MATH1302 Notes-Eric Hua

Pre-knowledge

General logic:

- If you say the statement may be false, you must give an explicit counterexample.
- If you say the statement is always true, then you CANNOT use an example to justify your response. You must give a clear explanation that works in all cases.

True/False: The line $L_1: \ x+y=2$ and the line $L_2: \ x+y=3$ are different.

True/False: The line $L_1: \ x+y=2$ and the line $L_2: \ x+y=3$ are same.

Set: A set is an unordered collection of objects.

- $\mathbb{Z}$ = the set of all integers.
- $\mathbb{R}$ = the set of all real numbers.
- Empty set $\emptyset$ or $\{ \ \ \}$.

For example, the set of integers greater than 1 and less than 5: $\{2,3,4\}$, or $\{2,4,3\}$, or ...

Elements of a set: If S is a set and x is an object in the set S, we write $x \in S$. If x is not in S, then we write $x \notin S$. We refer to the objects in a set as its elements.

Subset: A Subset T of a set S is another set which contains some of elements of the set S, we write $T \subset S$.

For example, let
$M = \{\text{MAT1302E students}\}$,
$A = \{\text{MAT1302E students with grade A+}\}$.
Then $A \subset M$.

Topic 1: Systems of Linear Equations

The basic idea of linear algebra is to solve systems of linear equations.

Example 1. *David inherited \$50,000 and invested part of it in a money market account which pays 6% annually, and part in a mutual fund which pays 5% annually. After one year, he received a total of \$2,700 in simple interest from the two investments. Find the amount David invested in each category.*

Solution: Let
$x =$ The amount of money invested in the money market account.
$y =$ The amount of money invested in a mutual fund.
Then

$$\begin{array}{rcl} x + y & = & 50000 \\ 0.06x + 0.05y & = & 2700 \end{array}$$

The solution is $x = 20,000$, $y = 30,000$.

Definition 1. *A **linear equation** in variables $x_1, x_2, \ldots, x_n$ has the form*

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

*where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the equation's coefficients and $b \in \mathbb{R}$ is a constant. An n-tuple $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ is a **solution** of, or satisfies, that equation if substituting the numbers $s_1, \ldots, s_n$ for the variables gives a true statement: $a_1s_1 + a_2s_2 + \ldots + a_ns_n = b$.*

*A **system of linear equations***

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations in the system.

Finding the set of all solutions is solving the system.

Example 2. *The ordered pair $(-1, 5)$ is a solution of this system.*

$$\begin{array}{ccccc} 3x_1 & + & 2x_2 & = & 7 \\ -x_1 & + & x_2 & = & 6 \end{array}$$

In contrast, $(1, 2)$ is not a solution.

Definition 2. *If we have two linear systems and they have the same solution set then the two linear systems are called **equivalent**.*

Theorem 1. *The linear system has,*

1. *no solution*
2. *one solution*
3. *infinitely many solutions.*

In case 1, the linear system is called inconsistent. In case 2 or 3, the linear system is called consistent.

Example 3. *The system*

$$\begin{array}{rcrcr} 3x_1 & + & 2x_2 & = & 7 \\ -x_1 & + & x_2 & = & 6 \end{array}$$

has only one solution (-1,5).

Example 4. *The system*

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 7 \\ -2x_1 & - & 4x_2 & = & -14 \end{array}$$

has infinite solutions (7-2k, k).

Example 5. *The system*

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 7 \\ -2x_1 & - & hx_2 & = & k \end{array}$$

has no solution when $h = 4$ and $k \neq -14$; one solution when $h \neq 4$; infinite solutions when $h = 4$ and $k = -14$.

Matrices

Definition 3. *An $m \times n$ (m by n)* matrix *A with m rows and n columns with entries in $\mathbb{R}$ is a rectangular array of the form*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where $\forall (i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}, \quad a_{ij} \in \mathbb{R}$.

As a shortcut, we often use the notation $A = [a_{ij}]$ to denote the matrix A with entries a_{ij}. Notice that when we refer to the matrix we put parentheses—as in "$[a_{ij}]$," and when we refer to a specific entry we do not use the surrounding parentheses—as in "a_{ij}."

Example 6.

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

is a 2×3 matrix and

$$B = \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$$

is a 3×2 matrix.

To solving linear systems, we put all the coefficients of each variable aligned in columns to get the **coefficient matrix**. By adding an additional column to the coefficient matrix consisting of the values on the right hand side of the equal sign to give the **augmented matrix**.

Example 7. *Consider this linear system*

$$\begin{array}{ccccccc} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

Solution:

$$\text{coefficient matrix} = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 5 & -2 \\ 1/3 & 2 & 0 \end{bmatrix}, \quad \text{augmented matrix} = \begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{bmatrix}.$$

Topic 2: Row reduction and echelon forms

A **leading entry of a row**: is the leftmost, nonzero entry in the row (nonzero row).

Definition 4. *A rectangular matrix is in* ***echelon form (EF)*** *if it has the following three properties.*

1. All non zero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zero.

Definition 5. *Given a matrix in echelon form, if it satisfies the following two conditions then it is in reduced echelon form* ***(REF)***

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

Example 8. *Classify row echelon form, reduced row echelon form, or not in echelon form:*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, E = \begin{bmatrix} 2 & 3 & 4 & 0 & 1 \\ 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, G = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Elementary row operations: There are three types of elementary row operations.

1. Replacement: Replace one row by the sum of itself and the multiple of another. We write "$R_i \to R_i + kR_j$"

2. Interchange: Interchange two rows. We write "$R_i \leftrightarrow R_j$"

3. Scaling: Multiply all entries in a row by a non zero constant. We write "$R_i \to kR_i$"

Definition 6. *Two matrices are row equivalent if one matrix can be transformed into another matrix by a sequence of elementary row operations.*

We are interested in performing row operations until one of these two matrix structures arises.
Uniqueness of the Reduced Echelon Form: Each matrix is row equivalent to one and only one reduced echelon matrix.

Example 9. *Consider the following matrix:*

$$\begin{bmatrix} 1 & h-1 & 1 & 0 & 1 \\ 0 & h & 1 & 0 & 1 \\ 0 & 0 & h-1 & 1 & 0 \end{bmatrix}$$

1) Find h such that the matrix is in reduced row echelon form;
2) Find h such that the matrix is in echelon form but not in reduced row echelon form;
3) Find h such that the matrix is not in row echelon form.

Solution: 1) $h = 1$. 2) $h \neq 0, 1$. 3) $h = 0$.

Definition 7. *A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.*

Algorithm to compute E.F.
Step 1. If there are no nonzero columns left, then stop.
Step 2. Determine the leftmost non-zero column.
Step 3. The top position of this column is a pivot position. If this pivot position contains a zero, use an interchange to make it nonzero.
Step 4. Use elementary row operations to put zeros (strictly) below the pivot position.
Step 5. If there are no more non-zero rows (strictly) below the pivot position, then stop.
Step 6. Apply Steps 2-5 to the submatrix consisting of the rows that lie (strictly below) the pivot position.
Step 7. The resulting matrix is in echelon form.

Algorithm to compute R.E.F. or REF
Step 8. Start from the rightmost pivot position.
Step 9. If the value of the pivot position is not equal to 1, use a scaling of its row to make it equal to 1. Use row operations to make all entries above the pivot position (in the same column) equal to zero.

Step 10. Repeat the previous step for the pivot position that was found immediately before the current pivot position.

