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Final Project Financial Engineering

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Abstract

This work investigates the pricing of credit tranches[1], focusing on the calibration and performance of various dependence models.
We begin with the double t-Student model, exploring two calibration strategies under the Large Homogeneous Portfolio (LHP) and Kullback-Leibler (KL) approximations. The model parameters—correlation and degrees of freedom ν—are optimized to best fit market-implied cumulative correlations.

To assess the impact of the LHP assumption, we compare the model prices obtained under exact, KL, and LHP methods across different portfolio sizes. Furthermore, we perform a comparative pricing analysis under alternative copula frameworks: Gaussian, double t-copula, and Archimedean (Clayton), using Monte Carlo simulation for each.

Our results show that the double t-Student model, particularly under the LHP approximation, provides the best fit to market-implied correlations across all tranches, outperforming both the Gaussian and Archimedean alternatives in terms of pricing accuracy and implied correlation structure.

[1] All prices are expressed as relative to the total portfolio notional $\mathcal{N}$, which is given by the number of obligors $I = 500$ multiplied by the notional of each obligor (2 million euros).

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Chapter 1

Calibration of the double t-Student model

1.1 Model framework

The model used for pricing and calibration is a one-factor latent variable model. Each entity in the portfolio is associated with a latent variable v_i, which drives the default event through a threshold mechanism. Specifically, the model assumes:

$$v_i = \sqrt{\rho}\, y + \sqrt{1-\rho}\, \varepsilon_i \tag{1.1}$$

where $y \sim t_\nu$ is the systemic risk factor shared by all obligors in the portfolio, and $\varepsilon_i \sim t_\nu$ are the idiosyncratic risk components, assumed to be independent and identically distributed (i.i.d.) across i, and independent of y. Both the systemic and idiosyncratic components follow a standardized t-Student distribution with the same number ν of degrees of freedom. This specification defines the so-called *double t-Student model*, originally introduced in Hull and White (2004).

Given the realization of the systemic factor y, the conditional default event for obligor i occurs if the latent variable v_i falls below the threshold k:

$$v_i \leq k \quad \iff \quad \sqrt{\rho} y + \sqrt{1-\rho}\, \varepsilon_i \leq k.$$

Given y, this is equivalent to:

$$\varepsilon_i \leq \frac{k - \sqrt{\rho}\, y}{\sqrt{1-\rho}}.$$

Since $\varepsilon_i \sim t_\nu$, the conditional probability of default given y becomes:

$$p(y) = \mathbb{P}(v_i \leq k \mid y) = \mathbb{P}\left(\varepsilon_i \leq \frac{k - \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right) = t_\nu\left(\frac{k - \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right),$$

where $t_\nu(\cdot)$ is the cumulative distribution function (CDF) of the t-Student distribution. The unconditional default probability is then obtained by integrating over the distribution of y:

$$\mathbb{P}(v_i \leq k) = \int_{-\infty}^{\infty} p(y)\, \phi_\nu(y)\, dy,$$

where $\phi_\nu(y)$ denotes the probability density function (PDF) of the standardized t-Student distribution with ν degrees of freedom. This integral equation is solved numerically to find the corresponding threshold k for a given p, ρ, and ν. In our implementation, this is done using a MATLAB function `calibration_K(k, rho, nu)`.

Once the threshold k is calibrated, we can derive the loss distribution of the portfolio under the Large Homogeneous Portfolio (LHP) assumption – i.e. that all names in the portfolio are identical, with the same notional, recovery rate, and default probability. In particular, the default ratio $z = \frac{m}{I}$, where m is the number of defaults out of I obligors, has cumulative distribution function:

$$\mathbb{P}(z \leq x) = t_\nu \left(\frac{1}{\sqrt{\rho}} \left(\sqrt{1-\rho}\, t_\nu^{-1}(x) - k \right) \right),$$

and probability density function:

$$f(z) = \frac{\phi_\nu \left(\dfrac{k - \sqrt{1-\rho} \cdot t_\nu^{-1}(z)}{\sqrt{\rho}} \right)}{\phi_\nu \left(t_\nu^{-1}(z) \right)} \cdot \frac{\sqrt{1-\rho}}{\sqrt{\rho}}.$$

To compute the expected tranche loss, we apply the loss function $\mathcal{L}(z)$:

$$\mathcal{L}(z) := \frac{\min \{ \max(z-d, 0),\, u-d \}}{u-d}, \quad z \in [0,1],$$

where $d := K_d/LGD$ and $u := K_u/LGD$ represent the normalized (with respect to the loss given default $LGD = 1 - recovery$) lower and upper bounds of the tranche, respectively.

The expected tranche loss ETL_{LHP} is then obtained by integrating $\mathcal{L}(z)$ against the density $f(z)$, derived earlier. The price of the tranche is given by:

$$Price_{LHP} = DF * (1 - ETL_{LHP})$$

The discount factor DF was computed through interpolating the ZC curve obtained via bootstrap ($T = 4$ years).

1.2 Calibration of optimal ν and ρ

In this part of the assignment, we calibrate the degrees-of-freedom parameter ν and the correlation parameter ρ of the double t-Student model to match market data. Calibration is performed using cumulative tranche prices, with the equity tranche—being the most liquid—serving as the benchmark.

For each fixed ν, we determine the value of ρ such that the model price of the equity tranche matches its market price, computed via the Vasicek model. The resulting pair (ν, ρ) is then used to price all tranches under the double t-Student model. We select the optimal value ν_{opt} as the one minimizing the mean squared error (MSE) between model and market prices across all tranches.

Finally, for each tranche, we compute the Vasicek-implied correlation that matches the price produced by the calibrated double t-Student model. By construction, the equity tranche's implied correlation coincides with the market-implied one.

