

# FUNCTIONAL ANALYSIS - NOTES

Gal Anton Gorše

# 1 Convexity

## 1.1 Locally convex spaces

Let  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$  be a field.

**Definition 1.1.** A *topological vector space* (TVS) is a  $\mathbb{F}$ -vector space that is also a topological space and the two structures are compatible. This means that the usual operations on vector spaces

$$V \times V \rightarrow V, (x, y) \mapsto x + y, \quad \mathbb{F} \times V \rightarrow V, (\lambda, x) \mapsto \lambda x$$

are continuous maps.

**Example 1.2.** Normed spaces are TVS.

**Definition 1.3.** Let  $V$  be a  $\mathbb{F}$ -space. Map  $p : V \rightarrow \mathbb{R}$  is a *seminorm* if:

- (1.)  $p(x) \geq 0, \forall x \in V$  (positivity);
- (2.)  $p(\lambda x) = |\lambda|p(x), \forall x \in V, \forall \lambda \in \mathbb{F}$  (positive homogeneity);
- (3.)  $p(x + y) \leq p(x) + p(y), \forall x, y \in V$  (triangle inequality).

A seminorm is therefore almost a norm, except that it's not necessarily positive definite.

Let  $V$  be a  $\mathbb{F}$ -vector space and  $\mathcal{P}$  a family of seminorms in  $V$ . Let  $\mathcal{T}$  be the topology in  $V$  with the following subbasis:

$$U(x_0, p, \varepsilon) = \{x \in V \mid p(x - x_0) < \varepsilon\}; \quad x_0 \in V, p \in \mathcal{P}, \varepsilon > 0.$$

Basis of  $\mathcal{T}$  are finite intersections of such sets. The set  $U \subseteq V$  is open iff for every  $x_0 \in U$  there exist seminorms  $p_1, \dots, p_n \in \mathcal{P}$  and  $\varepsilon_1, \dots, \varepsilon_n > 0$  such that

$$U \supset \bigcap_{j=1}^n U(x_0, p_j, \varepsilon_j).$$

The space  $(V, \mathcal{T})$  is then a TVS. If  $\mathcal{P}$  is a singleton and its element is a norm, then  $(V, \mathcal{T})$  is a normed space.

**Definition 1.4.** A TVS  $X$  is a *locally-convex space* (LCS) if its topology is generated by a family of seminorms  $\mathcal{P}$  satisfying

$$\bigcap_{p \in \mathcal{P}} \{x \in X \mid p(x) = 0\} = \{0\}.$$

Equivalently, for every  $x \in X \setminus \{0\}$  there exists a seminorm  $p \in \mathcal{P}$  such that  $p(x) \neq 0$ .

**Corollary 1.5.** Let  $X$  be a space with a topology generated by a family of seminorms  $\mathcal{P}$ . Then  $X$  is a LCS iff it is Hausdorff.

*Proof.* Start with  $(\Rightarrow)$ . Let  $x, y \in X$  be two distinct points. There exists a seminorm  $p \in \mathcal{P}$  such that  $p(x - y) = b \neq 0$ . Define the sets

$$V = U\left(x, p, \frac{b}{2}\right), \quad W = U\left(y, p, \frac{b}{2}\right).$$

By the triangle inequality property of a seminorm,  $V$  and  $W$  separate the points  $x, y$ . Now the converse  $(\Leftarrow)$ . Choose a point  $x \neq 0$ . Then there exist open sets  $0 \in V, x \in W$  that separate  $0$  from  $x$ . There exists an open basis set  $\bigcap_{j=1}^n U(0, p_j, \varepsilon_j) \subseteq V$ , so  $x \notin U(0, p_j, \varepsilon_j)$  for some index  $j$ . Hence,  $p_j(x - 0) = p_j(x) \geq \varepsilon > 0$ .  $\square$

LCS generally aren't first-countable, so we need to go beyond the usual sequences to describe the topology.

**Definition 1.6.** Partially ordered set  $(I, \leq)$  is *upwards-directed* if

$$\forall i', i'' \in I : \exists i \in I : i \geq i', i \geq i''.$$

**Example 1.7.** (1.) Every linearly ordered set is upwards-directed.

(2.) Let  $(X, \mathcal{T})$  be a topological space and  $x_0 \in X$ . Define a family of sets

$$\mathcal{U} = \{U^{\text{open}} \subseteq X \mid x_0 \in U\}$$

and a relation  $U \geq V \Leftrightarrow U \subseteq V$ . Then  $(\mathcal{U}, \leq)$  is an upwards-directed set.

(3.) Let  $S$  be a set and  $\mathcal{F}$  a family of all finite subsets of  $S$ . Define  $F_1 \geq F_2$  in  $\mathcal{F}$  if  $F_1 \supseteq F_2$ . Then  $(\mathcal{F}, \leq)$  is again an upwards-directed set.

**Definition 1.8.** A *generalized sequence (net)* is  $((I, \leq), x)$ , where  $(I, \leq)$  is upwards-directed and  $x : I \rightarrow X$  is a function. We usually write  $(x_i)_{i \in I}$  or  $(x(i))_{i \in I}$ .

**Example 1.9.** (1.) Every sequence is a net.

(2.) Let  $(X, \mathcal{T})$  be a topological space,  $x_0 \in X$  and  $\mathcal{U}$  a collection of all open sets which contain  $x_0$  (see example 1.7). For each  $U \in \mathcal{U}$  pick a  $x_U \in U$ . Then  $(x_U)_{U \in \mathcal{U}}$  is a net.

**Definition 1.10.** Let  $X$  be a topological space. A net  $(x_i)_{i \in I}$  *converges* to an  $x \in X$  if

$$\forall U^{\text{open}} \subseteq X, x \in U : \exists i_0 \in I : \forall i \geq i_0 : x_i \in U.$$

We write  $\lim_{i \in I} x_i = x$ , or alternatively,  $x_i \xrightarrow{i \in I} x$ . A point  $x \in X$  is called a *cluster point* of a net  $(x_i)_{i \in I}$  if

$$\forall U^{\text{open}} \subseteq X, x \in U : \forall i_0 \in I : \exists i \geq i_0 : x_i \in U.$$

**Example 1.11.** Take the net  $(x_U)_{U \in \mathcal{U}}$  from example 1.9. It follows from the definition that  $x_U \xrightarrow{U \in \mathcal{U}} x_0$ .

**Proposition 1.12.** (1.) Let  $X$  be a topological space and  $A \subseteq X$ . Then  $x \in \overline{A}$  iff there exists a net  $(a_i)_{i \in I}$  in  $A$  such that  $a_i \rightarrow x$ .  
(2.) Let  $X, Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is continuous at  $x_0 \in X$  iff  $f(x_i) \rightarrow f(x_0)$  for every net  $(x_i)_{i \in I}$  that converges to  $x_0$ .

*Proof.* (1.) We begin with the implication to the left ( $\Leftarrow$ ). Take any  $U^{\text{open}} \subseteq X$  such that  $x \in U$ . Since  $a_i \rightarrow x$ , there exists an index  $i_0 \in I$ , such that for every  $i \geq i_0$  we have  $a_i \in U$ . Hence  $a_i \in A \cap U \neq \emptyset$  and  $x \in \overline{A}$ . The converse ( $\Rightarrow$ ) is similar. Define  $\mathcal{U} = \{U^{\text{open}} \subseteq X \mid x \in U\}$ . Since  $x \in \overline{A}$ , for each  $U \in \mathcal{U}$ , we have  $A \cap U \neq \emptyset$ . Pick  $a_U \in A \cap U$ . Then the net  $(a_U)_{U \in \mathcal{U}}$  in  $A$  converges to  $x$ .  
(2.) Start with the implication ( $\Rightarrow$ ). Let  $f$  be a continuous function and let  $(x_i)_{i \in I}$  converge to  $x_0$ . Let  $f(x_0) \in U^{\text{open}} \subseteq Y$ . Then  $x_0 \in f^{-1}(U)^{\text{open}} \subseteq X$ , which means there exists an  $i_0 \in \mathbb{N}$  such that for every  $i \geq i_0$ ,  $x_i \in f^{-1}(U)$ . But that implies that for every  $i \geq i_0$ ,  $f(x_i) \in U$ , which is what we wanted. Now we prove the converse ( $\Leftarrow$ ). Let's say that for every net  $(x_i)_{i \in I}$  that converges to  $x_0$ , we have  $f(x_i) \xrightarrow{i \in I} f(x_0)$ . So for every set  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$  (using the first item), which proves that  $f$  is continuous.  $\square$

**Proposition 1.13.** (a) A net  $(x_i)_{i \in I}$  in a LCS converges to  $x_0$  iff a net  $(p(x_i - x_0))_{i \in I}$  converges to 0 for all  $p \in \mathcal{P}$ .

