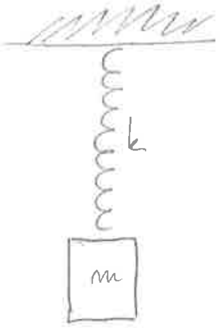
③ Vsiljeno (dušeno) nihanje

Vzmetno nihalo

- Spomnimo se: nedušeno nihanje



y: $F_y = -mg_0 - ky - c\dot{y}$; $-\frac{mg_0}{k} = y_0 \Rightarrow -mg_0 = ky_0$

$$F_y = ma_y = m\ddot{y}$$

$$m\ddot{y} = -k(y-y_0) - c\dot{y}$$

$$\boxed{\ddot{y} + \beta\dot{y} + \omega_0^2(y-y_0) = 0; \quad \omega_0^2 \equiv \frac{k}{m}, \beta \equiv \frac{c}{m}}$$ *

* Nehomogena diferencialna enačba za y:

$$\ddot{y} + \beta\dot{y} + \omega_0^2 y = \omega_0^2 y_0$$

S substitucijo $y \rightarrow y' = y - y_0$ jo predelamo v homogeno diferencialno enačbo za y':

$$\boxed{\ddot{y}' + \beta\dot{y}' + \omega_0^2 y' = 0}$$

Poznamo rešitve: za podkritično dušenje, npr., je

$$\boxed{y' = y'_{max} e^{-\frac{\beta}{2}t} \sin(\omega t + \delta); \quad \omega = \sqrt{\omega_0^2 - \left(\frac{\beta}{2}\right)^2}}$$

- harmonično vsiljeno nihanje: $F_y \rightarrow \boxed{F_y' = F_y + F_0 \sin(\omega_v t)}$

F_0: amplituda sile vzbujanja

$\nu_v = \frac{\omega_v}{2\pi}$: frekvenca vzbujanja ($t_v = \frac{1}{\nu_v} = \frac{2\pi}{\omega_v}$ nihajni čas vzbujanja)

$$F_y' = m\ddot{y}$$

$$\Rightarrow \boxed{\ddot{y} + \beta\dot{y} + \omega_0^2(y-y_0) = \frac{F_0}{m}\sin(\omega_v t)} \quad **$$

* ? običajno substitucijo $y \to y' = y - y_0$ enačba še vedno <u>nehomogena</u>!

$$\boxed{\ddot{y}' + \beta\dot{y}' + \omega_0^2 y' = \frac{F_0}{m}\sin(\omega_v t)}$$

Rešitev nehomogene enačbe: vsota rešitve y_h' homogene enačbe,

$$\boxed{y_h' = y_{h,max}' e^{-\beta/2t}\sin(\omega t + \delta)}$$ (za podkritično dušenje), • lastno nihanje, se zaduši

in partikularne rešitve y_p':

$$\boxed{y_p' = B \sin(\omega_v t + \delta_p)} = B\left[\cos\delta_p \sin(\omega_v t) + \sin\delta_p \cos(\omega_v t)\right]$$

(nastavek za partikularno rešitev)

$$\boxed{y' = y_h' + y_p'}$$

Vstavimo v diferencialno enačbo:

$$(\ddot{y}_h' + \ddot{y}_p') + \beta(\dot{y}_h' + \dot{y}_p') + \omega_0^2(y_h' + y_p') = \frac{F_0}{m}\sin(\omega_v t)$$

$$\Rightarrow \underbrace{\ddot{y}_h' + \beta\dot{y}_h' + \omega_0^2 y_h'}_{=0\ (y_h' \text{ rešitev homogene enačbe})} + \boxed{\ddot{y}_p' + \beta\dot{y}_p' + \omega_0^2 y_p' = \frac{F_0}{m}\sin(\omega_v t)}$$

$$\dot{y}_p' = \omega_v B \cos(\omega_v t + \delta_p) = \omega_v B\left[\cos\delta_p \cos(\omega_v t) - \sin\delta_p \sin(\omega_v t)\right]$$

$$\ddot{y}_p' = -\omega_v B \sin(\omega_v t + \delta_p) = -\omega_v^2 B\left[\cos\delta_p \sin(\omega_v t) + \sin\delta_p \cos(\omega_v t)\right]$$

$$\Rightarrow \boxed{B\cdot\left\{(\omega_0^2-\omega_v^2)\left[\cos\delta_p\sin(\omega_v t)+\sin\delta_p\cos(\omega_v t)\right]+\omega_v\beta\left[\cos\delta_p\cos(\omega_v t)-\sin\delta_p\sin(\omega_v t)\right]\right\} = \frac{F_0}{m}\sin(\omega_v t)\ ;\ \forall t}$$

a) $t_1=0$: $\omega_v t_1 = 0 \Rightarrow \sin(\omega_v t_1)=0,\ \cos(\omega_v t_1)=1$

$$\Rightarrow B\times\left\{(\omega_0^2-\omega_v^2)\sin\delta_p+\omega_v\beta\cos\delta_p\right\}=0$$

