# ECEg 3142:

# Network Analysis and Synthesis

## Chapter 8: Filter Design

# Chapter 8: Filter design

8.0. Introduction

8.1. Butterworth approximation

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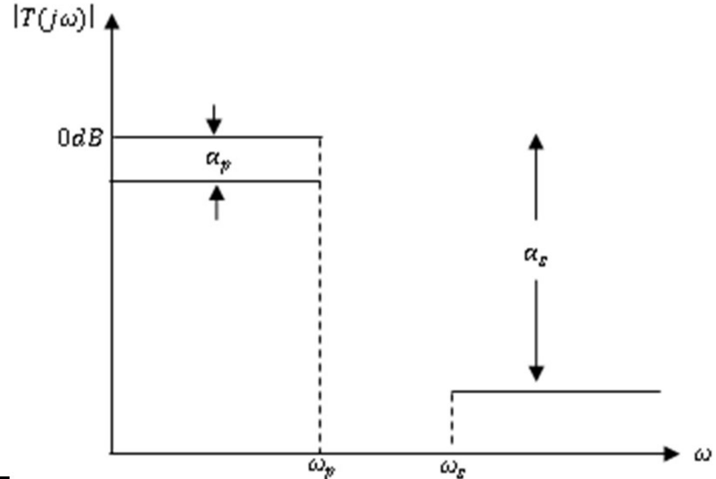
8.3 Comparison b/n Butterworth and Chebyshev approximation methods

8.4 Ideal vs. Real Low Pass Filter

# 8.0. Introduction

- Recall a LPF passes LF signals but attenuates signals with frequencies higher than the cutoff frequency.
- The magnitude and phase of an ideal LPF is




- $\omega_p \rightarrow PB\ edge$ freq.
- $\omega_s \rightarrow SB\ edge\ freq.$
- $\omega_p < \omega < \omega_s \rightarrow transtion\ band$
- $\alpha_p \ \rightarrow Maximum\ PB\ attenuation\ in\ dB$
- $\alpha_s \ \rightarrow Minimum\ SB\ attenuation\ in\ dB$

That is, $\alpha \leq \alpha_p,\quad \omega \leq \omega_p$ and $\alpha \geq \alpha_s,\quad \omega \geq \omega_s$

# 8.0. Introduction …

- The approximation methods solve the problem of selecting a realizable rational function whose frequency response approximates the given specification.
- There are different criterions for closeness of approximation.
- ***Minimizing the maximum error***

$$\varepsilon(\omega) = Max\left|F(\omega) - T(\omega)\right|$$

- ***Minimizing the mean square error***

$$\varepsilon(\omega) = \int_{\omega_{p1}}^{\omega_{p2}} \left|F(\omega) - T(\omega)\right|^2 d\omega$$

# 8.0. Introduction …

- In all approximation methods the ***normalized*** transfer function (of the LPF) is selected as

$$|T(j\omega)|^2 = \frac{1}{1+\varepsilon^2 K_n{}^2(\omega)}$$

With

$$\begin{aligned} 0 \le K_n{}^2(\omega) &<< 1, \quad 0 \le \omega \le \omega_p \\ K_n{}^2(\omega) &>> 1, \qquad\quad \omega \ge \omega_s \end{aligned}$$

- With this selection, $|T(j\omega)|$ is approximately 1 in the PB and approximately 0 in the SB.
- From the previous equation, the attenuation function is

$$\alpha(\omega) = 10\log\left[1+\varepsilon^2 K_n{}^2(\omega)\right]$$

# 8.0. Introduction …

- The constant $\varepsilon$ determines the PB and/or SB attenuation.
- We will discuss the two well-known methods of filter design.
- These are:
  - Butterworth
  - Chebyshev

# 8.1. Butterworth approximation

- Butterworth approximation:
  - minimizes the maximum error

$$\varepsilon(\omega) = Max\left|F(\omega) - T(\omega)\right|$$

  - The error at ω = 0 is zero.

i.e.

$$F(0) = T(0)$$

  - Maximally flat at ω=0.

$$\frac{d}{d\omega}K_n(\omega) = \frac{d^2}{d\omega^2}K_n(\omega) = \ldots = \frac{d^{n-1}}{d\omega^{n-1}}K_n(\omega) = 0$$

- In Butterworth approximation, $K_n(\omega)$ is selected as

$$K_n(\omega) = \beta_0 + \beta_1\omega + \beta_2\omega^2 + \ldots + \beta_n\omega^n$$

