Note Title 02-09-2024

Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

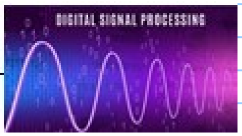
BS Electronic Systems programme

R. David Koilpillai

September – December 2024

Session # 15

November 4, 2024



4/11/24 Note Title

EE 3101 Digital Signal Processing

Session 15

EE3101
Session 15
03-01-2018

Outline

Last session
- Introduction to Z Transform

Today
- Z Transform properties

Week 7-8 O&S Chapter 3

Z-Transform: Generalized complex exponentials as eigen signals of LTI systems—z-transform definition—region of convergence (RoC)—properties of RoC—properties of the z-transform—pole-zero plots—time-domain responses of simple pole-zero plots—RoC implications of causality and stability

✓ Sec 1 Z Transform
✓ Sec 2 Properties of ROC (Region of Convergence)
Sec 3 Inverse Z Transform (Cover at end of week 8)
Sec 4 Z Transform Properties
Sec 5 Z Transform and LTI systems
Sec 6 Unilateral Z Transform (Omit)

Reading Assignment

O&S ch 3 The Z Transform

Ex 4. Obtain the DTFT $x_1[n] = \overbrace{\cos w_0 n \; u[n]}^{w[n]}$

Using Multiplication Property

$$X_1(e^{jw}) = \frac{1}{2\pi} W(e^{jw}) \circledast U(e^{jw})$$

$$\left.\begin{aligned} \cos w_0 n &\longleftrightarrow \pi[\delta(w-w_0) + \delta(w+w_0)] = W(e^{jw}) \\ u[n] &\longleftrightarrow \pi\delta(w) + \frac{1}{1-e^{-jw}} \end{aligned}\right\} \quad -\pi \leq w < \pi$$

Convolving with a Dirac Delta $\Rightarrow$ freq shift

$$W(e^{jw}) * U(e^{jw}) = \frac{1}{2\pi}\left[\pi[\delta(w-w_0) + \delta(w+w_0)]\right] \circledast \left[\pi\delta(w) + \frac{1}{1-e^{-jw}}\right]$$

$$\boxed{\begin{aligned} \frac{1}{2\pi}[\pi\delta(w-w_0) * \pi\delta(w)] &= \frac{\pi}{2}\delta(w-w_0) \\ \frac{1}{2\pi}\left[\pi\delta(w-w_0) * \frac{1}{1-e^{-jw}}\right] &= \frac{1/2}{1-e^{-jw_0}} \end{aligned}}$$

$$= \frac{\pi}{2}\left(\delta(w-w_0) + \delta(w+w_0)\right) + \frac{1}{2}\underbrace{\left[\frac{1}{1-e^{-jw_0}} + \frac{1}{1-e^{jw_0}}\right]}_{=1}$$

$$\boxed{X_1(e^{jw}) = \frac{1}{2} + \frac{\pi}{2}\left(\delta(w-w_0) + \delta(w+w_0)\right) \quad -\pi \leq w < \pi}$$

(ii) $x_2[n] = \sin w_0 n \, u[n]$

$$X_2(e^{jw}) = -\frac{1}{2}\cot\left(\frac{w_0}{2}\right) + \frac{\pi}{2}\left(\delta(w-w_0) - \delta(w+w_0)\right) \quad -\pi \leq w < \pi$$

Z Transform

Continuous time signals < Fourier Transf / Laplace Transform

Complex plane

s covers entire complex plane

$s = \sigma + j\Omega$

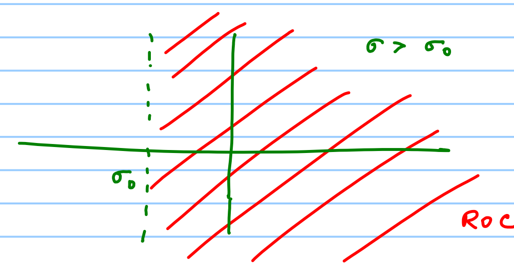
Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\Omega t}\, dt$$

