Note Title 02-09-2024

Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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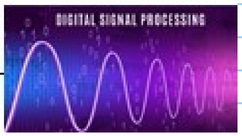
Session # 24

December 6, 2024



Note Title 03-01-2018

Session 23

Outline

Last session

- Properties of DFS
- Intro to DFT

Week 11-12 (DFT & FFT)

Discrete Fourier Transform (DFT): Definition of the DFT and inverse DFT—relationship to discrete-time Fourier series—matrix representation—DFT as the samples of the DTFT and the implied periodicity of the time-domain signal—circular convolution and its relationship with linear convolution—effect of zero padding—introduction to the Fast Fourier Transform (FFT) algorithm—decimation-in-time

Today

- DFT
- Properties
- Examples

Reading Assignment

O&S ch 8 Discrete Fourier Transform

O&S ch 8

✓ 8.1 Discrete Fourier Series (DFS) or (DTFS)

✓ 8.2 Properties of DFS

✓ 8.3 Fourier Transform of periodic signals

✓ 8.4 Sampling the FT

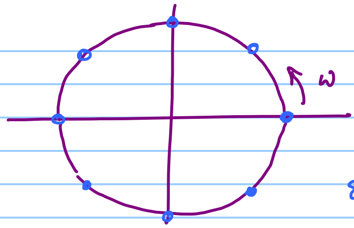
8.5 Discrete Fourier Transform (DFT) $\Rightarrow$ Fourier Representation of Finite Duration Sequences

8.6 Properties of DFT

Application of DFT $\longrightarrow$ Convolution

Using DFT we obtain periodic convolution

8.7 Computing Linear Convolution using the DFT



8 samples of the DTFT

DT Fourier Series

Inverse Synthesis Equation $\quad \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\frac{2\pi}{N}kn}$ (Inverse) $\quad *\ \tilde{x}[n]$ periodic with period = N

Forward Analysis Equation $\quad \tilde{X}[\ell] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\frac{2\pi}{N}\ell n}$ (Forward)

Viewing the DFS as an orthogonal transform

$$\underline{\tilde{X}} = \underbrace{\begin{bmatrix} \tilde{X}[0] \\ \tilde{X}[1] \\ \vdots \\ \tilde{X}[N-1] \end{bmatrix}}_{N\times 1 \text{ vector}} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 \cdots & W_N^{2(N-1)} \\ & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{\substack{N\times N \\ \underline{D_N}}} \underbrace{\begin{bmatrix} \tilde{x}[0] \\ \tilde{x}[1] \\ \vdots \\ \tilde{x}[N-1] \end{bmatrix}}_{N\times 1 \text{ vector}} \quad \underline{\tilde{x}}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\boxed{\begin{array}{ll} \text{Analysis eqn.} & \underline{\tilde{X}} = \underline{D_N} \cdot \underline{\tilde{x}} \\ \text{Synthesis eqn} & \underline{\tilde{x}} = \underline{D_N}^{-1}\, \underline{\tilde{X}} \end{array}}$$

$$\underline{D_N}^{-1} = \frac{1}{N} \underline{D_N}^{*}$$

Properties of DFS

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$$

Shift in time $\quad \tilde{x}[n-m] \longleftrightarrow e^{-j\frac{2\pi}{N}km}\,\tilde{X}[k] = W_N^{mk}\,\tilde{X}[k]$

