Solutions to Final Exam Review

1. Let S be the surface parameterized by $\vec{r}(u,v) = \langle 4-u^2, u-v, v\rangle$ with $0 \le v \le u \le 2$.

   (a) Find the equation of the tangent plane to the surface at the point $(0,1,1)$.

   The point $(0,1,1)$ occurs at $u=2, v=1$. The partial derivatives are $\vec{r}_u = \langle -2u, 1, 0\rangle$ and $\vec{r}_v = \langle 0,-1,1\rangle$ so $\vec{r}_u \times \vec{r}_v = \langle 1, 2u, 2u\rangle$ which is $\langle 1,4,4\rangle$ at $u=2, v=1$. The tangent plane goes through the point $(0,1,1)$ and has normal vector $\langle 1,4,4\rangle$ so the equation is $1(x-0)+4(y-1)+4(z-1)=0$ or $x+4y+4z=8$.

   (b) Find the surface area of S.

   The surface area is $\iint_D |r_u \times r_v|\ dA$ where D is the region of possible (u,v) values. Using that $r_u \times r_v = \langle 1,2u,2u\rangle$ from part (a), $|r_u \times r_v| = \sqrt{1+4u^2+4u^2} = \sqrt{1+8u^2}$. The region D is a triangle on the uv-plane bounded by $u=v, v=0$ and $u=2$, so the surface area is:

   $$\int_0^2 \int_0^u \sqrt{1+8u^2}\ dv\ du = \int_0^2 v\sqrt{1+8u^2}\Big|_0^u\ du$$

   $$\int_0^2 u\sqrt{1+8u^2}\ du = \frac{1}{24}(1+8u^2)^{3/2}\Big|_0^2 = \frac{33\sqrt{33}-1}{24}$$

   (c) Compute the flux of $\vec{F}(x,y,z) = \langle 2x, y, z\rangle$ across S.

   The flux is equal to $\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u(u,v)\times\vec{r}_v(u,v))\ dA$. Then $\vec{F}(\vec{r}(u,v)) = \langle 8-2u^2, u-v, v\rangle$ and $\vec{r}_u \times \vec{r}_v = \langle 1,2u,2u\rangle$, so the integral is $\int_0^2\int_0^u 8-2u^2+2u^2-2uv+2uv\ dvdu = \int_0^2\int_0^u 8\ dvdu$. The region on the uv-plane is a triangle with area 2, so the integral is equal to 16.

2. Find the absolute maximum and minimum of $f(x,y,z) = x+y^2+3z$ on the ellipsoid $x^2+2y^2+3z^2=36$.

   This is a closed and bounded region so there is an absolute max and min. There is no interior so we just need check for critical points on the ellipsoid using Lagrange multipliers. The Lagrange multiplier equations are $1 = \lambda 2x, 2y =$

$\lambda 4y, 3 = \lambda 6z, x^2 + 2y^2 + 3z^2 = 36$. Note that $\lambda \neq 0$ so the first and third equations can be written as $x = 1/(2\lambda)$ and $z = 1/(2\lambda)$ so $x = z$. The second equation gives us that $y = 0$ or $\lambda = 1/2$. We consider the two cases separately.

If $y = 0$, then as $x = z$ and $x^2 + 2y^2 + 3z^2 = 36$, we get that $x = z = \pm 3$ which gives us the critical points $(3, 0, 3)$ and $(-3, 0, -3)$. If $\lambda = 1/2$, then $x = z = 1/(2\lambda) = 1$. Plugging $x = z = 1$ into $x^2 + 2y^2 + 3z^2 = 36$ we get that $y = \pm 4$ which gives us the critical points $(1, 4, 1)$ and $(1, -4, 1)$.

To find the max and min, plug the critical points into f. The values of f at the critical points are $f(3, 0, 3) = 12$, $f(-3, 0, -3) = -12$, $f(1, 4, 1) = 20$, $f(1, -4, 1) = 20$ so the maximum is 20 and the minimum is -12.

3. Compute $\int_C (y^2 e^{xy})dx + (e^{xy} + xye^{xy})dy$ where C is the curve consisting of the two line segments from $(0, 0)$ to $(2, 2)$ and from $(2, 2)$ to $(0, 5)$.

   Let $\vec{F} = \langle y^2 e^{xy}, e^{xy} + xye^{xy} \rangle$. $\vec{F}$ is conservative with potential function $f(x, y) = ye^{xy}$, so by the fundamental theorem of line integrals, the integral is equal to $f(0, 5) - f(0, 0) = 5$.