Laplace Transform generalized form

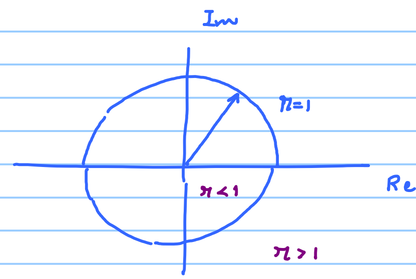
Fourier Transform is a special case of LT



Z Transform is a generalization of the DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$



$z = re^{j\omega}$

$z^{-n} = r^{-n} e^{-j\omega n}$

Z Transform is the DT counterpart of the LT

DTFT $\quad z = e^{j\omega} \Rightarrow |z| = 1$ or $r = 1$

General form $\quad z = re^{j\omega}$

Defn $\quad X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \left[x[n]\, r^{-n}\right] e^{-j\omega n} \equiv$ DTFT of $x[n]\, r^{-n}$

$\{r \in R \mid r \geq 0\}$

$$x[n] \xrightarrow{\;Z\;} X(z)$$

$$\xleftarrow[\text{Inv. ZT}]{}$$

Ex : $\quad x[n] = a^n\, u[n]$

$a = 3 \quad$ - growing exponential

- Not abs. summable

- DTFT does not exist

ZT of $x[n]$?

$$X(z) = \sum_{n=0}^{\infty} 3^n z^{-n} = \frac{1}{1 - 3z^{-1}}$$

$|3z^{-1}| < 1$

$|z| > 3$

$z = re^{j\omega}$

$|re^{j\omega}| > 3$

$r > 3$

ZT of $x[n] \equiv$ DTFT $\left(x[n] r^{-n} \right)$

Condition for existence (uniform convergence) — Absolutely summable

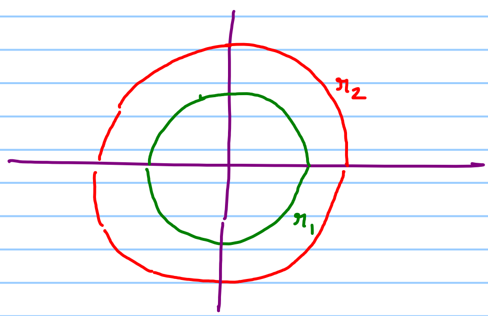
$$\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty$$

ROC of ZT $\equiv$ range of values of r for which $x[n] r^{-n}$ is abs summable

$\boxed{\text{ROC depends on the value of } r}$

If ZT exists for one value of r ie, $z = r e^{j\omega_0}$

then ZT exists for all points on circle with radius



ROC $|z| > r_1$ → right-sided (subset: Causal seq)

ROC cannot include any poles

$|z| < r_1$ → left-sided

ROC

$|z| > r_2$

$|z| < r_1$

$r_1 < |z| < r_2$ → Two-sided

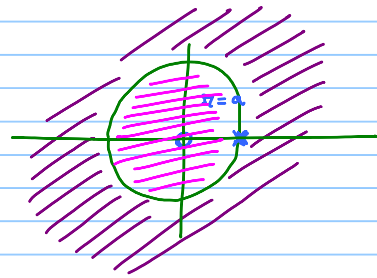
$H(x) = \dfrac{P(x)}{Q(x)}$ zeros of $P(x)$ = zeros of $H(x)$

Zeros of $Q(x)$ = poles of $H(x)$

Ex. $x[n] = a^n u[n]$ → causal seq.

$X(z) = \frac{1}{1-az^{-1}}$ $\quad |z| > a$

$= \frac{z}{z-a}$



ROC should not include pole

Part 2

$x[n] = -a^n u[-n-1]$ $\quad \left\{ \dots \ -\frac{1}{a^3} \ \overset{n=-2}{-\frac{1}{a^2}} \ \overset{n=-1}{-\frac{1}{a}} \ 0 \ 0 \ 0 \ \dots \right\}$

$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n}$

$x[n] = -a^n u[-n-1] \longleftrightarrow X(z) = \frac{1}{1-az^{-1}}$ $\quad |z| < a.$