Example 10. *Carry the following matrix to (1) Echelon form; (2) reduced echelon form; (3) Find the pivot positions and pivot columns.*

$$A = \begin{bmatrix} 0 & 2 & 6 & 3 & 10 & 4 \\ 3 & 0 & 6 & 0 & -3 & 9 \\ 3 & 2 & 12 & 1 & 11 & 11 \end{bmatrix}.$$

Solution: EF:

$$A = \begin{bmatrix} 0 & 2 & 6 & 3 & 10 & 4 \\ 3 & 0 & 6 & 0 & -3 & 9 \\ 3 & 2 & 12 & 1 & 11 & 11 \end{bmatrix} \xrightarrow{R_1 \to R_2} \begin{bmatrix} 3 & 0 & 6 & 0 & -3 & 9 \\ 0 & 2 & 6 & 3 & 10 & 4 \\ 3 & 2 & 12 & 1 & 11 & 11 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 3 & 0 & 6 & 0 & -3 & 9 \\ 0 & 2 & 6 & 3 & 10 & 4 \\ 0 & 2 & 6 & 1 & 14 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 3 & 0 & 6 & 0 & -3 & 9 \\ 0 & 2 & 6 & 3 & 10 & 4 \\ 0 & 0 & 0 & -2 & 4 & -2 \end{bmatrix}$$

REF:

$$\xrightarrow{R_3 \to \frac{1}{-2} R_3} \begin{bmatrix} 3 & 0 & 6 & 0 & -3 & 9 \\ 0 & 2 & 6 & 3 & 10 & 4 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_3} \begin{bmatrix} 3 & 0 & 6 & 0 & -3 & 9 \\ 0 & 2 & 6 & 0 & 16 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 3 & 0 & 6 & 0 & -3 & 9 \\ 0 & 1 & 3 & 0 & 8 & 1/2 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{3} R_1} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 3 \\ 0 & 1 & 3 & 0 & 8 & 1/2 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Theorem 2. *(Existence and Uniqueness Theorem) A linear system is consistent $\Leftrightarrow$ the rightmost column of the augmented matrix is not a pivot column.*

Properties:

- The augmented matrix of a consistent linear system is row equivalent to a matrix with the last non-zero row

$$\begin{bmatrix} \cdots & * & * \end{bmatrix}.$$

- The augmented matrix of a inconsistent linear system is row equivalent to a matrix with the last non-zero row

$$\begin{bmatrix} 0 & \cdots & 0 & * \end{bmatrix}.$$

Example 11. *Solve the following system by using elementary row operations:*

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & = & -2 \\ & & y & + & 3z & = & 7 \\ x & & & - & 5z & = & -1 \end{array}$$

Solution: We can start by going to echelon form:

$$\text{augmented matrix} = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 1 & 0 & -5 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

The last column is a pivot column, so the system is inconsistent.

Example 12. *Solve the following system by using elementary row operations:*

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 4 \\ & & 2x_2 & + & 2x_3 & = & 8 \end{array}$$

Solution: The general solution is

$$\begin{array}{ccc} x_1 & = & -7 - x_3 \\ x_2 & = & 4 - x_3 \\ x_3 & = & free \end{array}$$

Remark: The description of the solution above is called parametric description of the solution sets.

Definition 8. *Variables corresponding to pivot columns in the matrix are called basic variables; other variables are called free variables.*

Remark: In the example above, x_1, x_2 here are basic variables, x_3 is called a free variable.

Remark: For a consistent system, if it has free variables, it has infinite solutions; if no free variable, then only one solution.

Topic 3: Vector Equations, linear combinations, span

Definition 9. *A matrix with one column is called a column vector and a matrix with one row a row vector.*

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, x_1 \in \mathbb{R}, \cdots x_n \in \mathbb{R} \right\}.$$

A vector in $\mathbb{R}^n$ is denoted by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Example 13.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [\vec{a}_1 \ldots \vec{a_n}],$$

where $\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ldots, \vec{a_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m$ *are called column vectors.*

Definition 10. *Vector addition and scalar multiplication:*

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}, \quad k \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix}, k \in \mathbb{R}.$$

Properties: For $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^m$ and scalars c, d,

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + 0 = \vec{u} = 0 + \vec{u}$
4. $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = 0$ and $-\vec{u} = (-1)\vec{u}$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1\vec{u} = \vec{u}$

Definition 11. *Given a set of vectors* $\vec{v}_1, \cdots, \vec{v}_m$ *in* $\mathbb{R}^n$ *and scalars* $x_1, \ldots,$ x_m*, if*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{y},$$

then $\vec{y}$ *is called a linear combination of* $\vec{v}_1, \ldots, \vec{v}_m$*.*

Definition 12.

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b}$$

is called vector equation.

Sometimes we may be given a set of vectors and we want to express one as a linear combination of the others. In this situation our object is to find the coefficients that allow us to do this.

Example 14. *Let*

$$\vec{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \vec{v} = \begin{bmatrix}2\\3\\4\end{bmatrix}, \quad \vec{w} = \begin{bmatrix}-1\\2\\-5\end{bmatrix}, \quad \vec{y} = \begin{bmatrix}-4\\-2\\-10\end{bmatrix}.$$

Can $\vec{y}$ be a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$?

Solution: $\vec{y}$ is a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$ $\Leftrightarrow$ the vector equation

$$x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{y}$$

is consistent.

The augmented matrix of the vector equation:

$$\text{augmented matrix} = \begin{bmatrix}1 & 2 & -1 & -4\\2 & 3 & 2 & -2\\3 & 4 & -5 & -10\end{bmatrix} \xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-2R_1} \begin{bmatrix}1 & 2 & -1 & -4\\0 & -1 & 4 & 6\\0 & -2 & -2 & 2\end{bmatrix}$$

$$\xrightarrow[R_3\to R_3-2R_2]{R_1\to R_1+2R_2} \begin{bmatrix}1 & 0 & 7 & 8\\0 & -1 & 4 & 6\\0 & 0 & -10 & -10\end{bmatrix} \xrightarrow{R_3\to\frac{1}{-10}R_3} \begin{bmatrix}1 & 0 & 7 & 8\\0 & -1 & 4 & 6\\0 & 0 & 1 & 1\end{bmatrix}$$

$$\xrightarrow[R_2\to R_2-4R_3]{R_1\to R_1-7R_3} \begin{bmatrix}1 & 0 & 0 & 1\\0 & -1 & 0 & 2\\0 & 0 & 1 & 1\end{bmatrix} \xrightarrow{R_2\to -R_2} \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & -2\\0 & 0 & 1 & 1\end{bmatrix}.$$

$$\Rightarrow \vec{y} = \vec{u} - 2\vec{v} + \vec{w}.$$

Definition 13. *Given a set of vectors $\vec{v}_1, \cdots, \vec{v}_m$ in $\mathbb{R}^n$,*

$$Span\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m; \quad \text{where } c_1, c_2, ..., c_m \text{ are scalars}\}.$$

Example 15. *Are* $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ *in span*$\{\begin{bmatrix}1\\2\\3\end{bmatrix}\}$?

Solution:

Example 16. *span*$\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\} = \mathbb{R}^2$?

Solution:

Topic 4: Matrix equations

Definition 14. *Let* $A = [\vec{a}_1 \dots \vec{a}_n]$, $\quad \vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \dots, \vec{a_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}, \in \mathbb{R}^m, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. *Define*

$$A\vec{x} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n,$$

which is the linear combination of the columns of A.

$$A\vec{x} = \vec{b}$$

is called matrix equation.

Example 17. *The linear system :*

$$\begin{aligned} x_1 - x_4 &= 1 \\ x_1 - 2x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ x_2 + 5x_3 &= 4 \end{aligned}$$

$$\Leftrightarrow$$

The vector equation:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 3 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\Leftrightarrow$$

The matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Theorem 3. *Let* A *be an* $m \times n$ *matrix. Then the following statements are equivalent, (either all true or all false).*

1. *For each* $\vec{b} \in \mathbb{R}^m$, *the equation* $A\vec{x} = \vec{b}$ *has a solution.*
2. *Each* $\vec{b} \in \mathbb{R}^m$ *is a linear combination of the columns of* A.
3. *The columns of* A *span* $\mathbb{R}^m$.
4. A *has a pivot position in every row.*

Example 18. *Let* $A = [\vec{u}\ \vec{v}\ \vec{w}]$, *where*

$$\vec{u} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}.$$

Is the columns of A *span* $\mathbb{R}^3$, *i.e.,* $Span\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$?

Solution: Yes, since $A = [\vec{u}\ \vec{v}\ \vec{w}]$ has pivot in each row.

Properties: Some properties of matrix and vector multiplication are, if A is an $m \times n$ matrix, $\vec{u}, \vec{v} \in \mathbb{R}^n$, and c is a scalar, then,

1. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
2. $A(c\vec{u}) = c(A\vec{u})$

Example 19. *Let* $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. *Describe* $\vec{b}$ *for which* $A\vec{x} = \vec{b}$ *has a solution.*

Solution:

$$[A|\vec{b}] = \left[\begin{array}{cccc} 1 & -3 & -4 & x \\ -3 & 2 & 6 & y \\ 5 & -1 & -8 & z \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 + 3R_1} \left[\begin{array}{cccc} 1 & -3 & -4 & x \\ 0 & -7 & -6 & y+3x \\ 0 & 14 & 12 & z-5x \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{cccc} 1 & -3 & -4 & x \\ 0 & -7 & -6 & y+3x \\ 0 & 0 & 0 & x+2y+z \end{array}\right].$$

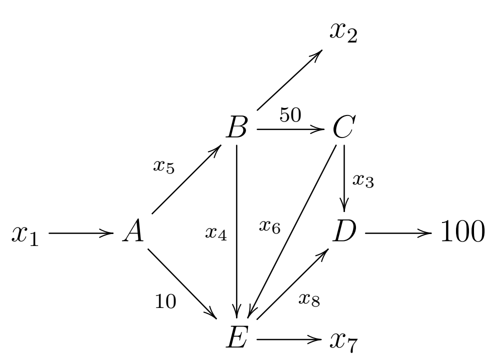
Thus, $A\vec{x} = \vec{b}$ has a solution $\Leftrightarrow x + 2y + z = 0$.

