Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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Session # 12

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EE 3101   Digital Signal Processing

Session 12

Outline

Last session
   - DTFT Symmetry Props
   - DTFT Theorems

O&S ch2 Section 7-9

Week 5-6

Discrete-Time Fourier Transform (DTFT): Complex exponentials as eigensignals of LTI systems—DTFT definition—inversion formula—properties—relationship to continuous-time Fourier series (CTFS)

✓ O&S   ch 2   Sec 7   Representation of Sequences by Fourier Transforms
         ch 2   Sec 8   Symmetry Properties of Fourier Transform ⎫ Properties
         ch 2   Sec 9   Fourier Transform Theorems   ⎬   &
                                                                 ⎭ Applications

Reading Assignment

O&S ch 2 Sec 7, Sec 8 and Sec 9

Discrete Time Fourier Transform (DTFT)

$$\text{DT seq } x[n] \xrightarrow[\text{DTFT}]{\mathcal{F}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \xleftarrow[\text{Inv DTFT}]{\mathcal{F}^{-1}} X(e^{j\omega})$$

Condition for existence of DTFT

$$\text{If } \sum_{n=-\infty}^{\infty} |x[n]| < \infty \implies |X(e^{j\omega})| < \infty \quad \forall \omega \qquad \text{Absolute Summability}$$

DTFT exists & Convergence is UNIFORM

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n] e^{-j\omega n}$$

uniform convergence $\lim_{M \to \infty} X_M(e^{j\omega}) = X(e^{j\omega})$

1. Sequences not absolutely summable, but Square Summable
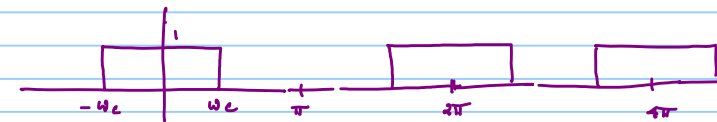
$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \qquad \sum_{n=-\infty}^{\infty} |x[n]| \longrightarrow \infty \qquad \text{DTFT converges in mean Square Sense}$$

Mean Square Convergence $\qquad X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n] e^{-j\omega n}$

$$\lim_{M \to \infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 \, d\omega \longrightarrow 0$$

$$X(e^{j\omega}) = rect\left(\frac{\omega}{2\omega_c}\right) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

Ideal LPF

$$\frac{\omega_c}{\pi} \, sinc\left(\frac{\omega_c n}{\pi}\right) \longleftrightarrow \sum_{m=-\infty}^{\infty} rect\left(\frac{\omega - 2\pi m}{2\omega_c}\right)$$
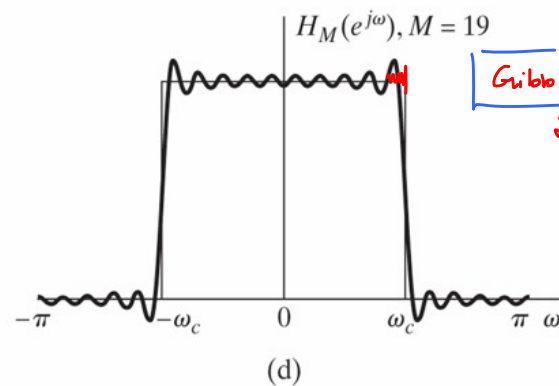
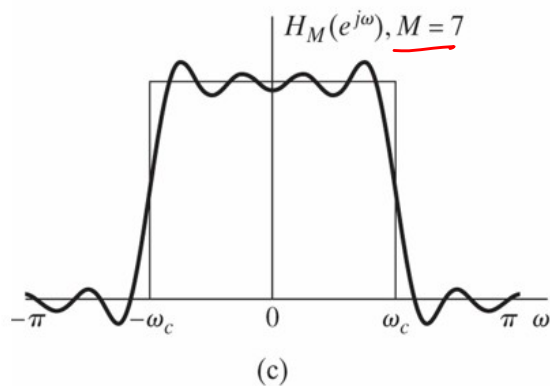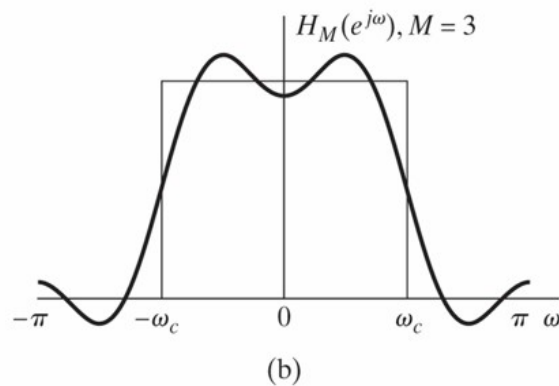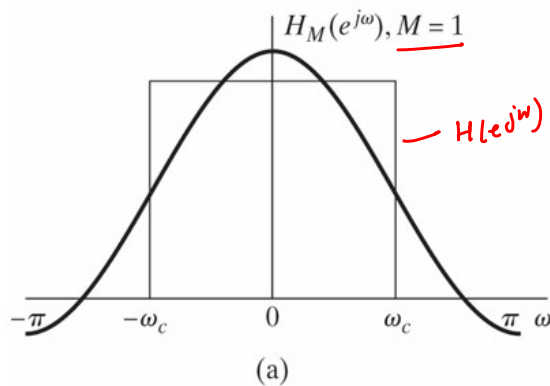2. DTFT can exist for some sequences that are not absolutely summable and not Square Summable