Causal sequences
(right-sided sequences)
ROC $|z| > r_R$

(Left sided sequences)
ROC $|z| < r_L$

Result

A DT signal $x[n]$ is uniquely represented by ZT & ROC

$X(z)$

Ex

$$x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u(-n-1)$$

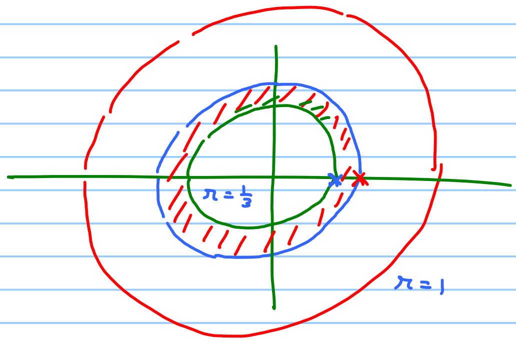
$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}$$

ROC_1: $|z| > \frac{1}{3}$ ROC_2: $|z| < \frac{1}{2}$

poles

@ $z = \frac{1}{3}$

$z = \frac{1}{2}$



$X(e^{j\omega})$ exists ? NO

ROC of $X(z)$ is intersection of ROC_1 & ROC_2

Finite Length Seq

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$$

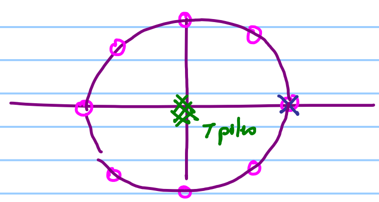
$X(z)$ is a polynomial of order $(N-1)$ with only negative powers of z

ROC <u>entire Z plane</u> except $z=0$

$$X(z) = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \; \frac{z^N - a^N}{z-a}$$

pole @ $z = a$

$N=8$



pole - zero cancellation $\Rightarrow$ ROC is entire Z plane excluding $Z=0$

Properties of ROC

Assume $X(z)$ is a rational form $\frac{P(z)}{Q(z)}$

\#1 ROC is a $\overbrace{\text{ring or a disc}}^{\text{annulus}}$ in Z plane centred @ origin

right-seq — Two-sided

$$0 \le r_R < |z| < r_L \le \infty$$

left-sided

\#2 DTFT converges uniformly if ROC includes unit circle $|z|=1$ or $r=1$ ✓

\#3 ROC cannot include any poles

\#4 If $x[n]$ is a finite duration seq

$$-\infty < N_1 \le n \le N_2 < \infty$$

Then ROC is the entire Z plane except possibly $z=0$ or $z=\infty$

$$X(z) = x[-1]z + x[0] + x[1]z^{-1}$$

$z=\infty$ pole — $z = 0$ pole

\#5 $x[n]$ is a right sided seq

$\Rightarrow$ seq is zero $n < N_1 < \infty$

ROC $|z| > r_R \leftarrow$ outermost pole of $X(z)$ (possibly exclude $z=\infty$)

\#6 $x[n]$ is a left sided seq

$\Rightarrow$ seq is zero $n > N_2 > -\infty$

ROC $|z| < r_L \leftarrow$ innermost pole of $X(z)$ (possibly exclude $z=0$)

#7 A two sided sequence is an ∞ duration seq that is neither left sided or right sided

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \underbrace{\sum_{n=0}^{\infty} x[n] z^{-n}}_{|z| > r_R} + \underbrace{\sum_{n=-\infty}^{-1} x[n] z^{-n}}_{|z| < r_L}$$

$$0 \leq r_R < |z| < r_L \leq \infty$$

#8 ROC must be a fully connected region
(no poles in ROC)

Props 1, 5, 6, 7

$$0 \leq \overbrace{r_R < |z|}^{\text{right-sided}} \underbrace{< r_L}_{\text{Left-sided}} \leq \infty$$

entire z plane except $z=0$ or/and $z=\infty$

Props 2, 3, 4, 8

Connected

No poles

If ROC incl. $|z|=1$, DTFT exists

Examples

Specify ROC for the following

1. $x[n] = \delta[n+2] + 3\,\delta[n] + \delta[n-2] = \{0, 0 \ldots 1, 0, \underset{\uparrow}{3}, 0, 1, 0 \ldots\}$

