Taylorjev razvoj: $\sin\phi = \phi + O(\phi^3) \Rightarrow \phi \ll 1$ $([\phi] = rd)$: $\sin\phi \simeq \phi$

$\Rightarrow \phi \ll 1$: $\boxed{\ddot{\phi} + \omega_0^2 \phi = 0}$; $\boxed{\omega_0^2 = \frac{g}{l}}$ (enačba sinusnega nihanja)

$\Rightarrow \boxed{t_0 = 2\pi\sqrt{\frac{l}{g}}}$ Preveri! (Neodvisno od amplitude!)

$\Rightarrow$ Rešitev: $\boxed{\phi = \phi_{max} \sin(\omega_0 t + \delta)}$; $[\phi_{max}] = /$ (rd)

Fizično nihalo

$\boxed{M_z = -mgl^* \sin\phi}$ * ; l^*: razdalja med osjo in težiščem

$\boxed{M_z = J_z \alpha} \Rightarrow -mgl^* \sin\phi = J_z \ddot{\phi}$

$\Rightarrow \phi \ll 1$: $\boxed{\ddot{\phi} + \frac{mgl^*}{J_z} \phi = 0}$

$\Rightarrow \boxed{\omega_0^2 = \frac{mgl^*}{J_z}}$ in $\boxed{t_0 = 2\pi \frac{1}{\omega_0}}$

J_z: vztrajnostni moment za vrtenje (razsežnega togega) telesa okoli dane osi.

Primer: homogena palica z osjo v enem izmed krajišč

$$J_z = J_z^* + ml^{*2} \quad (\text{Steinerjev izrek})$$

$$= \frac{1}{12} ml^2 + m\left(\frac{l}{2}\right)^2$$

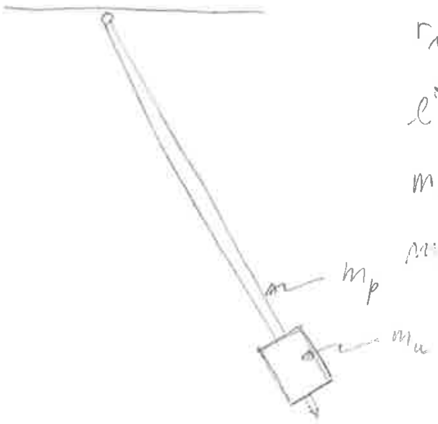
$$= \frac{1}{12} ml^2 + \frac{1}{4} ml^2$$

$$= \frac{1}{3} ml^2$$

* Navor sila v osi nihala je 0, ker je ročica 0

$$\Rightarrow \omega_0^2 = \frac{\cancel{m} g \frac{l}{2}}{\frac{1}{3}\cancel{m} l^{\cancel{2}}} = \frac{3}{2}\frac{g}{l} \Rightarrow t_0 = 2\pi\sqrt{\frac{2}{3}\frac{g}{l}} \;;\; \text{Preveri!}$$

Primer: Utež na togi aluminijasti palici



l_p : dolžina palice

r_u : razdalja uteži od osi

l^* : razdalja težišča od osi

m_p : masa palice

m_u : masa uteži

$$J_z = J_{z,p} + J_{z,u} = \frac{1}{3} m_p l_p^2 + m_u r_u^2$$

Energija nihanja

Vzmetno nihalo:
$$W_k = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{y}^2 = \frac{1}{2} m \omega_0^2 y_{max}'^2 \cos^2(\omega_0 t + \delta)$$
$$W_p = m g y = m g_0 (y_0 + y'_{max} \sin(\omega_0 t + \delta))$$
$$W_{pr} = \frac{1}{2} k y^2 = \frac{1}{2} k (y_0 + y'_{max} \sin(\omega_0 t + \delta))^2$$
$$= \frac{1}{2} k y_0^2 + y'_{max} k y_0 \sin(\omega_0 t + \delta) + \frac{1}{2} k y_{max}'^2 \sin^2(\omega_0 t + \delta)$$

