Note Title 02-09-2024

Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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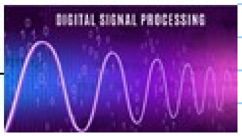
September – December 2024

Session # 19

November 19, 2024



19/11/24 Note Title

EE 3101 Digital Signal Processing

EE 3101
Session 19

03-01-2018

Session 19

Outline

Last session

- Intro to Ch 5
- $H(e^{j\omega})$ — magnitude, phase
  — Group delay
- Impact of GD
- Pole-zero plots

Today

- First order systems
- Allpass systems, properties
- Relationship between Mag & Phase

Week 9-10 O&S chapter 5

Frequency Domain Analysis of LTI Systems: Frequency response of systems with rational transfer function—definitions of magnitude and phase response—frequency response of single complex zero/pole—frequency response of simple configurations (second order resonator, notch filter, averaging filter, comb filter, allpass systems)—phase response—definition of principal phase—zero-phase response—conditions that have to be met for a filter to have generalized linear phase—four types of linear phase FIR filters—on the zero locations of a linear phase FIR filter—constrained zeros at z = 1 and at z = -1 and their implications on choice of filters Type I through Type IV (with focus on Type I)

Reading Assignment

O&S ch 5 Transform Analysis of LTI systems

Z Transform is a generalization of the DTFT

Defn $X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \left[x[n]\, r^{-n}\right] e^{-j\omega n}$

$x[n] \xrightarrow{Z} X(z)$ ROC

$\xleftarrow[\text{Inv. ZT}]{}$

$x[n] = \frac{1}{2\pi j} \oint_{C} X(z)\, z^{n-1}\, dz$

Scale factor

Cauchy Integral Theorem

Contour

- Closed contour in RoC
- encircle $z=0$
- Traversed in counter-clockwise

$X(z) = \frac{P(z)}{Q(z)}$

Evaluate Inv ZT using Cauchy Residue Theorem

Power Series

1 $$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$$

2 $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

3 $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

4 $$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

ZT of periodic signal (period = N) Commencing @ $n = 0$

One period $$X_1(z) = \sum_{n=0}^{N-1} x[n] z^{-n}$$

$$X(z) = X_1(z)\left[1 + z^{-N} + z^{-2N} + \cdots\right] = \frac{X_1(z)}{1 - z^{-N}} \qquad |z| > 1$$

O & S Ch 5 Transform Analysis of LTI System

✓ 5.1 Freq response of LTI systems $h[n] \longleftrightarrow H(e^{j\omega})$

✓ 5.2 Systems characterized by LCCDE

✓ 5.3 Freq response of Rational Transfer Function

5.4 Relationship between magnitude and phase

→ ✓ 5.5 Allpass System

5.6 Min. phase system

5.7 Linear phase ⇐

O&S ch 5

LTI system $h[n]$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = h[n] * x[n]$$

Using DTFT

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

$$H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{\text{Mag resp}}\; e^{j\,\overbrace{\arg[H(e^{j\omega})]}^{}}$$

phase response

$$|Y(e^{j\omega})| = |H(e^{j\omega})|\,|X(e^{j\omega})|$$

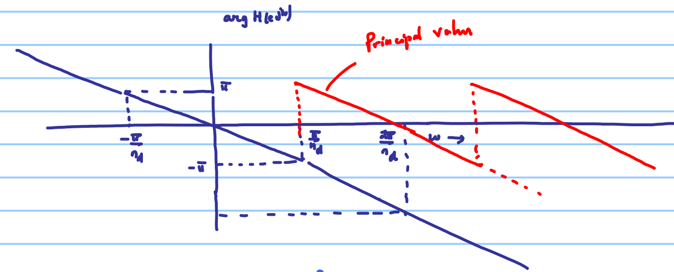
Mag resp, Phase resp } periodic in ω with period $= 2\pi$

