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Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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Session # 5

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EE3101 Digital Signal Processing

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Session 5

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Outline

Last session

- DT signals & properties
- LTI systems
- unit sample response
- Convolution

Week 1-2

Introduction to sampling - Review of Signals and Systems: Basic operations on signals - Discrete time complex exponentials and other basic signals – system properties (linearity, time-invariance, memory, causality, BIBO stability) -- impulse response and convolution

Reading Assignment

Oppenheim & Schafer : Sec 2.0 – 2.4

Rawat : Chapter 1

Gen representation of any DT sequence $x[n]$ $\quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$ $= x[0]\,\delta[n] + x[1]\,\delta[n-1] + \cdots$

$+ x[-1]\,\delta[n+1] + x[-2]\,\delta[n+2] + \cdots$

DT sinusoid $\quad A\cos(\omega_0 n + \phi)$

If Periodic with period $= N$ if $\boxed{\dfrac{2\pi}{\omega_0} = \dfrac{N}{m}}$ Rational function

N, m are integers

Sequence of operations on DT sequences

1. Shift
2. upsampling
3. downsamp
4. Time reversal

(2) and (3): Time Scaling

Even & Odd signals

$x_1[n] = x_1[-n]$ even

$x_1[n] = -x_1[-n]$ odd

Important Results

$x[n] \longrightarrow \boxed{\uparrow N} \longrightarrow \boxed{\downarrow N} \longrightarrow y[n] = x[n]$

$x[n] \longrightarrow \boxed{\uparrow N} \longrightarrow \boxed{\downarrow M} \longrightarrow y_1[n]$ $\qquad x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\uparrow N} \longrightarrow y_2[n]$

$y_1[n] = y_2[n]$ if and only if N & M are relatively prime

$x[n] \longrightarrow \boxed{\uparrow N} \longrightarrow \boxed{\downarrow N} \longrightarrow y_3[n]$ $\;\neq\;$ $x[n] \longrightarrow \boxed{\downarrow N} \longrightarrow \boxed{\uparrow N} \longrightarrow y_4[n]$

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DT System

$$x[n] \longrightarrow \boxed{T\{\quad\}} \longrightarrow y[n]$$

Properties of DT Systems

1. Memoryless property

2. Linearity

Systems that satisfy principle of Superposition → scaling, additivity

$T\{x_1[n]\} = y_1[n]$

$T\{x_2[n]\} = y_2[n]$

DT System is $\boxed{\text{linear}}$ if

Scaling

$$T\{a x_1[n] + b x_2[n]\} = \underbrace{a y_1[n] + b y_2[n]}_{\text{additivity}}$$

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Exercise

Check the following DT for linearity

1. $y[n] = n\,x[n]$

2. $y[n] = x^2[n]$

3 $y[n] = Re\{x[n]\}$

4. $y[n] = x[n]\;x[n-1]$

Time Invariance

$T\{x_1[n]\} \longrightarrow y_1[n]$

$T\{x_1[n-n_0]\} \longrightarrow y_1[n-n_0]$

System is Time Invariant

Check if the following DT Systems are Time / shift invariant.

Exercise

* ① $y[n] = x[-n]$

② $y[n] = a\,x[n]$

③ $y[n] = n\,x[n]$

* ④ $y[n] = x[2n]$

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LTI

$\boxed{\text{Linearity \& Time Invariance}}$ $\equiv$ Impulse Response

$$\boxed{T[\delta[n]] = h[n]}$$

unit impulse — impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{h[n-k]}_{h[-k+n]}$$

$$y[n] = x[n] * h[n] = h[n] * x$$

CONVOLUTION Operation

Computing the Output

1. Obtain $h[n]$ unit sample response
2. Time reversal $h[n]$
3. Shift the time-reversed seq
   → at each shift, obtain one output point

