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Laboratory Experiment 1 - Modelling and Control of an Inverted Pendulum on a Cart

MTRN3020 - Modelling and Control of Mechatronic Systems

I verify that the contents of this report are my own work

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1. Introduction

The inverted pendulum experiment aims to model and control an inverted pendulum mounted on top of a cart subjected to a 1-dimensional force along a track [1] as depicted in Figure 1. The system will be modelled using an Euler-Lagrangian approach from which a state feedback controller will be developed based on error dynamics. The controllers' gain values will be derived using a pole placement approach based on pre-determined settling times, percentage overshoots and separation constants to ensure desired time-domain performance. The controller will be applied to the Quanser High Fidelity Linear Cart (HFLC) System to position the cart to desired locations along the track whilst maintaining the vertical position of the pendulum.

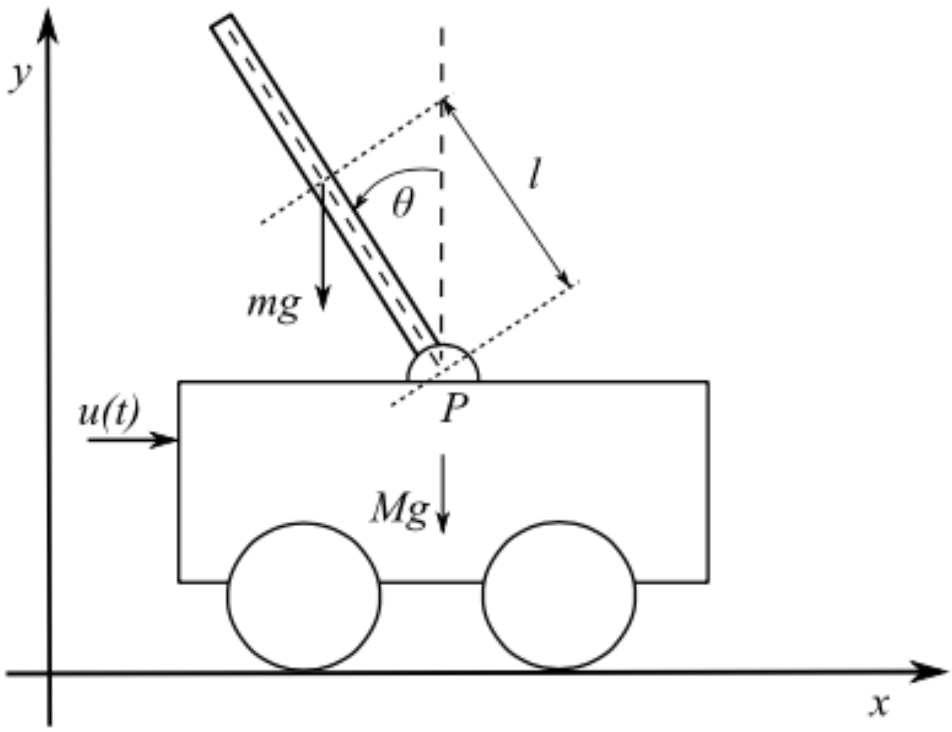


Figure 1. Inverted Pendulum Mounted on a Cart [1]

2. Aim

This experiment aims to model and control an inverted pendulum on a cart subjected to a 1-dimensional force. This will be accomplished by modelling the system using an Euler-Lagrangian approach from which the poles of the system will be calculated to meet time-domain requirements. The control gain vector will be calculated from said poles and applied to the Quanser High Fidelity Linear Cart (HFLC) system [1] to obtain both experimental and simulations results for the cart's position and pendulum angle to analyse and discuss.

3. Experimental Procedure

The following method was used to collect the experimental and simulation results:

1. Create a Control Gain Vector MATLAB Script to calculate suitable system poles based on time-domain requirements provided on Moodle.
2. Turn on the Quanser High Fidelity Linear Cart (HFLC) system, ensuring the track is free of obstacles and ensuring all power and data cables are securely fitted and free from getting caught on obstacles.
3. Load in the Control Gain Vector MATLAB script into the Quanser HFLC system.
4. Using the 'Step' and 'Step in' functions in the MATLAB Editor, step through the Control Gain Vector script with the given inputs and ensure the poles calculated are of appropriate values that will not cause damage to the Quanser HFLC system.
5. Position the cart in the centre of the rail/track with the guidance of the sensors on the Quanser HFLC system.
6. Let the pendulum come to rest in a vertically downright position.
7. Once at rest, calibrate the system to take the pendulum angle as 180-degrees.
8. Move the pendulum to the 0-degree position such that the pendulum is in vertically upright and the program begins.
9. Once the program is complete, gently bring the pendulum to rest and export the collected experimental and simulation data.

4. Derivation of Dynamic Equations

To model the complete system seen in Figure 1, the Euler-Lagrangian method will be used to derive the dynamic equations.