Parameter	Value
ν_{opt}	8.2915
ρ_{model}	0.2830

Table 1.1: Calibrated parameters of the double t-Student model under LHP

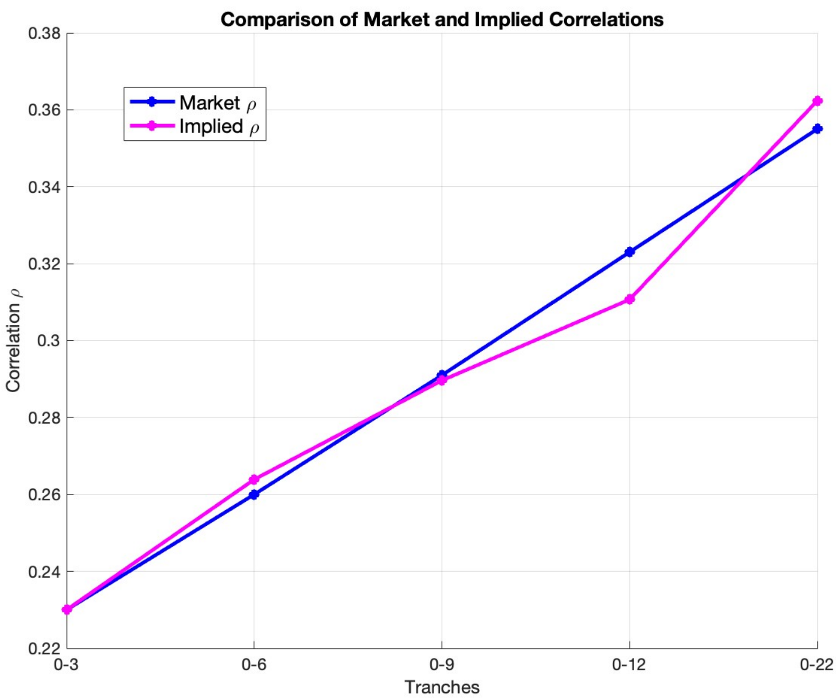

Figure 1.1: ρ implied in double t-Student model (LHP)

As shown in **Figure 1.1**, the double t-Student model accurately captures the market-implied cumulative correlations across all tranches. The fitted curve aligns closely with the market data, confirming the model's effectiveness in representing the observed correlation structure.

Observation. As $\nu \to \infty$, the double t-Student model converges to the Vasicek model. However, in practice, only a single correlation parameter ρ is calibrated and kept constant across all tranches, which limits the flexibility of the model. For this reason, the optimal value of ν is typically finite, often ranging between 3 and 15.

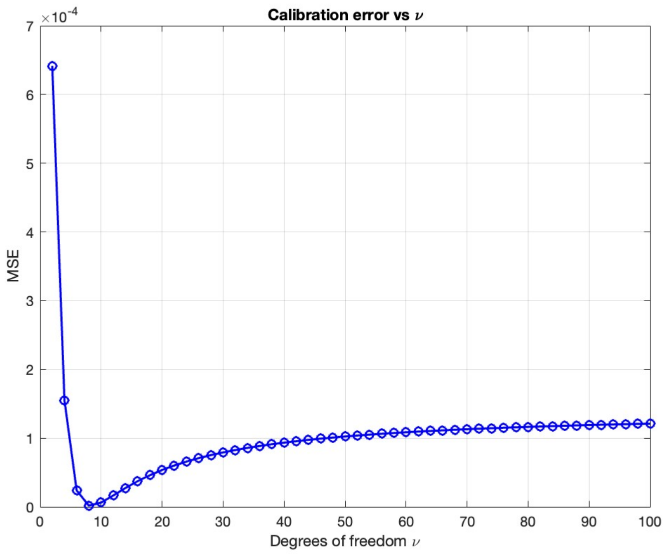

Figure 1.2: MSE of the double t-Student model (LHP) as a function of ν

In **Figure 1.2**, the mean squared error (MSE) associated with the calibration procedure is plotted as a function of the degrees of freedom ν of the double t-Student model, to verify that the minimization problem admits a well-defined and unique solution. Moreover, the minimum MSE is very close to zero ($2e-6$), indicating that the model prices closely match the market prices. We will frequently use the same graphical representation to comment on the numerical stability of the solution obtained through our calibration algorithm.

For completeness we repeat the same calibration and pricing analysis using a variant of the model in which the degrees of freedom differ between the systemic factor y and the idiosyncratic components ε_i, assigning separate values ν_y and ν_ε. The results show no meaningful difference in both tranche prices and implied correlations compared to the baseline case.

Chapter 2

Impact of the Large Homogeneous Portfolio approximation

2.1 Exact solution (HP) vs KL vs LHP

The Large Homogeneous Portfolio (LHP) approximation has already been discussed in the previous section.

In this section, we focus on the homogeneous portfolio (HP) and on the Kullback-Leibler (KL) approximation, which provides an efficient alternative when the exact HP solution becomes too computationally demanding.

As in the LHP approximation, the price of a tranche under the finite homogeneous portfolio (HP) model is computed as the discounted value of the expected payoff. Specifically, the tranche price is given by:

$$\text{Price}_{HP} = \text{DF} \cdot (1 - ETL_{HP}),$$

where DF is the discount factor and ETL_{HP} is the expected loss of the tranche. The expected loss $ETL_{HP}(K_d, K_u)$ is defined as:

$$ETL_{HP}(K_d, K_u) := \mathbb{E}[L_t(m; I, \dots)] = \mathfrak{N}_t \sum_{m=0}^{I} \mathcal{L}\left(\frac{m}{I}\right) f_{HP}(m; I),$$

where $\mathfrak{N}_t := \mathfrak{N}(K_u - K_d)$ is the tranche notional, and $\mathcal{L}(z)$ is the loss function described in Chapter 1, evaluated at the default ratio $z = \frac{m}{I}$.

The function $f_{HP}(m; I)$ represents the probability of observing m defaults out of I obligors and is given by:

$$f_{HP}(m; I) := \int_{-\infty}^{\infty} dy\, \phi(y) \binom{I}{m} p(y)^m [1 - p(y)]^{I-m},$$

where $\phi(y)$ is the density of the systemic factor y, and $p(y)$ is the conditional default probability given y. The exact form of $p(y)$ depends on the underlying model (e.g., Vasicek, double t-Student).

While the exact solution (HP) provides accurate results, it becomes computationally expensive as the number of obligors I increases. This is due to the evaluation of the binomial coefficient $\binom{I}{m}$ for each value of $m \in \{0, \dots, I\}$, which becomes numerically unstable or infeasible for large I, depending on the used machine. To overcome this limitation, we use the Kullback-Leibler (KL) approximation, which provides an asymptotically accurate estimation of the loss distribution. The KL method involves the Stirling formula:

$$m! \approx \sqrt{2\pi m}\, m^m e^{-m}$$

2.2 Visual Representation of the LHP Approximation as a Function of Portfolio Size

Following the project guidelines, we present the percentage prices of the cumulative tranches obtained under the HP, LHP, and KL methodologies, plotted against the number of obligors I, ranging from 10 to 10^4. This range reflects the realistic applicability of the LHP assumption under standard market subordination levels.