(b) The topology in a LCS  $X$  is the coarsest (smallest) topology in which all the maps  $x \mapsto p(x - x_0)$  are continuous for every  $x_0 \in X$  and  $p \in \mathcal{P}$ .

*Proof.* (a) Start with the implication ( $\Rightarrow$ ). Take any  $p \in \mathcal{P}$ . If we take  $U = U(x_0, p, \varepsilon)$  in the definition of a limit of a net, we get

$$\forall \varepsilon > 0 : \exists i_0 \in I : \forall i \geq i_0 : p(x_i - x_0) \in (-\varepsilon, \varepsilon).$$

This proves our claim. Now for the opposite direction ( $\Leftarrow$ ). For every  $p \in \mathcal{P}$  and  $\varepsilon_p > 0$  there exists an  $i_p$  such that for every  $i \geq i_p$ ,  $x_i \in U(x_0, p, \varepsilon_p)$ . Now let  $U$  be an arbitrary basis set that includes the point  $x_0$ . That means  $U$  is the finite intersection of the sets  $U(x_0, p, \varepsilon_p)$ . Now let  $i_0$  be greater than all indices  $i_p$ . By our assumption, for every  $i \geq i_0$  we have  $x_i \in U$ .

(b) Pick any point  $x_0 \in X$  and a seminorm  $p \in \mathcal{P}$ . Denote

$$f_{x_0, p} : X \rightarrow \mathbb{R}, \quad f_{x_0, p}(x) = p(x - x_0).$$

We essentially have to prove that the sets

$$f_{x_0, p}^{-1}(V), \quad V^{\text{open}} \subseteq \mathbb{R}, \quad x_0 \in X, \quad p \in \mathcal{P}$$

generate a subbasis for the seminorm topology of a LCS space. Since  $f_{p, x_0}$  are continuous functions (by the first item and Proposition 1.12), these are all open sets in the

seminorm topology. But on the other hand, all subbasis sets  $U(x_0, p, \varepsilon)$  of the seminorm topology are of this type, so the above subbasis generates the seminorm topology, thus concluding our proof.  $\square$

**Example 1.14.** Let  $X$  be a topological space. For every  $K^{compact} \subseteq X$  we define a seminorm

$$p_K : C(X) \rightarrow \mathbb{R}, \quad f \mapsto \sup_{x \in K} |f(x)|.$$

We endow  $C(X)$  with the topology induced by the family of seminorms  $\{p_K \mid K^{compact} \subseteq X\}$ . It's trivial to see that  $C(X)$  is then a LCS. Moreover, we notice that the induced seminorm topology coincides with the topology of compact convergence on  $X$ . In the future, we will require  $X$  to be locally compact Hausdorff (this implies complete regularity) so that  $C(X)$  has nice properties. There are examples of not completely regular spaces  $X$  such that the only elements of  $C(X)$  are constant maps.

**Example 1.15.** Let  $D^{open} \subseteq \mathbb{C}$  and let  $\mathcal{H}(D)$  be the set of all holomorphic functions on  $D$ . As in the example 1.14, we define  $\mathcal{P} = \{p_K \mid K^{compact} \subseteq D\}$ . This endows  $\mathcal{H}(D)$  with a topology and makes  $\mathcal{H}(D)$  into a LCS. Convergence in this topology coincides with the uniform convergence on compacts in  $D$ .

## 1.2 Weak topology

Let  $X$  be a normed space and let  $X^*$  be its dual. For every  $f \in X^*$  we define a seminorm

$$p_f : X \rightarrow \mathbb{R}, \quad x \mapsto |f(x)|.$$

We claim that  $\mathcal{P} = \{p_f \mid f \in X^*\}$  is a family of seminorms that induces a topology on  $X$  which makes  $X$  a LCS. Indeed, for any  $x \in X \setminus \{0\}$  define a nonzero linear functional

$$f : \text{span}(x) \rightarrow \mathbb{F}, \quad f(\lambda x) = \lambda$$

and extend it to  $F : X \rightarrow \mathbb{F}$  using Hahn–Banach. Then  $p_F(x) \neq 0$ .

The induced topology is the *weak topology* on  $X$ . We denote it as  $\sigma(X, X^*)$ . A net  $(x_i)_{i \in I}$  converges to  $x_0 \in X$  with regards to the weak topology iff  $p_f(x_i - x_0) \rightarrow 0, \forall f \in X^*$  which is then equivalent to  $f(x_i) \rightarrow f(x_0), \forall f \in X^*$ . We use the notation  $x_i \xrightarrow{w} x_0$ .

*Remark.* The closure of a set  $A \subseteq X$  in the weak topology will be denoted as  $\overline{A}^w$ .

**Example 1.16.** Let  $X = \mathbb{R}^n$ . Then  $X^* = \mathbb{R}^n$  and every linear functional  $f$  is of the form  $f(x) = \langle x, y \rangle$  for some  $y \in X$  (Riesz' representation theorem). The subbasis sets are

$$U(0, p_y, \varepsilon) = \{x \in \mathbb{R}^n \mid |\langle x, y \rangle| < \varepsilon\}.$$

Weak topology in this case coincides with Euclidean topology.

Let  $X$  again be a normed space. To  $x \in X$  we assign the seminorm

$$p_x : X^* \rightarrow \mathbb{R}, \quad f \mapsto |f(x)|.$$

The family  $\{p_x \mid x \in X\}$  defines a topology in  $X^*$  in which  $X^*$  becomes a LCS. This topology is called weak-\* topology and is denoted by  $\sigma(X^*, X)$ . We can easily check that  $f_i \xrightarrow{w^*} f$  iff  $f_i(x) \rightarrow f(x)$  for all  $x \in X$ . We can compare weak-\* topology on  $X^*$  with its weak topology. As a consequence of Hahn-Banach, we have for every  $x \in X$

$$\|x\| = \sup\{|f(x)| \mid f \in X^*, \|f\| \leq 1\},$$

which implies that the map

$$\iota : X \hookrightarrow X^{**}, \quad x \mapsto (f \mapsto f(x))$$

is an isometry and therefore injective. This means that every seminorm in the weak-\* topology is also a seminorm in a weak topology on  $X^*$ , so the weak topology is finer (stronger) than the weak-\* topology on  $X^*$ .

*Remark.* Weak and weak-\* topology can be defined even if  $X$  is merely a LCS. In that case,  $X^*$  is of course defined as the space of continuous linear functionals on  $X$ .

### 1.3 Banach-Alaoglu theorem

**Theorem 1.17** (Banach-Alaoglu).

*Let  $X$  be a normed space. Then the closed unit ball in  $X^*$  (denoted by  $(X^*)_1$ ) is compact in the weak-\* topology in  $X^*$ .*

*Proof.* To  $x \in X$  we assign  $D_x = \{z \in \mathbb{F} \mid |z| \leq \|x\|\}$  and endow  $D_x$  with the Euclidean topology. Then  $D_x$  is clearly compact. The set  $P = \prod_{x \in X} D_x$  is compact in the product topology (Tychonoff theorem). Now we construct a map

$$\Phi : (X^*)_1 \rightarrow P, \quad f \mapsto (f(x))_{x \in X} \in P.$$

Clearly,  $\Phi$  is well-defined and injective. We start by proving that  $\Phi$  is continuous. Let  $(f_i)_{i \in I}$  be a net in  $(X^*)_1$  that converges to  $f \in X^*$  in the weak-\* topology. Then  $f_i(x) \rightarrow f(x)$  for each  $x \in X$ . By the definition of the product topology in  $P$ , this means that  $\Phi(f_i) \mapsto \Phi(f)$  in  $P$ . Hence  $\Phi$  is continuous. Since  $\Phi$  is injective, it induces an inverse map

$$\Phi^{-1} : \text{im}(\Phi) \rightarrow (X^*)_1$$

that is also continuous (we read the previous argument backwards).

