13 Valovanje (zvok) v 3D

Primerjava:

- 1D: $u(x,t)$, $c^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

- 3D: $u(x,t) \longrightarrow \boxed{\vec{u}(\vec{r},t)}$

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \longrightarrow \boxed{c^2 \left(\frac{\partial^2 \vec{u}}{\partial x^2} + \frac{\partial^2 \vec{u}}{\partial y^2} + \frac{\partial^2 \vec{u}}{\partial z^2} \right) = \frac{\partial^2 \vec{u}}{\partial t^2}}$$ (brez izpeljave)

Primer: krogelno valovanje (zvočnik s krogelno membrano, ki niha radialno, krogelno simetrično)

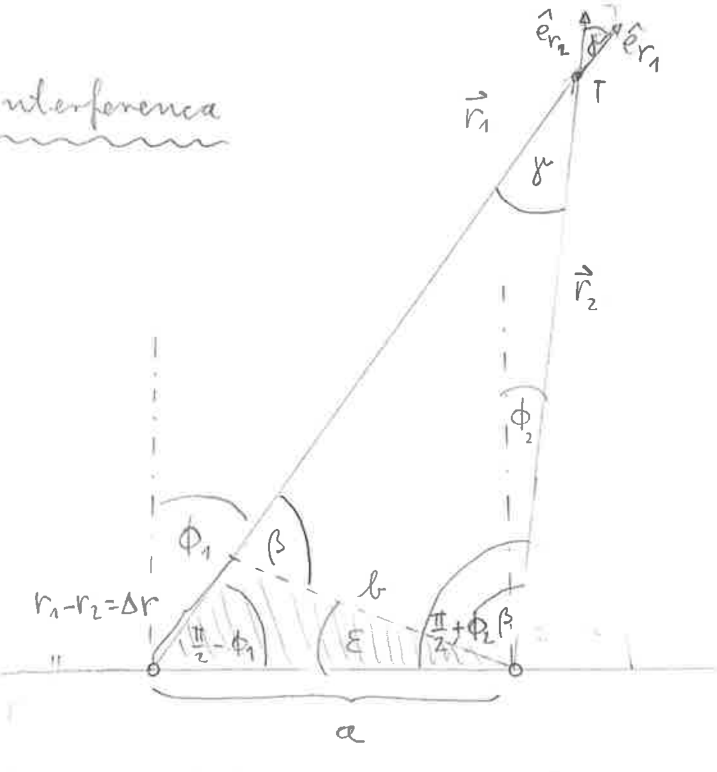
$$\vec{u}(\vec{r},t) = u(r,t)\,\hat{e}_r \; ; \quad \hat{e}_r = \frac{\vec{r}}{r}, \quad u(r,t) = \frac{u_0}{r} \sin(kr - \omega t + \delta) \; ; \quad u_0 = \text{konst.}$$

$$\left(\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; r = \sqrt{x^2+y^2+z^2} \right)$$

Domača naloga: pokaži, da zgornji $\vec{u}(\vec{r},t)$ res reši 3D valovno enačbo!

(Rešitev na seminarju – glej zadnje 3 strani pred poglavjem o električnem polju)

14 Interferenca



$$\vec{u}_1(\vec{r}_1,t) = \frac{u_0}{r_1} \sin(kr_1 - \omega t + \delta_1)\,\hat{e}_{r_1}$$

$$\vec{u}_2(\vec{r}_2,t) = \frac{u_0}{r_2} \sin(kr_2 - \omega t + \delta_2)\,\hat{e}_{r_2}$$

