Modeling and Control of Mechatronic Systems

Exercise 2: Mathematical Modeling and Simulation of an Inverted Pendulum

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Exercise 2

Modeling and Control of an Inverted Pendulum

The aim of this exercise is to model an inverted pendulum mounted on a cart subjected to a 1-D force $u(t)$. The arrangement is shown in Fig. 2.1. The controller designed must be capable of positioning the cart to a desired location, while maintaining the vertical position of the pendulum. First, we will model the complete system. We will be using two approaches, (i) Euler-Lagrange approach and (ii) Newton-Euler approach. In the former, we will be using the Lagrangian, and in the latter we will be using the free body diagrams. As a first approach, we will design the so-called state feedback controller that will drive the *state vector* to zero. This will ensure the pendulum resting upright while the cart is positioned at zero. In the final solution, we will augment the states of the system so that instead of the state vector, the *error vector* will be driven to zero. This will allow us to move the cart to any desired position while the pendulum remains upright.

2.1 Derivation of Dynamic Equations

2.1.1 Euler-Lagrange Approach

The Lagrangian is defined as the difference between the kinetic energy and potential energy. The kinetic energy K is,

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\frac{d}{dt}(x + l\sin\theta)\right]^2 + \frac{1}{2}m\left[\frac{d}{dt}(l\cos\theta)\right]^2 \tag{2.1}$$

The potential energy P is,

$$P = mgl\cos\theta \tag{2.2}$$

Therefore, Lagrangian is,

$$L = K - P \tag{2.3}$$

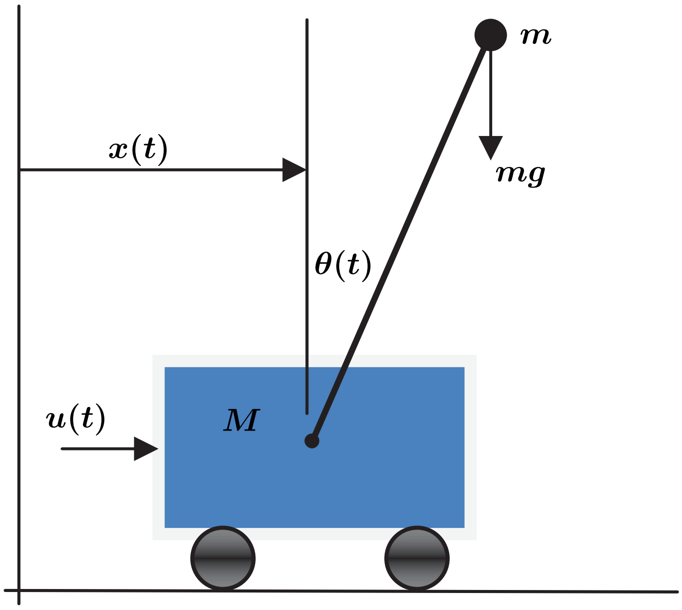

Figure 2.1: Inverted pendulum mounted on a cart

The dynamic equations can be obtained by using

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau} \tag{2.4}$$

where $\mathbf{q}$ are the so-called *generalized coordinates* and $\boldsymbol{\tau}$ are the so-called *generalized forces*. In our case $\mathbf{q} = \{x, \theta\}^T$ and the corresponding $\boldsymbol{\tau} = \{u, 0\}^T$. Thus, the two dynamic equations can be obtained by,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= u \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \end{aligned} \tag{2.5}$$

Applying the first equation of (2.5), the following can be obtained.

$$M\ddot{x} + m\ddot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = u \tag{2.6}$$

Applying the second equation of (2.5),

$$ml^2\ddot{\theta} + ml\cos\theta\ddot{x} - lmg\sin\theta = 0 \tag{2.7}$$

Obviously, these equations are non-linear time-varying. However, given the problem, a very good linear time invariant problem can be carved out of this non-linear problem. Note that the pendulum is not expected to deviate very much from zero and the pendulum velocities can also be considered small. Then the two equations can be simplified to give,

$$\begin{aligned} (M+m)\ddot{x} + ml\ddot{\theta} &= u \\ ml\ddot{\theta} + m\ddot{x} - mg\theta &= 0 \end{aligned} \tag{2.8}$$

