Mat: Stokesov izrek: $\boxed{\oint_{\partial S} \vec{E}\cdot d\vec{s} = \int_S (\nabla\times\vec{E})\cdot d\vec{S}}$

$$\oint_{\partial S}\vec{E}\cdot d\vec{s} = -U[\vec{r}(t_b)=\vec{r}(t_a), \vec{r}(t_a)] = -$$ napetost po (poljubni) zaključeni poti

$$\nabla\times\vec{E} = 0 \Rightarrow \boxed{\oint_{\partial S}\vec{E}\cdot d\vec{s} = 0} \quad \checkmark$$

Komentar: omejena veljavnost izreka o el. napetosti

- mirujoči naboji
- brez (časovno spreminjajočega se) magnetnega polja

Dopolnitev: (Faradayev) zakon o el. napetosti (indukcijski zakon)

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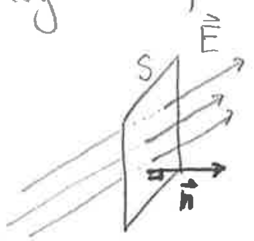
3 Električni pretok

Primer: homogeno $\vec{E}$, ravna ploskev



El. pretok - intuitivno: štejemo silnice, ki prebadajo ploskev

Formalno: $\vec{S} = \vec{n}S$; $\vec{n}$ normiran vektor, $\perp$ na S

$$\Phi_E \equiv \varepsilon_0 \vec{E}\cdot\vec{S} = \varepsilon_0 \vec{E}\cdot\vec{n}\, S$$

Splošna definicija (nehomogeno polje, kriva ploskev):

$$\boxed{\Phi_E \equiv \varepsilon_0\int_S \vec{E}\cdot d\vec{S} = \varepsilon_0\int_S \vec{E}\cdot\vec{n}\, dS}$$

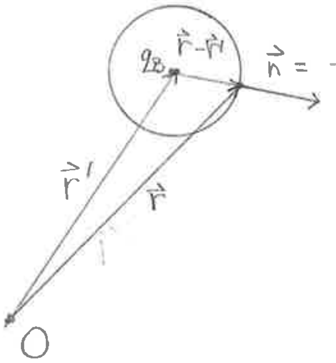
$$\boxed{[\Phi_E] = \frac{As}{Vm}\,\frac{V}{m}\,m^2 = As}$$

Primer: el. pretok skozi sfero z nabojem q v središču

$B(\vec r\,', \delta)$: krogla s središčem v $\vec r\,'$ in s polmerom δ

∂B : sfera (zaključena ploskev) ; $S = 4\pi\delta^2$

2D projekcija:



$$\vec n = \frac{\vec r - \vec r\,'}{|\vec r - \vec r\,'|} = \frac{\vec r - \vec r\,'}{\delta}$$ (kaže ven iz B)

$$\vec E(\vec r) = \frac{q_B}{4\pi\varepsilon_0}\frac{(\vec r - \vec r\,')}{|\vec r - \vec r\,'|^3} = \frac{q}{4\pi\varepsilon_0}\frac{(\vec r - \vec r\,')}{\delta^3}$$

$$\Rightarrow \boxed{\Phi_E(\partial B)} = \varepsilon_0 \oint_{\partial B} \frac{q_B}{4\pi\varepsilon_0}\frac{(\vec r - \vec r\,')}{|\vec r - \vec r\,'|^3}\cdot d\vec S = \oint \frac{q_B}{4\pi}$$

$$= \frac{q_B}{4\pi\delta^2}\oint_{\partial B} dS$$

$$= \frac{q_B}{4\pi\delta^2}\, S$$

$$= \frac{q_B}{4\pi\delta^2}\, 4\pi\delta^2$$

$$= \boxed{q_B}$$ ← zaobjeti naboj

Splošna resnica (izrek o Φ_E): poljubna zaključena ploskev, poljubna porazdelitev $\rho(\vec r\,')$ (mirujočih) nabojev

Mat : $\boxed{\operatorname{div} \vec{E} \equiv \nabla \cdot \vec{E} = \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z}$

Primer: $\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \Rightarrow E_x = \frac{q}{4\pi\varepsilon_0} \frac{(x-x')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$

