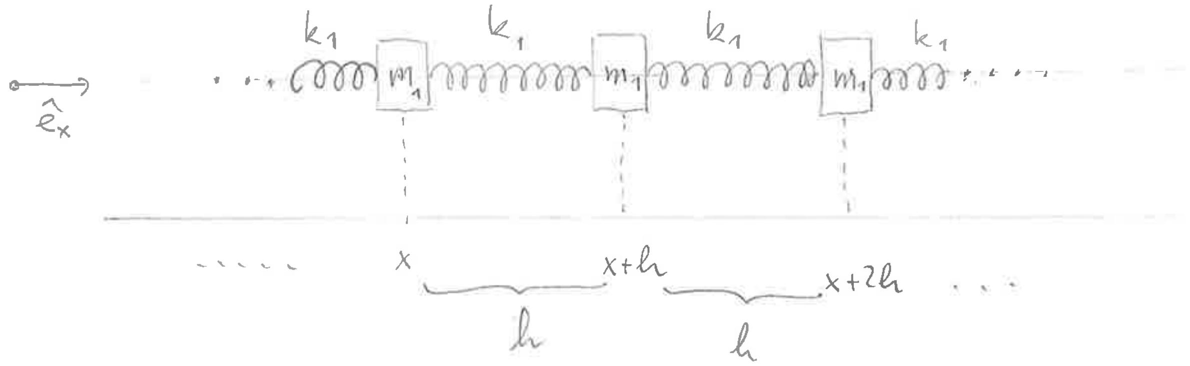
⑤ Mehansko valovanje - valovna enačba

Valovanje po vijačni vzmeti

- Pojem: valovanje - širjenje motnje (def.) po vijačni vzmeti
- Model masivne vzmeti: veliko število drobnih mas, povezanih z neskončno lahkimi vijačnimi vzmetmi (sklopljena vzmetna nihala).



m = masa celotne vzmeti

l = dolžina celotne vzmeti

n = število vseh mas

k = koeficient celotne vzmeti

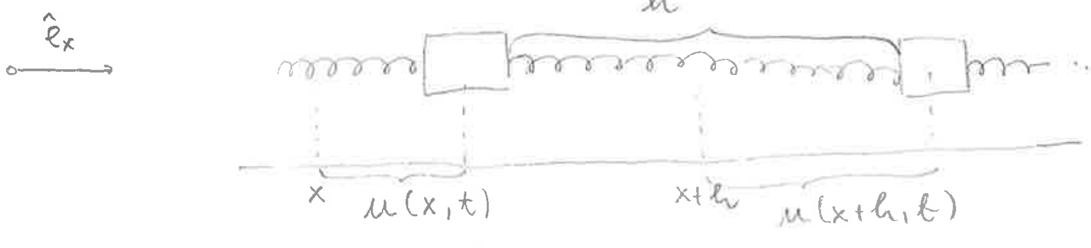
$$n = \frac{l}{h}$$

$$m_1 = \frac{m}{n} = \frac{m}{l} h$$

$$k_1 = k \frac{l}{h}$$

$u(x,t)$: odmik uteži, ki je v ravnovesni legi na položaju x, od ravnovesne lege v smeri $\hat{e}_x$, ob času t

$u(x+h,t)$, $u(x+2h,t)$; longitudinalno valovanje



$$u(x,t)+h' = h + u(x+h,t)$$

$$\Rightarrow h' = h + \underbrace{u(x+h,t)-u(x,t)}_{\Delta h}$$

$$\Rightarrow F_{x\to x+h} = -k_1 \Delta h = -k_1\left[u(x+h,t)-u(x,t)\right] = -\frac{kl}{h}\left[u(x+h,t)-u(x,t)\right]$$

$$\Rightarrow F_{x+2h\to x+h} = +k_1\left[u(x+2h,t)-u(x+h,t)\right] = +\frac{kl}{h}\left[u(x+2h,t)-u(x+h,t)\right]$$

$$\Rightarrow F_{x+h} = F_{x\to x+h} + F_{x+2h\to x+h} = kl\left[\frac{u(x+2h,t)-u(x+h,t)}{h} - \frac{u(x+h)-u(x)}{h}\right]$$