# 8.1. Butterworth approximation ...

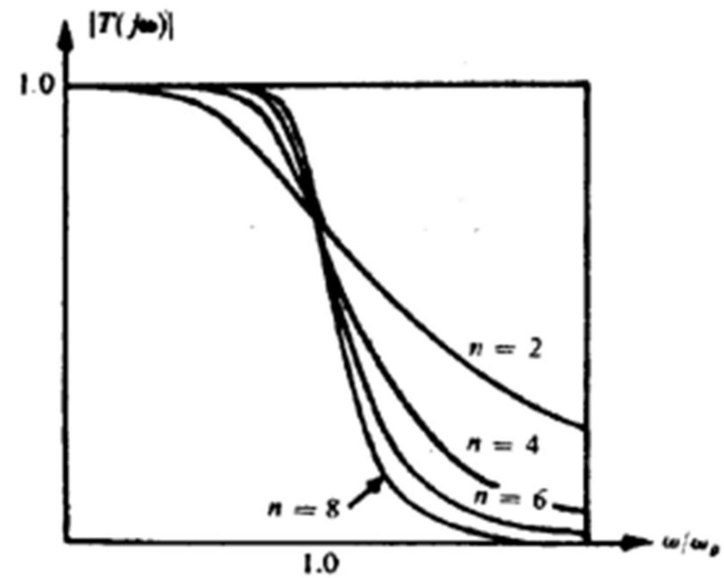
- To be maximally flat, all the derivatives of $K_n(\omega)$ must be zero at $\omega = 0$.
- This reduces $K_n(\omega)$ to:

$$K_n(\omega) = \beta_n \omega^n$$

- Therefore, the tr. fn and the attenuation function become:

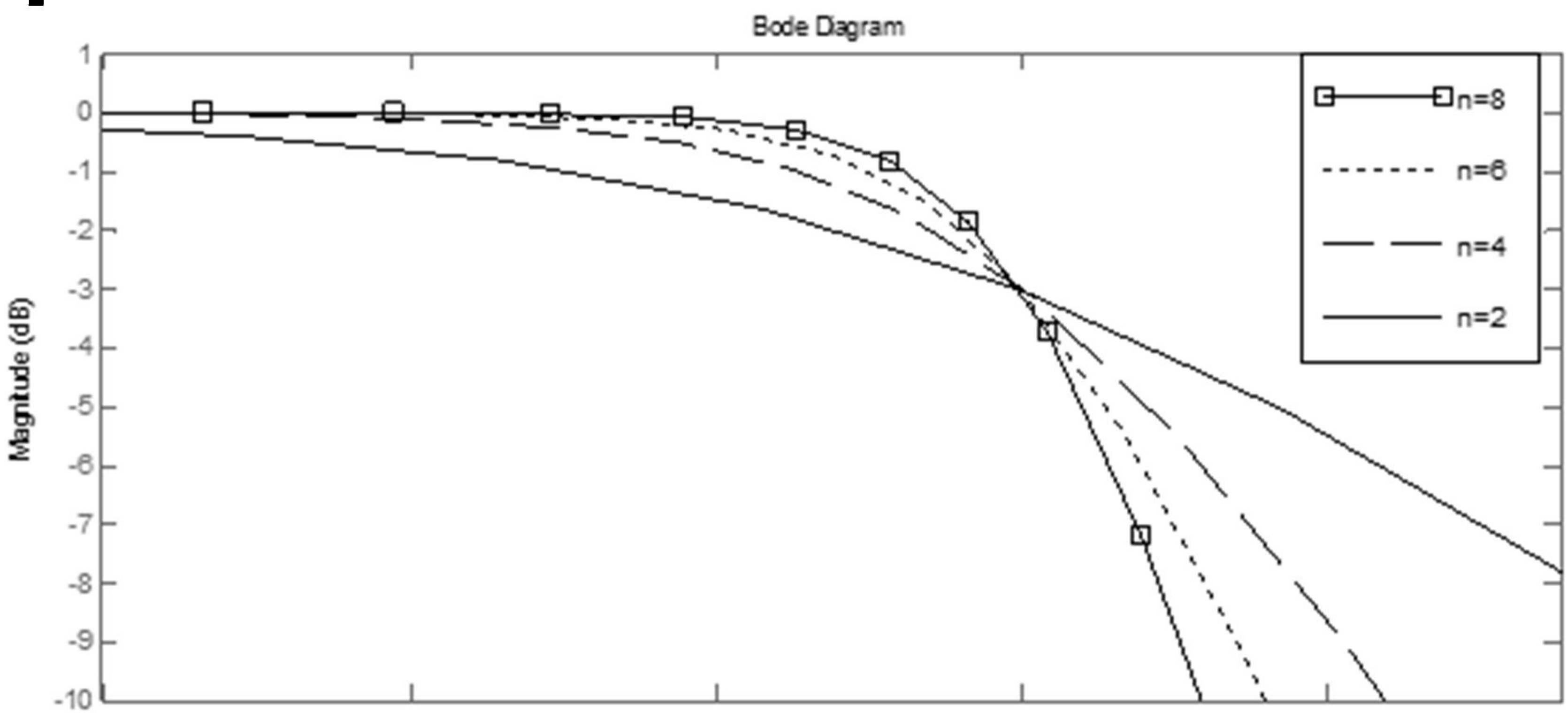
$$\left|T(j\omega)\right|^2 = \frac{1}{1 + c^2\left(\dfrac{\omega}{\omega_p}\right)^{2n}}$$

$$\alpha(\omega) = 10\log\left[1 + c^2\left(\frac{\omega}{\omega_p}\right)^{2n}\right]$$



