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ORTHONORMAL REPRESENTATIONS FOR OUTPUT SYSTEM PAIRS

Kurt S. Riedel[†] and Andrew P. Mullhaupt^{*}

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Abstract

A new class of canonical forms is given proposed in which (A, C) is in Hessenberg observer or Schur form and output normal: $\mathbb{I} - A^*A = C^*C$. Here, C is the $d \times n$ measurement matrix and A is the advance matrix. The (C, A) stack is expressed as the product of n orthogonal matrices, each of which depends on d parameters. State updates require only $\mathcal{O}(nd)$ operations and derivatives of the system with respect to the parameters are fast and convenient to compute. Restrictions are given such that these models are generically identifiable. Since the observability Grammian is the identity matrix, system identification is better conditioned than other classes of models with fast updates.

AMS Classifications: 93B10, 93B11 93B30

Key words. System representations, balanced representations, Hessenberg observer form, Schur representations, Stein's equation, discrete Lyapunov equation

^{*}S.A.C. Capital Management, LLC, 540 Madison Ave, New York, NY 10022, [†] Millennium Partners, 666 Fifth Ave, New York, 10103-0899

1 INTRODUCTION

Canonical forms are important in system identification, where an unique representation is desired to avoid identifiability problems [11, 19]. We consider the system (A, B, C) , where A is a real $n \times n$ matrix, B is a real $n \times m$ matrix, and C is $d \times n$ real matrix with $n \geq d$. Our systems are output normalized:

$$A^*A = \mathbb{I}_n - C^*C \quad . \quad (1.1)$$

In the next section, we describe the advantages of using an input normal or an output normal (ON) representation.

We consider real stable output pairs and show that a real output pair representation always exists where (A, C) is simultaneously output normal and in Hessenberg observer form or real Schur form. We give explicit parameterizations of the (C, A) stack as a product of orthogonal matrices of the form:

$$\begin{pmatrix} C \\ A \end{pmatrix} = \left[\prod_{i=1}^{nd} G_{j(i),k(i)}(\theta_i) \right]_{1:(n+d),1:n} \quad (1.2)$$

and related variants. Here $G_{j,k}$ is a Given's rotation in \mathcal{R}^{n+d} as defined at the end of this section. Our representations include the banded orthogonal filters of [18] as a special case (under the duality map $A \rightarrow A^*, C \rightarrow B$).

These orthogonal product representations are parameterized by the minimal number of free parameters and have no coordinate singularities. Our representation allow fast state updates in $\mathcal{O}(nd)$ operations and derivatives of the system with respect to the parameters are fast and convenient to compute.

Our results consider only the output pair, (A, C) , and are independent of B . Thus B and D may be treated as linear parameters in system identification or system synthesis (in contrast to (2.5)) and chosen separately from the parameters of A and C . In particular, the elements of B may be estimated with pseudo-linear regression. Corresponding controller representations exist for input pairs, (A, B) .

Any stable observable output pair may be transformed into one of our representations by the following three step process. First, we transform the output pair (A, C) to output normal form using the Cholesky factor of the solution of (2.2). Second, we orthogonally transform the output normal pair (A, C) to any of the three major output forms: Schur form, Hessenberg observer form, and observer triangular system form as defined in Section 3. Finally, we perform a series of Givens rotations to show that the transformed system must be of the form given by (1.2).

The final representations are given in Theorems 6.1, 7.1. For statistical estimation and numerical implementations, it is highly desirable to eliminate redundancy in the parameterization when possible. We address redundancy in two ways. First, we categorize when two distinct ON pairs in Hessenberg observer form are equivalent. Second, we impose constraints on the parameters in (1.2) to eliminate redundant parameterizations of the same ON pair generically. We repeat this analysis for Schur form and for observer triangular system form.

In Section 2, we give a brief overview of the advantages of input normal and output normal form. In the Section 3, we give the basic definitions and show every output pair is similar to an output pair in Schur ON form, to a Hessenberg ON pair and to an ON pair in triangular system form. In Section 4, we show that after standardization these Hessenberg ON systems uniquely parameterize transfer functions for *generic* systems. When A is reducible, we find an orthogonal transformation that preserve the output normal property. To construct the orthogonal product representations of the (C, A) stack, we need families of orthogonal matrices such as the set of Householder matrices. In Section 5, we give a general definition of orthogonal reduction families that includes Householder and Givens representation. In Section 6 and Section 7, we give explicit orthogonal product representations of Hessenberg output normal pairs.

Notation: The $n \times n$ identity matrix is \mathbb{I}_n and e_k is the unit vector in the k th coordinate. By $A_{i:j,k:m}$, we denote the $(j - i + 1) \times (m - k + 1)$ subblock of A from row i to row j and from column k to column m . We abbreviate $A_{i:j,1:n}$ by $A_{i:j,:}$. The matrix A has upper bandwidth d if $A_{i,j} = 0$ when $j > i + d$. A $k \times m$ matrix of zeros is denoted by $0_{k,m}$. The direct sum of matrices is denoted by \oplus . We denote the matrix transpose of A by A^* with no complex conjugation since we are interested in the real system case.

We denote the Given's rotation in the i th and j th coordinate by G_{ij} i.e. $g_{i,i} = g_{j,j} = \cos(\theta)$, $g_{i,j} = -g_{j,i} = \sin(\theta)$ and $g_{k,m} = \delta_{k,m}$ otherwise, where $g_{k,m}$ are the elements of G_{ij} . The symbol E denotes a signature matrix: $E_{i,j}^2 = \delta_{i,j}$.

Two systems (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are similar (equivalent) when $\tilde{A} \equiv T^{-1}AT$, $\tilde{C} \equiv CT$ and $\tilde{B} \equiv T^{-1}B$ for some invertible T . They are orthogonal equivalent if T is a real orthogonal matrix.

2 REPRESENTATIONS AND CONDITION NUMBERS

The goal of this paper is to propose system representations that are both well conditioned for system identification and are fast and convenient for numerical computation. We briefly discuss these issues in the context of existing alternative system representations. For more complete analysis of conditioning in system identification, we refer the reader to [16].

Let (A, B, C) be stable, observable and controllable. We define the observability Grammian, P_{A^*, C^*} and the controllability Grammian, $P_{A, B}$ by

$$P_{A, B} - AP_{A, B}A^* = BB^* \quad (2.1)$$

$$P_{A^*, C^*} - A^*P_{A^*, C^*}A = C^*C. \quad (2.2)$$

A popular class of system representations is balanced systems [12, 19, 20], where both the observability Grammian and the controllability Grammian are simultaneously diagonal: $P_{A, B} = P_{A^*, C^*} = \Sigma_{A, B, C}$. Balanced representations have many desirable theoretical properties. However, existing parameterizations of balanced models require $O(n^2)$ operations to update the state space system.