Duality $\quad \tilde{X}[n] \longleftrightarrow N\,\tilde{x}[-n]$

Shift in freq $\quad W_N^{-ln}\,\tilde{x}[n] \longleftrightarrow \tilde{X}[k-l]$

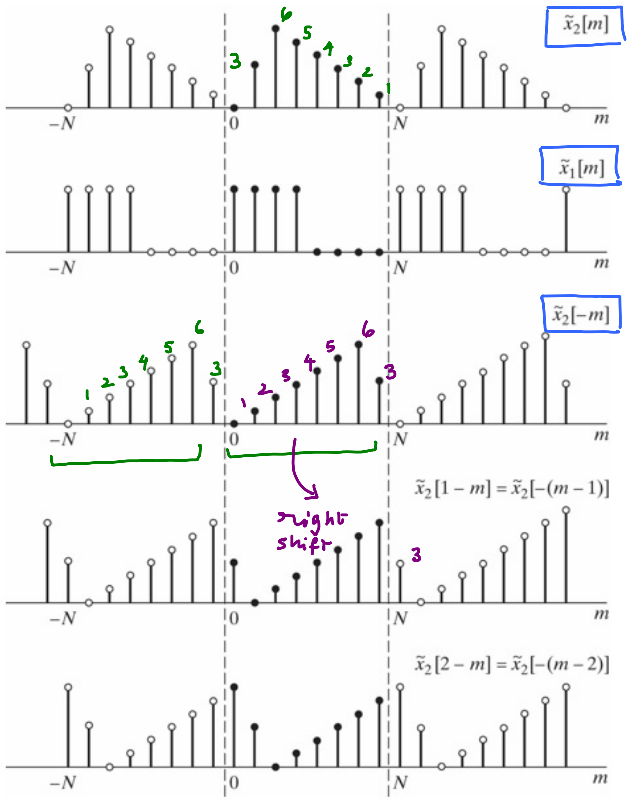
Periodic conv in time $\quad \tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\,\tilde{x}_2[n-m] \longleftrightarrow \tilde{X}_1[k]\,\tilde{X}_2[k]$

Periodic conv in freq $\quad \tilde{x}_3[n] = \tilde{x}_1[n]\,\tilde{x}_2[n] \longleftrightarrow \tilde{X}_3[k] = \frac{1}{N}\sum_{m=0}^{N-1} \tilde{X}_1[m]\,\tilde{X}_2[k-m]$

TABLE 8.1 SUMMARY OF PROPERTIES OF THE DFS

	Periodic Sequence (Period N)	DFS Coefficients (Period N)
Linearity	1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
	2. $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
	3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
Duality	4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
Shift in time	5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
	6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$ shift in freq
	7. $\underbrace{\sum_{m=0}^{N-1} \overbrace{\tilde{x}_1[m]}^{N}\overbrace{\tilde{x}_2[n-m]}^{N}}_{N}$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
	8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
	9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
	10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$

Periodic Convolution



Time - reversal

$N = 8$

$\tilde{x}_1[n] \circledast_N \tilde{x}_2[n]$

$N = 8$

$$\tilde{x}_3[n] = \sum_{m=0}^{7} \tilde{x}_1[m]\, \tilde{x}_2[n-m]$$

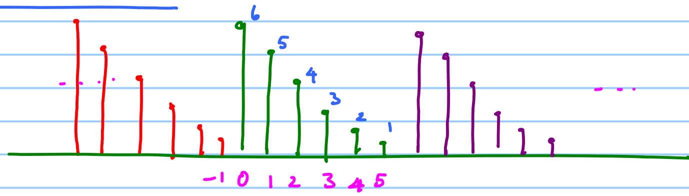
time-reversal & shifting

$$\tilde{x}_3[0] = \sum_{m=0}^{7} \tilde{x}_1[m]\, \tilde{x}_2[-m]$$

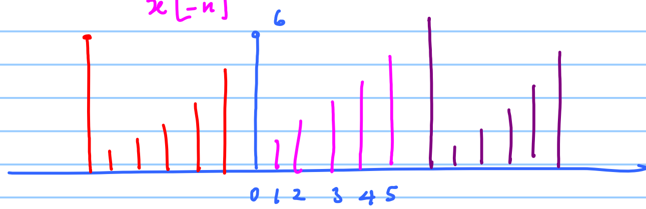
$$\tilde{x}_3[1] = \sum_{m=0}^{7} \tilde{x}_1[m]\, \tilde{x}_2[1-m]$$