Finally, we prove that  $\text{im}(\Phi)$  is closed in  $P$ . Suppose that  $(\Phi(f_i))_{i \in I}$  converges to  $p = (p_x)_{x \in X} \in P$ . By definition of the product topology, this means that  $f_i(x) \rightarrow p_x$  for all  $x \in X$ . Define

$$f : X \rightarrow \mathbb{F}, \quad x \mapsto p_x.$$

Then  $f$  is linear and  $f \in (X^*)_1$ . Thus  $p = \Phi(f) \in \text{im}(\Phi)$ . This in turn implies that  $(\text{im} \Phi)^{\text{closed}} \subseteq P^{\text{compact}}$ . But we know that  $(X^*)_1 \approx \text{im}(\Phi)$ , which implies that  $(X^*)_1$  is also compact.  $\square$

**Corollary 1.18.** *Every Banach space  $X$  is isometrically isomorphic to a closed subspace of  $C(K)$  for some compact  $T_2$  space  $K$ .*

*Proof.* Denote  $K = (X^*)_1$  endowed with the weak-\* topology. By the Banach-Alaoglu theorem,  $K$  is compact and  $T_2$ . We now define the map

$$\Delta : X \rightarrow C(K), \quad x \mapsto (f \mapsto f(x)).$$

First, we prove that  $\Delta$  is isometric. By Hahn-Banach, for every  $x \in X \setminus \{0\}$  there exists an  $f \in X^*$  such that  $\|f\| = 1$  and  $f(x) = \|x\|$ . Then we have

$$\|\Delta(x)\|_\infty = \sup_{g \in K} |g(x)| = \|x\|.$$

Since  $\Delta$  is an isometry, its image is complete and thus closed in  $C(K)$ . Obviously  $\Delta$  is a linear map, so we are done.  $\square$

## 1.4 Minkowski gauge

ADD MOTIVATION

**Definition 1.19.** Let  $X$  be a  $\mathbb{F}$ -vector space. A set  $A \subseteq X$  is

- balanced if:

$$\forall x \in A : \forall \alpha \in \mathbb{F}, |\alpha| \leq 1 : \alpha x \in A.$$

- absorbing if:

$$\forall x \in X : \exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : tx \in A.$$

- absorbing in  $a \in A$  if  $A - a = \{x - a \mid x \in A\}$  is absorbing.

**Example 1.20.** Let  $X$  be a vector space and  $p$  a seminorm in  $X$ . Then

$$V = \{x \in X \mid p(x) < 1\}$$

is convex, balanced, absorbing in each of its points.

**Theorem 1.21.**

Let  $X$  be a vector space and  $V \subseteq X$  convex, balanced and absorbing in each of its points. Then there exists a unique seminorm  $p$  on  $X$  such that

$$V = \{x \in X \mid p(x) < 1\}.$$

*Proof.* To  $V$  we associate the Minkowski gauge:

$$p(x) = \inf\{t \geq 0 \mid x \in t \cdot V\},$$

where  $t \cdot V = \{t \cdot v \mid v \in V\}$ . First we prove that  $p$  is well defined. Since  $V$  is absorbing, we have  $X = \bigcup_{n \in \mathbb{N}} n \cdot V$ , so for every  $x \in X$  the set  $\{t \geq 0 \mid x \in t \cdot V\}$  is nonempty. It's also

clear to see that  $p(0) = 0$ . Next we check for homogeneity. Suppose  $\alpha \neq 0$ . Then

$$\begin{aligned}
p(\alpha x) &= \inf\{t \geq 0 \mid \alpha x \in t \cdot V\} \\
&= \inf\left\{t \geq 0 \mid x \in \frac{t}{\alpha} \cdot V\right\} \\
&= \inf\left\{t \geq 0 \mid x \in \frac{t}{|\alpha|} \cdot V\right\} \\
&= \inf |\alpha| \left\{\frac{t}{|\alpha|} \geq 0 \mid x \in \frac{t}{|\alpha|} \cdot V\right\} \\
&= |\alpha| p(x).
\end{aligned}$$

Now we do the same for triangle inequality: let  $\alpha, \beta \geq 0$  so that  $\alpha + \beta > 0$ . Let  $a, b \in V$ . Then

$$\alpha a + \beta b = (\alpha + \beta) \left( \frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b \right) \in (\alpha + \beta) \cdot V.$$

This means that  $\alpha \cdot V + \beta \cdot V \subseteq (\alpha + \beta) \cdot V$ . Now let  $x, y \in X$  and  $p(x) = \alpha, p(y) = \beta$ . Take  $\delta > 0$ . Then  $x \in (\alpha + \delta) \cdot V, y \in (\beta + \delta) \cdot V$ . Hence

$$x + y \in (\alpha + \delta) \cdot V + (\beta + \delta) \cdot V \subseteq (\alpha + \beta + 2\delta) \cdot V,$$

and by definition,  $p(x + y) \leq \alpha + \beta + 2\delta$ . Since  $\delta > 0$  was arbitrary, we have  $p(x + y) \leq \alpha + \beta = p(x) + p(y)$ . Now that we have proved that  $p$  is a seminorm, we can show that

$$V = \{x \in X \mid p(x) < 1\}.$$

The inclusion  $(\supseteq)$  is easy: if  $p(x) < 1$ , then  $x \in (p(x) + \varepsilon) \cdot V$  for all  $\varepsilon > 0$ . By choosing  $\varepsilon = 1 - p(x) > 0$ , we get  $x \in V$ . Now we prove the other inclusion  $(\subseteq)$ . Let  $x \in V$ . Since  $V$  is absorbing in  $x$ , there exists an  $\varepsilon > 0$  such that  $y = x + tx \in V$  for all  $t \in (0, \varepsilon)$ . This means that  $x = \frac{1}{t+1}y$ , where  $y \in V$ . This implies that

$$p(x) = \frac{1}{t+1}p(y) \leq \frac{1}{1+t} \leq 1,$$

which proves the equality. Lastly, we prove the  $p$  is unique. Suppose there is some other seminorm  $q$  such that

$$\{x \in X \mid p(x) < 1\} = \{x \in X \mid q(x) < 1\}.$$

Suppose  $p \neq q$ . W.l.o.g. there exists an  $x \neq 0$  such that  $p(x) > q(x)$ . By homogeneity, we can assume that  $p(x) = 1 > q(x)$ , contradicting our assumption.  $\square$

*Remark.* If  $X$  is a TVS and  $V$  is an open subset, then  $V$  is absorbing at each of its points.

**Corollary 1.22.** *Let  $X$  be a TVS and  $\mathcal{U}$  a collection of all open convex balanced subsets of  $X$ . Then  $X$  is locally convex iff  $\mathcal{U}$  is a basis for the neighborhood system at 0.*



## 1.5 Applications of Hahn-Banach

Recall: if  $X$  is a  $\mathbb{R}$ -vector space then  $p : X \rightarrow \mathbb{R}$  is a sublinear functional if

$$p(x + y) \leq p(x) + p(y), \quad \forall x, y \in X$$

and

$$p(\alpha x) = \alpha x, \quad \forall x \in X, \quad \alpha > 0.$$

**Theorem 1.23** (Hahn-Banach theorem).

$\mathbb{R}$ : Suppose  $X$  is a  $\mathbb{R}$ -vector space and  $p : X \rightarrow \mathbb{R}$  is a sublinear functional. Given a linear functional  $f$  on  $Y \leq X$  such that  $f(y) \leq p(y)$  for every  $y \in Y$ ,  $f$  extends to a linear functional  $F : X \rightarrow \mathbb{R}$  such that  $F(x) \leq p(x)$  for every  $x \in X$ .

$\mathbb{C}$ : Suppose  $X$  is a  $\mathbb{C}$ -vector space and  $p : X \rightarrow \mathbb{R}$  is a seminorm. Given a linear functional  $f$  on  $Y \leq X$  such that  $|f(y)| \leq p(y)$  for every  $y \in Y$ ,  $f$  extends to a linear functional  $F : X \rightarrow \mathbb{R}$  such that  $|F(x)| \leq p(x)$  for every  $x \in X$ .

**Corollary 1.24** (Hahn-Banach extension theorem). Let  $X$  be a normed space,  $f \in X^*$  and  $Y \leq X$ . Then there exists an  $F \in X^*$  such that  $F|_Y = f$  and  $\|F\| = \|f\|$ .

**Corollary 1.25** (Hahn-Banach separation theorem). Suppose  $X$  is a LCS and  $A, B \subseteq X$  are disjoint closed convex sets. If  $B$  is compact then there exists an  $f \in X^*$  that separates  $A$  from  $B$ :

$$\exists \alpha, \beta \in \mathbb{R} : \forall a, b \in B : \operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(b).$$

**Theorem 1.26.**

Let  $X$  be a LCS and  $A \subseteq X$  convex. Then  $\overline{A} = \overline{A}^w$ .