$E_y = \dots$

$E_z = \dots$

Za $\vec{r} \neq \vec{r}'$: $\frac{\partial}{\partial x} E_x \propto \frac{1}{[\quad]^{3/2}} - (x-x')\left(-\frac{3}{2}\right)\frac{1}{[\quad]^{5/2}} \cdot 2(x-x')$

$$= \frac{1}{[\quad]^{3/2}} - \frac{3(x-x')^2}{[\quad]^{5/2}}$$

$$\frac{\partial}{\partial y} E_y \propto \frac{1}{[\quad]^{3/2}} - \frac{3(y-y')^2}{[\quad]^{5/2}}$$

$$\frac{\partial}{\partial y} E_z \propto \frac{1}{[\quad]^{3/2}} - \frac{3(z-z')^2}{[\quad]^{5/2}}$$

$$\Rightarrow \boxed{\nabla \cdot \vec{E}} \propto \frac{3}{[\quad]^{3/2}} - \frac{3[(x-x')^2+(y-y')^2+(z-z')^2]}{[\quad]^{5/2}}$$

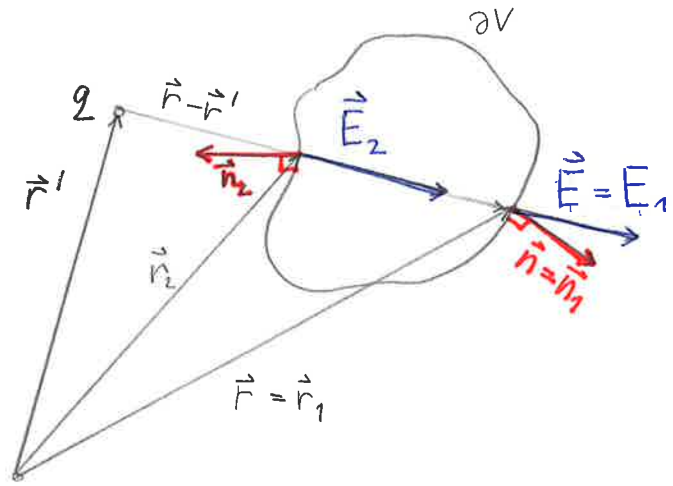
$$= \frac{3}{[\quad]^{3/2}} - \frac{3}{[\quad]^{3/2}} \boxed{= 0}$$

Mat : V območje v 3D

∂V : meja V (zaključena ploskev)

Izrek (Gauss, Ostrogradsky): $\boxed{\oint_{\partial V} \vec E\cdot d\vec S = \int_V \nabla\cdot\vec E\, dV}$ $\quad \left(\int_{\partial\Omega}\omega = \int_{\Omega} d\omega\right)$

Primer: Φ_E skozi ∂V, ki ne zaobjema nobenega naboja



$\forall \vec r\in V:\ \vec r\neq \vec r':\ \nabla\cdot\vec E=0$

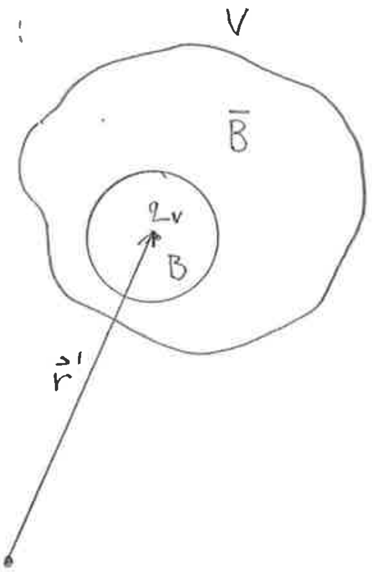
$$\Rightarrow \Phi_E = \varepsilon_0\oint_{\partial V}\vec E\cdot d\vec S = \varepsilon_0\int_V \nabla\cdot\vec E\, dV = \boxed{0}$$

Vsaka silnica, ki vstopi ($\vec n_2\cdot\vec E_2<0$), tudi izstopi ($\vec n_1\cdot\vec E_1>0$).