$$x[n] = 1 \quad \forall n \longleftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi \, \delta(\omega + 2\pi k)$$

$$x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi \, \delta(\omega - \omega_0 + 2\pi k)$$

Fig 21

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} h[n] e^{-j\omega n}$$

$$h[n] = \begin{cases} \dfrac{\omega_c}{\pi} \, \mathrm{sinc}\!\left(\dfrac{\omega_c \, n}{\pi}\right) & n \neq 0 \\[4mm] \dfrac{\omega_c}{\pi} & n = 0 \end{cases}$$

$H_M(e^{j\omega}), M = 1$

$H(e^{j\omega})$

$-\pi \qquad -\omega_c \qquad 0 \qquad \omega_c \qquad \pi \quad \omega$

(a)

$H_M(e^{j\omega}), M = 3$

$-\pi \qquad -\omega_c \qquad 0 \qquad \omega_c \qquad \pi \quad \omega$

(b)

$H_M(e^{j\omega}), M = 7$

$-\pi \qquad -\omega_c \qquad 0 \qquad \omega_c \qquad \pi \quad \omega$

(c)

$H_M(e^{j\omega}), M = 19$

$-\pi \qquad -\omega_c \qquad 0 \qquad \omega_c \qquad \pi \quad \omega$

(d)

Gibbs phenomenon

$\lim_{M \to \infty}$

Convergence in Mean-Square Sense

① Abrupt transition (discontinuity) cannot be supressed via uniform converge

② Convergence in Mean Square

✓

TABLE 2·1   SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$|X(e^{j\omega})| = |X(e^{-j\omega})|$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

① Conjugation

$$X^*(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{j\omega n}$$

② Replace $w \leftarrow -w$

$$X^*(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{-j\omega n}$$

③ Replace $m = -n$

$$X^*(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x^*[-m]\, e^{-j\omega m}$$

Results

$$x[n] \longleftrightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

✓

TABLE 2.2   FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$ *Time shift*
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$ *Freq shift*
4. $x[-n]$	$X(e^{-j\omega})$ *Time reversal* $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$ *Differentiation*
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$ *Convolution in Time*
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ *Multiplication in Time*

periodic convolution
Integrating over one period
Convolution of periodic functions

Parseval's theorem:

8. $\displaystyle\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

9. $\displaystyle\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

#3 $\underline{\text{Freq Shifting}}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$x[n]$