Although the assignment primarily focuses on the 3%, 6%, and 9% cumulative tranches, we extend the analysis to the remaining ones as well, in order to provide a more comprehensive and informative comparison.
The tranche prices are computed using the Vasicek model as reference. In **Figure 2.1**, we repeat the same comparison using the double t-Student model calibrated in **Chapter 1**.

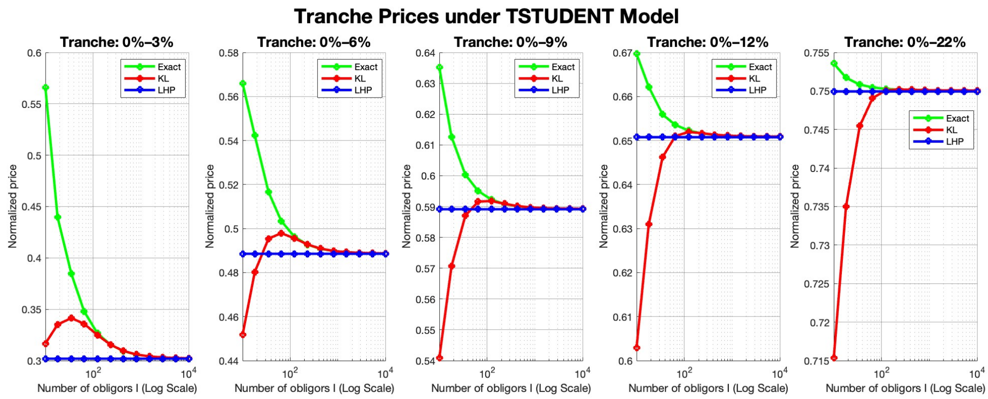

Figure 2.1: HP, LHP, KL prices of cumulative tranches using double t-Student model and ρ_{model} with respect to I.

As one can observe, all three methods (HP, LHP, and KL) converge to the same tranche price as the number of obligors I increases. However, while the LHP price remains constant with respect to I, the KL approximation significantly deviates from the HP benchmark for small portfolio sizes.
This behavior is mainly due to the nature of the analyzed tranches: although only the first tranche is technically the equity tranche, all the cumulative tranches plotted behave similarly to equity tranches. It is well known that the KL approximation becomes numerically unstable when the lower attachment point $K_d = 0$ and with small value of I: this is due to the Stirling formula approximation.

To overcome this issue, an alternative pricing approach is applied.

Tranche	HP (t-Student)	LHP (t-Student)	KL (t-Student)
0–3%	0.30821	0.30188	0.30842
0–6%	0.49005	0.48853	0.49066
0–9%	0.58998	0.58913	0.59004
0–12%	0.65121	0.65075	0.65126
0–22%	0.75012	0.74994	0.75014

Table 2.1: Tranche prices using t-Student model under HP, LHP and KL approximations

Alternative Approach for cumulative tranche pricing

Since the correlation parameter ρ_{model} is assumed constant across all tranches, we exploit a symmetry property: the price of the cumulative 0–X% tranche can be computed as the difference between the price of the full portfolio and that of the X%–100% cumulative tranche, the latter being numerically stable under the KL approximation:

$$Price_{[0-X]} = Price_{[0-100]} - Price_{[X-100]}$$

The improvement introduced by this method is clearly visible in **Figure 2.2**. In particular, **Figure 2.3** highlights the significantly better alignment of KL-based prices with the HP benchmark, with visibly reduced errors across all tranches.

While we explicitly illustrate the case of the equity tranche, it is important to note that the same issue—and consequently the same correction—applies to all cumulative tranches starting at 0%. These tranches are also sensitive to numerical instability in the KL approximation and benefit from being priced via the alternative method described above.

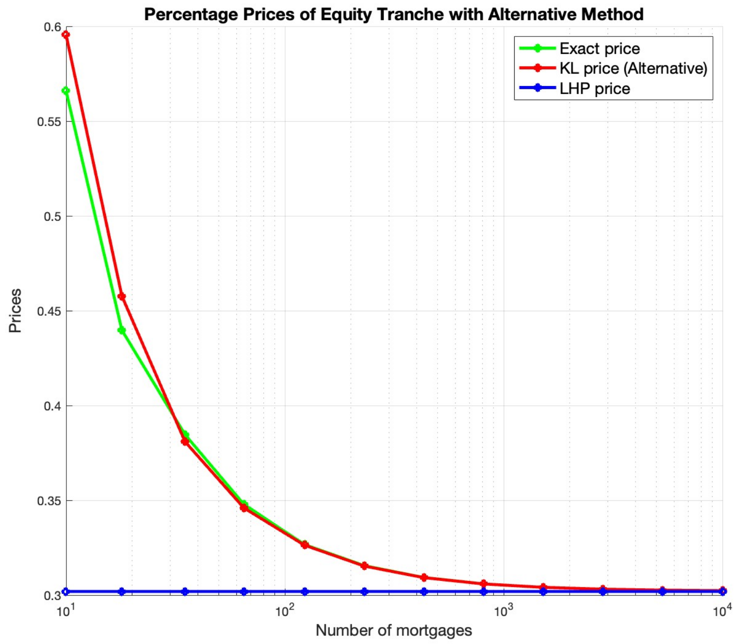

Figure 2.2: HP, LHP, alternative-KL prices of equity tranche using double t-Student model and ρ_{model} with respect to I.

