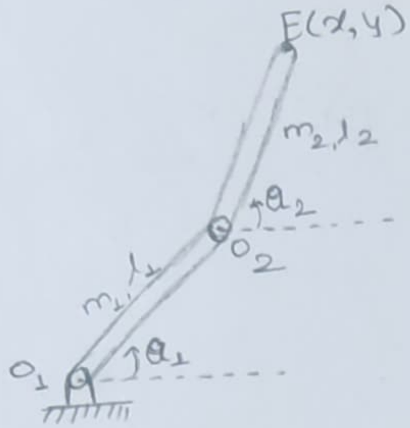


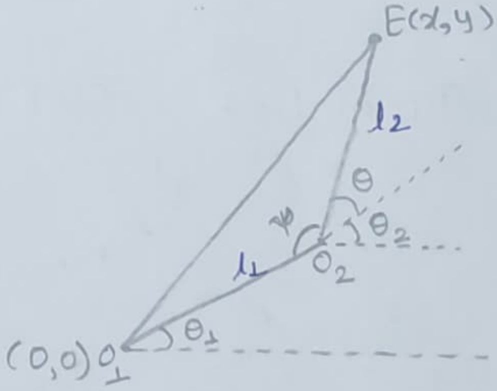
E = end effector
(x,y) = end effector position
(θ_1, θ_2) = angles w.r.t horizontal
- Assume origin at O_1
- let us assume motors are connected to each link at O_1 & O_2 respectively.

$$x = l_1 \cos\theta_1 + l_2 \cos\theta_2 \qquad \text{--- (1)}$$
$$y = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

Taking differential on both sides.

$$\dot{x} = -l_1 \dot{\theta}_1 \sin\theta_1 - l_2 \dot{\theta}_2 \sin\theta_2$$
$$\dot{y} = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 & -l_2 \sin\theta_2 \\ l_1 \cos\theta_1 & l_2 \cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \qquad \text{--- (2)}$$



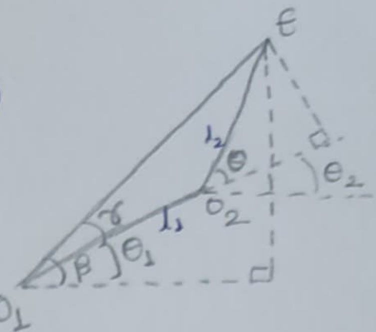
Taking inverse kinematics

In $\Delta O_1 O_2 E$,
using cosine rule,

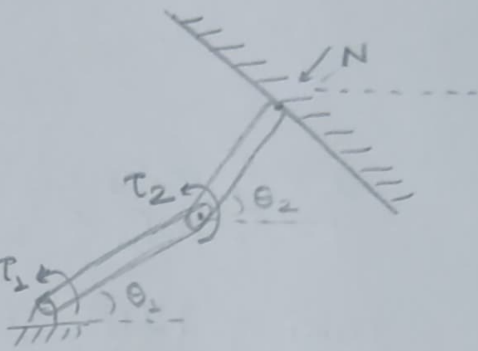
$$x^2 + y^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cos\psi$$
$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta$$
$$\cos\theta = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$
$$\theta = \cos^{-1}\left[\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right] \qquad \text{--- (3a)}$$

$$\theta_1 = \beta - \gamma$$

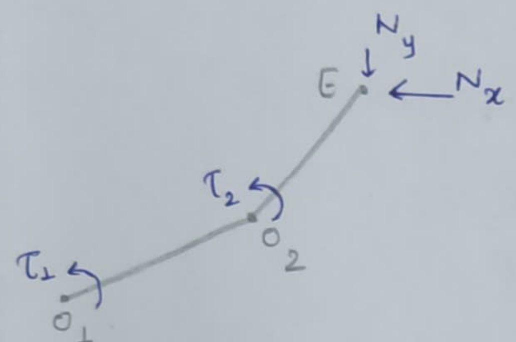
$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right) \quad \text{—— (3b)}$$



$$\theta_2 = \theta_1 + \theta \quad \text{—— (3c)}$$



FBD of 2R manipulator



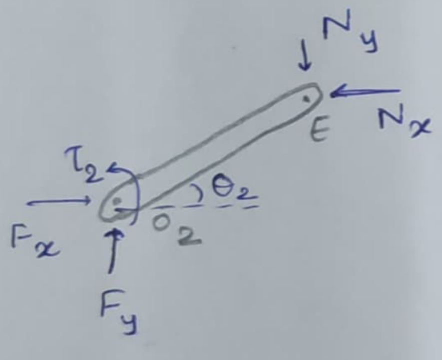
Force applied by manipulator (static equilibrium)

