Fall, 2015

MATH 2934 Section 110 and Section 070

Midterm Exam 2

Name: Solutions.

You have 75 minutes to complete the exam. If you have questions, be sure to ask. Be sure to completely answer each part of each question, and to **show all of your work**.

Problem	Points	Score
1	20	
2	20	
3	15	
4	15	
5	15	
6	15	
	100	

1. (*20 points*) Decide whether the following statements are TRUE or FALSE. If FALSE, give a short explanation of what is wrong.

a) The acceleration of a particle with trajectory on the curve $\mathbf{r}(t)$ is given by the vector $\mathbf{r}''(t)$.

True

b) The gradient of a function f at the point (a, b) is a vector that is tangent to the level curve of f containing the point (a, b).

False : ∇f is orthogonal to a level curve of f

c) If the function $f(x, y)$ has a point (a, b) where $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then $f(a, b)$ is either a maximum value or a minimum value for the function.

False : A critical point could also be neither a maximum nor a minimum, e.g. a saddle point.

d) The (nonzero) level curves of the function $f(x, y) = \sqrt{3x^2 + 8y^2}$ are ellipses.

True

$k = 3x^2 + 8y^2$

e) If $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, then we can say f is differentiable at the point (a, b).

False

The limit may exist without the function being continuous, so it would then also be non-differentiable at (a,b).

2. (*20 points*) Compute the following:

a) Compute the velocity and speed at time $t = 1$ of a particle moving on a path given by

$$\mathbf{r}(t) = 2t\mathbf{i} + e^{t^2}\mathbf{j}.$$

Velocity: $\vec{r}\,'(t) = 2\vec{i} + 2te^{t^2}\vec{j}$ $\quad \boxed{\vec{r}\,'(1) = 2\vec{i} + 2e\vec{j}}$

Speed $|\vec{r}\,'(1)| = \sqrt{4+4e^2} = \boxed{2\sqrt{1+e^2}}$

b) Compute $\dfrac{\partial^2}{\partial x \partial z}\left(\dfrac{1}{xyz}\right)$ $\quad \dfrac{\partial}{\partial z}\left(\dfrac{1}{xyz}\right) = -\dfrac{xy}{(xyz)^2} = \dfrac{-1}{xyz^2}$

Then $\dfrac{\partial}{\partial x}\left(\dfrac{-1}{xyz^2}\right) = \dfrac{yz^2}{(xyz^2)^2} = \boxed{\dfrac{1}{x^2yz^2}}$

c) Compute the derivative of the function $g(x, y, z) = xyz^2$ in the direction of the vector $\langle 1, 0, -3\rangle$.

$\vec{u} = \dfrac{1}{\sqrt{10}}\langle 1, 0, -3\rangle \qquad \nabla g = \langle yz^2, xz^2, 2xyz\rangle$

$D_{\vec{u}}\, g(x,y,z) = \nabla g \cdot \vec{u} = \boxed{\dfrac{1}{\sqrt{10}}\left(yz^2 - 6xyz\right)}$

d) The radius r of a right circular cylinder is increasing at a rate of 0.01 cm/min. and the height h is increasing at a rate of .02 cm./min. Find the rate at which the volume is increasing at the time at which the radius is 2 cm and the height is 5 cm. (Recall that the volume of such a cylinder is $V = \pi r^2 h$.)

By the Chain Rule: $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt}$

$= (2\pi r h)(.01) + \pi r^2(.02)$

$= 2\pi(2)(5)(.01) + \pi(2)^2(.02)$

$= (.2 + .08)\pi = \boxed{.28\pi \text{ cm}^3/\text{min}}$

3. (*15 points*) **a)** Find the equation of the tangent plane to the surface

$$x^3 - 2xy + z^3 + 7y = -6$$

at the point $P(1, 4, -3)$.

Let $F(x,y,z) = x^3 - 2xy + z^3 + 7y$, so our surface is a level surface of F.

