Fall, 2015

MATH 2934 Section 070

Midterm Exam 1

Name: Key.

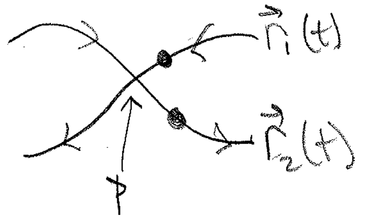
You have 50 minutes to complete the exam. If you have questions, be sure to ask. Be sure to completely answer each part of each question, and to **show all of your work**.

Problem	Points	Score
1	25	
2	15	
3	15	
4	15	
5	15	
6	15	
	100	

1. (*25 points*) Decide whether the following statements are TRUE or FALSE. If FALSE, give a short explanation of what is wrong.

a) Suppose one particle is moving along the curve given by $\mathbf{r}_1(t)$ and another particle is moving along a curve given by $\mathbf{r}_2(t)$. If the curves intersect, then the particles collide.

False



The particles could reach the point P of intersection at different parameter values t.

b) The cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ gives a vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

True

c) Two non-parallel planes (i.e. planes whose normal vectors are not parallel) always intersect in a line.

True

d) Two non-parallel lines always intersect in a point.

False

Skew lines are not parallel and also do not intersect.

For example: $\vec{r}_1(t) = \langle t, 0, 0 \rangle$

$\vec{r}_2(t) = \langle 0, 1, t \rangle$

2. (*15 points*) Find the equation of the line in **vector form** at the intersection of the planes $2x - y = 4$ and $x + z = 1$.

Explain carefully what you are doing in each step of your work.

Plane 1 has normal vector $\vec{n}_1 = \langle 2, -1, 0 \rangle$;
Plane 2 has normal vector $\vec{n}_2 = \langle 1, 0, 1 \rangle$.

The line of intersection lies in both planes, so it is orthogonal to both normal vectors $\vec{n}_1$ and $\vec{n}_2$.

$$\vec{V} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \langle -1, -2, 1 \rangle$$

Next, we need a point on both planos, so it is on the line.

P: $X = 0$ $\quad y = -4$ $\quad Z = 1$, for example, satisfies both $2x - y = 4$ and $x + Z = 1$.

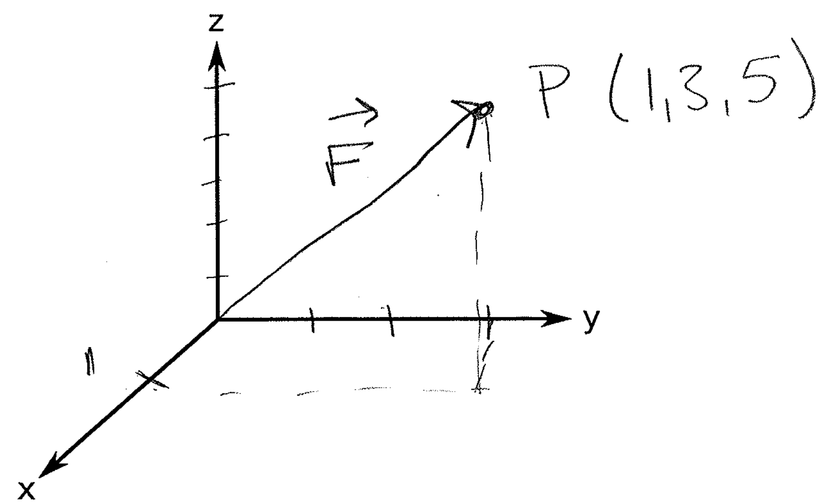
$\vec{r}_0 = \langle 0, -4, 1 \rangle$

This gives

$$\boxed{\begin{aligned} \vec{r}(t) = \vec{r}_0 + t\vec{v} &= \langle 0, -4, 1 \rangle + t \langle -1, -2, 1 \rangle \\ &= \langle -t, -4 - 2t, 1 + t \rangle \end{aligned}}$$

3. (*15 points*) Let $P(1,3,5)$ be a point in 3-dimensional space.