*Proof.* Since the weak topology is weaker than the original topology, we have  $\overline{A} \subseteq \overline{A}^w$ . Let  $x \notin \overline{A}$ . We now separate  $\overline{A}$  and the compact set  $\{x\}$ : there exists  $f \in X^*$  so that there exist  $\alpha, \beta \in \mathbb{R}$  and we have

$$\operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(x)$$

for all  $a \in \overline{A}$ . This means that

$$\overline{A} \subseteq \{y \in X \mid \operatorname{Re} f(y) \leq \alpha\} = (\operatorname{Re} f)^{-1}(-\infty, \alpha] = C.$$

Since  $C$  is closed in the weak topology, it follows from  $A \subseteq C$  that  $\overline{A}^w \subseteq \overline{C}^w = C$ . Since  $x \notin C$ , we have  $x \notin \overline{A}^w$ .  $\square$

**Corollary 1.27.** A convex set in a LCS is closed iff it is weakly closed.

**Proposition 1.28.** Let  $X$  be a TVS and  $f : X \rightarrow \mathbb{F}$  a linear functional. The following is equivalent:

(1.)  $f$  is continuous;

- (2.)  $f$  is continuous in 0;
- (3.)  $f$  is continuous in some point;
- (4.)  $\ker f$  is closed;
- (5.)  $x \mapsto |f(x)|$  is a seminorm.

If  $X$  is a LCS, then these are also equivalent to

- (6.)  $\exists \alpha_1, \dots, \alpha_n \in \mathbb{R}_{>0}$  and  $\exists p_1, \dots, p_n \in \mathcal{P}$  such that

$$|f(x)| \leq \sum_{k=1}^n \alpha_k p_k(x), \quad \forall x \in X.$$

*Proof.* Equivalence of the first five statements is routine. Assume that  $X$  is a LCS. We prove the equivalence of (2) and (6). We start with (6)  $\Rightarrow$  (2). Let  $(x_i)_{i \in I}$  be a net in  $X$  that converges to 0. Then we have

$$0 \leq |f(x_i)| \leq \sum_{k=1}^n \alpha_k p_k(x_i) \xrightarrow{i \in I} 0.$$

This implies that  $f(x_i) \xrightarrow{i \in I} 0$ , proving the implication. Now the opposite: (2)  $\Rightarrow$  (6). We know that  $f^{-1}(B_1^\circ(0)) = \{x \in X \mid |f(x)| < 1\}$  is an open neighborhood of 0 in  $X$ . Then there exist  $p_1, \dots, p_r \in \mathcal{P}$  and an  $\varepsilon > 0$  such that

$$0 \in \bigcap_{i=1}^r U(0, p_i, \varepsilon) \subseteq f^{-1}(B_1^\circ(0)).$$

If  $p_i(x) < \varepsilon$  for all  $i \leq r$ , then  $|f(x)| < 1$ . Pick any  $\delta > 0$ . Then

$$p_i \left( x \cdot \frac{\varepsilon}{\sum p_i(x) + \delta} \right) = \frac{\varepsilon}{\delta + \sum p_i(x)} \cdot p_i(x) < \varepsilon,$$

which implies

$$\left| f \left( x \cdot \frac{\varepsilon}{\sum p_i(x) + \delta} \right) \right| < 1.$$

From this we get  $|f(x)| < \frac{1}{\varepsilon} (\sum p_i(x) + \delta)$ . Since  $\delta > 0$  was arbitrary, we get

$$|f(x)| \leq \sum_{i=1}^r \frac{1}{\varepsilon} p_i(x).$$

□

Recall the following theorem from measure theory.

**Theorem 1.29** (Riesz-Markoff theorem).

Let  $X$  be a compact  $T_2$  space,  $\Phi \in C(X)^*$ . Then there exists a regular Borel measure  $\mu$  such that

$$\Phi(f) = \int_X f d\mu, \quad \forall f \in C(X).$$

Further,  $\|\Phi\| = \|\mu\| = |\mu|(X)$ .

*Remark.* The above also works if  $X$  is locally compact and  $\Phi \in C_0(X)^*$ .

As a corollary, we get the following proposition.

**Proposition 1.30.** Let  $X$  be completely regular. Endow  $C(X)$  with a topology induced by its seminorms. If  $L \in C(X)^*$  then there exists a compact  $K \subseteq X$  and a regular Borel measure on  $K$  such that

$$L(f) = \int_K f d\mu, \quad \forall f \in C(X).$$

Conversely, every such pair  $(K, \mu)$  defines  $L \in C(X)^*$  with the above equation.

*Proof.* Begin with the implication  $(\Leftarrow)$ . Given  $(K, \mu)$ , we just need to prove that the induced functional  $L$  is continuous on  $X$ . We have

$$|L(f)| = \left| \int_K f d\mu \right| \leq \|\mu\| \sup_K |f| = \|\mu\| p_K(f)$$

and  $L$  is continuous. Now the converse  $(\Rightarrow)$ . Let  $L \in C(X)^*$ . By the previous proposition, there exist compact sets  $K_1, \dots, K_p \subseteq X$  and  $\alpha_1, \dots, \alpha_p > 0$  such that

$$|L(f)| \leq \sum_{j=1}^p \alpha_j p_{K_j}(f).$$

Let  $K = \bigcup_{j=1}^p K_j$  and  $\alpha = \max\{\alpha_1, \dots, \alpha_p\}$ . Then  $\|f\| \leq \alpha p_K(f)$  for all  $f \in C(X)$ . Observe that if  $f \in C(X)$  and  $f|_K = 0$ , then  $L(f) = 0$ . We now define a map  $F : C(K) \rightarrow \mathbb{F}$ . Since  $X$  is completely regular, we have a Tietze-like extension theorem: for any compact  $K \subseteq X$  and a continuous function  $g \in C(K)$ , there exists an extension  $\tilde{g} \in C(X)$ . Define  $F(g) := L(\tilde{g})$ . First we need to check that  $F$  is well defined. Suppose we have two extensions  $\tilde{g}$  and  $\tilde{\tilde{g}}$  of  $g \in C(K)$ . Since  $\tilde{g} - \tilde{\tilde{g}}$  is evidently zero on  $K$ , we have

$$L(\tilde{g}) - L(\tilde{\tilde{g}}) = L(\tilde{g} - \tilde{\tilde{g}}) = 0$$

and  $F$  really is well defined. It is also clearly linear, so we just need to check continuity:

$$|F(g)| = |L(\tilde{g})| \leq \alpha \cdot p_K(\tilde{g}) = \alpha \cdot \|g\|_{\infty, K},$$

therefore  $\|F\| \leq \alpha$  and  $F$  is continuous. Lastly we apply Riesz-Markoff: there exists a regular Borel measure  $\mu$  on  $K$  so that  $F(g) = \int_K g d\mu$ . If  $f \in C(X)$ , then  $g := f|_K \in C(K)$  and we have

$$L(f) = F(g) = \int_K g d\mu = \int_K f d\mu. \quad \square$$

## 1.6 Krein-Milman theorem

**Definition 1.31.** Let  $X$  be a vector space and  $C \subseteq X$  a convex subset.

(a) A nonempty convex subset  $F \subseteq C$  is a *face* if for any  $x, y \in C$  we have

$$(\exists t \in (0, 1) : tx + (1 - t)y \in F) \Rightarrow x, y \in F.$$

(b) A point  $x \in C$  is called an *extreme point* if  $\{x\} \subseteq C$  is a face.

We use the notation  $\text{ext}(C)$  for the set of all extreme points of  $C$ .

**Example 1.32.** If we consider spaces of real sequences, we have

- $\text{ext}((\ell^\infty)_1) = \{(\pm 1, \pm 1, \dots)\}$ ;
- $\text{ext}((\ell^1)_1) = \{(0, 0, \dots, \pm 1, \dots)\}$ .

**Example 1.33.** We prove that for  $c_0$  (the space of complex sequences that converge to 0) we have  $\text{ext}(c_0)_1 = \emptyset$ . Indeed, let  $x = (x_n)_n \in (c_0)_1$ . Since  $\lim_n x_n = 0$ , there exists  $N \in \mathbb{N}$  such that  $|x_n| < \frac{1}{2}$  for all  $n > N$ . Now define  $y, z \in c_0$  by setting  $y_n = z_n = x_n$  for  $n \leq N$  and

$$y_n = x_n + \frac{1}{2^n}, \quad z_n = x_n - \frac{1}{2^n}$$

for  $n > N$ . Then  $y, z \in (c_0)_1$  and  $x = \frac{1}{2}(y + z)$ , so  $x \notin \text{ext}(c_0)_1$ .

**Example 1.34.** Let us show that  $\text{ext}(L^1[0, 1])_1 = \emptyset$ . Take any  $f \in (L^1[0, 1])_1$ . Then  $\int_0^1 |f(t)| dt = 1$ , so there must exist an  $x \in [0, 1]$  such that  $\int_0^x |f(t)| dt = 1/2$ . Now define  $g := 2 \cdot f \cdot \chi_{[0, x]}$  and  $h := 2 \cdot f \cdot \chi_{[x, 1]}$ . Now we have  $g, h \in (L^1[0, 1])_1$  and  $f = \frac{1}{2}g + \frac{1}{2}h$ , so  $f$  cannot be an extreme point.