$$\Rightarrow \boxed{\tan\delta_p = \frac{\omega_v\beta}{\omega_v^2-\omega_0^2}}$$

$$\Rightarrow \tan\delta_p = \frac{\sqrt{1-\cos^2\delta_p}}{\cos\delta_p} \Rightarrow \tan^2\delta_p = \frac{1-\cos^2\delta_p}{\cos^2\delta_p} = \tan^2\delta_p\cos^2\delta_p = 1-\cos^2\delta_p$$

$$\Rightarrow \cos^2\delta_p\,(1+\tan^2\delta_p)=1 \Rightarrow \cos^2\delta_p = \frac{1}{1+\tan^2\delta_p} = \frac{1}{1+\frac{\omega_v^2\beta^2}{(\omega_v^2-\omega_0^2)^2}}$$

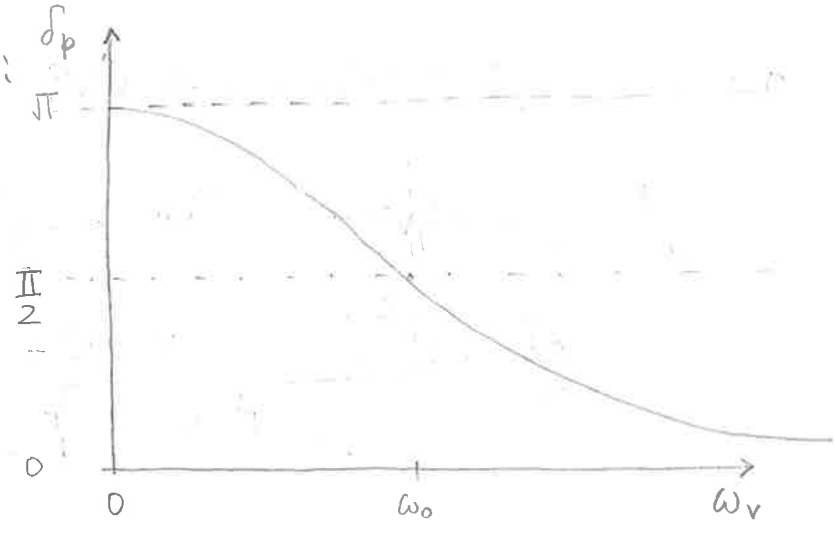
$$= \frac{(\omega_v^2-\omega_0^2)^2}{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}$$

$$\Rightarrow \boxed{\cos\delta_p = \pm\frac{(\omega_v^2-\omega_0^2)}{\sqrt{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}}}$$

$$\Rightarrow \sin^2\delta_p = 1-\cos^2\delta_p = 1-\frac{(\omega_v^2-\omega_0^2)^2}{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2} = \frac{\omega_v^2\beta^2}{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}$$

$$\Rightarrow \boxed{\sin\delta_p = \pm\frac{\omega_v\beta}{\sqrt{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}}}$$

Narisimo:



- $\omega_v\to 0$: $\tan\delta_p \to -0$
  $\Rightarrow \delta_p \to \pi$
- $\omega_v\to\omega_0$: $\tan\delta_p\to\infty$
  $\Rightarrow \delta_p\to\frac{\pi}{2}$
- $\omega_v\to\infty$: $\tan\delta_p\to +0$
  $\Rightarrow \delta_p\to 0$

$$\cos\delta_p = \pm \frac{(\omega_v^2-\omega_0^2)}{\sqrt{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}}$$

- $\omega_v \to 0$: $\delta_p \to \pi$, $\cos\delta_p \to -1$

$$\cos\delta_p \to \mp \frac{\omega_0^2}{\sqrt{\omega_0^4}} = \mp 1$$

- $\omega_v \to \infty$: $\delta_p \to 0$, $\cos\delta_p \to +1$

$$\cos\delta_p \to \pm \frac{\omega_v^2}{\sqrt{\omega_v^4}} = \pm 1$$

$$\Rightarrow \boxed{\cos\delta_p = + \frac{(\omega_v^2-\omega_0^2)}{\sqrt{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}}}$$

$$\sin\delta_p = \pm \frac{\omega_v\beta}{\sqrt{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}}$$

- $\delta_p \in [0,\pi] \Rightarrow \sin\delta_p > 0$

$$\Rightarrow \boxed{\sin\delta_p = + \frac{\omega_v\beta}{\sqrt{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}}}$$

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b) $\underline{t_2 = \frac{t_v}{4}}$: $\Rightarrow \omega_v t_2 = \frac{\pi}{2} \Rightarrow \sin(\omega_v t_2) = 1, \cos(\omega_v t_2) = 0$

$$\Rightarrow B \cdot \left\{ (\omega_0^2-\omega_v^2)\cos\delta_p - \omega_v\beta\sin\delta_p \right\} = \frac{F_0}{m}$$

$$\Rightarrow B \cdot \left\{ \frac{-(\omega_0^2-\omega_v^2)^2 - \omega_v^2\beta^2}{\sqrt{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}} \right\} = \frac{F_0}{m}$$

$$\Rightarrow \boxed{B = -\frac{F_0}{m}\frac{1}{\sqrt{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}}}$$

$$\Rightarrow \quad B = -\frac{F_0}{m\omega_v} \frac{1}{\sqrt{\frac{1}{\omega_v^2}(\omega_v^2-\omega_0^2)^2+\beta^2}}$$

$$\Rightarrow \quad \boxed{\left|\frac{F_0/m\omega_v}{B}\right| = Z = \sqrt{\frac{1}{\omega_v^2}(\omega_v^2-\omega_0^2)^2+\beta^2}} \quad : \text{impendanca}$$

$$|B| = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2}} \quad : \text{amplituda}$$