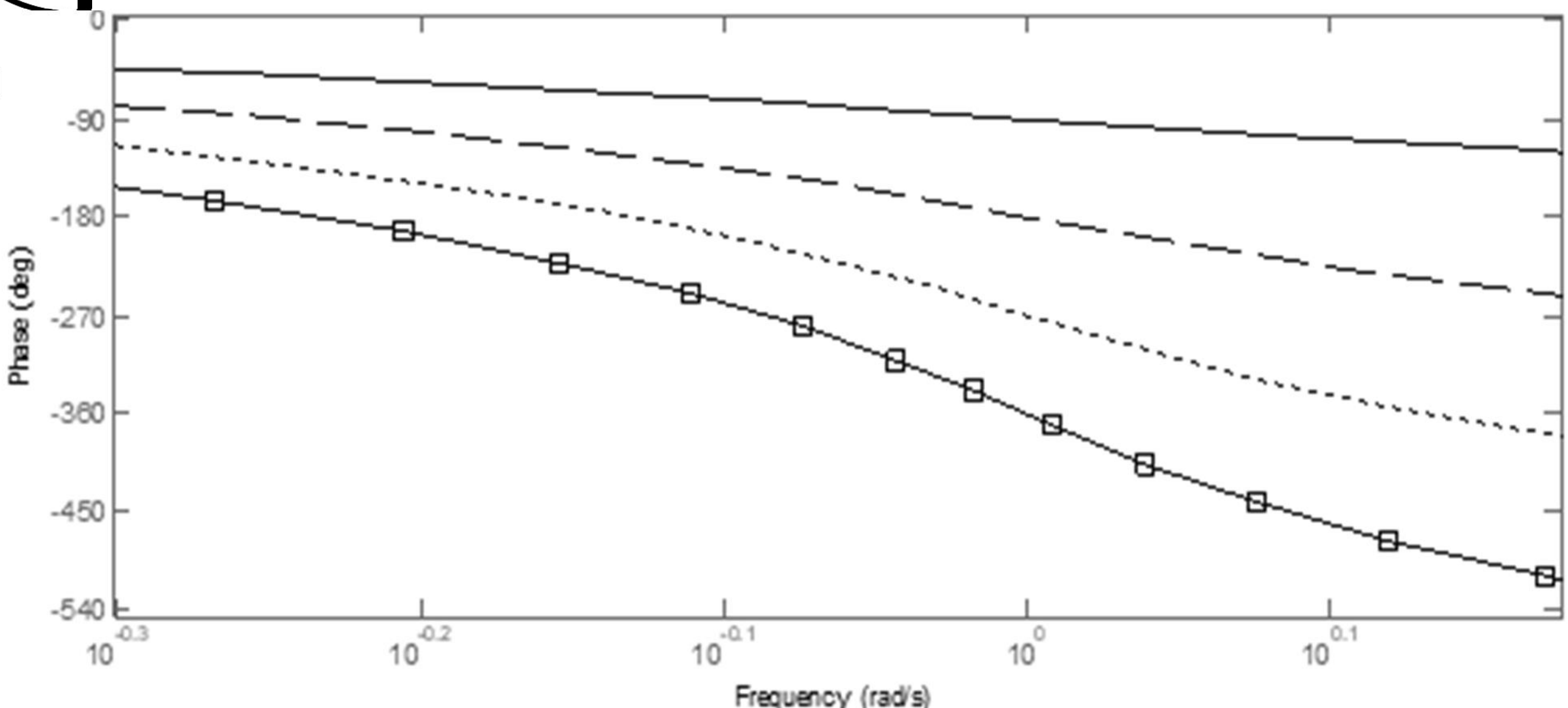
# 8.1. Butterworth approximation …

- Graphically, the gain function is



# 8.1. Butterworth approximation …

- And the phase function is



# 8.1. Butterworth approximation …

- If the specifications are barely satisfied, we will have:

$$\alpha(\omega_p) = \alpha_p = 10\log\left[1 + c^2\left(\frac{\omega_p}{\omega_p}\right)^{2n}\right]$$

$$\alpha_p = 10\log\left[1 + c^2\right]$$

$$\alpha(\omega_s) = \alpha_s = 10\log\left[1 + c^2\left(\frac{\omega_s}{\omega_p}\right)^{2n}\right]$$

- The factor $k = \frac{\omega_p}{\omega_s}$ is called the ***selectivity parameter.***
- Using k,

$$c^2 = 10^{0.1\alpha_p} - 1$$

$$\frac{c^2}{k^{2n}} = 10^{0.1\alpha_s} - 1$$

# 8.1. Butterworth approximation ...

- Let

$$k_1 = \left[ \frac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1} \right]^{1/2}$$

- Manipulating the above two equations

$$n = \frac{\log k_1}{\log k}$$

- Since n has to be integer, it is selected as the next higher integer

i.e.

$$n \geq \frac{\log\ k_1}{\log\ k}$$

- If n = $\frac{\log k_1}{\log k}$, then c obtained from $\alpha_p$ & $\alpha_s$ will be equal.
- If n is an integer > $\frac{\log k_1}{\log k}$, then c is chosen to satisfy either $\alpha_p$ or $\alpha_s$.