4. Let D be a region on the xy-plane. Let S be the part of the plane $4x - 6y + 2z = 5$ with (x, y) in D. If the area of S is 11, find the area of D.

   The surface S can be parametrized as $\vec{r}(x, y) = \langle x, y, (5/2) - 2x + 3y \rangle$ where the possible (x, y) values are exactly those in D. Then $r_x \times r_y = \langle 2, -3, 1 \rangle$ so $|r_x \times r_y| = \sqrt{14}$. Using the surface area formula we get that the surface area of S is $A(S) = \iint_D \sqrt{14}\, dA = \sqrt{14} A(D)$ where $A(D)$ is the area of D. Set $A(S) = 11$ and solve for $A(D)$ to get $A(D) = 11/\sqrt{14}$.

5. Let E be the region which is both inside the sphere $x^2 + y^2 + z^2 = 8$ and above the cone $z = \sqrt{x^2 + y^2}$. Let S be the boundary surface of E with inward orientation. Find $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}(x, y, z) = \langle xz + 5y^2, e^{\cos(xz)}, z^2 \rangle$.

   S has inward (negative) orientation, so by the divergence theorem the integral is $\iint_S \vec{F} \cdot d\vec{S} = -\iint_E \mathrm{div}(\vec{F})\, dV$. The divergence of $\vec{F}$ is $3z$. The region can be set up in either cylindrical or spherical coordinates. Note that the intersection of the cone and sphere is where $x^2 + y^2 + x^2 + y^2 = 8$ which is the circle $x^2 + y^2 = 4$

on the plane $z = 2$. The two set-ups for $\iint_E 3z\ dV$ are the following:

$$\int_0^{2\pi}\int_0^2\int_r^{\sqrt{8-r^2}} 3zr\ dz\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} 3\rho^3\sin(\phi)\cos(\phi)\ d\rho\ d\phi\ d\theta$$

Either of these integrals is reasonable to integrate by hand. Here we show the steps for the spherical integral.

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} 3\rho^3\sin(\phi)\cos(\phi)\ d\rho\ d\phi\ d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{3}{4}\rho^4\sin(\phi)\cos(\phi)\Big|_0^{\sqrt{8}}\ d\phi\ d\theta = \int_0^{2\pi}\int_0^{\pi/4} 48\sin(\phi)\cos(\phi)\ d\phi\ d\theta$$

$$= \int_0^{2\pi} 24\sin^2(\phi)\Big|_0^{\pi/4}\ d\theta = \int_0^{2\pi} 12\ d\theta = 24\pi$$

The triple integral is equal to 24π, so the surface integral is equal to -24π.

6. Find $\int_C \vec{F}\cdot d\vec{r}$ where C is the intersection of the plane $z = 1 - 2x - 3y$ and the cylinder $x^2 + y^2 = 4$ oriented clockwise when viewed from above and $\vec{F}(x, y, z) = \langle yz + \cos(x^2), -x^2, 3y\rangle$.

   By Stokes' Theorem, $\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\cdot\ d\vec{S}$ where S is the part of the plane which is inside the cylinder. To make the orientations of S and C match, we gives S the downward orientation. S can be parametrized as $\vec{r}(x, y) = \langle x, y, 1 - 2x - 3y\rangle$ where $x^2 + y^2 \leq 4$. Then $\vec{r}_x \times \vec{r}_y = \langle 2, 3, 1\rangle$ and we change this to $\langle -2, -3, -1\rangle$ to match the orientation. The curl of $\vec{F}$ is $\langle 3, y, -2x - z\rangle$ and on S this is $\langle 3, y, 3y - 1\rangle$. The dot product of the curl and the normal vector is $-5 - 6y$ so the integral is $\iint_{x^2+y^2\leq 4} -5 - 6y\ dA$. By symmetry, the integral of $-6y$ over the circle is 0. The integral of -5 over the circle is -5 times the area of the circle. Therefore the integral is $\iint_{x^2+y^2\leq 4} -5 - 6y\ dA = 0 - 5(\pi\cdot 2^2) = -20\pi$. Alternatively, you can evaluate by switching to polar to get $\int_0^{2\pi}\int_0^2 -5r - 6r^2\sin(\theta)\ drd\theta = -20\pi$.