Resonanca: $\frac{d}{d\omega_v}\left[(\omega_v^2-\omega_0^2)^2+\omega_v^2\beta^2\right] = 0 \quad (\text{min.})$

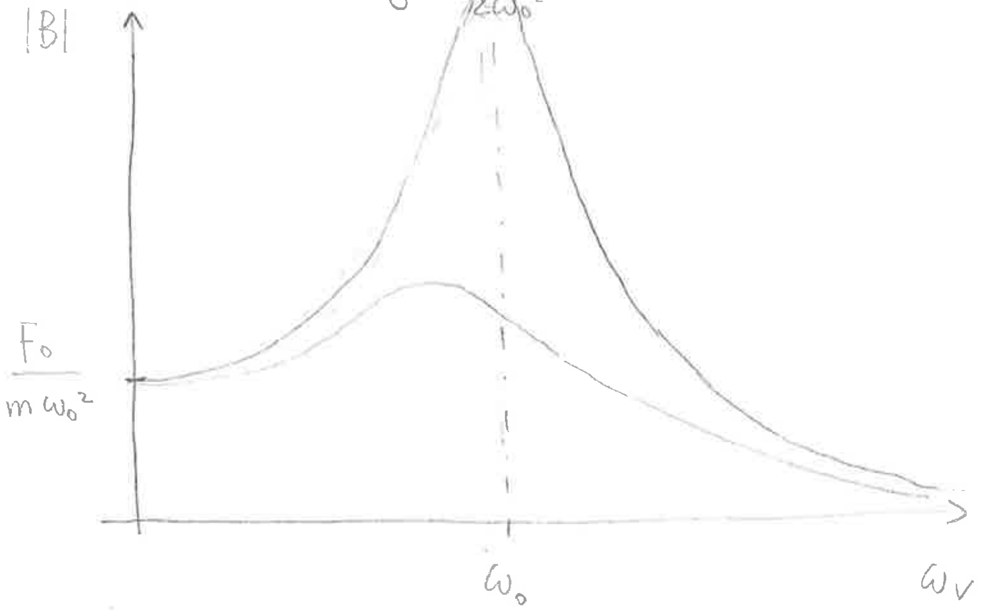
$$2(\omega_v^2-\omega_0^2)\cdot 2\omega_v + 2\omega_v\beta^2 = 0$$

$$2\omega_v\left[2(\omega_v^2-\omega_0^2)+\beta^2\right] = 0$$

$$\Rightarrow \omega_v^2 = \omega_0^2 - \frac{\beta^2}{2}$$

$$\boxed{\omega_v = \omega_0\sqrt{1-\frac{\beta^2}{2\omega_0^2}}}$$

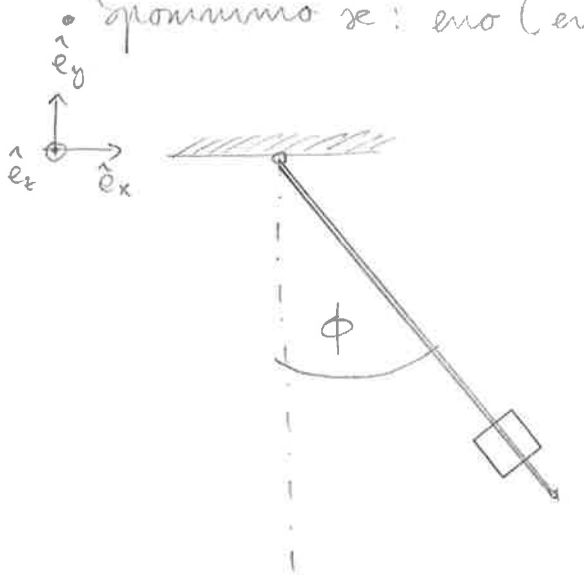
Zelo šibko dušenje $\left(\frac{\beta^2}{2\omega_0^2} \ll 1\right) \Rightarrow \omega_v \simeq \omega_0$