# 8.1. Butterworth approximation …

## Example 1

- Design a LPF with the following specification

$$\begin{array}{ll|l} \alpha \le 1dB, & for & f \le 3MHz \\ \alpha \ge 60dB, & for & f \ge 12MHz \end{array}$$

- Solution:

$$k = \frac{\omega_p}{\omega_s} = 0.25$$

$$k_1 = \left[\frac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1}\right]^{1/2} = \left[\frac{10^{0.1} - 1}{10^{6} - 1}\right]^{1/2} = 0.5089 * 10^{-3}$$

$$n \ge \frac{\log k_1}{\log k} = \frac{-3.2934}{-0.6021} = 5.4702$$

- Since the filter order n has to be integer, the next higher integer is taken $n = 6$

# 8.1. Butterworth approximation …

**Example 1 …**

- Since n is not exactly equal to $\frac{\log k_1}{\log k}$ , the value of c obtained from the PB and SB attenuation are different.
- In this case, c can be taken to satisfy either $\alpha_p$ or $\alpha_s$.
- To satisfy PB requirement

$$c = \sqrt{10^{0.1} - 1} = 0.5089$$

- To satisfy SB requirement

$$c = \sqrt{(0.25)^{12}\left(10^{6} - 1\right)} = 0.2441$$

- The magnitude function is then

$$\left|T(j\omega)\right|^2 = \frac{1}{1 + c^2\left(\frac{\omega}{\omega_p}\right)^{2n}}$$

$$\left|T(j\omega)\right|^2 = \frac{1}{1 + 0.259\left(\frac{\omega}{\omega_p}\right)^{12}}$$

# 8.1. Butterworth approximation …

**Example 1 …**

- To obtain the transfer function T(s),

$$\left|T(j\omega)\right|^2 = T(s)T(-s)\Big|_{s=j\omega}$$

$$T(s)T(-s) = \left|T(j\omega)\right|^2\Big|_{\omega=\frac{s}{j}}$$

- To obtain *T(s)* from *T(s)T(-s)*, the LHS poles of *T(s)T(-s)* are taken.
- We find zeros of polynomials of the form $s^n+a$ as discussed before.

# 8.1. Butterworth approximation …

**Example 2**

- The specifications of a band-pass filter are

$$\alpha_p \leq 3\text{dB} \qquad 50\text{k rad/s} < \omega < 72\text{k rad/s}$$

$$\alpha_s \leq 40\text{dB} \qquad \omega < 40\text{k rad/s} \qquad \omega > 120\text{k rad/s}$$

**Solution:**

- To synthesize BPF with Butterworth appx, we transform to LPF spec. first.
- The band-pass to low-pass transformation is

$$s_n = \frac{\omega_0}{BW}\left(\frac{s}{\omega_0} + \frac{\omega_0}{s}\right)$$

$s_n$ = the normailized LPF freq. variable

$s$ = the BPF freq. variable

$\omega_0$ = the center frequency of the BPF

# 8.1. Butterworth approximation …

**Example 2**

Solution: …

- The center frequency and bandwidth of the BPF are

$$\omega_0 = \sqrt{\omega_{p1}\omega_{p2}} = \sqrt{\omega_{s1}\omega_{s2}} = \sqrt{50krad * 72krad} = 60krad/s$$

$$BW = \omega_{p2} - \omega_{p1} = 22krad$$

Hence, the transformed low-pass specifications are

$$\Omega_p = \frac{1}{BW}\frac{\omega_p^2 - \omega_0^2}{\omega_p} = \frac{1}{22}\frac{72^2 - 60^2}{72} = 1$$

$$\Omega_s = \frac{1}{BW}\frac{\omega_s^2 - \omega_0^2}{\omega_s} = \frac{1}{22}\frac{120^2 - 60^2}{120} = 4.095$$

