Generative Models and Naïve Bayes

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Reading: [14.3, EA], [3.5, KPM], [1.5.4, CMB]

Outline

- Background and Probability Basics
- Probabilistic Classification Principle
  - Probabilistic discriminative models
  - Generative models and their application to classification
  - MAP and converting generative into discriminative
- Naïve Bayes – an generative model
  - Principle and Algorithms (discrete vs. continuous)
  - Example: Play Tennis
- Zero Conditional Probability and Treatment
- Summary

Background

- There are three methodologies:

  a) Model a classification rule directly

  Examples: k-NN, linear classifier, SVM, neural nets, ..

  b) Model the probability of class memberships given input data

  Examples: logistic regression, probabilistic neural nets (softmax),...

  c) Make a probabilistic model of data within each class

  Examples: naive Bayes, model-based ....

- Important ML taxonomy for learning models

  probabilistic models vs non-probabilistic models

  discriminative models vs generative models

Background

- Based on the taxonomy, we can see different the essence of learning models (classifiers) more clearly.

	Probabilistic	Non-Probabilistic
Discriminative	• Logistic Regression • Probabilistic neural nets • ……..	• K-nn • Linear classifier • SVM • Neural networks • ……
Generative	• Naïve Bayes • Model-based (e.g., GMM) • ……	N.A. (?)

Probability Basics

- Prior, conditional and joint probability for random variables
  - Prior probability: $P(x)$
  - Conditional probability: $P(x_1 | x_2), P(x_2 | x_1)$
  - Joint probability: $\mathbf{x} = (x_1, x_2), P(\mathbf{x}) = P(x_1, x_2)$
  - Relationship: $P(x_1, x_2) = P(x_2 | x_1)P(x_1) = P(x_1 | x_2)P(x_2)$
  - Independence:

$$P(x_2 | x_1) = P(x_2), P(x_1 | x_2) = P(x_1), P(x_1, x_2) = P(x_1)P(x_2)$$

- Bayesian Rule

$$P(c | \mathbf{x}) = \frac{P(\mathbf{x} | c)P(c)}{P(\mathbf{x})} \qquad Posterior = \frac{Likelihood \times Prior}{Evidence}$$

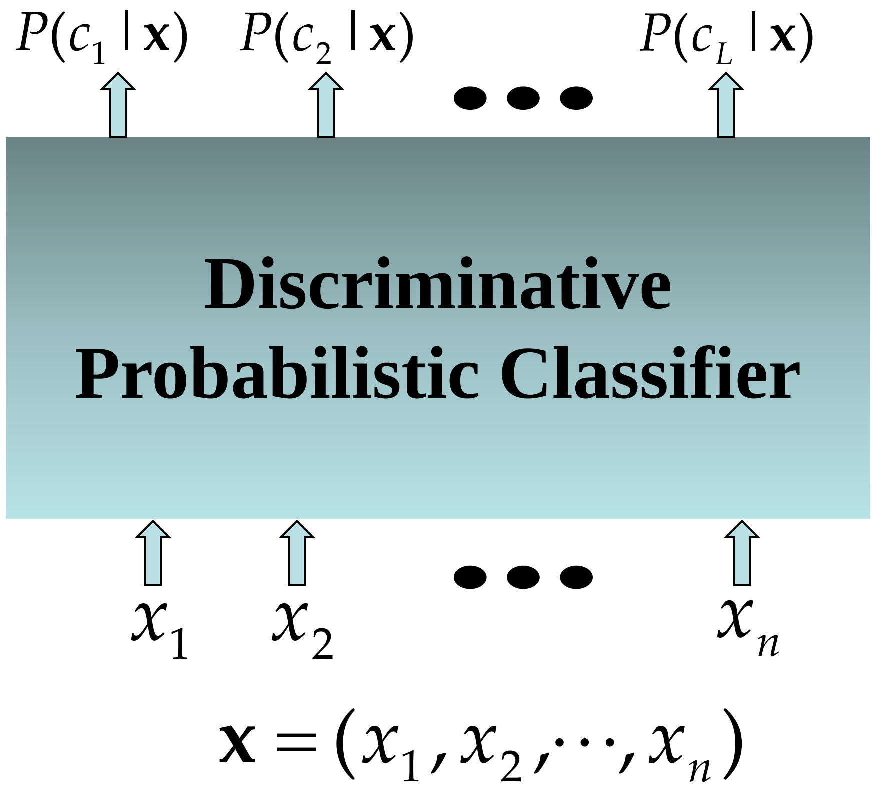
Discriminative

Generative

Probabilistic Classification Principle

- Establishing a probabilistic model for classification
  - **Discriminative model**

$$P(c \mid \mathbf{x}) \quad c = c_1, \cdots, c_L, \mathbf{x} = (x_1, \cdots, x_n)$$