DT Convolution

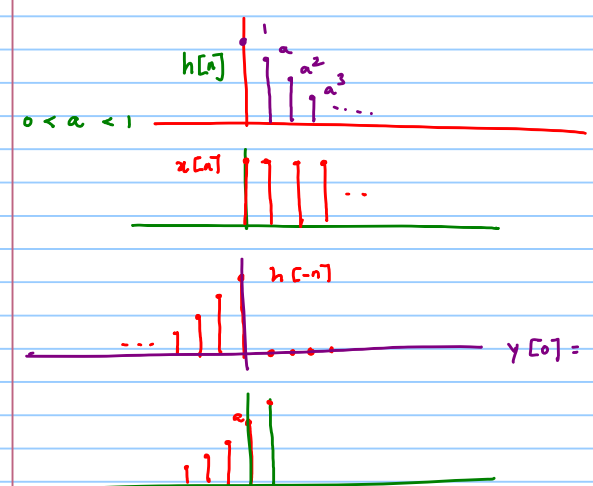
Ex LTI System $\Rightarrow$ impulse response

$$h[n] = a^n \, u[n]$$

$$x[n] = u[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{h[n-k]}_{h[-k+n]}$$

$0 < a < 1$



$y[0] = 1$

$y[0] = 1$ ✓

$y[1] = 1 + a$ ✓

$y[2] = 1 + a + a^2$ ✓

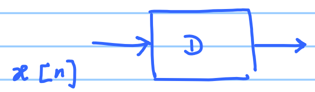
$y[-1] = 0$ ✓

$y[-2] = 0$ ✓

$\vdots$

$$y[n] = \left(1 + a + a^2 + \cdots a^n\right) u[n]$$

DT system



$y[n] = x[n-1]$

$x_1[n] \longrightarrow y_1[n] = x_1[n-1]$

$x_2[n] \longrightarrow y_2[n] = x_2[n-1]$

$x[n] = x_1[n] + x_2[n] \longrightarrow y[n] = x[n-1]$

$= \underbrace{x_1[n-1]}_{y_1[n]} + \underbrace{x_2[n-1]}_{y_2[n]}$

Linear system

✓ ① Scaling

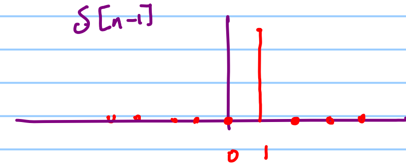
✓ ② Additivity

Linear

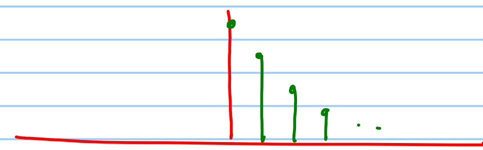
✓ Time Invariance

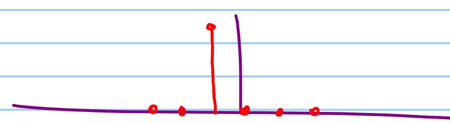
LTI → impulse response

$\delta[n-1]$



$x[n] = a^n u[n]$





$y[0] = 0$

$y[1] = 1$

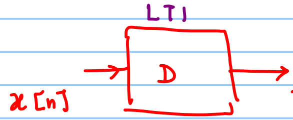
$y[2] = a$

$y[3] = a^2$

$\vdots$

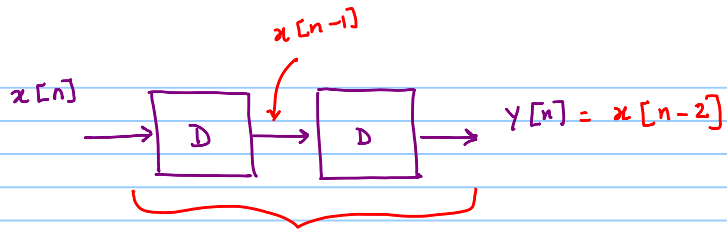
$x[n] * \delta[n] \longrightarrow x[n]$

$x[n] * \underbrace{\delta[n-1]}_{\text{LTI}} \longrightarrow x[n-1]$

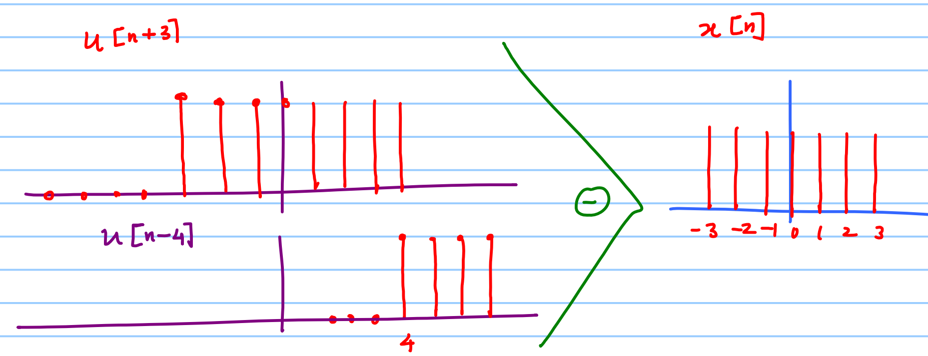


$y[n] = x[n-1]$

$h[n] = \delta[n-1]$



$h[n] = \delta[n-2]$



Example LTI system $x[n] = u[n+3] - u[n-4]$

$h[n] = x[n]$

$y[n] = x[n] * h[n]$

$= \sum_{-\infty}^{\infty} x[k]\, h[n-k]$

$y[0] = 7$		