Notch Filter

Principal value of the Phase

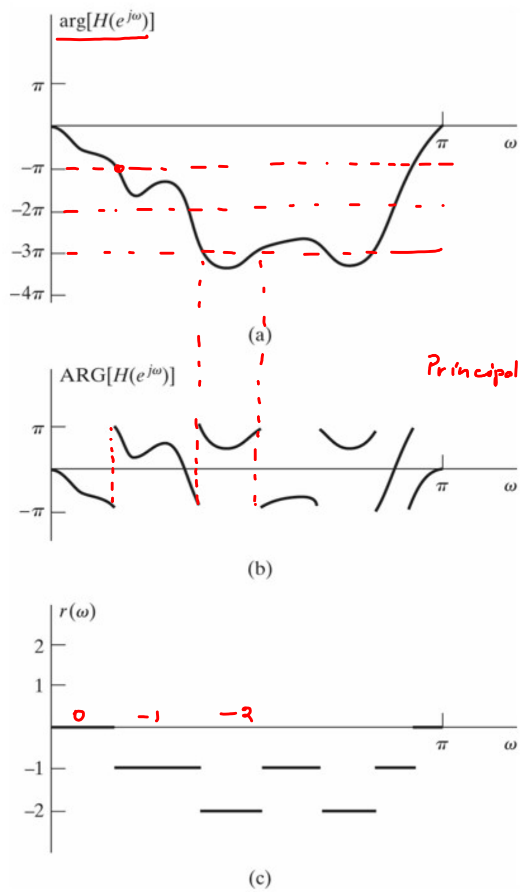
$$-\pi < \mathrm{ARG}\; H(e^{j\omega}) \le \pi$$

True phase $\underline{[H(e^{j\omega})]} = \mathrm{ARG}\; H(e^{j\omega}) + 2\pi\, r(\omega)$



True phase vs Principal value

✓ GD GD ×



Unwrapped Phase

Principal value

= phase wrap around

$$\underbrace{\arg H(e^{jw})}_{\text{phase unwrapping}} = \mathrm{ARG}\, H(e^{jw}) + 2\pi\, r(w)$$

$r(w) = \pm K \quad K = 0, 1, 2, \cdots$

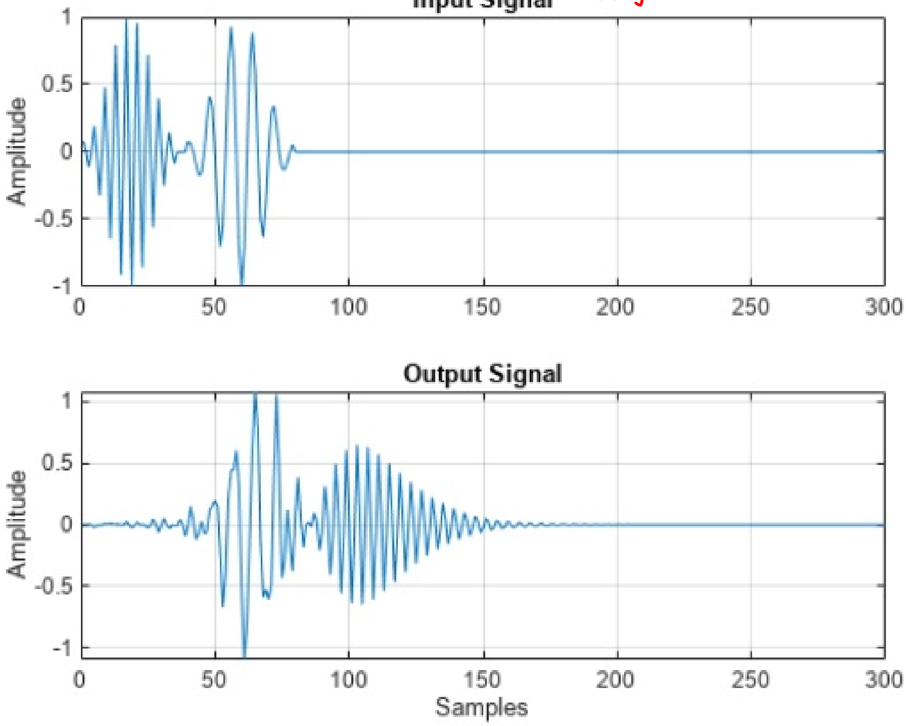
$$\text{Group Delay} = \tau(w) = -\frac{d}{dw} \underbrace{\arg H(e^{jw})}_{\text{True phase or unwrapped phase}}$$

Understand the GD distortion introduced by LTI system

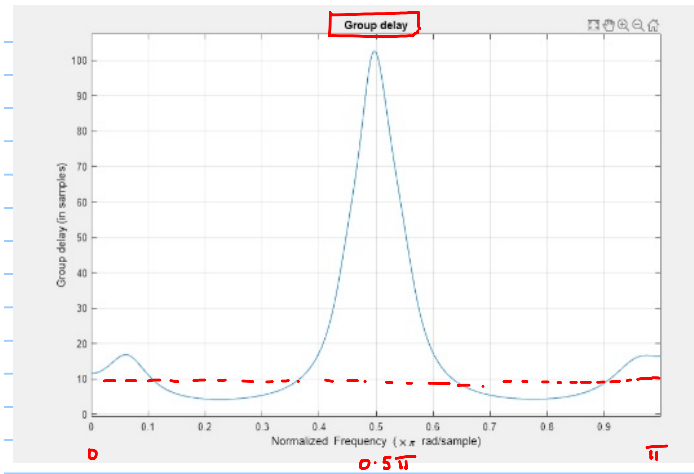
Choose LTI system with linear ϕ $\Rightarrow$ constant GD