④ Sklopljeno nihanje

Dve sklopljeni fizični nihali

• Spomnimo se: eno (enostavno) fizično nihalo



J_z: vztrajnostni moment za vrtenje nihala okrog osi $\hat{e}_z$

l^*: razdalja med osjo in težiščem nihala

m: masa celotnega nihala (utež + palica)

$$M_z = M_g = -ml^* g_0 \sin\phi \overset{\phi \ll 1}{\simeq} -ml^* g_0 \phi$$

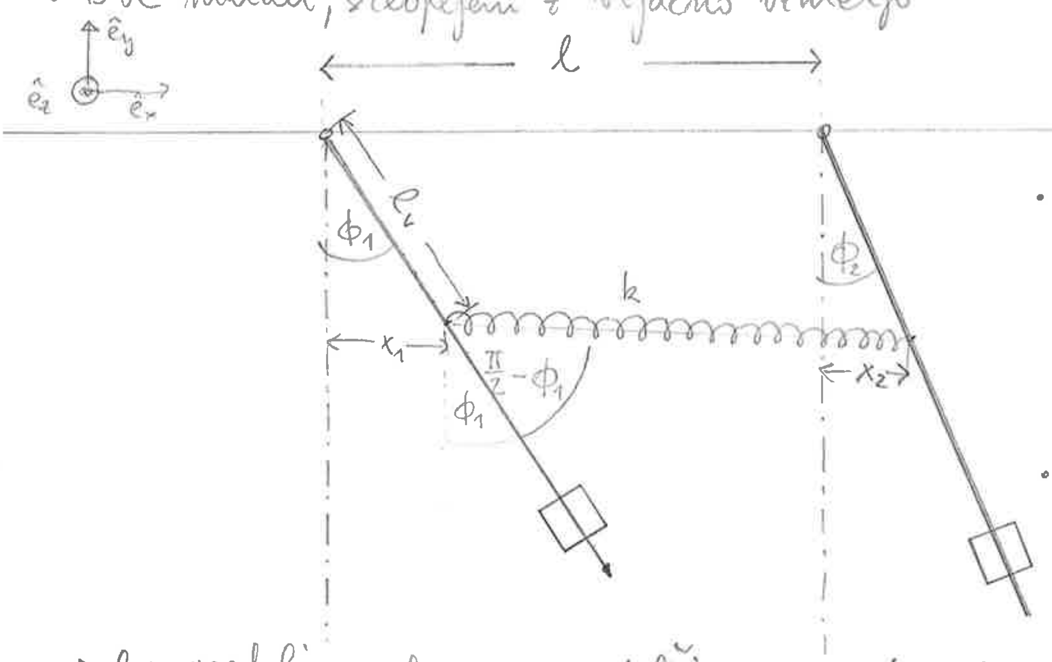
"Newtonov zakon" za vrtenje okoli dane osi:

$$M_z = J_z \ddot{\phi}$$

$$\Rightarrow -ml^* g_0 \phi = J_z \ddot{\phi}$$

$$\Rightarrow \boxed{\ddot{\phi} + \omega_0^2 \phi = 0} \quad \boxed{\omega_0^2 = \frac{ml^* g_0}{J_z}}$$

• Dve nihali, sklopljeni z vijačno vzmetjo



• enaki nihali:

$J_{z,1} = J_{z,2} = J_z$

$l^*_1 = l^*_2 = l^*$

$m_1 = m_2 = m$

• $l_{v,1} = l_{v,2} = l_v$: razdalja med osjo nihala in prijemališčem vzmeti

• l: razdalja med osema = dolžina neraztegnjene vzmeti

$$\frac{x_1}{l_v} = \sin\phi_1 \overset{\phi_1 \ll 1}{\simeq} \phi_1 \Rightarrow x_1 \simeq l_v \phi_1$$

$$\Rightarrow x_2 \simeq l_v \phi_2$$

Dolžina vzmeti : $l - x_1 + x_2 = l + (x_2 - x_1) = l + l_v(\phi_2 - \phi_1) = l + \Delta l$

$$\Rightarrow \Delta l = l_v(\phi_2 - \phi_1)$$

Hookov zakon : $F_{v1,1} = +k\Delta l = +k l_v(\phi_2 - \phi_1)$

$\Delta l > 0$: $F_{v1,1} > 0$ (v desno)

$\Delta l < 0$: $F_{v1,1} < 0$ (v levo)

$F_{v1,2} = -k\Delta l = -k l_v(\phi_2 - \phi_1) = -F_{v1,1}$ (3. Newtonov zakon)

$\Delta l > 0$: $F_{v1,2} < 0$ (v levo)

$\Delta l < 0$: $F_{v1,2} > 0$ (v desno)

$$\boxed{M_{v,1}} = l_v F_{v1,1} \sin\left(\frac{\pi}{2} - \phi_1\right) = +k l_v^2 (\phi_2 - \phi_1)\cos\phi_1 \overset{\phi_1 \ll 1}{\simeq} \boxed{+k l_v^2 (\phi_2 - \phi_1)}$$

$$\boxed{M_{v,2}} \simeq \boxed{-k l_v^2 (\phi_2 - \phi_1)}$$

$\phi_1, \phi_2 \ll 1$:

$$M_{z,1} = -mgl^*\phi_1 + k l_v^2(\phi_2 - \phi_1)$$

$$M_{z,1} = J_z \ddot{\phi}_1 \Rightarrow -mgl^*\phi_1 + k l_v^2 (\phi_2 - \phi_1) = J_z \ddot{\phi}_1$$

osnovni enačbi

$$\Rightarrow \boxed{\ddot{\phi}_1 + \omega_0^2 \phi_1 - \omega_k^2(\phi_2 - \phi_1) = 0} \quad ; \quad \omega_0^2 = \frac{mgl^*}{J_z}, \ \omega_k^2 = \frac{k l_v^2}{J_z}$$

$$\Rightarrow \boxed{\ddot{\phi}_2 + \omega_0^2 \phi_2 + \omega_k^2(\phi_2 - \phi_1) = 0}$$

