Note Title 02-09-2024

Electrical Engineering
IIT Madras

EE 3101
Digital Signal Processing

BS Electronic Systems programme

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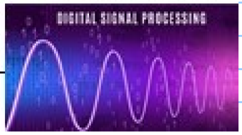
September – December 2024

Session # 20

November 25, 2024



EE 3101 Digital Signal Processing

Session 20

Outline

Last session

- Pole-zero plots
- First order systems
- Allpass systems, properties

Today

- Relationship between Mag & Phase
- Higher order TF
- Minimum Phase Systems

Week 9-10 O & S chapter 5

Frequency Domain Analysis of LTI Systems: Frequency response of systems with rational transfer function—definitions of magnitude and phase response—frequency response of single complex zero/pole—frequency response of simple configurations (second order resonator, notch filter, averaging filter, comb filter, allpass systems)—

phase response—definition of principal phase—zero-phase response—conditions that have to be met for a filter to have generalized linear phase—four types of linear phase FIR filters—on the zero locations of a linear phase FIR filter—constrained zeros at z = 1 and at z = -1 and their implications on choice of filters Type I through Type IV (with focus on Type I)

Reading Assignment

O & S ch 5 Transform Analysis of LTI systems

O&S Ch 5 Transform Analysis of LTI System

✓ 5.1 Freq response of LTI systems $h[n] \longleftrightarrow H(e^{j\omega})$

✓ 5.2 Systems characterized by LCCDE

✓ 5.3 Freq response of Rational Transfer Function

5.4 Relationship between magnitude and phase

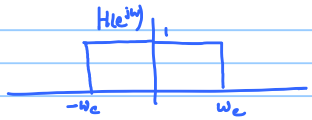
✓ 5.5 Allpass System

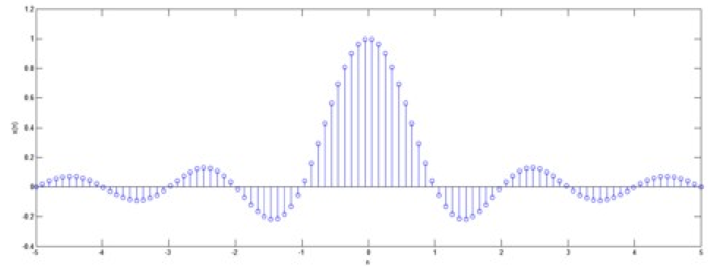
5.6 Min. phase system

5.7 Linear phase $\Leftarrow$

Ideal LPF

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases} \quad \longleftrightarrow \quad h[n] = \frac{\sin \omega_c n}{\pi n}$$





even symmetry $h[n] = h[-n]$

non-causal

infinite duration

$H(e^{j\omega})$ = real-valued

$\arg H(e^{j\omega}) = 0 \quad \forall \omega \implies$ zero phase function

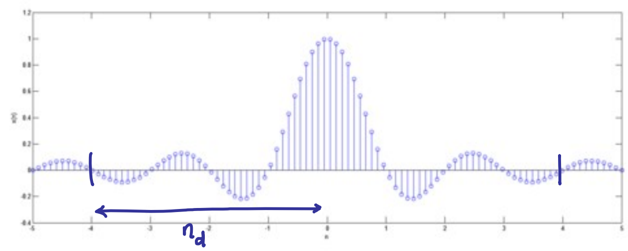
Zero-phase TF

→ non-causal

→ symmetry < even / odd

→ finite or inf duration

* LTI systems with Linear phase are realizable



Truncation + shift (Windowing)

$$h_{prac}[n] = \frac{\sin \cdot \omega_c (n-n_d)}{\pi (n-n_d)} \left[u[n] - u[n-(2n_d+1)] \right]$$

Linear phase $\quad -\omega n_d$

Group Delay $\quad n_d$ samples

$h[n]$ — finite length — Finite Impulse Response (FIR)

$h[n]$ — infinite length — Infinite Impulse Response (IIR)

Lin ϕ depends on the symmetry in $h[n]$

BIBO stability

$|H(e^{j\omega})| < \infty \quad \forall \omega \Rightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$ Absolute summability

Causal & stable $\Rightarrow$ all poles inside the unit circle

- FIR system (causal) $\Rightarrow$ poles are @ $z=0$ $\Rightarrow$ BIBO stable
- IIR system (causal) $\Rightarrow$ poles inside unit circle

FIR, causal
bounded $|h[n]| < \infty \ \forall n$
ROC entire Z plane except $z=0$

