Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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September – December 2024

Session #9

October 7, 2024

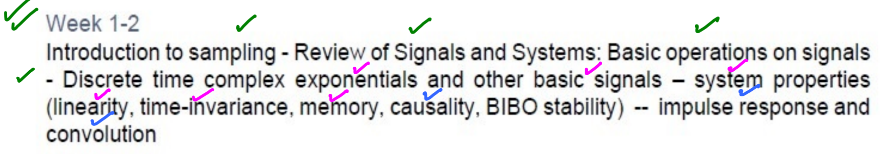
EE3101 Digital Signal Processing

EE3101
Session 8

Session 9

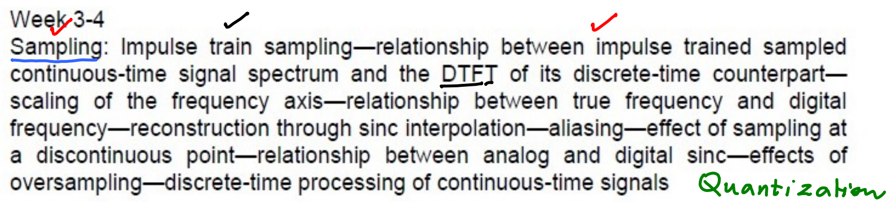
Outline

Last session
- Sampling
- Reconstruction
- Oversampling



Week 1-2

Introduction to sampling - Review of Signals and Systems: Basic operations on signals - Discrete time complex exponentials and other basic signals – system properties (linearity, time-invariance, memory, causality, BIBO stability) -- impulse response and convolution



Week 3-4

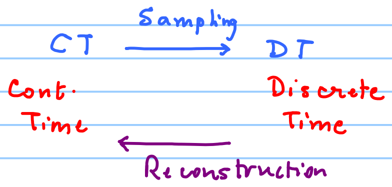
Sampling: Impulse train sampling—relationship between impulse trained sampled continuous-time signal spectrum and the DTFT of its discrete-time counterpart—scaling of the frequency axis—relationship between true frequency and digital frequency—reconstruction through sinc interpolation—aliasing—effect of sampling at a discontinuous point—relationship between analog and digital sinc—effects of oversampling—discrete-time processing of continuous-time signals

Quantization

Reading Assignment

O&S Chapter 4 : Sampling of CT signals

O&S ch 2 Sec 7 : Representation of Sequences by Fourier Transforms



Week 5-6 O&S ch 2 Section 7 - 9

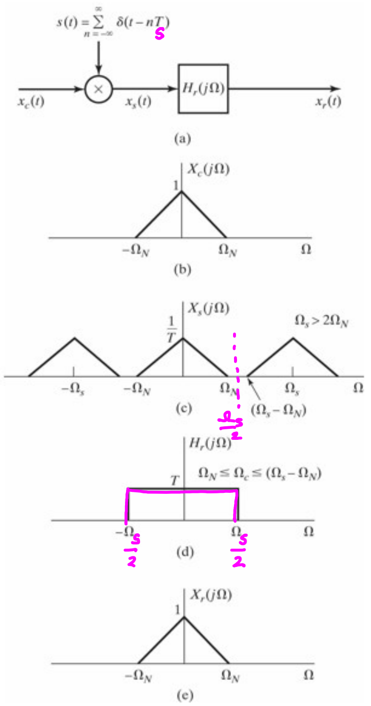
Discrete-Time Fourier Transform (DTFT): Complex exponentials as eigensignals of LTI systems—DTFT definition—inversion formula—properties—relationship to continuous-time Fourier series (CTFS)

O&S Ch 2 Sec 7 Representation of Sequences by Fourier Transforms

Ch 2 Sec 8 Symmetry Properties of Fourier Transform

Ch 2 Sec 9 Fourier Transform Theorems

Properties & Applications



Nyquist Sampling Theorem

Let $x_c(t)$ be a band-limited signal with $X_c(j\Omega) = 0$ $|\Omega| \geq \Omega_N$ rads/sec

$x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT_s)$ $n = 0, \pm 1, \pm 2, \ldots$

if $\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_N$

Nyquit rate $= 2\Omega_N$ $\Omega_s =$ rads/sec

$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \xleftrightarrow{\mathcal{F}} S(j\Omega) = \boxed{\frac{2\pi}{T_s}} \sum_{m=-\infty}^{\infty} \delta(\Omega - m\Omega_s)$

$$\boxed{\begin{aligned}\Omega_s &= 2\pi F_s \\ &= \frac{2\pi}{T_s}\end{aligned}}$$

