Note Title 02-09-2024

Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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September – December 2024

Session # 7

September 30, 2024

EE3101 Digital Signal Processing

EE3101
Session 7

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Outline

Last session

- Interconnection of LTI systems
- Examples

Week 1-2

Introduction to sampling - Review of Signals and Systems: Basic operations on signals - Discrete time complex exponentials and other basic signals – system properties (linearity, time-invariance, memory, causality, BIBO stability) -- impulse response and convolution

Week 3-4

Sampling: Impulse train sampling—relationship between impulse trained sampled continuous-time signal spectrum and the DTFT of its discrete-time counterpart—scaling of the frequency axis—relationship between true frequency and digital frequency—reconstruction through sinc interpolation—aliasing—effect of sampling at a discontinuous point—relationship between analog and digital sinc—effects of oversampling—discrete-time processing of continuous-time signals

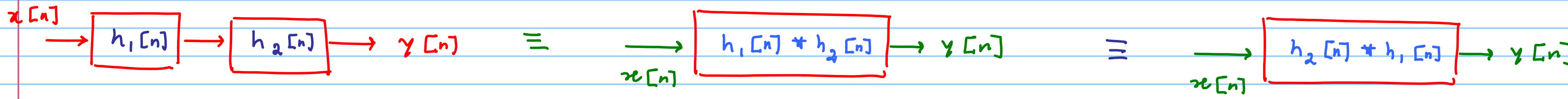
$\text{CT} \xrightarrow{\text{Sampling}} \text{DT}$

Cont. Time → Discrete Time

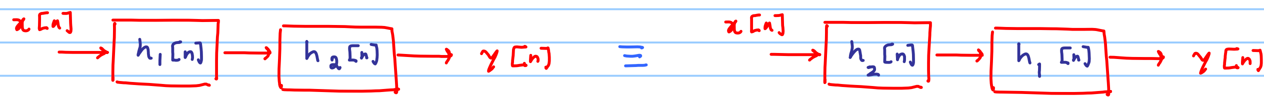
Reading Assignment

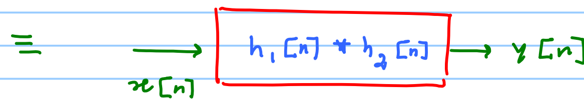
O&S Chapter 4: Sampling of CT signals

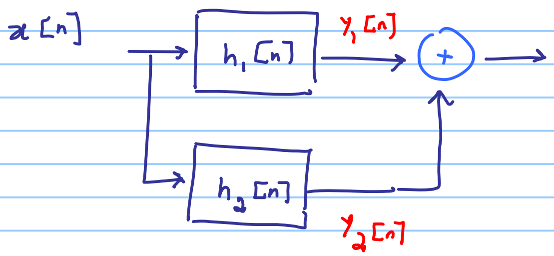
Rawat Chapter 2: Sampling and Quantization



Result:







$$y[n] = y_1[n] + y_2[n]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h_1[k] + \sum_{k=-\infty}^{\infty} x[n-k]\, h_2[k]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k] \underbrace{\left(h_1[k] + h_2[k] \right)}_{h[n]}$$

Session 5

Properties of LTI system $\longleftrightarrow h[n]$

1. DT LTI system is causal if $h[n] = 0 \quad n < 0$ i.e., $h[n]$ is causal

2. DT LTI System is BIBO stable if $\sum_n |h[n]| < \infty$ Absolute summable

Energy of DT seq $x[n]$ $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$

Power of DT seq $x[n]$ $P_x = \lim_{K \to \infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$

Periodic signal Period $= N$ $P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$

Periodic Signals $\mathcal{E}_x = \infty$, P_x finite Power signals

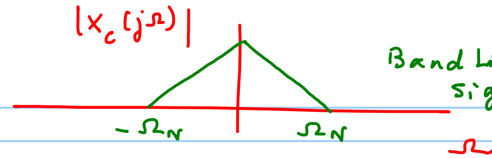
Finite duration signal $\mathcal{E}_x =$ finite $P_x \to 0$ Energy signals