An alternative to balanced models is output normal (ON) representations [13, 14], where the observability Grammian is required to be the identity matrix, but no structure on the controllability Grammian.

Definition 2.1 *An output pair, (A, C) , is output normal (ON) if and only if (1.1) holds. An input pair, (A, B) , is input normal (IN) if and only if*

$$AA^* = \mathbb{I}_n - BB^* \quad (2.3)$$

If A is stable, definition 2.1 is equivalent to $P_{A^*, C^*} = \mathbb{I}_n$ for output normal and $P_{A, B} = \mathbb{I}_n$ for input normal. In [19], Ober shows that stability plus a positive definite solution to the dual Stein equation, (2.2), implies that the output pair is observable. By Theorem 2.1 of [2], if the observability Grammian is positive definite and (A, C) is observable, then the output pair is stable. Thus for ON pairs, stability is equivalent to observability.

ON pairs are not required to be stable or observable. (From (1.1), A must be at least marginally stable.) In [12], ‘output normal’ has a more restrictive definition of (1.1) and the additional requirement that the controllability Grammian be diagonal. We do not impose any such condition on the controllability Grammian. In [13], we called condition (1.1) ‘output balanced’, whereas now we call (1.1) ‘output normal.’ We choose this language so that ‘normal’ denotes restrictions on only one Grammian while ‘balanced’ denotes simultaneous restrictions on both Grammians.

A measure of ill-conditioning in system identification is the condition number of $P_{A, B}$, $\kappa(P_{A, B}) \equiv$ largest singular value of $P_{A, B}$ divided by the smallest. In [16], we show that solving the Stein equation, $P_{A, B}$ is exponentially ill-conditioned in n/m for large classes of (A, B) pairs; i.e. $\kappa(P_{A, B}) \sim \exp(\alpha n/m)$ for some α . To avoid the possibility of ill-conditioning, we prefer to consider representations where either the observability or the controllability Grammian is the identity.

Let $\Sigma_{A,B,C}$ be the Grammian of the balanced system equivalent to (A, B, C) . In [9], it is shown that

$$\kappa(\Sigma_{A,B,C})^2 \leq \kappa(P_{A,B}) \kappa(P_{A^*,C^*}) , \quad (2.4)$$

where equality holds for balanced systems, input normal systems and output normal systems. For output balanced systems, the ill-conditioning is entirely in the controllability Grammian: $\kappa(P_{A,B}) = \kappa(\Sigma_{A,B,C})^2$. We interpret $\kappa(\Sigma_{A,B,C})^2$ as the intrinsic conditioning of a linear time invariant (LTI) system and $\kappa(P_{A,B}) \kappa(P_{A^*,C^*}) / \kappa(\Sigma_{A,B,C})^2$ as a measure of the excess ill-conditioning of a system representation.

Our representations resemble those based on embedded lossless systems [5, 22, 23]:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = P_1 \prod_{i=1}^{f_1} G_{k(i),m(i)}(\theta_i) P_2 \prod_{j=f_1+1}^{f_2} G_{k(j),m(j)}(\theta_j) P_3 , \quad (2.5)$$

where f_2 = number of free parameters, P_1 and P_3 are projections onto coordinate directions and P_2 is a prescribed permutation. In [23], the full system is first embedded in a lossless system (just as we transform the output pair (A, C) to output normal form). Next, these authors transform (A, B) to Hessenberg controller form (analogous to our transformation to Hessenberg observer form). We conjecture that there are analogous versions of (2.5), where A is in Schur form or (A, B) is in controller triangular system form. Finally, the authors perform a series of Givens rotations to show that the transformed system must be of the form given by (2.5). Our corresponding representations are given in Theorems 6.1, 7.1.

The main advantage of (1.2) over (2.5) is that the observability Grammian of ON models does not inflate the product condition number: $\kappa(P_{A,B}) \kappa(P_{A^*,C^*})$. A second advantage is that B and D may be treated as linear parameters in system identification or system synthesis, whereas (2.5) couples the parameterization of B and D to that of A and C in a nonlinear fashion. For these reasons, we recommend output normal representations over embedded lossless representations.

Another difference between our treatment and the analyses of [5, 22, 23] is that we try to impose constraints on the parameters to eliminate redundant representations whenever possible and to categorize when redundant representations can occur. If one is satisfied with having representations with a finite multiplicity of equivalent systems (at least generically), this last step may be too detailed. For numerical implementations, we believe that it is highly desirable to eliminate as much of the redundancy in representation as is possible.

Our representations include the banded orthogonal filters of [18] as a special case. Our analysis imposes additional constraints on representations of [18] to remove multiple representations of the same transfer function generically.

3 DEFINITIONS AND EXISTENCE

We now define observer triangular system form, Schur form and Hessenberg observer form and show that any stable observable output pair is equivalent to an output normal pair in any of these three forms. We denote the $(n + d) \times n$ matrix stack of C and A by Q :

$$Q \equiv \begin{pmatrix} C \\ A \end{pmatrix}. \quad (3.1)$$

Definition 3.1 *The output pair is in observer triangular system (OTS) form if the (C, A) stack, Q , satisfies $Q_{i,j} = 0$ for $j > i$. The output pair is unreduced if $Q_{i,i} \neq 0$ and is reducible if $Q_{k,k} = 0$ for some k . The output pair is in standard OTS form if $Q_{i,i} \geq 0$ and in strict OTS form if $Q_{i,i} > 0$.*

Thus strict is equivalent to unreduced and standard. The real Schur representation is defined and described in [4, 6, 7]. The diagonal subblocks of A may be placed in an arbitrary order. To ensure identifiability of our model, we must specify a particular standardization of the diagonal of the Schur form of A . Our choice, ‘ordered qd’ Schur form, is defined in Appendix A.

The OTS form includes the banded orthogonal filters of [18] as a special case under the duality map $A \rightarrow A^*$, $C \rightarrow B$. Our results correspond to a detailed analysis of the generic identifiability of the representations of [18].

Hessenberg observer (HO) form is a canonical form where A is Hessenberg. We impose the additional restriction that $C_{1,1} \geq 0$, $C_{1,j} = 0$ for $j > 1$.

