§1、不定积分的概念与性质

1、 原函数与不定积分

定义 1： 若 $F'(x)=f(x)$，则称 $F(x)$ 为 $f(x)$ 的原函数。

① 连续函数一定有原函数；

② 若 $F(x)$ 为 $f(x)$ 的原函数，则 $F(x)+C$ 也为 $f(x)$ 的原函数；

事实上，$(F(x)+C)'=F'(x)=f(x)$

③ $f(x)$ 的任意两个原函数仅相差一个常数。

事实上，由 $[F_1(x)-F_1(x)]'=F_1'(x)-F_2'(x)=f(x)-f(x)=0$，得 $F_1(x)-F_2(x)=C$

故 $F(x)+C$ 表示了 $f(x)$ 的所有原函数，其中 $F(x)$ 为 $f(x)$ 的一个原函数。

定义 2： $f(x)$ 的所有原函数称为 $f(x)$ 的不定积分，记为 $\int f(x)dx$，$\int-$积分号，$f(x)-$被积函数，$x-$积分变量。

$$\text{显然} \int f(x)dx=F(x)+C$$

例 1、求下列函数的不定积分

① $\int kdx=kx+C$

② $\int x^{\mu}dx=\begin{cases}\dfrac{1}{\mu+1}x^{\mu+1}+C & \mu\neq -1\\ \ln x+C & \mu=-1\end{cases}$

2、 基本积分表（共 24 个基本积分公式）

3、 不定积分的性质

① $\int[f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$

② $\int kf(x)dx=k\int f(x)dx \qquad (k\neq 0)$

例 2、求下列不定积分

① $\int\dfrac{dx}{x^2}=\int x^{-2}dx=\dfrac{1}{(-2)+1}x^{(-2)+1}+C=-\dfrac{1}{x}+C$

② $\int \frac{dx}{\sqrt{x}} = \int x^{-1/2}dx = \frac{1}{(-1/2)+1}x^{(-1/2)+1} + C = 2\sqrt{x} + C$

③ $\int \left(\frac{5}{\sqrt{1-x^2}} - \frac{3}{1+x^2} \right) dx = 5\arcsin x - 3\arctan x + C$

④ $\int \left(\pi^x e^x - \frac{1}{2x} \right) dx = \int (\pi e)^x dx - \frac{1}{2}\int \frac{dx}{x} = \frac{(\pi e)^x}{\ln(\pi e)} - \frac{1}{2}\ln x + C$

⑤ $\int \csc x(\csc x - \cot x)dx = \int \csc^2 xdx - \int \csc x \cot xdx = -\cot x + \csc x + C$

⑥

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x}dx = \int \csc^2 xdx + \int \sec^2 xdx = -\cot x + \tan x + C$$

⑦ $\int \cot^2 xdx = \int (\csc^2 x - 1)dx = -\cot x - x + C$

⑧ $\int \frac{x^4}{1+x^2}dx = \int \frac{x^4 - 1 + 1}{1+x^2}dx = \int \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx = \frac{1}{3}x^3 - x + \arctan x + C$

§2、不定积分的换元法

一、第一类换元法（凑微分法）

1、$\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)d(ax+b)$, 即$dx = \frac{1}{a}d(ax+b)$

例 1、求不定积分

① $\int \sin 5xdx = \frac{1}{5}\int \sin 5xd(5x) \underline{\underline{5x=u}} \frac{1}{5}\int \sin udu = -\frac{1}{5}\cos(5x) + C$

②

$$\int (1-2x)^7 dx = -\frac{1}{2}\int (1-2x)^7 d(1-2x) = -\frac{1}{2}\cdot\frac{1}{7+1}(1-2x)^{7+1} + C = -\frac{1}{16}(1-2x)^8 + C$$

③

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\int \frac{d(x/a)}{1+(x/a)^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad (20)$$

④

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{d(x/a)}{\sqrt{1-(x/a)^2}} = \arcsin\left(\frac{x}{a}\right)+C \tag{23}$$