$\Rightarrow$ No GD distortion

$w_1 = 0.5\pi$ $w_2 = 0.25\pi$ $x[n]$

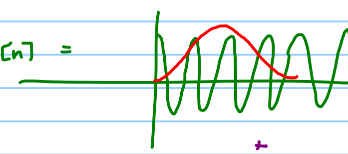


LTI

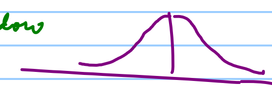


0 0.5π π

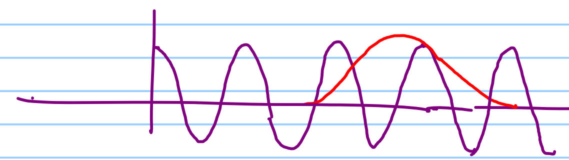
$x[n] =$



Hamming Window



n



Rule of Thumb

- GD is derivative arg $H(e^{jw})$
  - regions of rapid phase change $\Rightarrow$ large GD

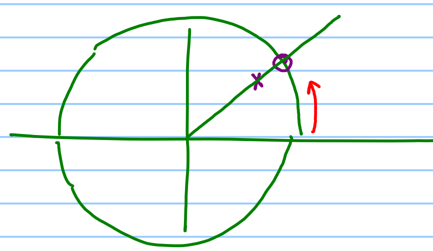
    near pole or zero
  - $\Rightarrow$ regions where change of magnitude

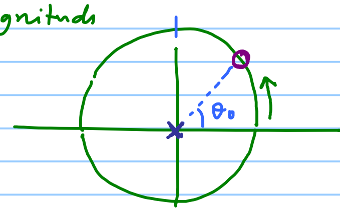
Ex

$(1 - z_0 z^{-1}) \Rightarrow$ zero @ z_0

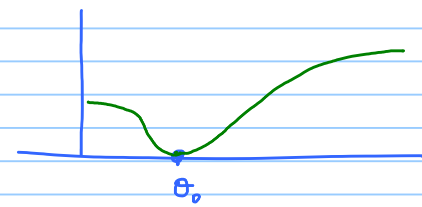
$\frac{z - z_0}{z}$

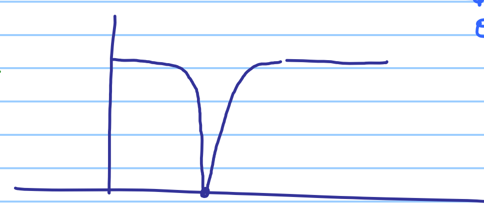
$\left(\frac{1 - z_0 z^{-1}}{1 - z_1 z^{-1}}\right)$





$H(e^{jw})$ $\quad 0 \leq w < 2\pi$



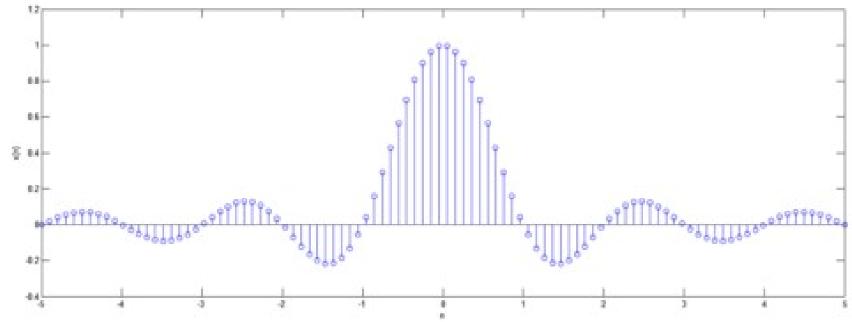


Magnitude response

1. Traverse the unit circle $w=0$ to $w=2\pi$ (counter clockwise)
2. At each point on unit circle, evaluate $|H(e^{jw})|$

Ideal LPF

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases} \quad \longleftrightarrow \quad h[n] = \frac{\sin \omega_c n}{\pi n}$$