- The LPF specification becomes

$$\alpha_p \le 3\text{dB} \quad \omega < 1$$

$$\alpha_s \le 40\text{dB} \quad \omega > 4.095$$

# 8.1. Butterworth approximation …

**Example 2**

Solution: …

- The normalized LPF is then designed using Butterworth appx. as

$$k = \frac{1}{4.095} = 0.2442$$

$$k_1 = \left[\frac{10^{0.3}-1}{10^4-1}\right] = 0.0100$$

and $$n \geqslant \frac{\log k_1}{\log k} = 3.27. \quad \text{Let } n = 4.$$

Since the attenuation at the pass-band edge is 3dB, $c = 1$. From Table 10.1 the low-pass transfer function (with $\omega_p = 1$ rad/sec) is

$$T_1(s) = \frac{1}{s^4 + 2.613s^3 + 3.414s^2 + 2.613s + 1}$$

# 8.1. Butterworth approximation …

**Example 2**

Solution: …

- The normalized LPF tr. fn is then transformed to BPF with

$$s_n = \frac{\omega_0}{BW}\left(\frac{s}{\omega_0} + \frac{\omega_0}{s}\right) = 0.0455 * 10^{-3}\,\frac{s^2 + (60 * 10^3)}{s}$$

$s_n$ is the normailize low-pass filter variable

$s$ is the band pass filter variable

- Substituting this eqn in the LPF transfer function

$$T_{BP}(s) = \frac{2.332 \times 100^{17} s^4}{\begin{gathered}(s^8 + 2.1978 \times 10^4 s^7 + 1.4883 \times 10^{10} s^6 + 2.4795 \times 10^{14} s^5 \\ + 8.1238 \times 10^{19} s^4 + 8.9272 \times 10^{23} s^3 + 1.9259 \times 10^{29} s^2 \\ + 1.0255 \times 10^{33} s + 1.6798 \times 10^{38})\end{gathered}}$$

# 8.2 Chebyshev approximation

- From the standpoint of obtaining the best approximation to the ideal filter characteristic from a polynomial of a given degree, the Butterworth approximation function doesn't do as well.
- This is so as it concentrates all approximating ability of the polynomial at $\omega = 0$, instead of distributing it over the range $0 < \omega < 1$.
- A better result is obtained if we look for a rational function that distributes the approximation in the range $0 < \omega < 1$.
- This leads to Chebyshev polynomials which are defined as linearly independent solutions of the differential equation.

$$(1 - \omega^2)\ddot{y} - \omega\dot{y} - n^2 y = 0$$

# 8.2 Chebyshev approximation …

- One of the solutions is

$$y = {T_n}^2(\omega) = cos(ncosh^{-1}\omega)$$

- Using binomial expansion & manipulation leads to the recursive formula for $T_n(w)$

$$T_{n+1}(\omega) = 2\omega T_n(\omega) - T_{n-1}(\omega)$$

$$T_1(\omega) = \omega \quad and \quad T_0(\omega) = 1$$

- Using the recursive formula will lead to the following table

| n | $T_n(\omega)$ |
|---|---|
| 0 | 1 |
| 1 | $\omega$ |
| 2 | $2\omega^2-1$ |
| 3 | $4\omega^3-3\omega$ |
| 4 | $8\omega^4-8\omega^2+1$ |
| 5 | $16\omega^5-20\omega^3+5\omega$ |
| 6 | $32\omega^6-48\omega^4+18\omega^2-1$ |

# 8.2 Chebyshev approximation …

- For Chebyshev approximation

$$|T(j\omega)|^2 = \frac{1}{1+\varepsilon^2 T_n^2\left(\frac{\omega}{\omega_p}\right)^{2n}}$$

and

$$\alpha(\omega) = 10\log\left[1+\varepsilon^2 T_n^2\left(\frac{\omega}{\omega_p}\right)^{2n}\right]$$

- If the specifications for LPF are barely satisfied,

$$\alpha(\omega_p) = \alpha_p = 10\log\left[1+\varepsilon^2 T_n^2\left(\frac{\omega_p}{\omega_p}\right)^{2n}\right]$$

- Or (noting also $T_n^2(1) = 1$ )

$$\alpha_p = 10\log\left[1+\varepsilon^2\right]$$

and

$$\alpha(\omega_s) = \alpha_s = 10\log\left[1+\varepsilon^2 T_n^2\left(\frac{\omega_s}{\omega_p}\right)^{2n}\right]$$

# 8.2 Chebyshev approximation …

- From the first approximation,

$$\varepsilon^2 = 10^{0.1\alpha_p} - 1$$

- Let,

$$k = \frac{\omega_p}{\omega_s} \quad \text{and} \quad k_1 = \left[\frac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1}\right]^{1/2}$$

- After mathematical manipulations, the filter order can be shown to be :

$$n \geq \frac{\cosh^{-1}\frac{1}{k_1}}{\cosh^{-1}\frac{1}{k}}$$

- As the filter order has to be an integer, it is taken as the next higher integer if the ratio is not exactly an integer.