Taking the coordinate frame from the centre of mass of the cart, the kinematical constraints for the system are found to be,

$$\boldsymbol{X_m = X - l\sin\theta} \tag{1}$$
$$\boldsymbol{Y_m = l\cos\theta} \tag{2}$$

where the velocity is given by the time derivative of equations (1) and (2),

$$\boldsymbol{\dot{X}_m = \dot{X} - l\dot{\theta}\cos\theta} \tag{3}$$
$$\boldsymbol{\dot{Y}_m = -l\dot{\theta}\sin\theta} \tag{4}$$

The Lagrangian is defined as the difference between the kinetic energy $\boldsymbol{T}$ and potential energy $\boldsymbol{V}$,

$$\boldsymbol{L = T - V} \tag{5}$$

where the kinetic energy is the summation of the kinetic energy in the cart and pendulum which is given as,

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{X}_m{}^2 + \dot{Y}_m{}^2\right)} \\ &= \boldsymbol{\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 - 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2\,(\cos^2\theta + \sin^2\theta)\right)} \\ &= \boldsymbol{\frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}\dot{x}\cos\theta} \end{aligned} \tag{6}$$

and the potential energy is the potential energy in the pendulum as the cart has 0 potential energy, which is given as,

$$\begin{aligned} \boldsymbol{V} &= \boldsymbol{mgY_m} \\ &= \boldsymbol{mgl\cos\theta} \end{aligned} \tag{7}$$

Hence giving the Lagrangian as,

$$\boldsymbol{L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}\dot{x}\cos\theta - mgl\cos\theta} \tag{8}$$

The dynamic equations can be obtained by using the 1-D Euler-Lagrange's equation,

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \tau} \tag{9}$$

where $\mathbf{q}$ are the generalised coordinates and $\boldsymbol{\tau}$ are the generalised forces. In our model, $\mathbf{q} = \{\boldsymbol{x}, \boldsymbol{\theta}\}^{\boldsymbol{T}}$ and $\boldsymbol{\tau} = \{\boldsymbol{u}, \mathbf{0}\}^{\boldsymbol{T}}$, thus giving the two equations,

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = u} \tag{10}$$

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0} \tag{11}$$

Applying equation (10) to equation (8), the first dynamic equation is derived as,

$$\boldsymbol{(M+m)\ddot{x} - \left(ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta\right) = u}$$

$$\boldsymbol{(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = u} \tag{12}$$

Applying equation (11) to equation (8), the second dynamic equation is derived as,

$$\boldsymbol{ml^2\ddot{\theta} - ml\ddot{x}\cos\theta + ml\dot{x}\dot{\theta}\sin\theta - ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta = 0}$$

$$\boldsymbol{ml^2\ddot{\theta} - ml\ddot{x}\cos\theta - mgl\sin\theta = 0}$$

Dividing both sides by $\boldsymbol{ml}$, the second dynamic equation is simplified to,

$$\boldsymbol{l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta = 0} \tag{13}$$

5. MATLAB Script Used to Derive the Control Gain Vector

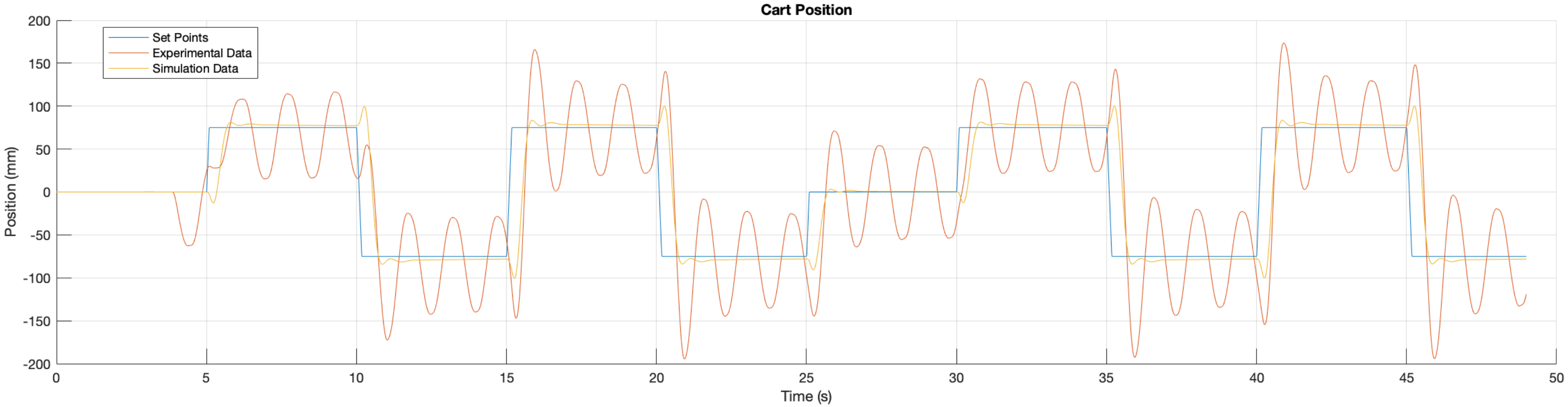
```
function K = z5260252(Ai,Bi)
% Time Domain Requirements:
P_OS = 6.16186473055022; % (%OS)
T_s = 1.12849502180248; % (secs)
Fs_1 = 2.13464519729045; % constant separation parameter
Fs_2 = 2.33213995220903; % constant separation parameter

% Equations:
DR = (-log(P_OS / 100)) / (sqrt(pi^2 + (log(P_OS / 100))^2)); % Damping ratio
NF = (-log(0.02 * sqrt(1 - (DR^2)))) / (T_s * DR); % Natural frequency
DNF = NF * sqrt(1 - (DR^2)); % Damped natural frequency

% Dominant poles:
p1 = -(DR * NF) + DNF*1j;
p2 = -(DR * NF) - DNF*1j;
% Non-dominant poles:
p3 = real(p1) * Fs_1;
p4 = real(p1) * Fs_2;
% p5 is given:
p5 = -0.1;
p = [p1, p2, p3, p4, p5];

% Derive control gains:
K = place(Ai, Bi, p);
return
```