$$\Rightarrow \boxed{W = W_k + W_p + W_{pr}}$$
$$= \frac{1}{2} k y_{max}'^2 [m \omega_0^2 \cos^2(\omega_0 t + \delta) + k^2 \sin^2(\omega_0 t)] \quad / \omega_0^2 = \frac{k}{m}$$
$$+ m g_0 y_0 + \frac{1}{2} k y_0^2 \quad / y_0 = -\frac{m g_0}{k}$$
$$+ y'_{max} \sin(\omega_0 t + \delta) \underbrace{[m g_0 + k y_0]}_{=0} \quad / m g_0 = -k y_0$$

$$= \frac{1}{2} y_{max}'^2 k^2 \underbrace{[\cos^2(\omega_0 t + \delta) + \sin^2(\omega_0 t + \delta)]}_{=1}$$
$$- \frac{m^2 g_0^2}{k} + \frac{1}{2} k \frac{m^2 g_0^2}{k^2}$$
$$\boxed{= \frac{1}{2} y_{max}'^2 k^2 - \frac{1}{2} \frac{m^2 g_0^2}{k} = \text{konst.}}$$

Nitno nihalo:
$$W_k = \frac{1}{2} m v^2 = \frac{1}{2} m (l \dot{\phi})^2 = \frac{1}{2} \phi_{max}^2 m l^2 \omega_0^2 \cos^2(\omega_0 t + \delta)$$
$$W_p = -m g l \cos\phi \overset{(\phi \ll 1)}{\simeq} -m g l \left(1 - \frac{\phi^2}{2!}\right)$$
$$= -m g l + \frac{1}{2} \phi_{max}^2 m g l \sin^2(\omega_0 t + \delta)$$

$$\Rightarrow \boxed{W = W_k + W_p} = \frac{1}{2} \phi_{max}^2 m [l^2 \omega_0^2 \cos^2(\omega_0 t + \delta) + g l \sin^2(\omega_0 t + \delta)] - m g l \quad / \omega_0^2 = \frac{g}{l}$$

$$= \frac{1}{2}\phi_{max}^2 m\left[gl\cos^2(\omega_0 t+\delta)+gl\sin^2(\omega_0 t+\delta)\right]-mgl$$

$$\boxed{= -mgl\left[1-\frac{1}{2}\phi_{max}^2\right] = \text{konst}}$$

Opomba: ohranitev $W = W_k + W_p$ velja tudi ko $\phi \not\ll 1$

$$A_v = \int_C \vec{F}_v \cdot d\vec{s} = \int_{t_1}^{t_2} \vec{F}_v \cdot \vec{v} \cdot dt = 0$$

$\vec{v} \perp \vec{F}_v \Rightarrow \vec{F}_v \cdot \vec{v} = 0$

$\Rightarrow A = A_g$

Spomnimo se (izrek o W_k in W_p):

$A' = \Delta W_k + \Delta W_p$; $A' = A - A_g$

V našem primeru torej velja, da je $A' = 0$

$\Rightarrow \Delta W_k + \Delta W_p = 0$

$W_k' - W_k + W_p' - W_p = 0$

$$\boxed{W_k' + W_p' = W_k + W_p \overset{(\forall t)}{=} \text{konst}} = -mgl\cos\phi_{max}$$

$$\Rightarrow \frac{1}{2}\cancel{m}l^{\cancel{2}}\dot{\phi}^2 - \cancel{m}g\cancel{l}\cos\phi = -\cancel{m}g\cancel{l}\cos\phi_{max}$$

$$\Rightarrow \dot{\phi}^2 = 2\frac{g}{l}(\cos\phi - \cos\phi_{max})$$

$$\Rightarrow \dot{\phi} = \pm\sqrt{\frac{2g}{l}(\cos\phi - \cos\phi_{max})} = \pm\sqrt{2}\,\omega_0\sqrt{\cos\phi - \cos\phi_{max}}$$

$$\Rightarrow \frac{d\phi}{dt} = \pm\sqrt{2}\,\omega_0\sqrt{\cos\phi - \cos\phi_{max}}$$

$$\phi = f(t) \Rightarrow t = f^{-1}(\phi) \Rightarrow \frac{df^{-1}}{dt} = 1$$

Po drugi strani: $\frac{df^{-1}}{dt} = \frac{df^{-1}}{d\phi} \cdot \frac{d\phi}{dt} = \frac{dt}{d\phi} \cdot \frac{d\phi}{dt}$

$$\Rightarrow \frac{dt}{d\phi} \cdot \frac{d\phi}{dt} = 1$$

$$\Rightarrow \frac{dt}{d\phi} = \frac{1}{\frac{d\phi}{dt}}$$

$$\Rightarrow \boxed{\frac{dt}{d\phi} = \pm \frac{1}{\sqrt{2}\,\omega_0} \frac{1}{\sqrt{\cos\phi - \cos\phi_{max}}}}$$