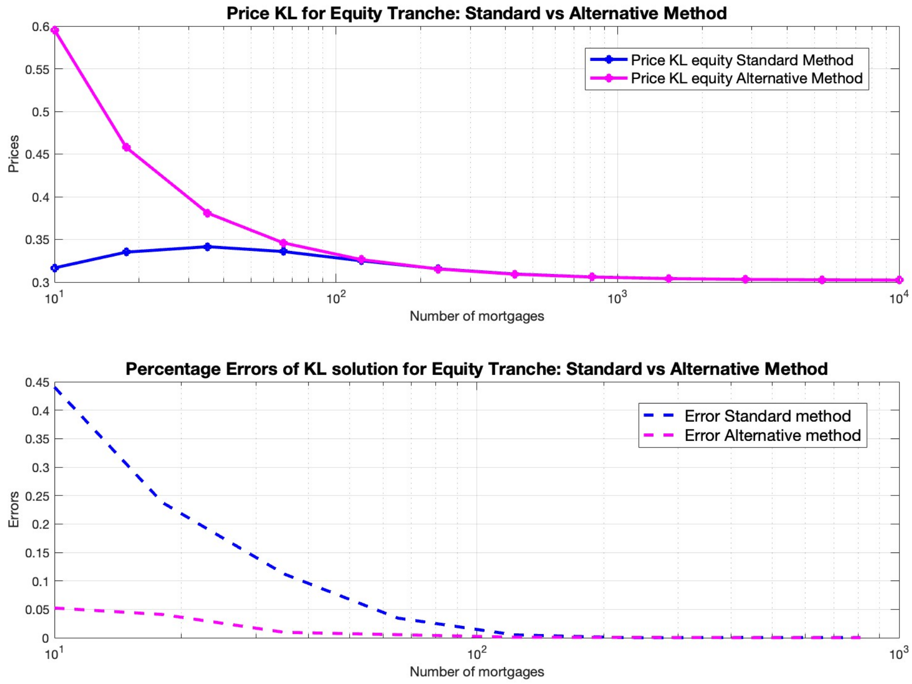

Figure 2.3: Top: standard-KL vs. alternative-KL prices of the equity tranche. Bottom: percentage errors of the two KL approaches with respect to HP prices.

As a final observation, we consider the pricing of non-cumulative (base) tranches (i.e. 0–3%, 3–6% and so on), which can be derived from cumulative tranche prices via weighted differencing. For example:

$$Price_{[0,6]} = \frac{1}{2}Price_{[0,3]} + \frac{1}{2}Price_{[3,6]} \quad \Rightarrow \quad Price_{[3,6]} = 2 \cdot Price_{[0,6]} - Price_{[0,3]}.$$

This inversion is necessary when working with market-implied correlations, which are only quoted for cumulative tranches. In contrast, models like the double t-Student assume a constant correlation ρ_{model}, allowing direct computation of base tranche prices.
Interestingly, when base tranche prices are reconstructed from cumulative ones, numerical errors (e.g., in the KL approximation) tend to cancel out. As a result, KL performs surprisingly well for base tranches, despite possible inaccuracies in the cumulative prices. This error-canceling effect, however, does not apply to the equity tranche (0–3%), which must always be priced directly and benefits most from dedicated methods.

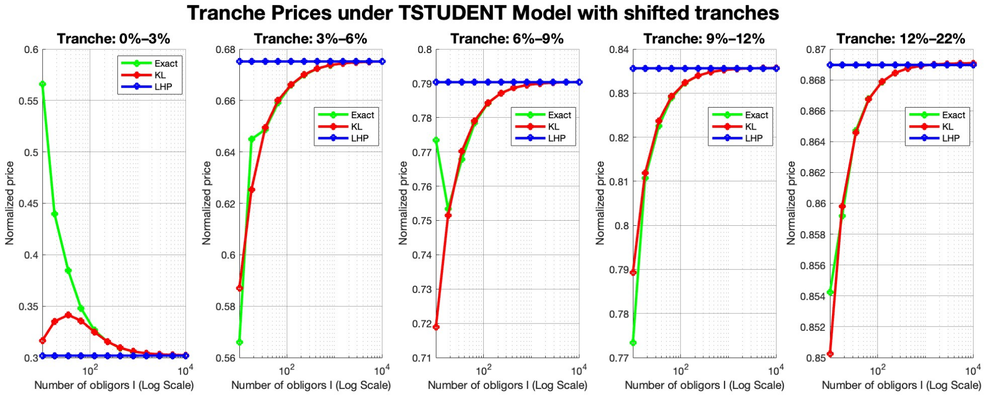

Figure 2.4: HP, LHP, KL prices of base tranches from cumulative ones using double t-Student model.

Chapter 3

Calibration under KL approximation

In this Chapter, we calibrate the degrees of freedom ν and correlation parameter ρ of the double t-Student model using the KL approximation, assuming a portfolio size of $I = 500$. Although the theoretically correct approach for pricing cumulative tranches under the KL framework—for equity tranche and cumulative tranches in general—would require the use of the alternative method (as discussed previously), we opted to proceed with the standard KL method. This decision is justified by the fact that, for $I = 500$, the numerical error introduced by the standard approach is negligible.

Specifically, the relative pricing error for the equity tranche with respect to the HP solution using double t-student model is shown in *Table 3.1*

Method	**Relative error (%)**
KL standard vs HP	0.0683
KL alternative vs HP	0.0066

Table 3.1: Relative errors between KL methods and the HP solution for equity @$I = 500$

The calibration procedure follows the same steps described more in details in **Section 1.2**: we match the price of the equity tranche to the reference market price (computed via the Vasicek model), and then minimize the mean squared error (MSE) between the model and market prices of the remaining tranches.

Finally, the implied correlations under the KL-calibrated model have also been computed within our codebase. These implied correlations turned out to be very similar to those obtained in the LHP calibration of the double t-Student model.

Parameter	**Value**
ν_{opt}	8.1761
ρ_{model}	0.2828

Table 3.2: Calibrated parameters of the double t-Student model under KL approximation

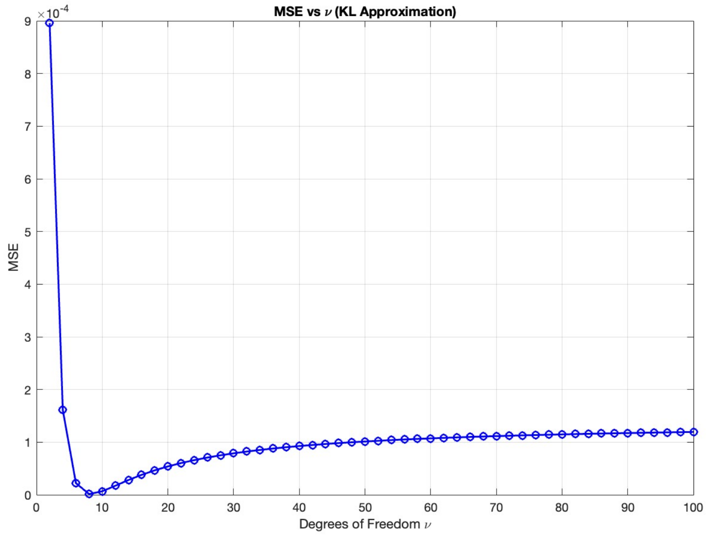

Figure 3.1: MSE of the double t-Student model (KL) as a function of ν.

