CS2603 Applied Logic Quiz Solutions

December 4, 2016

1. (Quiz 6) Prove that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

for any $n \geq 1$ by mathematical induction.

Answer: We proceed to prove this by mathematical induction on n.

The equality indeed holds when $n = 1$:

$$LHS(1)=\frac{1}{1\times 2}=1/2=\frac{1}{1+1}=RHS(1).$$

Inductive step

Let $n \geq 1$, and assume that the equality holds for such an n. That is, we assume that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

is true for the given n.

We now want to prove that it also holds for $n+1$. For this, we start from the left-hand side:

$$LHS(n+1)=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}+\frac{1}{(n+1)\times(n+2)}$$

Using the induction hypothesis (IH), we know that:

$$LHS(n+1)=\frac{n}{n+1}+\frac{1}{(n+1)\times(n+2)}.$$

And manipulating this expression we can get:

$$\begin{aligned}LHS(n+1)&=\frac{n}{n+1}+\frac{1}{(n+1)\times(n+2)}\\&=\frac{n(n+2)}{(n+1)(n+2)}+\frac{1}{(n+1)\times(n+2)}\\&=\frac{n(n+2)+1}{(n+1)(n+2)}\\&=\frac{n^2+2n+1}{(n+1)(n+2)}\\&=\frac{(n+1)^2}{(n+1)(n+2)}\\&=\frac{n+1}{n+2}\\&=RHS(n+1).\end{aligned}$$

Since we have proved that:

(a) The equality holds for $n = 1$,

(b) If it holds for some $n \geq 1$, then it holds for $n + 1$,

we conclude that the equality is true for every $n \geq 1$.

2. Let $P(x)$ be the predicate that x is a prime. Translate the statement "there is only one even prime number" into a logic formula. You need to use quantifiers $\forall$ and $\exists$. Assume that the universe of discourse is the set of natural numbers.

$$\exists x.(P(x) \wedge (\forall y.P(y) \wedge (\exists z.y = z + z) \implies y = x))$$

3. (Quiz 7) Suppose that the function c satisfies the axioms:

```
(c nil) = nil                                {c0}
(c (cons x xs)) = (cons x (c (rest xs)))     {c1}
```

Prove that

$$(\text{len } (\text{c } [x_1 \ x_2 \ \cdots \ x_{2n}])) = n$$

by mathematical induction. Write down the name of the axiom at every step.

Proof Base step: $n = 0$,

$$\begin{aligned} LHS =& (\text{len } (\text{c } nil)) & \\ =& (\text{len } nil) & \{\texttt{c0}\} \\ =& 0 & \{\texttt{len0}\} \\ =& RHS & \end{aligned}$$

Inductive step: Assume that for some $n \geq 0$, $(\text{len } (\text{c } [x_1 \ x_2 \ \cdots \ x_{2n}])) = n$.

$$\begin{aligned} & (\text{len } (\text{c } [x_1 \ x_2 \ \cdots \ x_{2(n+1)}])) & \\ =& (\text{len } (\text{c } (\text{cons } x_1 \ [x_2 \ \cdots \ x_{2(n+1)}]))) & \{\texttt{cons}\} \\ =& (\text{len } (\text{cons } x_1 \ (\text{c } (\text{rest } [x_2 \ \cdots \ x_{2(n+1)}])))) & \{\texttt{c1}\} \\ =& (+ \ 1 \ (\text{len } (\text{c } (\text{rest } [x_2 \ \cdots \ x_{2(n+1)}])))) & \{\texttt{len1}\} \\ =& (+ \ 1 \ (\text{len } (\text{c } (\text{rest } (\text{cons } x_2 \ [x_3 \ \cdots \ x_{2(n+1)}]))))) & \{\texttt{cons}\} \\ =& (+ \ 1 \ (\text{len } (\text{c } [x_3 \ \cdots \ x_{2(n+1)}]))) & \{\texttt{rst}\} \\ =& (+ \ 1 \ n) & \{\texttt{IH}\} \end{aligned}$$

So we prove the case of $n + 1$, and we are done.

4. (Quiz 8) Find the binary representation of the number 1323 using the function **bits**.