- $r_1, r_2 \gg a \Rightarrow$:

$$\gamma + \frac{\pi}{2} - \phi_1 + \frac{\pi}{2} + \phi_2 = \pi$$

$$\gamma - \phi_1 + \phi_2 = 0$$

$$\boxed{\gamma = \phi_2 - \phi_1 \xrightarrow{r_1, r_2 \gg a} 0} \Rightarrow \boxed{\phi_1 \simeq \phi_2} = \phi$$

$$\Rightarrow 2\beta + \gamma = \pi \Rightarrow \boxed{\beta \simeq \frac{\pi}{2}}$$

$$\varepsilon + \pi - \beta + \frac{\pi}{2} - \phi_1 = \pi$$

$$\Rightarrow \varepsilon = \phi_1 - \frac{\pi}{2} + \beta$$

$$\phi = \phi_1, \ \beta \simeq \frac{\pi}{2} \Rightarrow \boxed{\varepsilon = \phi}$$

$$\Rightarrow \boxed{\Delta r \simeq a \sin\varepsilon \simeq a \sin\phi}$$ (šrafirani trikotnik ~ pravokotni)

$$\Rightarrow \boxed{\Delta r \lesssim a \ll r_1, r_2}$$

$$r \to 0 \Rightarrow \boxed{\hat{e}_{r_1} \simeq \hat{e}_{r_2}} \equiv \hat{e}_r$$

$$\boxed{\frac{1}{r_2} = \frac{1}{r_1 - \Delta r} = \frac{1}{r_1\left(1 - \frac{\Delta r}{r_1}\right)} \simeq \frac{1}{r_1}\left(1 + \overset{\ll 1}{\frac{\Delta r}{r_1}}\right) \simeq \frac{1}{r_1}} \equiv \frac{1}{r}$$

$$\Rightarrow T: \vec{u}(t) = \vec{u}_1(\vec{r}_1, t) + \vec{u}_2(\vec{r}_2, t)$$

$$= \frac{u_0}{r}\left[\sin(kr_1 - \omega t + \delta_1) + \sin(kr_2 - \omega t + \delta_2)\right]\hat{e}_r$$

$$= \frac{2u_0}{r}\sin\left(k\frac{r_1 + r_2}{2} - \omega t + \frac{\delta_1 + \delta_2}{2}\right)\cos\left(k\frac{\Delta r}{2} + \frac{\delta_1 - \delta_2}{2}\right)\hat{e}_r$$

$$\simeq \tilde{u}_0 \sin\left(kr - \omega t + \frac{\delta_1 + \delta_2}{2}\right)\hat{e}_r \qquad \left(\frac{r_1 + r_2}{2} = \frac{2r_1 + \Delta r}{2} = r_1\left(1 + \frac{\Delta r}{2r_1}\right) \simeq r_1 = r\right)$$

pri čemer

$$\boxed{\tilde{u}_0 = \frac{2u_0}{r}\cos\left(\frac{k\Delta r}{2} + \frac{\delta_1 - \delta_2}{2}\right)} \simeq \boxed{\frac{2u_0}{r}\cos\left(\frac{ka\sin\phi}{2} + \frac{\delta_1 - \delta_2}{2}\right)}$$

Primer: $\delta_1 = \delta_2 \Rightarrow \tilde{u}_0 = \frac{2u_0}{r}\cos(ka\sin\phi)$

a) ojačitve: $\frac{ka\sin\phi}{2} = n\pi \quad (n \in \mathbb{Z})$

$$\frac{2\pi}{\lambda}\frac{a\sin\phi}{2} = n\pi \Rightarrow a\sin\phi = n\lambda \Rightarrow \boxed{\sin\phi_n = \frac{n\lambda}{a}}; \boxed{n \in \mathbb{Z}}$$

$|\sin\phi_n| \le 1$: $\frac{|n|\lambda}{a} \le 1 \Rightarrow \boxed{|n| \le \frac{a}{\lambda}}$

b) oslabitve : $\frac{ka\sin\phi}{2} = \frac{(2n+1)\pi}{2} = (n+\frac{1}{2})\pi \quad ; n \in \mathbb{Z}$

$$\Rightarrow \frac{2\pi a \sin\phi}{\lambda\, 2} = (n+\tfrac{1}{2})\pi$$

$$\Rightarrow a\sin\phi = (n+\tfrac{1}{2})\lambda \Rightarrow \boxed{\sin\phi_n = (n+\tfrac{1}{2})\frac{\lambda}{a}} \; ; \boxed{n \in \mathbb{Z}}$$

$$\boxed{|n+\tfrac{1}{2}| \le \frac{a}{\lambda}}$$

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(15) Dopplerjev pojav

ν: frekvenca oddajnika ($t_0 = \frac{1}{\nu}$)

