Fall, 2015

MATH 2934

Midterm Exam 3

Name: Solutions.

You have 75 minutes to complete the exam. If you have questions, be sure to ask. Be sure to completely answer each part of each question, and to **show all of your work**.

Problem	Points	Score
1	20	
2	20	
3	15	
4	15	
5	15	
6	15	
	100	

1. (*20 points*) State whether the following statements are TRUE or FALSE. If a statement is false, give a short explanation of what is wrong.

a) If a, b, c, d are constants, then $\int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x = -\int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

$\boxed{F}$ By Fubini's Thm, there is no "$-$" sign.

b) Green's theorem applies only to conservative vector fields.

$\boxed{F}$ Green's Thm. applies to any 2-dim. vector field w/ dfbl components.

c) A line integral on any vector field **F** is "independent of path", which means the path doesn't affect the integral, only the endpoints.

$\boxed{F}$ $\vec{F}$ must be conservative.

d) Given a function f and a smooth path C, $\int_{-C} f\,\mathrm{d}s = \int_C f\,\mathrm{d}s$.
(Recall that $-C$ denotes the same points as the path C but traversed in the opposite direction.)

$\boxed{T}$

e) The volume differential $\mathrm{d}V$ in cylindrical coordinates is $\mathrm{d}V = r^2\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

$\boxed{F}$ $dV = r\,dz\,dr\,d\theta$

2. (*20 points*) Complete the following problems.

a) $\int_0^1 \int_x^{2x} xy\, dy\, dx$

$$= \int_0^1 x \left. \frac{y^2}{2} \right|_x^{2x}$$

$$= \int_0^1 x \left(\frac{4x^2}{2} - \frac{x^2}{2} \right) dx$$

$$= \int_0^1 \frac{3x^3}{2}\, dx$$

$$= \left. \frac{3}{8} x^4 \right|_0^1 = \boxed{\frac{3}{8}}$$

b) $\int_C yz\, ds$ where C is given by $x(t) = \cos t$, $y(t) = \sin t$, and $z(t) = t\sqrt{8}$ for $0 \le t \le \pi$.

$$= \int_0^\pi \sqrt{8}\, t \sin t \sqrt{\cos^2 t + \sin^2 t + 8}\, dt$$

$$= 3\sqrt{8} \int_0^\pi t \sin t\, dt$$

$u = t$ $\quad dv = \sin t\, dt$

$du = dt$ $\quad v = -\cos t$

$$= 3\sqrt{8} \left(-t \cos t \Big|_0^\pi + \int_0^\pi \cos t\, dt \right)$$

$$=$$

c) Find a function f such that the vector field $\mathbf{F} = \langle e^y + 6x, xe^y + 4y^3 \rangle$ satisfies $\mathbf{F} = \nabla f$.

$$\frac{\partial f}{\partial x} = e^y + 6x \Rightarrow f(x,y) = xe^y + 3x^2 + g(y)$$

$$\frac{\partial f}{\partial y} = xe^y + 4y^3 \Rightarrow f(x,y) = xe^y + y^4 + h(x)$$

$$\boxed{f(x,y) = xe^y + 3x^2 + y^4 \quad (+ \text{ any constant})}$$

3. (*15 points*) You are given the double integral:

$$\int_0^1 \int_{-\sqrt{1-y}}^{2-2y} f(x, y)\, dx\, dy$$

a) Sketch the region of integration for this double integral.

$x = 2-2y$

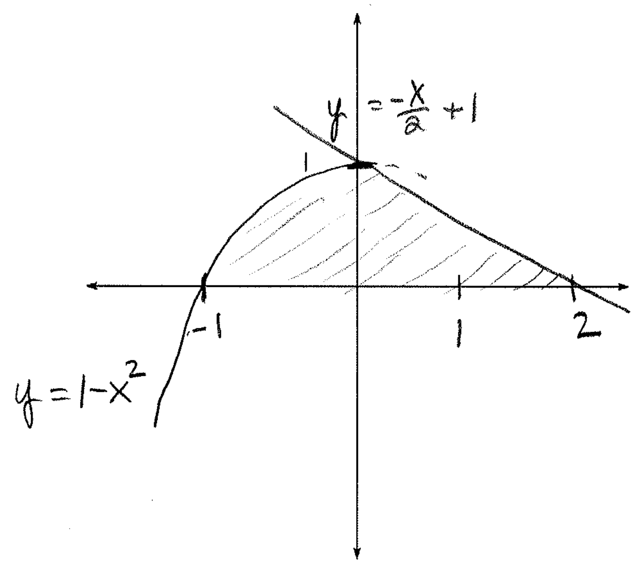
$y = -\frac{x}{2} + 1$

$x = -\sqrt{1-y}$

$x^2 = 1-y$

$y = 1 - x^2$

(x negative)