Mat: sklopljeni diferencialni enačbi (ϕ_1 in ϕ_2 v obeh enačbah)

Precej enostaven primer (simetrija, enaki nihali) → enačbi lahko neposredno razklopimo (splošen primer v seminarju)

$$\left.\begin{aligned}\phi_a &\equiv \phi_1+\phi_2\\ \phi_b &\equiv \phi_1-\phi_2\end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned}\phi_1 &= \tfrac{1}{2}(\phi_a+\phi_b)\\ \phi_2 &= \tfrac{1}{2}(\phi_a-\phi_b)\end{aligned}\right.$$

Diferencialni enačbi seštejemo:

$$\ddot{\phi}_1+\ddot{\phi}_2+\omega_0^2\phi_1+\omega_0^2\phi_2=0$$

$$(\ddot{\phi_1+\phi_2})+\omega_0^2(\phi_1+\phi_2)=0$$

$$\boxed{\ddot{\phi}_a+\omega_a^2\phi_a=0} \quad \boxed{\omega_a^2=\omega_0^2}$$

Podobno, če enačbi odštejemo, dobimo

$$\boxed{\ddot{\phi}_b+\omega_b^2\phi_b=0} \quad \boxed{\omega_b^2=\omega_0^2+2\omega_k^2}$$

Poznamo rešitvi:

$$\phi_a=\phi_{a,max}\sin(\omega_a t+\delta_a)$$

$$\phi_b=\phi_{b,max}\sin(\omega_b t+\delta_b)$$

$$\Rightarrow \boxed{\begin{aligned}\phi_1&=C_1\sin(\omega_a t+\delta_a)+C_2\sin(\omega_b t+\delta_b)\\ \phi_2&=C_1\sin(\omega_a t+\delta_a)-C_2\sin(\omega_b t+\delta_b)\end{aligned}} \left(C_1=\frac{\phi_{a,max}}{2},\ C_2=\frac{\phi_{b,max}}{2}\right)$$

C_1, C_2, δ_a in δ_b določimo iz "začetnih" pogojev.

Primer: $\phi_1(t=0)=\phi_0,\ \phi_2(t=0)=0$

$\dot{\phi}_1(t=0)=\dot{\phi}_2(t=0)=0$

$$\boxed{\begin{aligned}\dot{\phi}_1&=\omega_a C_1\cos(\omega_a t+\delta_a)+\omega_b C_2\cos(\omega_b t+\delta_b)\\ \dot{\phi}_2&=\omega_a C_1\cos(\omega_a t+\delta_a)-\omega_b C_2\cos(\omega_b t+\delta_b)\end{aligned}}$$

$$\Rightarrow \begin{aligned}C_1\sin\delta_a+C_2\sin\delta_b&=\phi_0\\ C_1\sin\delta_a-C_2\sin\delta_b&=0\end{aligned} \qquad \Rightarrow \begin{aligned}\omega_a C_1\cos\delta_a+\omega_b C_2\cos\delta_b&=0\\ \omega_a C_1\cos\delta_a-\omega_b C_2\cos\delta_b&=0\end{aligned}$$

$$\Rightarrow 2\omega_a C_1 \cos\delta_a = 0 \Rightarrow \delta_a = \pm\frac{\pi}{2}$$

$$2\omega_b C_2 \cos\delta_b = 0 \Rightarrow \delta_b = \pm\frac{\pi}{2}$$

$$\underline{\delta_a = \delta_b = +\frac{\pi}{2}} \Rightarrow \sin\delta_a = \sin\delta_b = 1$$

$$\left.\begin{aligned} C_1 + C_2 &= \phi_0 \\ C_1 - C_2 &= 0 \end{aligned}\right\} \Rightarrow C_1 = C_2 = \frac{\phi_0}{2}$$

$$\Rightarrow \boxed{\begin{aligned} \phi_1 &= \frac{\phi_0}{2}\left[\cos(\omega_a t) + \cos(\omega_b t)\right] \\ \phi_2 &= \frac{\phi_0}{2}\left[\cos(\omega_a t) - \cos(\omega_b t)\right] \end{aligned}}$$

Domača naloga: pokaži, da dobimo enaki rešitvi tudi za kombinacije:

- $\delta_a = +\frac{\pi}{2}$, $\delta_b = -\frac{\pi}{2}$
- $\delta_a = -\frac{\pi}{2}$, $\delta_b = +\frac{\pi}{2}$
- $\delta_a = \delta_b = -\frac{\pi}{2}$