$y[1] = 6$		$y[-1] = 6$
$y[2] = 5$		$y[-2] = 5$
$y[3] = 4$		$\vdots$
$y[4]$	3	
$y[5]$	2	
$y[6]$	1	
$y[7]$	0	

Convolution - Analytical Method

Example $x[n] = a^n u[n] \quad 0 < a < 1$

$x[n] = h[n]$

$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

Step 1 Substitute for $x[k]$

$= \sum_{k=-\infty}^{\infty} a^k u[k]\, h[n-k] = \sum_{k=0}^{\infty} a^k \underline{h[n-k]}$

Step 2 substitute for $h[n]$ $\quad h[n] = a^n u[n]$

$h[n-k] = a^{n-k} u[n-k]$

$y[n] = \sum_{k=0}^{\infty} a^k a^{n-k} \underline{u[n-k]}$ $\quad = \sum_{k=0}^{n} a^n = a^n \sum_{k=0}^{n} 1$

non-zero if $n-k \geq 0$

$\boxed{k \leq n}$ $\quad = a^n (n+1)\, u[n]$

$y[n] = (n+1)\, a^n u[n]$

$$\boxed{y[32] = 33\, a^{32}}$$

Homework

LTI system $h[n] = a^n u[n-1]$ $0 < a < 1$

$x[n] = u[n-1]$

$= \begin{cases} 1 & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$h[0] = 0$
$h[1] = a$
$h[2] = a^2$
$\vdots$

$\begin{cases} a^n & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} u[k-1]\, a^{n-k}\, \underbrace{u[n-k-1]}_{u[n-1-k]} = \sum_{k=1}^{n-1}$$

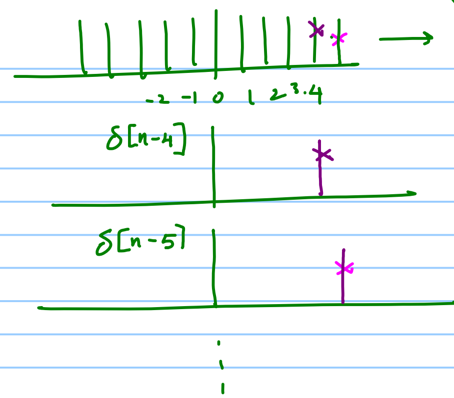
To be completed

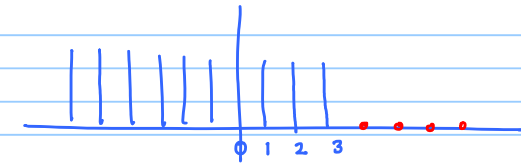
Example (Combination of Concept)

$$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k]$$

$$x[n] = u[Mn - n_0]$$

what is M, n_0





$$x_1[n] = u[n+3]$$

$$y[n] = x_1[-n] = u[-n+3]$$

$$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k] = \underbrace{u[-n+3]}_{A} = \underbrace{u[Mn-n_0]}_{B}$$