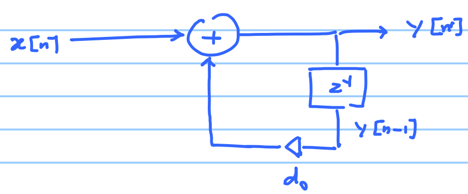
$$H(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

poles @ $z = d_k \quad k=1,\dots N$

zeros @ $z = c_k \quad k=1,\dots M$

$N \geq 1$ $h[n]$ will be IIR

Identifying IIR systems



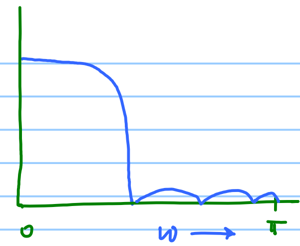
feed back loop

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-d_0 z^{-1}}$$

$$y[n] - d_0\, y[n-1] = x[n]$$

$$H(e^{j\omega}) \left\langle \begin{array}{l} |H(e^{j\omega})| \\ \arg H(e^{j\omega}) \end{array} \right.$$

$20 \log |H(e^{j\omega})|$



LTI System (IIR)

$$H(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$$20 \log_{10} |H(e^{j\omega})| = 20 \log \left|\frac{a_0}{b_0}\right| + \sum_{k=1}^{M} 20 \log |1 - c_k e^{-j\omega}| - \sum_{k=1}^{N} 20 \log |1 - d_k e^{-j\omega}|$$

$$\arg H(e^{j\omega}) = \sum_{k=1}^{M} \arg(1 - c_k e^{-j\omega}) - \sum_{k=1}^{M} \arg(1 - d_k e^{-j\omega})$$

Group delay $\quad \tau(\omega) = -\frac{d}{d\omega}\left[\underbrace{\arg H(e^{j\omega})}_{\text{unwrapped phase}}\right]$

$\boxed{\text{Use first order \& second order terms as building blocks.}}$

$H(z) = 1 - c_1 z^{-1}$ FIR First Order TF.

$\boxed{c_1 \text{ is real}}$ $H(e^{j\omega}) = 1 - c_1 e^{-j\omega} = 1 - c_1 \cos\omega + j\, c_1 \sin\omega$

$$|H(e^{j\omega})| = \sqrt{(1 - c_1\cos\omega)^2 + (c_1 \sin\omega)^2}$$

$$\arg H(e^{j\omega}) = \tan^{-1}\left[\frac{c_1 \sin\omega}{1 - c_1\cos\omega}\right]$$

$$\tau(\omega) = -\frac{d}{d\omega}\left(\arg H(e^{j\omega})\right) = -\frac{d}{d\omega} \tan^{-1}\left(\frac{c_1 \sin\omega}{1 - c_1 \cos\omega}\right) = \frac{c_1^2 - c_1\cos\omega}{1 + c_1^2 - 2c_1\cos\omega}$$

$\boxed{c_1 \text{ complex}}$ $c_1 = r_1 e^{j\theta_1}$ $H(e^{j\omega}) = 1 - r_1 \cos(\omega - \theta_1) + j\, r_1 \sin(\omega - \theta_1)$

$|H(e^{j\omega})|$

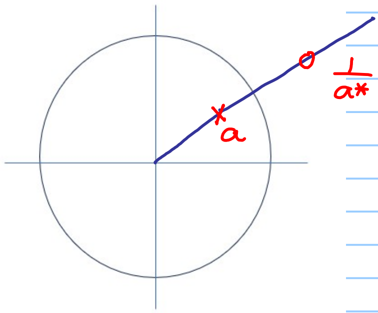
$\arg H(e^{j\omega})$

$$\tau(\omega) = \frac{|c_1|^2 - \mathrm{Re}\{c_1 e^{-j\omega}\}}{1 + |c_1|^2 - 2\mathrm{Re}\{c_1 e^{-j\omega}\}}$$

Task to Complete

5.5 Allpass Systems

$$\boxed{H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}} = \frac{z^{-1}(1 - a^* z)}{(1 - az^{-1})}$$



$$H_{ap}(e^{jw}) = e^{-jw}\frac{(1 - a^* e^{jw})}{(1 - a e^{-jw})} \Longrightarrow |H_{ap}(e^{jw})| = 1 \quad \forall w$$

Gen form of Allpass Systems

$$H_{ap}(z) = \frac{b_N^* + b_{N-1}^* z^{-1} + b_{N-2}^* z^{-2} + \dots + b_1^* z^{-(N-1)} + z^{-(N)}}{1 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_N z^{-N}}$$

Verify $|H_{ap}(e^{jw})| = 1 \quad \forall w$

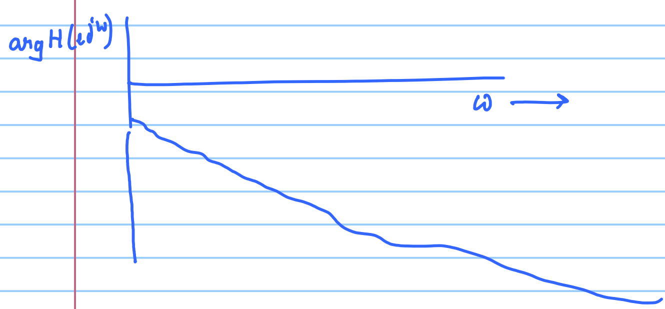
$$H_{ap}(e^{j\omega}) = e^{-j\omega} \frac{(1 - a^* e^{j\omega})}{(1 - a e^{-j\omega})}$$