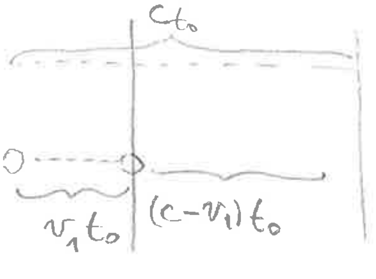
λ: valovna dolžina zvoka v snovi

c: hitrost valovanja (zvoka) v snovi (npr. zraku)

ν': frekvenca, ki jo sliši sprejemnik ($t_0' = \frac{1}{\nu'}$)

a) sprejemnik miruje glede na zrak ($v_2 = 0$), oddajnik pa se mu približuje s hitrostjo v_1

Ob času $t_1 = 0$ odda valovno čelo. Ob času t_0 valovno čelo prepotuje razdaljo ct_0, oddajnik pa $v_1 t_0$, in odda naslednje valovno čelo



$\Rightarrow$ razdalja med čeloma $\equiv \lambda = (c-v_1)t_0 = \frac{c-v_1}{\nu}$

$\Rightarrow$ na mestu sprejemnika je frekvenca ν' valovanja

$$\boxed{\nu' = \frac{c}{\lambda} = \nu \frac{c}{c-\frac{v_1}{c}} = \nu \frac{1}{1-\frac{v_1}{c}}}$$ (frekvenca se zviša)

Oddajnik se oddaljuje:

$$\boxed{\nu' = \nu \frac{1}{1+\frac{v_1}{c}}}$$ (frekvenca se zniža)

b) Oddajnik miruje glede na zrak, sprejemnik pa se mu približuje s hitrostjo v_2

Po zraku (proti sprejemniku) potujejo valovi dolžine λ s hitrostjo c.

Recimo, da 1. čelo pade na sprejemnik ob času $t_1 = 0$, drugo pa ob času $t_2 = t_0'$, ko 2. čelo prepotuje razdaljo ct_0', sprejemnik pa $v_2 t_0'$, skupaj

$$v_2 t_0' + c t_0' = \lambda = \frac{c}{\nu}$$

$$\frac{1}{\nu'}\left(1+\frac{v_2}{c}\right) = \frac{1}{\nu} \Rightarrow \boxed{\nu' = \nu\left(1+\frac{v_2}{c}\right)}$$

Sprejemnik se oddaljuje:

$$\boxed{\nu' = \nu\left(1-\frac{v_2}{c}\right)}$$

c) Oba, sprejemnik in oddajnik se gibljeta glede na zrak

$$\Rightarrow \boxed{\nu' = \nu \frac{1 \pm \frac{v_2}{c}}{1 \mp \frac{v_1}{c}}}$$

Seminar: funkcija $\vec{u}(\vec{r},t) = \frac{u_0}{r}\sin(kr-\omega t+\delta)\hat{e}_r$ je (asimptotska) rešitev valovne enačbe $\frac{\partial^2 \vec{u}(x,t)}{\partial t^2} = c^2 \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)}_{\nabla^2} \vec{u}(x,t)$

$\left(k = \frac{\omega}{c} = \frac{2\pi}{\lambda}\right)$.