(bits 1323)	= (cons (mod 1323 2)(bits (floor 1323 2)))	{`bits1`}
	= (cons 1 (bits 661))	{`math`}
	`Notation is informal for now on`	
	= [1 (cons (mod 661 2)(bits (floor 661 2)))]	{`bits1`}
	= [1 1 (bits 330)]	{`math`}
	= [1 1 (cons (mod 330 2)(bits (floor 330 2)))]	{`bits1`}
	= [1 1 0 (bits 165)]	{`math`}
	= [1 1 0 (cons (mod 165 2)(bits (floor 165 2)))]	{`bits1`}
	= [1 1 0 1 (bits 82)]	{`math`}
	= [1 1 0 1 (cons (mod 82 2)(bits (floor 82 2)))]	{`bits1`}
	= [1 1 0 1 0 (bits 41)]	{`math`}
	= [1 1 0 1 0 (cons (mod 41 2)(bits (floor 41 2)))]	{`bits1`}
	= [1 1 0 1 0 1 (bits 20)]	{`math`}
	= [1 1 0 1 0 1 (cons (mod 20 2)(bits (floor 20 2)))]	{`bits`}
	= [1 1 0 1 0 1 0 (bits 10)]	{`math`}
	= [1 1 0 1 0 1 0 (cons (mod 10 2)(bits (floor 10 2)))]	{`bits1`}
	= [1 1 0 1 0 1 0 0 (bits 5)]	{`math`}
	= [1 1 0 1 0 1 0 0 (cons (mod 5 2)(bits (floor 5 2)))]	{`bits1`}
	= [1 1 0 1 0 1 0 0 1 (bits 2)]	{`math`}
	= [1 1 0 1 0 1 0 0 1 (cons (mod 2 2)(bits (floor 2 2)))]	{`bits1`}
	= [1 1 0 1 0 1 0 0 1 0 (bits 1)]	{`math`}
	= [1 1 0 1 0 1 0 0 1 0 (cons (mod 1 2)(bits (floor 1 2)))]	{`bits1`}
	= [1 1 0 1 0 1 0 0 1 0 1 (bits 0)]	{`math`}
	= [1 1 0 1 0 1 0 0 1 0 1]	{`bits0`}

5. Find the hexdecimal representation of the number 1323.

   We take the binary representation of 1323, which is: 10100101011_2 and then take the bits in groups of 4 starting from the least significant bit:

$$101 \mid 0010 \mid 1011_2$$

   and then convert to hexadecimal every group of four bits. This results in the hexadecimal number:

$$5 \mid 2 \mid B.$$

   So we have that $1323_{10} = 52B_{16}$.

   More precisely,

$$\begin{aligned}
&1323_{10}\\
=&1 + 1\times 2^1 + 0\times 2^2 + 1\times 2^3 + 0\times 2^4 + 1\times 2^5 + 0\times 2^6 + 0\times 2^7 + 1\times 2^8 + 0\times 2^9 + 1\times 2^{10}\\
=&(1 + 1\times 2^1 + 0\times 2^2 + 1\times 2^3) + 2^4\times(0 + 1\times 2 + 0\times 2^2 + 0\times 2^3) + 2^8\times(1 + 0\times 2 + 1\times 2^2)\\
=&11 + 2\times 16 + 5\times 16^2\\
=&52B_{16}
\end{aligned}$$

6. Does the equation

$$(\text{len (prefix } n\ xs)) = n$$

hold in all the cases? If so, prove it. Otherwise give a counterexample.

Answer: The equation can not hold in all the case. A counterexample can be given when $n = 5$ and $xs = nil$.

7. (Quiz 9) A sequence of integers a_i is defined as

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= 3 \\
a_3 &= a_2 + 2a_1 + 2 = 3 + 2 * 1 + 2 = 7 \\
&\vdots \\
a_{i+2} &= a_{i+1} + 2a_i + 2 \text{ for all } i \geq 1.
\end{aligned}$$

Prove by the mathematical induction that

$$a_n = 2^n - 1$$

We proceed to prove this by mathematical induction on n.

Base step: The equality indeed holds when $n = 1$:

$$LHS(1) = a_1 = 1 = 2^1 - 1 = RHS(1).$$

And it also holds when $n = 2$

$$LHS(2) = a_2 = 3 = 2^2 - 1 = RHS(2).$$

Inductive step: Let $n \geq 2$, and assume that the equality holds for integers $n - 1$ and n. That is, we assume that

$$a_{n-1} = 2^{n-1} - 1, a_n = 2^n - 1$$

are true.

We now want to prove that it also holds for $n + 1$. For this, we start from the left-hand side:

$$LHS(n + 1) = a_{n+1} = a_n + 2a_{n-1} + 2,$$

We can write:

$$\begin{aligned}
LHS(n + 1) = a_{n+1} = a_n + 2a_{n-1} + 2 &= 2^n - 1 + 2 \times (2^{n-1} - 1) + 2 \\
&= 2^n - 1 + 2^n - 2 + 2 = (1 + 1) * 2^n - 1 = 2^{n+1} - 1 = RHS(n + 1).
\end{aligned}$$

So $LHS(n + 1) = RHS(n + 1)$ holds, therefore we have proved what we wanted to prove.