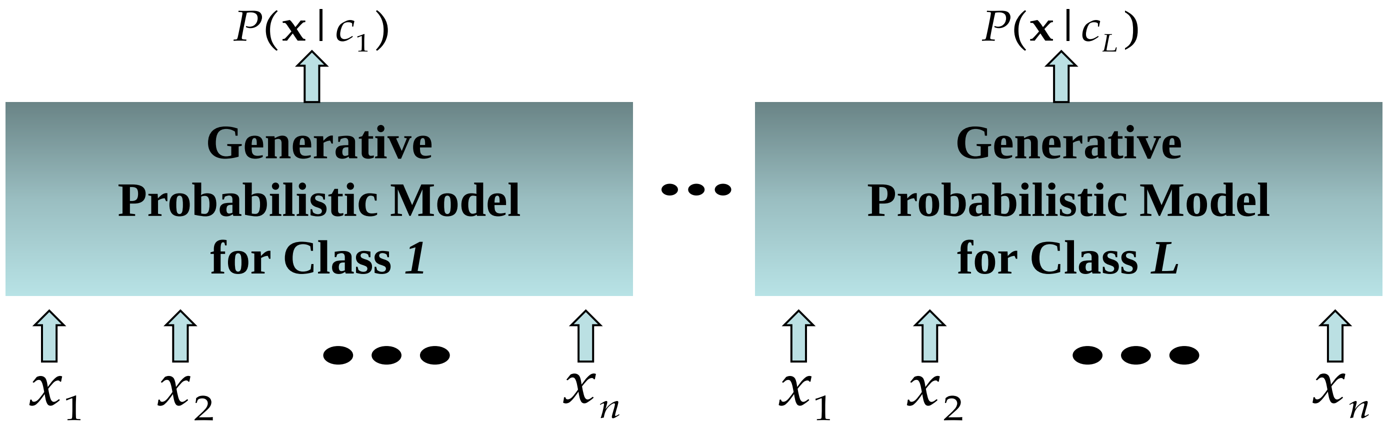


- To train a discriminative classifier regardless its probabilistic or non-probabilistic nature, all training examples of different classes must be jointly used to build up a single discriminative classifier.
- Output L probabilities for L class labels in a probabilistic classifier while a single label is achieved by a non-probabilistic classifier .

Probabilistic Classification Principle

- Establishing a probabilistic model for classification (cont.)
  - **Generative model (must be probabilistic)**

$$P(\mathbf{x} \mid c) \quad c = c_1, \cdots, c_L, \mathbf{x} = (x_1, \cdots, x_n)$$



$$\mathbf{x} = (x_1, x_2, \cdots, x_n)$$

- L probabilistic models have to be trained independently
- Each is trained on only the examples of the same label
- Output L probabilities for a given input with L models
- "Generative" means that such a model produces data subject to the distribution via sampling.

Probabilistic Classification Principle

- **Maximum A Posterior (MAP)** classification rule
  - For an input x, find the largest one from L probabilities output by a discriminative probabilistic classifier $P(c_1 | \mathbf{x}), \ldots, P(c_L | \mathbf{x})$.
  - Assign x to label c^* if $P(c^* | \mathbf{x})$ is the largest.
- Generative classification with the MAP rule
  - Apply Bayesian rule to convert them into posterior probabilities

$$P(c_i | \mathbf{x}) = \frac{P(\mathbf{x} | c_i)P(c_i)}{P(\mathbf{x})} \propto P(\mathbf{x} | c_i)P(c_i)$$

$$\text{for } i = 1, 2, \cdots, L$$

Common factor for all L probabilities

  - Then apply the MAP rule to assign a label

Naïve Bayes

- Bayes classification

$$P(c\,|\,\mathbf{x}) \propto P(\mathbf{x}\,|\,c)P(c) = P(x_1,\cdots,x_n\,|\,c)P(c) \quad \text{for } c = c_1,\ldots,c_L.$$

Difficulty: learning the joint probability $P(x_1,\cdots,x_n\,|\,c)$ is infeasible!