Reš: $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $r = \sqrt{x^2+y^2+z^2}$, $\hat{e}_r = \frac{\vec{r}}{r} \begin{bmatrix} x/r \\ y/r \\ z/r \end{bmatrix}$, $\hat{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\hat{e}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\hat{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x} = -\frac{1}{r^2}\cdot\frac{x}{r} = -\frac{x}{r^3}} \quad \left(\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right) = -\frac{nx}{r^{n+2}}\right)$$

$$\boxed{\frac{\partial}{\partial x}\vec{r} = \hat{e}_x}$$

$$\boxed{\frac{\partial}{\partial x}\hat{e}_r} = \frac{\partial}{\partial x}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\frac{\partial}{\partial x}\vec{r} + \vec{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\hat{e}_x}{r} - \frac{x\vec{r}}{r^3} = \boxed{\frac{\hat{e}_x}{r} - \frac{x\hat{e}_r}{r^2}}$$

$$\boxed{\frac{\partial^2}{\partial x^2}\hat{e}_r} = \frac{\partial}{\partial x}\left(\frac{\hat{e}_x}{r} - \frac{x\hat{e}_r}{r^2}\right) = \frac{\partial}{\partial x}\left(\frac{\hat{e}_x}{r}\right) - \frac{\partial}{\partial x}\left(\frac{x\hat{e}_r}{r^2}\right) =$$

$$= \hat{e}_x\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - \frac{\hat{e}_r}{r^2} - x\hat{e}_r\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right) - \frac{x}{r^2}\frac{\partial}{\partial x}(\hat{e}_r)$$

$$= -\frac{x\hat{e}_x}{r^3} - \frac{\hat{e}_r}{r^2} - x\hat{e}_r\left(-\frac{2x}{r^4}\right) - \frac{x}{r^2}\left(\frac{\hat{e}_x}{r} - \frac{x\hat{e}_r}{r^2}\right)$$

$$\boxed{= -\frac{2x\hat{e}_x}{r^3} - \frac{\hat{e}_r}{r^2} + \frac{3x^2\hat{e}_r}{r^4}}$$

$$\boxed{\frac{\partial}{\partial x}[\sin(kr\ \omega t+\delta)] = \frac{\partial}{\partial r}[\sin(\ldots)]\frac{\partial r}{\partial x} = \frac{kx}{r}\cos(\ )}$$

$$\boxed{\frac{\partial}{\partial x}[\cos(\ )] = -\frac{kx}{r}\sin(\ )}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x}\left[\frac{1}{r}\sin(\ )\right] = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\sin(\ ) + \frac{1}{r}\frac{\partial}{\partial x}[\sin(\ )] = -\frac{x}{r^3}\sin(\ ) + \frac{kx}{r^2}\cos(\ )}$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial x^2}\left[\frac{1}{r}\sin(\ )\right]} = \frac{\partial}{\partial x}\left[-\frac{x}{r^3}\sin(\ ) + \frac{kx}{r^2}\cos(\ )\right]$$

$$= -\frac{1}{r^3}\sin(\ ) - x\sin(\ )\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) - \frac{x}{r^3}\frac{\partial}{\partial x}[\sin(\ )]$$

$$+ \frac{k}{r^2}\cos(\ ) + kx\cos(\ )\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right) + \frac{kx}{r^2}\frac{\partial}{\partial x}[\cos(\ )]$$

$$= -\frac{1}{r^3}\sin(\ ) + \frac{3x^2}{r^5}\sin(\ ) - \frac{kx^2}{r^4}\cos(\ )$$

$$+ \frac{k}{r^2}\cos(\ ) - \frac{2kx^2}{r^4}\cos(\ ) - \frac{k^2x^2}{r^3}\sin(\ )$$

$$\boxed{= \left[-\frac{1}{r^3} + \frac{3x^2}{r^5} - \frac{k^2x^2}{r^3}\right]\sin(\ ) + \left[\frac{k}{r^2} - \frac{3kx^2}{r^4}\right]\cos(\ )}$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial x^2}\left\{\frac{1}{r}\sin(\ )\hat{e}_r\right\}} = \frac{\partial^2}{\partial x^2}\left[\frac{1}{r}\sin(\ )\right]\hat{e}_r + 2\frac{\partial}{\partial x}\left[\frac{1}{r}\sin(\ )\right]\frac{\partial}{\partial x}(\hat{e}_r) + \left[\frac{1}{r}\sin(\ )\right]\frac{\partial^2}{\partial x^2}(\hat{e}_r)$$