Izberimo začetne in končne pogoje:

$$t(\phi_{max}) = 0 \Leftarrow \quad \phi(t=0) = \phi_{max}$$

$$t(\phi=0) = \frac{t_0}{4} \Leftarrow \quad \phi\left(t_2 = \frac{t_0}{4}\right) = 0$$

$$\Big\} \Rightarrow \dot\phi < 0 \Rightarrow \dot\phi = -\sqrt{2}\,\omega_0 \sqrt{\cos\phi - \cos\phi_{max}}$$

$$\Rightarrow \boxed{\frac{dt}{d\phi} = -\frac{1}{\sqrt{2}\,\omega_0} \frac{1}{\sqrt{\cos\phi - \cos\phi_{max}}}}$$

$$\Rightarrow \boxed{\int_{\phi_{max}}^{0} \frac{dt}{d\phi}\, d\phi = -\frac{1}{\sqrt{2}\,\omega_0} \int_{\phi_{max}}^{0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_{max}}}}$$

$$\int_{\phi_{max}}^{0} \frac{dt}{d\phi}\, d\phi = t(\phi=0) - t(\phi_{max}) = \frac{t_0}{4}$$

$$\Rightarrow \boxed{\frac{\sqrt{2}\,\omega_0 t_0}{4} = -\int_{\phi_{max}}^{0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_{max}}}}$$

eliptični integral

$$\Rightarrow \frac{1}{\sqrt{\cos\phi - \cos\phi_{max}}} \simeq \frac{\sqrt{2}}{\sqrt{\phi_{max}^2 - \phi^2}\left[1 - \frac{1}{24}(\phi_{max}^2 + \phi^2)\right]}$$

$$\simeq \frac{\sqrt{2}\left[1 + \frac{1}{24}(\phi_{max}^2 + \phi^2)\right]}{\sqrt{\phi_{max}^2 - \phi^2}}$$

$$\Rightarrow -\int_{\phi_{max}}^{0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_{max}}} \simeq -\sqrt{2}\int_{\phi_{max}}^{0} \frac{\left[1 + \frac{1}{24}(\phi_{max}^2 + \phi^2)\right] d\phi}{\sqrt{\phi_{max}^2 - \phi^2}}$$

$$= -\sqrt{2}\left[\left(1 + \frac{\phi_{max}^2}{24}\right)\underbrace{\int_{\phi_{max}}^{0} \frac{d\phi}{\sqrt{\phi_{max}^2 - \phi^2}}}_{I_1} + \frac{1}{24}\underbrace{\int_{\phi_{max}}^{0} \frac{\phi^2 d\phi}{\sqrt{\phi_{max}^2 - \phi^2}}}_{I_2}\right]$$

$$I_1 = -\frac{\pi}{2}$$

(D.n.: preveri!)

$$I_2 = \int_{\phi_{max}}^{0} \frac{\phi^2 d\phi}{\sqrt{\phi_{max}^2 - \phi^2}} = \phi_{max}^2 \int_{1}^{0} \frac{\mu^2 du}{\sqrt{1 - \mu^2}} = \frac{\phi_{max}^2}{2}\left[\arcsin x - x\sqrt{1 - x^2}\right]_1^0$$

$$= -\frac{\phi_{max}^2}{2} \arcsin 1 =$$

$$= -\phi_{max}^2 \frac{\pi}{4}$$

$$\Rightarrow -\int_{\phi_{max}}^{0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_{max}}} \simeq \sqrt{2}\left[\left(1 + \frac{\phi_{max}^2}{24}\right)\frac{\pi}{2} + \frac{\phi_{max}^2}{24}\frac{\pi}{4}\right]$$

$$= \sqrt{2}\,\frac{\pi}{2}\left[1 + \frac{\phi_{max}^2}{24}\left(1 + \frac{1}{2}\right)\right]$$