Discrete Time Fourier Transform (DTFT)

$$x[n] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$X(e^{j\omega})$ is a periodic signal with period $= 2\pi$



Band Limited signal

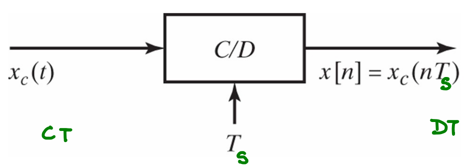
Nyquist Sampling Theorem

Let $x_c(t)$ be a band-limited signal with $X_c(j\Omega) = 0$ $\quad |\Omega| \geq \Omega_N$ rads/sec

$x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT_s)$ $\quad n = 0, \pm 1, \pm 2, \dots$

if $\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_N$

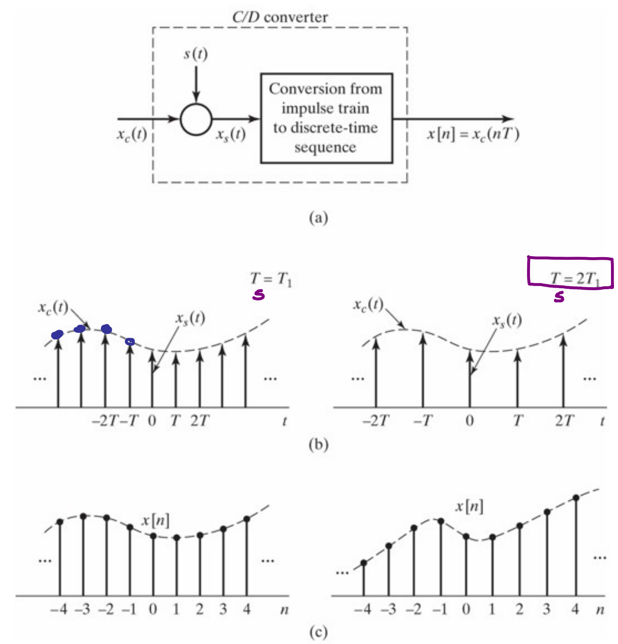
Nyquist rate $= 2\Omega_N$ $\quad \Omega_s$ = rads/sec



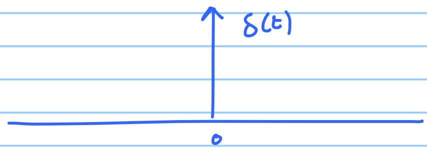
$x[n] = x_c(t)\Big|_{t = nT_s}$

BL Signal

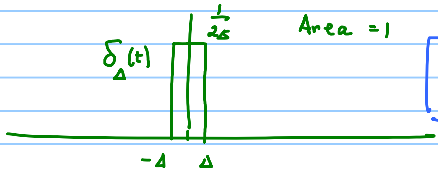
$BW = \Omega_N$



Properties of Dirac Delta function



$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undef} & t = 0 \end{cases}$$



Area = 1

$$\boxed{\delta(t) = \lim_{\Delta \to 0} \delta_{\Delta}(t)}$$

$$\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$$

$$\int_{-\infty}^{\infty} x(\tau)\, \delta(\tau)\, d\tau = x(0)$$

$$x(t)\, \delta(t - T_0) = x(T_0)\, \delta(t - T_0)$$

$$\int_{-\infty}^{\infty} x(\tau)\, \delta(\tau - T_0)\, d\tau = x(T_0)$$

O&S Ch 4

Periodic sampling
Uniform sampling

$$x[n] = x_c(nT_s) \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$x_c(t) \xrightarrow{C/D} x[n] \xrightarrow{Q} \hat{x}[n]$$