$x_s(t) = x_c(t) \cdot s(t) \longleftrightarrow \underline{X_s(j\Omega)} = \underline{\underline{\frac{1}{T_s}}} \sum_{m=-\infty}^{\infty} X_c(j(\Omega - m\Omega_s))$

Do not want copies to overlap @ $m=0$, $m=1$

$\Omega_N \leq \Omega_s - \Omega_N$ $\quad \boxed{2\Omega_N \leq \Omega_s}$

Nyquist Theorem

$\boxed{\Omega_s \geq 2\Omega_N}$

Ideal Reconstruction Filter

$\equiv$ Lowpass filter with cutoff $\Omega_c = \frac{\Omega_s}{2}$.

$X_s(j\Omega)$ Spectrum of the sampled signal

Reconstructed signal $x_r(t) \longleftarrow \quad \longleftarrow$

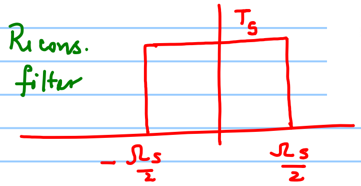
$x_c(t) \longrightarrow x_s(t) \longrightarrow x[n]$

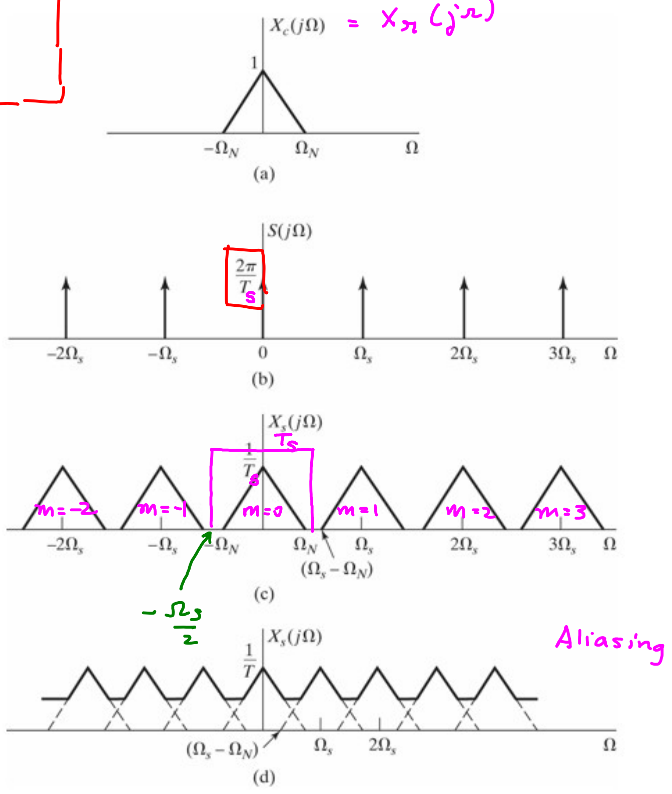
① Remove unwanted copies $m = \pm 1, \pm 2$

② Scale factor T_s

CT signal non-zero for all t

CT signal non-zero only @ nT_s

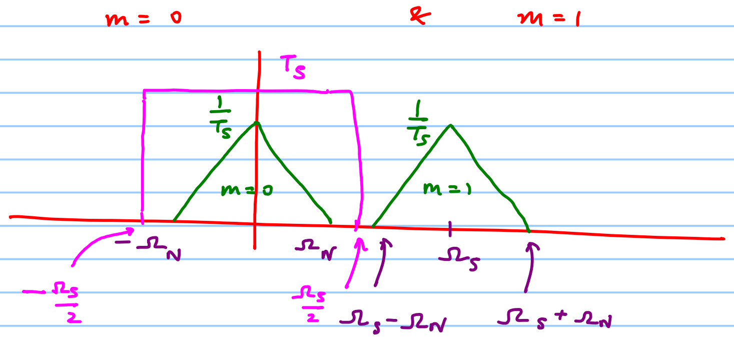




$$X_s(j\Omega) = \underbrace{\frac{1}{T_s}}_{\text{scale factor}} \sum_{m=-\infty}^{\infty} X_c(j(\Omega - m\Omega_s))$$