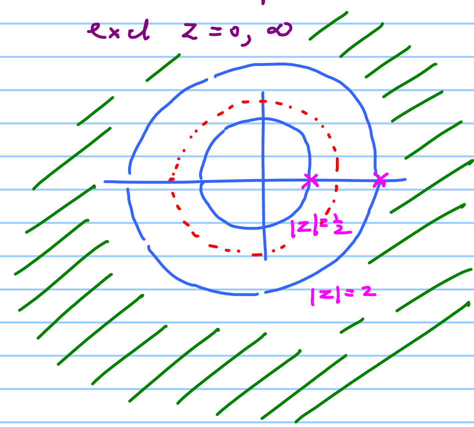
   $X(z) = z^{2} + 3 + z^{-2}$ ROC entire Z plane excl $z = 0, \infty$

   $0 < |z| < \infty$

2. $x[n] = 2^{n} u[n] + \left(\frac{1}{2}\right)^{n} u[n]$

   $X(z) = \dfrac{1}{1 - 2z^{-1}} + \dfrac{1}{1 - \frac{1}{2} z^{-1}}$

   $|z| > 2$ $\qquad$ $|z| > \frac{1}{2}$



ROC$_1$ $|z| > 2$

ROC$_2$ $|z| > \frac{1}{2}$

ROC$_1 \cap$ ROC$_2$

$|z| > 2$

Right sided? Yes $\quad$ Causal Yes

DTFT exist? $|z| = 1$ not in ROC

$\to$ Not absolutely summable
DTFT does not exist

(3) $x[n] = \alpha^{|n|} \qquad 0 < \alpha < 1$

decaying exponential $\rightarrow n \geq 0$, $n < 0$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^{|n|} z^{-n} = \sum_{n=0}^{\infty} \alpha^{n} z^{-n} + \underbrace{\sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n}} = \sum_{m=1}^{\infty} \alpha z = \frac{\alpha z}{1-\alpha z}$$

$$= \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z}{1-\alpha z}$$

$|z| > \alpha \qquad |z| < \frac{1}{\alpha}$

$|\alpha z| < 1$

$|z| < \frac{1}{\alpha}$

Ex

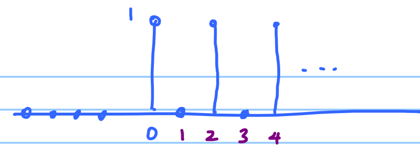
$\alpha = 0.5$

$ROC_1 \cap ROC_2$

$|z| > 0.5 \qquad |z| < 2$

$$\boxed{0.5 < |z| < 2}$$

4 $x[n] = \frac{1}{2}\left(1 + (-1)^n\right) u[n]$



$x[n] = \{0\ 0 \cdots 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \cdots\}$ ↑

1

$X(z) = 1 + z^{-2} + z^{-4} \cdots = \frac{1}{1 - z^{-2}}$ $\quad |z^{-2}| < 1$

$|z^2| > 1$

$|z| > 1$ ROC

$x[n] = \frac{1}{2} u[n] + \frac{1}{2}(-1)^n u[n]$

$X(z) = \frac{1}{2} \frac{1}{1 - z^{-1}} + \underbrace{\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n z^{-n}}_{\frac{1}{2} \frac{1}{1+z^{-1}}}$ $\qquad \frac{1}{2} \sum_{n=0}^{\infty} (-z)^{-n} = \frac{1}{1 + z^{-1}}$

$|z| > 1$ $\qquad |z| > 1$ $\qquad |-z^{-1}| < 1$

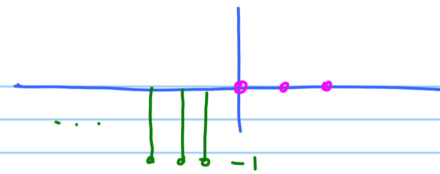
$|z| > 1$

2

$X(z) = \frac{1}{2}\left[\frac{1}{1 - z^{-1}} + \frac{1}{1 + z^{-1}}\right]$ $\quad |z| > 1$