Quantizer

- # bits
- Dynamic range
- Quantization level

- linearity
- Sample & Hold circuit

$$x_c(t) \longrightarrow x_s(t) \longrightarrow x[n]$$

$x_s(t)$: CT signal with sequence of Dirac Delta functions

Dirac Delta functions: Scaled & shifted in time in amplitude

Examples

a) $x(t) = \sin 200\pi t \sim \sin \Omega_0 t$ $\quad \Omega_0 = 200\pi$ rads/sec

$= 2\pi f_0$

Nyquist Rate $F_s = 200$ Hz $\quad \boxed{f_0 = 100 \text{ Hz}}$

$\Omega_s = 2\pi F_s$

$= 400\pi$

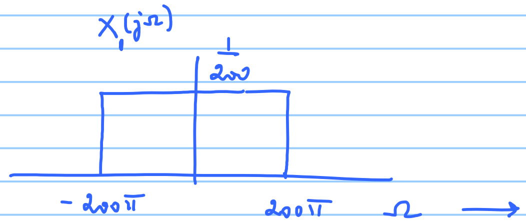
b) $x(t) = \sin^2 200\pi t = \frac{1 - \cos(400\pi t)}{2} = \frac{1}{2} - \frac{1}{2}\cos(400\pi t)$

Nyquist rate $= F_s = 400$ Hz $\quad f_0 = 200$ Hz

c) $x(t) = \cos^3 200\pi t$

$F_s = 600$ Hz

Ex 2 $\quad x(t) = \operatorname{sinc}(200t) = \dfrac{\sin(200\pi t)}{200\pi t}$



$$\longleftrightarrow \quad x_1(t) = \frac{1}{2\pi}\int_{-200\pi}^{200\pi} \frac{1}{200} e^{j\Omega t}\, d\Omega = \frac{\sin(200\pi t)}{200\pi t}$$

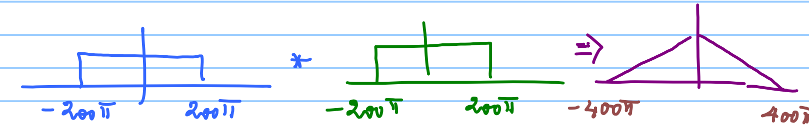
$$\longleftrightarrow \quad x_1(t) = \operatorname{sinc}(200t)$$

Highest freq component $\Omega_N = 200\pi$

$f_0 = 100\text{ Hz} \Rightarrow$ Nyquist rate $F_s = 200\text{ Hz}$

(b) $\quad x(t) = \operatorname{sinc}^2(200t)$

$\operatorname{sinc}(200t) \cdot \operatorname{sinc}(200t) \longleftrightarrow X(j\Omega) * X(j\Omega)$



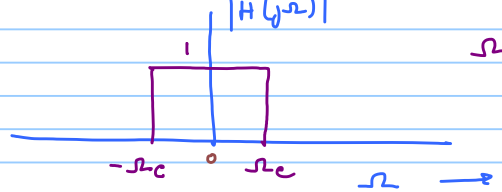
$\Omega_N = 400\pi$

$f_0 = 200\text{ Hz} \Rightarrow$ Nyquist rate $F_s = 400\text{ Hz}$

(c) $\quad x(t) = \operatorname{sinc}(200t) + \operatorname{sinc}^2(200t)$ $\qquad F_s = 400\text{ Hz}$

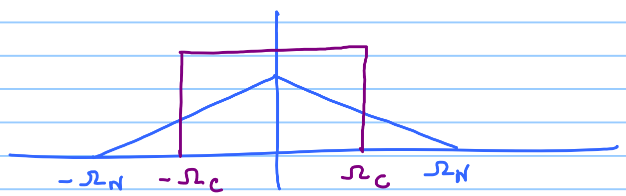
Ex 3

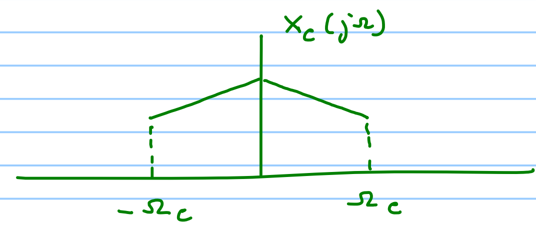
Low pass filter



Ω_c = cutoff frequency

CT signal $x_c(t)$ obtained at the output of an ideal lowpass filter $\Omega_c = 1000\pi$