Time Reversal

$\tilde{x}[n]$ $\quad N = 6$



$\tilde{x}[-n]$



$$\tilde{x}[n] = \{ \cdots 6, 5, 4, 3, 2, 1 \,|\, 6, 5, 4, 3, 2, 1 \,|\, 6, 5, 4, 3, 2, 1 \,|\, \cdots \}$$

$$|\, 6, 1, 2, 3, 4, 5 \,|\, 6, 1, 2, 3, 4, 5 \,|$$

Circular / Periodic Convolution

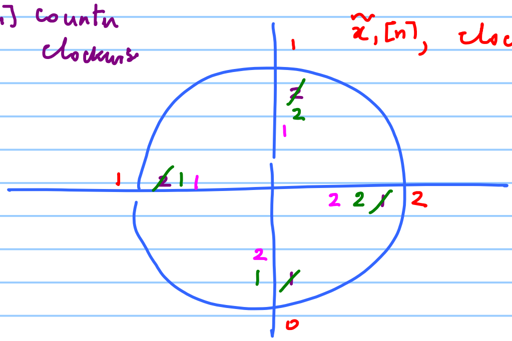
$\tilde{x}_1[n] = \{ \ldots \mid 1\ 2\ 0\ 1 \mid 1\ 2\ 0\ 1 \mid 1\ 2\ 0\ 1 \mid \ldots \}$

N = 4

$\tilde{x}_2[n] = \{ \ldots \mid 2\ 2\ 1\ 1 \mid 2\ 2\ 1\ 1 \mid 2\ 2\ 1\ 1 \mid \ldots \}$

$\tilde{x}_2[n]$ counter clockwise

$\tilde{x}_1[n]$, clockwise



$\tilde{x}_3[0] = 2 + 2 + 0 + 2 = 6$

$\tilde{x}_3[1] = 2 + 4 + 0 + 1 = 7$

$\tilde{x}_3[2] = 1 + 4 + 0 + 1 = 6$

$\tilde{x}_3[3] = 5$ Verify

$\tilde{x}_3[n] = \{6, 7, 6, 5\}$

Symmetry Properties (Verify)

11. $\mathcal{R}e\{\tilde{x}[n]\}$ — $\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{I}m\{\tilde{x}[n]\}$ — $\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$ — $\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$ — $j\mathcal{I}m\{\tilde{X}[k]\}$

Properties 15–17 apply only when $x[n]$ is real.

15. Symmetry properties for $\tilde{x}[n]$ real.

$$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ |\tilde{X}[k]| = |\tilde{X}[-k]| \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$$

16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$ — $\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$ — $j\mathcal{I}m\{\tilde{X}[k]\}$

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$$

$$\tilde{x}^*[n] \longleftrightarrow \tilde{X}^*[-k]$$

$$\tilde{x}^*[-n] \longleftrightarrow \tilde{X}^*[k]$$

$$\tilde{x}[N-n] = \tilde{x}[-n] \longleftrightarrow \tilde{X}[-k] = \tilde{X}[N-k]$$

Linear Convolution

$x_1[n] = \{1, 2, 0, 1\}$

$x_2[n] = \{2, 2, 1, 1\}$

Linear convolution

$\{1, 2, 0, 1\} * \{2, 2, 1, 1\} \longrightarrow y[n] = \{2, 6, 5, 5, 4, 1, 1\}$

$N = 4 \quad N = 4 \quad N = 7$

$$1 \quad 1 \quad 2 \quad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

Circular / Periodic convolution

$x_1[n] \; ④ \; x_2[n] = x_3[n]$

$$\boxed{1 \quad 1 \quad 2} \quad \begin{bmatrix} 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

$4 \times 4 \quad 4 \times 1 \quad 4 \times 1$

$x_1[n] = \{1, 2, 0, 1, 0, 0, 0\}$ $\quad N = 7$

$x_2[n] = \{2, 2, 1, 1, 0, 0, 0\}$ $\quad N = 7$

LTI

$y[n] = x[n] * h[n]$

Linear convolution

$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$

$\tilde{h}[n] \longleftrightarrow \tilde{H}[k]$

$\tilde{y}[n] \longleftrightarrow \underbrace{\tilde{X}[k]\,\tilde{H}[k]}_{\tilde{Y}[k]}$

000 11 2

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$x_1[n] \; (7) \; x_2[n]$

Observation

7-point output of Linear convolution of $x_1[n]$ and $x_2[n]$ $\equiv$ 7-pt circular convolution $x_1'[n]$ and $x_2'[n]$