$\boxed{a = r e^{j\theta}}$ location of pole

Group delay $\tau(\omega) = \dfrac{1 - r^2}{1 + r^2 - 2r\cos(\omega - \theta)}$

Causal, stable allpass $\Rightarrow$ all poles inside the unit circle $r < 1$

$\tau(\omega) > 0 \ \forall \omega \Rightarrow$ arg $H_{ap}(e^{j\omega})$ is a monotone decreasing function



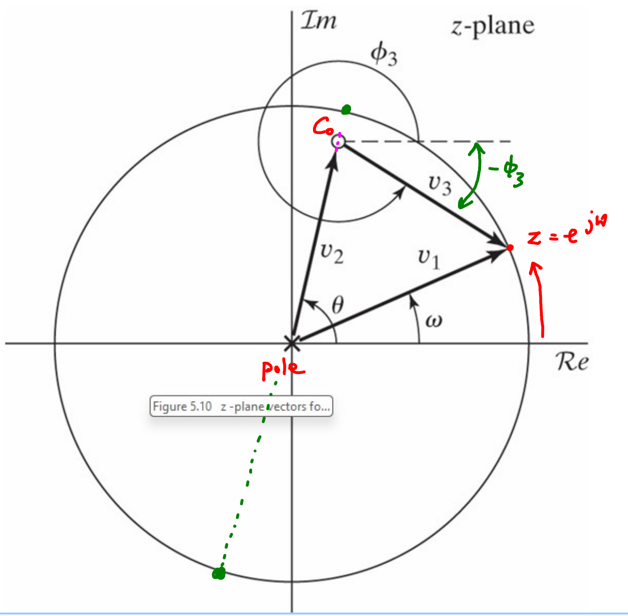
General result

Causal, stable Allpass $\Rightarrow$ all poles strictly inside unit circle

$\Rightarrow$ phase is monotone decreasing.

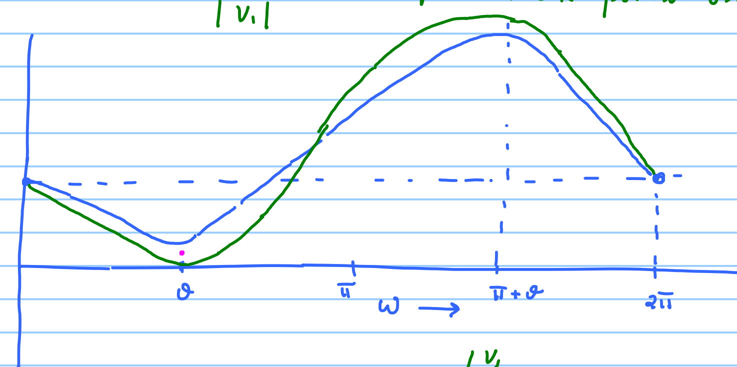
Magnitude response of first order zero

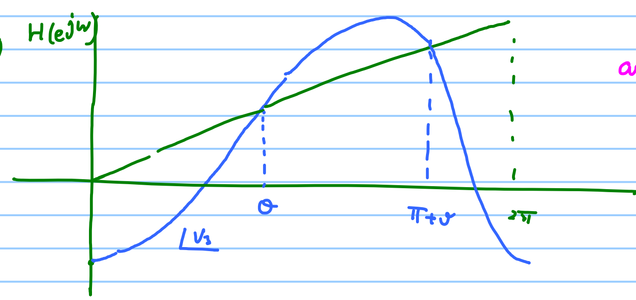
$$H(z) = (1 - c_0 z^{-1}) = \frac{z - c_0}{z} = \frac{v_3}{v_1}$$