$$= \sqrt{2}\,\frac{\pi}{2}\left[1 + \frac{\phi_{max}^2}{24}\,\frac{3}{2}\right]$$

$$= \sqrt{2}\,\frac{\pi}{2}\left[1 + \frac{1}{16}\phi_{max}^2\right]$$

$$\Rightarrow \frac{\sqrt{2}\,\omega_0 t_0}{\cancel{4}\,2} \simeq \frac{\sqrt{2}\,\pi}{\cancel{2}}\left[1 + \frac{1}{16}\phi_{max}^2\right]$$

$$\Rightarrow \boxed{t_0 \simeq \frac{2\pi}{\omega_0}\left[1 + \frac{1}{16}\phi_{max}^2\right]} = \boxed{2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{16}\phi_{max}^2\right]}$$

(Odvisno od amplitude)

$\phi_{max} \ll 1 \Rightarrow$

a) $\cos\phi \simeq 1 - \frac{\phi^2}{2!} = 1 - \frac{\phi^2}{2}$

$\cos\phi_{max} \simeq 1 - \frac{\phi_{max}^2}{2}$ $\Rightarrow \cos\phi - \cos\phi_{max} \simeq \frac{1}{2}(\phi_{max}^2 - \phi^2)$

$$\Rightarrow -\int_{\phi_{max}}^{0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_{max}}} \simeq -\int_{\phi_{max}}^{0} \frac{d\phi}{\frac{1}{\sqrt{2}}\sqrt{\phi_{max}^2 - \phi^2}} = -\sqrt{2}\int_{\phi_{max}}^{0} \frac{d\phi}{\sqrt{\phi_{max}^2 - \phi^2}}$$

(O.n.: preveri!)

$$\int_{\phi_{max}}^{0} \frac{d\phi}{\sqrt{\phi_{max}^2 - \phi^2}} = \int_{\phi_{max}}^{0} \frac{d\phi}{\phi_{max}\sqrt{1 - \left(\frac{\phi}{\phi_{max}}\right)^2}} = \int_1^0 \frac{du}{\sqrt{1-u^2}} = \operatorname{arctg}\left(\frac{u}{\sqrt{1-u^2}}\right)\Big|_1^0 =$$

$$= \operatorname{arctg} 0 - \operatorname{arctg} \infty$$

$$= 0 - \frac{\pi}{2}$$

$$= -\frac{\pi}{2}$$

$u \equiv \frac{\phi}{\phi_{max}} \Rightarrow du = \frac{d\phi}{\phi_{max}}$

$$\Rightarrow \frac{\sqrt{2}\,\omega_0 t_0}{4} = \frac{\sqrt{2}\,\pi}{2}$$

$$\Rightarrow \omega_0 t_0 = 2\pi \Rightarrow \boxed{t_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}}$$ ✓

b) En red višje:

$\cos\phi \simeq 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{24}$

$\cos\phi_{max} \simeq 1 - \frac{\phi_{max}^2}{2} + \frac{\phi_{max}^4}{24}$

$$\Rightarrow \cos\phi - \cos\phi_{max} \simeq \frac{1}{2}\left[\phi_{max}^2 - \phi^2 - \frac{1}{12}(\phi_{max}^4 - \phi^4)\right]$$

$$= \frac{1}{2}\left[\phi_{max}^2 - \phi^2 - \frac{1}{12}(\phi_{max}^2 - \phi^2)(\phi_{max}^2 + \phi^2)\right]$$

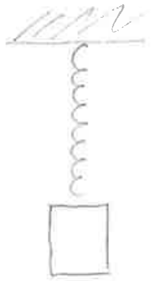
$$= \frac{1}{2}(\phi_{max}^2 - \phi^2)\left[1 - \frac{1}{12}(\phi_{max}^2 + \phi^2)\right]$$

$$\Rightarrow \sqrt{\cos\phi - \cos\phi_{max}} \simeq \frac{1}{\sqrt{2}}\sqrt{\phi_{max}^2 - \phi^2}\sqrt{1 - \frac{1}{12}(\phi_{max}^2 + \phi^2)}$$

$$\simeq \frac{1}{\sqrt{2}}\sqrt{\phi_{max}^2 - \phi^2}\left[1 - \frac{1}{24}(\phi_{max}^2 + \phi^2)\right]$$