$x_1[n]$ with 3 zeros appended

DFS $\longrightarrow$ DFT

$$\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k]$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, W_N^{-kn}$$

$$\tilde{X}(e^{j\omega}) = \sum_{K=-\infty}^{\infty} \frac{2\pi}{N}\tilde{X}[k]\,\delta\left(\omega - \frac{2\pi}{N}k\right)$$

$$\begin{array}{ccc} x[n] & \longleftrightarrow & X(e^{j\omega}) \\ \downarrow & & \\ \tilde{x}[n] & \longleftrightarrow & \tilde{X}[k] \end{array}$$

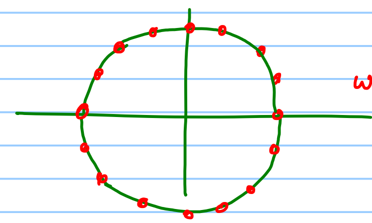
$$\rightarrow\ x[n] * p[n] = x[n] * \sum_{r=-\infty}^{\infty}\delta[n - rN] = \overbrace{\sum_{r=-\infty}^{\infty} x[n - rN]}^{\text{periodic signal}} = \tilde{x}[n] \quad \text{period} = N$$

Produces a periodic sequence from a finite length seq.

$x[n] = \tilde{x}[n] \quad 0 \le n \le N-1$

$$\tilde{X}(e^{j\omega}) = X(e^{j\omega})\,\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{N} X\left(e^{j\frac{2\pi}{N}k}\right)\delta\left(\omega - \frac{2\pi}{N}k\right)$$

$$\boxed{\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}}$$



$\omega = \frac{2\pi}{N}k$

$N = 16$

The periodic seq $\tilde{X}[k]$ DFS coefficients of periodic signal $\tilde{x}[n]$ $\equiv$ equi-spaced samples of DTFT of $x[n]$ which is one period of $\tilde{x}[n]$

Conclusion

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$\tilde{x}[n] \xleftrightarrow{\text{DFS}} \tilde{X}[k]$$

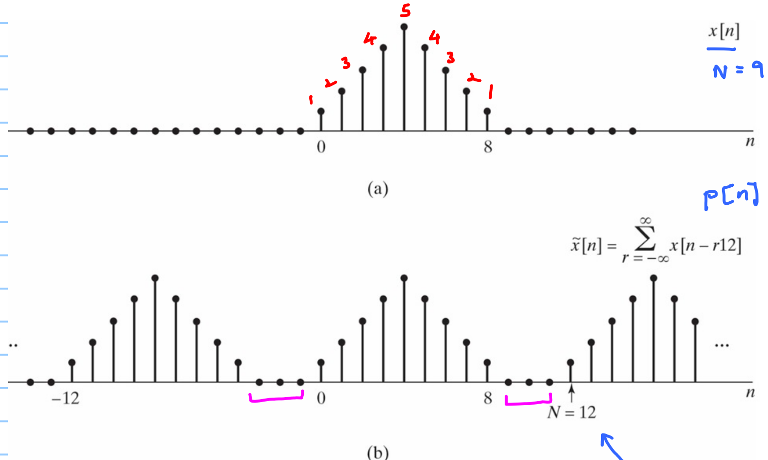
$$X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k} \longleftrightarrow \tilde{X}[k] \longleftrightarrow \tilde{x}[n] \longleftrightarrow x[n]$$

$$\boxed{\tilde{X}[k] \longleftrightarrow x[n]}$$

Sufficient to represent

DFT $\boxed{\underline{X} = \underline{D}_N \, \underline{x}}$

Inverse DFT $\underset{N\times 1}{\underline{x}} = \underset{N\times N}{\underline{D}_N^{-1}} \, \underset{N\times 1}{\underline{X}} = \frac{1}{N} D_N^{*} \underline{X}$

