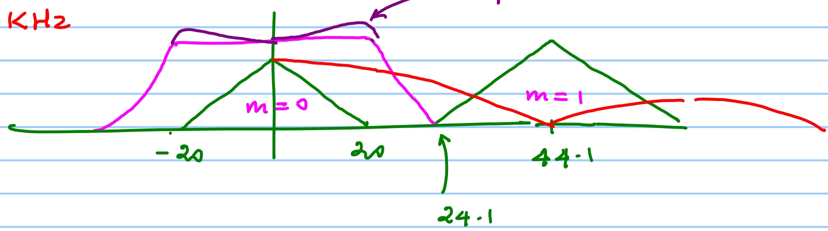
Sampling rate 44.1 KHz

Playback

DT ⟶ CT

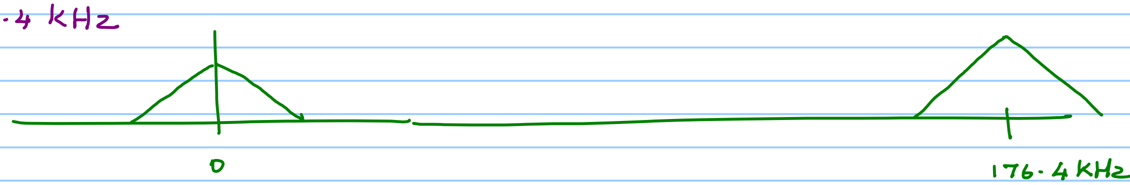
Reconstruction

compensation for the ZOH



Reconstruction

44.1 KHz ⟶ 176.4 KHz

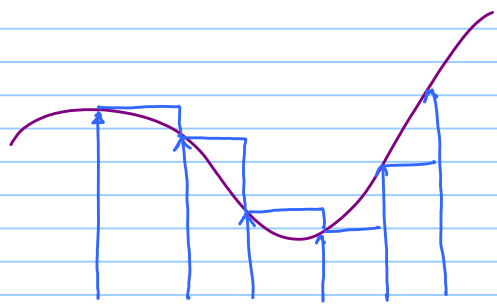


Ideal reconstruction process

$x[n] \longrightarrow x_s(t) \longrightarrow x_r(t)$

~~Dirac Delta~~

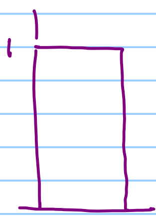
Zero Order Hold ~ Staircase approx



Old

Dirac Delta

1

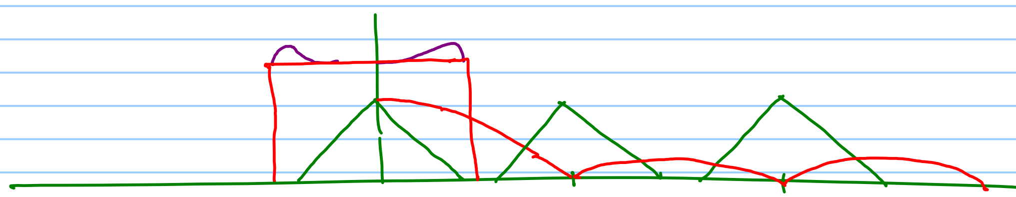


0 T_s

New

$$h_0(t) \longleftrightarrow H_0(j\Omega) = e^{-j\Omega \frac{T_s}{2}} \; \frac{2\sin\frac{\Omega T_s}{2}}{\Omega}$$

Spectrum $X_s(j\Omega) \cdot H_0(j\Omega)$



$\frac{2\pi}{T_s} = \Omega_s$ $\quad$ $\frac{4\pi}{T_s} = 2\Omega_s$

OKS

Ex 4.1

$\Omega_0 = 4000\pi = 2\pi f_0 \qquad f_0 = 2000\ Hz$

$x_c(t) = \cos 4000\pi t = \frac{1}{2}\left[e^{j4000\pi t} + e^{-j4000\pi t}\right]$