2、$\int f(x^n)x^{n-1}dx = \frac{1}{n}\int f(x^n)dx^n$，即$x^{n-1}dx = dx^n$

例 2、求不定积分

①

$$\int x\sqrt{1-x^2}dx = -\frac{1}{2}\int (1-x^2)^{1/2} d(1-x^2) = -\frac{1}{2}\cdot\frac{1}{\frac{1}{2}+1}(1-x^2)^{\frac{1}{2}+1}+C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C$$

② $\int x^2 e^{-x^3}dx = -\frac{1}{3}\int e^{-x^3}d(-x^3) = -\frac{1}{3}e^{-x^3}+C$

③ $\int \frac{1}{x^2}\cos\frac{1}{x}dx = -\int \cos\frac{1}{x}d\left(\frac{1}{x}\right) = -\sin\left(\frac{1}{x}\right)+C \quad \left(\frac{1}{x^2}dx = -d\left(\frac{1}{x}\right)\right)$

④ $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos\sqrt{x}d\sqrt{x} = 2\sin\sqrt{x}+C \qquad \left(\frac{1}{\sqrt{x}}dx = 2d\sqrt{x}\right)$

3、

$\frac{1}{x}dx = d\ln x$, $e^x dx = de^x$, $\sin xdx = -d\cos x$, $\cos xdx = d\sin x$, $\sec^2 xdx = d\tan x$,

$\sec x\tan xdx = d\sec x$, $\frac{1}{1+x^2}dx = d\arctan x$, $\frac{1}{\sqrt{1-x^2}}dx = d\arcsin x$,

$\frac{x}{\sqrt{a^2\pm x^2}}dx = \pm d\sqrt{a^2\pm x^2}$, …

例 3、求不定积分

①

$$\int \tan xdx = \int \frac{\sin x}{\cos x}dx = -\int \frac{d\cos x}{\cos x} = -\ln\cos x+C = \ln\sec x+C \tag{16}$$

②

$$\int \cot xdx = \int \frac{\cos x}{\sin x}dx = \int \frac{d\sin x}{\sin x} = \ln\sin x+C = -\ln\cos x+C \tag{17}$$

③

$$\int \sec x dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}dx = \int \frac{d(\sec x+\tan x)}{\sec x+\tan x} = \ln(\sec x+\tan x)+C \qquad (18)$$

④

$$\int \csc x dx = \int \frac{\csc x(\csc x-\cot x)}{\csc x-\cot x}dx = \int \frac{d(\csc x-\cot x)}{\csc x-\cot x} = \ln(\csc x-\cot x)+C \qquad (19)$$

⑤ $\int \frac{1}{x\ln x}dx = \int \frac{d\ln x}{\ln x} = \ln(\ln x)+C$

⑥ $\int \frac{dx}{\cos^2 x(1+\tan x)} = \int \frac{d(\tan x+1)}{\tan x+1} = \ln(\tan x+1)+C$

⑦ $\int \frac{e^x}{1+e^x}dx = \int \frac{d(1+e^x)}{1+e^x} = \ln(1+e^x)+C$

⑧ $\int \frac{dx}{1+e^x} = \int \frac{(1+e^x)-e^x}{1+e^x} = x-\ln(1+e^x)+C$

⑨ $\int \frac{e^x}{1+e^{2x}}dx = \int \frac{de^x}{1+(e^x)^2} = \arctan e^x + C$

⑩ $\int \frac{x}{\sqrt{1+x^2}}e^{-\sqrt{1+x^2}}dx = -\int e^{-\sqrt{1+x^2}}d\left(-\sqrt{1+x^2}\right) = -e^{-\sqrt{1+x^2}}+C$

例 4、求不定积分

① $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx = \frac{1}{2a}\left(\int\frac{d(x-a)}{x-a}-\int\frac{d(x+a)}{x+a}\right)$

$$= \frac{1}{2a}\ln\frac{x-a}{x+a}+C \qquad (21)(22)$$

② $\int \frac{x^2-x-2}{1+x^2}dx = \int \frac{x^2+1-x-3}{1+x^2}dx = \int\left(1-\frac{x+3}{1+x^2}\right)dx$

$$= x-\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}-3\int\frac{dx}{1+x^2} = x-\frac{1}{2}\ln(1+x^2)-3\arctan x+C$$