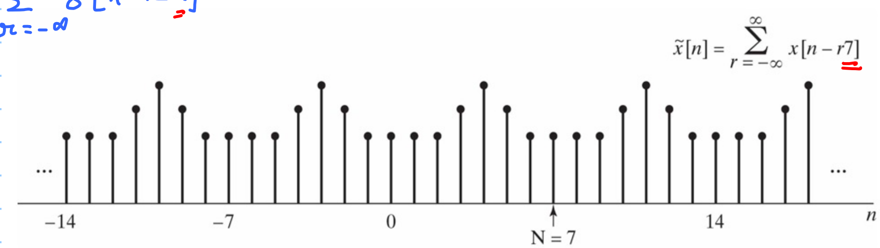


✓ No overlap $N = 12$

✗ overlap $\boxed{N = 7}$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$N = 12$	1	2	3	4	5	4	3	2	1	0	0	0	1	2	3	4 ..
$N = 7$	1	2	3	4	5	4	3	2	1							
								1	2	3	4	5	4	3	2	1
	3	3	3	4	5	4	3	3	3	3	4	5	4	3	3	3

$$p[n] = \sum_{r=-\infty}^{\infty} \delta[n - 12r]$$



$x[n] * p[n]$

$$p[n] = \sum_{r=-\infty}^{\infty} \delta[n - 7r]$$

$\Rightarrow$ distortion

Conclusion

Choose N such that there is "no overlap" of the copies of the signal $x[n]$

Discrete Fourier Transform (DFT)

* Finite length seq. $x[n]$ $\quad$ $x[n]$ is non-zero in the range $0 \le n \le N-1$

  length $= N$

* Can append zeros to increase the length to M

  Append $(M-N)$ zeros

* Associated periodic signal $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$

$$x[n] = \begin{cases} \tilde{x}[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{x}[n] = x[n \bmod N] = x[((n))_N]$$

$((n))_N \in [0, 1, \dots N-1]$

modulo operation

$$\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$$

* Choose the Fourier Coefficients from one period of $\tilde{X}[k]$

$$X[k] = \begin{cases} \tilde{X}[k] & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{X}[k] = X[K \bmod N] = X[((K))_N]$$

DFS

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

DFT

Analysis (Forward)

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn} & k = 0, \ldots N-1 \\ 0 & \text{otherwise} \end{cases}$$

Synthesis (Inverse)

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} & n = 0, \ldots N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}} X[k]$$

* Focus on DFT $\quad x[n] \longleftrightarrow X[k]$

* Underlying periodicity

$$x[n] \longleftrightarrow \boxed{\tilde{x}[n] \longleftrightarrow \tilde{X}[k]} \longleftrightarrow X[k]$$

Properties of DFT (Table 8.2)

Linearity

1. Two finite duration sequences $\overset{N_1}{x_1[n]}$ & $\overset{N_2}{x_2[n]}$ (of different lengths)

Append zeros to make them of same length $N = \max(N_1, N_2)$

$$x_1[n] \xrightarrow{DFT_N} X_1[k]$$
$$x_2[n] \xrightarrow{DFT_N} X_2[k]$$

$$x_3[n] = \alpha x_1[n] + \beta x_2[n]$$
$$\longleftrightarrow \quad \alpha X_1[k] + \beta X_2[k]$$

2. Circular shift

DFS

$$\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k]$$
$$\tilde{x}[n-m] \overset{DFS}{\longleftrightarrow} W_N^{km} \tilde{X}[k]$$

DFT

$$x[n] \overset{DFT}{\longleftrightarrow} X[k]$$
$$? \longleftrightarrow \underbrace{e^{-j\frac{2\pi}{N}mk} X[k]}_{X_1[k]}$$

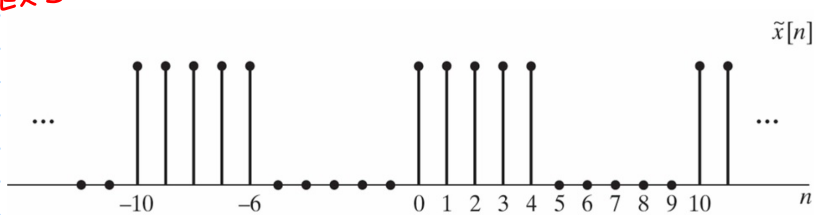
$x[n-m]$

$x[((n-m))_N]$

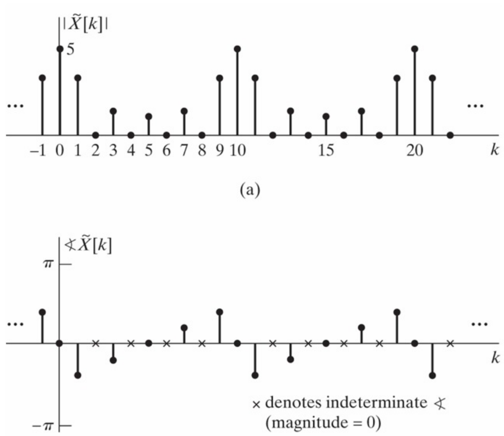
$$x_1[n] \overset{DFT}{\longleftrightarrow} X_1[k]$$
$$x_1[n] = x[((n-m))_N]$$