(2.) Dušeno nihanje

Vzmetno nihalo



$$\vec{F} = \begin{bmatrix} 0 \\ -mg_0 - ky \end{bmatrix}$$; $y\ (<0)$: odmik uteži od položaja, ko je vzmet neraztegnjena

$y_0 = -\frac{mg_0}{k}$: položaj uteži v ravnovesju

Upoštevamo še upor:

- $F_u \propto v^2\ (\dot{y}^2)$: kvadratni zakon upora
- $\vec{F}_u \propto \begin{bmatrix} 0 \\ -\dot{y} \\ 0 \end{bmatrix}$; linearni zakon upora (viskoznost) ✓

$$\Rightarrow \vec{F} \longrightarrow \vec{F} + \begin{bmatrix} 0 \\ -C\dot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg_0 - ky - C\dot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -k(y-y_0) - C\dot{y} \\ 0 \end{bmatrix} = \vec{F}' ;\ C > 0$$

$$\vec{a} = \begin{bmatrix} 0 \\ \ddot{y} \\ 0 \end{bmatrix}$$

$$\vec{F}' = m\vec{a} \Rightarrow \boxed{-k(y-y_0) - C\dot{y} = m\ddot{y}}$$

Spet: $\boxed{y' \equiv y - y_0} \Rightarrow \dot{y}' = \dot{y}$ in $\ddot{y}' = \ddot{y}$

$$\Rightarrow m\ddot{y}' + C\dot{y}' + ky' = 0$$

$$\Rightarrow \boxed{\ddot{y}' + \beta\dot{y}' + \omega_0^2 y' = 0}$$

$\omega_0^2 = \frac{k}{m} > 0$; $[\omega_0^2] = s^{-2}$

$\beta = \frac{C}{m} > 0$; $[\beta] = s^{-1}$; koeficient dušenja

Mat: je vedno homogena linearna diferencialna enačba 2. reda s konstantnimi koeficienti.

$\Rightarrow$ nastavek: $y' = C e^{\lambda t}$

$\Rightarrow$ $\boxed{\lambda^2 + \beta\lambda + \omega_0^2 = 0}$ (karakteristični polinom = kvadratna enačba)

$$\boxed{D = \beta^2 - 4\omega_0^2 = -4\omega^2 \Rightarrow \omega^2 = \omega_0^2 - \left(\frac{\beta}{2}\right)^2}$$

a) $D<0$ $(\omega^2>0)$ (podkritično dušenje)

$\Rightarrow$ $\lambda_{1,2} = -\frac{\beta}{2} \pm i\omega$

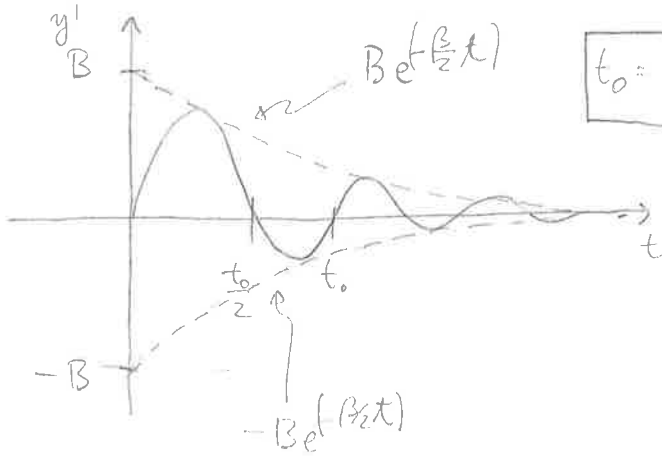
$$\Rightarrow y' = e^{-\frac{\beta}{2}t}\left(C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right)$$

$$= e^{-\frac{\beta}{2}t}\left(B_1 \cos(\omega t) + B_2 \sin(\omega t)\right)$$

$$= B e^{-\frac{\beta}{2}t} \sin(\omega t + \delta)\ ;\ B>0,\ \delta \text{ konst.}$$

↳ eksponentno dušenje amplitude

Primer: $\delta = 0$



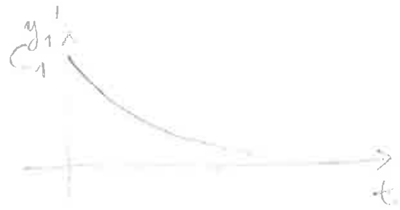
$\boxed{t_0 = \frac{2\pi}{\omega}}$

Amplituda se s časom manjša $\rightarrow$ energija nihanja (sistema) se manjša (uhaja iz sistema).