Definition 3.2 *The output pair is in Hessenberg observer (HO) form if A is a Hessenberg matrix and $C_{1,j} = 0$ for $j > 1$. A HO output pair is nondegenerate if $|C_{1,1}| < 1$. A HO output pair is unreduced if $A_{i+1,i} \neq 0$ for $1 \leq i < n$ and $C_{1,1} \neq 0$. A HO output pair is standard if $A_{i+1,i} \geq 0$ for $1 \leq i < n$, $0 \leq C_{1,1} < 1$. A HO output pair is strict if it is unreduced and standard. A HO output pair is in partial ordered Schur qd block form if $A_{i+1,i} = 0$ implies $A_{(i+1):n,(i+1):n}$ is in ordered Schur qd block form.*

Both Hessenberg observer output pairs and observer triangular system output pairs always can be transformed to a standard output pair using a signature matrix, E : $A \rightarrow EAE^{-1}$, $C \rightarrow CE^{-1}$. Generically, HO output pairs are unreduced and thereby unaffected by the requirement of partial Schur order. For both OTS form and HO form, the B matrix is unspecified. Dual definitions for controller forms reverse the roles of (A, B) and (A^*, C) . In our definitions, for $d = 1$, a OTS output pair has a lower Hessenberg matrix.

An important result in systems representation theory is

Theorem 3.3 [10, 24] *Any observable output pair is orthogonally equivalent to a system in real Schur form, to a system in observer triangular system form and to a system in Hessenberg observer form. The Hessenberg observer form can be chosen in partial ordered Schur qd block form.*

The standard proof of Theorem 3.3 begins by transforming C to its desired form and then defines Householder or Givens rotations which zero out particular elements in A in successive rows or columns [7].

Definition 3.4 *An output pair, (A, C) , is observer triangular system output normal (OT-SON) if it is in observer triangular system form and output normal. The output pair is Hessenberg observer output normal (HOON) if it is in Hessenberg observer form and output normal. The output pair is in Schur ON form if it is output normal and A is in real Schur form.*

Theorem 3.5 *Every stable, real observable output pair (A, C) , is similar to a real OTSON pair, to a real HOON pair, and to an ordered real Schur output pair with qd diagonal subblocks.*

Proof: The unique solution, P_{A^*, C^*} , of dual Stein equation, (2.2), is strictly positive definite. Let L be the unique Cholesky lower triangular factor of P with positive diagonal entries: $P = LL^*$. We set $T = L^{-*}$. Let U be orthogonal transformation that takes $(T^{-1}AT, CT)$ to the desired form (Schur, OTS or HO) as described in [10, 24]. Then UT is the desired transformation. ■

This result applies to any output pair with a positive definite solution to the dual Stein equation, (2.2). Observability and stability of (A, C) are sufficient but not necessary conditions for a positive definite solution.

Degenerate HOON pairs correspond to the direct sum of an identity matrix and a non-degenerate HOON system:

Lemma 3.6 *Every stable, real observable output pair (A, C) , is similar to a real HOON pair with a Q stack of the form $\tilde{Q} = \mathbb{I}_m \oplus \hat{Q}$ for some $m \leq d$, where \hat{Q} is a nondegenerate HOON stack.*

Thus we consider only HOON systems that are nondegenerate. Note that degenerate Hessenberg controller forms are excluded from [23] by their assumptions. If the HO pair is reducible, then it may be further simplified using orthogonal transformations as described in Theorem 4.4.

4 UNIQUENESS OF STRICT HOON AND OTSON REPRESENTATIONS

There are two main ways in which one of our system representations can fail to parameterize linear time invariant systems in a bijective fashion. First, there may be a multiplicity of equivalent HOON systems (or OTSON systems or Schur OB systems). Second, Givens product representation such as (1.2) may have multiple (or no) parameterizations of the same output pair.

For Schur OB pairs, the basic result is straightforward. If A has distinct eigenvalues and they are ordered in an unique fashion, then there is a parameterization that is globally bijective.

Each strict OTSON (HOON) pair generates 2^n distinct but equivalent OTSON (HOON) pairs using different signature matrices. If the OTS pair or the HO pair is reducible, then it may be further simplified using orthogonal transformations. For the HO pair, these reductions are described in Theorems 4.4. The representations of degenerate HO pairs reduce to a direct sum of a ‘nondegenerate’ HOON system and a trivial system and thus we consider only nondegenerate HOON pairs.

For OTSON pairs and nondegenerate HOON pairs, we find that the set of strict output pairs has a bijective representation in an easy to parameterize subset of Givens product representations. Our precise OTSON result is

Theorem 4.1 *If (A, C) is a strict OTSON pair, then there are no other equivalent strict OTSON pairs.*

This result and a generalization that reducible OTSON pairs is proven in [17]. For HOON representations, our uniqueness results are based on the following lemma that generalizes the Implicit Q theorem [4, 7] to HOON pairs:

Lemma 4.2 *Let (A, C) and (\tilde{A}, \tilde{C}) be equivalent standard nondegenerate HOON pairs ($\tilde{A} \equiv T^{-1}AT, \tilde{C} \equiv CT$). Let $A_{k+1,k} = 0$, $C_{1,1} > 0$ and $A_{j+1,j} > 0$ for $j < k$, then $T = \mathbb{I}_k \oplus U_{n-k}$, where U_{n-k} is an $(n-k) \times (n-k)$ orthogonal matrix. Furthermore, $k > 1$ and $\tilde{A}_{k+1,k} = 0$.*

Since $C_{1,j} = \tilde{C}_{1,j} = 0$, $j > 1$, $T_{j,1} = \delta_{j,1}$. The result follows from the Implicit Q theorem [4, 7]. ■

Corollary 4.3 *If (A, C) is a strict nondegenerate HOON pair, then there are no other equivalent strict HOON pairs.*

For reducible HOON pairs, we place the lower part of A in ordered Schur qd block form to remove redundant representations:

Theorem 4.4 *Let (A, C) be a nondegenerate HOON pair with $A_{k+1,k} = 0$ with $A_{j+1,j} \neq 0$ for $j < k$ and define $A^{(2,2)} \equiv A_{(k+1):n,(k+1):n}$. There exists an equivalent HOON pair (\tilde{A}, \tilde{C}) , $(\tilde{A} \equiv U^*AU, \tilde{C} \equiv CU)$, where \tilde{A} is in partial ordered Schur qd block form. If $\tilde{A}^{2,2}$ has distinct eigenvalues, \tilde{A} is uniquely defined.*

Proof: By results cited in Appendix A, there exists an $(n - k) \times (n - k)$ orthogonal transformation, U , such that $U^*A^{(2,2)}U$ is in partial ordered Schur qd block form. From Lemma 4.2, $V = \mathbb{I}_k \oplus U_{n-k}$ is the desired transformation and it is unique when $A^{(2,2)}$ has distinct eigenvalues. ■

5 ORTHOGONAL FAMILIES

We rewrite (1.1) as $Q^*Q = \mathbb{I}_n$, where Q is the $(n + d) \times n$ matrix stack. Thus Q is the first n columns of the product of n orthogonal $(n + d) \times (n + d)$ matrices. We parameterize each of the n matrices with d parameters for a Householder transformation or d Givens rotations. We denote the group of orthogonal $m \times m$ matrices by $O(m)$.