O&S

Ex 3



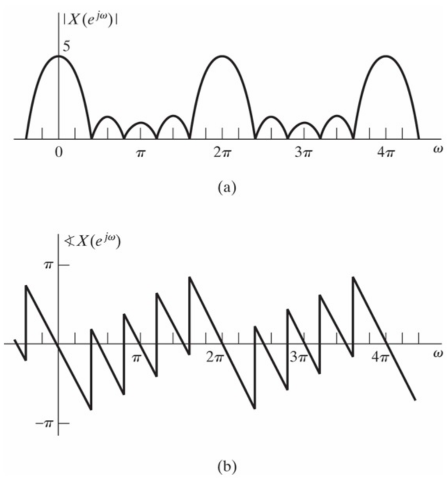
$\tilde{x}[n]$
...
...
–10
–6
0 1 2 3 4 5 6 7 8 9 10
n



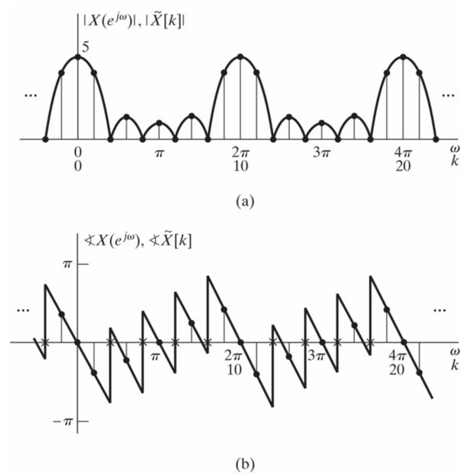
$|\tilde{X}[k]|$
5
–1 0 1 2 3 4 5 6 7 8 9 10
15
20
k
(a)
$\sphericalangle\tilde{X}[k]$
π
$-\pi$
k
× denotes indeterminate $\sphericalangle$
(magnitude = 0)

O&S

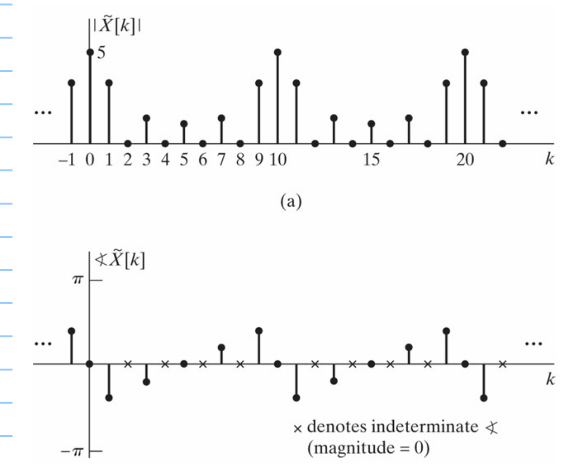
Ex 6



$|X(e^{j\omega})|$
5
0
π
2π
3π
4π
ω
(a)
$\sphericalangle X(e^{j\omega})$
π
$-\pi$
2π
3π
4π
ω
(b)



$|X(e^{j\omega})|, |\tilde{X}[k]|$
5
...
0
0
π
2π
10
3π
4π
20
ω
k
(a)
$\sphericalangle X(e^{j\omega}), \sphericalangle \tilde{X}[k]$
π
$-\pi$
2π
10
3π
4π
20
ω
k
(b)



$|\tilde{X}[k]|$
5
...
–1 0 1 2 3 4 5 6 7 8 9 10
15
20
k
(a)
$\sphericalangle \tilde{X}[k]$
π
$-\pi$
k
× denotes indeterminate $\sphericalangle$
(magnitude = 0)