③ $\int \frac{x-4}{x^2-2x+5}dx = \frac{1}{2}\int\frac{2x-2-6}{x^2-2x+5}dx = \frac{1}{2}\int\frac{d(x^2-2x+5)}{x^2-2x+5}-3\int\frac{dx}{(x-1)^2+4}$

$$=\frac{1}{2}\ln\left(x^2-2x+5\right)-\frac{3}{2}\arctan\frac{x-1}{2}+C$$

④ $\int\sin^2 xdx=\int\frac{1-\cos 2x}{2}dx=\frac{1}{2}x-\frac{1}{2}\cdot\frac{1}{2}\int\cos 2xd(2x)=\frac{1}{2}x-\frac{1}{4}\sin 2x+C$

⑤ $\int\sin 5x\cos 3xdx=\frac{1}{2}\int(\sin 8x+\sin 2x)dx=-\frac{1}{16}\cos 8x-\frac{1}{4}\cos 2x+C$

⑥ $\int\frac{\cot x}{\ln\sin x}dx=\int\frac{\cos xdx}{\sin\ln\sin x}=\int\frac{d\sin x}{\sin x\ln\sin x}=\int\frac{d\ln\sin x}{\ln\sin x}=\ln\ln\sin x+C$

⑦ $\int\frac{dx}{1+\sin x}=\int\frac{1-\sin x}{\cos^2 x}dx=\int\sec^2 xdx+\int\frac{d\cos x}{\cos^2 x}=\tan x-\frac{1}{\cos x}+C$

⑧ $\int\frac{dx}{\cos x+\sin x}=\int\frac{dx}{\sqrt{2}\sin(x+\pi/4)}=\frac{1}{\sqrt{2}}\int\csc\left(x+\frac{\pi}{4}\right)d\left(x+\frac{\pi}{4}\right)$

$$=\frac{1}{\sqrt{2}}\ln\left(\csc\left(x+\frac{\pi}{4}\right)-\cot\left(x+\frac{\pi}{4}\right)\right)+C$$

二、第二类换元法

1、三角代换

例 1、$\int\sqrt{a^2-x^2}dx$

解：令 $x=a\sin t$ (或$a\cos t$)，则

$\sqrt{a^2-x^2}=a\cos t,\ dx=a\cos tdt$

原式=$\int a\cos t\cdot a\cos tdt=a^2\int\frac{1+\cos 2t}{2}dt=\frac{a^2}{2}\left(\int dt+\frac{1}{2}\int\cos 2td(2t)\right)$

$$=\frac{a^2}{2}t+\frac{a^2}{4}\sin 2t+C=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{a^2}{4}\cdot 2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}+C$$

$$=\frac{1}{2}a^2\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$$

例 2、$\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{d(x/a)}{\sqrt{1-(x/a)^2}}=\arcsin\frac{x}{a}+C$

解：令 $x=a\sin t$

原式=$\int\frac{a\cos tdt}{a\cos t}=\int dt=t+C=\arcsin\frac{x}{a}+C$

例 3、 $\int \frac{dx}{\sqrt{a^2+x^2}}$

解： 令 $x = a\tan t$ (或$a\cot t$)，则 $\sqrt{a^2+x^2} = a\sec t$，$dx = a\sec^2 tdt$

$$原式=\int \frac{a\sec^2 tdt}{a\sec t} = \int \sec tdt = \ln(\sec t + \tan t) + C = \ln\left(\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right) + C$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C \tag{24}$$

例 4、 $\int \frac{dx}{x\sqrt{x^2+4}}$

解： 令 $x = a\tan t$ (或$a\cot t$)，则 $\sqrt{x^2+4} = 2\sec t$，$dx = 2\sec^2 tdt$

$$原式=\int \frac{a\sec^2 tdt}{a\sec t} = \int \sec tdt = \ln(\sec t + \tan t) + C = \ln\left(\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right) + C$$