Topic 5: Applications of Linear Systems

Network Flow

Principle: flow in = flow out at each intersection.

Example 20. *Consider the traffic flow described by the following diagram. The letters A through E label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per minute.*



(a) Write down a linear system describing the traffic flow, i.e., all constraints on the variables $x_i, i = 1, \ldots, 8$. *(Do not solve the linear system at this stage.)*

Solution: Set "flow in" = "flow out" at each intersection:

$$\begin{array}{lrl} A & x_1 & = x_5 + 10 \\ B & x_5 & = x_2 + x_4 + 50 \\ C & 50 & = x_3 + x_6 \\ D & x_3 + x_8 & = 100 \\ E & 10 + x_4 + x_6 & = x_7 + x_8 \\ \text{Total} & x_1 & = x_2 + x_7 + 100 \end{array}$$

Constraints: x_i $(i = 1, ..., 8)$ are non-negative integers.

(b) The reduced row echelon form of the augmented matrix corresponding to the linear system in part (a) is:

$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & -90 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 40 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find the general flow pattern (i.e., the general solution of the linear system).

Solution: x_1, x_2, x_3, x_4, x_6: basic; x_5, x_7, x_8: free.

$$\begin{cases} x_1 = x_5 + 10 \\ x_2 = x_5 - x_7 - 90 \\ x_3 = -x_8 + 100 \\ x_4 = x_7 + 40 \\ x_5 = \text{free} \\ x_6 = x_8 - 50 \\ x_7 = \text{free} \\ x_8 = \text{free} \end{cases}$$

(c) What is the minimum and maximum number of traffic along the road ED?

Solution: Since $x_6 = x_8 - 50 \geq 0$, $x_8 \geq 50$. From $x_3 = -x_8 + 100 \geq 0$ we imply that $x_8 \leq 100$. Thus $50 \leq x_8 \leq 100$.

(d) Suppose that due to road work, the flow along ED is limited to a maximum of 70 cars per minute. What is the maximum possible flow along CE?

Solution: Since $x_6 = x_8 - 50$, from $x_8 \leq 70$ it follows that $x_6 \leq 20$, i.e., the maximum number of cars along CE is 20 cars per minutes.

Topic 6: Solution sets of linear systems, vector parametric descriptions

1. Homogeneous System

Definition 15. *The linear system $A\vec{x} = \vec{b}$ is called homogeneous if $\vec{b} = \vec{0}$. Otherwise, it is non-homogeneous. Zero vector $\vec{0}$ is always a solution of $A\vec{x} = \vec{0}$, which is called a trivial solution; any non-zero solution is called a non-trivial solution.*

Example 21. *Consider the following system of linear equations*

$$\begin{aligned} x_1 + 4x_2 - 8x_3 &= 0 \\ 2x_1 + 5x_2 - 7x_3 &= 0 \\ -3x_1 - 7x_2 + kx_3 &= 0 \end{aligned}$$

(i) Find value(s) of k such that the system has only trivial solution.
(ii) Find value(s) of k such that the system has non-trivial solutions.
(iii) For the value(s) of k in (ii), describe the solution set.

Solution: (i)

$$\text{augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 2 & 5 & -7 & 0 \\ -3 & -7 & k & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & k-24 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 5 & k-24 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & k-9 & 0 \end{bmatrix}$$

Hence, for $k \neq 9$, the system has only trivial solution.
(ii) For $k = 9$, the system has non-trivial solution.
(iii) When $k = 9$, from the discussion in (i), we have

$$\text{augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 4R_2} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned} x_1 + 4x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

i.e.,

$$\begin{aligned} x_1 &= -4t \\ x_2 &= 3t \\ x_3 &= t \ (free) \end{aligned}$$

Then the general solution is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}.$$

2. Parametric Vector Form

Whenever a solution set is described explicitly with vectors, we say that the solution is in parametric vector form.

Example 22. *Consider the following system of linear equations*

$$\begin{aligned} x_1 - 2x_2 - 9x_3 + 5x_4 &= 0 \\ x_2 + 2x_3 - 6x_4 &= 0. \end{aligned}$$

Describe the solution set in parametric vector form.

Solution:

augmented matrix $= \begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \Rightarrow$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 + 7x_4 \\ -2x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 6x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}.$$

Example 23. *Find the line through two vectors $\vec{p} = (1, 2, 3)$ and $\vec{q} = (-1, -1, 0)$. Write the result in parametric vector form.*

Solution: The general equation of the line is

$$\vec{x} = \vec{p} + t\vec{v}, \quad t \in \mathbb{R},$$

where $\vec{v} = \vec{q} - \vec{p}$ is called direction vector of the line. We have

$$\vec{x} = (1, 2, 3) + t(-2, -3, -3), \quad \text{or} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ -3 \end{bmatrix} \quad t \in \mathbb{R},$$

Theorem 4. *If $\vec{p}$ is a solution of $A\vec{x} = \vec{b}$, and $\vec{v_h}$ is a solution of $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{v_h}$ is a solution of $A\vec{x} = \vec{b}$*

Example 24. *If $\begin{bmatrix} -14 \\ -6 \\ 1 \end{bmatrix}$ is a solution of*

$$A\vec{x} = \vec{b}$$

and $t \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}$ is the general solution of the corresponding equation

$$A\vec{x} = \vec{0},$$

describe all solutions of $A\vec{x} = \vec{b}$. List several of them.

Solution: The general solution of $A\vec{x} = \vec{b}$ is:

$$\begin{bmatrix} -14 \\ -6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Example 25. *Given the augmented matrix* $\begin{bmatrix} 1 & 5 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$.

(i) Is it consistent? If yes, write the solution in parametric vector form.
(ii) Solve the corresponding homogeneous system.

Solution: (i) Yes.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 - 5x_2 \\ x_2 \\ 12 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 12 \\ -3 \end{bmatrix} + \begin{bmatrix} -5x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 12 \\ -3 \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

(ii) The homogeneous system has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Topic 7: Linear Independence

Definition 16. *A set of vectors* $\{\vec{v}_1, \cdots, \vec{v}_m\}$ *in* $\mathbb{R}^n$ *is linearly independent if the vector equation*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{0}$$

has only the trivial solution. The set is said to be linearly dependent if there is a non-trivial solution to the vector equation.

Example 26. $\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ *is linearly independent,* $\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$ *is linearly dependent.*

Solution:

Example 27. *Given* $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$. *Show that* $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ *is linearly dependent and find the non-trivial linear combination.*

Solution: Let $x\vec{v_1} + y\vec{v_2} + z\vec{v_3} = \vec{0}$. By reducing the augmented matrix, we imply that $x = 2z, y = -z, z = free$. Thus we have non-trivial linear combination. So the set is linearly dependent.

Take $z = 1$, we have $2\vec{v_1} - \vec{v_2} + \vec{v_3} = \vec{0}$.

Theorem 5. *The columns of A are linearly independent if and only if $A\vec{x} = \vec{0}$ ONLY has the trivial solution.*

Example 28. *Analyse x values such that the columns of the matrix $\begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & x-3 \end{bmatrix}$ is linearly independent or dependent.*

Solution: Independent when $x \neq 3$; dependent when $x = 3$.

Theorem 6.

1. *A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other.*
2. *A set of two or more vectors is linearly dependent if and only if at least one vector may be written as a linearly combination of the others.*
3. *If a set contains more vectors than entries in each vector, then the set is linearly dependent.*
4. *If the zero vector is in a set of vectors, then the set of vectors is linearly dependent.*

Example 29. *The set $S = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \}$ is linearly dependent .*

The set $S = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \}$ is linearly independent .

The set $S = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \}$ is linearly dependent.

The set $S = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \}$ is linearly dependent .

Topic 8: Matrix operations

- Diagonal matrix: Except for entries on diagonal (main diagonal), all other entries are 0,
- Zero matrix: all entries are 0,
- Identity matrix I_n: all entries on diagonal are 1, other entries are 0.

Scalar multiplication and addition of matrices: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, r be a number. Then

$$rA = [ra_{ij}], \quad A + B = [a_{ij} + b_{ij}].$$

Remark. You can only add matrices of the same size. Also, two matrices are equal if they are the same size and corresponding entries are equal.