Verify Exp 1 $\equiv$ Exp 2

(5) $x[n] = -u[-n-1]$



$$X(z) = -\sum_{n=-\infty}^{-1} z^{-n}$$

$$= -\sum_{m=1}^{\infty} z^{m} = -\frac{z}{1-z} = \frac{1}{1-z^{-1}}$$

$|z| < 1$

z Transform of $u[n]$
$|z| > 1$

(6) Given $x[n]$ a finite length sequence, absolutely summable

$$\sum_{n=N_1}^{N_2} |x[n]| < \infty \overset{?}{\Longrightarrow} x[n] \text{ is a bounded sequence} \quad \text{Yes}$$

$|x[n]| < B_x$

Statements T/F

True (i) ROC is the entire Z plane except possibly $z=0$ and/or $z=\infty$

True (ii) $X(z)$ always exists

True (iii) DTFT of $x[n]$ has uniform convergence

Table 3.1 SOME COMMON z-TRANSFORM PAIRS

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $\boxed{na^n u[n]}$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

ROC

① $X(z) = 1$ entire Z plane

② $X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$ $|z|>1$

④ $m=2$ $\{0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \dots\}$

$X(z) = z^{-2}$ $\begin{cases}\text{excl. } z=0 \\ \text{entire } z \text{ plane}\end{cases}$

⑦

⑬ $x[n] = a^n$ $0 \le n \le N-1$

$X(z) = \dfrac{1-a^N z^N}{1-az^{-1}}$ pole-zero cancellation

ROC entire Z except $z=0$

$|z| > 0$

#7

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$\frac{d}{dz} X(z) = \frac{d}{dz}\left(\sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \right) = \sum_{n=-\infty}^{\infty} x[n]\,(-n)\, z^{-(n+1)}$$

$$\frac{d}{dz} X(z) = -z^{-1} \sum_{n=-\infty}^{\infty} n\, x[n]\, z^{-n}$$

$$-z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n\, x[n]\, z^{-n}$$

$$n\, x[n] \longleftrightarrow -z \frac{d}{dz} X(z)$$

$$a^{n} u[n] \longleftrightarrow \frac{1}{1 - a z^{-1}} \qquad |z| > a$$

$$n a^{n} u[n] \longleftrightarrow -z \frac{d}{dz}\left(\frac{1}{1 - a z^{-1}} \right) = \frac{a z^{-1}}{(1 - a z^{-1})^{2}} \qquad |z| > a$$

Ex $H(z) = \dfrac{K\,(1+4z^{-1})(1+2.5z^{-1})}{(1+3z^{-1})(1-1.5z^{-1})(1-0.9z^{-1})}$

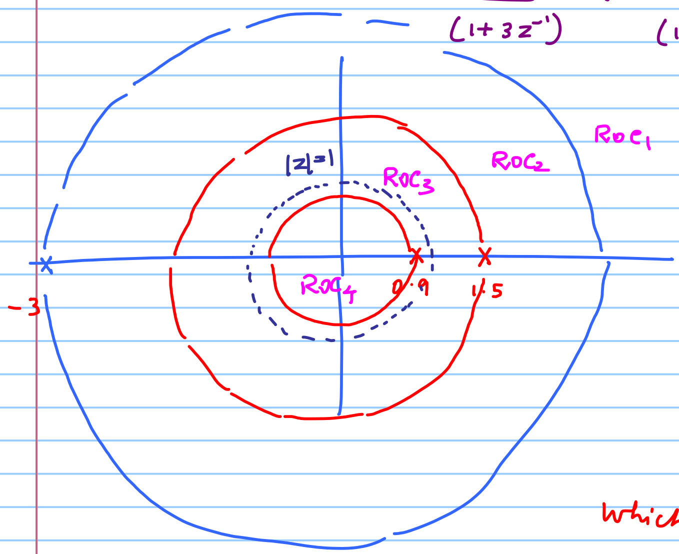
zeros $z = -4$, $z = -2.5$

poles $z = -3$, $z = 1.5$, $z = 0.9$

Partial Fraction Expansion

$$= \frac{A_1}{(1+3z^{-1})} + \frac{A_2}{(1-1.5z^{-1})} + \frac{A_3}{(1-0.9z^{-1})}$$

Assume A_1, A_2, A_3 are computed



ROC1 $|z| > 3$ $\quad A_1(-3)^n u[n] + A_2 (1.5)^n u[n] + A_3 (0.9)^n u[n]$

3 Right sided

ROC2 $1.5 < |z| < 3$ $\quad -A_1(-3)^n u[-n-1] + A_2 (1.5)^n u[n] + A_3 (0.9)^n u[n]$

2R, 1L

ROC3 $0.9 < |z| < 1.5$ 1R, 2L

ROC4 $|z| < 0.9$ 3L

Which of these seq. is BIBO stable? DTFT exists?