Comparing

$$M = -1$$
$$n_0 = -3$$

$$\Longrightarrow u[Mn-n_0] = u[-n+3]$$

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Causality

A DT system $y[n] = T\{x[n]\}$ is causal if for every choice of n_0 the output seq $y[n_0]$ depends only on the input seq values

$x[n_0], x[n_0-1], x[n_0-2] \dots$

$y[n_0]$ depends on $x[n]$ $n \le n_0$

LTI systems → characterized by unit sample response $h[n]$

→ causal if $\boxed{h[n] = 0 \qquad n < 0}$

Ex 1 $y[n] = x[-n]$

$x[n] \rightarrow \boxed{T\{\ \}} \rightarrow y[n] = x[-n]$

$y[0] = x[0]$ ✓

$y[1] = x[-1]$ ✓

$y[2] = x[-2]$ ✓

Causal ?

$y[-1] = x[1]$

not causal

Ex 2 $y[n] = x[n] \underbrace{\cos(n+1)}_{\text{scale factor}}$ causal

Energy of DT Seq $x[n]$

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Observation

If $x[n]$ is periodic $\mathcal{E}_x \to \infty$

Power of DT Seq $x[n]$

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$$

Periodic signal Period $= N$

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

Partial Energy $\mathcal{E}_{x,K} \triangleq \sum_{n=-K}^{K} |x[n]|^2$

$$P_x = \lim_{K\to\infty} \frac{\mathcal{E}_{x,K}}{2K+1}$$

Periodic Signals $\mathcal{E}_x = \infty$, P_x finite $P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$ Power signals

Finite duration signal $\mathcal{E}_x$ = finite $P_x \to 0$ Energy signals

Boundedness

A seq $\{x[n]\}$ is said to be bounded if $|x[n]| \leq B_x < \infty$

eg $A\cos(\omega_0 n + \phi) = x[n]$ $\quad |x[n]| \leq A \Rightarrow x[n]$ is bounded

$x_2[n] = A\alpha^n u[n] \quad \alpha > 1 \quad x[n]$ is not bounded

Absolutely Summable $\quad \sum_{n=-\infty}^{\infty} |x[n]| < \infty$

Square Summable $\quad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \Rightarrow$ Energy signal

Bounded seq $\qquad 0 < \rho < 1$

Ex $\quad y[n] = \sum_{k=0}^{\infty} \rho^k x[n-k] = x[n] + \rho\, x[n-1] + \rho^2 x[n-2] + \cdots \qquad$ BIBO stable

Given $x[n]$ is bounded $= B_x$

Is $y[n]$ bounded ?

$$\boxed{\begin{aligned} A &= B + C \\ |A| &= |B + C| \\ &\leq |B| + |C| \end{aligned}}$$

$$|y[n]| \leq \underbrace{|x[n]|}_{\leq B_x} + \rho \underbrace{|x[n+1]|}_{\leq B_x} + \rho^2 \underbrace{|x[n+2]|}_{\leq B_x} + \cdots$$

$$\leq B_x \left\{ 1 + \rho + \rho^2 + \cdots \right\}$$

$$|y[n]| \leq \underbrace{B_x \frac{1}{1-\rho}}_{B_y} \quad \forall n \quad \Rightarrow \quad y[n] \text{ is a bounded seq}$$

Ex. $y[n] = \frac{1}{3}\left[x[n] + x[n-1] + x[n-2]\right]$

- Averaging filter
- Causal ✓
- LTI $\longrightarrow$ $h[n] = \left[\underset{\uparrow}{0}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0 \ldots\right]$