- Naïve Bayes classification
  - Assume all input features are class conditionally independent!

$$\begin{aligned} P(x_1,x_2,\cdots,x_n\,|\,c) &= \underline{P(x_1\,|\,x_2,\cdots,x_n,c)}P(x_2,\cdots,x_n\,|\,c) \\ &= \underline{P(x_1\,|\,c)}P(x_2,\cdots,x_n\,|\,c) \\ &= P(x_1\,|\,c)P(x_2\,|\,c)\cdots P(x_n\,|\,c) \end{aligned}$$

Applying the independence assumption

  - Apply the MAP classification rule: assign $\mathbf{x}' = (a_1, a_2, \cdots, a_n)$ to c^* if

$$\underbrace{[P(a_1\,|\,c^*)\cdots P(a_n\,|\,c^*)]}_{\text{estimate of } P(a_1,\cdots,a_n\,|\,c^*)}P(c^*) > \underbrace{[P(a_1\,|\,c)\cdots P(a_n\,|\,c)]}_{\text{esitmate of } P(a_1,\cdots,a_n\,|\,c)}P(c), \quad c \neq c^*, c = c_1,\cdots,c_L$$

Naïve Bayes

- Algorithm: Discrete-Valued Features
  - Learning Phase: Given a training set S of F features and L classes,

    For each target value of c_i $(c_i = c_1, \cdots, c_L)$

    $\hat{P}(c_i) \leftarrow$ estimate $P(c_i)$ with examples in S;

    For every feature value x_{jk} of each feature x_j $(j = 1, \cdots, F; k = 1, \cdots, N_j)$

    $\hat{P}(x_j = x_{jk} | c_i) \leftarrow$ estimate $P(x_{jk} | c_i)$ with examples in S;

    Output: $F * L$ conditional probabilistic (generative) models
  - Test Phase: Given an unknown instance $\mathbf{x}' = (a'_1, \cdots, a'_n)$

    "Look up tables" to assign the label c^* to $\mathbf{X}'$ if

    $$[\hat{P}(a'_1 | c^*) \cdots \hat{P}(a'_n | c^*)]\hat{P}(c^*) > [\hat{P}(a'_1 | c_i) \cdots \hat{P}(a'_n | c_i)]\hat{P}(c_i), \quad c_i \neq c^*, c_i = c_1, \cdots, c_L$$

Example

- Example: Play Tennis

PlayTennis: training examples

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Example

- Learning Phase

Outlook	Play=*Yes*	Play=*No*
Sunny	2/9	3/5
Overcast	4/9	0/5
Rain	3/9	2/5

Temperature	Play=*Yes*	Play=*No*
Hot	2/9	2/5
Mild	4/9	2/5
Cool	3/9	1/5

Humidity	Play=*Yes*	Play=No
High	3/9	4/5
Normal	6/9	1/5

Wind	Play=*Yes*	Play=*No*
Strong	3/9	3/5
Weak	6/9	2/5

$P(\text{Play}=Yes) = 9/14$ $\quad$ $P(\text{Play}=No) = 5/14$

Example

- Test Phase
  - Given a new instance, predict its label

    $\mathbf{x}'$=(Outlook=*Sunny*, Temperature=*Cool*, Humidity=*High*, Wind=*Strong*)
  - Look up tables achieved in the learning phrase

    P(Outlook=*Sunny*|Play=*Yes*) = 2/9
    P(Temperature=*Cool*|Play=*Yes*) = 3/9
    P(Huminity=*High*|Play=*Yes*) = 3/9
    P(Wind=*Strong*|Play=*Yes*) = 3/9
    P(Play=*Yes*) = 9/14

    P(Outlook=*Sunny*|Play=*No*) = 3/5
    P(Temperature=*Cool*|Play==*No*) = 1/5
    P(Huminity=*High*|Play=*No*) = 4/5
    P(Wind=*Strong*|Play=*No*) = 3/5
    P(Play=*No*) = 5/14
  - Decision making with the MAP rule

    $P(Yes|\mathbf{x}') \approx [P(Sunny|Yes)P(Cool|Yes)P(High|Yes)P(Strong|Yes)]P(\text{Play}=Yes) = 0.0053$

    $P(No|\mathbf{x}') \approx [P(Sunny|\text{No})\, P(Cool|\text{No})P(High|No)P(Strong|No)]P(\text{Play}=No) = 0.0206$

Given the fact $P(Yes|\mathbf{x}') < P(No|\mathbf{x}')$, we label $\mathbf{x}'$ to be "*No*".

