Implicit formulation of Cohesive Cam Clay model

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January 15, 2026

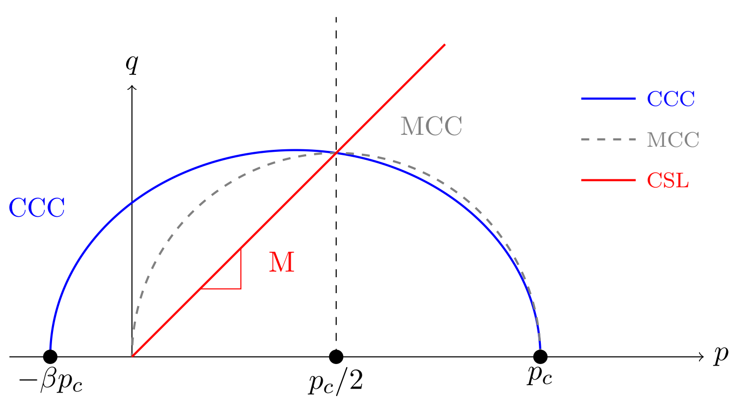
1 Cohesive Cam Clay equations

We define:

$$p = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}), \qquad q = \sqrt{\frac{3}{2}}\|\boldsymbol{\xi}\| \tag{1.1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{\xi} = \boldsymbol{\sigma} - \frac{1}{3}tr(\boldsymbol{\sigma})\mathbf{I}$ with $\mathbf{I}_{ij} = \delta_{ij}$ the Kronecker delta. The ellipsoid of the Cohesive Cam Clay (see[2]) model is of the form:

$$F(p, q) = q^2(1 + 2\beta) + M^2(p + \beta p_c)(p - p_c) = 0 \tag{1.2}$$



where $\mathbf{M}$ is the slope of the critical state line and p_c is the preconsolidation pressure. The hardening function in rate form is given by

$$\dot{p}_c = \vartheta p_c \dot{\varepsilon}_v^P, \quad \dot{\varepsilon}_v^P = \mathrm{tr}(\dot{\boldsymbol{\varepsilon}}^P), \quad \vartheta = \frac{1+e}{\lambda - \kappa}, \tag{1.3}$$

where $\dot{\boldsymbol{\varepsilon}}^P$ is the plastic strain rate tensor, e is the void ratio of the soil mass, λ is the virgin compression index and κ is the swell/recompression index. Both λ and κ are assumed constant. The soil's void ratio e is a state variable and, therefore, so is ϑ. The use of the associative flow rule enables one to express $\dot{\boldsymbol{\varepsilon}}^P$ as

$$\dot{\boldsymbol{\varepsilon}}^P = \dot{\phi}\frac{\partial F}{\partial \boldsymbol{\sigma}}, \tag{1.4}$$

with $\partial F/\partial \boldsymbol{\sigma}$ obtained from a direct application of the chain rule as follows

$$\frac{\partial F}{\partial \boldsymbol{\sigma}} = \frac{\partial F}{\partial p}\frac{\partial p}{\partial \boldsymbol{\sigma}} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial \boldsymbol{\sigma}} = \frac{1}{3}\left(\frac{\partial F}{\partial p}\right)\mathbf{I} + \sqrt{\frac{3}{2}}\left(\frac{\partial F}{\partial q}\right)\hat{\mathbf{n}}, \tag{1.5}$$

where $\hat{\mathbf{n}} = \boldsymbol{\xi}/\|\boldsymbol{\xi}\|$ and $\dot{\phi}$ is a factor. The derivatives of F with respect to p, q and p_c can be obtained from (1.2) as follows:

$$\frac{\partial F}{\partial p} = M^2[2p + p_c(\beta - 1)], \quad \frac{\partial F}{\partial q} = 2q(1+2\beta), \quad \frac{\partial F}{\partial p_c} = p(\beta - M^2) - 2\beta p_c \tag{1.6}$$

Equations (1.3) and (1.4) may be integrated over a finite time increment either analytically or by employing the generalized trapezoidal method with respect to the variable p_c. Integrating (1.3) and (1.4) over a finite time increment yields the following alternative incremental hardening laws:

$$(p_c)_{n+1} = (p_c)_n \exp\left(\vartheta \Delta\varepsilon_v^P\right), \quad \Delta\varepsilon_v^P = \Delta\Phi tr(\frac{\partial F}{\partial \boldsymbol{\sigma}}) \tag{1.7}$$

The conventional solution of incremental plasticity is based on the following rate constitutive equations (see [1]):

$$\dot{\boldsymbol{\sigma}} = c^{ep} : \dot{\varepsilon}, \quad c^{ep} = c^e - \langle c^p \rangle \tag{1.8}$$

where c^e is the elastic stress-strain tensor

$$c^e = K\mathbf{I}\otimes\mathbf{I} + 2G(\mathbb{I} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}) \tag{1.9}$$

and c^p is obtained as

$$c^p = \chi^{-1}(c^e : \frac{\partial F}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial F}{\partial \boldsymbol{\sigma}} : c^e), \quad \chi = \frac{\partial F}{\partial \boldsymbol{\sigma}} : c^e : \frac{\partial F}{\partial \boldsymbol{\sigma}} - \theta p_c \frac{\partial F}{\partial p}\frac{\partial F}{\partial p_c} \tag{1.10}$$

In eq. 1.9 $\mathbb{I}$ is the identity tensor defined such that $\mathbb{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. K is the bulk modulus and G is the shear modulus.