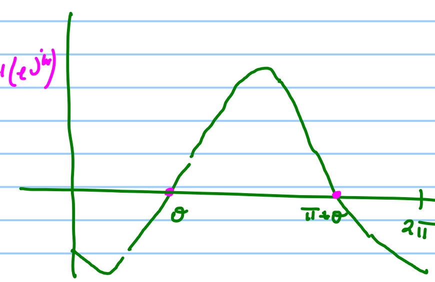


$$|H(e^{j\omega})| = \frac{|v_3|}{|v_1|}$$

compute at all points on the unit circle







$$\arg H(e^{j\omega}) = \angle v_3 - \angle v_1$$

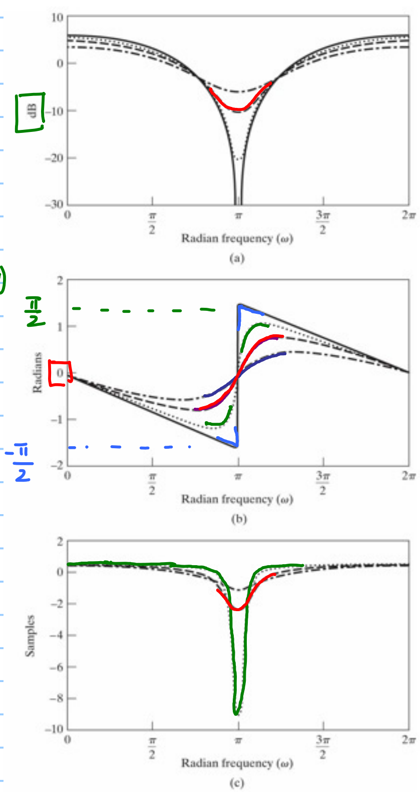
Zero on the unit circle

$|H(e^{j\omega})|$

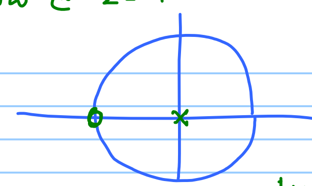
Arg $H(e^{j\omega})$

$r = 1$
$r = 0.9$
$r = 0.7$
$r = 0.5$

GD

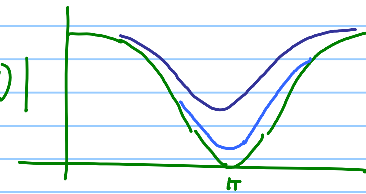


$H(z) = (1 + z^{-1})$ — zero @ $z = -1$

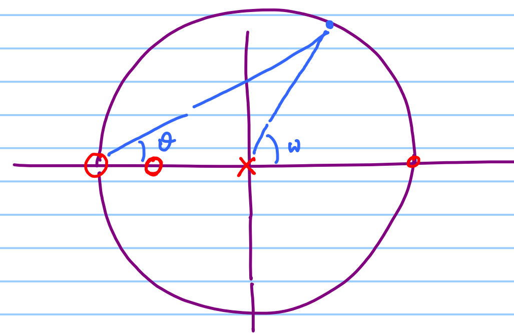


$\boxed{c_0 = -1}$

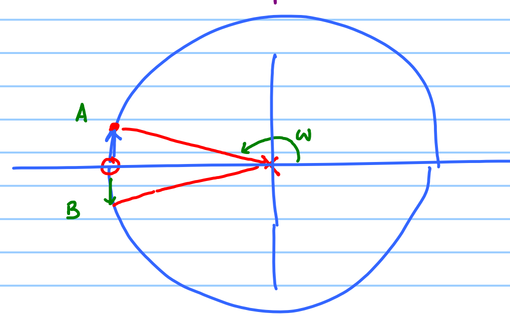
$|H(e^{j\omega})|$



$c_0 = -0.9$



① Magnitude response
② Phase response
③ Group response



$\arg H(e^{j\omega}) = \angle V_3 - \angle V_1$

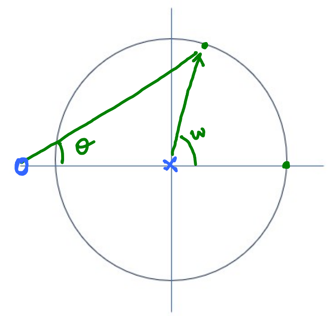
@A Near $\omega = \pi$ $\quad \arg H(e^{j\omega}) \sim 90^\circ - 180^\circ$
$\sim -90^\circ$

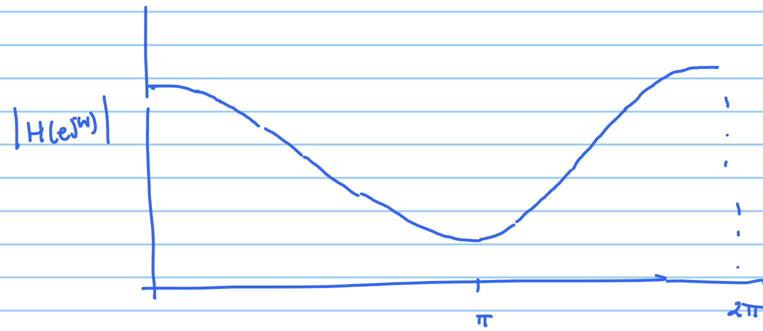
@B near $\omega = \pi$ $\quad \arg H(e^{j\omega}) \sim -90 - 180$
$\sim -270^\circ$