7. Evaluate the surface integral $\iint_S xy\ dS$ where S is the triangular region with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

The surface is a plane so we first find the equation of the plane. It contains vectors $\langle -1, 2, 0\rangle$ and $\langle -1, 0, 2\rangle$ so the cross product $\langle 4, 2, 2\rangle$ is a normal vector to the plane, as is any multiple of this vector including $\langle 2, 1, 1\rangle$. Using the point $(1, 0, 0)$ and normal vector $\langle 2, 1, 1\rangle$ we get that the plane is $2(x-1)+y+z=0$ or $2x+y+z=2$.

To evaluate the surface integral, parameterize the plane as $\vec{r}(x,y) = \langle x, y, 2-2x-y\rangle$. Then $\vec{r}_x \times \vec{r}_y = \langle 2, 1, 1\rangle$ so $dS = |\vec{r}_x \times \vec{r}_y| dA = \sqrt{6} dA$. We next need to figure out which values of (x, y) correspond to the triangular region. If we draw the triangular region in 3 dimensions we can see that it's projection down to the xy-plane is the triangle with vertices $(1, 0), (0, 2), (0, 0)$ so the integral will be taken over this triangle. The surface integral becomes $\iint_S xy\, dS = \int_0^1 \int_0^{2-2x} xy\sqrt{6}\, dydx = \sqrt{6}\int_0^1 \frac{1}{2}xy^2\Big|_0^{2-2x} dx = \sqrt{6}\int_0^1 \frac{1}{2}x(2-2x)^2\, dx = \sqrt{6}\int_0^1 2x^3 - 4x^2 + 2x\, dx = \sqrt{6}(\frac{1}{2}x^4 - \frac{4}{3}x^3 + x^2\Big|_0^1) = \frac{\sqrt{6}}{6}$.

8. Evaluate the limit or show it does not exist.

   (a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3y^2 - xy^4}{x^5 + y^5}$.

   Along the paths $x = 0, y = 0, y = x$ the limit is 0, however along $y = 2x$ the limit is $-\frac{12}{33}$ so the limit does not exist.

   (b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{x^2 + 2y^2}$.

   Switching to polar we get that this equals

$$\lim_{r\to 0} \frac{r^2\cos^2(\theta)\sin^2(r\sin(\theta))}{r^2\cos^2(\theta) + 2r^2\sin^2(\theta)} = \lim_{r\to 0} \frac{\cos^2(\theta)\sin^2(r\sin(\theta))}{\cos^2(\theta) + 2\sin^2(\theta)}.$$

   The bottom of this fraction is never 0 and the top approaches 0 as $r \to 0$ so the limit is 0.

   (c) $\displaystyle \lim_{(x,y)\to(1,2)} \frac{y - 2x}{4 - xy^2}$.

   Along the path $x = 1$, this is $\displaystyle \lim_{y\to 2} \frac{y-2}{4-y^2} = \lim_{y\to 2} -\frac{1}{2+y} = -\frac{1}{4}$. Along the path $y = 2$, this is $\displaystyle \lim_{x\to 1} \frac{2-2x}{4-4x} = \frac{1}{2}$. So the limit does not exist.

9. Find all critical points of the function f and determine if each critical point is a local max, local min, or saddle point.

(a) $f(x, y) = x^3y + 12x^2 - 8y$

The partial derivatives are $f_x(x, y) = 3x^2y + 24x$ and $f_y(x, y) = x^3 - 8$. The critical points are where both partial derivatives are 0 so $x^3 - 8 = 0$ which implies that $x = 2$ and $3x^2y + 24x = 0$ so $12y + 48 = 0$ and $y = -4$. So f has one critical point at $(2, -4)$.

To determine what type of point it is, use the second derivative test. The second order partial derivatives are $f_{xx}(x, y) = 6xy + 24$, $f_{yy}(x, y) = 0$, $f_{xy}(x, y) = 3x^2$ which are -24, 0, and 12 respectively when $x = 2, y = -4$. Then $D = f_{xx}f_{yy} - (f_{xy})^2 = -144$ so it is a saddle point.

(b) $f(x, y) = e^{4y-x^2-y^2}$

The partial derivatives are $f_x(x, y) = -2xe^{4y-x^2-y^2}$, $f_y(x, y) = (4-2y)e^{4y-x^2-y^2}$. As $e^{4y-x^2-y^2}$ is never 0, these are both 0 when $x = 0$ and $y = 2$ so f has one critical point at $(0, 2)$.