# 8.2 Chebyshev approximation …

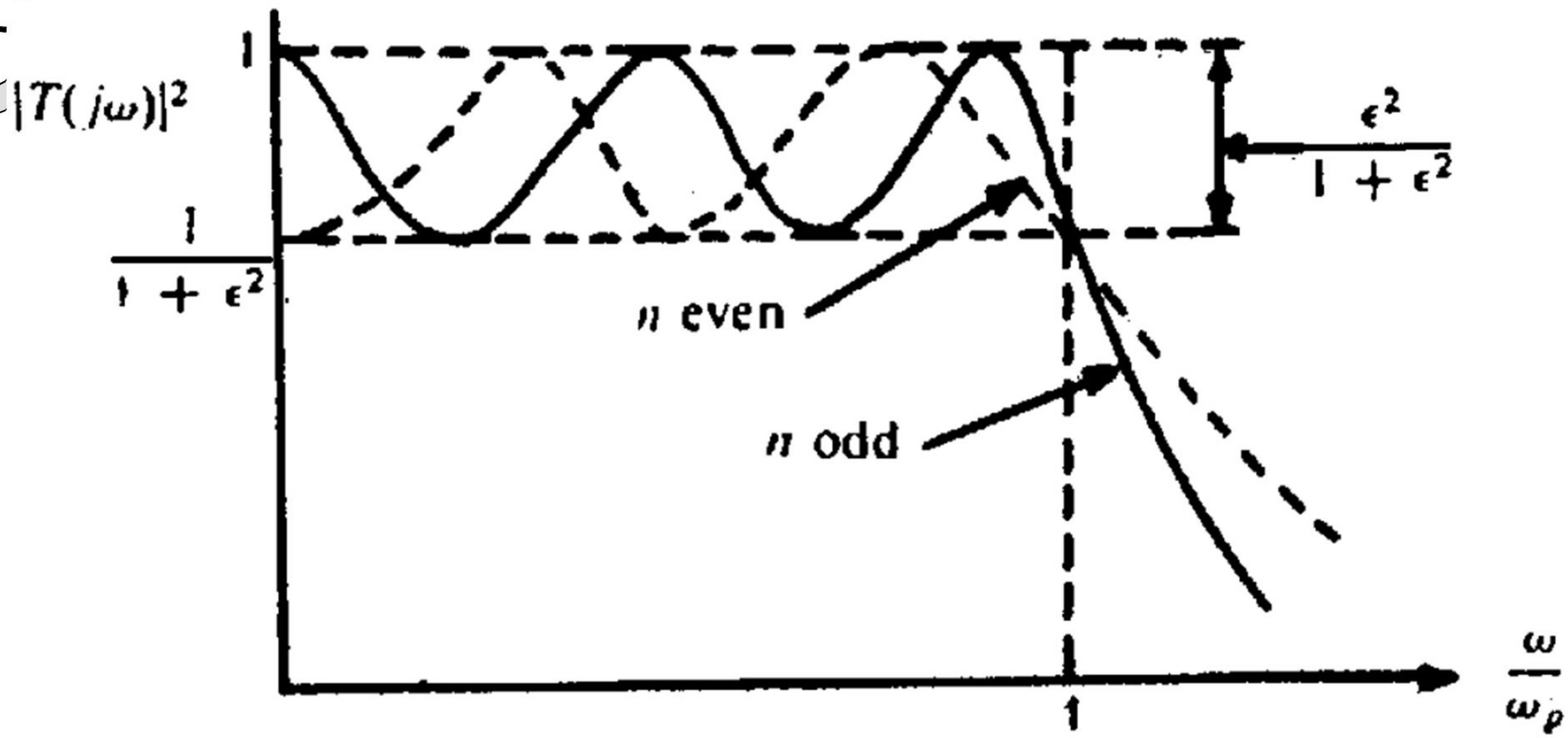
- The transfer function T(s) can be obtained as in the case of Butterworth approximation by determining its poles.

$$|H(j\omega)|^2 = 1 + \varepsilon^2 {T_n}^2(\omega)$$

- The zeroes of H(s)H(-s) appear on a quadrantal symmetry (they lien on an ellipse).
- We take the LHS zeros to H(s) and the rest for H(-s).
- It can be shown that the poles of T(s) are given as
  - $s_k = \sigma_k + j\varphi_k, \quad k = 0,1,2,\ldots,2n-1$
  - $\sigma_{k=}\sinh(v)\sin\left(\frac{2k+1}{2n}\pi\right)$
  - $\varphi_{k=}\cosh(v)\cos\left(\frac{2k+1}{2n}\pi\right)$
  - $v = \frac{1}{n} sinh^{-1}\frac{1}{\varepsilon}$
- We select the poles on the left hand side of the s plane.

# 8.2 Chebyshev approximation …

- Plot of a Chebyshev transfer function is



# 8.2 Chebyshev approximation …

**Example 3**

- The specifications of a low-pass filter are

➢ Pass-band ripple: 1dB

➢ Pass-band : 0 < ω < 1.75MHz

➢ Stop-band ripple at least 20dB at 2.5MHz and above.