Izrek o Φ_E za točkast naboj: $\boxed{\Phi_E(\partial V) = \varepsilon_0\oint_{\partial V}\vec E\cdot d\vec S = q_V}$ zaobjeti naboj v V

Reš: • $q_V=0$: glej zgornji primer

• $q_V\neq 0$:



$V;\ B(\vec r',\sigma)\subset V;\ \bar B = V\setminus B\ \ (B\cap\bar B=\emptyset,\ B\cup\bar B=V)$

$q_B=q_V,\ q_{\bar B}=0\ (\nabla\cdot\vec E=0)$

$$\Rightarrow \boxed{\varepsilon_0\oint_{\partial V}\vec E\cdot d\vec S} = \varepsilon_0\int_V \nabla\cdot\vec E\, dV$$
$$= \varepsilon_0\int_B \nabla\cdot\vec E\, dV + \varepsilon_0\int_{\bar B}\overbrace{\nabla\cdot\vec E}^{0}\, dV$$
$$= \varepsilon_0\int_B \nabla\cdot\vec E\, dV = \dots$$

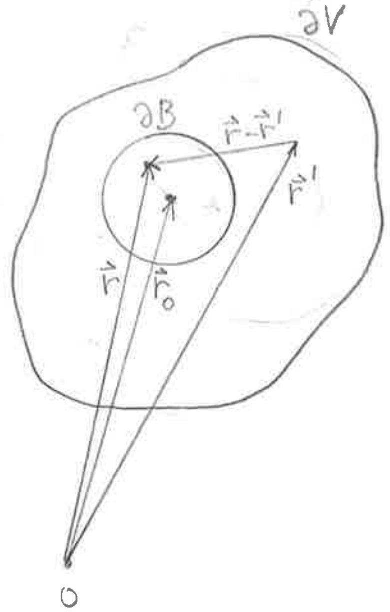
$$\dots = \varepsilon_0 \int_{\partial B} \vec{E}\cdot d\vec{S}$$

$$= q_B$$

$$= \boxed{q_V}$$

• <u>Komentar</u>: velja za poljubno območje, tudi za kroglo $B(\vec{r}_0, \delta)$, ki ni centrirana v $\vec{r}\,'$

<u>Splošen izrek o Φ_E!</u>



Ω: nabito telo

∂V: ploskev, skozi katero gledamo Φ_E

$\rho(\vec{r}\,')$: gostota naboja v okolici $\vec{r}\,'$

$$q_V = \int_V \rho(\vec{r}\,')\, dV'$$

<u>Izrek</u>: $\boxed{\Phi_E(\partial V) = \varepsilon_0 \oint_{\partial V} \vec{E}\cdot d\vec{S} = q_V}$

<u>Res</u>: $B(\vec{r}_0, \delta)$: krogla s središčem v $\vec{r}_0$ in s polmerom δ

$$\left.\begin{array}{l} B \subset V \\ \bar{B} \equiv V \setminus B \end{array}\right\} B \cup \bar{B} = V, \quad B \cap \bar{B} = \emptyset$$

Coulombov zakon (+ načelo superpozicije)

<u>$\vec{r} \in B$</u>; $$\vec{E}(\vec{r}) = \int_\Omega \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, dV' = \underbrace{\int_B \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, dV'}_{} + \int_{\bar{B}} \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, dV'$$

$$= \vec{E}_B + \vec{E}_{\bar{B}}$$

$$\Phi_E(\partial B) = \varepsilon_0 \oint_{\partial B} \vec{E}(\vec{r})\, dV = \varepsilon_0 \int_B \nabla\cdot\vec{E}\, dV = \varepsilon_0 \int_B \nabla\cdot(\vec{E}_B + \vec{E}_{\bar{B}})\, dV =$$

$$= \varepsilon_0 \int_B \nabla \cdot \vec{E}_B \, dV + \varepsilon_0 \int_B \nabla \cdot \vec{E}_{\bar{B}} \, dV$$

$$\vec{E}_{\bar{B}} = \int_{\bar{B}} \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \, dV' \quad ; \quad \vec{r}\,' \in \bar{B} \Rightarrow \vec{r}\,' \notin B \Rightarrow \vec{r}\,' \neq \vec{r}$$

$$\Rightarrow \nabla \cdot \vec{E}_{\bar{B}} = \nabla \cdot \int_{\bar{B}} \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \, dV'$$