The second order partial derivatives are $f_{xx}(x, y) = e^{4y-x^2-y^2}(4x^2-2)$, $f_{yy}(x, y) = e^{4y-x^2-y^2}((4-2y)^2-2)$, and $f_{xy}(x, y) = e^{4y-x^2-y^2}(-2x)(4-2y)$. These are $-2e^4, -2e^4$, and 0 respectively at $x = 0, y = 2$ so $D = 4e^8$. As $D > 0$ and $f_{xx} < 0$ this is a local maximum.

10. Evaluate the following double integrals.

(a) $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{y^2}{(x^2+y^2)^{3/2}}\, dydx.$

The curve $y = \sqrt{2x - x^2}$ can be rewritten as $x^2 - 2x + y^2 = 0$ and completing the square makes it $(x-1)^2 + y^2 = 1$ so it is the circle of radius 1 with center $(1, 0)$. The region we are integrating over is the upper half of this circle. This integral will be much easier in polar. The circle $x^2 + y^2 = 2x$ becomes $r^2 = 2r\cos(\theta)$ or $r = 2\cos(\theta)$. The θ values which trace out the upper half of the circle are from 0 to $\pi/2$ so the integral becomes

$$\int_0^{\pi/2} \int_0^{2\cos(\theta)} \frac{r^2\sin^2(\theta)}{r^3} r\, drd\theta = \int_0^{\pi/2} \int_0^{2\cos(\theta)} \sin^2(\theta)\, drd\theta$$

$$= \int_0^{\pi/2} 2\cos(\theta)\sin^2(\theta)\, d\theta = \frac{2}{3}\sin^3(\theta)\Big|_0^{\pi/2} = \frac{2}{3}.$$

(b) $\displaystyle\int_0^1 \int_1^4 x\sqrt{3 + x^2/y}\, dydx + \int_1^2 \int_{x^2}^4 x\sqrt{3 + x^2/y}\, dydx.$

This would be difficult to integrate in this order, but could be integrated in the order $dxdy$ using u-substitution. These two regions can be combined into one $dxdy$ region as $\int_1^4 \int_0^{\sqrt{y}} x\sqrt{3 + x^2/y}\, dxdy$. Using the u-substitution $u = 3 + x^2/y$, $du = (2x/y)dx$ the inside integral is $\int_0^{\sqrt{y}} x\sqrt{3 + x^2/y}\; dx = \int_3^4 (y/2)\sqrt{u}\; du = (y/3)u^{3/2}\Big|_3^4 = (\frac{8}{3} - \sqrt{3})y$. So the double integral is $\int_1^4 (\frac{8}{3} - \sqrt{3})y\; dy = (\frac{1}{2})(\frac{8}{3} - \sqrt{3})y^2\Big|_1^4 = (\frac{1}{2})(\frac{8}{3} - \sqrt{3})(16 - 1) = \frac{40 - 15\sqrt{3}}{2}$.

11. Let $f(x, y, z)$ be differentiable. Suppose $f(1, 3, 5) = 7$ and $\nabla f(1, 3, 5) = \langle 2, -3, 1\rangle$.

(a) Compute the directional derivative of f at the point $(1, 3, 5)$ in the direction of the point $(-1, 4, 7)$.

The direction vector is $\langle -2, 1, 2\rangle$ and the unit direction vector is $\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\rangle$. The directional derivative is the dot product of the unit direction vector with the gradient so it is $\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\rangle \cdot \langle 2, -3, 1\rangle = -\frac{5}{3}$.

(b) In what direction is the directional derivative at $(1, 3, 5)$ the largest? What is the directional derivative in that direction?

The directional derivative is largest in the direction of the gradient $\langle 2, -3, 1\rangle$. The value of the directional derivative in this direction is the magnitude of the gradient which is $\sqrt{14}$.

(c) Find the equation of the tangent plane to the surface $f(x, y, z) = 7$ at the point $(1, 3, 5)$.

The is the plane through the point $(1, 3, 5)$ with normal $\nabla f(1, 3, 5) = \langle 2, -3, 1\rangle$ so the equation is $2(x-1)-3(y-3)+(z-5) = 0$ or $2x-3y+z = -2$.

(d) Use linear approximation to estimate $f(.9, 3.2, 5.1)$.

