Some Formulas

- **Line Equations:** The equation of the line segment connecting $\mathbf{r}_0$ and $\mathbf{r}_1$:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

  The equation of the line containing $\mathbf{r}_0$ in the direction $\mathbf{v}$:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} \qquad -\infty < t < \infty$$

- **Plane Equations:** The equation of the plane with normal vector $\mathbf{n}$ and containing $\mathbf{r}_0$:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

  The equation of the plane in scalar form, when $\mathbf{n} = \langle a, b, c \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$:

$$ax + by + cz + d = 0 \qquad d \text{ constant.}$$

- **Arc Length Integrals:** The length of the curve C given by $\mathbf{r}(t)$ for $a \le t \le b$:

$$\int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t$$

  An integral of a function f with respect to arc length on C is :

$$\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)| \, \mathrm{d}t = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

  where $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

- **Tangent Planes to Surfaces:** If a surface S is of the form $z = g(x, y)$, the tangent plane at $P(x_0, y_0, z_0)$ is:

$$z - z_0 = g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0).$$

  If S is a level surface $f(x, y, z) = k$, the tangent plane at $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ has normal vector ∇f, so the equation is:

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0, \qquad \text{or} \qquad \nabla f \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

  If S is parameterized in vector form by $\mathbf{r}(u, v)$, the tangent plane at $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$ has normal vector $\mathbf{r}_u \times \mathbf{r}_v$, so the equation is:

$$(\mathbf{r}_u \times \mathbf{r}_v) \cdot (\mathbf{r} - \mathbf{r}_0)$$

- **Implicit Function Theorem:** If a curve satisfies the equation $F(x, y) = 0$, if F_x and F_y are continuous, and $F_y \ne 0$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}.$$

  If a surface satisfies $F(x, y, z) = 0$ with F_x, F_y, and F_z continuous with $F_z \ne 0$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

- **Cylindrical Coordinates:** Volume differential: $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \qquad x^2 + y^2 = r^2, \qquad \tan\theta = \frac{y}{x}$$

- **Spherical Coordinates:** Volume differential: $\mathrm{d}V = \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\cos\theta, \qquad z = \rho\cos\phi, \qquad \rho^2 = x^2 + y^2 + z^2$$

- **Line Integrals:** C a curve parameterized by $\mathbf{r}(t)$, $a \le t \le b$. $f(x,y,z)$ a function.

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,\mathrm{d}t$$

  $\mathbf{F} = \langle P, Q, R\rangle$ a vector field:

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z.$$

- **Surface Integrals:** S parameterized by $\mathbf{r}(u,v)$ where $(u,v) \in D$, $f(x,y,z)$ a function.

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A.$$

  $\mathbf{F}$ a vector field:

$$\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}A.$$

- **Fundamental Theorem of Line Integrals:**

$$\int_C (\nabla f)\cdot d\mathbf{r} \;=\; f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

  where the curve C is the curve given by $\mathbf{r}(t)$ where $a \le t \le b$.

- **Green's Theorem:** $\mathbf{F} = \langle P, Q\rangle$ is a 2-dimensional vector field.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \oint_C P\,dx + Q\,dy \;=\; \iint_D (Q_x - P_y)\,dA$$

  where C is the positively oriented boundary of the 2-dimensional region D.

- **Stokes' Theorem:** $\mathbf{F} = \langle P, Q, R\rangle$ is a 3-dinmensional vector field.

$$\oint_C \mathbf{F}\cdot\,d\mathbf{r} \;=\; \iint_S \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S}$$

  where C is the positively oriented boundary of the oriented surface S in 3-dimensional space.

- **Divergence Theorem:** $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field.

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} \;=\; \iiint_E \mathrm{div}(\mathbf{F})\,dV$$

  where S is the positively oriented boundary of the 3-dimensional region E.