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Naïve Bayes

- Algorithm: Continuous-valued Features
  - Numberless values taken by a continuous-valued feature
  - Conditional probability often modeled with the normal distribution

$$\hat{P}(x_j \mid c_i) = \frac{1}{\sqrt{2\pi}\sigma_{ji}} \exp\left(-\frac{(x_j - \mu_{ji})^2}{2\sigma_{ji}^2} \right)$$

μ_{ji} : mean (avearage) of feature values x_j of examples for which $\mathrm{c} = c_i$

σ_{ji} : standard deviation of feature values x_j of examples for which $c = c_i$

  - Learning Phase: for $\mathbf{X} = (X_1, \cdots, X_F)$, $C = c_1, \cdots, c_L$

    Output: $F \times L$ normal distributions and $P(C = c_i)$ $i = 1, \cdots, L$
  - Test Phase: Given an unknown instance $\mathbf{X}' = (a'_1, \cdots, a'_n)$
    - Instead of looking-up tables, calculate conditional probabilities with all the normal distributions achieved in the learning phrase
    - Apply the MAP rule to assign a label (the same as done for the discrete case)

Naïve Bayes

- Example: Continuous-valued Features
  - Temperature is naturally of continuous value.

    Yes: 25.2, 19.3, 18.5, 21.7, 20.1, 24.3, 22.8, 23.1, 19.8

    No: 27.3, 30.1, 17.4, 29.5, 15.1

  - Estimate mean and variance for each class

$$\mu = \frac{1}{N}\sum_{n=1}^{N} x_n, \quad \sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - \mu)^2$$

$$\mu_{Yes} = 21.64, \ \sigma_{Yes} = 2.35$$
$$\mu_{No} = 23.88, \ \sigma_{No} = 7.09$$

  - **Learning Phase**: output two Gaussian models for P(temp|C)

$$\hat{P}(x \mid Yes) = \frac{1}{2.35\sqrt{2\pi}} \exp\left(-\frac{(x-21.64)^2}{2\times 2.35^2}\right) = \frac{1}{2.35\sqrt{2\pi}} \exp\left(-\frac{(x-21.64)^2}{11.09}\right)$$

$$\hat{P}(x \mid No) = \frac{1}{7.09\sqrt{2\pi}} \exp\left(-\frac{(x-23.88)^2}{2\times 7.09^2}\right) = \frac{1}{7.09\sqrt{2\pi}} \exp\left(-\frac{(x-23.88)^2}{50.25}\right)$$

Zero conditional probability

- If no example contains the feature value
  - In this circumstance, we face a zero conditional probability problem during test

$$\hat{P}(x_1|c_i)\cdots\hat{P}(a_{jk}|c_i)\cdots\hat{P}(x_n|c_i)=0 \quad \text{for } x_j=a_{jk},\ \hat{P}(a_{jk}|c_i)=0$$

  - For a remedy, class conditional probabilities re-estimated with

$$\hat{P}(a_{jk}|c_i)=\frac{n_c+mp}{n+m}$$ **(m-estimate)**

n_c : number of training examples for which $x_j=a_{jk}$ and $c=c_i$

n : number of training examples for which $c=c_i$

p : prior estimate (usually, $p=1/t$ for t possible values of x_j)

m : weight to prior (number of "virtual" examples, $m\geq 1$)

Zero conditional probability

- Example: P(outlook=overcast|no)=0 in the play-tennis dataset
  - Adding m "virtual" examples (m: up to 1% of #training example)
    - In this dataset, # of training examples for the "no" class is 5.
    - We can only add $m=1$ "virtual" example in our m-esitmate remedy.
  - The "outlook" feature can takes only 3 values. So $p=1/3$.
  - Re-estimate P(outlook|no) with the m-estimate

$$\mathrm{P(overcast|no)} = \frac{0+1*\left(\frac{1}{3}\right)}{5+1} = \frac{1}{18}$$

$$\mathrm{P(sunny|no)} = \frac{3+1*\left(\frac{1}{3}\right)}{5+1} = \frac{5}{9} \qquad \mathrm{P(rain|no)} = \frac{2+1*\left(\frac{1}{3}\right)}{5+1} = \frac{7}{18}$$

Summary

- Probabilistic Classification Principle
  – Discriminative vs. Generative models: learning P(c|x) vs. P(x|c)
  – Generative models for classification: MAP and Bayesian rule
- Naïve Bayes: the conditional independence assumption
  – Training and test are very efficient.
  – Two different data types lead to two different learning algorithms.
  – Working well sometimes for data violating the assumption!
- Naïve Bayes: a popular generative model for classification
  – Performance competitive to most of state-of-the-art classifiers even in presence of violating independence assumption
  – Many successful applications, e.g., spam mail filtering
  – A good candidate of a base learner in ensemble learning
  – Apart from classification, naïve Bayes can do more…