例 5、 $\int \frac{dx}{\sqrt{x^2-a^2}}$

解： 令 $x = a\sec t$ (或$a\csc t$)，则

$\sqrt{x^2-a^2} = a\tan t$，$dx = a\sec t\tan tdt$

$$原式=\int \frac{a\sec t\tan tdt}{a\tan t} = \int \sec tdt = \ln(\sec t + \tan t) + C = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + c$$

$$= \ln\left(x + \sqrt{x^2-a^2}\right) + C \tag{25}$$

例 6、 $\int \frac{\sqrt{x^2-9}}{x}dx$

解： 令 $x = a\sec t$，则 $\sqrt{x^2-9} = 3\tan t$，$dx = 3\sec t\tan tdt$

$$原式=\int \frac{3\tan t}{3\sec t}\cdot 3\sec t\tan tdt = 3\int \tan^2 tdt = 3\int\left(\sec^2 t - 1\right) = 3(\tan t - t) + C$$

$$= 3\left(\frac{\sqrt{x^2-9}}{3} - \arccos\frac{3}{x}\right) + C = \sqrt{x^2-9} - 3\arccos\frac{3}{x} + C$$

小结： $f(x)$ 中含有 $\begin{cases}\sqrt{a^2-x^2}\\ \sqrt{x^2+a^2}\\ \sqrt{x^2-a^2}\end{cases}$ 可考虑用代换 $\begin{cases}x=a\sin t\\ x=a\tan t\\ x=a\sec t\end{cases}$

2、无理代换

例 7、 $\int\frac{dx}{1+\sqrt[3]{x+1}}$

解： 令 $\sqrt[3]{x+1}=t$，则 $x=t^3-1$，$dx=3t^2dt$

$$\text{原式}=\int\frac{3t^2dt}{1+t}=3\int\frac{t^2-1+1}{1+t}dt=3\int\left(t-1+\frac{1}{1+t}\right)dt=3\left(\frac{t^2}{2}-t+\ln(1+t)\right)+C$$

$$=\frac{3}{2}\sqrt[3]{(x+1)^2}-3\sqrt[3]{x+1}+3\ln\left(1+\sqrt[3]{x+1}\right)+C$$

例 8、 $\int\frac{dx}{\sqrt{x}\left(1+\sqrt[3]{x}\right)}$

解： 令 $\sqrt[6]{x}=t$，则 $x=t^6$，$dx=6t^5dt$

$$\text{原式}=\int\frac{6t^5dt}{t^3\left(1+t^2\right)}=6\int\frac{t^2}{1+t^2}dt=6\int\left(1-\frac{1}{1+t^2}\right)dt=6(t-\arctan t)+C$$

$$=6\left(\sqrt[6]{x}-\arctan\sqrt[6]{x}\right)+C$$

例 9、 $\int\frac{1}{x}\sqrt{\frac{1+x}{x}}dx$

解： 令 $\sqrt{\frac{1+x}{x}}=t$，则 $x=\frac{1}{t^2-1}$，$dx=-\frac{2tdt}{\left(t^2-1\right)^2}$

原式

$$=\int\left(t^2-1\right)t\left(-\frac{2tdt}{\left(t^2-1\right)^2}\right)=-2\int\frac{t^2}{t^2-1}dt=-2\int\left(1+\frac{1}{t^2-1}\right)dt=-2\left(t+\frac{1}{2}\ln\frac{t-1}{t+1}\right)+C$$

$$=-2\sqrt{\frac{1+x}{x}}-\ln\frac{\sqrt{1+x}-\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}+C$$

例 10、 $\int\frac{dx}{\sqrt{1+e^x}}$

解： 令 $\sqrt{1+e^x}=t$，则$x=\ln\left(t^2-1\right)$，$dx=\dfrac{2tdt}{t^2-1}$

$$原式=\int\frac{1}{t}\cdot\frac{2t}{t^2-1}dt=2\int\frac{dt}{t^2-1}=2\cdot\frac{1}{2}\ln\frac{t-1}{t+1}+C=\ln\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}+C$$