6. Experimental and Simulation Results



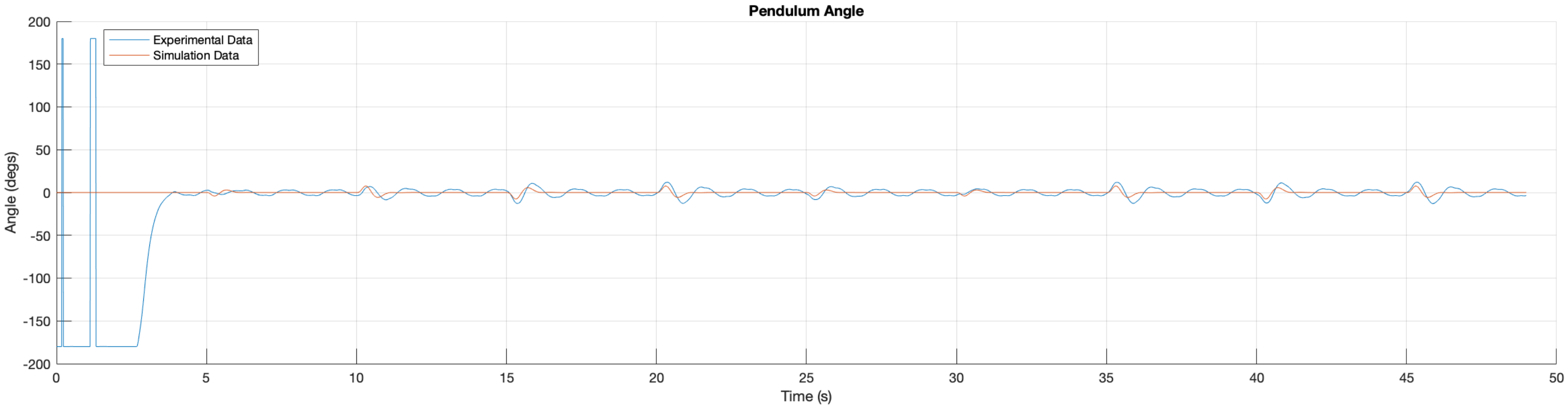


Figure 2. Experimental and Simulation Plots for Cart Position and Pendulum Angle

7. Discussion

Figure 2 shows that there is a variation between the simulated and experimental data. For the cart position, the simulation data closely follows the setpoint curve, only deviating off when the set position changes. However, the experimental data shows the cart position oscillates 50-70mm around the set position to maintain a small pendulum angle with cases of slight damping in the oscillations evident through the reducing amplitudes on the oscillations. For pendulum angle, the experimental data shows the angle varies 3-5 degrees in an oscillatory manner from the expected simulation result.

There are several factors that the systems models do not account for that may be causing these deviations. The first is friction, acting to resist the motion of the cart along the track/rail as well as the pendulum when a voltage is applied to move the cart to a setpoint. Hence, this impacts both the system's ability to respond to the current state of the system as well as the ability to reach the required setpoints. Additional friction is present in the system through the power and data cables connected to the cart that drag along the desk with varying levels of friction and slack based on the position on the track the cart is in relation to the data acquisition system. The second is the altered distance from the pivot point to the centre of mass and the mass of the pendulum. It was observed during the experiment, the pendulum rod was attached to the encoder using an additional piece of aluminium which may have altered the values of the mass of the pendulum and the position of the centre of mass. The third factor that needs to be accounted for is the backlash from the pinion of the motor [2], where the delay from the pinion's movement to the interaction with the rack on the track may lead to inaccurate data on the position of the cart as well as delayed responses to the errors in angle of the pendulum.

References

[1] J. Katupitiya and H. Jayakody, *Laboratory Experiment I : Modeling and Control of an Inverted Pendulum on a Cart,* 2022.

[2] Motors & Drives, "Separating rack and pinion myths from reality," 1 November 2011. [Online]. Available: https://www.machinedesign.com/motors-drives/article/21833478/separating-rack-and-pinion-myths-from-reality. [Accessed 20 March 2022].