Tangent plane: $F_x = 3x^2 - 2$ $F_y = -2x + 7$ $F_z = 3z^2$

$F_x(1,4,-3)(x-1) + F_y(1,4,-3)(y-4) + F_z(1,4,3)(z+3) = 0$

$(x-1) + 5(y-4) + 27(z+3) = 0$

b) Find the equation of the normal line to the surface from part (a) above at the same point $P(1, 4, -3)$.

Direction $\nabla F(1,4,-3) = \langle 1, 5, 27 \rangle$

Position vector $\langle 1, 4, -3 \rangle$

Normal Line $\boxed{\vec{r}(t) = \langle 1, 4, -3 \rangle + t\langle 1, 5, 27 \rangle}$

4. (*15 points*) Use differentials to estimate the change in the volume V of a rectangular box if the side lengths l, w and h change respectively from $(2, 3, 5)$ to $(2.2, 3.1, 4.7)$.

Explain carefully what you are doing in each step of your work.

$$V = lwh$$

$$dV = wh\,dl + lh\,dw + lw\,dh$$

$$(l, w, h) = (2, 3, 5)$$

$$dl = .2$$

$$dw = .1$$

$$dh = -.3$$

$$dV = 15(.2) + 10(.1) + 6(-.3)$$

$$= 3 + 1 - 1.8 = \boxed{2.2}$$

5). (*15 points*) The depth (in feet) of a lake at a point (x, y) on its surface is given by the function

$$D(x, y) = 100 - 2x^2 - 3y^2.$$

A turtle is swimming at the point $(4, 2)$ and wants to head for deeper water.
a) In which direction should the turtle swim to increase the depth of the water most rapidly?

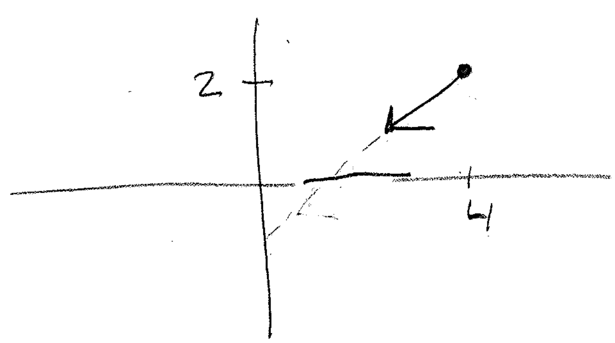
b) What is the rate of change of the water depth in that direction?

Explain carefully what you are doing in each step of your work.

Direction of the gradient ∇D will give steepest change in D.

$$\nabla D = \langle -4x, -6y \rangle$$

(a) $\boxed{\nabla D(4,2) = \langle -16, -12 \rangle}$ Direction the turtle should swim



(b) Rate of change of water depth (slope) in direction ∇D is $|\nabla D(4,2)|$

$$\sqrt{(-16)^2 + (-12)^2} = \sqrt{400} = \boxed{20}$$

6). (*15 points*) Find the location of all local maxima and minima (and determine which is which) for the function

$$f(x, y) = x^4 + y^3 + 32x - 9y.$$

$f_x(x,y) = 4x^3 + 32 = 4(x^3+8)$

$f_x(x,y) = 0$ when $x^3+8=0$

$x = -2$

$f_y(x,y) = 3y^2 - 9 = 3(y^2-3)$

$f_y(x,y) = 0$ when $y = \pm\sqrt{3}$

Critical points are $(-2, \sqrt{3})$ and $(-2, -\sqrt{3})$

Second Derivative Test is used to Classify.

$f_{xx} = 12x^2 \geq 0$ for all x

$f_{yy} = 6y$

$f_{xy} = 0$

$D = (12x^2)(6y) - 0$

	f_{xx}	D	
$(-2, \sqrt{3})$	$+$	$+$	local minimum
$(-2, -\sqrt{3})$	$+$	$-$	Saddle point - neither max. nor min.