$H(e^{j\omega})$ = real-valued

$\arg H(e^{j\omega}) = 0 \quad \forall \omega \Rightarrow$ zero phase function

even symmetry $h[n] = h[-n]$

non-causal

infinite duration

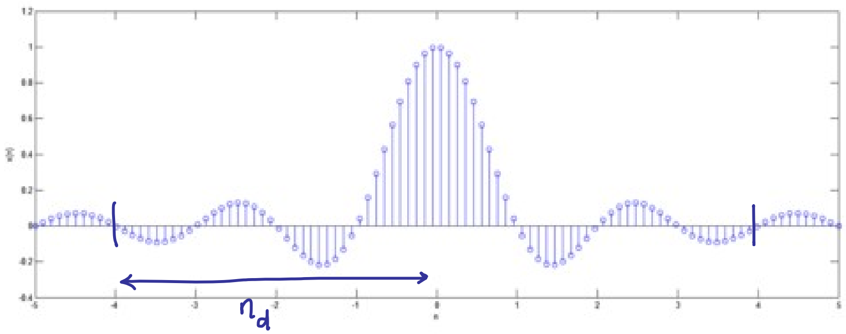
Zero-phase TF

→ non-causal

→ symmetry < even / odd

→ finite or inf duration

* LTI systems with Linear phase are realizable



$h_{prac}[n] = \dfrac{\sin \cdot w_c(n-n_d)}{\pi(n-n_d)} \left[u[n] - u[n-(2n_d+1)]\right]$

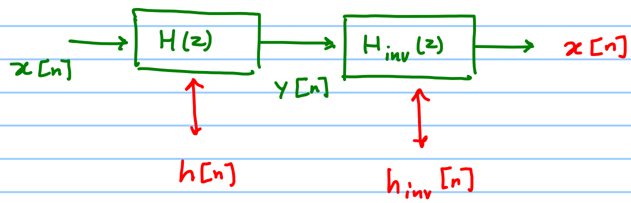
Linear phase $-wn_d$

Group Delay n_d samples

Truncation + shift to make it realizable
(Windowing)

Application

Inverse System



$$h[n] * h_{inv}[n] = \delta[n]$$

$$H(z)\, H_{inv}(z) = 1 \quad \Rightarrow \quad H_{inv}(z) = \frac{1}{H(z)}$$

Suppose $H(z)$ causal & stable $\rightarrow$ all poles inside unit circle $|z| > r_R$ $\quad r_R < 1$

$\rightarrow$ no constraint on zeros

— What about $H_{inv}(z)$?

$\rightarrow$ poles & zeros

poles of $H_{inv}(z)$ inside unit circle

zeros of $H(z)$ inside unit circle

$h[n]$ — finite length — Finite Impulse Response (FIR)

$h[n]$ — infinite length — Infinite Impulse Response (IIR)

Linear ϕ

1. Can IIR system have linear ϕ ?

   Yes. Ideal LPF

2. Can FIR system have linear ϕ ? Yes.

Lin ϕ depends on the symmetry in $h[n]$

Causality

3. Can an IIR filter be causal? Yes

$$h[n] = a^n u[n]$$

4. Can a causal IIR filter have linear ϕ No.

BIBO stability

$$|H(e^{j\omega})| < \infty \quad \forall \omega$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad \text{Absolute summability}$$

$\Rightarrow$ ZT having ROC that includes unit circle

Causal & stable $\Rightarrow$ all poles inside the unit circle

FIR system (causal) $\Rightarrow$ poles are @ $z=0$ — BIBO stable

IIR system (causal) $\Rightarrow$ poles inside unit circle

$$H(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

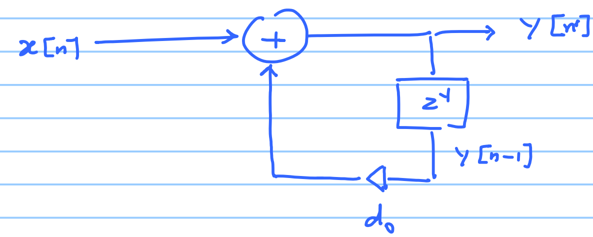
poles @ $z = d_k$ $\quad k = 1, \dots N$

zeros @ $z = c_k$ $\quad k = 1, \dots M$

$N \geq 1$ $\quad h[n]$ will be IIR

Example

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - d_0 z^{-1}} \quad (1)$$



(2) feed back loop

$\rightarrow$ $y[n] - d_0\, y[n-1] = x[n]$ (A)

(3) Recursive relation

$y[n-1] = d_0\, y[n-2] + x[n-1]$ (B)

Substitute (B) in (A)

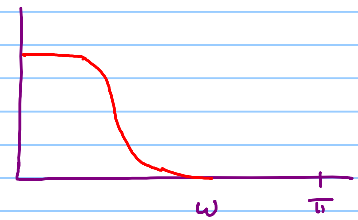
$y[n] = d_0^2\, y[n-2] + x[n] + d_0\, x[n-1]$

$$H(e^{j\omega}) \begin{cases} |H(e^{j\omega})| \\ \arg H(e^{j\omega}) \end{cases}$$