$$= \int_{B} \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0} \underbrace{\nabla \left\{ \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \right\}}_{= 0 \, (\vec{r} \neq \vec{r}\,')} dV'$$

$$= 0$$

$$\Rightarrow \varepsilon_0 \int \nabla \cdot \vec{E}_{\bar{B}} \, dV = 0$$

(naboj izven B ne prispeva k el. pretoku skozi ∂B; velja splošno za poljubno območje V z mejno ploskvijo ∂V)

$$\Rightarrow \Phi_E(\partial B) = \varepsilon_0 \int_B \nabla \cdot \vec{E}_B \, dV$$

$$= \cancel{\varepsilon_0} \int_B \nabla \cdot \left\{ \int_B \frac{\rho(\vec{r}\,')}{4\pi\cancel{\varepsilon_0}} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \, dV' \right\} dV$$

$$= \int_B \left\{ \int_B \frac{\rho(\vec{r}\,')}{4\pi} \nabla \cdot \left\{ \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \right\} dV' \right\} dV$$

$$= \int_B \rho(\vec{r}\,') \left\{ \int_B \frac{1}{4\pi} \nabla \cdot \left\{ \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \right\} dV \right\} dV'$$

$$\int_B \frac{1}{4\pi} \nabla \cdot \left\{ \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \right\} dV = \int_{\partial B} \frac{1}{4\pi} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \, dV$$

$$= \frac{1}{q} \underbrace{\oint_{\partial B} \frac{q}{4\pi} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \, dS}$$

električni pretok skozi sfero z nabojem q v središču (glej zgoraj!)

$$= \frac{1}{q} \cdot q$$

$$= 1$$

$$\Rightarrow \Phi_E(\partial B) = \varepsilon_0 \int_B \nabla\cdot\vec{E}\, dV = \int_B \rho(\vec{r}\,')\, dV'$$

$$\varepsilon_0 \int_B \nabla\cdot\vec{E}\, dV = \varepsilon_0\, \nabla\cdot\vec{E}(\vec{r}_c)\, V_B \quad ; \vec{r}_c \in B$$

$$\int_B \rho(\vec{r}\,')\, dV' = \rho(\vec{r}_c\,')\, V_B \quad ; \vec{r}_c\,' \in B$$

$$\Rightarrow \varepsilon_0 \nabla\cdot\vec{E}(\vec{r}_c) = \rho(\vec{r}_c\,') \quad ; \forall \delta$$

$$\delta \to 0 \Rightarrow \vec{r}_c, \vec{r}_c\,' \to \vec{r}_0$$

$$\Rightarrow \boxed{\varepsilon_0 \nabla\cdot\vec{E}(\vec{r}_0) = \rho(\vec{r}_0) \quad ; \forall \vec{r}_0 \in V}$$

$\boxed{\varepsilon_0 \nabla\cdot\vec{E}(\vec{r}) = \rho(\vec{r})}$ izrek o Φ_E v diferencialni obliki.

$$\Rightarrow \Phi_E(\partial V) = \varepsilon_0 \oint_{\partial V} \vec{E}\cdot d\vec{S} = \varepsilon_0 \int_V \nabla\cdot\vec{E}\, dV = \int_V \rho(\vec{r})\, dV = q_V$$ (dokazan tudi v integralni obliki) ✓

- Komentar: izrek temelji na Coulombovem zakonu, ki (striktno) velja za mirujoče naboje. Eksperimenti pa kažejo na to, da trditev (tako v integralni kot tudi diferencialni obliki) velja splošno - tudi za premikajoče se naboje ⇒ izrek o električnem pretoku → zakon o električnem pretoku.