many copies

shifted by multiples of Ω_s



No overlap of spectra $\Omega_s - \Omega_N \geq \Omega_N$

$$\boxed{\Omega_s \geq 2\Omega_N}$$

Special case

Nyquist rate $\Omega_s = 2\Omega_N$

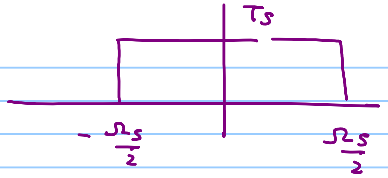
Oversampled $\Omega_s > 2\Omega_N$

undersampled $\Omega_s < \Omega_N$ ↔ aliasing

$$h_r(t) = \mathrm{sinc}\left(\frac{t}{T_s}\right)$$

$$\begin{cases} = 1 & @ \; t = 0 \\ = 0 & @ \; t = nT_s \quad n = \pm 1, \pm 2 \dots \end{cases}$$

→ important property for reconstruction



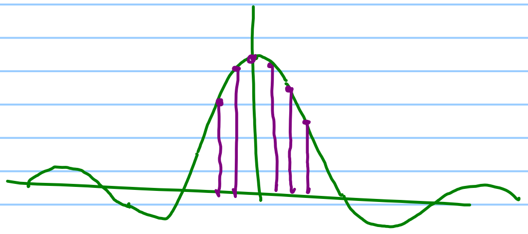
$h_r(t) \longleftrightarrow H_r(j\Omega)$

Is $h_r(t)$ is BL ? Yes.

↑

Can apply Nyquist sampling

$$h_r[n] = h_r(t)\Big|_{t = n\frac{T_s}{8}} = \left.\frac{\sin\frac{\pi t}{T_s}}{\frac{\pi t}{T_s}}\right|_{t = n\frac{T_s}{8}} = \frac{\sin\frac{\pi n}{8}}{\frac{\pi n}{8}}$$



Oversampled system

Factor of 8

$$h_r[n] = \frac{\sin\frac{\pi n}{8}}{\frac{\pi n}{8}} \qquad -\infty < n < \infty$$

non-causal.

∞ duration

Is $h_r[n]$ is BIBO stable ?

$$\sum_{n=-\infty}^{\infty} |h_r[n]| < \infty \text{ (finite)}$$

$$h_r[n] = \frac{\sin\left(\frac{\pi n}{8}\right)}{\frac{\pi n}{8}}$$

$h_r[0]$ L'Hospital's rule

$$\lim_{n \to 0} \frac{\sin \frac{\pi n}{8}}{\frac{\pi n}{8}} = \lim_{n \to 0} \frac{\cancel{\frac{\pi}{8}} \cos \frac{\pi n}{8}}{\cancel{\frac{\pi}{8}}} = 1$$

$= 1 \quad n = 0$

$$\sum_{n=-\infty}^{\infty} |h_r[n]| = 1 + 2 \sum_{n=1}^{\infty} \left| \frac{\sin \frac{\pi n}{8}}{\frac{\pi n}{8}} \right| \leq 1 + 2 \sum_{n=1}^{\infty} \left| \frac{1}{\frac{\pi n}{8}} \right|$$

$\left|\sin \frac{\pi n}{8}\right| \leq 1$

$$\leq 1 + 2 \cdot \frac{8}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$$

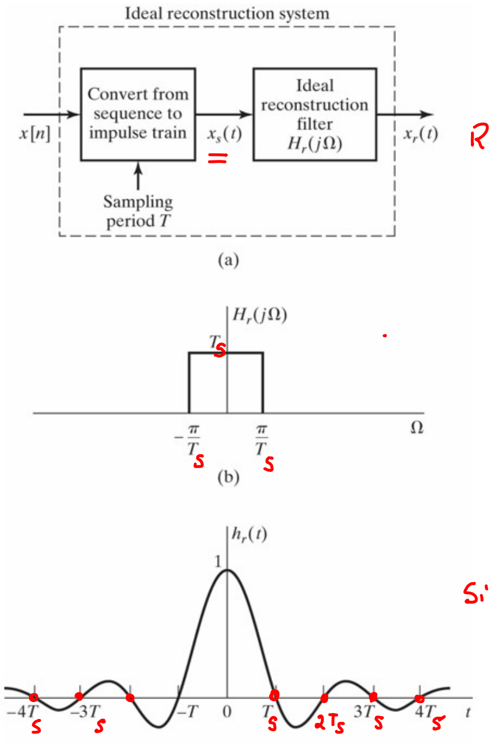
Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{> \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{> \frac{1}{2}} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

does not converge



(a) (b) (c)