**Example 1.35.** Finally, let us prove that  $\text{ext}(C[0, 1])_1 = \{\pm 1\}$  for real valued functions. Take any  $f \in (C[0, 1])_1$ . Then define functions  $g(t) = \min\{2f(t) + 1, 1\}$  and  $h(t) = \max\{2f(t) - 1, -1\}$ . Clearly  $g, h \in (C[0, 1])_1$  and  $f = \frac{1}{2}g + \frac{1}{2}h$ . If  $f$  is an extreme point, then  $g = h$ , which happens only if  $f = \pm 1$ .

**Definition 1.36.** For a vector space  $X$  and  $A \subseteq X$ , define a *convex hull*  $\text{co} A$  as the intersection of all convex sets in  $X$  that contain  $A$ . If  $X$  is a TVS, then define a *closed convex hull*  $\overline{\text{co}} A$  as the intersection of all closed convex sets that contain  $A$ .

Convex hull of a set  $A$  can be given explicitly:

$$\text{co} A = \left\{ \sum_{i=0}^n \alpha_i x_i \mid n \in \mathbb{N}, \alpha_i \geq 0, \sum_{i=0}^n \alpha_i = 1, x_i \in A \right\}.$$

If  $X$  is a TVS, then  $\overline{\text{co}} A = \overline{\text{co} A}$ .

**Lemma 1.37.** If  $C \subseteq X$  is a convex subset of a vector space and  $a \in C$ , then the following is equivalent.

- (a)  $a \in \text{ext } C$ .
- (b) If  $x_1, x_2 \in C$  and  $a = \frac{1}{2}(x_1 + x_2)$ , then  $x_1 = x_2 = a$ .
- (c) If  $x_1, x_2 \in C$ ,  $t \in (0, 1)$  and  $a = tx_1 + (1 - t)x_2$ , then  $x_1 = x_2 = a$ .
- (d)  $C \setminus \{a\}$  is a convex set.
- (e) If  $x_1, \dots, x_n \in C$  and  $a \in \text{co}\{x_1, \dots, x_n\}$ , then  $a = x_k$  for some index  $k$ .

*Proof.* Items (a) and (c) are equivalent by definition.

(b)  $\Rightarrow$  (c): Let  $a = tx_1 + (1 - t)x_2$ . Then

$$a = \frac{1}{2}(2tx_1 + (1 - 2t)x_2) + \frac{1}{2}x_2,$$

so we get  $2tx_1 + (1 - 2t)x_2 = x_2$ , which gives us  $x_1 = x_2$ .

(c)  $\Rightarrow$  (d): Take any  $x_1, x_2 \in C \setminus \{a\}$ . Since  $C$  is convex,  $tx_1 + (1 - t)x_2 \in C$ . Now if  $a = tx_1 + (1 - t)x_2 \in \text{co}\{x_1, x_2\}$ , then  $a = x_1 = x_2$ , which contradicts our assumption.

So  $tx_1 + (1 - t)x_2 \in C \setminus \{a\}$  and  $C \setminus \{a\}$  is convex.

(d)  $\Rightarrow$  (e) If  $x_1, \dots, x_n \in C \setminus \{a\}$ , then  $\text{co}\{x_1, \dots, x_n\} \subseteq C \setminus \{a\}$  by convexity, contradiction.

(e)  $\Rightarrow$  (b): Suppose  $a = \frac{1}{2}(x_1 + x_2)$ . Then either  $x_1 = a$  or  $x_2 = a$  by our assumption. W.l.o.g. assume  $x_1 = a$ . Then  $a = \frac{1}{2}(a + x_2)$ , which implies  $a = x_2$ .  $\square$

**Lemma 1.38.** Let  $X$  be a TVS and  $C \subseteq X$  a nonempty compact convex set. Then for  $\Phi \in X^*$  the set

$$F = \{x \in C \mid \text{Re } \Phi(x) = \min_C \text{Re } \Phi\}$$

is a closed face of  $C$ .

*Proof.* Since  $C$  is compact and  $x \mapsto \text{Re } \Phi(x)$  is continuous, it attains its minimum on  $C$ . Hence  $F$  is nonempty. Since  $F$  is a continuous preimage of a point, it is also closed. By the linearity of  $\Phi$ ,  $F$  is convex. Now suppose that  $t \in (0, 1)$  and  $x, y \in C$  are such that  $tx + (1 - t)y \in F$ . Then

$$\begin{aligned} \min_C \text{Re } \Phi &= \text{Re } \Phi(tx + (1 - t)y) \\ &= t \cdot \text{Re } \Phi(x) + (1 - t) \text{Re } \Phi(y) \\ &\geq t \cdot \min_C \text{Re } \Phi + (1 - t) \min_C \text{Re } \Phi \\ &= \min_C \text{Re } \Phi. \end{aligned}$$

Since we have the equality in the second-to-last line, we have  $\text{Re } \Phi(x) = \min_C \text{Re } \Phi$  and  $\text{Re } \Phi(y) = \min_C \text{Re } \Phi$ , meaning that  $x, y \in F$ .  $\square$

*Remark.* Not all closed convex faces are of this form.

ADD A PIC-  
TURE

**Theorem 1.39** (Krein-Milman).

Let  $X$  be a LCS and  $C \subseteq X$  a nonempty compact convex subset. Then  $C = \overline{\text{co}}(\text{ext } C)$ . In particular,  $\text{ext } C \neq \emptyset$ .

*Proof.* Let  $\mathcal{F} = \{\text{closed faces in } C\}$  be ordered with  $\supseteq$ . Since  $C \in \mathcal{F}$ , it is nonempty. The set  $\mathcal{F}$  is then partially ordered. Since any increasing chain in  $\mathcal{F}$  has the finite intersection property,  $\mathcal{F}$  has a nonempty intersection due to  $C$  being compact. As a result, any increasing chain in  $\mathcal{F}$  has an upper bound. This tells us that we can apply Zorn's lemma to obtain a maximal element  $F_0 \in \mathcal{F}$ .

We prove that  $F_0 = \{p\}$  for some  $p \in X$ . Assume for a contradiction that there are distinct  $x, y \in F_0$ . By Hahn-Banach, there exists a  $\Phi \in X^*$  such that  $\Phi(x) \neq \Phi(y)$ . W.l.o.g. we assume that  $\text{Re } \Phi(x) < \text{Re } \Phi(y)$ . Define a set

$$F_1 = \{z \in F_0 \mid \text{Re } \Phi(z) = \min_{F_0} \text{Re } \Phi\}.$$

Then  $F_1 \subsetneq F_0$ , since  $y \notin F_1$ . By the previous lemma,  $F_1$  is a closed face in  $F_0$ , so it is a closed face in  $C$ , contradicting maximality of  $F_0$ . As a result,  $F_0 = \{p\}$ , which implies that  $p \in \text{ext}(C)$  and the set of extreme points of  $C$  is non-empty.

Since we have  $C \supseteq \text{ext } C$ , we also have  $C = \overline{\text{co}}(C) \supseteq \overline{\text{co}}(\text{ext } C)$ . Suppose  $x \in C \setminus \overline{\text{co}}(\text{ext } C)$ . By Hahn-Banach, there exists a  $\Psi \in X^*$  such that  $\text{Re } \Psi(x) < \min_{\overline{\text{co}}(\text{ext } C)} \text{Re } \Psi$ . So the set

$$F = \{z \in C \mid \text{Re } \Psi(z) = \min_C \text{Re } \Psi\}$$

is a closed face in  $C$ . By the first part of this proof, there exists a  $z \in \text{ext } F \subseteq \text{ext } C$ . Hence

$$\min_C \text{Re } \Psi = \text{Re } \Psi(z) = \min_{\overline{\text{co}}(\text{ext } C)} \text{Re } \Psi > \text{Re } \Psi(x) \geq \min_C \text{Re } \Psi,$$

which leads to a contradiction. Therefore  $\overline{\text{co}}(\text{ext } C) = C$ .  $\square$

**Example 1.40.** Let  $\mathcal{H}$  be a Hilbert space. Then

$$\text{ext}(\mathcal{H})_1 = \{v \in \mathcal{H} \mid \|v\| = 1\}.$$

First we prove the inclusion ( $\supseteq$ ). Suppose that  $\|v\| = 1$  and  $v = tx + (1-t)y$ , where  $t \in (0, 1)$  and  $x, y \in (\mathcal{H})_1$ . We have

$$\begin{aligned} 1 &= \|v\|^2 \\ &= \|tx + (1-t)y\|^2 \\ &= \langle tx + (1-t)y, tx + (1-t)y \rangle \\ &= t^2\|x\|^2 + (1-t)^2\|y\|^2 + 2t(1-t)\text{Re}\langle x, y \rangle \\ &\leq t^2 + (1-t)^2 + 2t(1-t) = 1. \end{aligned}$$

We get equality in the Cauchy-Schwartz inequality, so  $x, y$  are linearly dependent and there-

fore equal. For the reverse inclusion, let  $v \in \text{ext}(\mathcal{H})_1$ . If  $\|v\| < 1$ , then

$$v = \frac{1}{2} \cdot \frac{v}{\|v\|} + \frac{1}{2} \cdot (2\|v\| - 1) \frac{v}{\|v\|},$$

so  $v$  cannot be an extreme point of  $(\mathcal{H})_1$ .