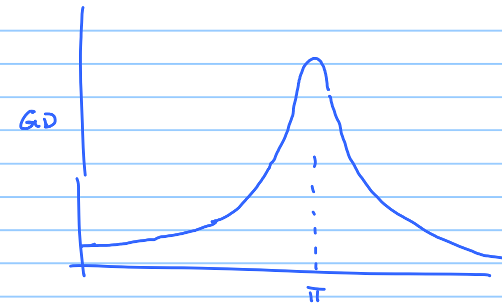
zero outside the unit circle

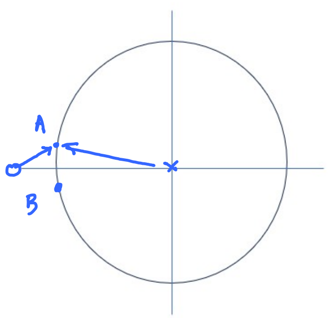
$(1 + a_0 z^{-1})$

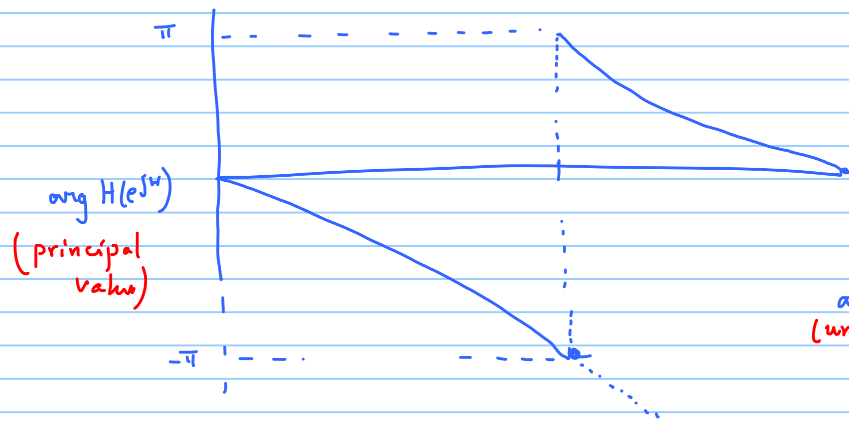
$a_0 = +1.2$







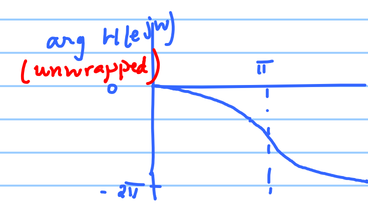




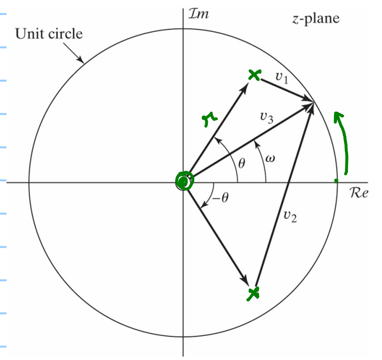
near $\omega = \pi$

@A $\arg H(e^{j\omega}) \sim 0 - 180^\circ$

@B $\arg H(e^{j\omega}) \sim \underbrace{-\psi - 180}_{< -180^\circ}$



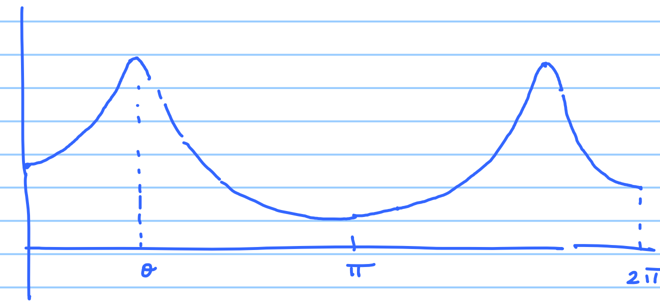
O&S Example 5.6



$$H(z) = \frac{1}{(1 - a z^{-1})(1 - a^* z^{-1})}$$

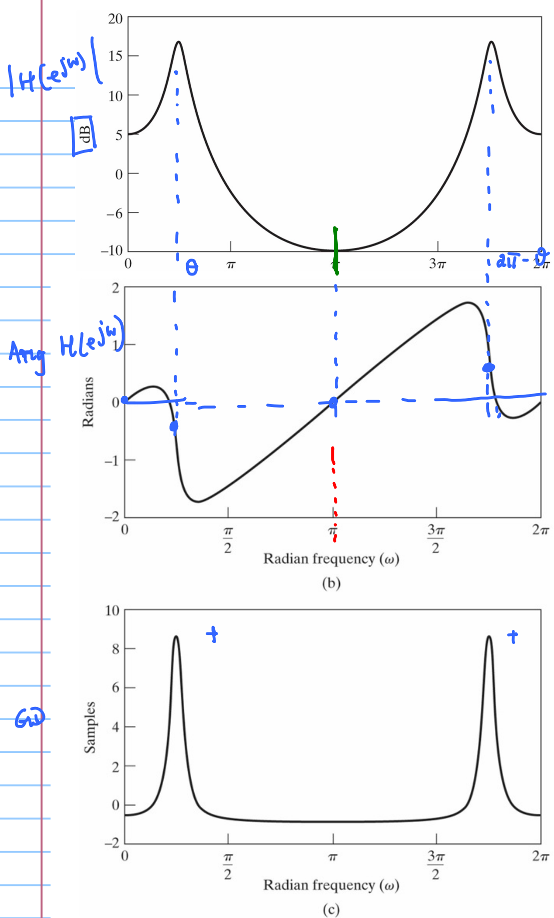
$$a = re^{j\theta}$$

$$|H(e^{j\omega})| = \frac{|v_3|^2}{|v_1||v_2|} = \frac{1}{|v_1||v_2|}$$



$$\text{Arg } H(e^{j\omega}) = 2\omega - \angle v_1 - \angle v_2$$