2 Associative flow rule: closest point projection

We define some preliminaries quantities that can be found in detail in [1].

$$\frac{\partial p_{n+1}^{tr}}{\partial \varepsilon_{n+1}^{tr}} = K\mathbf{I}, \quad \frac{\partial q_{n+1}^{tr}}{\partial \varepsilon_{n+1}^{tr}} = 2G\sqrt{\frac{2}{3}}\hat{\mathbf{n}}, \quad \frac{\partial \hat{\mathbf{n}}}{\partial \varepsilon_{n+1}^{tr}} = \frac{2G}{\|\boldsymbol{\xi}_{n+1}^{tr}\|}(\mathbb{I} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I} - \hat{\mathbf{n}}\otimes\hat{\mathbf{n}}) \tag{2.1}$$

Applying the normality rule over a finite time increment on $\Delta\varepsilon^p$ results in the following scalar equations:

$$p(\Delta\phi) = p_{n+1}^{tr} - K\Delta\phi\frac{\partial F}{\partial p} = p_{n+1}^{tr} - K\Delta\phi M^2[2p + p_c(\beta - 1)] \tag{2.2}$$

$$q(\Delta\phi) = q_{n+1}^{tr} - K\Delta\phi\frac{\partial F}{\partial q} = q_{n+1}^{tr} - K\Delta\phi 2q(1+2\beta) \tag{2.3}$$

$$p_c(\Delta\phi) = (p_c)_n \exp(\theta\Delta\phi\frac{\partial F}{\partial p}) = (p_c)_n \exp\{\theta\Delta\phi M^2[2p + p_c(\beta - 1)]\} \tag{2.4}$$

3 Determination of the scalar consistency parameter

The parameter $\Delta\phi$ in eq. 2.2-2.4 is computed by imposing the consistency requirement

$$F(\Delta\phi) = q^2(1+2\beta) + M^2(p+\beta p_c)(p-p_c) = 0 \tag{3.1}$$

This can be solved with Newton-Rapshon algorithm

$$F'(\Delta\phi) = \frac{\partial F}{\partial p}\frac{\partial p}{\partial \Delta\phi} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial \Delta\phi} + \frac{\partial F}{\partial p_c}\frac{\partial p_c}{\partial \Delta\phi} \tag{3.2}$$

$$\frac{\partial p}{\partial \Delta\phi} = \frac{a_2/a_1 - [a_4/(a_5 a_1)]}{1 + a_4/(a_5 a_1)} \tag{3.3}$$

$$\frac{\partial q}{\partial \Delta\phi} = \frac{a_6}{1 + 6G\Delta\phi(1+2\beta)} \tag{3.4}$$

$$\frac{\partial p_c}{\partial \Delta\phi} = \frac{a_7 + a_8}{a_9} \tag{3.5}$$

$$a_1 = 1 + 2K\Delta\phi M^2 \tag{3.6}$$

$$a_2 = -KM^2(2p + \beta p_c - p_c) \tag{3.7}$$

$$a_3 = \theta\Delta\phi M^2(2p + \beta p_c - p_c) \tag{3.8}$$

$$a_4 = K\Delta\phi M^2 (p_c)_n exp(a_3)\theta M^2\Delta\phi \tag{3.9}$$

$$a_5 = 1 - (p_c)_n exp(a_3) M^2\theta\Delta\phi(\beta - 1) \tag{3.10}$$

$$a_6 = -6Gq(1+2\beta) \tag{3.11}$$

$$a_7 = (p_c)_n exp(a_3)\theta M^2(2p + \beta p_c - p_c) \tag{3.12}$$

$$a_8 = (p_c)_n exp(a_3)\theta M^2\Delta\phi 2\frac{\partial p}{\partial \Delta\phi} \tag{3.13}$$

$$a_9 = 1 - (p_c)_n exp(a_3) M^2\theta\Delta\phi(\beta - 1) \tag{3.14}$$

$$\Delta\phi_{n+1} = \Delta\phi_n - \frac{F}{F'} \tag{3.15}$$

Since p and p_c are coupled through 2.2 and 2.4 in non-linear equation, they will have to be solved iteratively as:

$$R(p_c) = (p_c)_n exp(g_3) - p_c \tag{3.16}$$

$$g_1 = p^{tr}_{n+1} + K\Delta\phi M^2 p_c(1-\beta) \tag{3.17}$$

$$g_2 = 1 + K\Delta\phi 2M^2 \tag{3.18}$$

$$g_3 = \theta\Delta\phi(2M^2\frac{g_1}{g_2} + \beta p_c M^2 - M^2 p_c) \tag{3.19}$$