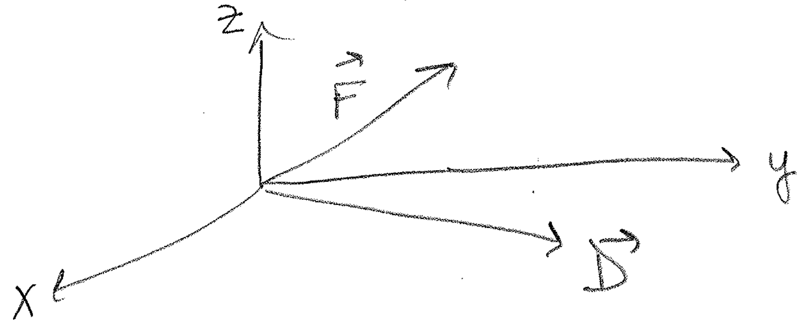
a) On the axes below, label P, draw the position vector of P, and label the position vector **F**.



b) Suppose the vector **F** above is a force (in newtons) being applied by a student to pull their heavy bag of dirty clothes along the floor to the laundry. If the path to the washing machine is given by the displacement vector

$$\mathbf{D} = \langle 5, 20, 0 \rangle$$

where the distance is measured in meters, find the work expended in moving the laundry bag.



Work = $\vec{F} \cdot \vec{D}$, a Scalar quantity

$\vec{F} \cdot \vec{D} = \langle 1,3,5 \rangle \cdot \langle 5,20,0 \rangle$

$= 5 + 60 + 0 = \boxed{65 \text{ N-m}}$ or Joule

4. (*15 points*) Given the vector $\mathbf{c} = \langle 2, -5, 3 \rangle$, find two vectors $\mathbf{a}$ and $\mathbf{b}$ that are both orthogonal to $\mathbf{c}$ AND are orthogonal to each other.

Hint: Find these one at a time – first find $\mathbf{a}$ orthogonal to $\mathbf{c}$, then find $\mathbf{b}$. There are many possible solutions.

Explain carefully what you are doing in each step of your work.

$\vec{c} = \langle 2, -5, 3 \rangle$

Here are some options based on choice of $\vec{a}$.

$\vec{a} = \langle 1, 1, 1 \rangle \qquad \vec{b} = \vec{a} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & -5 & 3 \end{vmatrix} = \langle 8, -1, -7 \rangle$

$\vec{a} = \langle 3, 0, -2 \rangle \qquad \vec{b} = \vec{a} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -2 \\ 2 & -5 & 3 \end{vmatrix} = \langle -10, -13, -15 \rangle$

5a). (*15 points*) Find a vector function $\mathbf{r}(t)$ describing the curve of intersection of the cylinder $y^2 + z^2 = 16$ and the plane $x + y + z = 4$.

Explain carefully what you are doing in each step of your work.

The y z components are on the cylinder,
So parameterize with $y = 4\cos t$ $z = 4\sin t$.
Then, points on the plane satisfy

$$x = 4 - y - z$$

This gives $\boxed{\vec{r}(t) = \langle 4 - 4\cos t - 4\sin t, 4\cos t, 4\sin t \rangle}$

b) Find the unit tangent $\mathbf{T}(t)$ to $\mathbf{r}(t)$ above.

$$\vec{r}\,'(t) = \langle 4\sin t - 4\cos t, -4\sin t, 4\cos t \rangle$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 4\sin t - 4\cos t, -4\sin t, 4\cos t \rangle}{4\sqrt{(\sin t - \cos t)^2 + \sin^2 t + \cos^2 t}}$$

(Cancel 4's)

$$= \boxed{\frac{\langle \sin t - \cos t, -\sin t, \cos t \rangle}{\sqrt{2 - 2\sin t \cos t}}}$$

6a). (*15 points*) Given the vector function

$$\mathbf{r}(t) = t^2\,\mathbf{i} + e^{3t}\,\mathbf{j} - t\cos t\,\mathbf{k}$$

write, but *do not evaluate* the expression giving the arc length s of the path from $t = -2$ to $t = 7$.

$$\vec{r}\,'(t) = \langle 2t,\ 3e^{3t},\ -\cos t + t\sin t \rangle$$

$$S = \int_{-2}^{7} |\vec{r}\,'(t)|\,dt = \boxed{\int_{-2}^{7} \sqrt{4t^2 + 9e^{6t} + (-\cos t + t\sin t)^2}\,dt}$$

b) Let $s(t)$ be the arc length function for the curve $\mathbf{r}(t)$ from part (a), starting from $t = -2$. Find the rate of change $\dfrac{ds}{dt}$.

$$S(t) = \int_{-2}^{t} |\vec{r}\,'(u)|\,du$$

$$\frac{ds}{dt} = |\vec{r}\,'(t)| = \boxed{\sqrt{4t^2 + 9e^{6t} + (-\cos t + t\sin t)^2}}$$