ROC3 $0.9 < |z| < 1.5$

Non-causal sequence

Ex

$$x[n] = r_0^n \sin w_0 n \, u[n] = \frac{1}{2j}\left[r_0^n e^{jw_0 n} - r_0^n e^{-jw_0 n} \right] u[n]$$

$$X(z) = \underbrace{\frac{1}{2j}\sum_{n=0}^{\infty} r_0^n e^{jw_0 n} z^{-n}}\; - \;\underbrace{\frac{1}{2j}\sum_{n=0}^{\infty} r_0^n e^{-jw_0 n} z^{-n}}$$

$$= \frac{1}{2j}\left[\frac{1}{1 - r_0 e^{jw_0} z^{-1}} - \frac{1}{1 - r_0 e^{-jw_0} z^{-1}} \right] \qquad \text{ROC } |z| > r_0$$

$$|r_0 e^{jw_0} z^{-1}| < 1 \qquad\qquad |r_0 e^{-jw_0} z^{-1}| < 1$$

$$|z| > r_0 \qquad\qquad |z| > r_0$$

$$= \frac{1}{2j}\left[\frac{\cancel{1} - r_0 e^{-jw_0} z^{-1} - (\cancel{1} - r_0 e^{jw_0} z^{-1})}{(1 - r_0 e^{jw_0} z^{-1})(1 - r_0 e^{-jw_0} z^{-1})} \right] = \frac{\overbrace{\boxed{\frac{1}{2j}\left(r_0 e^{jw_0} - r_0 e^{-jw_0}\right)}}^{r_0 \sin w_0} z^{-1}}{1 - 2 r_0 \cos w_0 \, z^{-1} + r_0^2 z^{-2}}$$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n] \longleftrightarrow$	$X(z)$	R_x
		$x_1[n] \longleftrightarrow$	$X_1(z)$	R_{x_1} or R_1
		$x_2[n] \longleftrightarrow$	$X_2(z)$	R_{x_2} or R_2
Linearity 1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $\boxed{R_{x_1} \cap R_{x_2}}$ Linearity
Shift 2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
Multip. by expon. seq 3	3.4.3	$z_0^n x[n]$	$\boxed{X(z/z_0)}$	$\|z_0\|R_x$ $\quad \|z_0\| r_R < \|z\| < \|z_0\| r_L$
Differentiation 4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

$$aX(z) = \sum_{n=-\infty}^{\infty} a\,x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(x_1[n] + x_2[n]\right) z^{-n}$$

$$= X_1(z) + X_2(z)$$

PFE $\leftrightarrow$ Linearity Property

$$\frac{A_1}{1-a_1 z^{-1}} + \frac{A_2}{1-a_2 z^{-1}} + \frac{A_3}{1-a_3 z^{-1}}$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$a_1^n u[n] + a_2^n u[n] + a_3^n u[n]$$

$$\text{ROC}_1 \qquad \text{ROC}_2 \qquad \text{ROC}_3$$

$$\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2 \cap \text{ROC}_3$$

Ex $x[n] = r_0^n \sin w_0 n \; u[n]$

$$= \frac{1}{2j}\left[r_0^n e^{jw_0 n} - r_0^n e^{-jw_0 n} \right] u[n]$$

Property of Linearity

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

x

$$= a^n \left(u[n] - u[n-N]\right)$$

$$x[n] = a^n u[n] - a^n u[n-N]$$

$$X(z) = ZT\left[a^n u[n]\right] - ZT\left[a^n u[n-N]\right]$$

$$= \frac{1}{1-az^{-1}} - \frac{a^N z^{-N}}{1-az^{-1}} = \frac{1 - a^N z^{-N}}{1-az^{-1}}$$