$$R'(p_c) = R(p_c)(g_5 + \beta M^2 - M^2) - 1 \tag{3.20}$$

$$g_4 = \theta\Delta\phi^2 2KM^4(1-\beta) \tag{3.21}$$

$$g_5 = \frac{g_4}{g_2} \tag{3.22}$$

$$p_{c(n+1)} = p_{c(n)} - \frac{R(p_c)}{R'(p_c)} \tag{3.23}$$

4 Determination of the consistent tangential moduli

The incremental response function corresponding to the strain tensor increment can be written in the following form:

$$\boldsymbol{\sigma}^{\boldsymbol{k}}_{\boldsymbol{n+1}} = \frac{1}{3} tr(\boldsymbol{\sigma}^{\boldsymbol{k}}_{\boldsymbol{n+1}})\mathbf{I} + \|\boldsymbol{\xi}^{k}_{n+1}\| \hat{\boldsymbol{n}} = p\mathbf{I} + \sqrt{\frac{2}{3}} q\hat{\boldsymbol{n}} \tag{4.1}$$

The consistent tangential moduli can be obtained by directly evaluating the variation

$$c^{k}_{n+1} = \frac{\partial \boldsymbol{\sigma}^{\boldsymbol{k}}_{\boldsymbol{n+1}}}{\partial \varepsilon^{k}_{n+1}} = \mathbf{I} \otimes \frac{\partial p}{\partial \varepsilon^{k}_{n+1}} + \sqrt{\frac{2}{3}} q \frac{\partial \hat{\boldsymbol{n}}}{\partial \varepsilon^{k}_{n+1}} + \sqrt{\frac{2}{3}} \hat{\boldsymbol{n}} \otimes \frac{\partial q}{\partial \varepsilon^{k}_{n+1}} \tag{4.2}$$

Evaluating the derivatives implicitly

$$\frac{\partial p}{\partial \varepsilon^{k}_{n+1}} = \frac{K}{b_0 b_7}\mathbf{I} - \frac{b_1 b_6 b_2 b_4}{b_0 b_4 b_7} \frac{\partial \Delta\phi}{\partial \varepsilon^{k}_{n+1}} \tag{4.3}$$

$$\frac{\partial q}{\partial \varepsilon^{k}_{n+1}} = \frac{b_8}{b_{10}} \sqrt{\frac{3}{2}} \hat{\boldsymbol{n}} - \frac{b_9}{b_{10}} \frac{\partial \Delta\phi}{\partial \varepsilon^{k}_{n+1}} \tag{4.4}$$

$$\frac{\partial \Delta\phi}{\partial \varepsilon^{k}_{n+1}} = b_{16}\hat{\boldsymbol{n}} + b_{17}\mathbf{I} + b_{18}\mathbf{I} \tag{4.5}$$

$$b_0 = 1 + K\Delta\phi 2M^2 \tag{4.6}$$

$$b_1 = K\Delta\phi M^2(\beta - 1) \tag{4.7}$$

$$b_3 = \theta\Delta\phi(2M^2 p - M^2\beta p_c + M^2\beta p_c) \tag{4.8}$$

$$b_4 = 1 - (p_c)_n exp(b_3)(M^2\beta - M^2)\theta\Delta\phi \tag{4.9}$$

$$b_5 = (p_c)_n exp(b_3)\theta\Delta\phi 2M^2 \tag{4.10}$$

$$b_6 = (p_c)_n exp(b_3)(2M^2 p - M^2 p_c + M^2\beta p_c) \tag{4.11}$$

$$b_7 = 1 + \frac{b_1 b_5}{b_0 b_4} \tag{4.12}$$

$$b_8 = 2G\sqrt{\frac{3}{2}} \tag{4.13}$$

$$b_9 = 3G2q(1 + 2\beta) \tag{4.14}$$

$$b_{10} = 1 + 3G\Delta\phi 2(1 + 2\beta) \tag{4.15}$$

$$b_{11} = 2q(1 + 2\beta) \tag{4.16}$$

$$b_{12} = M^2(2p - p_c + \beta p_c) \tag{4.17}$$

$$b_{13} = M^2(-p + \beta p - 2\beta p_c) \tag{4.18}$$

$$b_{14} = -\frac{b_{11} b_9}{b_{10}} - \frac{b_{12} b_1 b_6 b_2 b_4}{b_0 b_4 b_7} \tag{4.19}$$

$$b_{15} = b_{14} + \frac{b_{13} b_6 b_4 b_7 - b_5 b_{13} b_1 b_6 b_2 b_4}{b_4^2 b_7} \tag{4.20}$$

$$b_{16} = -\frac{b_{11}b_8}{b_{10}b_{15}} \tag{4.21}$$

$$b_{17} = -\frac{b_{12}K}{b_0 b_7 b_{15}} \tag{4.22}$$

$$b_{18} = \frac{b_5 b_{13} K}{b_0 b_7 b_{15}} \tag{4.23}$$

References

1. Ronaldo I. Borja, Cam-Clay plasticity, part I : implicit integration of elasto-plastic constitutive relations, 1989

2. Johan Gaume, Dynamic anticrack propagation in snow, 2018

3. UMAT subroutine