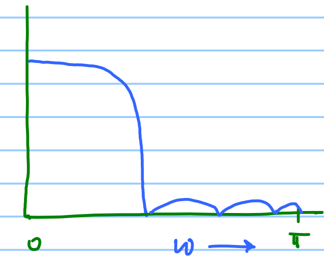
Example

LPF

$|H(e^{j\omega})|$



$20 \log |H(e^{j\omega})|$



LTI System (IIR)

$$H(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

FIR if $N = 0$

$$20 \log_{10} |H(e^{j\omega})| = 20 \log \left|\frac{a_0}{b_0}\right| + \sum_{k=1}^{M} 20 \log |1 - c_k e^{-j\omega}| - \sum_{k=1}^{N} 20 \log |1 - d_k e^{-j\omega}|$$

$$\arg H(e^{j\omega}) = \sum_{k=1}^{M} \arg(1 - c_k e^{-j\omega}) - \sum_{k=1}^{M} \arg(1 - d_k e^{-j\omega})$$

Group delay $\tau(\omega) = -\frac{d}{d\omega} \left[\underbrace{\arg H(e^{j\omega})}_{\text{unwrapped phase}} \right]$

$H(z) = 1 - c_1 z^{-1}$ $\boxed{c_1 \text{ is real}}$

$H(e^{j\omega}) = 1 - c_1 e^{-j\omega} = 1 - c_1\cos\omega + j\, c_1 \sin\omega$

$|H(e^{j\omega})|^{\cancel{2}} = \sqrt{(1-c_1\cos\omega)^2 + (c_1\sin\omega)^2}$

$\arg H(e^{j\omega}) = \tan^{-1}\left[\dfrac{c_1\sin\omega}{1-c_1\cos\omega}\right]$

$\tau(\omega) = -\dfrac{d}{d\omega}\left(\arg H(e^{j\omega})\right) = -\dfrac{d}{d\omega}\tan^{-1}\left(\dfrac{c_1\sin\omega}{1-c_1\cos\omega}\right) = \dfrac{c_1^2 - c_1\cos\omega}{1+c_1^2 - 2c_1\cos\omega}$

$\boxed{c_1 \text{ complex}}$ $c_1 = r_1 e^{j\theta_1}$

$H(e^{j\omega}) = 1 - r_1 e^{j\theta_1} e^{-j\omega} = 1 - r_1 e^{-j(\omega-\theta_1)}$

$= 1 - r_1\cos(\omega-\theta_1) + j\, r_1 \sin(\omega-\theta_1)$

$|H(e^{j\omega})|$

$\arg H(e^{j\omega})$

$\tau(\omega) = \dfrac{|c_1|^2 - \mathrm{Re}\{c_1 e^{-j\omega}\}}{1+|c_1|^2 - 2\mathrm{Re}\{c_1 e^{-j\omega}\}}$

Task to Complete

5.5 Allpass Systems

$$\boxed{H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}} = \frac{z^{-1}(1 - a^* z)}{(1 - az^{-1})}$$

$|H_{ap}(e^{jw})| = 1 \quad \forall w$ Allpass filter

	$n=-1$	$n=0$	$n=1$
Dr		1	$-a$
Nr	$-a^*$	1	

Time-reversal + conjugation

$$H_{ap}(e^{jw}) = e^{-jw}\frac{(1 - a^* e^{jw})}{(1 - ae^{-jw})}$$

$$|H_{ap}(e^{jw})| = \underbrace{|e^{-jw}|}_{=1} \underbrace{\frac{|1 - a^* e^{jw}|}{|1 - ae^{-jw}|}} \qquad \frac{N(e^{jw})}{D(e^{jw})}$$

$$N(e^{jw}) = D^*(e^{jw})$$

$$= \frac{|c^*|}{|c|} = 1$$

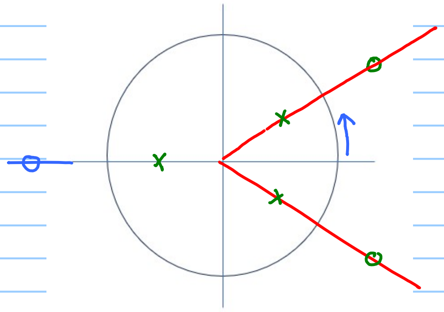
Gen form of Allpass Systems

$$H_{ap}(z) = \frac{b_N^* + b_{N-1}^* z^{-1} + b_{N-2}^* z^{-2} + \cdots + b_1^* z^{-(N-1)} + z^{-(N)}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}} = \frac{z^{-N}\left[1 + b_1^* z + b_2^* z^2 + \cdots + b_N^* z^N\right]}{\left[1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}\right]}$$