4、 倒代换

例 11、 $\displaystyle\int\frac{dx}{x\left(x^6+4\right)}$

解： 令 $x=\dfrac{1}{t}$，则$\dfrac{1}{x\left(x^6+1\right)}=\dfrac{t^7}{1+4t^6}$，$dx=-\dfrac{dt}{t^2}$

$$原式=-\int\frac{t^6dt}{1+4t^6}=-\frac{1}{24}\int\frac{d\left(4t^6+1\right)}{4t^6+1}=-\frac{1}{24}\ln\left(4t^6+1\right)+C=\frac{1}{24}\ln\frac{x^6}{x^6+4}+C$$

$$=\frac{1}{4}\ln x-\frac{1}{24}\ln\left(x^6+4\right)+C$$

§3、分部积分法

分部积分公式：$(UV)'=U'V+UV'$，$UV'=(UV)'-U'V$

$$\int UV'dx=\int(UV)'dx-\int U'Vdx，故\int UdV=\ UV\ -\int VdU$$

（前后相乘）（前后交换）

例 1、 $\displaystyle\int x\cos xdx$

$$=\int xd\sin x=x\sin x-\int\sin xdx=x\sin x+\cos x+C$$

例 2、 $\displaystyle\int xe^xdx$

$$=\int xde^x=xe^x-\int e^xdx=xe^x-e^x+C$$

例 3、 $\displaystyle\int\ln xdx=x\ln x-\int xd\ln x=x\ln x-\int x\cdot\frac{1}{x}dx=x\ln x-x+C$

或解： 令 $\ln x=t,\ x=e^t$

$$原式=\int tde^t=te^t-\int e^tdt=te^t-e^t+C=x\ln x-x+C$$

例 4、 $\displaystyle\int\arcsin xdx$

$$= x\arcsin x - \int xd\arcsin x = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= x\arcsin x + \frac{1}{2}\int \frac{d\left(1-x^2\right)}{\sqrt{1-x^2}} = x\arcsin x + \sqrt{1-x^2} + C$$

或解： 令 $\arcsin x = t,\ x = \sin t$

原式$= \int td\sin t = t\sin t - \int \sin tdt = t\sin t + \cos t + C = x\arcsin x + \sqrt{1-x^2} + C$

例 5、 $\int e^x \sin xdx$

$$= \int \sin xde^x = e^x \sin x - \int e^x \cos xdx = e^x \sin x - \int \cos xde^x$$

$$= e^x \sin x - e^x \cos x + \int e^x d\cos x = e^x(\sin x - \cos x) - \int e^x \sin xdx$$

故 $\int e^x \sin xdx = \frac{1}{2}e^x(\sin x - \cos x) + C$

例 6、 $\int \frac{x}{\cos^2 x}dx$

$$= \int xd\tan x = x\tan x - \int \tan xdx = x\tan x - \ln\sec x + C$$

例 7、 $\int \ln\left(x+\sqrt{1+x^2}\right)dx$

$$= x\ln\left(x+\sqrt{1+x^2}\right) - \int x\cdot\frac{1+x/\sqrt{1+x^2}}{x+\sqrt{1+x^2}}dx = x\ln\left(x+\sqrt{1+x^2}\right) - \int\frac{x}{\sqrt{1+x^2}}dx$$

$$= x\ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2} + C$$

§4、两种典型积分

一、有理函数的积分

有理函数 $R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$ 可用待定系数法化为部分分式，然后积分。

例 1、 将 $\frac{x+3}{x^2-5x+6}$ 化为部分分式，并计算 $\int \frac{x+3}{x^2-5x+6}dx$

解： $\frac{x+3}{x^2-5x+6} = \frac{x+3}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{(A+B)x-(3A+2B)}{(x-2)(x-3)}$

$$\begin{cases} A+B=1 \\ 3A+2B=-3 \end{cases} \qquad \begin{cases} A=-5 \\ B=6 \end{cases}$$

故 $\int \frac{x+3}{x^2-5x+6}dx = -5\int \frac{dx}{x-2} + 6\int \frac{dx}{x-3} = -5\ln(x-2) + 6\ln(x-3) + C$