Example 30.

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 0+4 & -1+5 & 1+6 \\ 1+7 & 2+8 & 3+9 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 7 \\ 8 & 10 & 12 \end{bmatrix}.$$

$$3 \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & 6 \\ 0 & 9 \end{bmatrix}.$$

Properties:

Let A, B, C be matrices of the same size and let r and s be scalars.

1. A+B = B+A
2. (A+B)+C = A+(B+C)
3. A+0 = A
4. r(A+B) = rA + rB
5. (r+s)A = rA + sA
6. r(sA) = (rs)A

Matrix multiplication: Let $A = [a_{ij}]_{m\times r}$ and $B = [b_{ij}]_{r \times n}$. Then

$$AB = [c_{ij}]_{m \times n},$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

Example 31.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 1(7)+2(9)+3(11) & 1(8)+2(10)+3(12) \\ 4(7)+5(9)+6(11) & 4(8)+5(10)+6(12) \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Remark. In order to have the product AB of two matrices A and B, the number of columns of A must equal the number of rows of B. So, if A is an $m \times r$ and B is an $s \times n$ matrix, in order to have the product AB, we need r = s. The resulting matrix AB will be an $m \times n$ matrix.

Properties of matrix multiplication: Let A, B, C be matrices for which sums and products are defined.

1. A(BC) = (AB)C (associativity)
2. A(B+C)= AB + AC (Left distributivity)
3. (B+C)A = BA + CA (Right distributivity)
4. r(AB) = (rA)B = A(rB)
5. $I_m A = A = AI_n$, here A is m x n.
6. In general, $AB \neq BA$.
7. $AB = AC$ can not imply $B = C$.

Example 32. *Let* $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 4 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$. *Then* $AB = AC$, *but* $B \neq C$.

Powers of a matrix

$$A^k = AAA \cdots A.$$

Example 33. *Let* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 6 & 5 & 3 & 0 \end{bmatrix}$. *Compute* A^2, A^3, A^4, A^{2012}.

Solution:

$$\begin{aligned} A^2 &= AA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 17 & 6 & 0 & 0 \end{bmatrix}, \\ A^3 &= A^2A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 17 & 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 6 & 5 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}, \\ A^4 &= A^3A = 0 \end{aligned}$$

Therefore $A^{2012} = A^4A^{2008} = 0A^{2008} = 0$.

Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. Compute A^2, A^3, A^4, then predict A^{2012}.

Solution:

$$
\begin{aligned}
A^2 &= AA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}, \\
A^3 &= A^2A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}, \\
A^4 &= A^3A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Therefore $A^{2012} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2012 & 1 & 0 \\ 0 & -2012 & 0 & 1 \end{bmatrix}$.

Transpose of a matrix: Given a matrix A then transpose of A is a matrix denoted by A^T, whose rows are the columns of A and whose columns are the rows of A.

Example 34.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties of Transpose:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(rA)^T = rA^T$ where r is a scalar.
4. $(AB)^T = B^TA^T$.

Example 35. *Let* $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. *Calculate* AB *and* AB^T.

Solution: AB not possible.

$$AB^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} a-b+2c & d-e+2f \\ 3a+b & 3d+e \end{bmatrix}.$$

Remark. A **and** B **commute if** $AB = BA$.

Example 36. *Find all 2 by 2 matrices that are commutative with* $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be commutative with B. Then $a = c + d$, $b = 0$.

Topic 9: The Inverse of a Matrix

Definition 17. *Given an $n \times n$ matrix A, the inverse of A is an $n \times n$ matrix B such that*

$$BA = AB = I,$$

where I is the $n \times n$ identity. The inverse of A is denoted by A^{-1}.

Example 37. *The inverse of 2×2 matrix: If $ad - bc \neq 0$, then*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 38. *Let $B = \begin{bmatrix} 4 & 4 \\ 1 & 3 \end{bmatrix}$. Calculate B^{-2}.*

Solution:

$$B^{-2} = B^{-1}B^{-1} = \frac{1}{8}\begin{bmatrix} 3 & -4 \\ -1 & 4 \end{bmatrix} \frac{1}{8}\begin{bmatrix} 3 & -4 \\ -1 & 4 \end{bmatrix} = \frac{1}{64}\begin{bmatrix} 13 & -28 \\ -7 & 20 \end{bmatrix} = \begin{bmatrix} 13/64 & -7/16 \\ -7/64 & 5/16 \end{bmatrix}$$

Theorem 7. *If A is an invertible $n \times n$ matrix, then for each $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.*

Example 39. *Given $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix}$. Solve $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.*

Solution: $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \\ 19 \end{bmatrix}$.

Properties of inverses:

1. If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. If A and B are invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
4. If A is invertible, $AB = AC$, then $B = C$.

Elementary Matrices:

An elementary matrix is a matrix obtained by performing one elementary row operation onto an identity matrix. Every elementary matrix is invertible. The inverse of an elementary matrix E is again an elementary matrix E^{-1} and represents the elementary row operation that transforms E into the identity matrix.

Theorem 8. *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n and in this case, any sequence of elementary row operations that reduce A to I_n also transforms I_n into A^{-1}.*

Example 40. *Find A^{-1}, where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix}.$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & -2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} 14 & -2 & -3 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Example 41. *Given the matrix equation*

$$A(B^{-1} + DX)C^T = I,$$

where A, B, C, D and X are $n \times n$ invertible matrices. Solve for X in terms of A, B, C, D.

Solution:

$$X = D^{-1}[A^{-1}(C^T)^{-1} - B^{-1}].$$

Theorem 9. *(The Invertible Matrix Theorem) Let A be a square $n \times n$ matrix. Then the following statements are equivalent.*

1. *A is an invertible matrix.*
2. *A is row equivalent to the identity matrix.*
3. *A has n pivot positions.*
4. *The equation $A\vec{x} = \vec{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b} \in \mathbb{R}^n$.*

7. *The columns of A span $\mathbb{R}^n$.*

8. *There is an $n \times n$ matrix C such that $CA = I$.*

9. *There is an $n \times n$ matrix D such that $AD = I$.*

10. *A^T is invertible.*

Remark. Another name for an invertible matrix is non singular and similarly another name for a non-invertible matrix is singular.

Example 42. *Is* $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 10 \end{bmatrix}$ *invertible?*

Solution: Yes, 3 pivot positions.

Topic 10: The Leontief Input-output Model

The Leontief Input-output Model is an economic model measuring how changes in one sector affect other sectors.
Suppose a nation's economy is divided into n sectors that produce goods or services.

- $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$: A production vector in $\mathbb{R}^n$ that lists the units produced by each sector (output).

- $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$: A final demand vector that lists the values of goods or services demanded from the various sectors by the nonproductive part of economy.

- C: Consumption matrix (input-output matrix). Each column of C lists units consumed by all sectors from producing 1 unit of each sector. For example, the first column of C lists units consumed by all sectors from producing 1 unit of sector 1.

- $C\mathbf{x}$: Intermediate demand created by the producers themselves for goods they need as inputs for their own production.

The Leontief Input-output Model: $C\mathbf{x} + \mathbf{d} = \mathbf{x}$.

Theorem 10. *If C and $\boldsymbol{d}$ have nonnegative entries and if each column sum of C is less than 1, then $(I - C)^{-1}$ exists, and the production vector $\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d}$ has nonnegative entries and is the unique solution of the Leontief Input-output Model.*

Example 43. *Consider an economy with three industries: coal-mining operation, electricity-generating plant and an auto-manufacturing plant. To produce \$1 of coal, the mining operation must purchase \$0.5 of its own production, \$0.2 of electricity and \$1 worth of automobile for its transportation. To produce \$1 of electricity, it takes \$0.8 of electricity and \$0.4 of automobile. Finally, to produce \$1 worth of automobile, the auto-manufacturing plant must purchase \$0.25 of coal, \$0.1 of electricity. Assume also that during a period of one week, the economy has an exterior demand of \$100 worth of coal, \$500 worth of electricity, and \$700 worth of automobile.*

(i) Construct the input-output table.

(ii) What amounts (intermediate demands) will be consumed by the electricity sector if it decides to produce \$1000?

(iii) Find the production level of each of the three industries in that period of one week in order to exactly satisfy both the internal and the external demands.

Solution: (i)

Inputs consumed per unit of output (\$1):			
coal	electricity	auto	Purchased from:
0.5	0.0	0.25	coal
0.2	0.8	0.1	electricity
1.0	0.4	0.0	auto
$\uparrow$	$\uparrow$	$\uparrow$	
$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$	

(ii) Compute

$$1000\vec{c}_2 = 1000 \begin{bmatrix} 0 \\ 0.8 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0 \\ 800 \\ 400 \end{bmatrix}.$$

To produce \$1000 electricity, it will consume \$800 of electricity and \$400 of auto-manufacturing.