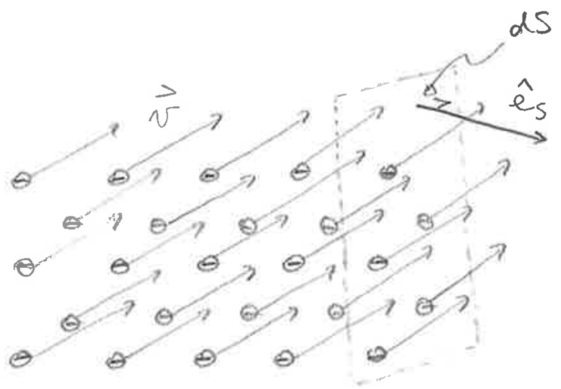
$\boxed{\text{III. ELEKTRIČNI TOK}}$

① gostota električnega toka, električni tok, kontinuitetna enačba

Analogija (konceptualna):

- Električni pretok Φ_E: pretakanje (silnic) el. polja skozi ploskev
- Električni tok I: pretakanje el. nabojev skozi ploskev

Prosti (nevezani) naboji:



- $n = \frac{N}{V}$; številska gostota prostih nabojev (>0)
- q_1 (v našem primeru $q_1 < 0$)

$\Rightarrow \rho_e = n q_1$; gostota naboja; $[\rho_e] = \frac{As}{m^3}$

- $\vec{v}$: (povprečna) hitrost nabojev (npr. pod vplivom el. sile zunanjega el. polja)

Def: $\boxed{\vec{j}_e \equiv \rho_e \vec{v} = q_1 n \vec{v}}$ $\longleftrightarrow$ Analogija: $\varepsilon_0 \vec{E}$ = vektor gostote el. polja

vektor gostote el. toka

$\boxed{[\vec{j}_e] = \frac{As}{m^3}\frac{m}{s} = \frac{A}{m^2}}$

$\vec{v} = v\hat{e}_v$, kjer je $v = |\vec{v}| > 0$

$\Rightarrow \vec{j}_e = q_1 n v \hat{e}_v$

$\Rightarrow \boxed{j_e} = |\vec{j}_e| = \boxed{|q_1| n v} > 0$

$\Rightarrow \boxed{\vec{j}_e} = |q_1| n v \frac{q_1}{|q_1|}\hat{e}_v = \boxed{j\hat{e}_j}$, kjer je $\boxed{\hat{e}_j \equiv \frac{q_1}{|q_1|}\hat{e}_v = \begin{cases} +\hat{e}_v; & q_1 > 0 \\ -\hat{e}_v; & q_1 < 0 \end{cases}}$

Ploskev $d\vec{S} = \hat{e}_s dS$:

Površina ploskve dS "dovolj" majhna, da je

a) ploskev ~ ravna

b) $\vec{j}_e$ v neposredni okolici ~ konst. (homogeno vektorsko polje)

Def: $\boxed{dI \equiv \vec{j}_e \cdot d\vec{S}}$ $\longleftrightarrow$ Analogija : $d\Phi_e \equiv \varepsilon_0 \vec{E} \cdot d\vec{S}$ = električni pretok skozi $d\vec{S}$

El. tok skozi $d\vec{S}$

$$\boxed{[dI] = \frac{A}{\cancel{m^2}} \cancel{m^2} = A}$$

Splošno : kriva ploskev, nehomogeno polje $\vec{j}_e$:

Def: $\boxed{I \equiv \int_S \vec{j}_e \cdot d\vec{S}}$ $\longleftrightarrow$ Analogija : $\Phi_E \equiv \int_S \varepsilon_0 \vec{E} \cdot d\vec{S}$

Primer: homogeno polje $\vec{j}_e$, ravna ploskev $\vec{S}$

$$\Rightarrow \boxed{I} = \int_S \vec{j}_e \cdot d\vec{S} = \int_S j_e \cos\phi \, dS = j_e \cos\phi \int_S dS = j_e \cos\phi \, S = \boxed{\vec{j}_e \cdot \vec{S}}$$

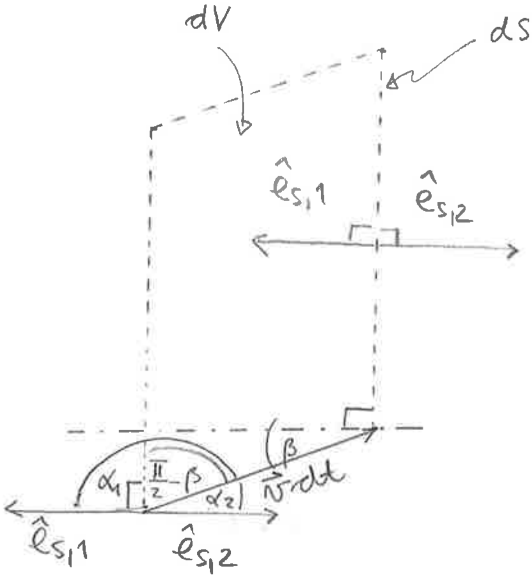
$\phi : \sphericalangle \vec{j}_e, \vec{S}$ = konst.

j_e = konst.

