Laboratory Experiment I : Modeling and Control of an Inverted Pendulum on a Cart

MTRN3020 Modelling and Control of Mechatronic Systems

1 Introduction

The aim of this experiment is to model and control an inverted pendulum mounted on a cart subjected to a 1-D force u. The pendulum-cart arrangement is shown in Fig. 1. The controller designed must be capable of positioning the cart to a desired location, while maintaining the vertical position of the pendulum. We will develop a state feedback controller based on error dynamics to achieve this goal. The gain values of the controller will be derived using a pole placement approach. The pole placement approach will ensure that the system exhibits desired time domain performance. The derived controller is then applied to Quanser High Fidelity Linear Cart (HFLC) System.

2 Aims of the Experiment

- Calculate suitable values for the system poles based on time domain requirements provided to you on Moodle.
- Calculate control gain vector $\mathbf{K}'$ based on the poles derived in 1. above.
- Apply the controller to Quanser High Fidelity Linear Cart (HFLC) system and record the cart position and pendulum angle outputs.

3 Mathematical Modelling of Inverted Pendulum and Cart System

We will use the knowledge we gathered in Exercise 2: Modeling and Control of an Inverted Pendulum to develop the dynamic model of the inverted pendulum and cart system. Either Euler-Lagrange approach or Newton-Euler approach can be used to derive the dynamic equations of the system.

For simplicity, we assume that the total mass of the pendulum is concentrated at the centre of gravity of the pendulum. This assumption would lead to the following dynamic equations.

$$(M+m)\ddot{x} + ml\dot{\theta}^2 \sin\theta - ml\ddot{\theta}\cos\theta = u \tag{1}$$

$$l\ddot{\theta} - g\sin\theta - \ddot{x}\cos\theta = 0 \tag{2}$$

Note that the pendulum is not expected to deviate very much from zero and the pendulum velocities can also be considered small. This simplifies (1) and (2) to,

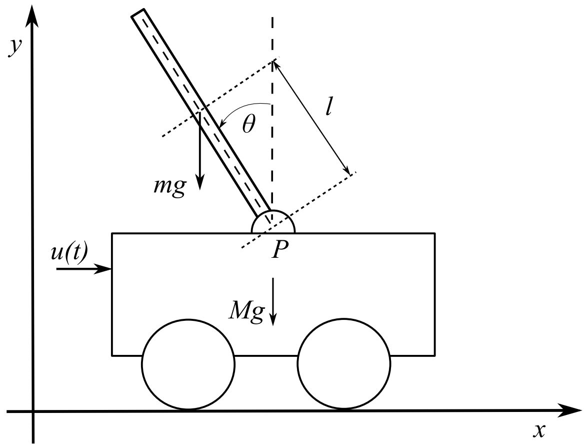

Figure 1: An inverted pendulum on a cart

$$(M+m)\ddot{x} - ml\ddot{\theta} = u \tag{3}$$

$$l\ddot{\theta} - g\theta - \ddot{x} = 0 \tag{4}$$

Rearranging (3) and (4) results in,

$$\ddot{\theta} = \frac{(M+m)g}{Ml}\theta + \frac{1}{Ml}u \tag{5}$$

$$\ddot{x} = \frac{mg}{M}\theta + \frac{1}{M}u \tag{6}$$

The control input, u, defined in these equations represents the force applied to the cart system. We control the force applied to the cart by controlling the current applied to the motor attached to the cart. Hence, in order to use the derived model in controlling the Quanser HFLC system, we need to modify the control input as,

$$u = \frac{\eta_m K_t}{r_{mp}} I_m \tag{7}$$

where η_m, K_t, r_{mp} and I_m represents efficiency of the motor, motor torque constant, motor pinion radius and input current to the motor, respectively. Therefore, the final linear dynamic equations of the inverted pendulum and cart system can be written as,

$$\ddot{\theta} = \frac{(M+m)g}{Ml}\theta + \frac{\eta_m K_t}{Mlr_{mp}} I_m \tag{8}$$

$$\ddot{x} = \frac{mg}{M}\theta + \frac{\eta_m K_t}{Mr_{mp}} I_m \tag{9}$$

3.1 State Space Model

The next step is to derive the state space model of the inverted pendulum and cart system for controller design.

Introduce state variables as follows.

$$
\begin{aligned}
x_1 &= x && (10)\\
x_2 &= \theta && (11)\\
x_3 &= \dot{x} && (12)\\
x_4 &= \dot{\theta} && (13)
\end{aligned}
$$

Note that the order of the state variables are different from Exercise 2. The states are rearranged to match Quanser HFLC software. This results in the following state space system,

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u_m && (14)\\
\mathbf{y} &= \mathbf{C}\mathbf{x} && (15)
\end{aligned}
$$

where,

$$
\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{gm}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{lm} & 0 & 0 \end{pmatrix} \tag{16}
$$

$$
\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ \frac{\eta_m K_t}{M r_{mp}} \\ \frac{\eta_m K_t}{M l r_{mp}} \end{pmatrix} \tag{17}
$$

$$
\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{18}
$$

and

$$
u_m = I_m \tag{19}
$$

This state space model is used for the controller design.