**Solution:**

$$\varepsilon^2 = 10^{0.1\alpha_p} - 1 = 0.2589$$

$$\varepsilon = 0.5088$$

- To find filter order

$$k = \frac{\omega_p}{\omega_s} = \frac{1.75}{2.5} \quad , \quad \frac{1}{k_1} = \left[\frac{10^2 - 1}{10^{0.1} - 1}\right]^{\frac{1}{2}} = 19.56$$

Thus, $$n \geq \frac{\cosh^{-1} 19.56}{\cosh^{-1} 1.43} = 4.0914 \qquad \Rightarrow n = 5$$

# 8.2 Chebyshev approximation …

**Example 3 …**

- To find T(s), we use the poles on the left hand side

$$s_k = \sigma_k + j\psi_k$$

$$\sigma_k = \sinh\nu \sin\left(\frac{2k+1}{2n}\pi\right), k = 0,1,2,\ldots,2n-1$$

$$\psi_k = \cosh\nu \cos\left(\frac{2k+1}{2n}\pi\right)$$

$$\nu = \frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}$$

- That is $\sigma_k$ should be negative

$$\Rightarrow \frac{2n-1}{2} \le k \le 2n-1$$

# 8.2 Chebyshev approximation …

**Example 3 …**

- To find the poles

$$\nu = \frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon} = \frac{1}{5}\sinh^{-1}\frac{1}{0.5088} = 0.2856$$

$5 \le k \le 9$

Inserting these values in the equation for finding the zeros,

$$-0.2895, \quad -0.2342 \pm j0.612, \quad -0.0895 \pm j0.9902$$

- Then T(s) becomes

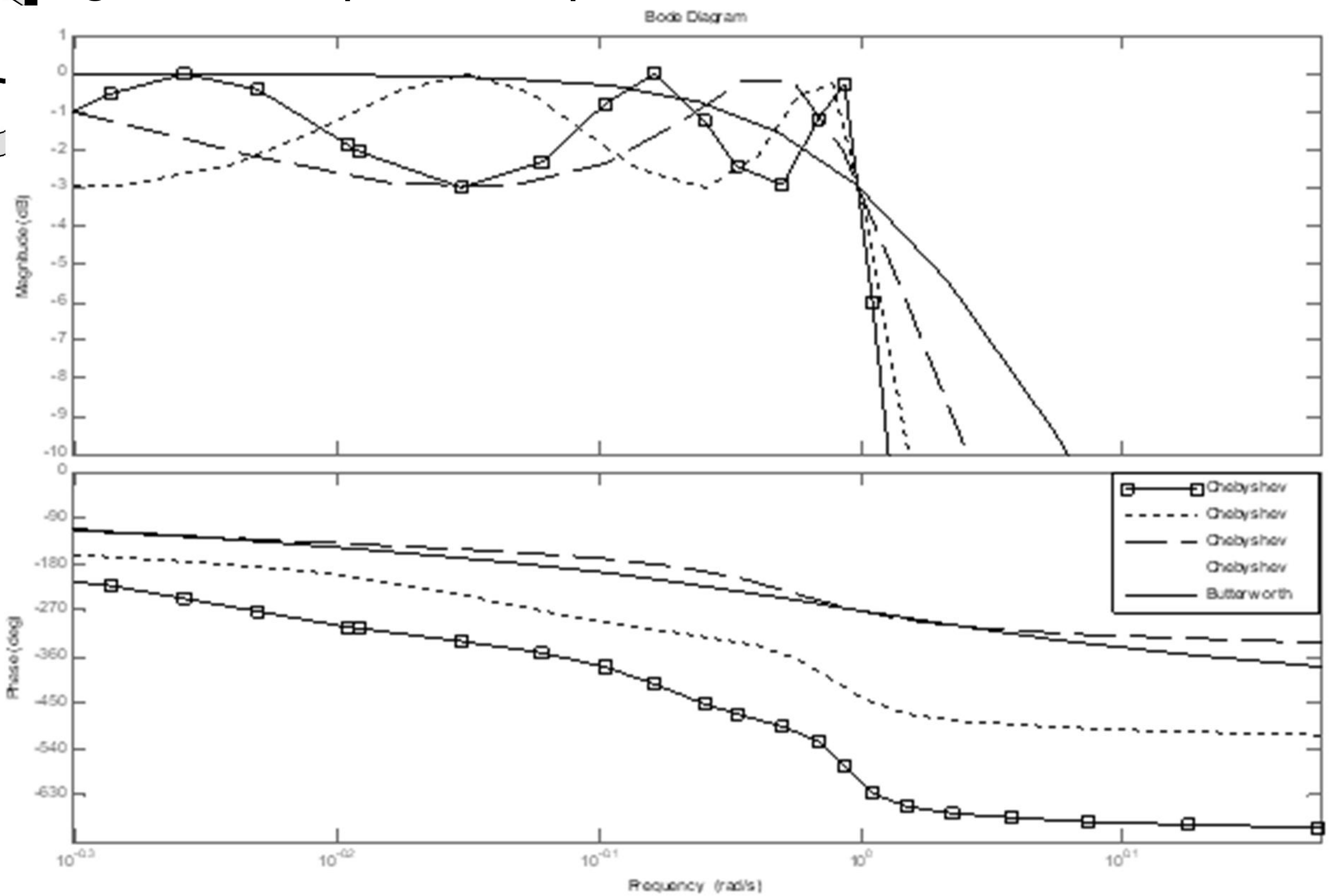
$$T(s) = \frac{1}{(s+0.2895)(s+0.2342+j0.612)(s+0.2342-j0.612)(s+0.0895+j0.9902)(s+0.0895-j0.9902)}$$