Primer: enako, kot v prejšnjem primeru, in $\hat{e}_s = \pm \hat{e}_j$ ($\vec{S}$ pravokotna na $\vec{j}$)

$$\Rightarrow I = \begin{cases} +j_e S \;\; (>0) \; ; \; \hat{e}_s = +\hat{e}_j \\ -j_e S \;\; (<0) \; ; \; \hat{e}_s = -\hat{e}_j \end{cases}$$

Interpretacija el. toka



- β: vpadni kot naboja (kot med $\vec v$ in $\perp$ na dS); $\cos\beta \geq 0$
- $dV = dS\, v\, dt \cos(\beta)$ prostornina, iz katere v času $[0, dt]$ vsi naboji prečkajo (desno) ploskev dS

$\Rightarrow dN = n\, dV = n v \cos\beta\, dS\, dt$

$\Rightarrow \boxed{d^3 q = q_1 \cdot dN = q_1 n v \cos\beta\, dS\, dt}$

= naboj, ki se v časovnem intervalu dolžine dt pretoči skozi ploskev dS v smeri $\vec v$

$$\Rightarrow \boxed{d^2\left(\frac{\partial q}{\partial t}\right) = q_1 n v \cos\beta\, dS}$$

Kako se to izrazi z $dI = \vec j_e \cdot d\vec S$?

$\alpha : \sphericalangle\, \vec e_v, \hat e_s$

(1) $\alpha_1 = \frac{\pi}{2} + \frac{\pi}{2} - \beta = \pi - \beta$

a) $q_1 > 0 \Rightarrow \hat e_j = \hat e_v \Rightarrow \boxed{\vec j_e \cdot d\vec S} = j_e dS\, \hat e_j \cdot \hat e_s = \overset{q_1}{\overset{\shortparallel}{|q_1|}} n v\, dS\, \overbrace{\cos(\pi-\beta)}^{-\cos\beta}$

$= -q_1 n v \cos\beta\, dS = \boxed{-d^2\left(\frac{\partial q}{\partial t}\right)}$

b) $q_1 < 0 \Rightarrow \hat e_j = -\hat e_v \Rightarrow \boxed{\vec j_e \cdot d\vec S} = j_e dS\, \hat e_j \cdot \hat e_s = -j_e dS\, \hat e_v \cdot \hat e_s = -\overbrace{|q_1|}^{-q_1} n v\, dS\, \overbrace{\cos(\pi-\beta)}^{-\cos\beta}$

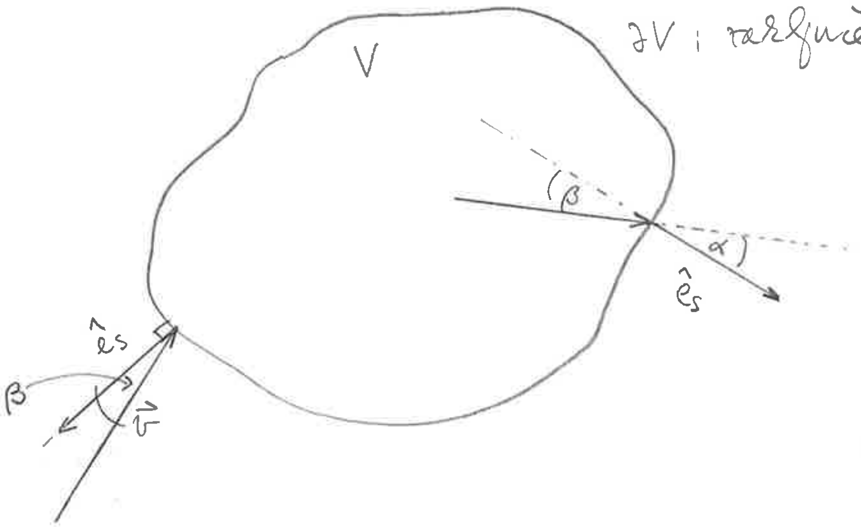
$= -q_1 n v \cos\beta\, dS = \boxed{-d^2\left(\frac{\partial q}{\partial t}\right)}$