Our basic building block is a d dimensional parameterization $\{\tilde{Q}(\theta)\}$ of these orthogonal reduction transformations. Here θ is the d -dimensional parameter vector.

Definition 5.1 *An orthogonal reduction parameterization (ORP) of $O(m)$ to e_k is a $m - 1$ dimensional family $\mathcal{Q} = \{\tilde{Q}(\theta)\}$ of $m \times m$ orthogonal matrices such that for every m vector, h , there exists a unique $\theta(h)$ such that $\tilde{Q}(\theta)^*h = \|h\|e_k$. A family of orthogonal matrices is an unsigned orthogonal reduction parameterization (ORP) of $O(m)$ to e_k if for every m vector, $h \neq 0$, there exists a unique $\theta(h)$ such that $\tilde{Q}(\theta)^*h$ is in the e_k direction.*

Unsigned ORPs require that $\theta(-h) = \theta(h)$ while standard ORPs require that $Q(\theta(-h)) \neq Q(\theta(h))$. The K th column of $\tilde{Q}(\theta)$ is equal to $\pm h/\|h\|$. Thus $\tilde{Q}(\theta)$ may be determined by the k th column of $\tilde{Q}(\theta)$. For OTSON representations, we will use ORPs of $O(d + 1)$ to e_1 . For HOON representations, we will use ORPs of $O(d + 1)$ to e_{d+1} and ORPs of $O(d)$ to e_d .

The traditional vector reduction families are the set of Householder transformations and families of Given's transformations. For ORPs from $O(d + 1)$ to e_1 , the two traditional Given's ORPs are

$$\tilde{\mathcal{Q}}_1 = \{\tilde{Q}(\theta) = G_{1,d+1}(\theta_d)G_{1,d}(\theta_{d-1}) \cdots G_{1,2}(\theta_1)\}, \quad (5.1)$$

$$\tilde{\mathcal{Q}}_2 = \{\tilde{Q}(\theta) = G_{d,d+1}(\theta_d)G_{d-1,d}(\theta_{d-1}) \cdots G_{2,3}(\theta_2)G_{1,2}(\theta_1)\}. \quad (5.2)$$

For both \tilde{Q}_1 and \tilde{Q}_2 , we restrict the Givens angles: $-\pi/2 < \theta_i \leq \pi/2$ for $1 < i \leq d$ and $-\pi < \theta_1 \leq \pi$. The rightmost Givens rotation has twice the angular domain since it is used to make $e_1^* Q^*(\theta) h$ positive.

Let $\tilde{Q}(\theta)$ be a ORP from $O(d+1)$ to e_1 with the block representation:

$$\tilde{Q}(\theta) = \begin{pmatrix} \mu & y^* \\ x & \tilde{O} \end{pmatrix}, \quad (5.3)$$

where μ is a scalar and x and y are d -vectors. The orthogonality of \tilde{Q} implies $\mu^2 + \|x\|^2 = \mu^2 + \|y\|^2 = 1$, $\mu x = -\tilde{O} y$, $\mu y = -\tilde{O}^* x$ and $\mathbb{I}_d = \tilde{O} \tilde{O}^* + x x^* = \tilde{O} [\mathbb{I}_d + y y^* / \mu^2] \tilde{O}^*$. Thus \tilde{O} is invertible if $\mu \neq 0$.

We embed $\tilde{Q}(\theta)$ in the space of $(n+d) \times (n+d)$ matrices.

$$Q^{(k)}(\theta) = \mathbb{I}_{k-1} \oplus \begin{pmatrix} \mu_k & 0_{1,n-k} & y_k^* \\ 0_{n-k,1} & \mathbb{I}_{n-k} & 0_{n-k,d} \\ x_k & 0_{d,n-k} & \tilde{O}_k \end{pmatrix}, \quad (5.4)$$

where μ_k , x_k , y_k and \tilde{O}_k are subblocks of (5.3). For the Givens rotations of class \mathcal{Q}_1 , we have

$$Q^{(k)}(\theta) = G_{k,n+d}(\theta_d) G_{k,n+d-1}(\theta_{d-1}) \cdots G_{k,n+1}(\theta_1). \quad (5.5)$$

An ORP from $O(d)$ to e_d is

$$\tilde{Q}_3 = \{ \tilde{Q}(\theta) = G_{1,2}(\theta_1) G_{2,3}(\theta_2) \cdots G_{d-2,d-1}(\theta_{d-2}) G_{d-1,d}(\theta_{d-1}) \}, \quad (5.6)$$

where now the angular restrictions are $-\pi/2 < \theta_i \leq \pi/2$ for $1 \leq i < d-1$ and $-\pi < \theta_{d-1} \leq \pi$.

6 OTSON representations.

The key to our OTSON representation is the recognition that the (C, A) stack is column orthonormal. These results include the representation of [18] as an important special case. Our fundamental representation for OTSON pairs is

Theorem 6.1 *Every real OTSON pair has the representation:*

$$\begin{pmatrix} C \\ A \end{pmatrix} = Q^{(n)}(\theta_n) Q^{(n-1)}(\theta_{n-1}) \cdots Q^{(2)}(\theta_2) Q^{(1)}(\theta_1) \begin{pmatrix} \mathbb{I}_n \\ 0_{d,n} \end{pmatrix} \quad (6.1)$$

for some set of n d -vectors $\{\theta_1, \theta_2 \dots \theta_n\}$. Here the $Q^{(k)}$ are given by (5.4), where μ_k , x_k , y_k and \tilde{O}_k are subblocks of a ORP of $O(d+1)$ to e_1 .

We successively determine $\theta_n, \theta_{n-1}, \dots, \theta_1$. At the $(n - k + 1)$ th stage, θ_k is determined to zero out the d of the $d + 1$ nonzero entries in the k th column. By orthogonality the other entries in the k th row must be zero.