**Example 1.41.** It holds that

$$\text{ext}(\mathcal{B}(\mathcal{H}))_1 = \{V \in \mathcal{B}(\mathcal{H}) \mid V \text{ or } V^* \text{ is an isometry}\}.$$

Here, we will just prove the inclusion  $(\supseteq)$ . Let  $V \in \mathcal{B}(\mathcal{H})$  be an isometry and suppose  $V = tS + (1-t)T$  for  $t \in (0, 1)$  and  $S, T \in (\mathcal{B}(\mathcal{H}))_1$ . For  $x \in \mathcal{H}$  we have:

$$\begin{aligned} \|x\| &= \|Vx\| \\ &= \|tSx + (1-t)Tx\| \\ &\leq t\|Sx\| + (1-t)\|Tx\| \\ &\leq t\|S\|\|x\| + (1-t)\|T\|\|x\| \\ &\leq t\|x\| + (1-t)\|x\| = \|x\|. \end{aligned}$$

Since we have equality, we get  $\|S\| = \|T\| = 1$  and  $\|Sx\| = \|Tx\| = \|x\|$ . So  $S, T$  are isometries. For every  $x \in \partial(\mathcal{H})_1 = \text{ext}(\mathcal{H})_1$ , we have

$$Vx = t \cdot Sx + (1-t)Tx$$

and by the previous example that implies  $Tx = Sx = Vx$ , so we really have  $S = T = V$ . We use the same argument if  $V^*$  is an isometry. For now, we lack some tools to prove the reverse inclusion. We will prove the equality in Corollary .

**Example 1.42.** If  $X$  be a Banach space, then  $(X^*)_1$  is weak-\* compact (by Banach-Alaoglu), so Krein-Milman gives us  $(X^*)_1 = \overline{\text{co}}(\text{ext}(X^*)_1)$ . Hence  $(X^*)_1$  has a lot of extreme points. As a corollary,  $c_0$ ,  $L^1[0, 1]$  and  $C[0, 1]$  are not duals of Banach spaces.

**Theorem 1.43** (Milman).

Let  $X$  be a LCS,  $K \subseteq X$  compact and assume  $\overline{\text{co}}(K)$  is compact. Then  $\text{ext}(\overline{\text{co}}(K)) \subseteq K$ .

*Proof.* Assume there exists  $x_0 \in \text{ext}(\overline{\text{co}}(K)) \setminus K$ . Then there exists a basis neighborhood  $V$  of 0 in  $X$  such that  $(x_0 + \overline{V}) \cap K = \emptyset$ , or equivalently,  $x_0 \notin K + \overline{V}$ . If we write  $K \subseteq \bigcup_{x \in K} (x + V)$ , we get

$$K \subseteq \bigcup_{j=1}^n (x_j + V).$$

Form  $K_j = \overline{\text{co}}(K \cap (x_j + V))$ . Then  $K_j$  is convex and compact since  $K_j \subseteq \overline{\text{co}}(K)$ . We also have  $K_j \subseteq x_j + \overline{V} = x_j + \overline{V}$  since  $V$  is convex. Also,  $K \subseteq K_1 \cup \dots \cup K_n$ . Next we prove

that  $\text{co}(K_1 \cup \dots \cup K_n)$  is compact. Define

$$\Sigma = \{(t_1, \dots, t_n) \in [0, 1]^n \mid \sum_{j=1}^n t_j = 1\}$$

and the function

$$f : \Sigma \times K_1 \times \dots \times K_n \rightarrow X, \quad (t, k_1, \dots, k_n) \mapsto \sum_{j=1}^n t_j k_j.$$

Denote  $C := \text{im } f$ . Obviously,  $C \subseteq \text{co}(K_1 \cup \dots \cup K_n)$  and  $C$  is a convex compact set. Furthermore,  $C \supset K_j$  for each  $j$ , so  $C = \text{co}(K_1 \cup \dots \cup K_n)$ . From there, we get

$$\overline{\text{co}}(K) \subseteq \overline{\text{co}}(K_1 \cup \dots \cup K_n) = \text{co}(K_1 \cup \dots \cup K_n).$$

But since  $K_j \subseteq \overline{\text{co}}(K)$  for all  $j$ , we deduce  $\overline{\text{co}}(K) = \overline{\text{co}}(K_1 \cup \dots \cup K_n)$ . We know that  $x_0 \in \overline{\text{co}}(K)$ , so

$$x_0 = t_1 y_1 + \dots + t_n y_n$$

for some  $t_i \in [0, 1]$ ,  $\sum t_i = 1$  and  $y_j \in K_j$ . But  $x_0 \in \text{ext}(\overline{\text{co}})(K)$ , so  $y_j = x_0$  for some  $j$ . So we get  $x_0 \in K_j \subseteq x_j + \bar{V} \subseteq K + \bar{V}$ , a contradiction.  $\square$

*Remark.* (1.) In finite dimensions, the convex hull of a compact set is compact. In infinite dimensions this fails.

(2.) The set  $\text{ext}(C)$  is not always closed, even if  $C \subseteq \mathbb{R}^3$  is convex and compact.

ADD A PICTURE



## 2 $C^*$ -algebras and continuous functional calculus

### 2.1 Spectrum

Let  $A$  be a complex algebra with a unit 1 and

$$\mathrm{GL}(A) = \{a \in A \mid a \text{ is invertible}\}.$$

If  $x \in A$ , we define the spectrum

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \cdot 1 \notin \mathrm{GL}(A)\}.$$

**Proposition 2.1.** *Let  $A$  be a complex algebra with unity 1 and  $x, y \in A$ . Then*

$$\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}.$$

*Proof.* Suppose  $1 - xy \in \mathrm{GL}(A)$ . Formally, we can write

$$(1 - xy)^{-1} = 1 + xy + (xy)^2 + \dots$$

and

$$(1 - yx)^{-1} = 1 + yx + (yx)^2 + \dots = 1 + y(1 - xy)^{-1}x.$$

From this, we claim that indeed  $1 - yx \in \mathrm{GL}(A)$  and

$$(1 - yx)^{-1} = 1 + y(1 - xy)^{-1}x.$$

The proof is straightforward: we have

$$\begin{aligned} (1 + y(1 - xy)^{-1}x)(1 - yx) &= (1 - yx) + y(1 - xy)^{-1}(x - xyx) \\ &= (1 - yx) + y(1 - xy)^{-1}(1 - xy)x \\ &= (1 - yx) + yx = 1 \end{aligned}$$

and

$$\begin{aligned} (1 - yx)(1 + y(1 - xy)^{-1}x) &= (1 - yx) + (y - yxy)(1 - xy)^{-1}x \\ &= (1 - yx) + y(1 - xy)(1 - xy)^{-1}x \\ &= (1 - yx) + yx = 1. \end{aligned}$$

Now the proof of the statement is at hand: if  $\lambda \in \sigma_A(xy) \setminus \{0\}$ , then

$$\lambda - xy \notin \mathrm{GL}(A) \Rightarrow 1 - \frac{x}{\lambda}y \notin \mathrm{GL}(A) \Rightarrow 1 - y\frac{x}{\lambda} \notin \mathrm{GL}(A) \Rightarrow \lambda - yx \notin \mathrm{GL}(A).$$

Thus,  $\lambda \in \sigma_A(yx)$ . Similarly, if  $\lambda \in \sigma_A(yx) \setminus \{0\}$ , then  $\lambda \in \sigma_A(xy)$ . □

**Example 2.2.** *Let  $S, S^* \in \mathcal{B}(\ell^2)$  be the right and left shift operators, respectively. Then  $SS^* = I$ , but*

$$SS^*(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

This implies that  $0 \in \sigma(SS^*)$ , but  $0 \notin \sigma(S^*S)$ .