或解： $I = \frac{1}{2}\int \frac{2x-5+11}{x^2-5x+6}dx = \frac{1}{2}\int \frac{d(x^2-5x+6)}{x^2-5x+6} + \frac{11}{2}\int \frac{dx}{x^2-5x+6}$

$$= \frac{1}{2}\ln(x^2-5x+6) + \frac{11}{2}\int\left(\frac{1}{x-3} - \frac{1}{x-2}\right)dx$$

$$= \frac{1}{2}\ln(x^2-5x+6) + \frac{11}{2}\ln\frac{x-3}{x-2} + C$$

例 2、 $\int \frac{dx}{x(x-1)^2} = -\int \frac{x-1+x}{x(x-1)^2}dx = -\int\left(\frac{1}{x(x-1)} - \frac{1}{(x-1)^2}\right)dx$

$$= -\int\left(\frac{1}{x-1} - \frac{1}{x} - \frac{1}{(x-1)^2}\right)dx = \ln\frac{x}{x-1} - \frac{1}{x-1} + C$$

例 3、 $\int \frac{x^2+1}{x^4+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}} + C$

例 4、 $\int \frac{dx}{x^4+1} = \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx = \frac{1}{2}\left(\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx - \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx\right)$

$$= \frac{1}{2}\left(\int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} - \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right) + C$$

$$= \frac{1}{2\sqrt{2}}\left(\arctan\frac{x^2-1}{\sqrt{2}x} - \frac{1}{2}\ln\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right) + C$$

二、三角函数有理式的积分

对三角函数有理式积分 $I = \int R(\sin x, \cos x)dx$，令 $u = \tan\frac{x}{2}$，则 $x = 2\arctan u$，

$\sin x = \frac{2u}{1+u^2}$，$\cos x = \frac{1-u^2}{1+u^2}$，$dx = \frac{2}{1+u^2}du$，故 $I = \int R\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{2}{1+u^2}du$，

三角函数有理式积分即变成了有理函数积分。

例 5、 $\int \frac{dx}{3+5\cos x}$

解： 令 $u=\tan\frac{x}{2}$，则$x=2\arctan u$， $\cos x=\frac{1-u^2}{1+u^2}$， $dx=\frac{2}{1+u^2}du$

$$\text{原式}=\int \frac{1}{3+5\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}du=\int\frac{du}{4-u^2}=\frac{1}{2\cdot 2}\ln\frac{2+u}{2-u}+C=\frac{1}{4}\ln\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}+C$$

例 6、 $\int \frac{dx}{2\sin x-\cos x+5}$

解： 令 $u=\tan\frac{x}{2}$，则$x=2\arctan u$， $\sin x=\frac{2u}{1+u^2}$， $\cos x=\frac{1-u^2}{1+u^2}$， $dx=\frac{2}{1+u^2}du$

$$\text{原式}=\int \frac{1}{2\frac{2u}{1+u^2}-\frac{1-u^2}{1+u^2}+5}\frac{2}{1+u^2}du=\int\frac{du}{3u^2+2u+2}$$

$$=\frac{1}{3}\int\frac{d\left(u+\frac{1}{3}\right)}{\left(u+\frac{1}{3}\right)^2+\frac{5}{9}}=\frac{1}{3}\cdot\frac{3}{\sqrt{5}}\arctan\frac{u+\frac{1}{3}}{\frac{\sqrt{5}}{3}}+C=\frac{1}{\sqrt{5}}\arctan\frac{3\arctan\frac{x}{2}+1}{\sqrt{5}}+C$$

例 7、 $\int\frac{1+\sin x}{1-\cos x}dx=\int\frac{1+\frac{2u}{1+u^2}}{1-\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}du=\int\frac{1+u^2+2u}{u^2(u^2+1)}du$

$$=\int\left(\frac{1}{u^2}+\frac{2}{u(1+u^2)}\right)du=-\frac{1}{u}+2\int\left(\frac{1}{u}-\frac{u}{1+u^2}\right)du$$

$$=-\frac{1}{u}+2\ln u-\ln\left(1+u^2\right)+C=-\cot\frac{x}{2}+2\ln\sin\frac{x}{2}+C$$