$x_1[n] = e^{j\omega_0 n} x[n]$

$$e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

$|X(e^{j\omega})|$

$|X(e^{j(\omega-\omega_0)})|$

ω_0
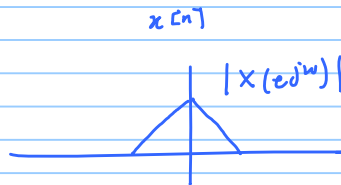
#6 $\underline{\text{Convolution Theorem}}$

$x[n] \longrightarrow$

DT LTI

$h[n]$

$\longrightarrow y[n] = x[n] * h[n]$

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$h[n] \longleftrightarrow H(e^{j\omega})$$

$$x[n] * h[n] \longleftrightarrow X(e^{j\omega}) H(e^{j\omega})$$

$Y(e^{j\omega})$
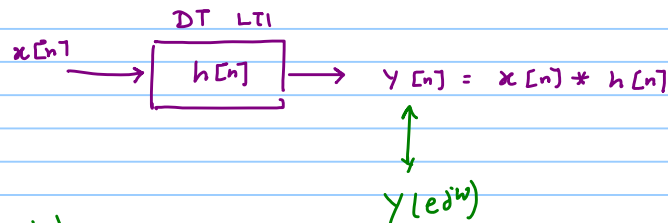
#8,9 $\underline{\text{Parseval Theorem}}$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$$\sum_{n=-\infty}^{\infty} x[n] y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) y^*(e^{j\omega}) d\omega \quad \left(\text{Generalized Parseval's Theorem} \right)$$
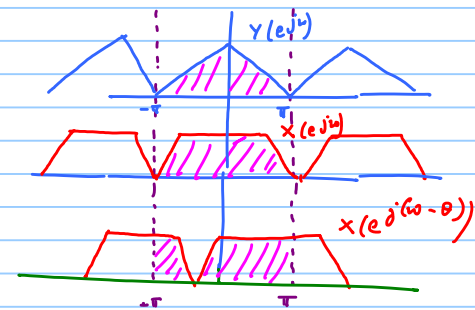
#5 Differentiation

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$n\,x[n] \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

#7 Multiplication $x[n] \longleftrightarrow X(e^{j\omega})$ $y[n] \longleftrightarrow Y(e^{j\omega})$
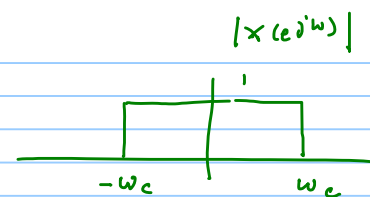
$$w[n] = x[n]\,y[n] \longleftrightarrow W(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\theta})\,X(e^{j(\omega-\theta)})\,d\theta$$

Periodic Convolution

$$x[n] = \frac{w_c}{\pi} \, \text{sinc}\left(\frac{w_c n}{\pi}\right) \longleftrightarrow \sum_{m=-\infty}^{\infty} \text{rect}\left(\frac{w - 2\pi m}{2 w_c}\right)$$

$|X(e^{jw})|$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad \text{Parseval's thm} \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{jw})|^2 \, dw$$

$$x[n]$$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{w_c}{\pi} \qquad = \frac{1}{2\pi} \int_{-w_c}^{w_c} 1 \cdot dw \qquad = \frac{w_c}{\pi}$$

Ex. 1

$$x[n] = 1 \quad \forall n \qquad \longleftrightarrow \qquad 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)$$

Ex 2.   Given $a^n u[n] \longleftrightarrow \dfrac{1}{1 - a e^{-j\omega}}$

$|a| < 1$

Method 1

$$x_1[n] = (-a)^n u[n] \longleftrightarrow X_1(e^{j\omega}) = \dfrac{1}{1 + a e^{-j\omega}}$$