$f(.9, 3.2, 5.1) \approx f(1, 3, 5)+f_x(1, 3, 5)(.9-1)+f_y(1, 3, 5)(3.2-3)+f_z(1, 3, 5)(5.1-5) = 7 + 2(-.1) + (-3)(.2) + 1(.1) = 6.3$.

(e) Compute $\nabla g(3, 2)$ where $g(x, y) = f(2y - x, xy - 3, x + y)$.

Since we are using x, y for the variables plugged into g, we will use u, v, w for the variables in f. Then $g = f(u, v, w)$ where $u = 2y-x$, $v = xy-3$, $w =$

$x + y$. We need to find $g_x(3,2)$ and $g_y(3,2)$. By the chain rule,

$$\begin{aligned} g_x(x,y) &= f_u(u,v,w)\frac{\partial u}{\partial x} + f_v(u,v,w)\frac{\partial v}{\partial x} + f_w(u,v,w)\frac{\partial w}{\partial x} \\ &= f_u(u,v,w)(-1) + f_v(u,v,w)(y) + f_w(u,v,w)(1) \end{aligned}$$

and

$$\begin{aligned} g_y(x,y) &= f_u(u,v,w)\frac{\partial u}{\partial y} + f_v(u,v,w)\frac{\partial v}{\partial y} + f_w(u,v,w)\frac{\partial w}{\partial y} \\ &= f_u(u,v,w)(2) + f_v(u,v,w)(x) + f_w(u,v,w)(1)\ . \end{aligned}$$

When $x = 3, y = 2$ we have that $u = 1, v = 3, w = 5$ so $g_x(3,2) = 2(-1)+(-3)(2)+(1)(1) = -7$ and $g_y(3,2) = (2)(2)+(-3)(3)+(1)(1) = -4$. So $\nabla g(3,2) = \langle -7, -4 \rangle$.

12. Let $w = x\sqrt{y} - x - y$. Find the maximum and minimum values of w and where they occur on the triangular region bounded by the x-axis, the y-axis, and the line $x + y = 12$.

This is a closed and bounded region so we will find all critical points and find w at each. First check for interior critical points. The partial derivatives are $\partial w/\partial x = \sqrt{y} - 1$ and $\partial w/\partial y = \frac{x}{2\sqrt{y}} - 1$ which are both 0 at the point $(2,1)$. This point is in the region it is an interior critical point. Also, $\partial w/\partial y$ is undefined when $y = 0$ so the points on the line $y = 0$ are also critical. This is one of the boundary curves, so we will check what is happening at these points anyway.

Next check for critical points on each of the three boundary lines. On $x = 0$, $w = -y$ and $w' = -1$ so there are no critical points. Similarly on $y = 0$, $w = -x$ so there are no critical points. For the line $x + y = 12$ we will use Lagrange multipliers. The equations we get are $\sqrt{y} - 1 = \lambda, \frac{x}{2\sqrt{y}} - 1 = \lambda, x + y = 12$. Combining the first two equations we get that $x = 2y$ and plugging this into $x + y = 12$ we get that $y = 4$ and $x = 8$. This point is in our region so we have a critical point at $(8,4)$.

Finally, we must also include corner points where two boundary curves meet. This gives us three more critical points: $(0,0), (0,12)$, and $(12,0)$. We check the value of w at all 5 critical points.

Point	w
$(2,1)$	-1
$(8,4)$	4
$(0,0)$	0
$(12,0)$	-12
$(0,12)$	-12

The maximum is 4 at $(8,4)$ and the minimum is -12 at $(0,12)$ and $(12,0)$.

13. Find $\int_C y^2(e^x+1)dx + 2y(e^x+1)dy$ where C is the closed path formed of three parts: the curve $y = x^2$ from $(0,0)$ to $(2,4)$, the line segment from $(2,4)$ to $(0,2)$ and the line segment from $(0,2)$ to $(0,0)$.

    C is closed and oriented counterclockwise so we can use Green's Theorem to evaluate this with $P = y^2(e^x+1), Q = 2y(e^x+1)$. We integrate $Q_x - P_y = (2ye^x) - (2y(e^x+1)) = -2y$. So the integral becomes $\int_0^2 \int_{x^2}^{x+2} -2y \; dydx = \int_0^2 -y^2\Big|_{x^2}^{x+2} = \int_0^2 -(x+2)^2 + x^4 \; dx = -\frac{1}{3}(x+2)^3 + \frac{1}{5}x^5\Big|_0^2 = -\frac{184}{15}$.