$F_x = -N_x$

$F_y = -N_y$

(Neglecting gravity)

FBD of Link-2 (O_2E)

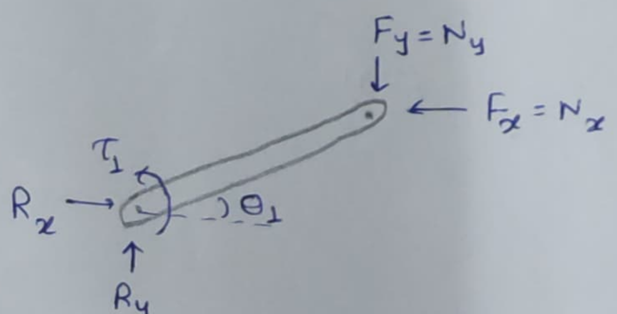


$$\Sigma M_{O_2} = 0$$

$$-N_y l_2 \cos\theta_2 + N_x l_2 \sin\theta_2 + \tau_2 = 0$$

$$\tau_2 = N_y l_2 \cos\theta_2 - N_x l_2 \sin\theta_2$$

FBD of Link-1 (O_1O_2)



$$\Sigma M_{O_1} = 0$$

$$N_y l_1 \cos\theta_1 - N_x l_1 \sin\theta_1 = \tau_1$$

$$\tau_1 = N_y l_1 \cos\theta_1 - N_x l_1 \sin\theta_1$$

$$\tau_1 = N_y l_1 \cos\theta_1 - N_x l_1 \sin\theta_1$$
$$\tau_2 = N_y l_2 \cos\theta_2 - N_x l_2 \sin\theta_2$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 & l_1 \cos\theta_1 \\ -l_2 \sin\theta_2 & l_2 \cos\theta_2 \end{bmatrix} \begin{bmatrix} N_x \\ N_y \end{bmatrix} \quad \text{--- (4)}$$

Lagrange's equation & Lagrangian

$\mathcal{L} = K - V$ $\qquad$ K = kinetic energy
V = Potential energy

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = Q_i' \quad \text{--- (5)}$$

Q_i' = Generlized force derived using principle of virtual work.

So kinetic energy given by

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right) \dot{\theta}_1^2}_{\text{K.E. of Link-1}} + \underbrace{\frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right) \dot{\theta}_2^2 + \frac{1}{2} m_2 V_{c_2}^2}_{\text{K.E. of link-2}}$$

$$V_{c_2}^2 = (l_1 \dot{\theta}_1)^2 + \left(\frac{l_2}{2} \dot{\theta}_2\right)^2 + 2 l_1 \dot{\theta}_1 \cdot \frac{l_2}{2} \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

Potential energy given by

$$V = m_1 g \frac{l_1}{2} \sin\theta_1 + m_2 g \left(l_1 \sin\theta_1 + \frac{l_2}{2} \sin\theta_2\right)$$

Putting values of potential & kinetic energy in eq^n (5) gives

$$\tau_1 = \frac{1}{3} m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + \frac{m_2 l_1 l_2}{2} \ddot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$- \frac{m_2 l_1 l_2}{2} \dot{\theta}_2 (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_2 - \theta_1) + m_1 g \cdot \frac{l_1}{2} \cos\theta_1 + m_2 g l_1 \cos\theta_1$$

$$\text{— } (6)$$

$$\tau_2 = \frac{1}{3} m_2 l_2^2 \ddot{\theta}_2 + \frac{m_2 l_2^2}{4} \ddot{\theta}_2 + \frac{m_2 l_1 l_2}{2} \ddot{\theta}_1 \cos(\theta_2 - \theta_1) - \frac{m l_1 l_2}{2} \dot{\theta}_1 (\dot{\theta}_2 - \dot{\theta}_1).$$

$$\sin(\theta_2 - \theta_1) + \frac{m_2 g l_2}{2} \sin\theta_2$$

$$\text{— } (6)$$