Nyquist rate $F_s \geq \cancel{4000}\ Hz$

$6000\ Hz$

$X_c(j\Omega) = \pi\left[\delta(\Omega - 4000\pi) + \delta(\Omega + 4000\pi)\right]$

Sample $\boxed{\Omega_s = 12000\pi}$

$T_s = \frac{1}{6000}$ sec

$x_c(t) = \cos 4000\pi t$

$x[n] = x_c(t)\Big|_{t = nT_s} = \cos \frac{4000\pi n}{6000} = \cos \frac{2\pi}{3} n \qquad \omega_0 = \frac{2\pi}{3}$

Sampling process

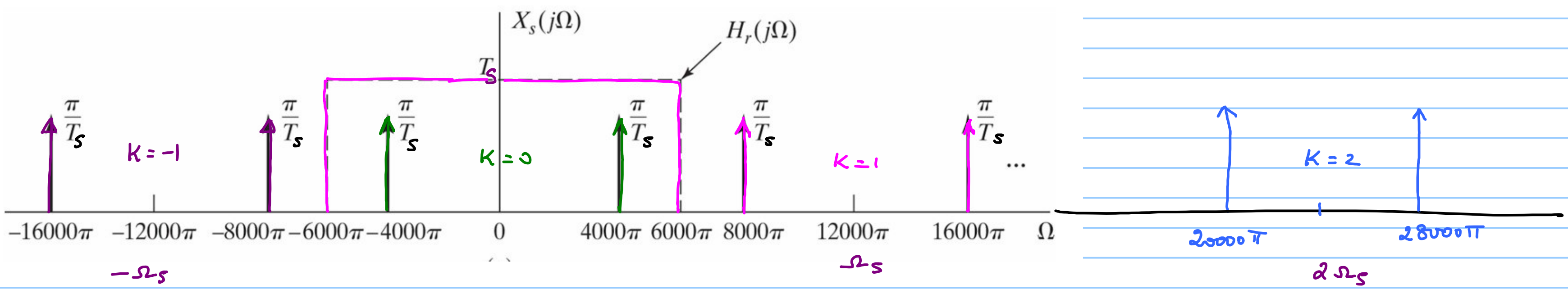
$X_c(j\Omega) = \pi\left[\delta(\Omega - 4000\pi) + \delta(\Omega + 4000\pi)\right]$

$S(j\Omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\Omega - \Omega_s k)$

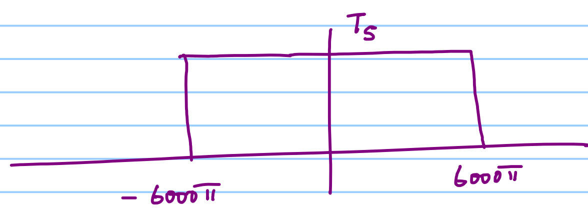
$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) = \frac{\pi}{T_s}\left[\sum_{k=-\infty}^{\infty}\delta(\Omega - 4000\pi - \Omega_s k) + \delta(\Omega + 4000\pi - \Omega_s k)\right]$

$\Omega_s = 12000\pi$

	$\delta(\Omega - 4000\pi - k\Omega_s)$	$\delta(\Omega + 4000\pi - k\Omega_s)$
$k=0$	4000π	-4000π
$k=1$	16000π	8000π
$k=2$	28000π	20000π
$k=-1$	-8000π	-16000π



$X_s(j\Omega)$
$H_r(j\Omega)$
T_s
$\frac{\pi}{T_s}$
K = -1
K = 0
K = 1
K = 2
−16000π
−12000π
−8000π
−6000π
−4000π
0
4000π
6000π
8000π
12000π
16000π
Ω
$-\Omega_s$
Ω_s
20000π
28000π
$2\Omega_s$



T_s
− 6000π
6000π