$|x[n]| \leq B_x \quad \forall n$

$$|y[n]| \leq \frac{1}{3}\left[\underbrace{|x[n]|}_{\leq B_x} + \underbrace{|x[n-1]|}_{\leq B_x} + \underbrace{|x[n-2]|}_{\leq B_x}\right]$$

$$\leq \frac{1}{3} B_x [1+1+1]$$

$$\boxed{|y[n]| \leq B_x}$$

$y[n]$ is a bounded

DT System (Bounded Input, Bounded Output) Stability

BIBO

$y[n] = \frac{1}{3}\left[x[n] + x[n-1] + x[n-2]\right]$

BIBO stable

LTI system $\longrightarrow$ impulse response $h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$x[n]$ is bounded $\quad |x[n]| \leq B_x < \infty \quad \forall n$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} \left| h[k]\, x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]| \underbrace{|x[n-k]|}_{\leq B_x}$$

If $\sum_{k=-\infty}^{\infty} |h[k]| \leq A < \infty$ Absolute summability

$$|y[n]| \leq B_x \underbrace{\sum_{k=-\infty}^{\infty} |h[k]|}$$

$|y[n]| \leq A \cdot B_x \quad \Rightarrow y[n]$ is also bounded

Linear Constant Coefficient Difference Eqn (LCCDE)

Ex

$$\underbrace{y[n] - 2y[n-1] = 4x[n]}_{\text{recursive system}}$$

$y[n] = 2y[n-1] + 4x[n]$ $= 4y[n-2] + 4x[n] + 8x[n-1]$

$y[n-1] = 2y[n-2] + 4x[n-1]$

$y[n-2]$

$y[n-3]$

LTI ?

$x_1[n] \longrightarrow y_1[n]$

$x_2[n] \longrightarrow y_2[n]$

$y_1[n] - 2y_1[n-1] = 4x_1[n]$

$y_2[n] - 2y_2[n-1] = 4x_2[n]$

$x[n] = ax_1[n] + bx_2[n] \longrightarrow \underline{y[n] \overset{?}{=} ay_1[n] + by_2[n]}$

$y[n] - 2y[n-1] = 4x[n]$ $= a\,4x_1[n] + b\,4x_2[n]$

$= a\left(y_1[n] - 2y_1[n-1]\right) + b\left(y_2[n] - 2y_2[n-1]\right)$

$= ay_1[n] + by_2[n] - 2\left(ay_1[n-1] + by_2[n-1]\right)$ ✓

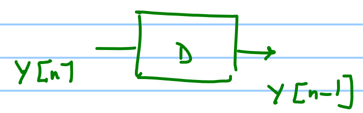
$y[n] - 2y[n-1] = 4x[n]$

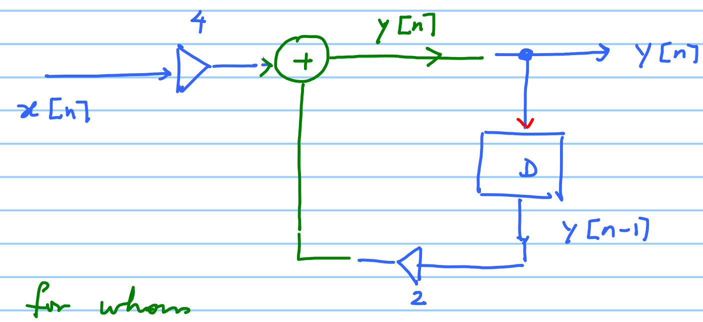
is LTI ✓

$y[n] = 4x[n] + 2y[n-1]$

$y[n] - 2y[n-1] = 4x[n]$ LTI system

Implement





An important class of LTI are those for whom the input/out relationship is given

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{m=0}^{M} b_m \, x[n-m]$$

a_k's & b_k's constants

$a_0 = 1$ $b_0 = 4$

$a_1 = -2$ $\left.\begin{array}{l} b_1 \\ b_2 \\ \vdots \end{array}\right\} = 0$

$\left.\begin{array}{l} a_2 \\ \vdots \end{array}\right\} = 0$

$$\frac{d}{dt} x(t) = \lim_{\Delta \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