Highest freq. component $\Omega_c = 1000\pi$

$f_0 = 500$ Hz

$F_s = 1000$ Hz

$T_s = \frac{1}{F_s} = 10^{-3}$ sec $= 1$ msec

Which of the following values of T_s will guarantee that $x_c(t)$ will be recovered (without loss of information) in the process of sampling

(a) $T_s = 0.5 \times 10^{-3} = 0.5$ msec Yes

(b) $T_s = 2 \times 10^{-3} = 2$ msec No → violate Nyquist thm

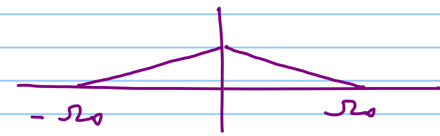
(c) $T_s = 10^{-4}$ sec $= 0.1$ msec Yes

Example 5

$$x_c(t) \longleftrightarrow X_c(j\Omega)$$

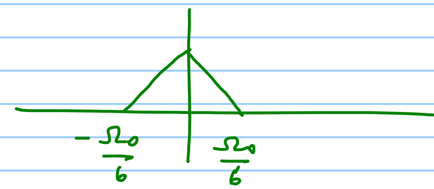
Time Scaling $\quad x_c(at) \longleftrightarrow \frac{1}{a} X_c\left(j\frac{\Omega}{a}\right)$

1. $x_c(2t) \longleftrightarrow \frac{1}{2} X_c\left(j\frac{\Omega}{2}\right)$



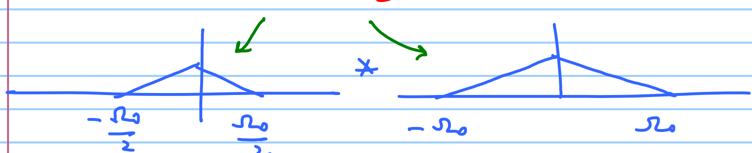
$\boxed{\Omega_s = 2\Omega_0}$

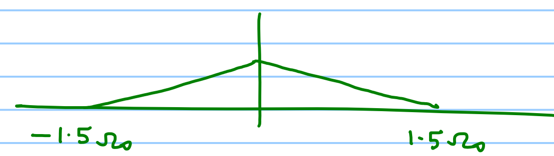
2. $x_c\left(\frac{t}{3}\right) \longleftrightarrow 3X_c(j3\Omega)$



$\Omega_s = 2 \times \frac{\Omega_0}{6} = \frac{\Omega_0}{3}$

3. $x_c(t)\, x_c(2t) \longleftrightarrow$

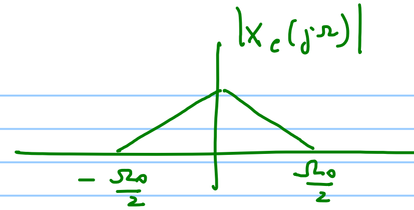




$\boxed{\Omega_s = 3\Omega_0}$

Ex 4

Let $x_c(t)$ be a BL signal with Nyquist rate Ω_0



Using FT, determine the Nyquist rate for the following signal

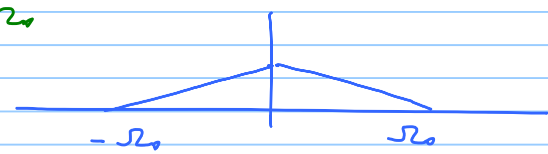
(a) $x_c(t+1) \longleftrightarrow e^{j\Omega} X_c(j\Omega)$ $\qquad \Omega_s = \Omega_0$