## 2.2 Banach and $C^*$ -algebras

- Definition 2.3.**
- A Banach algebra is a Banach space  $A$  that is also an algebra, satisfying  $\|xy\| \leq \|x\|\|y\|$  for all  $x, y \in A$ . If a Banach algebra has a unit 1, we also demand  $\|1\| = 1$ .
  - An involution on a Banach algebra  $A$  is a skew-linear map

$$*: A \rightarrow A, \quad a \mapsto a^*$$

satisfying

$$(xy)^* = y^*x^*, \quad (x^*)^* = x, \quad \|x^*\| = \|x\|.$$

A  $C^*$ -algebra is a Banach  $*$ -algebra  $A$  that also satisfies  $\|x^*x\| = \|x\|^2$  for all  $x \in A$ .

Unless otherwise mentioned, all algebras in this section are unital.

**Proposition 2.4.** *We collect some basic properties of Banach algebras.*

(1.) *If  $A$  is a Banach  $*$ -algebra, then  $(x^*)^{-1} = (x^{-1})^*$  and  $\sigma_A(x^*) = (\sigma_A(x))^*$ .*

(2.) *Let  $A$  be a Banach algebra. If  $\|x\| < 1$ , then  $1 - x \in \text{GL}(A)$  and*

$$(1 - x)^{-1} = 1 + x + x^2 + \dots$$

*As a consequence, if  $\|1 - x\| < 1$ , then  $x \in \text{GL}(A)$ .*

(3.) *Let  $A$  be a Banach algebra. Then  $\text{GL}(A) \subseteq A$  is open, and the map  $x \mapsto x^{-1}$  is continuous on  $\text{GL}(A)$ .*

(4.) *If  $A$  is a Banach algebra and  $x \in A$ , then  $\sigma_A(x)$  is a nonempty compact set.*

*Proof.* (1.) Suppose that the inverse  $(x^*)^{-1}$  exists. Then  $(x^*)^{-1} \cdot (x^*) = 1$ , so starring gives us  $(x^*)^* \cdot ((x^*)^{-1})^* = 1$  and  $x \cdot ((x^*)^{-1})^* = 1$ . Similarly, we have  $(x^*) \cdot (x^*)^{-1} = 1$ , which implies  $((x^*)^{-1})^* \cdot x = 1$ . This means that  $x$  is invertible and  $((x^*)^{-1})^* = x^{-1}$ . Starring this equation now gives us  $(x^*)^{-1} = (x^{-1})^*$ . For the opposite direction, suppose that  $x$  is invertible. Then

$$(x^{-1})^* \cdot x^* = (x \cdot x^{-1})^* = 1^* = 1$$

and

$$x^* \cdot (x^{-1})^* = (x^{-1} \cdot x)^* = 1^* = 1,$$

which means that  $x^*$  is invertible and  $(x^*)^{-1} = (x^{-1})^*$ . The rest is a matter of simple computation:

$$\begin{aligned} \lambda \in \sigma_A(x^*) &\Leftrightarrow x^* - \lambda \notin \text{GL}(A) \Leftrightarrow (x - \bar{\lambda})^* \notin \text{GL}(A) \\ &\Leftrightarrow (x - \bar{\lambda}) \notin \text{GL}(A) \Leftrightarrow \bar{\lambda} \in \sigma_A(x) \\ &\Leftrightarrow \lambda \in (\sigma_A(x))^*. \end{aligned}$$

Are all our Banach algebras complex? We probably need that for nonempty spectra.

If  $\|x\| \leq 1$ , then the series  $\sum_{n=0}^{\infty} x^n$  converges in norm to some  $x'$ . Since multiplication between elements of a Banach algebra is norm-continuous, we get

$$(1-x)x' = (1-x) \cdot \lim_{k \rightarrow \infty} \sum_{n=1}^k x^n = \lim_{k \rightarrow \infty} (1-x) \cdot \sum_{n=1}^k x^n = \lim_{k \rightarrow \infty} 1 - x^{k+1} = 1$$

and similarly for  $x'(1-x)$ .

Let  $y \in \text{GL}(A)$ . If  $\|x - y\| \leq \frac{1}{\|y^{-1}\|}$ , then

$$\|1 - xy^{-1}\| = \|(y-x)y^{-1}\| \leq \|y-x\|\|y^{-1}\| \leq 1,$$

which implies that  $xy^{-1} \in \text{GL}(A)$ , and thus  $x = xy^{-1} \cdot y \in \text{GL}(A)$ . We have shown that  $\text{GL}(A)$  is open. Using the same notation and noting that  $(xy^{-1})^{-1} = (1 - (1 - xy^{-1}))^{-1}$ , we get

$$\|(xy^{-1})^{-1}\| \leq \sum_{n=0}^{\infty} \|(1 - xy^{-1})\|^n \leq \sum_{n=0}^{\infty} \|y^{-1}\|^n \|x - y\|^n \leq \frac{1}{1 - \|y^{-1}\| \cdot \|x - y\|}.$$

Now,

$$\begin{aligned} \|x^{-1} - y^{-1}\| &= \|x^{-1}(y-x)y^{-1}\| \\ &\leq \|y^{-1}(xy^{-1})^{-1}\| \|y-x\| \|y^{-1}\| \\ &\leq \|(xy^{-1})^{-1}\| \|y-x\| \|y^{-1}\|^2 \\ &\leq \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\| \cdot \|x - y\|} \|y-x\|. \end{aligned}$$

Since the function  $t \mapsto \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\| \cdot t} t$  is continuous at  $t = 0$ , the map  $x \mapsto x^{-1}$  is continuous.

First, we prove compactness by showing that  $\sigma_A(x)$  is bounded and closed. Suppose there exists  $\lambda \in \sigma_A(x)$ , such that  $|\lambda| > \|x\|$ . Then  $(1 - \frac{x}{\lambda})$  is invertible by (2.), so  $(-\lambda) \cdot (1 - \frac{x}{\lambda}) = x - \lambda$  is invertible as well. But this contradicts the fact that  $\lambda \in \sigma_A(x)$ , so we have shown that  $\sigma_A(x) \subseteq \overline{B(0, \|x\|)}$ . Next, we prove that the spectrum is closed. Define a continuous map

$$\mathbb{C} \rightarrow A, \quad \lambda \mapsto x - \lambda$$

and notice that the inverse image of  $\text{GL}(A)$  (which is open by (3.)) is exactly  $\mathbb{C} \setminus \sigma_A(x)$ . This means that  $\mathbb{C} \setminus \sigma_A(x)$  is open and  $\sigma_A(x)$  is closed. For non-emptiness, we have to employ some standard Banach algebra techniques. We say that a function  $f$  from a domain  $\Omega \subseteq \mathbb{C}$  to a Banach space  $X$  is analytical if there exists a limit

$$f'(z_0) := \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

for every  $z_0 \in \Omega$  and the function  $f'$  is continuous on  $\Omega$ . A lot of theory for complex analytic functions also applies to Banach space-valued analytic functions; in particular, we have Cauchy's integral formula, Liouville's theorem and the fact that every vector valued analytic function can be locally expressed as a power series with coefficients in  $X$ . Now we can define the resolvent function

$$F : \mathbb{C} \setminus \sigma_A(x) \rightarrow A, \quad F(z) = (z - x)^{-1}.$$

It's routine to show that  $F$  is analytic and its derivative is  $F'(z) = (z - x)^{-2}$ . Now for  $z \in \mathbb{C} \setminus \overline{B(0, \|x\|)}$ , we have  $F(z) = z^{-1} \cdot (1 - a/z)$ , which goes to 0 as  $z \rightarrow \infty$ . Now if  $\sigma_A(x) = \emptyset$ , then  $F$  would be an entire function that vanishes at  $\infty$ . By Liouville's theorem,  $F$  is constant and so  $F' = 0$ . This is a contradiction.  $\square$

**Theorem 2.5** (Gelfand-Mazur).

*If  $A$  is Banach algebra that is also a division ring, then  $A = \mathbb{C}$ .*

*Proof.* Let  $x \in A$  and  $\lambda \in \sigma_A(x)$ . Then  $x - \lambda \cdot 1 \notin \text{GL}(A)$ , implying  $x - \lambda = 0$ , hence  $x = \lambda \in \mathbb{C}$ .  $\square$

**Definition 2.6.** If  $f(x) = \sum_{j=0}^n a_j x^j$  is a polynomial and  $a \in A$ , we define  $f(a) = \sum_{j=0}^n a_j a^j \in A$ .