2. Newtonov zakon: $$m_1 \frac{\partial^2 u(x+h,t)}{\partial t^2} = kl\left[\ \dots\ \right]$$

$$\Rightarrow \frac{\partial^2 u(x+h,t)}{\partial t^2} = \frac{kl^2}{m}\, \frac{\left[\frac{u(x+2h,t)-u(x+h,t)}{h} - \frac{u(x+h)-u(x)}{h}\right]}{h}$$

$h\to 0$ $(n\to\infty)$: $$\frac{u(x+2h,t)-u(x+h,t)}{h} \to u'(x+h,t) \quad \left(= \frac{\partial u(x+h,t)}{\partial x}\right)$$

$$\frac{u(x+h,t)-u(x,t)}{h} \to u'(x,t) \quad \left(= \frac{\partial u(x,t)}{\partial x}\right)$$

$$\frac{u'(x+h,t)-u'(x,t)}{h} \to u''(x,t) \quad \left(= \frac{\partial^2 u(x,t)}{\partial x^2}\right)$$

$$\Rightarrow \boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}} \; ; \; \boxed{c^2 = \frac{kl^2}{m}} \qquad [c^2] = \frac{N}{m}\frac{m^2}{kg} = \frac{kg}{s^2}\frac{m^2}{kg} = \frac{m^2}{s^2}$$

valovna enačba v 1 D

Valovanje po elastični palici

- E : elastični modul $\left([E] = \frac{N}{m^2} \right)$

$$\frac{F}{S} = E \frac{\Delta l}{l}$$

Δl ← raztezek palice

l ← dolžina palice

$\frac{F}{S}$ ← natezna napetost

(S = prečni presek palice

F = natezna sila)

$$\Rightarrow F = \underbrace{\frac{ES}{l}}_{k} \Delta l$$

k (Hookov zakon)

⇒ enak model, kot pri masivni vrvi!

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}} \; ; \; c^2 = \frac{ES\, l^2}{l\, m} = \frac{EV}{m} = \boxed{\frac{E}{\rho}}$$