difference
for
DT seq

$$\Delta_1 x[n] = x[n] - x[n-1]$$

$$\Delta_1 x[n-1] = x[n-1] - x[n-2]$$

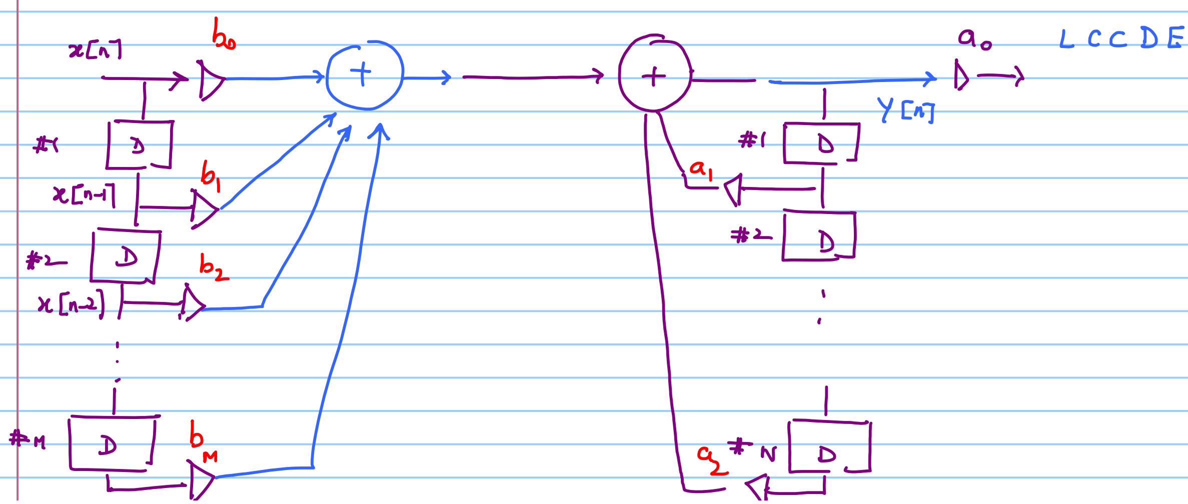
$$\vdots$$

$$\boxed{\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{m=0}^{M} b_m \, x[n-m]}$$

LCCDE

constants

Solved via Transform domain



Time Domain $\longrightarrow$ Freq Domain

Freq. Domain Representation of DT signals & System

$$\boxed{x[n] = e^{j\omega_0 n}} \quad -\infty < n < \infty$$

LTI

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,e^{j\omega_0 (n-k)}$$

$$= \underbrace{e^{j\omega_0 n}}_{\text{same as input}} \underbrace{\sum_{k=-\infty}^{\infty} h[k]\,e^{-j\omega_0 k}}_{H(e^{j\omega_0})}$$

eigenfunctions of all LTI systems

$H(e^{j\omega_0})$ Freq response of the LTI system

$$\underline{A}\,\underline{v} = \lambda\,\underline{v}$$

matrix, vector, eigenvalue

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j\,H_I(e^{j\omega})$$

$$= \left|H(e^{j\omega})\right| e^{j(\arg H(e^{j\omega}))}$$

Ex $y[n] = x[n-n_0]$

$h[n] = \delta[n-n_0]$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k} = e^{-j\omega n_0}$$

$$H(e^{j\omega}) = e^{-j\omega n_0}$$

$$\left|H(e^{j\omega})\right| = \left|e^{-j\omega n_0}\right| = 1 \quad \forall\, \omega$$

$$\arg H(e^{j\omega}) = -\omega n_0$$ Linear phase

Discrete Time Fourier Transform

FS coefficients

$$x[n] \xrightarrow{\mathcal{F}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$\xleftarrow[\mathcal{F}^{-1}]{}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underline{X(e^{j\omega})}\, e^{j\omega n}\, d\omega$$

$X(e^{j\omega})$ is a periodic signal with period $= 2\pi$

Oppenheim & Schafer Section 2.6