(b) $x_c(t) + x_c(t-1) \longleftrightarrow X_c(j\Omega)\left[1 + e^{-j\Omega}\right]$

$\Omega_s = \Omega_0$

(c) $\frac{d}{dt} x_c(t) \longleftrightarrow j\Omega X_c(j\Omega)$ $\qquad \Omega_s = \Omega_0$

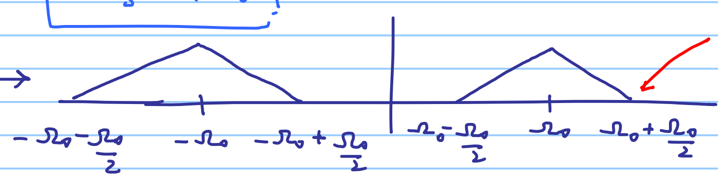
(d) $x^2(t) \longleftrightarrow \frac{1}{2\pi} X_c(j\Omega) * X_c(j\Omega)$



$\boxed{\Omega_s = 2\Omega_0}$

(e) $x_c(t)\cos\Omega_0 t \longleftrightarrow$

Modulated Signal

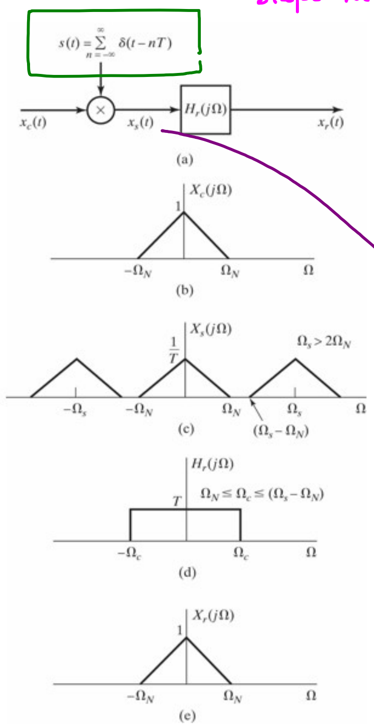


Highest freq component $1.5\Omega_0$

$\boxed{\Omega_s = 3\Omega_0}$

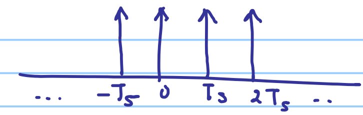
Mathematical Framework for Sampling

Steps in sampling



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

train of Dirac Delta



$$x_c(t)\cdot s(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT_s)\,\delta(t-nT_s)$$

$$x_c(t)\,\delta(t-mT_s) = x_c(mT_s)\,\delta(t-mT_s)$$

$$x_s(t) = x_c(t)\cdot s(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

evaluate

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

periodic → Yes

period → T_s

$$s(t) = \sum_{m=-\infty}^{\infty} a_m e^{j\frac{2\pi}{T_s}mt}$$

$$a_m = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)\, e^{-j\frac{2\pi}{T_s}nt}\, dt = \frac{1}{T_s} \quad \forall m$$