$(-1)^n a^n u[n]$

$$e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega - \omega_0)})$$
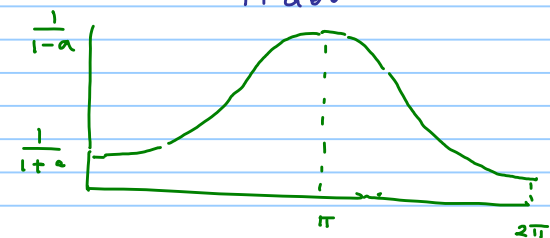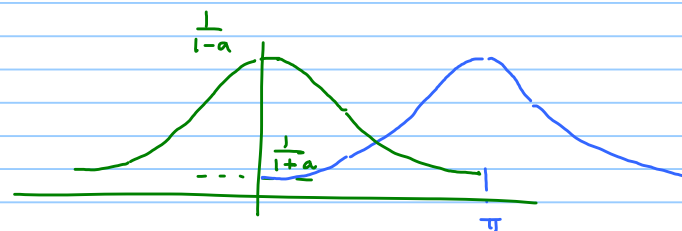
Method 2   Freq shift property

$(-1)^n = e^{+j\pi n}$

$$(-1)^n a^n u[n] = e^{j\pi n} a^n u[n]$$

$$e^{j\pi n} a^n u[n] \longleftrightarrow \dfrac{1}{1 + a e^{-j\omega}}$$

$\dfrac{1}{1-a}$

$\dfrac{1}{1+a}$

π 2π

High Pass Filter

$\dfrac{1}{1-a}$

$\dfrac{1}{1+a}$

π

Ex3

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$$

$$n x[n] \longleftrightarrow j\frac{d}{d\omega} X(e^{j\omega})$$

$$n a^n u[n] \longleftrightarrow j\frac{d}{d\omega}\left(\frac{1}{1-ae^{-j\omega}}\right) = \frac{a e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} \quad \textcircled{1}$$

(3b)

$$\begin{array}{c} ? \\ x_1[n] \end{array} \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^2} \qquad \text{Calculation of Inverse DTFT}$$

$$a\, x_1[n] \longleftrightarrow \frac{a}{(1-ae^{-j\omega})^2}$$

$$a\, x_1[n-1] \longleftrightarrow \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2} \quad \textcircled{2} \qquad \textcircled{1} = \textcircled{2} \qquad \text{LHS} \quad \cancel{a}\, x_1[n-1] = n a^{n-1} u[n]$$

Replace $n \leftarrow n+1$

$$(n+1)\, a^n u[n+1] \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^2} \qquad\qquad x_1[n] = (n+1)\, a^n u[n+1]$$

Ex 4.   Property of convolution

$$x_1[n] = x_2[n] = \{1 \quad 1 \quad 1\}$$

$$x_1[n] * x_2[n] = \{1 \quad 2 \quad 3 \quad 2 \quad 1\}$$

$$X_1(e^{j\omega}) = X_2(e^{j\omega}) = e^{j\omega} + 1 + e^{-j\omega}$$

$$x_1[n] * x_2[n] = X_1(e^{j\omega}) X_2(e^{j\omega})$$

$$= \left(e^{j\omega} + 1 + e^{-j\omega}\right)\left(e^{j\omega} + 1 + e^{-j\omega}\right)$$

$$= e^{j2\omega} + e^{j\omega} + 1$$

$$e^{j\omega} + 1 + e^{-j\omega}$$

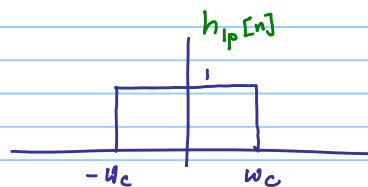$$+ 1 + e^{-j\omega} + e^{-j2\omega}$$

$$\overline{e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega}}$$

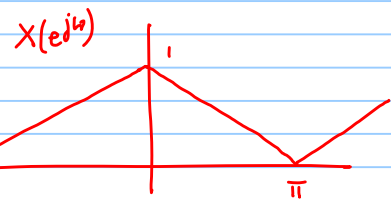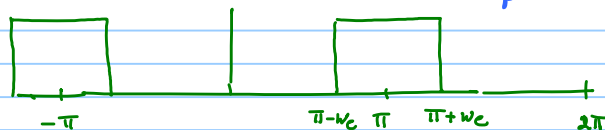$$x_1[n] * x_2[n] = \{1, 2, 3, 2, 1\}$$
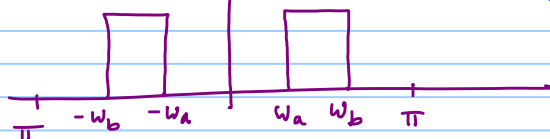
Ideal Filters