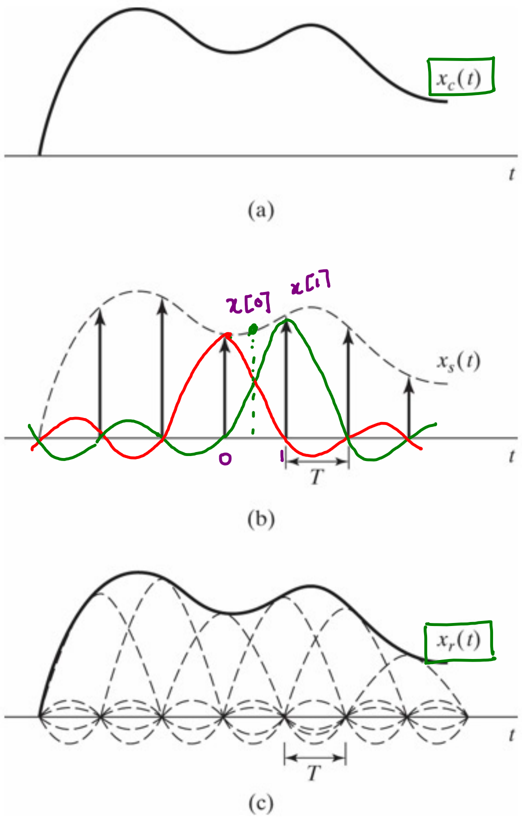
Reconstructed signal

$\text{Sinc}\left(\frac{t}{T_s}\right)$

impulse response of reconstruction filter

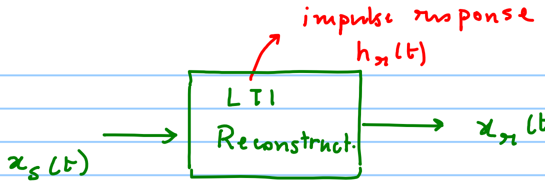
$$h_r(t) = \frac{1}{2\pi}\int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} T_s \, e^{j\Omega t} \, d\Omega = \frac{\sin\frac{\pi t}{T_s}}{\frac{\pi t}{T_s}}$$

$$\boxed{x_r(t) = x_s(t) * h_r(t)} \longleftrightarrow X_s(j\Omega)\, H_r(j\Omega)$$



Dirac Delta functions

Reconstruction



$$x_r(t) = x_s(t) * h_r(t)$$

$$h_r(t) = \frac{\sin \frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} = \text{sinc}\left(\frac{t}{T_s}\right)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\, \delta(t-nT_s)$$

Scaled Dirac Delta func

$$x_r(t) = x_s(t) * h_r(t)$$

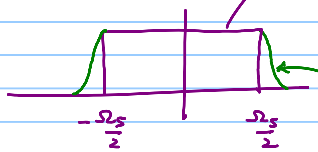
$$= \underbrace{\left[\sum_{n=-\infty}^{\infty} x[n]\, \delta(t-nT_s)\right]}_{x_s(t)} * h_r(t)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\, h_r(t-nT_s)$$

Oversampling

Ω_s > Nyquist Rate

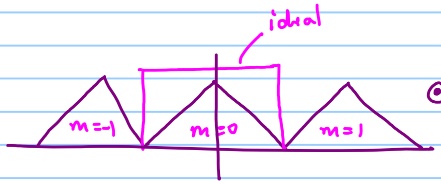
Ω_s = Nyquist Rate



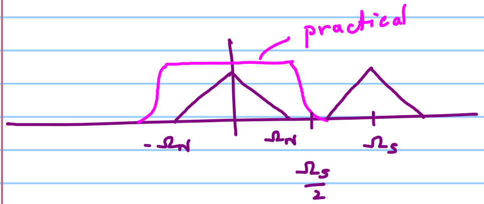
ideal filter

ideal filter
(Not realizeable)