2.1.2 Newton-Euler Approach

The same dynamic equations can be obtained by resolving forces in horizontal direction and taking moments of forces acting on the pendulum alone about the pendulum pivot point. Resolving forces horizontally for the entire system gives,

$$M\ddot{x} + m\frac{d^2}{dt^2}(x + l\sin\theta) = u \tag{2.9}$$

Taking moments about the pivot point for the pendulum gives,

$$m\left\{\frac{d^2}{dt^2}(x + l\sin\theta)\right\} l\cos\theta - m\left\{\frac{d^2}{dt^2}(l\cos\theta)\right\} l\sin\theta = mgl\sin\theta \tag{2.10}$$

When (2.9) and (2.10) are simplified, we obtain (2.6) and (2.7), respectively. They can then be simplified to obtain (2.8). By re-arranging the two equations in (2.8) we obtain,

$$\begin{aligned} Ml\ddot{\theta} &= (M+m)g\theta - u \\ M\ddot{x} &= -mg\theta + u \end{aligned} \tag{2.11}$$

which are in fact,

$$\ddot{\theta} = \frac{(M+m)g}{Ml}\theta - \frac{1}{Ml}u \tag{2.12}$$

$$\ddot{x} = -\frac{mg}{M}\theta + \frac{1}{M}u \tag{2.13}$$

2.2 State Space Approach

Introduce the state variables as follows.

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \\ x_3 &= x \\ x_4 &= \dot{x} \end{aligned}$$

Then (2.12) and (2.13) can be put into matrix form as follows,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{pmatrix} u \tag{2.14}$$

The output equation is,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.15}$$

The state equations can now be represented as,

$$\dot{\mathrm{x}} = \mathbf{A}\mathrm{x} + \mathbf{B}u \tag{2.16}$$

$$\mathrm{y} = \mathbf{C}\mathrm{x} \tag{2.17}$$

2.2.1 State Feedback Controller Design

In a state feedback controller, we feedback *all states* to generate the control input. This obviously means that we assume that all the states are available for feedback. However, in practice, it is impossible to measure all the states. Even in this pendulum case it is possible for us to put encoders to measure the cart position and the pendulum position, however, although possible, mounting sensors to measure the cart velocity and the pendulum velocity is an over kill. In such situations we will assume that a state estimator can be designed so that all states can be estimates and hence can be fed back.

The design task we undertake here is to drive the state vector to $\mathbf{0}$. Hence, this type of controllers are called *regulators* in which the set point does not change. Let the desired state vector be x_d and the state feedback matrix be $\mathbf{K}$. Therefore the error signal is $(\mathrm{x}_d - \mathrm{x})$. Therefore, the control input is $u = \mathbf{K}(\mathrm{x}_d - \mathrm{x})$. If $\mathrm{x}_d = \mathbf{0}$,

$$u = -\mathbf{K}\mathrm{x} \tag{2.18}$$

with $K = \{k_1 \quad k_2 \quad k_3 \quad k_4\}$. Substituting into (2.16),

$$\begin{aligned} \dot{\mathrm{x}} &= \mathbf{A}\mathrm{x} - \mathbf{B}\mathbf{K}\mathrm{x} \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathrm{x} \end{aligned} \tag{2.19}$$

Taking Laplace transforms of both sides,

$$[s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})]\, \mathbf{X}(s) = \mathbf{0} \tag{2.20}$$

Recall the scalar equations we have seen earlier in the course, for example,

$$D(s)X(s) = 0$$

$D(s) = 0$, represented the characteristic equation and for the system to be stable the roots of the characteristic equation must have negative real parts. In a matrix equation such as (2.20), for the system to be stable the matrix $[s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})]$ must have eigen values with negative real parts. The characteristic polynomial of (2.20) is,

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})| = 0 \tag{2.21}$$

If the desired characteristic equation is given, then by equating the coefficients of (2.21) to those of the desired characteristic equation, k_1, k_2, k_3 and k_4 can be found.

Task 1

(a). Generate the matrices $\mathbf{A}$ and $\mathbf{B}$ using (2.16) and (2.17). Use $M = 2$ kg., $m = 0.1$ kg., $l = 0.5$ m and $g = 9.81$.