Faktorizacija:

$$\cos(\omega_a t) + \cos(\omega_b t) = 2\cos\left(\frac{\omega_a+\omega_b}{2}t\right)\cos\left(\frac{\omega_a-\omega_b}{2}t\right)$$

$$\cos(\omega_a t) - \sin(\omega_b t) = 2\sin\left(\frac{\omega_a+\omega_b}{2}t\right)\sin\left(\frac{\omega_a-\omega_b}{2}t\right)$$

$$\Rightarrow \boxed{\begin{aligned} \phi_1 &= \phi_0 \cos\left(\frac{\omega_a+\omega_b}{2}t\right)\cos\left(\frac{\omega_a-\omega_b}{2}t\right) = \phi_0 \cos\left(\frac{\omega_a+\omega_b}{2}t\right)\cos\left(\frac{\omega_b-\omega_a}{2}t\right) \\ \phi_2 &= \phi_0 \sin\left(\frac{\omega_a+\omega_b}{2}t\right)\sin\left(\frac{\omega_a-\omega_b}{2}t\right) \end{aligned}}$$

Šibka sklopitev: $\omega_k^2 \ll \omega_0^2$

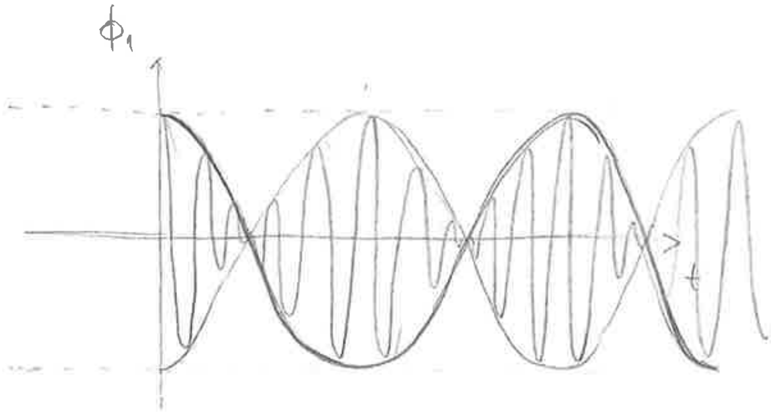
$$\frac{k l_v^2}{J_z} \ll \frac{m g_0 l^*}{J_z} \Rightarrow k \ll \frac{m g_0 l^*}{l_v^2}$$

$$\Rightarrow \omega_b = \sqrt{\omega_0^2 + 2\omega_k^2} = \omega_0 \sqrt{1 + \underbrace{2\frac{\omega_k^2}{\omega_0^2}}_{\ll 1}} \simeq \omega_0 + \frac{\omega_k^2}{\omega_0}$$

$$\Rightarrow \frac{\omega_b - \omega_a}{2} = \frac{1}{2}\frac{\omega_k^2}{\omega_0}$$

$$\Rightarrow \frac{\omega_a + \omega_b}{2} \simeq \omega_a = \omega_0$$

$$\Rightarrow 2\frac{\omega_k^2}{\omega_0^2} \ll 1 \Rightarrow \frac{1}{2}\frac{\omega_k^2}{\omega_0^2} \ll 1 \Rightarrow \frac{1}{2}\frac{\omega_k^2}{\omega_0} \ll \omega_0$$



Domača naloga:

a) $\phi_1(t=0) = \phi_2(t=0) = \phi_0$

$\dot{\phi}_1(t=0) = \dot{\phi}_2(t=0) = 0$

b) $\phi_1(t=0) = -\phi_2(t=0) = \phi_0$

$\dot{\phi}_1(t=0) = \dot{\phi}_2(t=0) = 0$

V splošnem rešitev linearna kombinacija $\sin(\omega_a t + \delta_a)$ in $\sin(\omega_b t + \delta_a)$.

Trdim: $\nu_a = \frac{\omega_a}{2\pi}$ in $\nu_b = \frac{\omega_b}{2\pi}$ sta (edini) lastni frekvenci.

Reš: Iščemo lastne frekvence ν_L, ko je

$$\phi_1 = \phi_{1,max}\sin(\omega_L t + \delta_1) \qquad ; \ \phi_{1,max}, \phi_{2,max} > 0, \ \omega_L = 2\pi\nu_L$$

$$\phi_2 = \phi_{2,max}\sin(\omega_L t + \delta_2)$$

$$\Rightarrow \ddot{\phi}_1 = -\omega_L^2 \phi_{1,max} \sin(\omega_L t + \delta_1) = -\omega_L^2 \phi_1$$

$$\ddot{\phi}_2 = -\omega_L^2 \phi_{2,max} \sin(\omega_L t + \delta_2) = -\omega_L^2 \phi_2$$

Vstavimo v osnovni enačbi:

$$\boxed{\begin{aligned} &(-\omega_L^2 + \omega_0^2 + \omega_k^2)\phi_1 - \omega_k^2 \phi_2 = 0 \\ &(-\omega_L^2 + \omega_0^2 + \omega_k^2)\phi_2 - \omega_k^2 \phi_1 = 0 \end{aligned}}$$