4 Controller Design

We will apply a state feedback controller designed using error dynamics to ensure stability of the pendulum and bring the cart to the desired position. The controller is developed based on the approach presented in Section 2.3.1 of Exercise 2: Modelling and Control of an Inverted Pendulum. Please refer to Exercise 2 notes to understand how the controller is developed. In this document, we will focus on deriving the final poles of the system which will help us achieve specific performance criteria. The poles we derive would help us calculate the required control gain vector $\mathbf{K}'$ (Please refer to Exercise 2: Section 2.3.1).

4.1 Pole Placement

Poles of a dynamic system affect factors such as settling time, peak time and system overshoot. In this section, we will learn to calculate the poles of the system such that we can achieve desired time domain performance criteria. Time domain performance criteria such as settling time and overshoot are commonly used in determining the poles of the system. The calculation procedure is as follows.

Let T_s, $OS\%$, ζ and ω_n be the settling time, percentage overshoot, damping ratio and the natural frequency of the final system (including the controller). The dominant poles of the system have a direct impact on T_s and $OS\%$. Hence, let us first focus on determining the dominant poles of the system by deriving relevant relationships.

The relationship between the percentage overshoot, $OS\%$, and damping ratio, ζ can be derived as,

$$\zeta = \frac{-ln(OS\%/100)}{\sqrt{\pi^2 + ln^2(OS\%/100)}} \tag{20}$$

Utilizing the value of damping ratio, ζ, and settling time, T_s, we can calculate the natural frequency, ω_n, of the system as,

$$\omega_n = \frac{-ln(0.02\sqrt{1-\zeta^2})}{T_s\zeta} \tag{21}$$

Given that the characteristic equation for the dominant poles can be expressed as,

$$s^2 + 2\omega_n\zeta s + \omega_n^2 \tag{22}$$

the dominant poles would take the form of,

$$p_1 = -\sigma + j\omega_d \tag{23}$$

$$p_2 = -\sigma - j\omega_d \tag{24}$$

where,

$$\sigma = \zeta\omega_n \tag{25}$$

and

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{26}$$

The next step is to determine the non-dominant poles of the system. The non-dominant poles should be faster than the dominant poles (i.e.: the real part of the non-dominant poles should be moved further left), and should not introduce any oscillations as the dominant poles bring the system to equilibrium. Thus, using constant separation parameters $f_{s1} > 0$ and $f_{s2} > 0$ the non-dominant poles can be calculated as,

$$p_3 = -f_{s1}.\sigma \tag{27}$$

$$p_4 = -f_{s2}.\sigma \tag{28}$$

Selecting large separation parameters would require large control inputs, so special care must be exercised in selecting these values so that they do not exceed the maximum available control input. The fifth and last pole of the system p_5 has a direct impact on the integral gain of the position error. As large integral gains could result in output oscillation, the corresponding pole, p_5 is placed close to the origin. In this experiment, we will use values between $-0.02 > p_5 > -0.1$ for p_5.

Once the values of poles are calculated you can use Matlab's '`place`' function to derive the corresponding control gains. You are required to write your own Matlab script file for this purpose. Make sure the Matlab script uses the values assigned to you. Name this file `z1234567.m` with `1234567` replaced by your student number. You will need to submit this file as a part of your report.

5 Report Content

The report must have the following. If items 1. and 2. below are not adhered to, your report will not be marked. All reports must be submitted electronically to Moodle by **11.59 pm on Monday 28 March 2022**.

1. A cover page containing title of the experiment (16pt font size centred), the course code and course name(14pt centred) and the statement "I verify that the contents of this report are my own work" in 12 pt font centred. Your electronic submission will be taken as your signature to this statement. Then mention your name(12pt), student number(12pt) and date(12pt) at the bottom right hand corner of the cover page on three separate lines (0.5 marks).
2. All sections, pages, figures and tables must be numbered. Equations may be numbered as required (0.5 mark).
3. Introduction - briefly describe what the experiment is about (1 mark).
4. Aim - what you are supposed to learn from this experiment (0.5 mark).
5. Brief description of the experimental procedure (0.5 mark).
6. Using Euler-Lagrangian method or Newton-Euler method, derive Equations (1) and (2) (2 marks)
7. Write a Matlab script to derive the control gain vector $\mathbf{K}'$. (1.5 marks)
8. Plot the results you obtained by applying your controller to Quanser HFLC system. Create separate plots for cart position and pendulum angle, where each plot contains both experimental and simulation data. (2 marks)
9. Discuss the reasons for the differences in experimental and simulation results. (1.5 marks)

6 Late submission policy

As per the School policy, you will lose 20% of the maximum marks you can get for this assignment per each calendar day.

Appendix: System Parameters

Parameter	Symbol	Value
Cart system mass	M	3.22 kg
Distance from pivot point to pendulum's centre of mass	l	0.1778 m
Pendulum mass	m	0.127 kg
Motor efficiency	μ_m	80 $\%$
motor torque constant	K_t	0.36 Nm/A
Motor pinion radius	r_{mp}	1.11×10^{-2} m