realizeable
practical filter



@ Nyquist rate
need ideal reconstruction filter

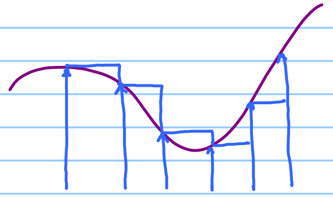


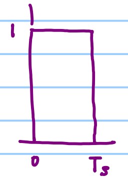
> Nyquist rate

$\frac{\Omega_s}{2} > \Omega_N \Rightarrow \Omega_s > 2\Omega_N$

Ω_s > Nyquist Rate

Practical Reconstruction

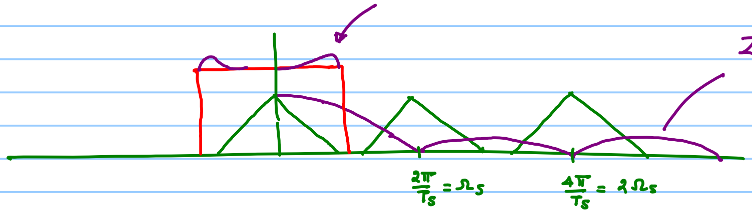




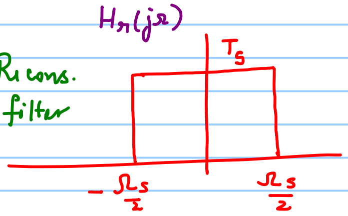
New

$h_0(t) \longleftrightarrow H_0(j\Omega) = e^{-j\Omega\frac{T_s}{2}} \frac{2\sin\frac{\Omega T_s}{2}}{\Omega}$

Compensation in Reconstruction filter



Recons. filter



$$h_r(t) = \text{Sinc}\left(\frac{t}{T_s}\right)$$

$$H_0(j\Omega) = e^{-j\Omega\frac{T_s}{2}} \frac{2\sin\frac{\Omega T_s}{2}}{\Omega}$$

Zero Order Hold

Modified Reconstruction Filter

$$H_r'(j\Omega) = \frac{H_r(j\Omega) \rightarrow \text{original}}{H_0(j\Omega) \rightarrow \text{ZoH}}$$

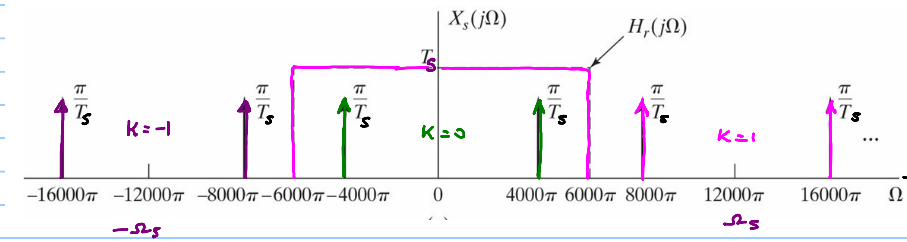
Ok5
4.1

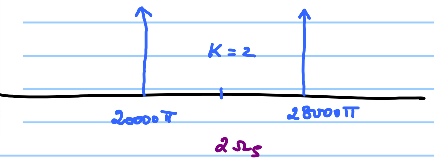
$x_c(t) = \cos 4000\pi t$ Sample $\boxed{\Omega_s = 12000\pi}$ $\Rightarrow T_s = \frac{1}{6000}$

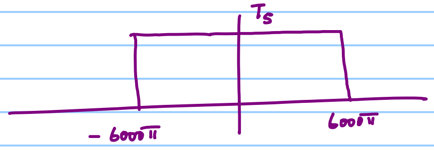
$\Omega_s = \frac{2\pi}{T_s}$

$\delta(\Omega - 4000\pi - k\Omega_s)$ $\delta(\Omega + 4000\pi - k\Omega_s)$

$k=0$	4000π	-4000π
$k=1$	16000π	8000π
$k=2$	28000π	20000π
$k=-1$	-8000π	-16000π







$x[n] = x_c(t)\Big|_{t = nT_s}$

$= \cos 4000\pi n T_s \Big|_{T_s = \frac{1}{6000}}$

$= \cos \frac{4000\pi n}{6000}$

$\boxed{x[n] = \cos \frac{2\pi}{3} n}$

Part (b)