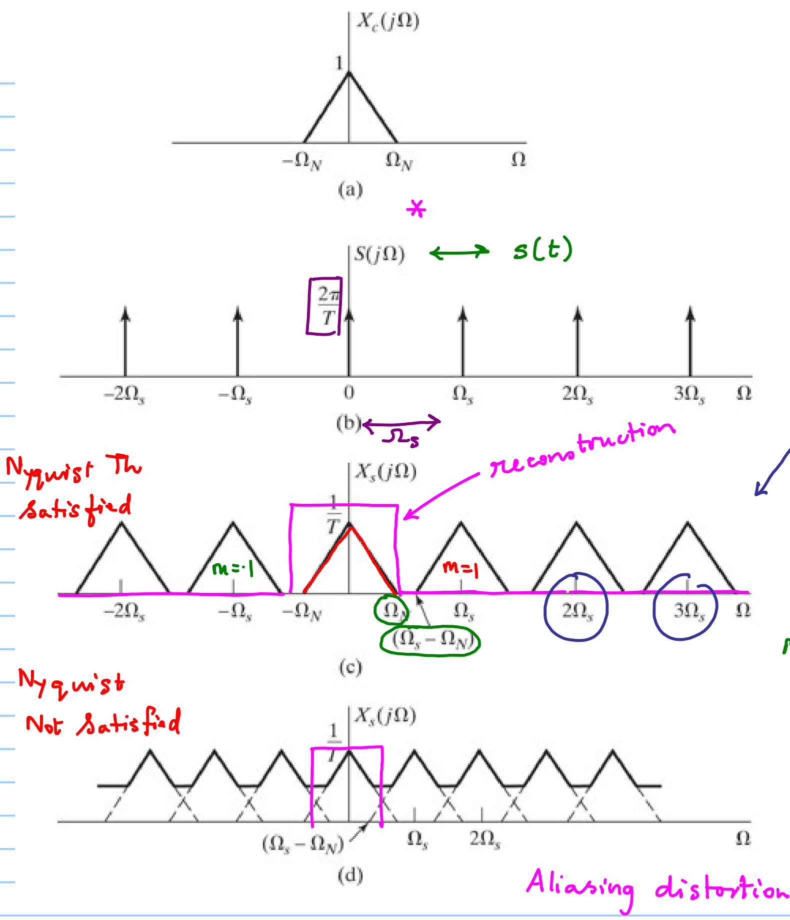
$$s(t) = \frac{1}{T_s}\sum_{m=-\infty}^{\infty} e^{j\frac{2\pi}{T_s}mt}$$

Fourier Series

$$s(t) = \frac{1}{T_s}\sum_{m=-\infty}^{\infty} e^{j\frac{2\pi}{T_s}mt}$$

$$e^{j\Omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\Omega-\Omega_0) \quad (1)$$

$$S(j\Omega) = \frac{2\pi}{T_s}\sum_{m=-\infty}^{\infty} \delta(\Omega - m\Omega_s)$$



$$X_c(j\Omega) * S(j\Omega)$$

$$\frac{1}{\cancel{2\pi}} X_c(j\Omega) * \frac{\cancel{2\pi}}{T_s} \sum_{m=-\infty}^{\infty} \delta(\Omega - m\Omega_s)$$

$$\boxed{\frac{1}{T_s} \sum_{m=-\infty}^{\infty} X_c(j(\Omega - m\Omega_s)) = X_s(j\Omega)}$$

scale factor

repetition of spectrum

shifts multiples of Ω_s

Result #1

Original signal @ $\Omega = 0$ $\quad [-\Omega_N, \Omega_N]$

copy m=1 @ $\Omega = \Omega_s$ $\quad [\Omega_s - \Omega_N, \Omega_s + \Omega_N]$