Verify $|H_{ap}(e^{jw})| = 1 \quad \forall w$

Zeros & Poles of $H_{ap}(z)$

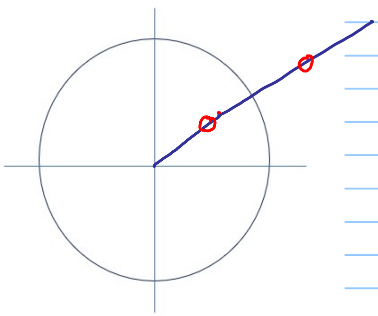
$H_{ap}(z)$ causal & stable $\Rightarrow$ poles must be within unit circle



$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

pole @ $z = a$

zero @ $z = \frac{1}{a^*}$



$H_1(z) = \left(1 - az^{-1}\right)$

$H_2(z) = \left(z^{-1} - a^*\right)$

$$H_{ap}(e^{j\omega}) = e^{-j\omega}\,\frac{(1-a^* e^{j\omega})}{(1-a e^{-j\omega})}$$

$\boxed{a = r e^{j\theta}}$ location of pole

Verify $|H_{ap}(e^{j\omega})| = 1 \quad \forall\, \omega$

$$\arg\left[H_{ap}(e^{j\omega})\right] = -\omega + \underbrace{\tan^{-1}\left[\frac{-r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]}_{Nr} - \underbrace{\tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]}_{Dr}$$

Group delay $\tau(\omega) = -\frac{d}{d\omega}\left[\arg H_{ap}(e^{j\omega})\right] = +1 - 2\frac{d}{d\omega}\tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$