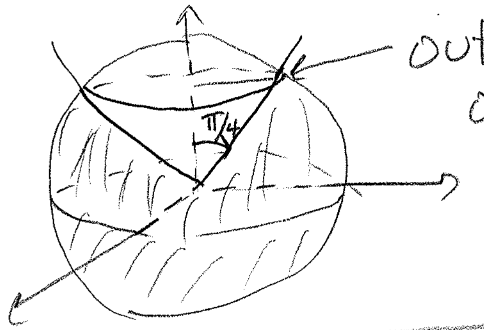
b) Determine the expression you would use to integrate over the same region, but with the order of integration reversed. Note that you do NOT need to compute any integrals here.

$$\int_0^1 \int_{-\sqrt{1-y}}^{2-2y} f(x,y)\, dx\, dy =$$

$$\boxed{\int_{-1}^{0} \int_0^{1-x^2} f(x,y)\, dy\, dx + \int_0^2 \int_0^{-\frac{x}{2}+1} f(x,y)\, dy\, dx}$$

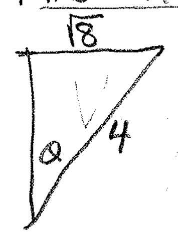
4. (*15 points*) Write, but DO NOT evaluate, the following integrals.

a) Write an integral in spherical coordinates that will compute the volume of the region that is outside the cone $z = \sqrt{x^2 + y^2}$ but is inside the sphere $x^2 + y^2 + z^2 = 16$.



outside of cone, inside sphere

Find interior cone angle:



The intersection of cone and sphere is

$$2x^2 + 2y^2 = 16$$

$$x^2 + y^2 = 8$$

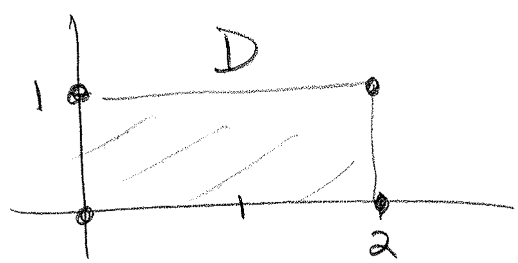
radius $\sqrt{8}$

$$\sin \varphi = \frac{\sqrt{8}}{4} = \frac{\sqrt{2}}{2}$$

$$\boxed{\varphi = \frac{\pi}{4}}$$

$$\boxed{\int_{\pi/4}^{\pi} \int_0^{2\pi} \int_0^4 \underbrace{\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi}_{dV}}$$

b) Write an integral that gives the surface area of the part of the surface $z = 3x^2 + ye^x$ that lies above the rectangle with vertices $(0,0), (2,0), (2,1), (0,1)$. Find the limits of integration and the integrand, but you do not need to solve the integral.



$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

$$= \boxed{\int_0^1 \int_0^2 \sqrt{1 + (6x + ye^x)^2 + (e^x)^2} \, dx \, dy}$$

5. (*15 points*) Write down the definite integral with respect to the variable t which will compute the work expended by the vector field

$$\mathbf{F} = \langle xy, xz, e^x \rangle$$

to move an object on the path C given in vector form by

$$\mathbf{r}(t) = \langle t, \cos t, \sin t \rangle, \qquad 0 \leq t \leq \pi.$$

You do NOT need to compute this integral.

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{\pi} \langle t\cos t, t\sin t, e^t \rangle \cdot \langle 1, -\sin t, \cos t \rangle\, dt$$

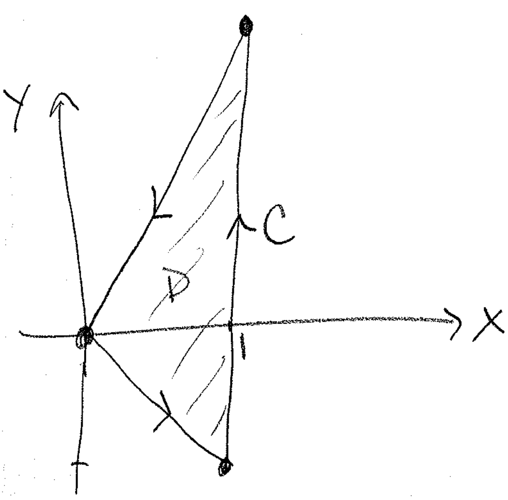
$$= \boxed{\int_0^{\pi} t\cos t - t\sin^2 t + e^t \cos t \; dt}$$

6. (*15 points*) Use Green's Theorem to compute $\int_C \mathbf{F} \cdot \mathbf{dr}$ where

$$\mathbf{F} = \langle 3xy, 2x^2 \rangle$$

and C is the triangular path from $(0,0)$ to $(1,-1)$, then to $(1,4)$, and back to $(0,0)$. Hint: sketch the path C.

Green's Thm: $\oint_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$



$Q_x = 4x$

$P_y = 3x$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^1 \int_{-x}^{4x} 4x - 3x \; dy\,dx$$

$$= \int_0^1 xy \Big|_{-x}^{4x} dx$$

$$= \int_0^1 x(4x + x)\, dx$$

$$= \frac{5x^3}{3}\Big|_0^1 = \boxed{\frac{5}{3}}$$