b) $D = 0$ $(\omega = 0)$ (kritično dušenje)

$\lambda = -\frac{\beta}{2}$

$\Rightarrow \boxed{y_1' = C_1 e^{-\beta/2\, t}}$



Trditev: če je y_1' rešitev, potem je (v primeru kritičnega dušenja)

$y_2' = C_2\, t\, y_1'$ tudi rešitev

Res (D.n.): y_1' rešitev $\Rightarrow \dot{y}_1' = -\frac{\beta}{2} y_1'$

$\Rightarrow \ddot{y}_1' + \beta \dot{y}_1' + \omega_0^2 y_1' = 0$

$\Rightarrow \dot{y}_2' = C_2 y_1' + C_2 t \dot{y}_1'$

$\Rightarrow \ddot{y}_2' = C_2 \dot{y}_1' + C_2 \dot{y}_1' + C_2 t \ddot{y}_1' = C_2 t \ddot{y}_1 + 2 C_2 \dot{y}_1' = C_2 t \ddot{y}_1' - \beta C_2 \dot{y}_1'$

$\Rightarrow \ddot{y}_2 + \beta \dot{y}_2' + \omega_0^2 y_2 =$

$C_2 t \ddot{y}_1' \cancel{- \beta C_2 \dot{y}_1'} + \beta C_2 t \dot{y}_1' \cancel{+ \beta C_2 \dot{y}_1'} + \omega_0^2 y_1' =$

$C_2 t (\ddot{y}_1' + \beta \dot{y}_1' + \omega_0^2 y_1') =$

0 ✓

$\Rightarrow$ Splošna rešitev: $\boxed{y' = (C_1 + C_2 t)\, e^{-\beta/2\, t}}$

c) $D > 0$ ($\omega^2 < 0$) (nadkritično dušenje)

$$D = -4\omega^2 > 0 \Rightarrow \omega^2 < 0 \Rightarrow \omega = \pm i|\omega| \Rightarrow -\omega^2 = i^2(\pm i)^2|\omega|^2 = i^4|\omega|^2 = |\omega|^2$$

$$\Rightarrow D = 4|\omega|^2 \Rightarrow \boxed{\lambda_{1/2} = -\frac{\beta}{2} \pm |\omega|}$$

$$\omega^2 = \omega_0^2 - \left(\frac{\beta}{2}\right)^2 \Rightarrow -\omega^2 = \left(\frac{\beta}{2}\right)^2 - \omega_0^2 = |\omega|^2 \Rightarrow |\omega| = \sqrt{\left(\frac{\beta}{2}\right)^2 - \omega_0^2}$$

$$= \frac{\beta}{2}\sqrt{1 - \frac{4\omega_0^2}{\beta^2}} < \frac{\beta}{2}$$

$$\Rightarrow \boxed{\lambda_{1/2} = -\frac{\beta}{2} \pm |\omega| < 0}$$

$$\boxed{y' = C_1 \exp\left\{-\frac{\beta}{2}\left[1 + \sqrt{1 - \frac{4\omega_0^2}{\beta^2}}\right]t\right\} + C_2 \exp\left\{-\frac{\beta}{2}\left[1 - \sqrt{1 - \frac{4\omega_0^2}{\beta^2}}\right]t\right\}}$$

Vsota dveh eksponentno padajočih prispevkov.

V primeru zelo močnega dušenja, ko velja $\frac{4\omega_0^2}{\beta^2} \ll 1$

$$\Rightarrow \sqrt{1 - \frac{4\omega_0^2}{\beta^2}} \simeq 1 - \frac{2\omega_0^2}{\beta^2}$$

$$\Rightarrow y' \simeq C_1 \exp\{-\beta t\} + C_2 \exp\left\{-\frac{\beta}{2}\left(\frac{2\omega_0^2}{\beta^2}\right)t\right\}$$

$\ll 1$

- prvi člen: zelo hitro padanje k nič (hitreje, kot pri kritičnem dušenju)
- drugi člen: zelo počasno približevanje ravnovesni legi!