## 8.3 Comparison b/n Butterworth and Chebyshev approximation methods

- Both produce all-pole transfer functions.
- The degree of Chebyshev estimation is always lower than Butterworth estimation for the same specifications.
- Example: The specification of a LPF is
  - Pass-band attenuation is 1dB, Pass-band edge is 150 k rad/s
  - Stop-band attenuation is 60dB, Stop-band edge is 200 k rad/s
- It can be shown that the filter orders are such that:
  - For Butterworth $n = 27$
  - For Chebyshev $n = 11$
- Thus, the filter order is much lower for Chebyshev appx.
- Chebyshev appx. will have a lower value in the SB as long as $c = \varepsilon$ and hence better attenuation.
- Butterworth approximation leads to a more linear phase.

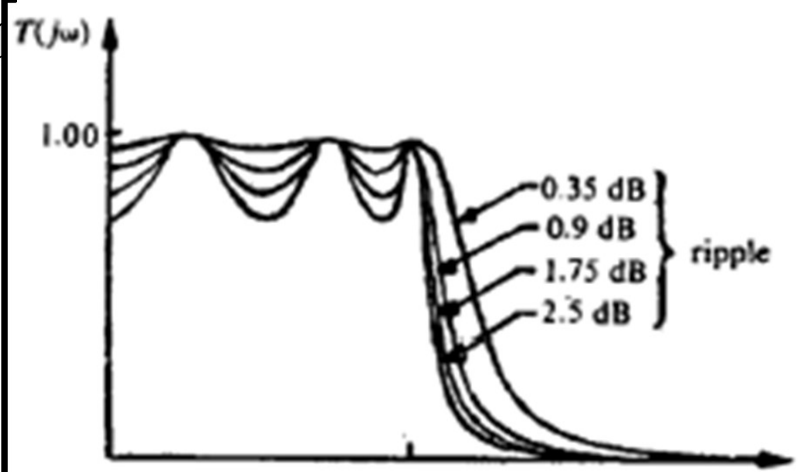
# 8.3 Comparison b/n Butterworth ….

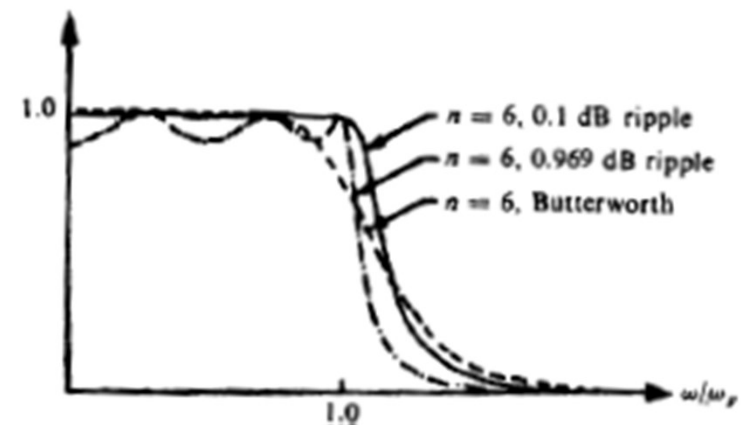
- Magnitude and phase comparisons:



# 8.3 Comparison b/n Butterworth ….

- A small pass-band ripple and sharper cut off frequencies cannot be achieved at the same time for both approximations.



# 8.4 Ideal vs. Real Lowpass Filter

- Comparing the ideal LPF with the realizable filter order, the ideal LPF spec is the limit as the following three are satisfied.
  - Pass-band attenuation approaches zero,
  - Stop-band attenuation approaches infinity,
  - Transition band approaches zero,
- To approach the ideal LPF, investigate the effect of PB & SB attenuations and transition band on the filter order.

# 8.4 Ideal vs. Real Lowpass Filter…

**Pass-band attenuation vs. filter order**

- From Butterworth filter design, it can be shown as the PB attenuation is decreased, the filter order increases.
- In the limit as PB attenuation => 0, filter order approaches ∞.

**Stop-band attenuation vs. filter order**

- As the SB attenuation is increased, the filter order increases.
- In the limit as the SB attenuation => ∞, filter order approaches ∞.

**Transition band vs. filter order**

- As the transition band is decreased, the filter order increases.
- In the limit the transition band => 0, the filter order approaches ∞.