8. (Quiz 10) Evaluate `(nmb '( 0 1 0 1 1))`

$$
\begin{aligned}
(\text{nmb}\ '(0\ 1\ 0\ 1\ 1)) &= 0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 && \{\texttt{Horner2}\} \\
&= 26 && \{\texttt{math }\}
\end{aligned}
$$

9. Assume the word size $w = 16$. Add two twos-complement numerals

```
0100 1011 1110 0101
1101 0000 0101 0101
```

The answer in twos-complement representation is

`0001 1100 0011 1010`

10. The following is a list of numbers represented by twos-complement numerals with $w = 8$. Put them in the sorted order, starting from the smallest number.

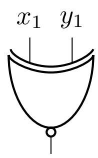
$$
\begin{aligned}
a_1 &= 01001100 \\
a_2 &= 11010011 \\
a_3 &= 10110011 \\
a_4 &= 01011001
\end{aligned}
$$

We know that since these numbers are in twos-complement, then the most significant bit corresponds to the sign of the number: a 1 in the MSB means the number is negative, a 0 in the MSB means the number is positive. So we know that the largest numbers will be those that have a MSB with a 0.

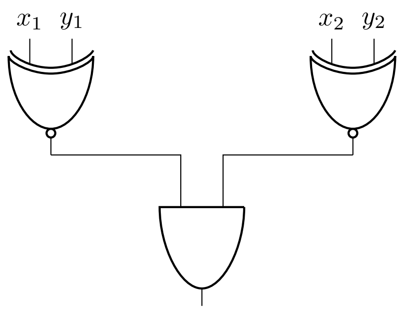
The order is:

$$a_3 < a_2 < a_1 < a_4.$$

11. (Quiz 11) Diagram a circuit that tests whether two bits are equal or not. The output bit should be 1 if the two input bits are equal, and 0 otherwise.



12. Diagram a circuit that tests whether two 2-bit numbers are equal or not. Note that there should be only one output bit.



13. What are the absolute values of the following twos-complement numerals? The answers should be in binary representation. Assume that $w = 8$.

    (a) 0101 1100
    (b) 1001 0011
    (c) 1010 1101

    Answer:

    (a) 0101 1100
    (b) 0110 1101
    (c) 0101 0011

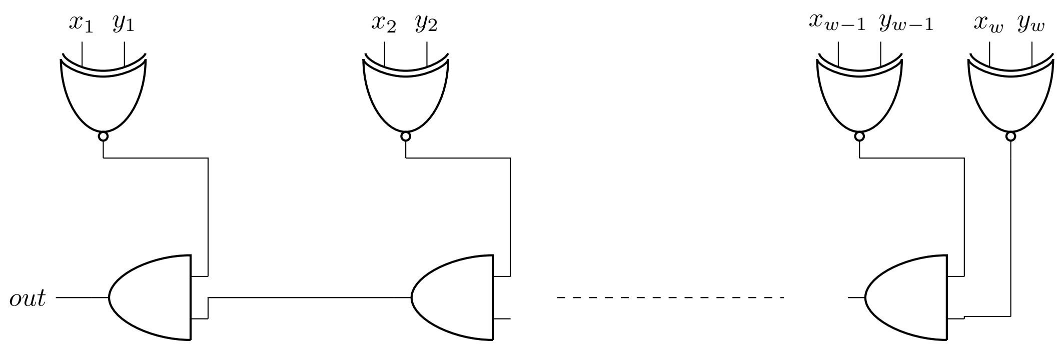
14. (Quiz 12) Multiply the binary numerals 111011 and 11101. The result should be in binary.

					1	1	1	0	1	1
						1	1	1	0	1
					1	1	1	0	1	1
			1	1	1	0	1	1		
		1	0	0	1	0	0	1	1	1
		1	1	1	0	1	1			
	1	0	1	1	1	1	1	1	1	1
	1	1	1	0	1	1				
1	1	0	1	0	1	0	1	1	1	1

15. Diagram a circuit that tests whether two w-bit numbers are equal or not.

    The answer is in the next question.

16. (Quiz 13) Define an ACL2 model for the following circuit that tests equality of two binary numerals.




```
(defun xnor (x y)
   (if (= x y)
       1
       0
   )
)

(defun test_equal (xs ys)
    (let* ((s1 (xnor (first xs) (first ys))))
          ( if (= (len xs) 1)
               s1
               (and-gate s1
                  (test_equal  (rest xs) (rest ys))
               )
          )
    )
)
```

Note that we assume that the initial *xs* and *ys* are not nil, and they are bit strings of the same length.

17. Write a theorem in ACL2 that states the correctness of your program in the first problem. You donot need to prove it.

```
(defthm test_equal_ok
   (= (test_equal xs ys) (= (nmb xs) (nmb ys))))
```