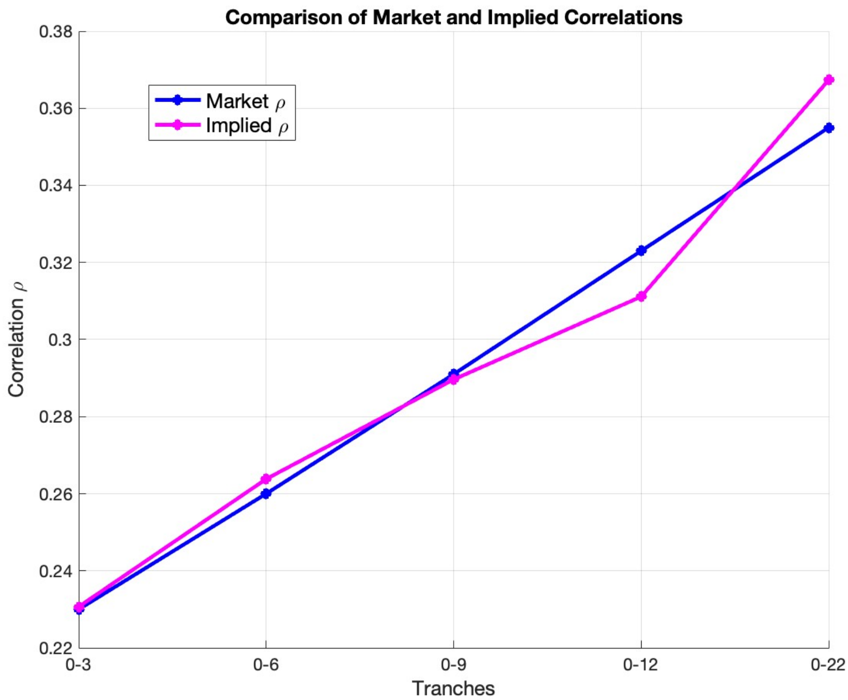

Figure 3.2: ρ implied in double t-Student model (KL)

As shown in Figure 3.1, the optimal solution for the parameter ν in the KL calibration of the double t-Student model is both stable and unique, consistent with what was observed in **Section 1.2**. Moreover, from Figure 3.2, we observe that the implied correlations obtained from this model are almost perfectly aligned with those computed by inverting the market Vasicek model using tranche prices from the double t-Student model under the Large Homogeneous Portfolio (LHP) assumption.

Chapter 4

Pricing using the Vasicek model

In this section, we use the Vasicek model under the LHP assumption to analyze the impact of using the equity tranche correlation ρ_e to price all tranches. The resulting prices are compared with those obtained using the correct market-implied correlations ρ_{mkt} for each tranche.

Tranche	**Price (ρ_e)**	**Price (ρ_{mkt})**	**Relative error (%)**
$[0\% - 3\%]$	0.301884	0.301884	0.0000
$[0\% - 6\%]$	0.472025	0.486661	3.0075
$[0\% - 9\%]$	0.573016	0.589503	2.7967

Table 4.1: Relative errors between LHP prices with ρ_{mkt} and ρ_e for all tranches

- **Tranche price sensitivity.** As shown in *Table 4.1*, using the equity-implied correlation ρ_e across all tranches introduces non-negligible pricing errors. This occurs because each tranche reacts differently to changes in correlation, as illustrated in **Figure 4.1**.

  The equity tranche, by construction, shows no pricing error since ρ_e is calibrated to match its market price. Moreover, it exhibits the highest sensitivity to ρ, as it absorbs the first losses and is most affected by changes in tail risk: increasing ρ raises the likelihood of joint defaults, amplifying the impact on the equity tranche value.

- **Convergence at high correlation.** As $\rho \to 1$, all tranche prices converge. In this limit, defaults are perfectly correlated, leading to an all-or-nothing loss distribution. The tranche structure becomes irrelevant: either no losses occur, or the entire portfolio defaults—affecting all tranches equally.

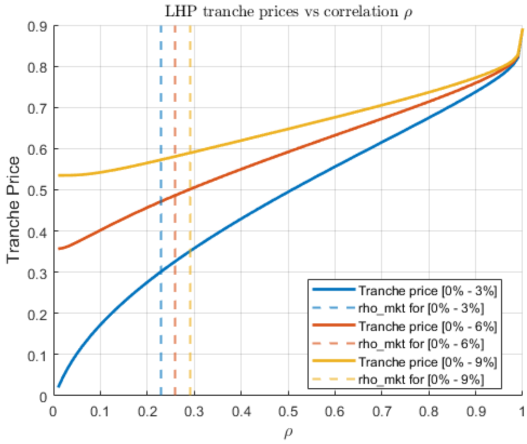

Figure 4.1: Tranche prices with respect to ρ

Chapter 5
Pricing using the Gaussian Copula

5.1 Introduction: Copulas in Finance

Copulas are widely used in finance to model default dependence, as they separate the joint distribution from the marginals.

In our analysis, we first price CDO tranches using a Gaussian copula via Monte Carlo simulation,[1] employing a constant correlation ρ_{model} calibrated from the double t-Student model (**Section 1.2**). We then compare these model-based prices to those obtained using market-implied correlations ρ_{market}.
To assess the impact of different dependence structures, we extend the framework to the double t-copula and the Clayton copula (from the Archimedean family). Each copula is calibrated to match the market price of the equity tranche, after which we price the full set of tranches and computed the implied correlations by inverting the Vasicek model, following the method introduced in **Chapter 3**.