14. A particle is moved in the plane from the origin to the point $(1,1)$. While it is moving, it is acted on by the force $\vec{F} = \langle y^2 - ye^x + xy, 2xy - e^x + x^2\rangle$. This experiment is done twice. The first time the particle is moved in a straight line and the second time is it moved along the curve $y = x^3$. The work done by the force the first time is W_1 and the second time it is W_2. Determine which of W_1 and W_2 is bigger and by how much.

    Write C_1 for the line segment from $(0,0)$ to $(1,1)$ which is along the line $y = x$ and C_2 for the curve $y = x^3$ from $(0,0)$ to $(1,1)$. Note that for $0 \le x \le 1$, the curve C_2 is below C_1. Then $C = C_2 \cup -C_1$ is a closed curve oriented counterclockwise. If W is the work done over C then $W = W_2 - W_1$. We don't need to know W_1 and W_2, just their difference so this is exactly what we want to calculate. We can calculate W using Green's theorem with $P = y^2 - ye^x + xy$ and $Q = 2xy - e^x + x^2$. Then as $Q_x = 2y - e^x + 2x, P_y = 2y - e^x + x$ the difference $Q_x - P_y$ is x. Then $W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \int_{x^3}^{x} x \; dydx = \int_0^1 x^2 - x^4 \; dx = \frac{2}{15}$. So W_2 is larger than W_1 by $2/15$.

15. (a) Find a number c such that the force field $\vec{F} = \langle ye^x + 3x^2 + 3y^2, e^x + cxy + 3y^2\rangle$ is conservative.

If $P = ye^x + 3x^2 + 3y^2$, $Q = e^x + cxy + 3y^2$ then F will be conservative if and only if $P_y = Q_x$. The derivatives are $P_y = e^x + 6y$ and $Q_x = e^x + cy$ so $c = 6$.

(b) Suppose the constant c has the value found in part a. Find a function $f(x, y)$ such that $\vec{F} = \nabla f$.

We are trying to find f with $f_x = ye^x + 3x^2 + 3y^2$ and $f_y = e^x + 6xy + 3y^2$. Integrating f_x with respect to x we get that $f(x, y) = ye^x + x^3 + 3xy^2 + g(y)$. The y partial derivative of this is $f_y(x, y) = e^x + 6xy + g'(y)$. We set this equal to our formula for $f_y = e^x + 6xy + 3y^2$ to get that $g'(y) = 3y^2$ so $g(y) = y^3$. Plug this into the formula for $f(x, y)$ to get that $f(x, y) = ye^x + x^3 + 3xy^2 + y^3$.

(c) Continuing to assume that c has the value found in part a, find the work done by $\vec{F}$ on a particle moving from $(1, 0)$ to $(0, 1)$ along the circle of radius 1 centered at the origin.

Using the fundamental theorem of line integrals, this will be $f(0, 1) - f(1, 0) = 2 - 1 = 1$.

16. Let S be the union of the three surfaces S_1, S_2, S_3 where S_1 is the part of the cylinder $x^2 + y^2 = 16$ with $0 \le z \le 4$ oriented outwards, S_2 is the disk $x^2 + y^2 \le 16$ on the plane $z = 4$ oriented up, and S_3 is the hemisphere $z = \sqrt{16 - x^2 - y^2}$ oriented down. Find $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle e^{\cos(z)}, 2y + 3x, 1/(x^2 + y^2) \rangle$.

If computed directly, this would be 3 surface integrals and the $e^{\cos(z)}$ component of $\vec{F}$ would be messy and difficult to integrate. The three surfaces together form a closed surface with positive orientation, so we can apply the divergence theorem to get that $\iint_S \vec{F} \cdot \, d\vec{S} = \iiint_E \operatorname{div}(\vec{F}) \, dV$ where E is the region enclosed by S.

The divergence of $\vec{F}$ is $\operatorname{div}(\vec{F}) = 2$. The triple integral is equal to 2 times the volume of E. E looks like a cylinder of radius 4 and height 4 with a hemisphere of radius 4 removed from it. The volume of E is therefore $64\pi - \frac{128}{3}\pi = \frac{64\pi}{3}$ so the integral is equal to $\frac{128\pi}{3}$.