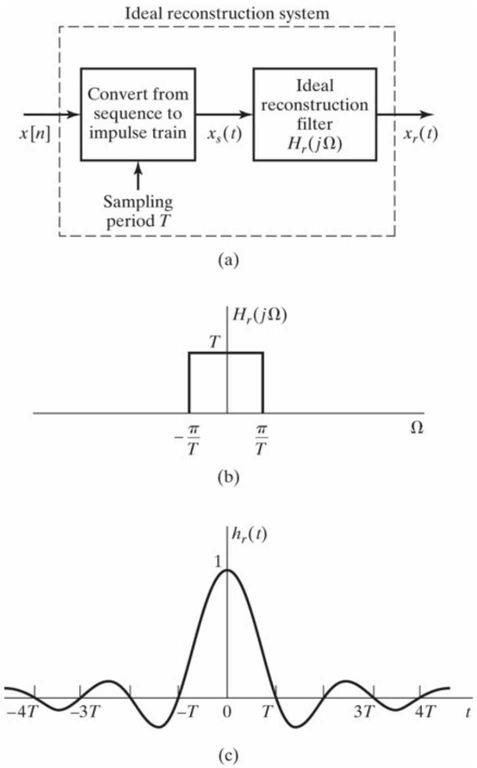
Copy @ m=-1 @ $\Omega = -\Omega_s$

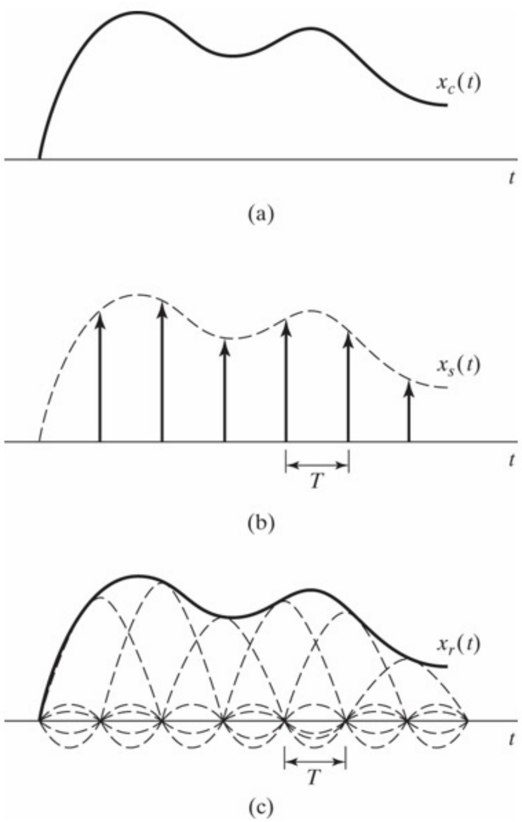
Do not want copies to overlap @ m=0, m=1

$\Omega_N \leq \Omega_s - \Omega_N \qquad 2\Omega_N \leq \Omega_s$

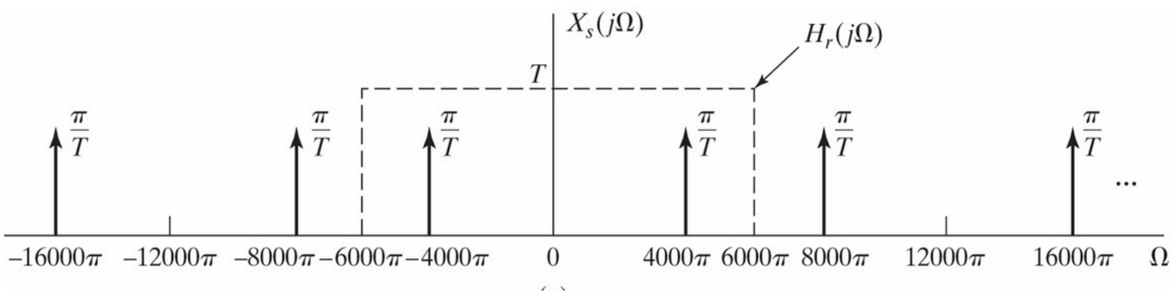
Nyquist Theorem

$$\boxed{\Omega_s \geq 2\Omega_N}$$


Ideal reconstruction system
Convert from sequence to impulse train
Ideal reconstruction filter $H_r(j\Omega)$
$x[n]$
$x_s(t)$
$x_r(t)$
Sampling period T
(a)
$H_r(j\Omega)$
T
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
Ω
(b)
$h_r(t)$
1
$-4T$
$-3T$
$-T$
0
T
$3T$
$4T$
t
(c)



$x_c(t)$
t
(a)
$x_s(t)$
t
T
(b)
$x_r(t)$
t
T
(c)



$X_s(j\Omega)$
$H_r(j\Omega)$
T
$\frac{\pi}{T}$
...
-16000π
-12000π
-8000π
-6000π
-4000π
0
4000π
6000π
8000π
12000π
16000π
Ω