5.2 Gaussian Copula

For the Gaussian copula, in particular, we also test an alternative approach using the market-implied correlations ρ_{market} directly in the simulation. The Monte Carlo simulation is based on the same latent variable formulation introduced in Equation (1.1):

$$v_i = \sqrt{\rho} \cdot y + \sqrt{1-\rho} \cdot \varepsilon_i,$$

where both the systemic factor y and the idiosyncratic components ε_i are simulated as independent standard normal random variables. For each simulation, we count the number of defaults across the portfolio and compute the corresponding tranche losses, from which the expected tranche prices are derived.

Moreover, as in the subsequent copula models, we observe that the calibration procedure yields a stable and well-defined solution for the optimal correlation parameter ρ, further supporting the robustness of the methodology.

[1] An efficient alternative to Monte Carlo simulation is discussed in Hull and White (2004), where the authors develop two fast procedures for pricing CDO tranches under a factor copula model.

All prices obtained via Monte Carlo simulation are reported along with their 95% confidence intervals. A total of 10^6 simulations is used when sampling standard normal variables, as they are faster to generate and allow for higher precision. For the t-Student and Gamma distributions, which are significantly slower to sample from, we use 10^4 simulations instead.

Observation. As shown in *Table 5.1*, the mean squared error (MSE) between the model prices and those generated by the HP Vasicek model—used here as a market proxy—decreased significantly when using ρ_{market} instead of ρ_{model}, confirming the importance of an accurate correlation input.

Tranche	Gaussian Copula (ρ_{model})	Gaussian Copula (ρ_{market})	Exact HP
0–3%	[0.3493, 0.3500, 0.3506]	[0.3071, 0.3078, 0.3084]	0.3076
0–6%	[0.4992, 0.4998, 0.5005]	[0.4884, 0.4890, 0.4896]	0.4887
0–9%	[0.5880, 0.5885, 0.5891]	[0.5901, 0.5907, 0.5913]	0.5904

MSE (Gaussian Copula with ρ_{model} vs. Exact): 6.40×10^{-4}
MSE (Gaussian Copula with ρ_{market} vs. Exact): 6.14×10^{-8}

Table 5.1: Comparison of Gaussian Copula (with ρ_{model} and ρ_{market}) vs. Exact HP model.

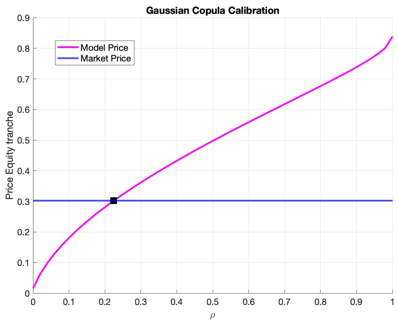

Figure 5.1: Stability of solution: Gaussian Copula

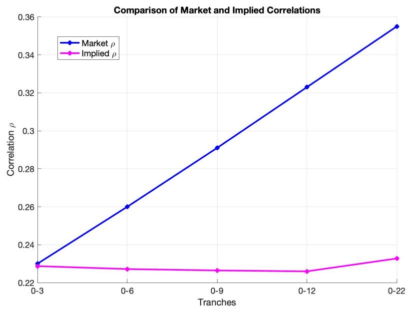

Figure 5.2: Implied ρ:Gaussian Copula

5.3 Double t-Student Copula

We now perform a Monte Carlo simulation under the double t-Student model, using the same latent variable framework as in Equation (1.1). In this case, both the systemic factor and the idiosyncratic components are simulated from a standardized t-distribution with ν degrees of freedom, as calibrated in **Section 1.2**. The pricing procedure remains the same as for the Gaussian copula: defaults are counted across scenarios and used to compute the expected tranche losses.

Observation. As shown in *Table 5.2*, the results exhibit the opposite behavior compared to the Gaussian copula: tranche prices are closer to the market benchmark when using the correlation ρ_{model} calibrated from the double t-Student model, rather than the market-implied correlation ρ_{market}. This is expected, as the market-implied correlations are specifically consistent with the Gaussian copula framework, and do not necessarily provide accurate inputs for a model with heavier tails such as the double t-Student.

Tranche	t-Copula (ρ_{model})	t-Copula (ρ_{market})	Exact HP
0–3%	[0.2927, 0.2986, 0.3045]	[0.2524, 0.2579, 0.2634]	0.3076
0–6%	[0.4772, 0.4829, 0.4887]	[0.4668, 0.4725, 0.4781]	0.4887
0–9%	[0.5788, 0.5839, 0.5891]	[0.5808, 0.5860, 0.5912]	0.5904

MSE (t-Copula with ρ_{model} vs. Exact): 5.22×10^{-5}
MSE (t-Copula with ρ_{market} vs. Exact): 9.20×10^{-4}

Table 5.2: Comparison of t-Copula (with ρ_{model} and ρ_{market}) vs. Exact HP model.

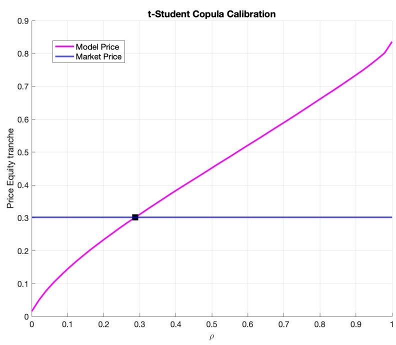

Figure 5.3: Stability of solution. t-Copula

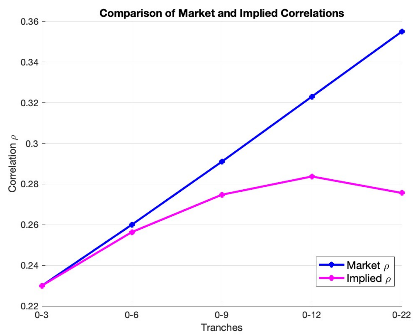

Figure 5.4: Implied ρ: t-Copula

5.4 Archimedean Copula — Clayton

As a final experiment, we consider an Archimedean copula model, and in particular the Clayton copula. In general, Archimedean copulas are defined through a generator function ϕ, such that:

$$C(\mathbf{u}) = \phi^{-1}\left(\sum_{i=1}^{I} \phi(u_i)\right),$$

where $\phi : [0,1] \to \mathbb{R}^+$ is a strictly decreasing function with $\phi(0) = \infty$ and $\phi(1) = 0$. The dependence structure is fully characterized by this generator.