(iii) Let x_1, x_2 and x_3 be the dollar values of outputs of coal-mining, electricity, auto-manufacturing, respectively. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the production vector.

The input-output matrix of this economy and the demand vector are

$$C = \begin{bmatrix} 0.5 & 0 & 0.25 \\ 0.2 & 0.8 & 0.1 \\ 1 & 0.4 & 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 100 \\ 500 \\ 700 \end{bmatrix}.$$

Then

$$I - C = \begin{bmatrix} 0.5 & 0 & -0.25 \\ -0.2 & 0.2 & -0.1 \\ -1 & -0.4 & 1 \end{bmatrix}, \quad (I-C)^{-1} = \begin{bmatrix} 16 & 10 & 5 \\ 30 & 25 & 10 \\ 28 & 20 & 10 \end{bmatrix},$$

which gives

$$\mathbf{x} = (I-C)^{-1}\mathbf{d} = \begin{bmatrix} 10100 \\ 22500 \\ 19800 \end{bmatrix}.$$

So, the total output of the coal-mining operation must be \$10100, the total output for the electricity-generating plant is \$22500 and the total output for the auto-manufacturing plant is \$19800.

Example 44. *Consider an economy with three industries: coal-mining operation, electricity-generating plant and an auto-manufacturing plant. To produce \$1 of coal, the mining operation must purchase \$0.1 of its own production, \$0.30 of electricity and \$0.1 worth of automobile for its transportation. To produce \$1 of electricity, it takes \$0.25 of coal, \$0.4 of electricity and \$0.15 of automobile. Finally, to produce \$1 worth of automobile, the auto-manufacturing plant must purchase \$0.2 of coal, \$0.5 of electricity and consume \$0.1 of automobile. Assume also that during a period of one week, the economy has an exterior demand of \$50,000 worth of coal, \$75,000 worth of electricity, and \$125,000 worth of autos. Find the production level of each of the three industries in that period of one week in order to exactly satisfy both the internal and the external demands.*

Example 45. *An economy consists of two sectors: Auto and Bike. For each unit of output, Auto requires 0.20 units from Auto and 0.40 units from Bike. For each unit of output, Bike uses 0.30 units from Auto and 0.60 units from Bike.*

(a) What is the consumption matrix C for this economy?

(b) Determine what intermediate demands are created if Auto plans to produce 1000 units.

(c) Find the production levels that will satisfy the final demand of 1000 units from Auto and 2000 units from Bike.

Solution: (a) The consumption matrix of this economy is: $C = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$.

(b) $1000 \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 200 \\ 400 \end{bmatrix}$.

(c) $I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.3 \\ -0.4 & 0.4 \end{bmatrix}$.

$(I - C)^{-1} = \frac{1}{0.32-0.12} \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.8 \end{bmatrix} = \begin{bmatrix} 2 & 1.5 \\ 2 & 4 \end{bmatrix}$.

Let x_1 and x_2 be the outputs of Auto and Bike respectively. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the production vector. Note that $\mathbf{d} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$.

$$\mathbf{x} = (I - C)^{-1}\mathbf{d} = \begin{bmatrix} 2 & 1.5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 5000 \\ 10000 \end{bmatrix}.$$

So, the production levels are: Auto 5000, Bike 10000.

Topic 11: Subspaces of $\mathbb{R}^n$

Definition 18. *A subspace is any subset V of $\mathbb{R}^n$ that satisfies,*

1. $\vec{0} \in V$
2. *For each* $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$ *(Closed under addition).*
3. *For each* $\vec{x} \in V$ *and scalar* c, $c\vec{x} \in V$ *(Closed under scalar multiplication.)*

Example 46.

1. $S = \{\vec{0}\}$ *is a subspace.*

2. *Given a set of vectors* $\vec{v}_1, \cdots, \vec{v}_m$ *in* $\mathbb{R}^n$,

$$Span\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m | \quad c_1, c_2, ..., c_m \in \mathbb{R}\}$$

is a subspace.

3. $H = \left\{ \begin{bmatrix} x \\ x \\ 3x - 2y \end{bmatrix} | x, y \in \mathbb{R} \right\}$ *is a subspace of* $\mathbb{R}^3$;

Solution: $\begin{bmatrix} x \\ x \\ 3x-2y \end{bmatrix} = \begin{bmatrix} x \\ x \\ 3x \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2y \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$. *Thus*

$$H = span\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \}.$$

4. $H = \left\{ \begin{bmatrix} a \\ a+1 \\ a-1 \end{bmatrix} | a \in \mathbb{R} \right\}$ *is not a subspace of* $\mathbb{R}^3$;

Solution: *Proof. (1)* $\vec{0} \notin H$. *In fact, if* $\begin{bmatrix} a \\ a+1 \\ a-1 \end{bmatrix} = 0$, *then* $a = 0$, $a - 1 = 0$, $a + 1 = 0$, *a contradiction.*

(2) H is not closed under addition: for example, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \in H$, *but*

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 3+1 \\ 3-1 \end{bmatrix}.$$

(3) H is not closed under scalar multiplication: for example, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \in H$, *but*

$$3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 3+1 \\ 3-1 \end{bmatrix}.$$

5. $H = \left\{ \begin{bmatrix} a-1 \\ b \\ c \end{bmatrix} | a,b,c \in \mathbb{R} \right\}$ *is a subspace of* $\mathbb{R}^3$.

Definition 19. *Let* $A = [\vec{a}_1 \dots \vec{a_n}]_{m \times n}$. *The columns of* A *span a subspace of* $\mathbb{R}^n$ *called the column space of* A *and is denoted by ColA:*

$$ColA = Span\{\vec{a}_1, \dots, \vec{a_n}\}.$$

The null space of a matrix A *is the set of all solutions to the homogeneous equation* $A\vec{x} = \vec{0}$ *and is a subspace of* $\mathbb{R}^n$. *The null space of* A *is denoted by NulA.*

$$NulA = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}.$$

Definition 20. *A basis for a subspace* H *is a linearly independent set of vectors that spans* H. *We denote it by* $\mathcal{B}_H$. *When* H *is clear, we just write the basis as* $\mathcal{B}$. *The number of vectors in a basis for a subspace* H *is called the dimension of* H *and is denoted by* $\dim H$.

Properties of basis:

- There is more than one basis for a subspace.
- A basis is the largest spanning set of linearly independent vectors for a subspace.
- Pivot columns can be a basis for ColA.

Example 47. *Let*

$$A = [\vec{a}_1\, \vec{a}_2\, \vec{a}_3\, \vec{a}_4\, \vec{a_5}] = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 2 & -6 & 4 & 10 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find $\mathcal{B}_{ColA}$ *and* $\mathcal{B}_{NulA}$.

Solution: Pivot columns are $\vec{a}_1, \vec{a}_4$. So $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 10 \end{bmatrix} \right\}$ is a basis of ColA.

To find a basis for NulA , consider $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \vec{0}$.

$$Augmented\ matrix = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 & -17 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_3 + 17x_5 \\ x_2 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 17 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Therefore, a basis of Nul A is

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 17 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

Remark. The set

$$\{e_1, \cdots, e_n\}$$

is called the standard basis for $\mathbb{R}^n$.

Topic 12: Dimension and Rank

Definition 21. *The rank of a matrix A is equal to the number of pivot positions in A. This in turn equals the number of pivot columns of A which equals the number of vectors in the basis for $ColA$. $rankA = \dim ColA$.*

The Rank Theorem: If a matrix A has n columns then

$$dimColA + dimNulA = n$$

or

$$rankA + \dim NulA = n.$$

Example 48. *Let* $A = [\vec{a}_1\, \vec{a}_2\, \vec{a}_3\, \vec{a}_4\, \vec{a}_5] = \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 4 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. *Find* $\dim ColA$ *and* $\dim NulA$.

Solution: Pivot columns are $\vec{a}_1$, $\vec{a}_3$. So a basis for $ColA$ is $\{\vec{a}_1, \vec{a}_3\}$. $\dim ColA = 2$.

$\dim NulA = 5 - \dim ColA = 5 - 2 = 3$.

