浙江大学

2024～2025学年离散数学期中测验

测试方式：闭卷 测试时间：90分钟 卷面总分：100分

姓名：____________ 学号：_____________ 分数：____________.

1 A witch invites you to play a game. In the game, there are three propositions: **p,q,r**.

Game Rules:

- Each round, you can propose a compound proposition composed of:
  · Each of **p, q, r** appearing **exactly once**,
  · Logical operators:$\neg$**(NOT),** $\wedge$ **(AND),**$\vee$**(OR)**,
  · **Parentheses** may be used to explicitly group sub-expressions (e.g., $(\mathbf{p} \wedge \mathbf{q}) \vee \neg \mathbf{r}$)
  (For example, $\mathbf{p} \wedge \mathbf{q}$ is illegal because $\mathbf{r}$ does not appear;$(\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r} \wedge (\neg \mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r})$ is illegal because **p,q,r** appearing more than once.)
- The witch then answers whether the compound proposition is true or false.
- After several rounds, if you can uniquely determine the truth values of **p,q,r,** the game ends.

Assumptions:

-The witch is **perfectly logical** and will always ensure that:
· There exists at least one possible truth assignment for **p,q,r** that allows the game to end.
· She aims to **maximize the number of rounds** before you can uniquely determine the solution.
Example:
-In small example, there only two propositions:**p,q.**
Your propose Witch's answer
-Round1: $p \wedge q$ False
(Answering 'false' leaves three possible truth-value combinations for (p, q), whereas answering 'true' would only permit one. Consequently, this strategy maximizes the number of rounds in the game.)
-Round2: $\neg p \wedge q$ False
-Round3: $p \wedge \neg q$ False
-Round4: $\neg p \wedge \neg q$ True
(Now the game ends. It takes 4 rounds.You can uniquely determine the truth values of $\mathbf{p}, \mathbf{q}, : \mathbf{p}$ is False and $\mathbf{q}$ is False)

Problem:

-Provide a strategy to end the game as quickly as possible.
-Determine the minimum number of rounds required to guarantee the game ends.(10')

2 An ordered pair (a,b) can be represented as the set {a,{a,b}].Please express the following expressions in set notation(can't use ordered pair notation like (a,b)).(10')
i. $\{1\} \times \{2\}$
ii. $\{1,2\} \times \{2,3\}$
iii. $\emptyset \times \{1,2\}$
iv. $(1,2) \times \{1,2\}$
V. $\{\emptyset\} \times \{\emptyset, \{\emptyset\}\}$

3 i.Construct a bijection $(0,1) \to R$.(6')
ii. Construct an injection $(0,1) \times (0,1)$ → (0,1).(Hint: Elements of (0,1) can be represented as $0.a_1a_2...$)(4')

4 i. Express an algorithm in pseudocode to find all numbers that appear **exactly twice** in a sequence of length n (where numbers in the sequence are **no greater than 1000**) (8')
ii. **Directly write** the number of comparisons your algorithm would use in the **worst-case** (excluding comparisons in **for/while** conditions).(2')

5 Show that if $\mathbf{a}$ and $\mathbf{m}$ are relatively prime positive integers, then the inverse of $\mathbf{a}$ modulo $\mathbf{m}$ is unique modulo $\mathbf{m}$.(10')

6 Show that if $\mathbf{a}$ and $\mathbf{b}$ are both positive integers, then $(2^a - 1) mod (2^b - 1) = 2^{a mod b} - 1$.(10')

7 i. Compute 3^{2003} **mod** 7.(6')
ii. Compute the lowest digit (e.g.2 is the lowest digit of 8762; 3 is the lowest digit of 123) of $\lfloor 3^{2003} div 7 \rfloor$in base-10.(4')

8 Show that$[(\mathbf{p}_1 \to \mathbf{p}_2) \wedge (\mathbf{p}_2 \to \mathbf{p}_3) \wedge \cdots \wedge (\mathbf{p}_{n-1} \to \mathbf{p}_n)]$
$\to [(\mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \cdots \wedge \mathbf{p}_{n-1}) \to \mathbf{p}_\mathrm{n}]$ is a **tautology** (always true) using **mathematical induction** whenever $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ are propositions, where n ≥ 2.(10')

9 Which amounts of money can be formed using just two dollar bills and five-dollar bills? Prove your answer using **strong induction**.(10')

10 i. Give **iterative** and **recursive** algorithms for finding the **n**th term of the sequence defined by $\mathbf{a}_0 = 1, \mathbf{a}_1 = 3, \mathbf{a}_2 = 5, and \quad \mathbf{a}_n = \mathbf{a}_{n-1} \cdot \mathbf{a}_{n-2}^2 \cdot \mathbf{a}_{n-2}^3$.(8')

ii. Which is more efficient?(2')