Another way to compute the integral is to set it up in cylindrical coordinates. This is

$$\iiint_E \operatorname{div} F \, dV = \int_0^{2\pi} \int_0^4 \int_{\sqrt{16-r^2}}^4 2r \, dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^4 8r - 2r\sqrt{16-r^2}\ dr d\theta$$

$$= \int_0^{2\pi} 4r^2 + \frac{2}{3}(16-r^2)^{3/2}\Big|_0^4\ d\theta = \int_0^{2\pi} \frac{64}{3}\ d\theta = \frac{128\pi}{3}\ .$$

17. Let E be the region inside the cylinder $x^2+y^2=4$, below the cone $z=\sqrt{3x^2+3y^2}$, and above the xy-plane. Set up the integral $\iiint_E z\ dV$ in 3 coordinate systems: rectangular, cylindrical, and spherical. Pick one and evaluate it.

    Rectangular: $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{3x^2+3y^2}} z\ dz\ dy\ dx$

    Cylindrical: $\displaystyle\int_0^{2\pi} \int_0^2 \int_0^{\sqrt{3}r} zr\ dz\ dr\ d\theta$

    Spherical: $\displaystyle\int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{2/\sin(\phi)} \rho^3 \sin(\phi)\cos(\phi)\ d\rho\ d\phi\ d\theta$

    Both cylindrical and spherical are reasonable integrals to compute. The value of the integral is 12π.

18. Evaluate $\int_C x^2 y dx + \frac{1}{3}x^3 dy + xy dz$ where C is the curve of intersection of the hyperbolic paraboloid $z=y^2-x^2$ and the cylinder $x^2+y^2=1$ oriented counterclockwise when viewed from above.

    We will use Stokes' Theorem with the surface S equal to the part of $z=y^2-x^2$ which is inside the cylinder $x^2+y^2=1$ oriented upwards. The surface S can be parametrized as $\vec{r}(x,y)=\langle x,y,y^2-x^2\rangle$ for $x^2+y^2\le 1$. The partial derivatives are $\vec{r}_x=\langle 1,0,-2x\rangle$ and $\vec{r}_y=\langle 0,1,2y\rangle$ so the cross product $\vec{r}_x\times\vec{r}_y=\langle 2x,-2y,1\rangle$. This matches our orientation.

    Take $\vec{F}=\langle x^2y,\frac{1}{3}x^3,xy\rangle$ so $\text{curl}(\vec{F})=\langle x,-y,0\rangle$. This is already just in terms of x,y which are the variables in our parametrization of S. Then $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}(\vec{F})\cdot\vec{S}$ which is equal to

    $$\iint_{x^2+y^2\le 1} \langle x,-y,0\rangle\cdot\langle 2x,-2y,1\rangle\ dA = \iint_{x^2+y^2\le 1} 2x^2+2y^2\ dA\ .$$

    Changing to polar this becomes

    $$\int_0^{2\pi}\int_0^1 2r^3\ drd\theta = \int_0^{2\pi} \frac{1}{2}r^4\Big|_0^1\ d\theta = \int_0^{2\pi}\frac{1}{2}\ d\theta = \pi\ .$$

19. Let S be the part of the sphere $x^2+y^2+z^2=4$ which is above the plane $z=1$ oriented upwards. Let $\vec{F}(x,y,z)=\langle yz,-2z,4z^2-e^{x^2+y^2}\rangle$. Compute $\iint_S \vec{F}\cdot d\vec{S}$ and $\iint_S \text{curl}(\vec{F})\cdot d\vec{S}$.

These surface integrals are both messy to compute directly, so we will use Stokes' Theorem and the Divergence Theorem to compute them more easily.

$\iint_S \text{curl}(\vec{F})\cdot d\vec{S}$:

For this integral, we can apply Stokes' Theorem and there are two possible ways to do this. One possibility is to replace $\iint_S \text{curl}(\vec{F})\cdot d\vec{S}$ with $\int_C \vec{F}\cdot d\vec{r}$ where C is the boundary curve of S. The curve C is the circle of radius $\sqrt{3}$ on the plane $z=1$. The upward orientation of S induces the counterclockwise (when viewed from above) orientation on C. C can be parametrized as $\vec{r}(t)=\langle\sqrt{3}\cos(t),\sqrt{3}\sin(t),1\rangle$ where $0\leq t\leq 2\pi$. The line integral is then

$$\int_0^{2\pi}\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\ dt=\int_0^{2\pi}\langle\sqrt{3}\sin(t),-2,4-e^3\rangle\cdot\langle-\sqrt{3}\sin(t),\sqrt{3}\cos(t),0\rangle\ dt$$