In the case of the Clayton copula, the generator function is:

$$\phi(t) = t^{-\theta} - 1, \quad \theta \geq 0,$$

with inverse:

$$\phi^{-1}(s) = (1+s)^{-1/\theta}.$$

The simulation procedure involves generating i.i.d. variables $\varepsilon_i \sim \mathcal{U}[0,1]$ and an independent random variable $y \sim \text{Gamma}\left(\frac{1}{\theta}, 1\right)$. The copula samples u_i are then constructed as:

$$u_i = \phi^{-1}\left(\frac{-\ln \varepsilon_i}{y}\right).$$

As in the other models, we calibrat the Clayton copula by matching the equity tranche price to its market value, optimizing over the parameter θ. Using the resulting calibrated parameter, all tranche prices are computed, and the implied correlations are derived by inverting the Vasicek model as in previous sections.

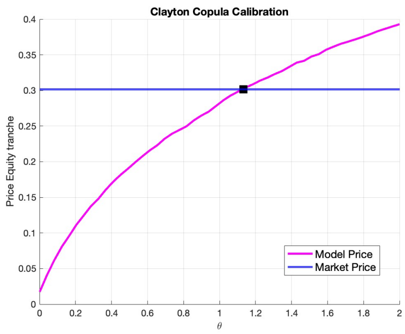

Figure 5.5: Stability of solution: Archimedean (Clayton) Copula

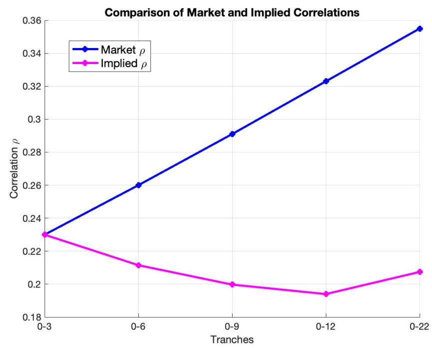

Figure 5.6: Implied ρ: Archimedean (Clayton) Copula

As an alternative to calibrate θ by matching the equity tranche price, we also test a global approach that minimizes the mean squared error (MSE) across all tranches. As expected, this leads to a model-implied correlation for the equity tranche that no longer matches the market-implied value (*Figure 5.7*), due to the shift in calibration objective.

Despite yielding a lower global MSE — as expected, since the calibration was no more designed for that purpose — we decided not to pursue this approach, in order to stay consistent with the methodology of Burtschell, Gregory, and Laurent (2009), where all tranche-implied correlations are derived from the market-implied equity correlation.

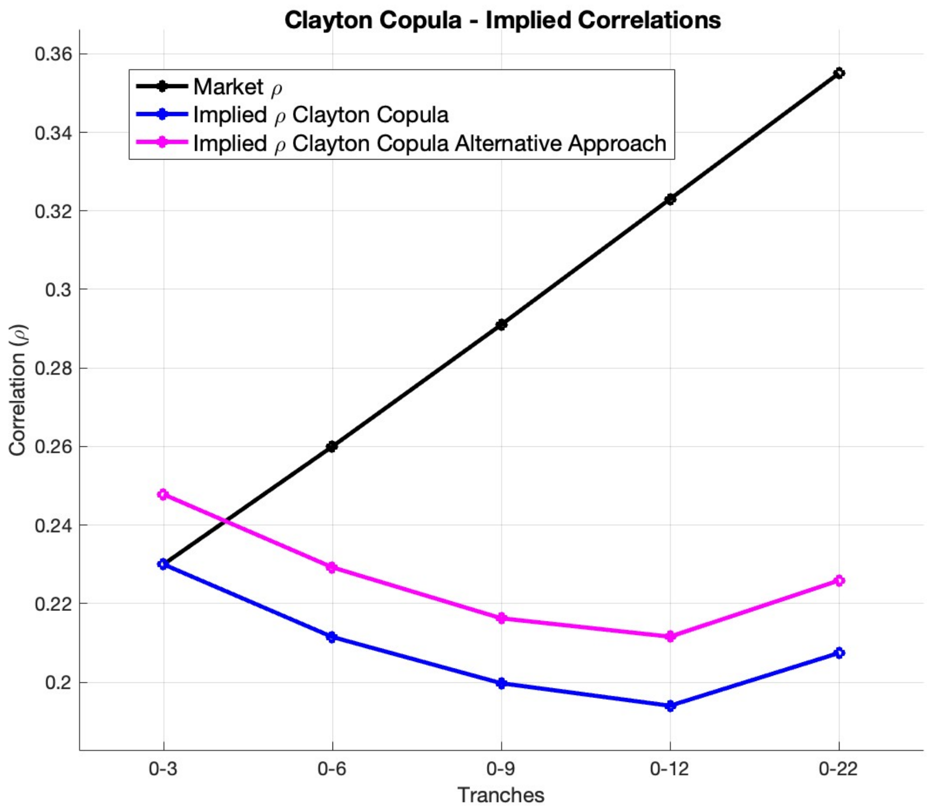

Figure 5.7: ρ implied in Archimedean (Clayton) Copula model alternative approach

Chapter 6

Conclusions

From the comparison presented in **Figure 6.1** and *Table 6.1*, it is evident that the *double t-Student copula* provides the best match to market-implied base correlations across all tranches. In particular, the semi-analytical calibration of the double t model (under the LHP approximation with a single constant correlation ρ across all tranches) aligns remarkably well with market-implied ρ values. This is a strong indication that the heavy tails of the t-distribution capture default dependence more effectively than the Gaussian benchmark when ρ is assumed constant.
Even in the more general setting using Monte Carlo simulation, the double-t copula remains the closest to market data among all models tested. This further supports the double-t copula as a robust modelling choice for pricing tranches with realistic dependence structures.