$$= \left[-\frac{1}{r^3} + \frac{3x^2}{r^5} - \frac{k^2x^2}{r^3}\right]\sin(\ )\hat{e}_r + \left[\frac{k}{r^2} - \frac{3kx^2}{r^4}\right]\cos(\ )\hat{e}_r$$

$$+ 2\left[-\frac{x}{r^3}\sin(\ ) + \frac{kx}{r^2}\cos(\ )\right]\left[\frac{\hat{e}_x}{r} - \frac{x\hat{e}_r}{r^2}\right]$$

$$+ \frac{1}{r}\sin(\ )\left[-\frac{2x\hat{e}_x}{r^3} - \frac{\hat{e}_r}{r^2} + \frac{3x^2\hat{e}_r}{r^4}\right]$$

$$\boxed{\begin{aligned}= &\left[-\frac{1}{r^3} + \frac{3x^2}{r^5} - \frac{k^2x^2}{r^3}\right]\sin(\ )\hat{e}_r + \left[\frac{k}{r^2} - \frac{3kx^2}{r^4}\right]\cos(\ )\hat{e}_r \\ &+ 2\left[\frac{1}{r^3}\sin(\ ) - \frac{k}{r^2}\cos(\ )\right]\left[\frac{x\hat{e}_x}{r} - \frac{x^2\hat{e}_r}{r^2}\right] \\ &+ \frac{1}{r}\sin(\ )\left[-\frac{2x\hat{e}_x}{r^3} - \frac{\hat{e}_r}{r^2} + \frac{3x^2\hat{e}_r}{r^4}\right]\end{aligned}}$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial y^2}\left\{\frac{1}{r}\sin(\ )\hat{e}_r\right\} =}$$

$$\boxed{\begin{aligned} &= \left[-\frac{1}{r^3}+\frac{3y^2}{r^5}-\frac{k^2y^2}{r^3}\right]\sin(\ )\hat{e}_r + \left[\frac{k}{r^2}-\frac{3ky^2}{r^4}\right]\cos(\ )\hat{e}_r \\ &+2\left[\frac{1}{r^3}\sin(\ )-\frac{k}{r^2}\cos(\ )\right]\left[\frac{y\hat{e}_y}{r}-\frac{y^2\hat{e}_r}{r^2}\right] \\ &+\frac{1}{r}\sin(\ )\left[-\frac{2y\hat{e}_y}{r^3}-\frac{\hat{e}_r}{r^2}+\frac{3y^2\hat{e}_r}{r^4}\right] \end{aligned}}$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial z^2}\left\{\frac{1}{r}\sin(\ )\hat{e}_r\right\} =}$$

$$\boxed{\begin{aligned} &= \left[-\frac{1}{r^3}+\frac{3z^2}{r^5}-\frac{k^2z^2}{r^3}\right]\sin(\ )\hat{e}_r + \left[\frac{k}{r^2}-\frac{3kz^2}{r^4}\right]\cos(\ )\hat{e}_r \\ &+2\left[\frac{1}{r^3}\sin(\ )-\frac{k}{r^2}\cos(\ )\right]\left[\frac{z\hat{e}_z}{r}-\frac{z^2\hat{e}_r}{r^2}\right] \\ &+\frac{1}{r}\sin(\ )\left[-\frac{2z\hat{e}_z}{r^3}-\frac{\hat{e}_r}{r^2}+\frac{3z^2\hat{e}_r}{r^4}\right] \end{aligned}}$$