$|z| > a$ $\qquad$ $|z| > a$

RoC with pole-zero cancellation

RoC entire Z plane excl. $z=0$

$|z| > 0$

~~(To be completed.)~~

2 Time-Shift Property

$x[n] \longleftrightarrow X(z) \qquad ROC = R_x$

$x_1[n] = x[n-n_0]$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n} = \sum_{l=-\infty}^{\infty} x[l] z^{-(l+n_0)} = z^{-n_0} \underbrace{\sum_{l=-\infty}^{\infty} x[l] z^{-l}}_{X(z)}$$

$l = n - n_0$

$x_1[n] \longleftrightarrow z^{-n_0} X(z) \qquad ROC = R_x$

If n_0 is positive, $z = 0$ excl. from ROC

If n_0 is negative, $z = \infty$ excl.

$$X(z) = \frac{a + b z^{-1}}{1 - \alpha z^{-1}} = \frac{a}{1 - \alpha z^{-1}} + b\, z^{-1} \frac{1}{1 - \alpha z^{-1}}$$

ROC $|z| > \alpha$

$$x[n] = a \cdot \alpha^n u[n] + b\, \alpha^{n-1} u[n-1]$$

#3 Multiplication by exponential seq

$x[n] \longleftrightarrow X(z)$ $\quad r_R < |z| < r_L$

$$z_0^n \, x[n] \longleftrightarrow \sum_{n=-\infty}^{\infty} z_0^n \, x[n] \, z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{z_0}\right)^{-n}$$

RoC $\quad r_R < \left|\frac{z}{z_0}\right| < r_L$

$|z_0| r_R < |z| < |z_0| r_L$

Scale the RoC

$|z_0| > 1$ expanding

< 1 shrinking of RoC

Case 1 $\quad z_0 = e^{j\omega_0}$

$$e^{j\omega_0 n} \, x[n] \longleftrightarrow X\left(e^{j(\omega-\omega_0)}\right) \quad \text{DTFT}$$

$$X(z) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} \, x[n] \, z^{-n}$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} x[n] \left(z e^{-j\omega_0}\right)^{-n}}_{X\left(z e^{-j\omega_0}\right)}$$

Case 2 $\quad z_0 = r_0$ (real)

$$r_0^n \, x[n] \longleftrightarrow X\left(\frac{z}{r_0}\right)$$

RoC

$r_0 r_R < |z| < r_0 r_L$

Case 3 $\quad z = r_0 e^{j\omega_0 n} \longrightarrow$ Combination of scaling of RoC & freq. shift

Ex

$$X(z) = \frac{1}{1 - 1.1 z^{-1}} = 1 + 1.1 z^{-1} + (1.1)^2 z^{-2} + \cdots$$

$|z| > 1.1$ RoC does not include the unit circle

$z_0^n x[n] \longleftrightarrow X_1(z) = 1 + \left(\frac{1.1}{1.2}\right) z^{-1} + \left(\frac{1.1}{1.2}\right)^2 z^{-2} + \cdots$

$z_0 = \frac{1}{1.2}$

$$= \frac{1}{1 - \underbrace{\left(\frac{1.1}{1.2}\right)}_{\beta} z^{-1}}$$

RoC $|z| > \beta = \frac{1.1}{1.2}$

includes unit circle

$$X\left(\frac{z}{z_0}\right) = X(1.2 z)$$

Conjugation

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$X^*(z) = \left(\sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \right)^* = \sum_{n=-\infty}^{\infty} x^*[n]\, (z^{-n})^*$$

$$(z^{-n})^* = (z^*)^{-n}$$

$$(z^{-n})^* = \left(\frac{1}{z^n}\right)^* = \left(\frac{1}{z^*}\right)^n = (z^*)^{-n}$$

$$z = re^{j\omega} \qquad z^* = re^{-j\omega}$$

$$z^{-n} = r^{-n} e^{-j\omega n}$$

$$(z^*)^{-n} = r^{-n} e^{j\omega n}$$

$$(z^{-n})^* = r^{-n} e^{j\omega n}$$