Theorem 11. *(The Invertible Matrix Theorem) Let A be a square $n \times n$ matrix. Then the following statements are equivalent.*

1. *A is an invertible matrix.*
2. *A is row equivalent to the identity matrix.*
3. *A has n pivot positions.*
4. *The equation $A\vec{x} = \vec{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b} \in \mathbb{R}^n$.*
7. *The columns of A span $\mathbb{R}^n$.*
8. *There is an $n \times n$ matrix C such that $CA = I$.*
9. *There is an $n \times n$ matrix D such that $AD = I$.*
10. *A^T is invertible.*
11. *The columns of A form a basis for $\mathbb{R}^n$.*
12. *$Col\ A = \mathbb{R}^n$*
13. *$dim\ ColA = n$*
14. *$rank\ A = n$*

15. *Nul* $A = \{0\}$

16. *dim* $NulA$ = *0.*

Example 49. *(a) Is the REF of an* $n \times n$ *matrix* I_n*?*
(b) Is A *invertible if* $A\vec{x} = \vec{0}$ *has infinitely many solutions?*

Solution: (a) T; (b) F.

Topic 13: Determinants

Definition 22. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *The determinant of* A *is defined as*

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a $n \times n$ *matrix* A*, let* A_{ij} *be the matrix obtained from* A *by deleting the i-th row and j-th column. The* $(i,j)^{th}$ *cofactor of* A *is the number,*

$$c_{ij} = (-1)^{i+j} \det A_{ij}.$$

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + \ldots + a_{in}c_{in},$$

which is called a cofactor expansion across the i-th row. Similarly,

$$\det A = a_{1j}c_{1j} + a_{2j}c_{2j} + \ldots + a_{nj}c_{nj},$$

which is called a cofactor expansion across the j-th column.

Example 50. *Calculate* $\det A$*, where*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}.$$

Solution: We do cofactor expansion across the 2nd row.

$$\begin{aligned} \det A &= a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} \\ &= 2(-1)^{2+1} \det \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} + (-1)^{2+2} \det \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} + (-1)^{2+3} \det \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \\ &= 2(14) + (-13) + 5 = 20. \end{aligned}$$

Definition 23. *A triangular matrix is a matrix that is all zeros either above or below the diagonal. An upper triangular matrix means all entries below the main diagonal are zero; an lower triangular matrix means all entries above the main diagonal are zero.*

Theorem 12. *If* A *is a triangular matrix then* $detA$ *is the product of the entries on the main diagonal of* A.

Example 51. *Calculate* $\det A$*, where*

$$A = \begin{bmatrix} 5 & 3 & 5 & 7 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 12 \end{bmatrix}.$$

Solution: A is an upper triangular matrix. $\det A = 5(1)(2)(12) = 120$.

Theorem 13. *If A is a 3×3 matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then*

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Remark. This comes from the three main diagonals and three other diagonals by repeating the first two columns.

Properties of Determinants

Theorem 14. *(Row Operations)*

1. *If $A \xrightarrow{R_i \to R_i + kR_j} B$, then $\det A = \det B$.*

2. *If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det A = -\det B$.*

3. *If $A \xrightarrow{R_i \to kR_i} B$, then $\det A = \dfrac{1}{k}\det B$.*

Example 52. *Let*

$$A = \begin{bmatrix} 1 & -3 & 2 & -4 & 1 \\ -4 & 12 & -4 & 5 & -4 \\ 2 & -5 & 4 & -3 & 2 \\ -3 & 10 & -1 & 7 & -3 \\ 1 & -3 & 2 & -4 & 2 \end{bmatrix}.$$

(a) Calculate $\det A$ by using row reduction. (b) Find C_{23}, the (2,3)-cofactor of A.

Solution:

$$\det A = \begin{vmatrix} 1 & -3 & 2 & -4 & 1 \\ -4 & 12 & -4 & 5 & -4 \\ 2 & -5 & 4 & -3 & 2 \\ -3 & 10 & -1 & 7 & -3 \\ 1 & -3 & 2 & -4 & 2 \end{vmatrix} \overset{\substack{R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + 3R_1 \\ R_5 \to R_5 - R_1}}{=\!=\!=\!=} \begin{vmatrix} 1 & -3 & 2 & -4 & 1 \\ 0 & 0 & 4 & -11 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 1 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{R_2 \leftrightarrow R_4}{=\!=\!=} - \begin{vmatrix} 1 & -3 & 2 & -4 & 1 \\ 0 & 1 & 5 & -5 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 4 & -11 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \overset{R_3 \to R_3 - R_2}{=\!=\!=\!=} - \begin{vmatrix} 1 & -3 & 2 & -4 & 1 \\ 0 & 1 & 5 & -5 & 0 \\ 0 & 0 & -5 & 10 & 0 \\ 0 & 0 & 4 & -11 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{R_3 \to -\frac{1}{5}R_3}{=\!=\!=\!=\!=\!=} -(-5)\begin{vmatrix} 1 & -3 & 2 & -4 & 1 \\ 0 & 1 & 5 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & -11 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \overset{R_4 \to R_4 - 4R_3}{=\!=\!=\!=\!=\!=\!=\!=} 5\begin{vmatrix} 1 & -3 & 2 & -4 & 1 \\ 0 & 1 & 5 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 5(1)(1)(-3) = -15.$$

(b)

$$C_{23} = (-1)^{2+3} \det A_{23} = -\det \begin{bmatrix} 1 & -3 & -4 & 1 \\ 2 & -5 & -3 & 2 \\ -3 & 10 & 7 & -3 \\ 1 & -3 & -4 & 2 \end{bmatrix}$$

$$\overset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1}}{=\!=\!=\!=\!=\!=} -\begin{vmatrix} 1 & -3 & -4 & 1 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 0 \\ 1 & -5 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\overset{R_2 \to R_2 - R_1}{=\!=\!=\!=\!=} -\begin{vmatrix} 1 & 5 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1(-10)(1) = 10.$$

Example 53. *Given that* $\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, *what is* $\begin{vmatrix} a-2g & b-2h & c-2i \\ 3g & 3h & 3i \\ 5d & 5e & 5f \end{vmatrix}$

?

Solution: $\begin{vmatrix} a-2g & b-2h & c-2i \\ 3g & 3h & 3i \\ 5d & 5e & 5f \end{vmatrix} = \text{-15}\begin{vmatrix} a-2g & b-2h & c-2i \\ d & e & f \\ g & h & i \end{vmatrix}$

$= \text{-15}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -15(2) = -30.$

Properties of determinants:

1. If A is a square matrix then $\det A^T = \det A$.

2. $\det(AB) = \det A \det B$.

3. If A is an $n \times n$ matrix and c a scalar then $\det(cA) = c^n \det A$.

4. If A is invertible then $det(A^{-1}) = \frac{1}{\det A}$.

5. A square matrix A is invertible $\Leftrightarrow \det A \neq 0$.

6. If $A \xrightarrow{C_i \to C_i + kC_j} B$, then $\det A = \det B$.

7. If $A \xrightarrow{C_i \leftrightarrow C_j} B$, then $\det A = -\det B$.

8. If $A \xrightarrow{C_i \to kC_i} B$, then $\det A = \dfrac{1}{k} \det B$.

Example 54. *Let A, B and C be 3×3 invertible matrices, $det(A) = 3$, $det(B) = 5$, $det(C) = 6$. Calculate $det(A^{-1}C(-2B))$.*

Solution:

$$det(A^{-1}C(-2B)) = det(A^{-1})det(C)(-2)^3 det(B) = -8\left(\frac{1}{det(A)}\right) det(C)det(B)$$

$$= -8\left(\frac{1}{3}\right)(6)5 = -80.$$

Topic 14: Complex Numbers

Complex number is

$$z = a + bi, \quad a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1.$$

Addition and multiplication:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

The conjugate of $z = a + bi$ is: $\bar{z} = a - bi$. We have

$$z\bar{z} = a^2 + b^2.$$

The modulus of z is defined by

$$|z| = \sqrt{a^2 + b^2}.$$

Properties

1. $\bar{z} = z$ if and only if z is real,
2. $\overline{z + w} = \bar{z} + \bar{w}$,
3. $\overline{zw} = \bar{z}\bar{w}$,
4. $|wz| = |w||z|$,

Example 55. *Simplify* $\frac{2+3i}{4-5i}$.

Solution:

$$\frac{2 + 3i}{4 - 5i} = \frac{(2 + 3i)(4 + 5i)}{(4 - 5i)(4 + 5i)} = \frac{8 + 10i + 12i - 15}{4^2 + 5^2} = \frac{-7}{41} + i\frac{22}{41}.$$

Example 56. *Let* $z = \sqrt{3} + i$. *Calculate* z^6.

Solution: We write z as:

$$z = 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2e^{i\pi/6}.$$

$$z^6 = 2^6 e^{(i\pi/6)6} = 2^6 e^{i\pi} = 64(\cos\pi + i\sin\pi) = -64.$$

Fundamental Theorem of Algebra: Any polynomial equation of degree $n \geq 1$ has at least one root.