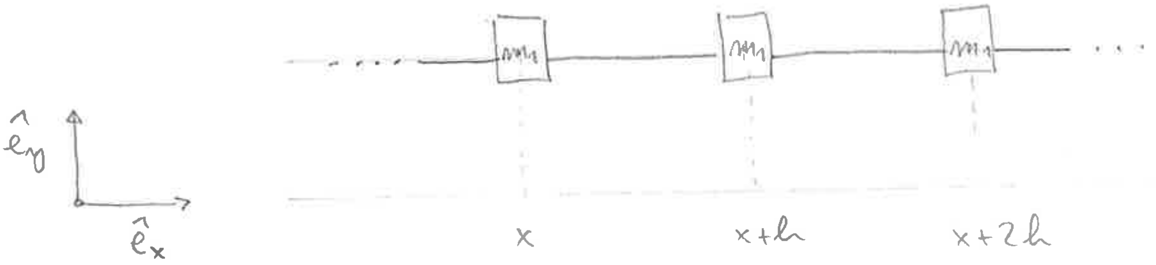
V ← Sl = prostornina palice

Valovanje po napeti vrvi

- F : sila, s katero je napeta masivna vrv
- m : masa vrvi

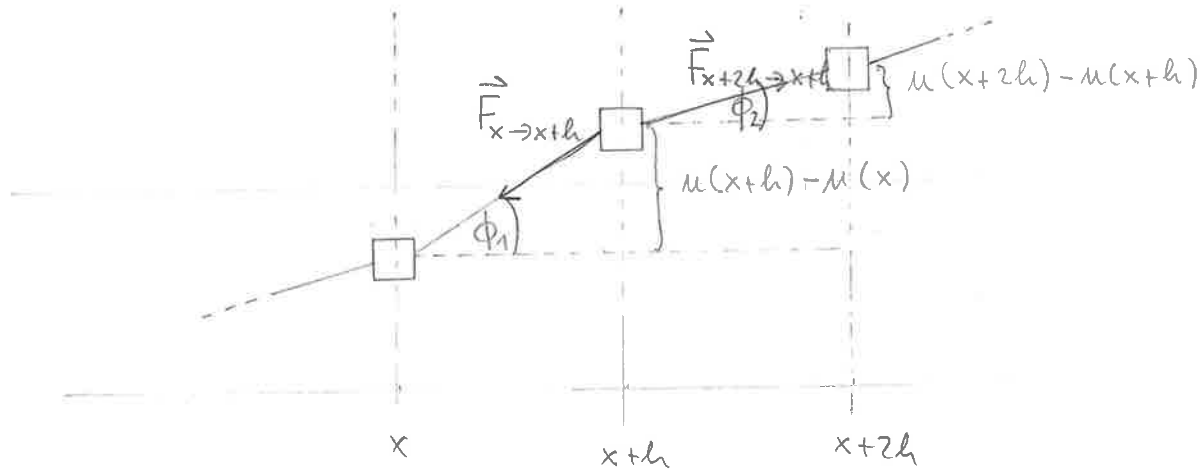
- l : dolžina vrvi

• Model masivne vrvi: veliko število drobnih uteži, povezanih z neskončno lahkimi (brezmasnimi) ravnimi vrvicami



Spet: $n = \frac{l}{h}$, $m_1 = \frac{m}{n} = \frac{mh}{l}$

Predpostavka: $u(x,t), u(x+h,t), u(x+2h,t)$: odmiki v navpični smeri (vzdolž $\hat{e}_y$) – transverzalno valovanje



$$\vec{F}_{x \to x+h} = \begin{bmatrix} -F \\ -F \cdot \operatorname{tg}\phi_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -F \\ -F \frac{u(x+h)-u(x)}{h} \\ 0 \end{bmatrix}$$

$$\vec{F}_{x+2h \to x} = \begin{bmatrix} F \\ F \operatorname{tg}\phi_2 \\ 0 \end{bmatrix} = \begin{bmatrix} F \\ F \frac{u(x+2h)-u(x)}{h} \\ 0 \end{bmatrix}$$

posamezne uteži se ne premikajo vzdolž $\hat{e}_x$

$$\Rightarrow \vec{F}_{x+h} = \begin{bmatrix} 0 \\ F \left[\frac{u(x+2h)-u(x+h)}{h} - \frac{u(x+h)-u(x)}{h} \right] \\ 0 \end{bmatrix}$$

• 2. Newtonov zákon v smere $\hat{e}_y$:

$$m_1 \frac{\partial^2 u(x+h,t)}{\partial t^2} = F\left[\frac{u(x+2h,t)-u(x+h,t)}{h} - \frac{u(x+h,t)-u(x,t)}{h}\right]$$

$$\Rightarrow \frac{\partial^2 u(x+h,t)}{\partial t^2} = c^2 \frac{\left[\frac{u(x+2h,t)-u(x+h,t)}{h} - \frac{u(x+h,t)-u(x,t)}{h}\right]}{h} \; ; \boxed{c^2 = \frac{F}{m/\ell}}$$

$[c^2] = m^2/s^2$

• $h \to 0$: $\dfrac{\partial^2 u(x+h,t)}{\partial t^2} \longrightarrow \dfrac{\partial^2 u(x,t)}{\partial t^2}$

$$\frac{\left[\frac{u(x+2h,t)-u(x+h,t)}{h} - \frac{u(x+h,t)-u(x,t)}{h}\right]}{h} \longrightarrow \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}} \; ; \boxed{c^2 = \frac{F}{m/\ell}}$$

Vlnová enačba v 3D

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u} \; ; [c^2] = m^2/s^2$$

← parciálna diferenciálna enačba

ali

$$\boxed{\Box u = 0} \text{ , pri čemer } \boxed{\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2}$$

6 Rešitve 1D valovne enačbe

- poljubna (dvakrat odvedljiva) funkcija $\boxed{f(x-ct)}$

$$u = f[\eta(x,t)] \; ; \; \eta = x-ct$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial \eta}\underbrace{\frac{\partial \eta}{\partial t}} = -c\frac{\partial f}{\partial \eta} \Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \eta}\left(-c\frac{\partial f}{\partial \eta}\right)\cdot\underbrace{(-c)}_{\frac{\partial \eta}{\partial t}} = c^2\frac{\partial^2 f}{\partial \eta^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial f}{\partial \eta} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial f}{\partial \eta}\right)\cdot\underbrace{1}_{\frac{\partial \eta}{\partial x}} = \frac{\partial^2 f}{\partial \eta^2}$$

$$\Downarrow$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 f}{\partial \eta^2} \checkmark$$

Interpretacija rešitve

- Velja: $\boxed{\begin{array}{l} a)\; u(x,t) = u(x-ct,0) \\ b)\; u(x,t) = u(0, t-\frac{x}{c}) \end{array}}$