Proof: We determine θ_n so that $(Q^{(n)*}(\theta_n)Q)_{i,n} = \delta_{i,n}$. By orthonormality, $(Q^{(n)*}(\theta_n)Q)_{n,i} = \delta_{i,n}$. Let $\Omega^{(n+1)}$ be the (C, A) stack and set

$$\Omega^{(k)}(\theta_k, \theta_{k+1}, \dots, \theta_n) \equiv Q^{(k)*}(\theta_k)Q^{(k+1)*}(\theta_{k+1}) \dots Q^{(n)*}(\theta_n) \begin{pmatrix} C \\ A \end{pmatrix}. \quad (6.2)$$

Assume that $\Omega^{(k)}$ has its last $(n - k + 1)$ columns satisfying $\Omega_{ij}^{(k)} = \delta_{ij}$. Since $\Omega^{(k)}$ has orthonormal columns, $\Omega_{k:n,1:(k-1)}^{(k)} = 0$. Select θ_{k-1} such $\Omega_{(n+1):(n+d),(k-1)}^{(k-1)} = 0$. Then $\Omega_{j,(k-1)}^{(k-1)} = 0$ for $j \neq k - 1$ and therefore the last $(n - k + 1)$ columns satisfy $\Omega_{ij}^{(k-1)} = \delta_{ij}$. ■

For the Givens rotations of class \mathcal{Q}_1 , we have

$$\begin{pmatrix} C \\ A \end{pmatrix} = G_{n,n+d}(\theta_{n,d})G_{n,n+d-1}(\theta_{n,d-1}) \dots G_{n,n+1}(\theta_{n,1}) \dots G_{1,n+d}(\theta_{1,d}) \dots G_{1,n+1}(\theta_{1,1}) \begin{pmatrix} \mathbb{I}_n \\ 0_{d,n} \end{pmatrix} \quad (6.3)$$

We now show that every matrix of the form given in the righthand side of (6.1) is a OTSON matrix. We define

$$\Gamma^{(k)} \equiv Q^{(k)}(\theta_k)Q^{(k-1)}(\theta_{k-1}) \dots Q^{(1)}(\theta_1). \quad (6.4)$$

Lemma 6.2 *Let $Q^{(k)}$ have the structure given by (5.3) and (5.4), then $\Gamma^{(k)}$ has the structure:*

$$\Gamma^{(k)} \equiv \begin{pmatrix} L_k & 0_{k,n-k} & N_k \\ 0_{n-k,k} & \mathbb{I}_{n-k} & 0_{n-k,d} \\ M_k & 0_{d,n-k} & P_k \end{pmatrix}, \quad (6.5)$$

where L_k is a lower triangular $k \times k$ matrix and the following recurrence relations hold:

$$L_k = \begin{pmatrix} L_{k-1} & 0 \\ y_k^* M_{k-1} & \mu_k \end{pmatrix}, \quad N_k = \begin{pmatrix} N_{k-1} \\ y_k^* P_{k-1} \end{pmatrix} \quad (6.6)$$

$$M_k = \begin{pmatrix} \tilde{O}_k M_{k-1} & x_k \end{pmatrix}, \quad P_k = \tilde{O}_k P_{k-1}. \quad (6.7)$$

Proof: Assume (6.5) for $k - 1$ and multiply $Q^{(k)}\Gamma^{(k-1)}$. ■

Lemma 6.2 does not use the fact that $\tilde{Q}^{(k)}$ is orthogonal. Lemma 6.2 is a special case of a more general theory of matrix subblock products [15].

The last $(d + 2)$ rows of A may be rewritten as

$$A_{(n-d-1):n,1:n} = \begin{pmatrix} y_{n-1}^* M_{n-2} & \mu_{n-1} & 0 \\ y_n^* \tilde{O}_{n-1} M_{n-2} & y_n^* x_{n-1} & \mu_n \\ \tilde{O}_n \tilde{O}_{n-1} M_{n-2} & \tilde{O}_n x_{n-1} & x_n \end{pmatrix}. \quad (6.8)$$

Lemma 6.2 implies that L_n is lower triangular and thus $\Gamma_{:,1:n}^{(n)}$ corresponds to the (C, A) stack of an observer Hessenberg system stack:

Corollary 6.3 *Every (C, A) stack of the form (6.1) is a OTSON pair when the $Q^{(k)}$ are orthogonal matrices satisfying (5.4).*

A parameterization of state space models is identifiable when only one parameter vector corresponds to each transfer function; i.e. the map from parameters to input-output behavior is injective. We now show that the mapping between standard OTSON pairs and orthogonal product representation given in Theorem 6.1 is one to one and onto.

Theorem 6.4 *Let each $Q^{(k)}(\theta_k)$ be an embedding of a ORP of $O(d + 1)$ to e_1 as given by (5.3). Then there is a one to one correspondence between strict OTSON pairs and the orthogonal product parameterization of Lemma 6.2 with $\{\mu_k > 0\}$. There is a one to one correspondence between unreduced OTSON pairs and the orthogonal product parameterization restricted to $\{\mu_k \neq 0\}$.*

Proof: Theorem 6.1 shows that every OTSON pair has such a representation. From (6.7), we represent the last d rows of A as $M_n = (\tilde{O}_n \tilde{O}_{n-1} \cdots \tilde{O}_2 x_1, \dots, \tilde{O}_n x_{n-1}, x_n)$. For $\mu_k \neq 0$, \tilde{O}_k may be determined and inverted and x_{k-1} is determinable from $A_{k-1:n,n-d+1:n}$. We let $\{x_1, x_2 \dots x_n\}$ vary over $|x_k| < 1$. Thus the mapping of OTSON pairs into the product ORP representation is *onto*. ■

For our parameterization of output pairs to be truly identifiable, we need to restrict our parameter space, $\theta \in \Theta$, such that no two output pair representations, $Q(\theta_1)$ and $Q(\theta_2)$, are equivalent. We prefer to restrict our parameterizations to $\{\theta | \mu_k \geq 0\}$. This set has redundant representations only when at least one $\mu_k = 0$.

7 HESSENBERG OBSERVER OUTPUT NORMAL FORM

In this section, we give representation results for HOON pairs. The first row of C satisfies $C_{1,1} = \sqrt{1 - \gamma^2}$, $C_{1,j} = 0$ for $j > 1$. We do not transform this row and treat γ as a free parameter. We use Givens rotations to zero out the lower diagonal of A and the row 2

through row d of C . For each column of the (C, A) stack, we use d Givens rotations except for the final column which requires only $d - 1$.