Example 57. *Solve the following equations (a)* $z^2+4=0$*, (b)* $x^2-2x+2=0$.

Solution: (a): $z=\pm 2$. (b) $x=1\pm i$.

Example 58. *Calculate* $\begin{bmatrix} 3 & 2-i \\ i+1 & i \end{bmatrix}\begin{bmatrix} i & i-1 \\ 2+i & 1 \end{bmatrix}$

Solution: $\begin{bmatrix} 5+3i & -1+2i \\ -2+3i & -2+i \end{bmatrix}$

Example 59. *Calculate* $\det A$*, where*

$$A=\begin{bmatrix} 1 & 3 & 5 \\ 2 & 3+i & 1 \\ 3 & 9 & 12-3i \end{bmatrix}.$$

Solution: By row reduction,

$$A=\begin{bmatrix} 1 & 3 & 5 \\ 2 & 3+i & 1 \\ 3 & 9 & 12-3i \end{bmatrix} \xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-2R_1} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -3+i & -9 \\ 0 & 0 & -3-3i \end{bmatrix}.$$

Thus

$$\det A = 1(-3+i)(-3-3i)=12+6i.$$

Topic 15: Eigenvectors and Eigenvalues

Definition 24. *An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. $\vec{x}$ is called the eigenvector corresponding to λ.*

To determine whether a given value λ is an eigenvalue of a matrix A we need to find a non-zero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. This is the same as determining whether the matrix equation

$$(A - \lambda I)\vec{x} = 0$$

has a non-trivial solution.

Example 60. *Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$,$\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that*

$$A\vec{u} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\vec{u}, \quad A\vec{v} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\vec{v}, \quad A\vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \neq \lambda\vec{w}.$$

Thus $\vec{u}$ is an eigenvector corresponding to $\lambda = 7$, $\vec{v}$ is an eigenvector corresponding to $\lambda = -4$, $\vec{w}$ is not an eigenvector.

Definition 25. *The set of all eigenvectors for a particular eigenvalue λ of a matrix A is a subspace and so is called the eigenspace of A corresponding to λ. We denote it by $E_\lambda(A)$.*

Example 61. *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

Theorem 15. *If $\vec{v_1}, ..., \vec{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ . \ . \ . \ , \lambda_r$ of an $n \times n$ matrix A, then the set $\{\vec{v_1}, ..., \vec{v_r}\}$ is linearly independent.*

Example 62. *Let $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix}$. Let $\lambda = 1$ be an eigenvalue. Find a basis to the corresponding eigenspace.*

Solution: Solve the equation $(A - \lambda I_3)\vec{x} = \vec{0}$.

Topic 16: The Characteristic Equation

Characteristic equation: $\det(A - \lambda I)$ is called the characteristic polynomial of A and

$$\det(A - \lambda I) = 0$$

is called the characteristic equation.

Theorem 16. *The solutions of the characteristic equation are the eigenvalues of A.*

Example 63. *Let* $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$. *Find the eigenvalues and eigenvectors.*

Solution:

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 1 & -5 & 5-\lambda \end{vmatrix} \overset{R_3\to R_3-R_2}{=\!=\!=} \begin{vmatrix} 3-\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 0 & -3+\lambda & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 3-\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 0 & -1 & 1 \end{vmatrix} = (3-\lambda)\left\{(3-\lambda)\begin{vmatrix} -2-\lambda & 2 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} -4 & 2 \\ -1 & 1 \end{vmatrix}\right\}$$

$$= (3-\lambda)\{(3-\lambda)(-\lambda)+2\} = (3-\lambda)(\lambda-1)(\lambda-2).$$

Thus the eigenvalues are 1, 2, 3 with each multiplicity 1.

When $\lambda = 3$,

$$A-3I = \begin{bmatrix} 0 & -4 & 2 \\ 1 & -5 & 2 \\ 1 & -5 & 2 \end{bmatrix} \xrightarrow{R_3\to R_3-R_2} \begin{bmatrix} 0 & -4 & 2 \\ 1 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A-3I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}x_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

When $\lambda = 1$,

$$A-I = \begin{bmatrix} 2 & -4 & 2 \\ 1 & -3 & 2 \\ 1 & -5 & 4 \end{bmatrix} \xrightarrow{R_1\to \frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & 1 \\ 1 & -3 & 2 \\ 1 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A-I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

When $\lambda = 2$,

$$A-2I = \begin{bmatrix} 1 & -4 & 2 \\ 1 & -4 & 2 \\ 1 & -5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A-2I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Multiplicity:

- The multiplicity of an eigenvalue is equal to the number of times it is a root of the characteristic equation.

Example 64. *Let*

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

The eigenvalue 3 has multiplicity 2 and the eigenvalue 1 has multiplicity 1. Find the corresponding eigenspaces and bases.

Solution: When $\lambda = 3$,

$$A - \lambda I = A - 3I = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - 3I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenspace has a basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

When $\lambda = 1$,

$$A - \lambda I = A - I = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The eigenspace has a basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

The Invertible Matrix Theorem (continued): Let A be an $n \times n$ matrix. Then the following statements are equivalent to, "A is an invertible matrix."

19. The number 0 is not an eigenvalue of A.
20. The determinant of A is not zero.

Example 65. *Is A invertible if the characteristic equation is* $-\lambda^7+5\lambda^6-4\lambda^5=0$*? Find all eigenvalues.*

Solution: No, since 0 is an eigenvalue.

Similar matrices: Two matrices A and B are similar if there is an invertible matrix P such that,

$$A = PBP^{-1}.$$

Theorem 17. *If matrices A and B are similar, then they have the same characteristics polynomial and hence the same eigenvalues (with the same multiplicities).*

Solution:
Proof.

$$\begin{aligned}
\det(A-\lambda I) &= \det(PBP^{-1}-\lambda PP^{-1}) \\
&= \det[P(B-\lambda I)P^{-1}) \\
&= \det(P)\det(B-\lambda I)\det(P^{-1}) \\
&= \det(P)\det(B-\lambda I)\frac{1}{\det(P)} \\
&= \det(B-\lambda I).
\end{aligned}$$

Topic 17: Diagonalization

A diagonal matrix is a matrix with zero values on all its off diagonal entries.
Definition 26. *If A is a square* $n\times n$ *matrix and A is similar to a diagonal matrix D then A is said to be diagonalizable, i.e.,* $A = PDP^{-1}$ *or* $P^{-1}AP = D$.

Theorem 18. *(Diagonalization Theorem) Let A be an* $n \times n$ *matrix.*

- *A is diagonalizable if and only if A has n linearly independent eigenvectors. If* $A = PDP^{-1}$*, where D is a diagonal matrix, the the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.*

- *If A has n distinct eigenvalues, then A is diagonalizable.*

- *A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n, if and only if the dimension of the eigenspace for each eigenvalue equals the algebraic multiplicity of the eigenvalue. (Generally, the dimension of the eigenspace for each eigenvalue is less than or equal to the algebraic multiplicity of the eigenvalue).*

For an $n\times n$ matrix A, if A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to the eigenvalue λ_k, $k = 1, ..., p$, then the total collection of vectors in the sets $\mathcal{B}_1$,..., $\mathcal{B}_p$ forms an eigenvector basis of $\mathbb{R}^n$.

Example 66. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ *is diagonalizable: 4 distinct eigenvalues.*

$B = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ *is not diagonalizable:* $\lambda = 4$, *one eigenvector* $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example 67. *Let* $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

1) Find P and D such that $A = PDP^{-1}$.
2) Calculate A^4.

Solution: 1) $\det(A - \lambda I) = (\lambda - 5)(\lambda + 2)$.
When $\lambda = 5$: $\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;
When $\lambda = -2$: $\vec{x} = \frac{x_2}{4} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Thus

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}; \text{ or } P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.$$

2) Let $P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$, then $P^{-1} = \frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$.

$$A^4 = \left\{PDP^{-1}\right\}^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}^4 \frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 364 & 261 \\ 348 & 277 \end{bmatrix}.$$

Example 68. *Consider the matrix* $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

1. *Find the eigenvalues of A.*
2. *For each eigenvalue, find a basis for the corresponding eigenspace.*
3. *Find an invertible matrix P and a diagonal matrix D such that*

$$A = PDP^{-1}, \quad \text{or} \quad P^{-1}AP = D.$$

Solution: (a). Expanding along the third column, we have

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 4\lambda + 3).$$

Thus the eigenvalues of A are 1 (multiplicity=2) and 3 (multiplicity=1).