$$\tau(\omega) = \frac{1-r^2}{1+r^2-2r\cos(\omega-\theta)}$$ Verify

Causal, stable allpass

$\Rightarrow$ all poles inside the unit circle

$r < 1$

$\tau(\omega) = \frac{Nr}{Dr}$

$\tau(\omega) > 0 \quad \forall\, \omega$

$r < 1$	$Nr > 0$
$r < 1$	$Dr > 0$

Smallest value of Dr

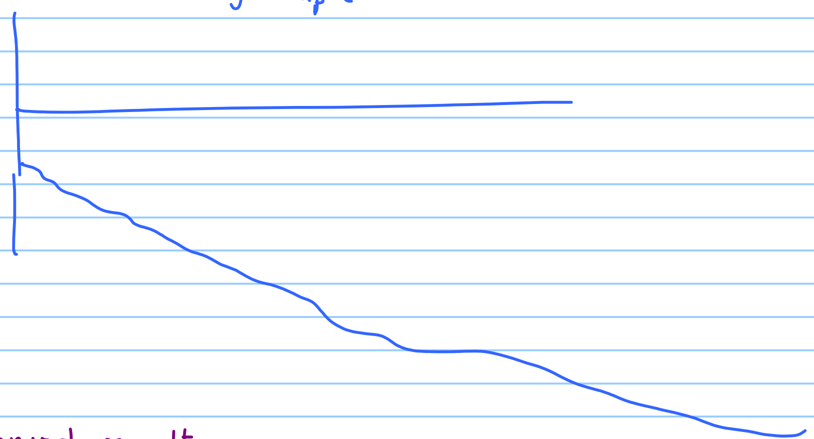
$\cos(\omega-\theta) = +1$

Dr $1 + r^2 - 2r = (1-r)^2$

$$\tau(\omega) = -\frac{d}{d\omega} \arg H_{ap}(e^{j\omega})$$

$\tau(\omega) > 0 \quad \forall \omega$

$\Rightarrow \arg H_{ap}(e^{j\omega})$ is a monotone decreasing function



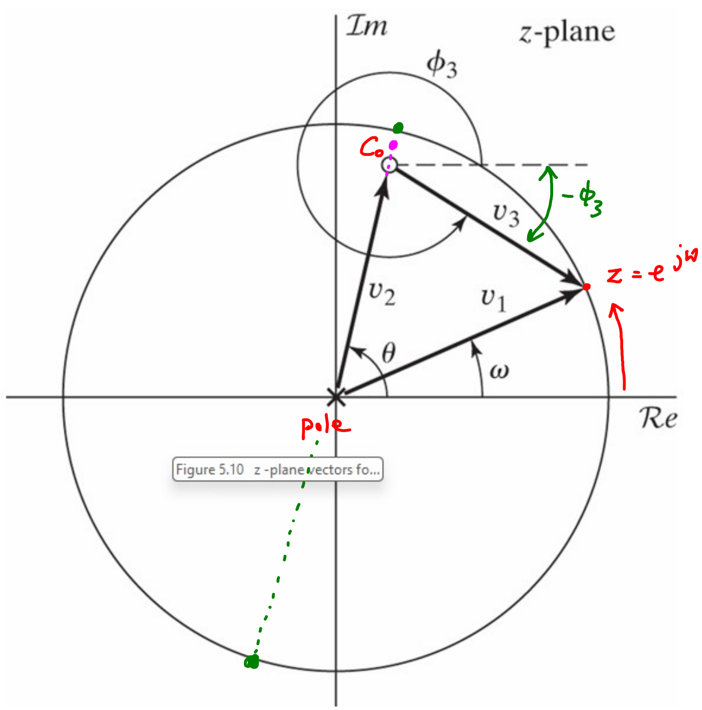
<u>General result</u>

Causal, stable Allpass $\Rightarrow$ all poles strictly inside unit circle

$\Rightarrow$ phase is monotone decreasing.

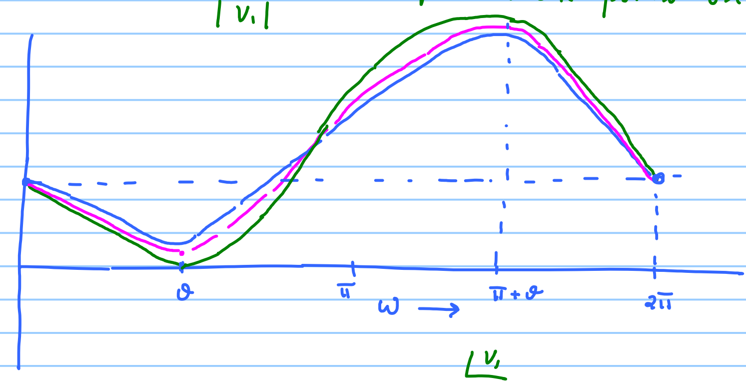
Magnitude response of first order zero

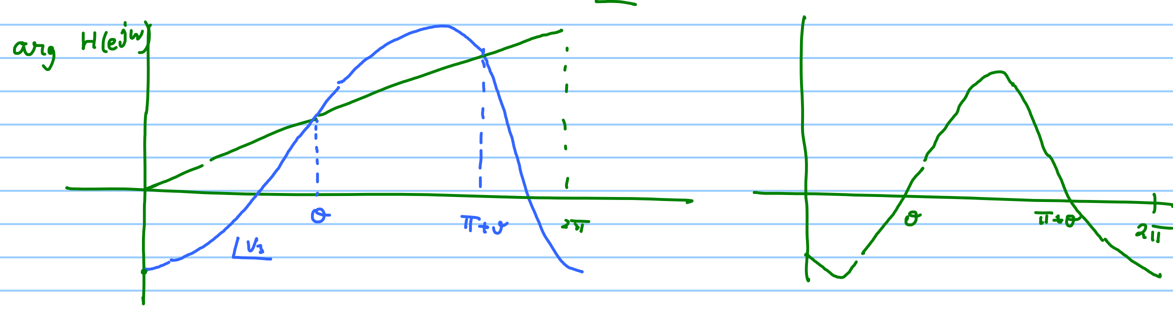
$$H(z) = (1 - c_0 z^{-1}) = \frac{z - c_0}{z} = \frac{v_3}{v_1}$$



$$|H(e^{j\omega})| = \frac{|v_3|}{|v_1|}$$

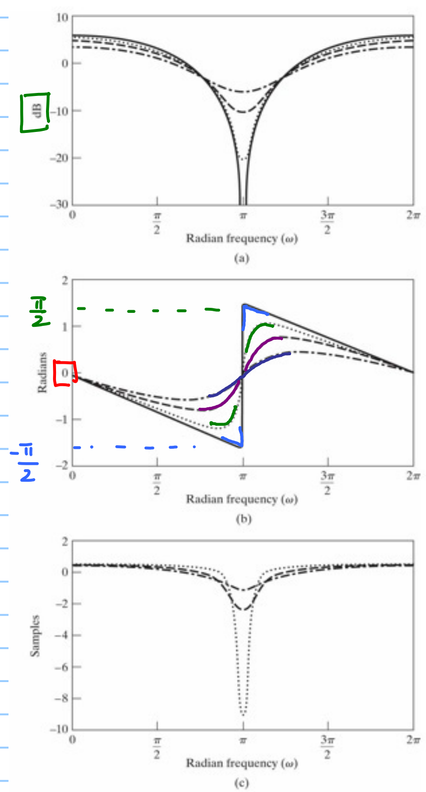
compute at all points on the unit circle