We embed orthogonal reduction parameterizations of $O(d + 1)$ to e_{d+1} into the space of $(n + d - 1) \times (n + d - 1)$ matrices. We define $\tilde{V}(\theta)$ in $(n + d - 1) \times (n + d - 1)$ dimensional matrices $V^{(k)}$:

$$V^{(k)}(\theta) = \begin{pmatrix} \tilde{O}_k & 0_{d,k-1} & x_k \\ 0_{k-1,d} & \mathbb{I}_{k-1} & 0_{k-1,1} \\ y_k^* & 0_{1,k-1} & \mu_k \end{pmatrix} \oplus \mathbb{I}_{n-k-1}, \quad (7.1)$$

for $1 \leq k < n$. Here x_k, y_k are d -vectors. Thus $V^{(k)}$ alters only the rows $1 : d$ and row $k + d$. We require that the $(d + 1) \times (d + 1)$ orthogonal matrix,

$$\tilde{V}(\theta) \equiv \begin{pmatrix} \tilde{O}_k & x_k \\ y_k^* & \mu_k \end{pmatrix} \quad (7.2)$$

be a member of a ORP from $O(d + 1)$ to e_{d+1} . For $V^{(n)}(\theta)$, we define.

$$V^{(n)}(\theta) = \tilde{V}_d(\theta) \oplus \mathbb{I}_{n-1}, \quad (7.3)$$

where $\tilde{V}_d(\theta)$ is a ORP from $O(d)$ to e_d . Thus $\{\theta_1, \dots, \theta_{n-1}\}$ are d -vectors while θ_n is a $d-1$ -vector. Our parameterization of HOON pairs uses a scalar, $0 \leq \gamma < 1$ and $\{\theta_1, \dots, \theta_n\}$. We denote the bottom $(d - 1)$ rows of C by \hat{C} .

Theorem 7.1 *Every real nondegenerate HOON pair has the representation:*

$$\begin{pmatrix} \hat{C} \\ A \end{pmatrix} = V^{(1)}(\theta_1)V^{(2)}(\theta_2)\dots V^{(n-1)}(\theta_{n-1})V^{(n)}(\theta_n) \begin{pmatrix} 0_{d-1,n} \\ P(\gamma) \end{pmatrix} \quad (7.4)$$

for some set of parameters, $\{\gamma, \theta_1, \theta_2, \dots, \theta_n\}$, with $0 < |\gamma| < 1$ and $C_{1,1} = \sqrt{1 - \gamma^2}$. Here $P(\gamma)$ is the $n \times n$ scaled permutation matrix: $P_{2,1} = \gamma$, $P_{k+1,k} = 1$ for $2 \leq k < n$, $P_{1,n} = 1$, and $P_{i,j} = 0$ otherwise. The $V^{(k)}(\theta_k)$ are defined in (7.1)-(7.3) and are members of the appropriate ORPs.

Proof: Let Ω_{n+1} be the (\hat{C}, A) stack and set

$$\Omega^{(k)}(\theta_1, \theta_2, \dots, \theta_k) \equiv V^{(k)*}(\theta_k)V^{(k-1)*}(\theta_{k-1})\dots V^{(1)*}(\theta_1) \begin{pmatrix} \hat{C} \\ A \end{pmatrix}. \quad (7.5)$$

Assume that $\Omega^{(k)}$ has its first k columns satisfying $\Omega_{i,j}^{(k)} = \gamma_j \delta_{i-d,j}$, where $\gamma_1 = \gamma$ and $\gamma_j = 1$ for $1 < j \leq k$. Since $\Omega^{(k)}$ has orthonormal columns, $\Omega_{(d+1):(d+k),1:n}^{(k)} = \gamma_j \delta_{i-d,j}$.

Select θ_{k+1} such $\Omega_{1:d,k+1}^{(k+1)} = 0$. Then $\Omega_{j,(k+1)}^{(k+1)} = 0$ for $j \neq k+1$, and therefore the first $k+1$ columns satisfying $\Omega_{ij}^{(k+1)} = \delta_{ij}$. ■

For $d = 1$ and $C_{1,:} = 0$, (7.4) is the well-known expression of an unitary Hessenberg matrix as a product of n Givens rotations [1]. To show that every matrix of the form given by the righthand side of (7.4) is a HOON pair, we define

$$X^{(k)}(\theta_1, \theta_2, \dots, \theta_k) \equiv V^{(1)}(\theta_1)V^{(2)}(\theta_2) \dots V^{(k)}(\theta_k). \quad (7.6)$$

Lemma 7.2 *Let $V^{(k)}$ have the structure given by (7.1) - (7.3), then $X^{(k)}$ has the structure:*

$$X^{(k)} \equiv \begin{pmatrix} N_k & H_k \end{pmatrix} \oplus \mathbb{I}_{n-k-1}, \quad (7.7)$$

where N_k is $(d+k) \times d$ and H_k is a $(d+k) \times k$ upper triangular matrix and the following recurrence relations hold:

$$N_k = \begin{pmatrix} N_{k-1} \tilde{O}_k \\ y_k^* \end{pmatrix}, \quad H_k = \begin{pmatrix} H_{k-1} & N_{k-1} x_k \\ 0_{1,k-1} & \mu_k \end{pmatrix}. \quad (7.8)$$

This result follows from multiplying out the matrix product.

Corollary 7.3 *Let $V^{(k)}$ have the structure given by (7.1) - (7.3) and let $P(\gamma)$ be the scaled permutation matrix. The righthand side of (7.4) defines an HOON pair with $C_{1,j} = \sqrt{1 - \gamma^2} \delta_{1,j}$ and $C_{2:d,:} = \hat{C}$.*

Theorem 7.4 *Under the definitions of Theorem 7.1, there is a one to one correspondence between strict HOON pairs and the parameterization of Theorem 7.1 restricted to $\{\mu_k > 0\}$.*

Proof: From Lemma 7.2, $A_{2,1} = \gamma \mu_1$ and $A_{k+1,k} = \mu_k$ for $2 \leq k < n$. The first d rows of A as $H_{1:d,1:n}^{(n)} = (x_1, \tilde{O}_1 x_2, \tilde{O}_1 \tilde{O}_2 x_3, \dots, \tilde{O}_1 \dots \tilde{O}_{n-1} x_n)$. Thus we can determine x_{k+1} from A and $\{x_1, \dots, x_k\}$. The proof is now identical to the proof of Theorem 6.4. ■

8 DISCUSSION

For each of the three output pairs, Schur ON form, observer triangular system ON and Hessenberg observer ON pairs, we have examined the uniqueness/identifiability of the representation in Section 4. We then express each of these output pairs in terms of an orthogonal product representation (OPR) as the product of orthogonal matrices involving a total of nd parameters (Theorems 6.1, 7.1). A similar representation is possible for Schur

ON form [17]. We have shown how to place restrictions on the parameters such that the orthogonal product representations are in one to one correspondence with sets of generic transfer functions. For OTSON and HOON representations, we recommend restricting the Given's rotations in Theorems 6.1 and 7.1 such that $\{\theta | \mu_k \geq 0\}$. This set has redundant representations only when at least one $\mu_k = 0$.

In practice, these orthogonal product representations are implemented with either Given's rotations or Householder transformations. Our definition of ORPs allows us to treat all the standard cases similarly. We do not explicitly store or multiply by $Q^{(k)}$ or $V^{(k)}$. Instead we store only the Given's or Householder parameters and we perform the matrix multiplication implicitly. For an n -vector v , we compute Av and Cv using the orthogonal product representation.