Solution: (b) For the eigenvalue 1, we row reduce

$$\left[\begin{array}{c|c} A-I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenspace consists of the vectors

$$\vec{x} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_2, x_3 \text{are free.}$$

Therefore, a basis of this eigenspace $E_1(A)$ is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

For the eigenvalue 3, we row reduce

$$\left[\begin{array}{c|c} A-3I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenspace consists of the vectors

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 \text{ free.}$$

Therefore, a basis of this eigenspace $E_3(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Solution: (c). Since A is 3×3, and we have three linearly independent 3 eigenvectors, the matrix A is diagonalizable and we have $P^{-1}AP = D$, where

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Topic 18: Complex Eigenvalues

Example 69. *Let*

$$A = \begin{bmatrix} 3+5i & 2i & -1 \\ 0 & 1 & i-2 \\ 0 & 0 & \frac{8+2i}{1-i} \end{bmatrix}.$$

Find the eigenvalues and their multiplicities.

Solution: Since $\frac{8+2i}{1-i} = 3+5i$, the eigenvalue $3+5i$ has multiplicity 2 and the eigenvalue 1 has multiplicity 1.

Example 70. *Let* $A = \begin{bmatrix} 4 & 5 \\ -1 & 0 \end{bmatrix}$. *Find all eigenvalues and corresponding eigenvectors.*

Solution: $\det(A - \lambda I) = \lambda^2 - 4\lambda + 5 \Rightarrow \lambda = 2 \pm i$.
When $\lambda = 2+i$,

$$\left[\begin{array}{c|c} A-(2+i)I & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2-i & 5 & 0 \\ -1 & -2-i & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 5 & 5(2+i) & 0 \\ -1 & -2-i & 0 \end{array}\right]$$

$$\xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 2+i & 0 \\ -1 & -2-i & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 2+i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $x_1 + (2+i)x_2 = 0$, $x_1 = -(2+i)x_2$, x_2 free.

$$E_{2+i} = \{c \begin{bmatrix} -2-i \\ 1 \end{bmatrix} \mid c \in \mathbb{R}\}.$$

Therefore, a basis of

$$E_{2+i} = \{\begin{bmatrix} -2-i \\ 1 \end{bmatrix}\}.$$

When $\lambda = 2+-$,

$$\left[\begin{array}{c|c} A-(2-i)I & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2+i & 5 & 0 \\ -1 & -2+i & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 5 & 5(2-i) & 0 \\ -1 & -2+i & 0 \end{array}\right]$$

$$\xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 2-i & 0 \\ -1 & -2+i & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 2-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $x_1 + (2-i)x_2 = 0$, $x_1 = -(2-i)x_2$, x_2 free.

$$E_{2-i} = \{c \begin{bmatrix} -2+i \\ 1 \end{bmatrix} \mid c \in \mathbb{R}\}.$$

Therefore, a basis of

$$E_{2+i} = \{\begin{bmatrix} -2+i \\ 1 \end{bmatrix}\}.$$

More Properties:

(1) Let $\vec{x}$ be an eigenvector of the matrix A corresponding to the eigenvalue a. For any positive integer n, a^n is an eigenvalue of A^n with corresponding eigenvector $\vec{x}$.

(2) Let $\vec{x}$ be an eigenvector of the matrix A corresponding to the eigenvalue a. If A is invertible, then $\frac{1}{a}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\vec{x}$.

(3) Let $\vec{x}$ be an eigenvector of the matrix A corresponding to the eigenvalue a. If A is invertible, then for any integer n, a^{-n} is an eigenvalue of A^{-n} with corresponding eigenvector $\vec{x}$.

(4) Let $\vec{x}$ be an eigenvector of both the matrices A and B associated with eigenvalues a and b. $\vec{x}$ is an eigenvector of $A+B$ associated with eigenvalue $a+b$.

(5) Let $\vec{x}$ be an eigenvector of both the matrices A and B associated with eigenvalues a and b. $\vec{x}$ is an eigenvector of AB associated with eigenvalue ab.

(6) A is invertible if and only if 0 s not an eigenvalue.

Solution:

Proof.

(1) $A^n\vec{x} = a^n\vec{x}$.

(2) $A\vec{x} = a\vec{x} \Rightarrow \vec{x} = A^{-1}(a\vec{x}) = aA^{-1}\vec{x}$. Thus $A^{-1}\vec{x} = \frac{1}{a}\vec{x}$.

(3) By (1) and (2), let $B = A^n$. Then

$$B\vec{x} = a^n\vec{x}.$$

$$A^{-n}\vec{x} = B^{-1}\vec{x} = \frac{1}{a^n}\vec{x}.$$

(4) $(A+B)\vec{x} = A\vec{x} + B\vec{x} = a\vec{x} + b\vec{x} = (a+b)\vec{x}$.

(5) $(AB)\vec{x} = Ab\vec{x} = bA\vec{x} = ba\vec{x}$.

(6) Assume A is not invertible. Then Ax = 0 = 0x has non-trivial solution by invertible matrix theorem. So by definition, 0 is an eigenvalue.

Assume 0 is an eigenvalue. Thus there is some nontrivial solution to Ax = 0x = 0. By the invertible matrix theorem, A is not invertible.

Topic 19: Difference equation, Markov Chains

- Probability vector: A vector with nonnegative entries that add up to 1.
- Stochastic matrix: A square matrix whose columns are probability vectors.
- Markov chain: A sequence of probability vectors $x_0, x_1, \cdots$ together with a stochastic matrix P such that $x_{k+1} = Px_k, k = 0, 1, 2, \cdots$
- Steady-State Vectors (which gives long-term distribution): If P is a stochastic matrix, then a steady-state vector for P is a probability vector $\vec{q}$ such that
$$P\vec{q} = \vec{q}, \Rightarrow (P - I)\vec{q} = \vec{0}.$$
- Convergence: A stochastic matrix P is called regular if, for some integer k, all entries of P^k are strictly positive. If P is a regular stochastic matrix, then P has a unique steady-state vector $\vec{q}$. Further, Markov chain $\{x_k\}$ converges to $\vec{q}$ as $k \to \infty$.

Example 71. *Mat1302 Tutoring Ottawa has two branches in Ottawa: DowntownB, and EastB, with market portion 70% and 30% respectively. According to market statistics, due to variety of tutors, after each year, 10% of DowntownB's customer will switch to EastB, while 20% of EastB's customer will switch to DowntownB.*

(a) Find the migration matrix M. ***Solution:*** $M = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$

(b) Determine the market share of each of the two branches after a year according to the market statistics.

Solution: *Let* $x_k = \begin{bmatrix} d_k \\ e_k \end{bmatrix}$ *be the market share at the time of the kth measurement ($k = 0 \leftrightarrow$ initial market share), where d_k is the market share in DowntownB, e_k is the market share in EastB,* $d_0 = 0.7$, $e_0 = 0.3$. $\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .69 \\ .31 \end{bmatrix}$

Therefore the market shares are $\begin{cases} \text{DowntownB:} & 69\% \\ \text{EastB:} & 31\% \end{cases}$

(c) Assume that the marketing strategies do not change. What will be the long term market share of the two branches?

Solution:

$M\mathbf{x} = \mathbf{x} \;\Rightarrow\; (M - I_2)\mathbf{x} = \mathbf{0} \;\Rightarrow\; \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -.1 & .2 & 0 \\ .1 & -.2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = 2x_2 \\ x_1 + x_2 = 1 \end{cases}$$

$\Rightarrow\ x_1 = \frac{2}{3}$ *and* $x_2 = \frac{1}{3}$

Therefore the long term market shares are: $\begin{cases} \textit{DowntownB:} & 67\% \\ \textit{EastB:} & 33\% \end{cases}$

Example 72. *Let*

$$P = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}.$$

Find a steady-state vector for P.

Solution:

$$[P - I|\vec{0}] = \begin{bmatrix} -0.7 & 0.4 & 0.5 & | & 0 \\ 0.3 & -0.6 & 0.3 & | & 0 \\ 0.4 & 0.2 & -0.8 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & -7 & | & 0 \\ 0 & 5 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

By solving (P-I)x=0 we have: $x_1 = (7/5)x_3, x_2 = (6/5)x_3$. From $x_1+x_2+x_3 = 1$ we imply that $x_3 = \frac{5}{18}$. So $\vec{q} = \begin{bmatrix} 7/18 \\ 6/18 \\ 5/18 \end{bmatrix}$.

Example. $P = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}$ is regular. $\vec{q} = \begin{bmatrix} 7/18 \\ 6/18 \\ 5/18 \end{bmatrix}$. This shows that after a large number of deliveries, it no longer matters which location we were in when we started, we have (approximately) a 33.3% Chance of being at location B.