$x_c(t) = \cos 4000\pi t$

max freq = 2000 Hz

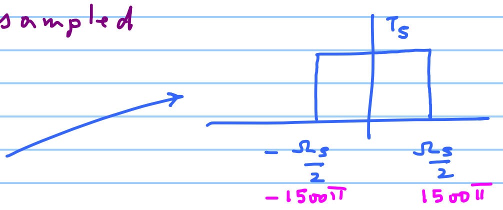
Nyquist rate = 8000π

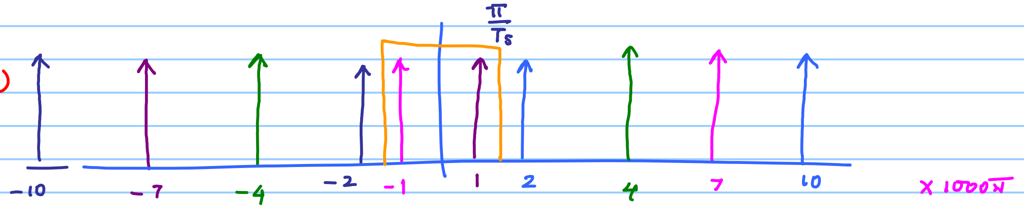
$\Omega_s = 12000\pi$ (part (a)) → oversampled

$X_c(j\Omega)$

	$\delta(\Omega - 4000\pi - k\Omega_s)$	$\delta(\Omega + 4000\pi - k\Omega_s)$
$k=0$	4000π	-4000π
$k=1$	7000π	-1000π
$k=2$	10000π	2000π
$k=-1$	1000π	-7000π
$k=-2$	-2000π	-10000π

$\Omega_s = 3000\pi$





$x_r'(t) = \cos 1000\pi t$

$x_c(t) = \cos 1000\pi t \Big|_{t = nT_s} \quad T_s = \frac{1}{1500}$

$x[n] = \cos \frac{1000\pi n}{1500} = \cos \frac{2\pi}{3} n$

① Below Nyquist rate → aliasing

② W. aliasing ⟶ will not get the original signal via reconstruction

③ In reconstruction process, T_s plays a very important role

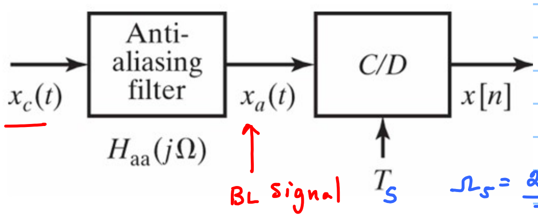
④ Same DT signal

$$\begin{tikzcd} & \cos 4000\pi t \quad T_s = \frac{1}{6000} \\ x[n] = \cos \frac{2\pi}{3} n \arrow[ur] \arrow[dr] & \\ & \cos 1000\pi t \quad T_s = \frac{1}{1500} \end{tikzcd}$$

Quantization

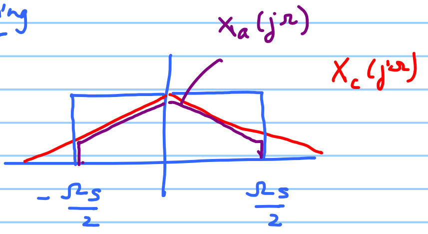
$$x_c(t) \xrightarrow{\text{Samp.}} x[n] \xrightarrow{\text{Quant}} \hat{x}[n]$$

$x[n]$	$\hat{x}[n]$
Disc. in time	Disc. in time
Cont in amplitude (∞ bits)	Disc in amplitude (finite # bits)