Iz zgornje enačbe: $\phi_2 = \dfrac{(-\omega_L^2 + \omega_0^2 + \omega_k^2)}{\omega_k^2} \phi_1$

V spodnjo enačbo: $(-\omega_L^2 + \omega_0^2 + \omega_k^2)^2 \phi_1 = \omega_k^4 \phi_1 \; ; \forall t$

$$\Rightarrow (-\omega_L^2 + \omega_0^2 + \omega_k^2)^2 = \omega_k^4$$

$$\Rightarrow \omega_L^4 - 2\omega_L^2(\omega_0^2 + \omega_k^2) + 2\omega_0^2\omega_k^2 + \omega_0^4 + \cancel{\omega_k^4} = \cancel{\omega_k^4}$$

$$\Rightarrow \omega_L^4 - 2\omega_L^2(\omega_0^2 + \omega_k^2) + \underbrace{\omega_0^2}_{\omega_a^2}\underbrace{(\omega_0^2 + 2\omega_k^2)}_{\omega_b^2} = 0$$

$$\Rightarrow \omega_L^4 - \omega_L^2(2\omega_0^2 + 2\omega_k^2) + \omega_a^2\omega_b^2 = 0$$

$$\Rightarrow \boxed{\omega_L^4 - \omega_L^2(\omega_a^2 + \omega_b^2) + \omega_a^2\omega_b^2 = 0}$$

Kvadratna enačba za ω_L^2 (dve rešitvi!)

$$D = (\omega_a^2 + \omega_b^2)^2 - 4\omega_a^2\omega_b^2 = \omega_a^4 + \omega_b^4 + 2\omega_a^2\omega_b^2 - 4\omega_a^2\omega_b^2$$

$$= (\omega_a^2 - \omega_b^2)^2$$

$$\Rightarrow \boxed{\omega_{L;1,2}^2} = \frac{\omega_a^2 + \omega_b^2 \pm (\omega_a^2 - \omega_b^2)}{2} = \boxed{\begin{cases} + : \omega_a^2 \\ - : \omega_b^2 \end{cases}}$$

$\Rightarrow$ lastni frekvenci sta ν_a in ν_b ✓

$$(-\omega_L^2+\omega_0^2+\omega_k^2)\,\phi_1=\omega_k^2\,\phi_2$$

$$(-\omega_L^2+\omega_0^2+\omega_k^2)\,\phi_{1,max}\sin(\omega_L t+\delta_1)=\omega_k^2\,\phi_{2,max}\sin(\omega_L t+\delta_2)$$

a) $\omega_L^2=\omega_a^2=\omega_0^2 \Rightarrow \cancel{\omega_k^2}\,\phi_{1,max}\sin(\omega_L t+\delta_1)=\cancel{\omega_k^2}\,\phi_{2,max}\sin(\omega_L t+\delta_2)$

$$\Rightarrow \phi_{1,max}\sin(\omega_L t+\delta_1)=\phi_{2,max}\sin(\omega_L t+\delta_2)\ \forall t$$

$$\Rightarrow t_1=0:\ \phi_{1,max}\sin\delta_1=\phi_{2,max}\sin\delta_2$$

$$\Rightarrow t_2=\frac{t_L}{4}:\ \phi_{1,max}\cos\delta_1=\phi_{2,max}\cos\delta_2$$

$$\Rightarrow \operatorname{tg}\delta_1=\operatorname{tg}\delta_2$$

$$\Rightarrow \boxed{\delta_1=\delta_2} \Rightarrow \boxed{\phi_{1,max}=\phi_{2,max}}\ \checkmark$$

b) $\omega_L^2=\omega_b^2=\omega_a^2+2\omega_k^2 \Rightarrow -\omega_L^2+\omega_0^2+\omega_k^2=-\omega_a^2-2\omega_k^2+\omega_a^2+\omega_k^2=-\omega_k^2$

$$\Rightarrow -\omega_k^2\,\phi_{1,max}\sin(\omega_L t+\delta_1)=\omega_k^2\,\phi_{2,max}\sin(\omega_L t+\delta_2)$$

$$\Rightarrow \phi_{1,max}\sin(\omega_L t+\delta_1)=\phi_{2,max}\cos\pi\,\sin(\omega_L t+\delta_2)$$

$$\Rightarrow \phi_{1,max}\sin(\omega_L t+\delta_1)=\phi_{2,max}\sin(\omega_L t+\delta_2+\pi);\ \forall t$$

$$\Rightarrow t_1=0:\ \phi_{1,max}\sin\delta_1=\phi_{2,max}\sin(\delta_2+\pi)$$

$$\Rightarrow t_2=\frac{t_L}{4}:\ \phi_{1,max}\cos\delta_1=\phi_{2,max}\cos(\delta_2+\pi)$$

$$\Rightarrow \operatorname{tg}\delta_1=\operatorname{tg}(\delta_2+\pi) \Rightarrow \boxed{\delta_1=\delta_2+\pi}\ \checkmark$$

$$\Rightarrow \phi_{1,max}\sin(\delta_2+\pi)=\phi_{2,max}\sin(\delta_2+\pi) \Rightarrow \boxed{\phi_{1,max}=\phi_{2,max}}$$