(2) $\alpha_2 = \beta$

a) $q_1 > 0 \Rightarrow \hat e_j = \hat e_v \Rightarrow \boxed{\vec j_e \cdot d\vec S} = \overbrace{|q_1|}^{=q_1} n v\, dS \cos\beta = q_1 n v \cos\beta\, dS = \boxed{d^2\left(\frac{\partial q}{\partial t}\right)}$

b) $q_1 < 0 \Rightarrow \hat{e}_j = -\hat{e}_v \Rightarrow \boxed{\vec{j}_e \cdot d\vec{S} = -\overbrace{|q_1|}^{-q_1} n v \cos\beta \, dS = q_1 n v \cos\beta \, dS = d^2\left(\frac{\partial q}{\partial t}\right)}$

$\Rightarrow \quad dI = \vec{j}_e \cdot d\vec{S} = \pm d^2\left(\frac{\partial q}{\partial t}\right)$; $\pm$ naboj, ki se na enoto časa pretoči skozi dS (interpretacija dI)



∂V : zaključena ploskev; $\hat{e}_s$ ven iz V

$\alpha = \beta$

$$d^2\left(\frac{\partial q_V}{\partial t}\right) = -d^2\left(\frac{\partial q}{\partial t}\right) = -\vec{j}_e \cdot d\vec{S}$$

pretočeni naboj skozi dS gre iz V

$\alpha = \pi - \beta$:

$d^2\left(\frac{\partial q_V}{\partial t}\right)$: sprememba naboja v V na enoto časa

$$d^2\left(\frac{\partial q_V}{\partial t}\right) = +d^2\left(\frac{\partial q_V}{\partial t}\right) = -\vec{j}_e \cdot d\vec{S}$$

pretočeni naboj skozi dS preide v V

V obeh primerih: $\boxed{d^2\left(\frac{\partial q_V}{\partial t}\right) = -\vec{j}_e \cdot d\vec{S}}$

$$\Rightarrow \boxed{\left.\frac{\partial q_V}{\partial t}\right|_{\partial V} - \oint_{\partial V} \vec{j}_e \cdot d\vec{S}}$$

Sprememba naboja v V na enoto časa zaradi pretakanja naboja skozi ∂V

Zakon o ohranitvi (električnega) naboja - kontinuitetna enačba

Fizikalni zakon: (neto) električnega naboja ne moremo ne ustvariti ne izničiti.

Primeri:

$$\gamma \rightarrow e^+ e^-$$
$$q=0 \quad \underbrace{+q_0 \quad -q_0}_{q=0}$$

$$n \rightarrow p + e^- + \bar{\nu}_e$$
$$q=0 \quad \underbrace{+q_0 \quad -q_0 \quad 0}_{q=0}$$

$$p \rightarrow n + e^+ + \nu_e$$
$$+q_0 \quad \underbrace{0 \quad +q_0 \quad 0}_{+q_0}$$

⇒ Če hočemo v nekem območju V (neto) naboj povečati (zmanjšati), ga moramo (skozi ∂V) pripeljati iz okolice (odpeljati v okolico). Oz.: neto naboj v V se (na enoto časa) spremeni le za toliko, kolikor ga (na enoto časa) skozi ∂V pripeljemo iz okolice (ali odpeljemo v okolico).

Formalno:

$$q_V = \int_V \rho_e(\vec{r})\, dV$$

$$\Rightarrow \boxed{\frac{\partial q_V}{\partial t}} = \frac{\partial}{\partial t} \int_V \rho_e(\vec{r})\, dV = \boxed{\int_V \frac{\partial \rho_e}{\partial t}\, dV}$$

$$\boxed{\frac{\partial q_V}{\partial t} = \left.\frac{\partial q_V}{\partial t}\right|_{\partial V}}$$ zakon o ohranitvi naboja