These orthogonal product representations have several advantageous properties:

- 1) $\frac{d}{d\theta_k} \begin{pmatrix} C \\ A \end{pmatrix}$ is easy to compute.
- 2) Vector multiplication by Q and by $\frac{d}{d\theta_k}Q$ require $O(6nd)$ and $O(8nd)$ operations, where Q is the (C, A) stack.
- 3) Observability and stability are equivalent and $\|A\| \leq 1$ is automatically satisfied
- 4) The controllability matrix, B , may be parameterized by its elements, $B_{i,j}$, separately from the parameters of (A, C) .
- 5) The observability Grammian is perfectly conditioned.

The final advantage is key for us. Many of the other well-known representation are very ill-conditioned [16]. A measure of the conditioning of a representation is the product of the condition number of the observability Grammian and the condition number of the controllability Grammian. As discussed in [16], balanced, input normal and output normal representations minimize this product of the condition numbers.

The fast filtering methods of [24] may be further sped up when (A, C) or (A^*, B) has the orthogonal product representations of this article. To transform a specific output pair to ON form, the dual Stein equation must be solved. The numerical conditioning of this problem can be quite poor [21, 16].

Which orthogonal product representation is most appropriate for my problem? Schur ON representations naturally display the eigenvalues of A while the spectrum of A must be numerically calculated when A is OTSON or HOON. If the parameterization evolves in time, the form of the Schur representation changes when eigenvalues coalesce and the block structure of A changes. Thus, for evolving representations, we prefer the OTSON and HOON representations. It is straightforward to impose the restrictions that $\text{rank } C = d$ in the OTSON form. If the problem requires derivatives of $A(\theta)$ and $C(\theta)$, the Givens rotation parameterization of ORPs is usually simpler than Householder reflections.

In summary, these orthogonal product representations offer the best possible conditioning while having a convenient representation with fast matrix multiplication. Corresponding controller representations exist for input pairs, (A, B) , that are input normalized.

9 APPENDIX A: SPECIFYING THE REAL SCHUR FORM

The real Schur representation is defined and described in [4, 6, 7]. We denote the number of complex conjugate pairs of eigenvalues by ℓ and the number of real eigenvalues by $n - 2\ell$. Let $m(k) = 2k - 1$ for $k \leq \ell$ and $m(k) = k + 2\ell$ for $\ell < k \leq n - \ell + 1$ with $m(0) = 0$, and define $M = n - \ell$. The Schur form is

$$\begin{pmatrix} Z_1 & R_{1,2} & R_{1,3} & \dots & R_{1,M} \\ 0 & Z_2 & R_{2,3} & \ddots & R_{2,M} \\ 0 & \dots & \ddots & \ddots & \vdots \\ \vdots & \dots & 0 & Z_{M-1} & R_{M-1,M} \\ 0 & \dots & 0 & 0 & Z_M \end{pmatrix}, \quad (9.1)$$

where Z_i are 2×2 matrices for $i \leq \ell$ and real scalars for $\ell < i \leq n - \ell$. Here $R_{i,j} \equiv A_{m(i):(m(i+1)-1), m(j):(m(j+1)-1)}$. Thus we explicitly require the complex conjugate eigenvalues to be placed ahead of the real eigenvalues for a matrix to be in Schur form. For identifiability, we need to uniquely specify the order of the blocks and the form of each block. Let $\{\lambda_i\}$ be the eigenvalues of A with $\lambda_{m(k)}$ being an eigenvalue of Z_k and λ_{2j} being an eigenvalue of Z_j for $j \leq \ell$.

Definition 9.1 *Let A be in real Schur form, (9.1), with ordered eigenvalues $\{\lambda_j\}$ as described above. Then A is in ordered Schur form if 1) $|\lambda_j| \geq |\lambda_i|$ for $i < j \leq \ell$ and for $2\ell < i < j$; 2) If $|\lambda_m(j)| = |\lambda_m(i)|$, then $\text{Re } \lambda_m(j) \leq \text{Re } \lambda_m(i)$ for $i < j \leq \ell$ and for $2\ell < i < j$;*

Definition 9.1 can be replaced by any other complete specification of the eigenvalue block order. Note that A may be transformed by a product of ℓ Givens rotations: $G_{1,2}G_{3,4} \dots G_{2\ell-1,2\ell}$ and still stay in Schur form. For identifiability, we also need to specify the form of each 2×2 diagonal subblock. Let Z_j denote a 2×2 diagonal subblock of a Schur A :

$$Z_j = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}. \quad (9.2)$$

A common standardization of Z_j is to require $z_{11} = z_{22}$, with $z_{12}z_{21} < 0$ and $z_{12} + z_{21} > 0$. We refer to this standardization of the two by two subblocks as λ_r block form since $z_{11} =$

$z_{22} = \lambda_r$, the real part of the eigenvalues. The λ_r form is also known as standardized form [3].

Theorem 9.2 *Let A and \hat{A} be $n \times n$ matrices in real Schur form with ordered eigenvalues. Let A and \hat{A} be orthogonally similar: $\hat{A}U = UA$ with U orthogonal. Let m be the number of distinct eigenvalue pairs plus the number of distinct real eigenvalues. Partition A , \hat{A} and U into m blocks corresponding to the repeated eigenvalue blocks. Then U has block diagonal form: $U = U_1 \oplus U_2 \oplus \dots \oplus U_m$, where U_i is orthogonal.*

Proof: From $\hat{A}_{m,m}U_{m,1} = U_{m,1}A_{1,1}$. If $m > 1$, then $\hat{A}_{m,m}$ and $A_{1,1}$ have no common eigenvalues. By Lemma 7.1.5 of [7], $U_{m,1} \equiv 0$. Repeating this argument shows $U_{m-k,1} \equiv 0$ for $k = 0, 1, \dots, m-1$. By orthogonality, $U_{1,j} = 0$ for $1 < j < m$. We continue this chain showing that $U_{i,2} = 0$ for $i \neq 2$, etc. Proof by finite induction. ■

When A has the distinct eigenvalues the block decomposition is precisely that of (9.1). When A has eigenvalue with multiplicity greater than one, the block decomposition groups the repeated eigenvalue blocks together. In the repeated eigenvalue case, not every block orthogonal transformation, $U = U_1 \oplus U_2 \oplus \dots \oplus U_m$, preserves the Schur form. We use the freedom of the 2×2 orthogonal blocks to standardize the diagonal of the real Schur form:

Corollary 9.3 *Let A be a $n \times n$ matrix with distinct eigenvalues. The A is orthogonally similar to a matrix, \hat{A} , in ordered λ_r real Schur form. and \hat{A} is unique up to diagonal unitary similarities: $\hat{A} \leftarrow E\hat{A}E^*$, where $|E_{i,j}|^2 = \delta_{i-j}$ and $E_{j,j} = 1$ for $j \leq 2\ell$.*

Proof: Existence of the orthogonal transformation is proven in Theorem 2.3.4 of [8] and by Theorem 9.2, it is unique up to block orthogonal transformations. The λ_r standardization uniquely determines the 2×2 diagonal subblocks of A [6]. ■

We propose the alternative standardization of the 2×2 blocks:

Definition 9.4 *Let A be a Schur matrix as given by (9.1). It is in qd block form if each 2×2 diagonal subblock satisfies $Z_j = Q_j D_j$, where Q_j is a 2×2 orthogonal matrix and D_j is a nonnegative diagonal matrix:*

$$Z_j = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad (9.3)$$

with $s \geq 0$ and $d_1 \geq d_2$.