Property of Convolution

Ideal LPF

$h_{lp}[n]$

$-w_c$ w_c

Ideal HPF $\longleftrightarrow$ $h_{hp}[n]$

$-\pi$ $\pi-w_c$ π $\pi+w_c$ 2π

$X(e^{jw})$

$-\pi$ π

Bandpass filter $\longleftrightarrow$ $h_{bp}[n]$

π $-w_b$ $-w_a$ w_a w_b π

Bandstop filter $\longleftrightarrow$ $h_{bs}[n]$

π π

$x[n] * h_{lp}[n]$ $\longleftrightarrow$

$X_{lp}(e^{jw})$

$-w_c$ w_c

$x[n] * h_{hp}[n]$ $\longleftrightarrow$

$X_{hp}(e^{jw})$

$-\pi$ $-\pi+w_c$ $\pi-w_c$ π

$x[n] * h_{bp}[n]$ $\longleftrightarrow$

$X_{bp}(e^{jw})$

$-w_b$ $-w_a$ w_a w_b

$x[n] * h_{bs}[n]$ $\longleftrightarrow$

$X_{bs}(e^{jw})$
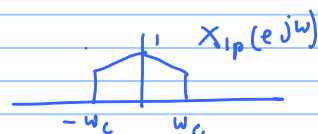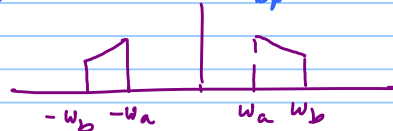
<u>Ex 6</u>

Freq down conversion

$X(j\Omega)$

$-f_0$ f_0
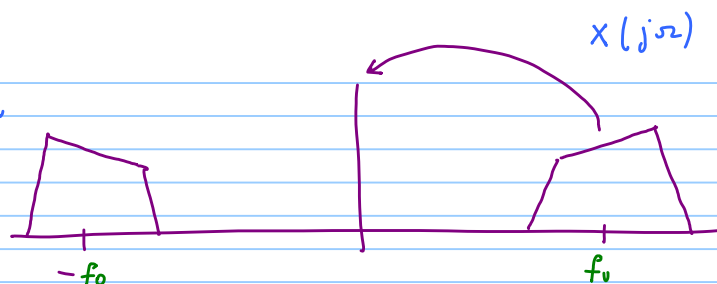
$$X_{lp}(e^{j\omega}) = X\left(e^{j(\omega + \underline{\underline{\omega_v}})}\right)$$

$$= X\left(e^{j(\omega + \frac{\pi}{2})}\right)$$

Sample signal

$$\Omega_S = 2\Omega_N \times 2 \qquad\qquad T_S = \frac{1}{4f_0}$$

$$x_{lp}[n] = \underline{e^{-j\frac{\pi}{2}n}}\ \ \underset{\uparrow}{x[n]}$$

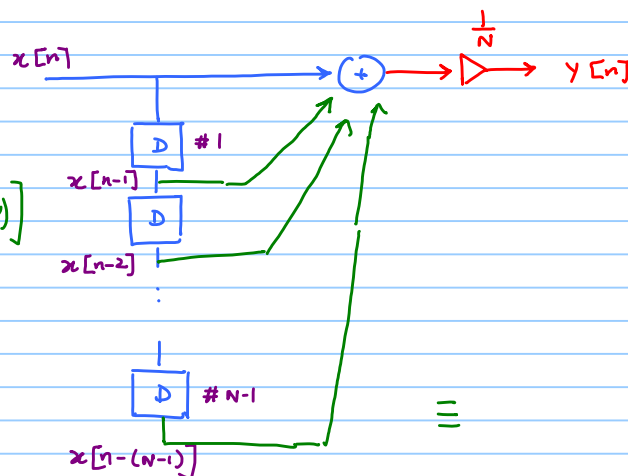| n	0	1	2	3	4	5	6	7
$e^{-j\frac{\pi}{2}n}$	1	$-j$	-1	j	1	$-j$	-1	j