$$\arg H(e^{j\omega}) = \angle v_3 - \angle v_1$$

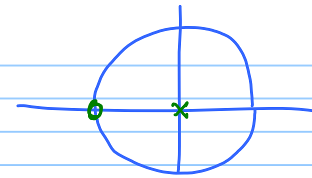
$r = 1$
$r = 0.9$
$r = 0.7$
$r = 0.5$

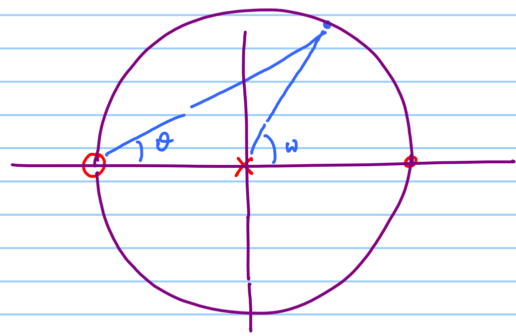


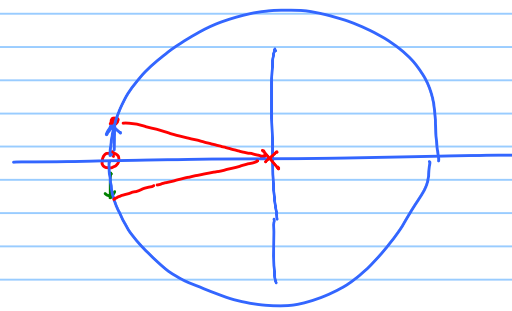
$H(z) = (1 + z^{-1})$

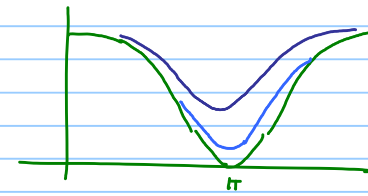
$\boxed{C_0 = -1}$

$C_0 = -0.9$



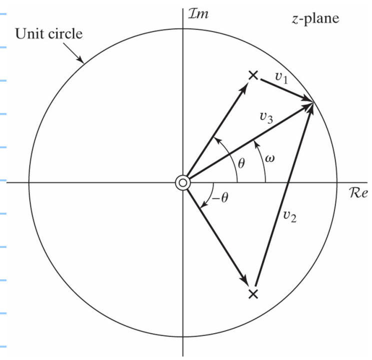


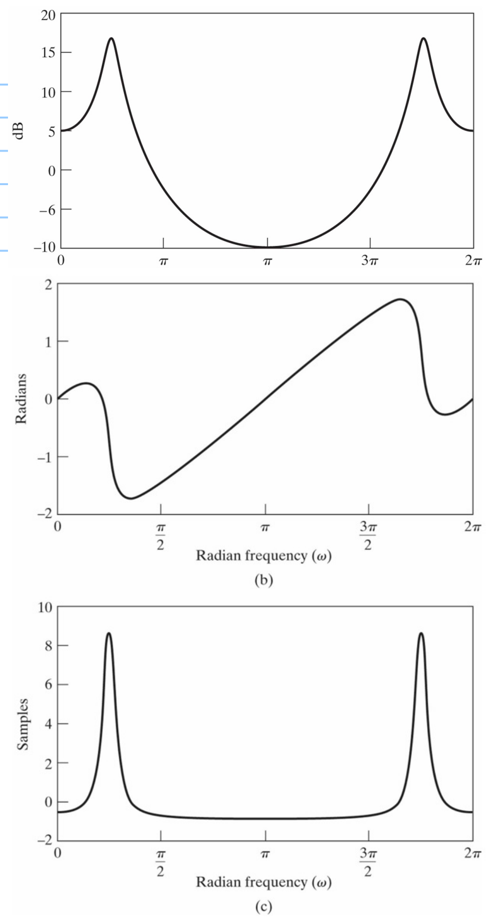




① Magnitude response
② Phase response
③ Group response

O&S Example 5.6




20
15
10
5
0
−6
−10
dB
0
π
π
3π
2π
2
1
0
−1
−2
Radians
0
π/2
π
3π/2
2π
Radian frequency (ω)
(b)
10
8
6
4
2
0
−2
Samples
0
π/2
π
3π/2
2π
Radian frequency (ω)
(c)

O & S Example 5.8