• Iskanje lastnih frekvenc se lahko lotimo tudi nekoliko drugače:

Nastavek: $\phi_1 = C\, e^{i\lambda t}$, $\phi_2 = B\, e^{i\lambda t}$ $\Rightarrow$ $\ddot{\phi}_1 = -\lambda^2 \phi_1 = -\lambda^2 C e^{i\lambda t}$, $\ddot{\phi}_2 = -\lambda^2 \phi_2 = -\lambda^2 B e^{i\lambda t}$

$$\Rightarrow \left[(-\lambda^2+\omega_0^2+\omega_k^2)C - \omega_k^2 B\right] e^{i\lambda t} = 0;\ \forall t \Rightarrow (-\lambda^2+\omega_0^2+\omega_k^2)C - \omega_k^2 B = 0$$

$$\left[-\omega_k^2 C + (-\lambda^2+\omega_0^2+\omega_k^2)B\right] e^{i\lambda t} = 0 \qquad \Rightarrow -\omega_k^2 C + (-\lambda^2+\omega_0^2+\omega_k^2)B = 0$$

Matrična enačba:

$$\begin{bmatrix} \omega_0^2+\omega_k^2-\lambda^2 & , & -\omega_k^2 \\ -\omega_k^2 & , & \omega_0^2+\omega_k^2-\lambda^2 \end{bmatrix} \begin{bmatrix} C \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- iskanje lastnih vrednosti λ^2 in lastnih vektorjev $\begin{bmatrix} C \\ B \end{bmatrix}$ matrike

$$\begin{bmatrix} \omega_0^2+\omega_k^2 & , & -\omega_k^2 \\ -\omega_k^2 & , & \omega_0^2+\omega_k^2 \end{bmatrix}$$

$$\det \begin{bmatrix} \omega_0^2+\omega_k^2-\lambda^2 & , & -\omega_k^2 \\ -\omega_k^2 & , & \omega_0^2+\omega_k^2-\lambda^2 \end{bmatrix} = 0 \Rightarrow \boxed{(\omega_0^2+\omega_k^2-\lambda^2)^2 - \omega_k^4 = 0}$$

$$\text{Rešitve: } \lambda^2 = \begin{cases} \omega_a^2 \\ \omega_b^2 \end{cases} \Rightarrow \lambda = \begin{cases} \pm\omega_a \\ \pm\omega_b \end{cases}$$

a) $\lambda^2 = \omega_a^2 = \omega_0^2$:

$\Rightarrow \lambda_{1,2} = \pm \omega_a$

$\Rightarrow \lambda_1 = +\omega_a \Rightarrow \phi_1 = C_1 e^{i\omega_a t}$

$\phi_2 = B_1 e^{i\omega_a t}$

$$\begin{bmatrix} \omega_k^2 , & -\omega_k^2 \\ -\omega_k^2 , & \omega_k^2 \end{bmatrix} \begin{bmatrix} C_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left. \begin{aligned} C_1 - B_1 = 0 \\ -C_1 + B_1 = 0 \end{aligned} \right\} \Rightarrow C_1 = B_1$$

$$\lambda_2 = -\omega_a \Rightarrow \left. \begin{aligned} \phi_1 = C_2 e^{-i\omega_a t} \\ \phi_2 = B_2 e^{-i\omega_a t} \end{aligned} \right\} \Rightarrow \ldots C_2 = B_2$$

$$\phi_1 = C_1 e^{i\omega_a t} + C_2 e^{-i\omega_a t} = \phi_{max,1} \sin(\omega_a t + \delta_1)$$

$$\phi_2 = B_1 e^{i\omega_a t} + B_2 e^{-i\omega_a t} = \phi_{max,2} \sin(\omega_a t + \delta_2)$$

$B_1 = C_1$, $B_2 = C_2 \Rightarrow \phi_{max,1} \sin(\omega_a t + \delta_1) = \phi_{max,2} \sin(\omega_a t + \delta_2)$

$$\Rightarrow \boxed{\delta_1 = \delta_2 \text{ in } \phi_{max,1} = \phi_{max,2}}$$

b) $\lambda^2 = \omega_b^2 = \omega_0^2 + 2\omega_k^2$

$\Rightarrow \lambda_1 = +\omega_b$

$$\begin{bmatrix} -\omega_k^2 , & -\omega_k^2 \\ -\omega_k^2 , & -\omega_k^2 \end{bmatrix} \begin{bmatrix} C_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow B_1 = -C_1 \; ; \; B_2 = -C_2$$

$$\Rightarrow \phi_{max,2} \sin(\omega_a t + \delta_2) = -\phi_{max,1} \sin(\omega_a t + \delta_1) = \phi_{max,1} \sin(\omega_a t + \delta_1 + \pi)$$

$$\Rightarrow \boxed{\phi_{max,2} = \phi_{max,1}} , \boxed{\delta_2 = \delta_1 + \pi}$$