The qd condition implies as $z_{11}z_{12} + z_{21}z_{22} = 0$, $z_{12} \geq 0$. An uniqueness result is

Lemma 9.5 *Every 2×2 nonsingular matrix, M , is orthogonally similar to a unique matrix $Z = QD$ in qd block form.*

Proof: Let M have the singular value decomposition, $M = U\Lambda V^*$ and set $Q = V^*U$ and $D = \Lambda$. If $d_1 < d_2$, then permute the rows and columns of Z : $D \rightarrow PDP$, $Q \rightarrow PQP$, where P is the 2×2 permutation matrix: $P_{1,2} = P_{2,1} = 1$ and $P_{1,1} = P_{2,2} = 0$. If $s < 0$, orthogonally transform Z by $T = \text{diag}(1, -1)$. Now suppose Z_1 and Z_2 are both in qd block form and are both orthogonally similar to M . Then Let $Z_i = Q_i D_i$ for $i = 1, 2$. Then there are two orthogonal matrices U_1 and U_2 such that $D_1 = U_1^* D_2 U_1$. From $D_1^2 = U_1^* D_2^2 U_1$, $D_1 = D_2$. If $D_{i,1,1} \neq D_{i,2,2}$ for $i = 1, 2$, then $U_1 = U_2 = \mathbb{I}_2$. If $D_{i,1,1} = D_{i,2,2}$, then by direct computation, $Q_1 = Q_2$. ■

Since we can always rotate the 2×2 diagonal subblocks from λ_r form to qd form, we have

Corollary 9.6 *Let A be a stable $n \times n$ matrix with distinct eigenvalues. The A is orthogonally similar to a unique matrix \hat{A} , where \hat{A} is ordered qd real Schur form.*

References

- [1] G. Ammar, W. Gragg, L. Reichel. Constructing a Unitary Hessenberg Matrix from Spectral Data. in *Numerical Linear Algebra, Digital Signal Processing, and Parallel Algorithms*, G. Golub and P. Van Dooren, eds., NATO ASI, Vol. 70, Springer Verlag, Berlin, pp. 385-305, 1991.
- [2] B.D.O. Anderson and J.B. Moore, *Optimal filtering*, Prentice-Hall, Englewood Cliffs, NJ 1979.
- [3] Z. Bai and J. W. Demmel, On swapping diagonal blocks in real Schur form, *Linear Algebra and its Applic.* 186,(1993) 73-95.
- [4] J. W. Demmel, *Applied Numerical Linear Algebra*, SIAM Press, Philadelphia, 1996.
- [5] U. B. Desai A state-space approach to orthogonal digital filters, *IEEE Trans. Circuits and Systems*, 38, (1991), 160-169.
- [6] A. Edelman, The circular law and the probability that a random matrix has k real eigenvalues, *J. Multivariate Anal.*, 60, (1997) 203-232.
- [7] G. H. Golub and C. F. Van Loan, *Matrix Computations*, John Hopkins University Press, Baltimore, 1996.
- [8] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

- [9] A. Laub, M. T. Heath, C. C. Paige and R. C. Ward, Computation of system balancing transformations and other applications of simultaneous diagonalization algorithms. *IEEE Trans. Automatic Control*, 32, (1987), 115-124.
- [10] A. J. Laub and A. Linnemann. Hessenberg and Hessenberg triangular form in linear system theory, *Int. J. Control*, 44, (1986), 1523-1547.
- [11] L. Ljung and T. Söderström. *Theory and practice of recursive identification*, The MIT Press, Cambridge, MA, 1983.
- [12] B. Moore. Principal components analysis in linear systems, *IEEE Trans. Aut. Cont.* 29 (1981) pp. 17-32, 1981.
- [13] A. Mullhaupt and K. S. Riedel, Fast identification of innovations filters, *IEEE Trans. Signal Proc.*, 45, (1997), 2616-2619.
- [14] A. Mullhaupt and K. S. Riedel, Banded matrix fraction representations of triangular input normal pairs. *IEEE Trans. Automatic Control*, vol. 49, pp. 2018-2022, Dec. 2001.
- [15] A. Mullhaupt and K. S. Riedel, Low grade matrices and matrix fraction representations. *Linear Algebra and its Applic.*, 342, (2001), 187-201.
- [16] A. Mullhaupt and K. S. Riedel, Bounds on the condition number of solutions of the discrete Lyapunov equation. *IEEE Trans. Signal Proc.*, vol. 52, pp. ?, 2004.
- [17] A. Mullhaupt and K. S. Riedel, Hessenberg and Schur output normal pair representations, manuscript.
- [18] C. T. Mullis and R. A. Roberts, *Digital Signal Processing*, Addison Wesley, Reading, MA, 1987.
- [19] R. J. Ober. "Balanced canonical forms" in *Identification, Adaption and Learning*, eds. S. Bittani and G. Picci, Springer Verlag, Berlin, 1996, 120-183.
- [20] R. J. Ober, Balanced parametrization of classes of linear systems. *SIAM J. Contr. Optim.*, 29, (1991), 1251-1287.
- [21] T. Penzl, Eigenvalue decay bounds for solutions of Lyapunov equations: The symmetric case, *System and Control Letters*. 40, (2000), 139-144.
- [22] P.P. Vaidyanathan. The discrete-time Bounded-Real Lemma in digital filtering. *IEEE Trans. CAS*, 38, (1991), 160-169.

- [23] View publication stats A.J. van der Veen and M. Viberg. Minimal continuous state-space parametrizations. In *Proc. Eusipco*, Trieste (Italy), (1996), 523-526.
- [24] P. Van Dooren and M. Verhaegen. Condensed forms for efficient time-invariant Kalman filtering. *SIAM J. Sci. Stat. Comp.*, 9, (1988), 516-530.