(b). Design a state feedback controller that will place the closed loop poles at $-1.25 \pm j5.0$, -4.5 and -3.0. Note that these four poles are chosen to have negative real parts and therefore, the response must be stable.

You can first workout the $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices by substituting the numerical values given above. Then use the '`place`' function (type '`help place`' in Matlab command window to read about '`place`') in Matlab to get the gain values. The gain values you should get are;

$k_1 = -97.6904, \quad k_2 = -21.8740, \quad k_3 = -36.5539, \quad k_4 = -23.7481$

These $\boldsymbol{k}$ values can then be used in $\boldsymbol{u} = -\mathbf{Kx}$ to obtain the control input in your simulation.

(c). Implement the controller and plot the response of $\boldsymbol{\theta}$ for a desired value of 0 degrees. Assume an initial condition of 2 degrees for the pendulum and $x = 0$ for the cart with all velocities zero. Also plot the response of cart position.

2.3 Transfer Function Approach

Note that (2.12) has no x in it. Therefore, we can design a controller that will give stable $\boldsymbol{\theta}$ response. This may or may not give the desired response of x.

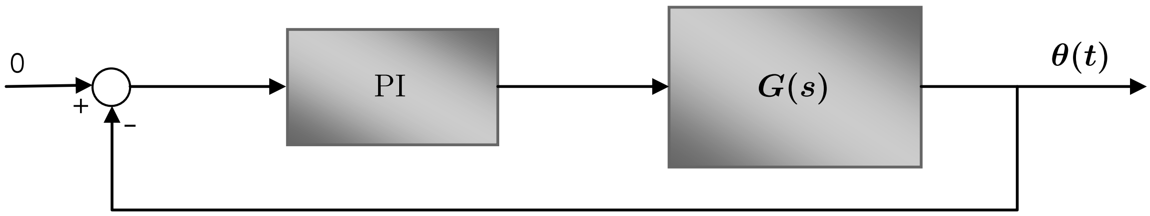

Figure 2.2: PI controller for pendulum angle only

Task 2

(a). Generate a transfer function out of (2.12). Use $M = 2$ kg., $m = 0.1$ kg., $l = 0.5$ m and $g = 9.81$.

(b). Plot root-locus.

(c). Design a controller that will give stable $\boldsymbol{\theta}$ response.

(d). Plot the response of θ for a desired value of 0 degrees. Assume an initial condition of 2 degrees for the pendulum and $x = 0$ for the cart with all velocities zero. Also plot the response of the cart.

(e). Compare the plots you obtained with those you obtained under **Task 1**

Task 2 - Solution

(a). Let $\Theta(s)$ be the Laplace transform of the pendulum position and $U(s)$ be the Laplace transform of the force applied to the cart. Using (2.12),

$$\frac{\Theta(s)}{U(s)} = \frac{-\dfrac{1}{Ml}}{\left(s^2 - \dfrac{(M+m)g}{Ml}\right)}$$

when the numerical values are substituted, this becomes,

$$\frac{\Theta(s)}{U(s)} = \frac{-1}{(s^2 - 20.601)}$$

2.3.1 Controlling Cart Position Through Error Dynamics

Instead of dynamic equations that represent states, we need to develop dynamic equations that represent error. Then we can drive error to zero just as we drove state to zero in the case we discussed earlier. Obviously for errors to be zero we need to incorporate an integrator for systems that do not have built-in integrators. In the case of the inverted pendulum system, there is no built-in integrator and hence we need to incorporate an integrator. Our approach is to keep the state feedback as it was and then to implement integral control to eliminate the steady state error in the cart position. Define a new state equation,