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left\{\frac{1}{r}\sin(\ )\hat{e}_r\right\} =}$$

$$= \left[-\frac{\cancel{3}}{\cancel{r^3}}+\frac{3\cancel{(x^2+y^2+z^2)}}{\cancel{r^5}}-\frac{k^2\overset{r^2}{(x^2+y^2+z^2)}}{r^3}\right]\sin(\ )\hat{e}_r + \left[\frac{3k}{\cancel{r^2}}-\frac{3k(x^2+y^2+z^2)}{\cancel{r^4}}\right]\cos(\ )\hat{e}_r$$

$$+2\left[\frac{1}{r^3}\sin(\ )-\frac{k}{r^2}\cos(\ )\right]\left[\underbrace{\frac{\overset{\vec{r}}{\cancel{x\hat{e}_x+y\hat{e}_y+z\hat{e}_z}}}{r}}_{=\hat{e}_r}-\frac{\cancel{(x^2+y^2+z^2)}\hat{e}_r}{\cancel{r^2}}\right]$$

$$+\frac{1}{r}\sin(\ )\left[\underbrace{-\frac{2(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z)}{r^3}}_{\frac{\hat{e}_r}{r^2}}-\frac{3\hat{e}_r}{\cancel{r^2}}+\frac{3\cancel{(x^2+y^2+z^2)}\hat{e}_r}{\cancel{r^4}}\right]$$

$$= \left[-\frac{k^2}{r}-\frac{2}{r^3}\right]\sin(\ )\hat{e}_r$$

$$\boxed{= -\left[1+\frac{2}{k^2r^2}\right]\frac{k^2}{r}\sin(\ )\hat{e}_r} \quad ; \quad \frac{2}{k^2r^2}=\frac{2\lambda^2}{4\pi^2r^2}=\frac{1}{2\pi^2}\frac{\lambda^2}{r^2}$$

$$\Rightarrow \boxed{r>\lambda :\ \frac{2}{k^2r^2}\ll 1}$$

$$\Rightarrow \boxed{\nabla^2\vec{u}(\vec{r},t) = -\left[1+\frac{2}{k^2r^2}\right]k^2\vec{u}(\vec{r},t) \overset{r>\lambda}{\simeq} -k^2\vec{u}(\vec{r},t)}$$

$$\Downarrow$$

$$\boxed{\frac{\partial^2\vec{u}(\vec{r},t)}{\partial t^2} = -\omega^2\vec{u}(\vec{r},t)} \Rightarrow -\omega^2\vec{u}(\vec{r},t) = -\overset{\omega^2}{k^2c^2}\,\vec{u}(\vec{r},t) \quad ✓$$

II. ELEKTRIČNO POLJE

- statično, počasi se spreminjajoče (elektrostatika)

① (Električni) naboj, električna sila, električno polje

Vzemimo dva protona (vodikovi jedri) in ju približamo na razdaljo 1 cm. Med njima deluje (privlačna) gravitacijska sila; njena velikost je

$$\boxed{F_g = \frac{\kappa m_p^2}{r^2} \simeq 2\cdot 10^{-60}\,N}$$

- $m_p \simeq 1{,}7\cdot 10^{-27}\,kg$

  $\kappa \simeq 6{,}7\cdot 10^{-11}\,Nm^2/kg^2 \quad (m^3/kg\,s^2)$

  $r = 1\,cm$

Opazimo, da je sila med protonoma odbojna (!?) in mnogo večja od gravitacijske,

$$\boxed{F_{el} \simeq 2\cdot 10^{-24}\,N} \quad \text{(električna sila)}$$

$$\boxed{\frac{F_{el}}{F_g} \simeq 10^{36}}$$

Izvor gravitacijske sile ("gravitacijski naboj") je masa m_p obeh protonov, izvor nove (električne) sile pa je električni naboj protonov, e_p. Električna sila tudi med drugimi telesi, npr. e^-, z nabojem e_e. Meritve pokažejo, da je el. sila med e^- (na enaki razdalji) enaka sili med dvema p, med p in e^- pa po velikosti enaka, po smeri pa nasprotna (privlačna).