$\Omega_s = \frac{2\pi}{T_s}$

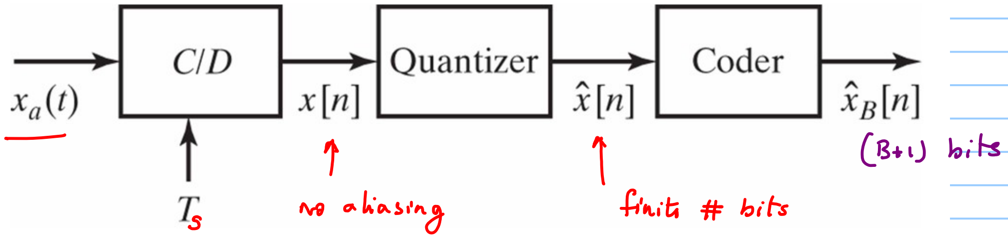
No aliasing



B+1 bit representation

fractional

sign



2's complement representation

X - axis   input DT signal $x[n]$

max value $[-X_m, X_m]$

Total range $2X_m$

Uniform Quantizer via rounding

3 Bits $\underbrace{a_0}_{\text{sign}} | \underbrace{a_1 \; a_2}_{\text{fraction}}$

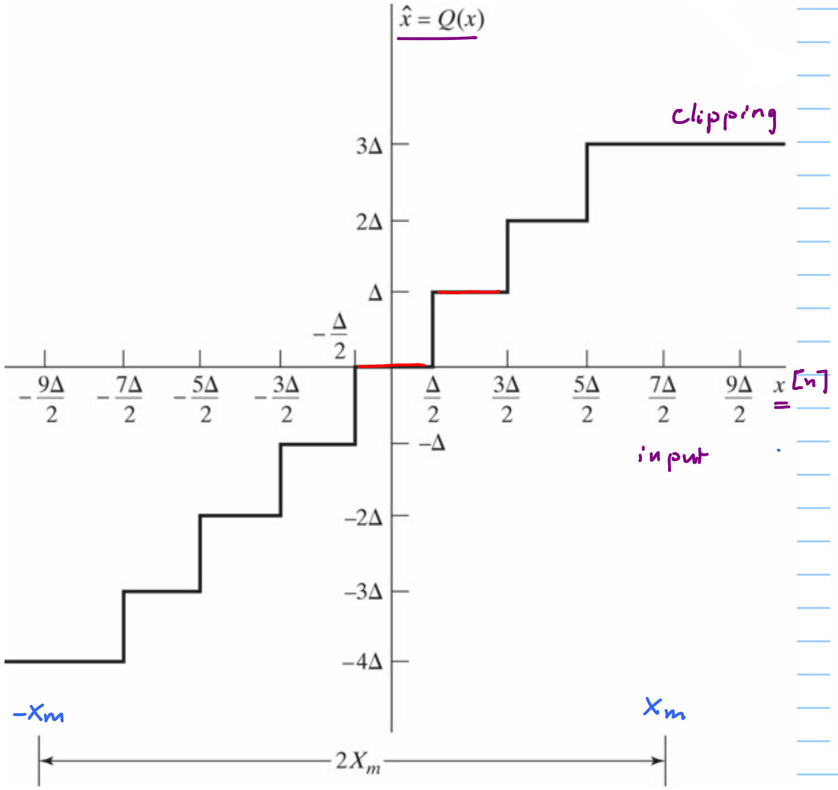
value

$$-a_0 2^0 + a_1 2^{-1} + a_2 \cdot 2^{-2} + \cdots$$

Output

$-\frac{\Delta}{2} \le x[n] < \frac{\Delta}{2}$ 000

$\frac{\Delta}{2} \le x[n] < \frac{3\Delta}{2}$ 001

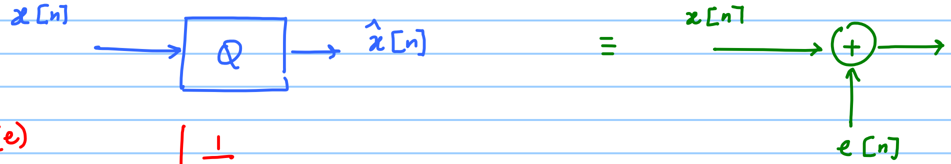


$B+1 = 3$

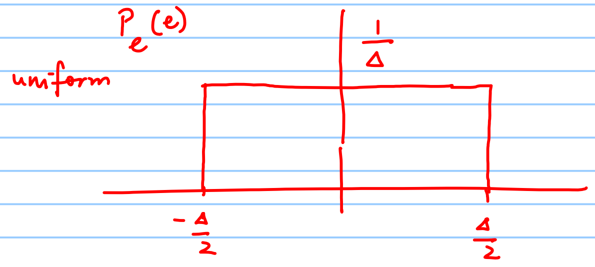
sign

Two's-complement code	
011	3/4
010	1/2
001	1/4
000	0
111	$-\frac{1}{4}$
110	$-\frac{1}{2}$
101	$-\frac{3}{4}$
100	-1

Stepsize $\Delta = \frac{\cancel{2}X_m}{2^{B+\cancel{1}}} = \frac{X_m}{2^B}$



$\hat{x}[n] = x[n] + e[n]$ — max $\frac{\Delta}{2}$, min $\frac{\Delta}{2}$



Characteristics of e[n]

① e[n] is a sample seq of a Wide Sense Stationary (WSS) Random Process

② e[n] error seq is uncorrelated with x[n]

③ e[n] is a white noise random noise

④ PDF of e[n] $P_e(e)$ is uniform

① $$\mu_e = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e\, P_e(e)\, de$$

$$= \frac{1}{\Delta}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e\, de = \frac{1}{\Delta}\left.\frac{e^2}{2}\right|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = 0$$