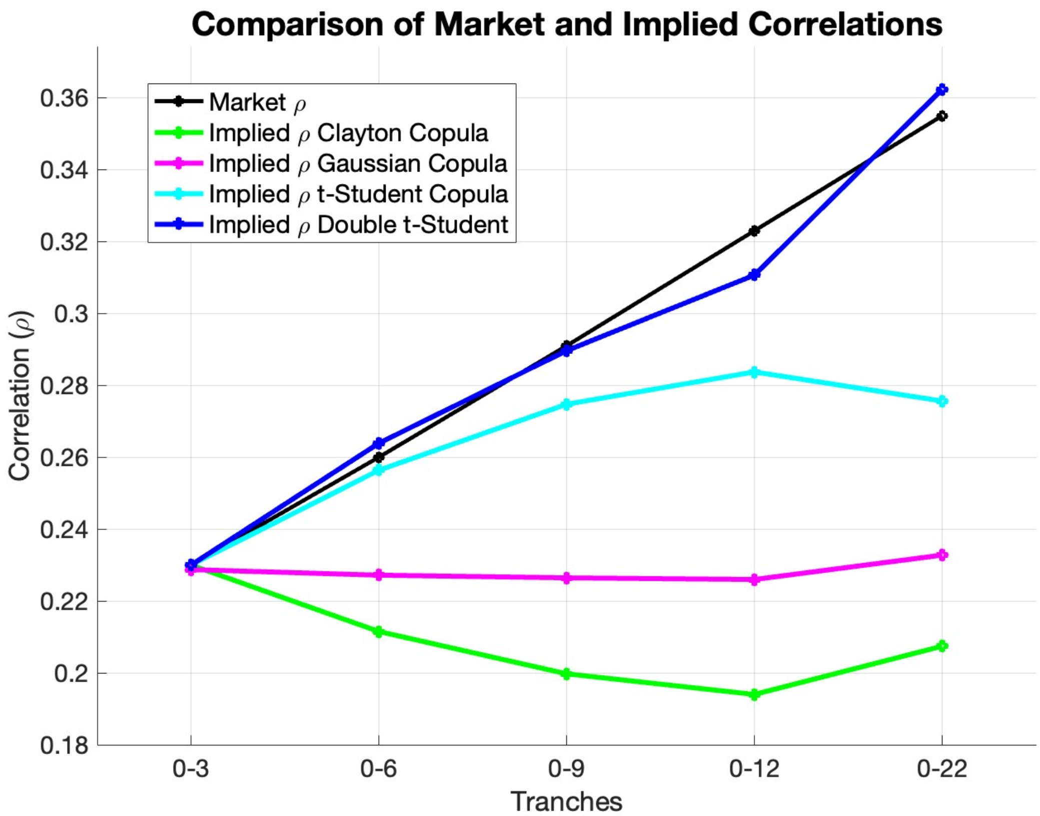

Figure 6.1: ρ implied in all models

Method	[0 - 3]	[0 - 6]	[0 - 9]
Market (Vasicek model under LHP)	**0.3019**	**0.4867**	**0.5895**
double t-Student model (calibrated under LHP)	0.3019	0.4885	0.5891
Gaussian copula (ρ from point b)	0.3500	0.4998	0.5885
Gaussian copula (implied ρ per tranche)	0.3078	0.4890	0.5907
Gaussian copula (calibrated)	0.3019	0.4711	0.5724
double t-Copula (calibrated)	0.3019	0.4849	0.5851
Archimedean Clayton copula (calibrated)	0.3019	0.4628	0.5651

Table 6.1: Tranche prices under different dependence modelling approaches

In contrast, the Archimedean Clayton copula exhibits a *decreasing* pattern of implied ρ across tranches — a clearly inconsistent trend compared to the upward-sloping market base correlation curve. Moreover, the instability in inverting the Vasicek model to extract implied ρ values under the Clayton specification (especially for the senior tranche) is evident in **Figure 6.2**, where the flatness of the objective function near the market value leads to poorly defined or unstable roots in the numerical inversion (via `fzero`).

Lastly, the Gaussian copula shows a relatively flat correlation structure when calibrated globally, and only slightly improves when ρ is implied tranche-by-tranche. Nevertheless, both Gaussian variants fail to reproduce the upward shape observed in market base correlations — confirming the known limitations of the Gaussian copula in capturing tail dependence.

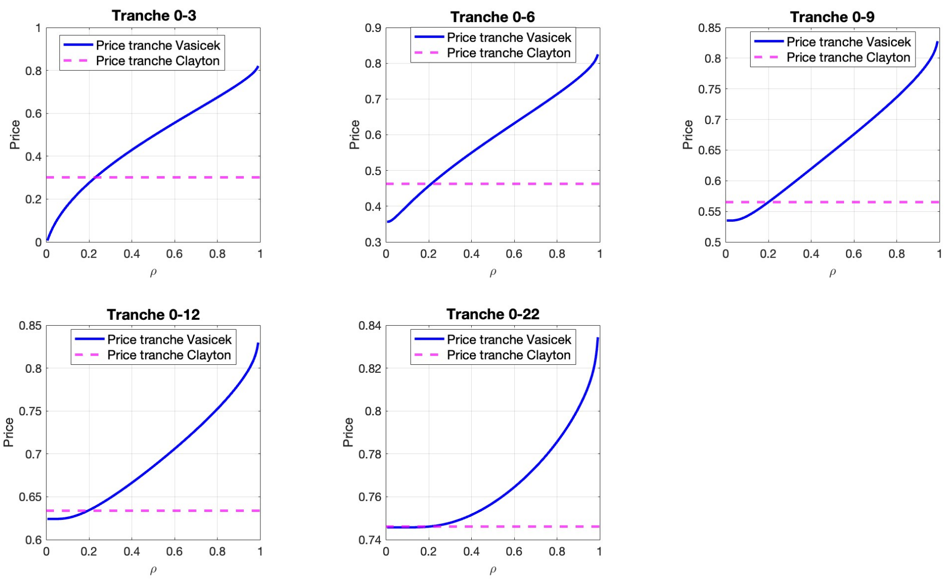

Figure 6.2: check for stability of the solution for the invertion of vasicek model

Initially, our analysis has been conducted in MATLAB, then a parallel PYTHON implementation has been developed to verify the numerical consistency of results, using an alternative programming environment. Minor and negligible differences observed are attributable to solutions' instability, variations in numerical integration techniques, internal libraries used by each language (e.g. the use of custom grids in Python versus MATLAB's `quadgk` function) and random number generation.

References

Burtschell, X., J. Gregory, and J.-P. Laurent (Feb. 2009). *A Comparative Analysis of CDO Pricing Models*. Tech. rep. This version: 20th February 2009. Working Paper.

Hull, John and Alan White (2004). "Valuation of a CDO and an n^{th} to Default CDS Without Monte Carlo Simulation". In: *Journal of Derivatives* 12.2, pp. 8–23.