@ $\omega = 0$ $\quad$ arg $H(e^{j\omega}) \sim 0$



$H(z) \longleftrightarrow h[n]$

$\boxed{\text{If } h[n] \text{ real}}$ $\Rightarrow$ $H(e^{j\omega}) = \sum_n h[n] e^{-j\omega n}$

$= \underbrace{\sum_n h[n] \cos \omega n}_{H_R(e^{j\omega})} - j \underbrace{\sum_n h[n] \sin \omega n}_{-H_I(e^{j\omega})}$

$H_R(e^{-j\omega}) = H_R(e^{j\omega})$ even funcn of ω

$H_I(e^{-j\omega}) = -H_I(e^{j\omega})$ odd funcn of ω

$|H(e^{j\omega})| = \sqrt{\underbrace{H_R^2(e^{j\omega})}_{\text{even}} + \underbrace{H_I^2(e^{j\omega})}_{\text{even}}}$

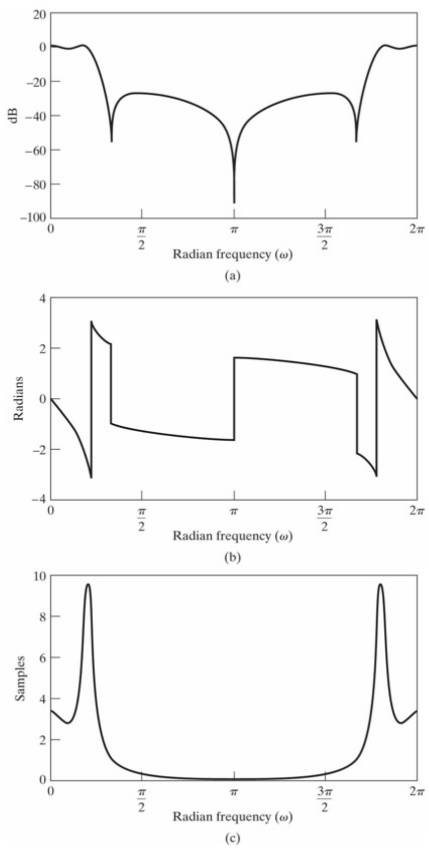
even

$\text{Arg } H(e^{j\omega}) = \tan^{-1} \dfrac{H_I(e^{j\omega}) \text{ (odd)}}{H_R(e^{j\omega}) \text{ (even)}}$