② $\sigma_e^2 = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 P_e(e)\, de = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 de = \frac{1}{\Delta} \frac{e^3}{3} \Big|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{\Delta^2}{12}$

$\Delta = \frac{X_m}{2^B} = 2^{-B} X_m$

$\boxed{\sigma_e^2 = 2^{-2B} \frac{X_m^2}{12}}$ ①

$\boxed{SQNR = 10 \log_{10} \frac{\sigma_x^2}{\sigma_e^2}}$ ②

Signal to Quantiz. Noise Ratio

Substituting ① in ② $SQNR = 10 \log_{10} 12 \cdot 2^{2B} \left(\frac{\sigma_x^2}{X_m^2} \right)$

σ_x^2 = energy of Input signal

X_m = Range of input signal

-6dB/bit Quantization

Rule-of-Thumb for Quantiz

Typically $\sigma_x = \frac{X_m}{4}$ ③

$SQNR = 10 \log_{10} \left(2^{2B}\right) + 10 \log_{10} (12) - 20 \log_{10} \left(\frac{X_m}{\sigma_x} \right)$ ④

Substitute ③ in ④ $\boxed{SQNR = \left[6.02B - 1.25 \right] dB}$

$B = 16$ bits $SQNR = 96 - 1.25$

$= 94.75$ dB

Discrete Time Fourier Transform

DT seq $x[n] \xrightarrow[\text{DTFT}]{\mathcal{F}} X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$

Continuous function of w

w is periodic function with period $= 2\pi$

$X(e^{jw})$ is also periodic function

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw \xleftarrow[\text{Inv DTFT}]{\mathcal{F}^{-1}} X(e^{jw})$$

$$\begin{tikzcd} x_c(t) \arrow[r] \arrow[d] & x_s(t) \\ & X_s(jw) \arrow[d, leftrightarrow] \\ x[n] \arrow[r] & X(e^{jw}) \end{tikzcd}$$

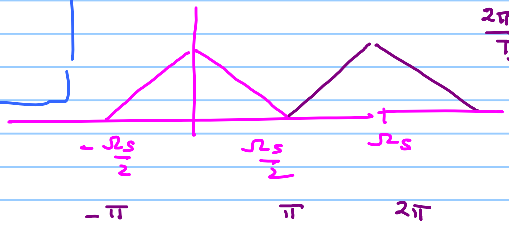
$X_s(jw)$ periodic

period $= \Omega_s$

$\Omega_s T_s = \omega_s$

$\frac{2\pi}{T_s} \cdot T_s = w_s$

$X(e^{jw})$



$x[n]$ can be real valued or complex-valued

$X(e^{jw})$ is complex valued (in general)

Fourier Transform

or

Fourier Spectrum

or

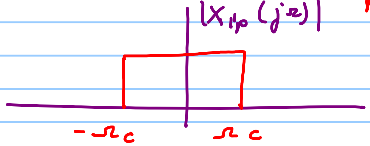
DTFT

$$X(e^{jw}) = X_R(e^{jw}) + j X_I(e^{jw})$$

$$= \underbrace{|X(e^{jw})|}_{\text{Magnitude Spectrum}}\, e^{j \underbrace{\arg\{X(e^{jw})\}}_{\text{phase spectrum}}}$$

Low pass filter

Magnitude Spectrum



LTI system $\longleftrightarrow$ Impulse Response $\longleftrightarrow$ DTFT

$h[n]$ $\qquad$ $H(e^{jw})$

example

$$h[n] = \delta[n-2] \longleftrightarrow H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-jwn}$$

$$y[n] = x[n] * h[n] = x[n-2]$$

$$H(e^{jw}) = e^{-j2w}$$

$$|H(e^{jw})| = 1 \quad \forall w \qquad \Big| \qquad \arg(H(e^{jw})) = -2w$$

Proof of Inv. DTFT

Given $$X(e^{jw}) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-jwk}$$

Suppose $$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underline{X(e^{jw})}\, e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-jwk}\right) e^{jwn}\, dw = \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{+jw(n-k)}\, dw}_{\begin{array}{ll} =1 & k=n \\ 0 & k\neq n \end{array}}$$

$$\hat{x}[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k] = x[n]$$

$$\boxed{\hat{x}[n] = x[n]}$$

DTFT pair

$$\boxed{\begin{aligned} X(e^{jw}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} && \text{DTFT (forward)} \\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw && \text{Inv DTFT (Inverse)} \end{aligned}}$$