Ex 7   MA filter

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[n-k]$$

Apply DTFT

$$Y(e^{j\omega}) = \frac{1}{N}\left[X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega}) + \cdots + e^{-j\omega(N-1)} X(e^{j\omega})\right]$$

$$= \frac{X(e^{j\omega})}{N}\left[1 + e^{-j\omega} + \cdots + e^{-j\omega(N-1)}\right]$$

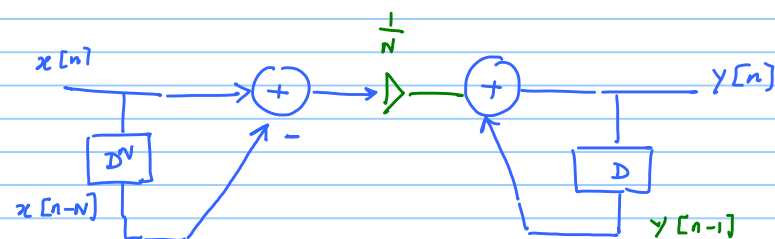$$Y(e^{j\omega}) = \frac{X(e^{j\omega})}{N}\left(\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}\right)$$

LCCDE

$$Y(e^{j\omega})\left(1 - e^{-j\omega}\right) = \frac{1}{N} X(e^{j\omega})\left(1 - e^{-j\omega N}\right)$$

Inv. DTFT

$$y[n] - y[n-1] = \frac{1}{N}\left(x[n] - x[n-N]\right)$$

$x[n]$ $\frac{1}{N}$ $y[n]$

$x[n-1]$
$x[n-2]$
$x[n-(N-1)]$

D #1
D
D #N-1

$\equiv$

$x[n]$ $\frac{1}{N}$ $y[n]$

D^N $x[n-N]$ $-$

D $y[n-1]$

Canonical form (# computation)

Ex 7b

LTI system represented by the LCCDE

$$\longrightarrow \quad y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$$

Apply DTFT

$$Y(e^{j\omega}) - \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) - \frac{1}{4}e^{-j\omega}X(e^{j\omega})$$

$$Y(e^{j\omega})\left(1 - \frac{1}{2}e^{-j\omega}\right) = X(e^{j\omega})\left(1 - \frac{1}{4}e^{-j\omega}\right)$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - \frac{1}{4}\frac{e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}} = H(e^{j\omega})$$

$$\updownarrow$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1}u[n-1]$$

$$y[n] = h[n] * x[n]$$

$$DTFT \implies Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\dfrac{1}{1-e^{-j\omega}} + \displaystyle\sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{(1-ae^{-j\omega})^2}$
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\lvert r\rvert < 1)$	$\dfrac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases}1, & \lvert\omega\rvert < \omega_c,\\ 0, & \omega_c < \lvert\omega\rvert \le \pi\end{cases}$ Ideal LPF
9. $x[n] = \begin{cases}1, & 0\le n\le M\\ 0, & \text{otherwise}\end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\displaystyle\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k) + \pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

$$\frac{1}{2\pi} \longleftrightarrow \sum_{k=-\infty}^{\infty} \delta(\omega+2\pi k)$$

\#11

$$\cos(\omega_0 n + \phi) = \frac{1}{2}\left[e^{j(\omega_0 n+\phi)} + e^{-j(\omega_0 n+\phi)}\right]$$

$$= \frac{e^{j\phi}}{2}\, e^{j\omega_0 n} + \frac{e^{-j\phi}}{2}\cdot e^{-j\omega_0 n}$$

Apply \#10

$$\cos(\omega_0 n + \phi) \longleftrightarrow \pi e^{j\phi}\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0+2\pi k)$$

$$+ \pi e^{-j\phi}\sum_{k=-\infty}^{\infty}\delta(\omega+\omega_0+2\pi k)$$