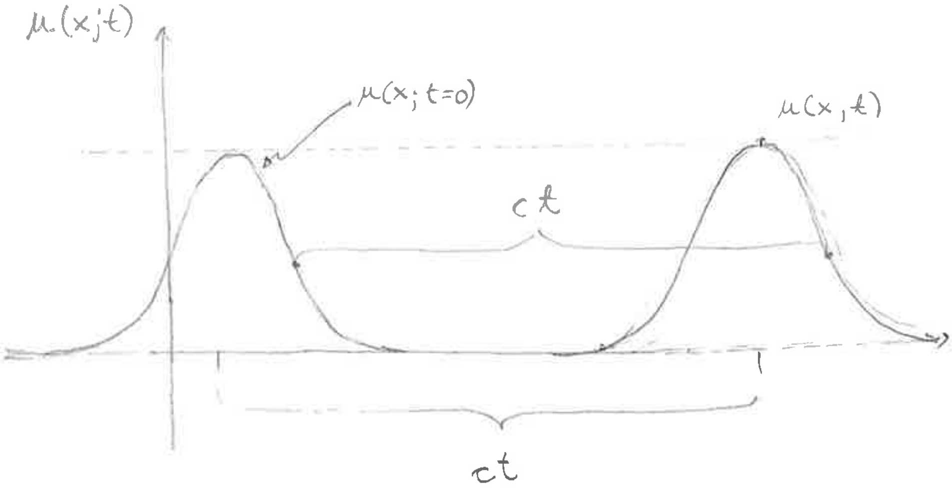
Res: $u(x,t) = f(x-ct)$

$\Rightarrow$ a) $u(x-ct,0) = f(x-ct-c\cdot 0) = f(x-ct) = u(x,t)$ ✓

$\Rightarrow$ b) $u(0,t-\frac{x}{c}) = f(0-c(t-\frac{x}{c})) = f(x-ct) = u(x,t)$ ✓

Zveza a): Točka vrvi (vrvi, elastične palice, ...), ki se v ravnovesju nahaja na položaju x, je ob času t zmaknjena iz x za $u(x,t)$.
To je enako odmiku $u(x-ct,0)$ od ravnovesne lege v točki $x-ct$ ob času $t=0$.

Primer: motnjo Gaussove oblike, ki se širi po elastični vrvi, slikamo ob časih $t=0$ in $t>0$



⇒ Motnja (val) je v času t prepotoval razdaljo ct

$\Delta x = ct$ ⇒ $\boxed{c = \text{hitrost razširjanja motnje (valovanja)}}$

premo-enakomerno potovanje motnje

Pozor: Hitrost c potovanja motnje (valovanja) v splošnem ni enaka hitrosti $v = \frac{\partial u(x,t)}{\partial t}$ posameznega dela vrvi (vzmeti, elastične palice, ...)

$$v = \left|\frac{\partial u(x,t)}{\partial t}\right| = \left|\frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial t}\right| = c\left|\frac{\partial f}{\partial \eta}\right| \quad \left(\neq c\ ;\ \left|\frac{\partial f}{\partial \eta}\right| \neq 1\right)$$

- Vsak odmik se vzdolž vrvi (vzmeti, palice, ...) giblje z enako hitrostjo (c), zato se oblika vala ne spreminja.

- Če poznamo odmik od ravnovesne lege v vsaki točki ob času $t=0$ ⇒ poznamo odmik od ravnovesne lege od vsake točke ob poljubnem času t; val točno določen iz začetnega pogoja

Zveza b): Če poznamo odmik v točki $\underline{x=0}$ ob vsakem času ⇒ poznamo odmik v vsaki točki x ob vsakem času; val točno določen iz robnih pogojev.

- Rešitev tudi vsaka (dvakrat odvedljiva) funkcija oblike $\boxed{g(x+ct)}$: motnja, ki se s hitrostjo c širi v smeri $-\hat{e}_x$.

Komentar: vsa izvajanja se nanašajo na "neskončne" vzmeti, (palice, vrvi, ...), ko lahko druge sile zanemarimo.
V naših modelih masivne vzmeti (vrvi) je imela vsaka drobna utež levega in desnega soseda.
Na robu to ne drži več → drugačna enačba in rešitve!