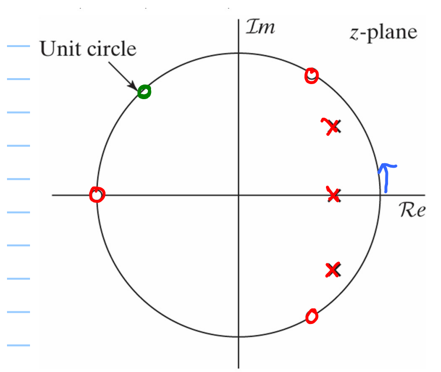
odd

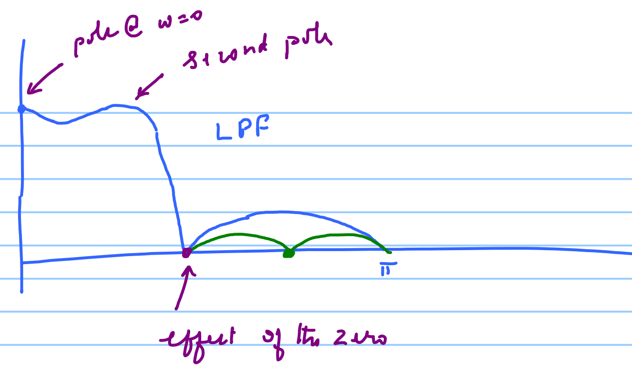
O & S Example 5.8

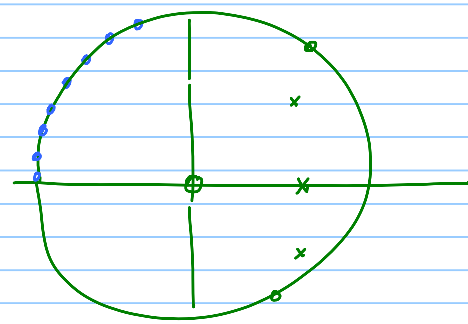
Verify

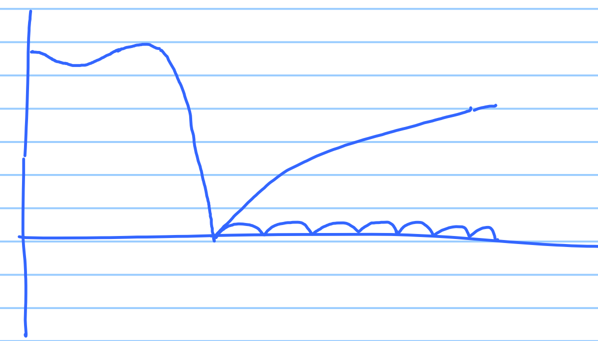


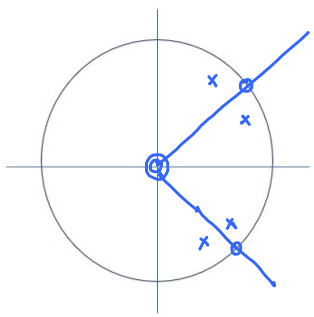
Verify

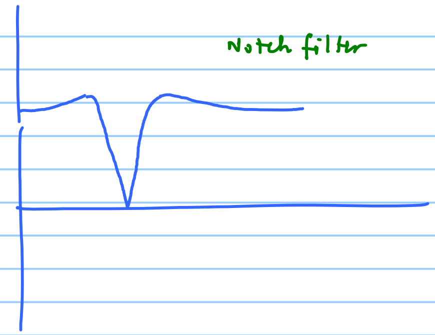


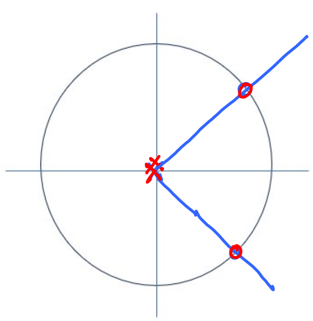


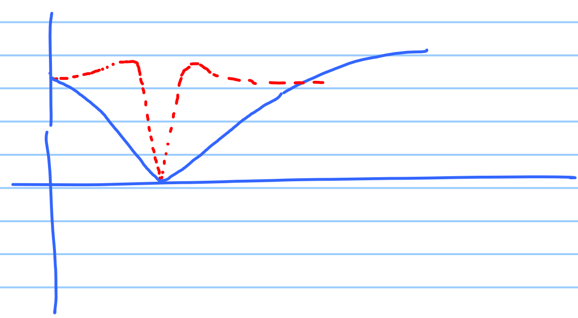







Notch filter





Magnitude Squared Response

$h[n] \longleftrightarrow H(e^{j\omega})$

$h^*[-n] \longleftrightarrow H^*(e^{j\omega})$

$$|H(e^{j\omega})|^2 = \underset{\updownarrow\; h[n]}{H(e^{j\omega})} \; \underset{\updownarrow\; h^*[-n]}{H^*(e^{j\omega})}$$

$$\underbrace{h[n] * h^*[-n]}_{\text{autocorrelation}} \longleftrightarrow |H(e^{j\omega})|^2$$

$$h[n] * h^*[-n] \longleftrightarrow H(z) \quad H^*\left(\frac{1}{z^*}\right)$$

$z = e^{j\omega}$

$$H^*\left(\frac{1}{z^*}\right)\Bigg|_{z = e^{j\omega}} = H^*(e^{j\omega})$$

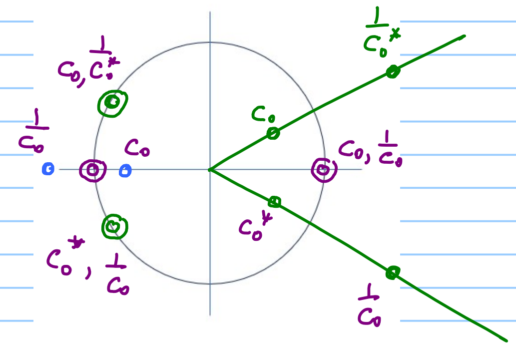
$$H(z)\, H^*\left(\frac{1}{z^*}\right)$$

$$|H(e^{j\omega})|^2 = H(z)\, H^*\left(\frac{1}{z^*}\right)\Big|_{z=e^{j\omega}}$$