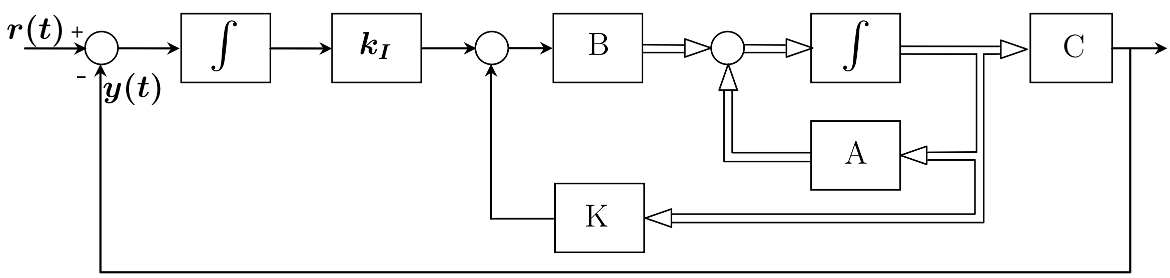

Figure 2.3: State feedback controller with error dynamics

$$\dot{\xi} = r(t) - y(t) \tag{2.22}$$

where $r(t)$ is the desired cart position and $y(t)$ is the actual cart position. Therefore, this equation represents error in the cart position. Then $\xi(t)$, is the integral of the cart position error. By summing up the state feedback control effort and the integral control effort we get,

$$u = -\mathbf{K}\mathbf{x} + k_I \xi \tag{2.23}$$

where k_I is the integral constant. The system state equations are,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + \mathbf{D}u \end{aligned} \tag{2.24}$$

Note that as $y(t)$ represents cart position only, $\mathbf{C} = [0\ 0\ 1\ 0]$ and $\mathbf{D} = \mathbf{0}$. Combining the above equation and (2.22) we can write,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ \dot{\xi} &= -\mathbf{C}\mathbf{x} + r(t) \end{aligned} \tag{2.25}$$

A combined equation can be formed as,

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \dot{\xi}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & 0 \\ -\mathbf{C} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}(t) \\ \xi(t) \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r(t) \tag{2.26}$$

Substituting steady state values,

$$\begin{pmatrix} \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(\infty) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & 0 \\ -\mathbf{C} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}(\infty) \\ \xi(\infty) \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ 0 \end{pmatrix} u(\infty) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r(\infty) \tag{2.27}$$

Subtracting (2.27) from (2.26)and noting that $r(\infty) = r(t)$ (for a step change in cart position)

$$\begin{pmatrix} \dot{\mathbf{x}}_e(t) \\ \dot{\xi}_e(t) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & 0 \\ -\mathbf{C} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ 0 \end{pmatrix} u_e(t) \tag{2.28}$$

where the subscript e refer to error in each quantity. With the error vector defined as,

$$\mathbf{e}(t) = \begin{pmatrix} \dot{\mathbf{x}}_e(t) \\ \xi_e(t) \end{pmatrix}$$

a new error dynamics equation can be written as,

$$\dot{\mathbf{e}} = \mathbf{A}'\mathbf{e} + \mathbf{B}'u_e \tag{2.29}$$

Choosing,

$$u_e = -\mathbf{K}'\mathbf{e} \tag{2.30}$$

The error dynamics become,

$$\dot{\mathbf{e}} = [\mathbf{A}' - \mathbf{B}'\mathbf{K}']\mathbf{e} \tag{2.31}$$

The matrix $\mathbf{K}'$ can be determined by equating the coefficients of the characteristic equation of error dynamics to the desired characteristic equation. Note that $\mathbf{K}'$ is,

$$\mathbf{K}' = [\mathbf{K} \quad \vdots - k_I] \tag{2.32}$$

Therefore, the control input is,

$$u(t) = -\mathbf{K}\mathbf{x}(t) + k_I \xi(t) \tag{2.33}$$

Task 3

(a). Generate the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ out of (2.24). Use $\boldsymbol{M = 2}$ kg., $\boldsymbol{m = 0.1}$ kg., $\boldsymbol{l = 0.5}$ m and $\boldsymbol{g = 9.81}$.

(b). Design a state feedback controller that will place the closed loop poles at $\boldsymbol{-1.25 \pm j5.0}$, single pole at $\boldsymbol{-4.5}$, single pole at $\boldsymbol{-3.5}$ and a single pole at $\boldsymbol{-3.0}$.

(c). Implement the controller and plot the response of $\boldsymbol{\theta}$ for a desired value of $\boldsymbol{0}$ degrees and a cart position of 1.0 m. Assume an initial condition of $\boldsymbol{2}$ degrees for the pendulum and $\boldsymbol{x = 0}$ for the cart with all velocities zero. Also plot the response of the cart.

(d). Compare the plots you obtained with those you obtained under **Tasks 1 & 2**