7 Razširjanje valovanja v neskončnem sredstvu

- $\left.\begin{aligned} u_1(x,t) &= f(x-ct) \\ u_2(x,t) &= g(x+ct) \end{aligned}\right\}$ rešitvi valovne enačbe

⇒ tudi $u(x,t) = u_1(x,t) + u_2(x,t) = f(x-ct) + g(x+ct)$ rešitev

Res: $u_1(x,t)$ rešitev ⇒ $\ddot{u}_1 - c^2 u_1'' = 0$

$u_2(x,t)$ rešitev ⇒ $\ddot{u}_2 - c^2 u_2'' = 0$

$$\begin{aligned}
\Rightarrow\ \ddot{u} - c^2 u'' &= \\
(\ddot{u_1 + u_2}) - c^2 (u_1 + u_2)'' &= \\
\ddot{u}_1 + \ddot{u}_2 - c^2 u_1'' - c^2 u_2'' &= 0
\end{aligned}$$

$$\boxed{u(x,t) = f(x-ct) + g(x+ct)}$$ splošna rešitev (seštevek valov poljubnih oblik, ki potujeta eden desno ($\to \hat{e}_x$) in drugi levo ($-\hat{e}_x$)

Dana začetna pogoja:

$$u(x,0) = \boxed{f(x) + g(x) = A(x)}$$

$$\frac{\partial u}{\partial t}(x, t=0) = \frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial t}\Big|_{t=0} + \frac{\partial g}{\partial \xi}\,\frac{\partial \xi}{\partial t}\Big|_{t=0} \qquad (\eta = x-ct,\ \xi = x+ct)$$

$$= -c\,\frac{\partial f}{\partial \eta}\Big|_{t=0} + c\,\frac{\partial g}{\partial \xi}\Big|_{t=0}$$

$$= -c\,\frac{\partial f(x)}{\partial x} + c\,\frac{\partial g(x)}{\partial x} = B(x)$$

$$\Rightarrow\ -\frac{\partial f(x')}{\partial x'} + \frac{\partial g(x')}{\partial x'} = \frac{\partial}{\partial x'}\left[-f(x') + g(x')\right] = \frac{1}{c}B(x')$$

$$\Rightarrow\ \int_{x_0}^{x} \frac{\partial}{\partial x'}\left[-f(x') + g(x')\right]dx' = \frac{1}{c}\int_{x_0}^{x} B(x')\,dx'$$

$$\Rightarrow\ -f(x) + g(x) \overbrace{- f(x_0) + g(x_0)}^{-D} = \frac{1}{c}\int_{x_0}^{x} B(x')\,dx'$$

$$\Rightarrow\ \boxed{-f(x) + g(x) = \frac{1}{c}\int_{x_0}^{x} B(x')\,dx' + D}$$

$$\Rightarrow\ \boxed{\begin{aligned} f(x) &= \frac{1}{2}\left[A(x) - \frac{1}{c}\int_{x_0}^{x} B(x')\,dx'\right] - D/2 \\ g(x) &= \frac{1}{2}\left[A(x) + \frac{1}{c}\int_{x_0}^{x} B(x')\,dx'\right] + D/2 \end{aligned}}$$

$$\Rightarrow\ \boxed{u(x,t)} = \frac{1}{2}\left[A(x-ct) + A(x+ct)\right] + \frac{1}{2c}\left[-\int_{x_0}^{x-ct} B(x')\,dx' + \int_{x_0}^{x+ct} B(x')\,dx'\right]$$

- D/2 se odšteje
- $x_0 = -\infty$

$$= \frac{1}{2}\left[A(x-ct) + A(x+ct)\right] + \frac{1}{2c}\left[-\int_{x_0}^{x-ct} + \int_{x_0}^{x-ct} B(x')\,dx' + \int_{x-ct}^{x+ct} B(x')\,dx'\right]$$

$$\boxed{= \frac{1}{2}\left[A(x-ct) + A(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} B(x')\,dx'}\quad \text{(D'Alambert)}$$

Primer: $u(x,t=0) = y_0 \exp\left\{-\frac{x^2}{2\sigma^2}\right\} = A(x)$

$\dot{u}(x,t=0) = B(x) = 0$

$\Rightarrow u(x,t) = \frac{y_0}{2} \exp\left\{-\frac{(x-ct)^2}{2\sigma^2}\right\} + \frac{y_0}{2} \exp\left\{-\frac{(x+ct)}{2\sigma^2}\right\}$