$$H(z) = (1 - c_0 z^{-1})$$

Mag. squared $(1 - c_0 z^{-1})(1 - c_0^* z)$

$\downarrow$ $z = c_0$ $\qquad \downarrow$ $z = \frac{1}{c_0^*}$



From property of allpass functions

$$\frac{z^{-1}(1 - a^* z)}{(1 - a z^{-1})}$$

Observations about Magnitude Squared function

① zeros/poles   If c_0 is a zero/pole, $\frac{1}{c_0^*}$ is also a zero/pole

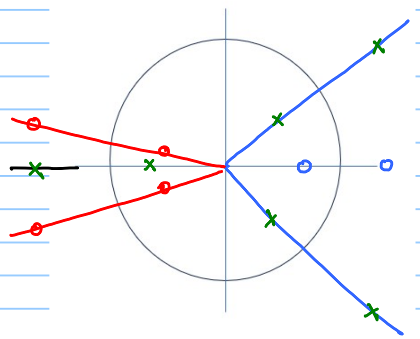
② If $h[n]$ is real valued, then c_0, c_0^* $\quad \frac{1}{c_0^*}, \frac{1}{c_0}$ $\longrightarrow$ $c_0, \frac{1}{c_0^*}, \quad c_0^*, \frac{1}{c_0}$

③ If zero or pole is on real axis

④ on unit circle, but not on real axis c_0, c_0^* will occur as double zeros

⑤ zero/pole @ ±1, double zero/pole

Example $H(z)\, H^*\left(\frac{1}{z^*}\right)$



$H(z)$ 3 poles, 3 zeros

$H(z)\, H^*\left(\frac{1}{z}\right) \sim$ 6 poles, 6 zeros

possible values for TF $H(z)$ $2^6 = 64$

possible TF with real coefficients $2^4 = 16$

possible TF $\underbrace{\text{causal, stable}}_{\text{all poles inside } |z|=1}$ $2^3 = 8$

possible TF causal, stable, real coeff $2^2 = 4$

Min phase Property

$H(z) = z^{-1} - z_0^*$ $\quad |z_0| < 1$ $\quad$ zero @ $z = \frac{1}{z_0^*}$ outside unit circle

$$= \frac{(1 - z_0 z^{-1})}{(1 - z_0 z^{-1})} \cdot (z^{-1} - z_0^*) = \underbrace{(1 - z_0 z^{-1})}_{H'(z)} \underbrace{\frac{z^{-1} - z_0^*}{1 - z_0 z^{-1}}}_{\text{AP, stable, causal}}$$

pole @ $z = z_0$

$H(z) = H'(z) \; H_{ap}(z)$

causal stable AP

$$|H(e^{j\omega})| = |H'(e^{j\omega})| \underbrace{|H_{ap}(e^{j\omega})|}_{=1 \; \forall \omega} = |H'(e^{j\omega})|$$

unwrapped phase

$$\arg H(e^{j\omega}) = \arg H'(e^{j\omega}) + \underbrace{\arg H_{ap}(e^{j\omega})}_{\leq 0}$$

$\leq 0 \begin{cases} = 0 & @ \; \omega = 0 \\ < 0 & \omega \neq 0 \end{cases}$

$\longrightarrow$ $GD\{H(e^{j\omega})\} > GD\{\underbrace{H'(e^{j\omega})}_{\text{min } \phi \text{ system}}\}$

$\arg H(e^{j\omega}) \leq \arg H'(e^{j\omega})$

phase lag $H(e^{j\omega}) = -\arg H(e^{j\omega})$

Phase lag $H(e^{j\omega}) \geq$ phase lag $H'(e^{j\omega})$

TF with poles & zeros inside unit circle (causal, stable) $\rangle$ min ϕ ~~lag~~ system

min ϕ system

Min ϕ TF   causal & stable $\Rightarrow$ all poles & zeros inside unit circle

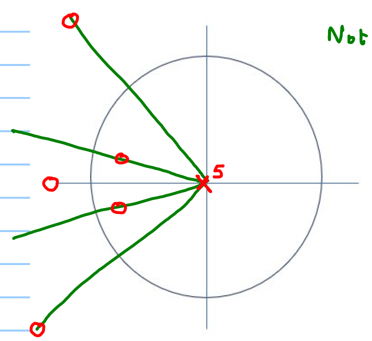
causal & stable → all poles inside unit circle

Min ϕ TF → all zeros inside unit circle

Example

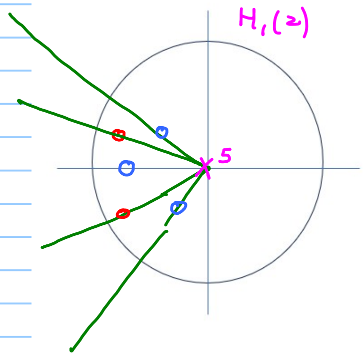
Fifth order $H(z)$

Not min ϕ



Min. phase TF

$H_1(z)$ $H'(z)$



$$|H(e^{j\omega})| = |H_1(e^{j\omega})|$$

Properties Min ϕ systems

(Min ϕ lag)

* Causal, stable $\longrightarrow$ all poles & zeros inside unit circle

* Min ϕ lag compared to any other TF with same mag response

* Min GD "

* Max partial energy "