$$=\int_0^{2\pi}-3\sin^2(t)-2\sqrt{3}\cos(t)\ dt=\int_0^{2\pi}-\frac{3}{2}+\frac{3}{2}\cos(2t)-2\sqrt{3}\cos(t)\ dt$$

$$=-\frac{3}{2}t+\frac{3}{4}\sin(2t)-2\sqrt{3}\sin(t)\Big|_0^{2\pi}=-3\pi$$

Another way to compute this integral using Stokes' Theorem is to use that if two surfaces have the same boundary curve with the same induced orientation, then the surface integral of $\text{curl}(\vec{F})$ is the same on both surfaces. So we can replace S with a simpler surface with the same boundary curve, such as the disk with boundary C. Let S_1 be the disk $x^2+y^2\leq 3$ on the plane $z=1$, so S_1 also has boundary curve C. For S and S_1 to induce the same orientation on C, their orientations should match so we take the upward orientation on S_1. Then S_1 can be parametrized as $\vec{r}(x,y)=\langle x,y,1\rangle$ where $x^2+y^2\leq 3$. Then $\vec{r}_x\times\vec{r}_y=\langle 0,0,1\rangle$ which matches the upward orientation. The curl of $\vec{F}$ is $\langle -2ye^{x^2+y^2}+2,2xe^{x^2+y^2}+y,-z\rangle$ so $\text{curl}(\vec{F})\cdot(\vec{r}_x\times\vec{r}_y)=-z$ which is -1 on S_2. The surface integral thus becomes $\iint_{x^2+y^2\leq 3}-1\ dA$ which is -1 times the area of a circle of radius $\sqrt{3}$, which is -3π.

$\iint_S \vec{F}\cdot d\vec{S}$:

The surface S is not closed, so we cannot immediately apply the Divergence Theorem to S. However, if we again take S_1 to be the disk $x^2+y^2\leq 3$ on the plane $z = 1$, then the surfaces S and S_1 combine to give you a closed surface. If we take the downward orientation on S_1, then $S \cup S_1$ will be closed with positive (outward) orientation. Let S_2 be the union of S_1 and S. The surface integrals of $\vec{F}$ over S_1 and S_2 are both easier to compute than the surface integral over S, so we will find these and use them to get the surface integral over S.

The surface S_2 is closed and positively oriented, so by the Divergence Theorem, $\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_E \operatorname{div}(\vec{F})\, dV$ where E is the region inside the sphere $x^2+y^2+z^2 = 4$ and above the plane $z = 1$. The divergence of $\vec{F}$ is $8z$, so we are integrating $8z$ over E. If we set up the triple integral using cylindrical coordinates it becomes:

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} 8zr \; dz \; dr \; d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} 4z^2 r \Big|_{z=1}^{z=\sqrt{4-r^2}} z \; dr \; d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} 12r - 4r^3 \; dr \; d\theta = \int_0^{2\pi} 6r^2 - r^4 \Big|_0^{\sqrt{3}} \; d\theta = \int_0^{2\pi} 9 \; d\theta = 18\pi$$

We use direct computation to find the surface integral over S_1. We can again parametrize S_1 as $\vec{r}(x, y) = \langle x, y, 1\rangle$ where $x^2 + y^2 \leq 3$. Then $\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1\rangle$ which does not match the orientation so instead we use $\langle 0, 0, -1\rangle$. On S_1, the vector field has $z = 1$ so it becomes $\vec{F}(\vec{r}(x, y)) = \langle y, -2, 4-e^{x^2+y^2}\rangle$. The surface integral is therefore equal to:

$$\iint_{x^2+y^2\leq 3} -4+e^{x^2+y^2} \; dA = \int_0^{2\pi} \int_0^{\sqrt{3}} (-4+e^{r^2})r \; dr \; d\theta = \int_0^{2\pi} -2r^2+\frac{1}{2}e^{r^2} \Big|_0^{\sqrt{3}} \; d\theta$$

$$= \int_0^{2\pi} -6 + \frac{1}{2}e^3 - \frac{1}{2} \; d\theta = -12\pi + \pi e^3 - \pi = \pi(e^3 - 13)$$

Finally, we use these two computations as well as the fact that $\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S_1} \vec{F} \cdot d\vec{S}$ to find $\iint_S \vec{F} \cdot d\vec{S} = 18\pi - \pi(e^3 - 13) = \pi(31 - e^3)$.