**Theorem 2.7** (Spectral mapping theorem for polynomials).

*Let  $A$  be a complex unitary algebra and  $f \in \mathbb{C}[x]$ . Then  $f(\sigma_A(a)) = \sigma_A(f(a))$  for all  $a \in A$ .*

*Proof.* First, we prove the inclusion  $(\subseteq)$ . If  $\lambda \in \sigma_A(a)$  and  $f(x) = \sum_{j=0}^n a_j x^j$ , then

$$f(x) - f(\lambda) = \sum_{j=1}^n a_j (x^j - \lambda^j) = (x - \lambda) \cdot \sum_{j=1}^n a_j \sum_{k=0}^{j-1} x^k \lambda^{j-1-k}.$$

Substituting  $x = a$ , we obtain

$$f(a) - f(\lambda) = (a - \lambda) \left( \sum_{j=1}^n a_j \sum_{k=0}^{j-1} a^k \lambda^{j-1-k} \right).$$

Since  $a - \lambda$  commutes with the second factor,  $f(a) - f(\lambda)$  is not invertible and  $f(\lambda) \in \sigma_A(f(a))$ . For the converse inclusion  $(\supseteq)$ , assume  $\mu \notin f(\sigma_A(a))$ . We factor

$$f(x) - \mu = a_n (x - \lambda_1) \cdots (x - \lambda_n).$$

Since  $f(\lambda) - \mu \neq 0$  for any  $\lambda \in \sigma_A(a)$ , it follows that  $\lambda_i \notin \sigma_A(a)$  for all  $i$ . Therefore,  $f(a) - \mu \in \text{GL}(A)$ .  $\square$

**Definition 2.8.** Let  $A$  be a Banach algebra and  $x \in A$ . The spectral radius of  $x$  is

$$r(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|.$$

*Remark.* By proposition 2.1, we have  $r(xy) = r(yx)$ .

In the introductory course, we proved the following.

**Theorem 2.9** (Spectral radius formula).

*Let  $A$  be a Banach algebra and  $x \in A$ . Then  $\lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}}$  exists and is equal to  $r(x)$ .*

**Definition 2.10.** Let  $A$  be a Banach  $*$ -algebra and  $x \in A$ .

- $x$  is *normal* iff  $xx^* = x^*x$ .
- $x$  is *self-adjoint* iff  $x^* = x$ .
- $x$  is *skew self-adjoint* iff  $x^* = -x$ .

The set of all self-adjoint operators is denoted as  $A_{\text{sa}}$ .

*Remark.* Every  $a \in A$  can be uniquely expressed as a sum of a self-adjoint and skew self-adjoint element:

$$a = \frac{a + a^*}{2} + \frac{a - a^*}{2}.$$

Alternatively, we can uniquely write it in the form of

$$a = \left( \frac{a + a^*}{2} \right) + i \cdot \left( \frac{a - a^*}{2i} \right)$$

where both terms in parentheses are self-adjoint.

**Corollary 2.11.** *Let  $A$  be a Banach  $*$ -algebra and  $x \in A$  normal. Then  $r(x^*x) \leq r(x)^2$ . If  $A$  is a  $C^*$ -algebra, then  $r(x^*x) = r(x)^2$ .*

*Proof.* We use the spectral radius formula:

$$\begin{aligned} r(x^*x) &= \lim_{n \rightarrow \infty} \|(x^*x)^n\|^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \|(x^*)^n x^n\|^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \|(x^n)^* x^n\|^{\frac{1}{n}} \\ &\leq \lim_{n \rightarrow \infty} \|x^n\|^{\frac{2}{n}} = r(x)^2. \end{aligned}$$

If  $A$  is a  $C^*$ -algebra, we have an equality in the last line of the above calculation.  $\square$

**Proposition 2.12.** *Let  $A$  be a  $C^*$ -algebra and  $x \in A$  normal. Then  $r(x) = \|x\|$ .*

*Proof.* First, assume  $x$  is self-adjoint. Then

$$\|x^2\| = \|xx^*\| = \|x\|^2.$$

By induction, we get  $\|x^{2^n}\| = \|x\|^{2^n}$  for every  $n \in \mathbb{N}$ . Therefore,

$$r(x) = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|.$$

If  $x$  is only normal, then

$$\|x\|^2 = \|x^*x\| = r(x^*x) = r(x)^2,$$

which implies  $\|x\| = r(x)$ .  $\square$

**Corollary 2.13.** *Let  $A, B$  be  $C^*$ -algebras and  $\Phi : A \rightarrow B$  a  $*$ -homomorphism ( $\Phi(x^*) = \Phi(x)^*$ ). Then  $\Phi$  is a contraction. Furthermore, if  $\Phi$  is a  $*$ -isomorphism, then it is isometric.*

*Proof.* Clearly,  $\Phi$  maps invertible elements to invertible elements, so  $\Phi(\text{GL}(A)) \subseteq \text{GL}(B)$ . This implies  $\sigma_B(\Phi(x)) \subseteq \sigma_A(x)$ , hence  $r(\Phi(x)) \leq r(x)$ . Then

$$\begin{aligned} \|\Phi(x)\|^2 &= \|\Phi(x)\Phi(x)^*\| = \|\Phi(x)\Phi(x^*)\| \\ &= \|\Phi(xx^*)\| = r(\Phi(xx^*)) \\ &\leq r(xx^*) = \|xx^*\| = \|x\|^2. \end{aligned}$$

If  $\Phi$  is a  $*$ -isomorphism, we apply the same reasoning to its inverse, which implies that  $\Phi$  must be an isometry.  $\square$

**Corollary 2.14.** *If  $A$  is a  $*$ -algebra, then there exists at most one norm on  $A$  that makes it into a  $C^*$ -algebra.*

*Proof.* Considering the identity map

$$(A, \|\cdot\|_1) \rightarrow (A, \|\cdot\|_2),$$

it is a  $*$ -isomorphism, so it preserves the norm by the previous corollary.  $\square$

**Lemma 2.15.** *Let  $A$  be a  $C^*$ -algebra and  $x \in A$  self-adjoint. Then  $\sigma_A(x) \subseteq \mathbb{R}$ .*

*Proof.* Suppose  $\lambda = \alpha + i\beta \in \sigma_A(x)$  for some  $\alpha, \beta \in \mathbb{R}$ . Define  $y = x - \alpha + it$  for  $t \in \mathbb{R}$ . Then  $i(\beta + t) \in \sigma_A(y)$  and  $y$  is normal. Thus,

$$\begin{aligned} |i(\beta + t)|^2 &= (\beta + t)^2 \leq r(y) \\ &= \|y\|^2 = \|yy^*\| \\ &= \|(x - \alpha)^2 + t^2\| \leq \|x - \alpha\|^2 + t^2. \end{aligned}$$

Simplifying, we get  $\beta^2 + 2\beta t \leq \|x - \alpha\|^2$ , and since  $t \in \mathbb{R}$  was arbitrary, we have  $\beta = 0$ .  $\square$

**Lemma 2.16.** *Let  $A$  be a Banach algebra and  $x \notin \text{GL}(A)$ . If  $(x_n)_n \subseteq \text{GL}(A)$  satisfies  $x_n \rightarrow x$ , then  $\|x_n^{-1}\| \rightarrow \infty$ .*

*Proof.* If the sequence  $\|x_n^{-1}\|$  is bounded, then

$$\|1 - xx_n^{-1}\| = \|(x_n - x)x_n^{-1}\| \leq \|x_n - x\| \cdot \|x_n^{-1}\| \rightarrow 0.$$

In particular, there exists some  $n \in \mathbb{N}$  such that  $\|1 - xx_n^{-1}\| < 1$ , which implies  $xx_n^{-1} \in \text{GL}(A)$  and therefore  $x = (xx_n^{-1})x_n \in \text{GL}(A)$ , a contradiction.  $\square$

**Proposition 2.17.** *Let  $B$  be a  $C^*$ -algebra and  $A \subseteq B$  a unital  $C^*$ -subalgebra. Then for all*

$x \in A$ , we have  $\sigma_A(x) = \sigma_B(x)$ .

*Proof.* Obviously,  $\text{GL}(A) \subseteq \text{GL}(B)$ . For a self adjoint  $x \in A \setminus \text{GL}(A)$ , we have  $it \notin \sigma_A(x)$  for  $t \in \mathbb{R}$ . So there exists  $(x - it)^{-1} \in A$ . Clearly,

$$x - it \in \text{GL}(A) \xrightarrow{t \rightarrow 0} x \notin \text{GL}(A),$$

thus  $\|(x - it)^{-1}\| \rightarrow \infty$ . Since the inverse function is continuous, this immediately yields  $x \notin \text{GL}(B)$ . For general  $x \in A$ : if  $x \in \text{GL}(B)$ , then  $x^*x \in \text{GL}(B)$  is self-adjoint. By the first part of the proof,  $x^*x \in \text{GL}(A)$ . It follows that

$$x^{-1} = (x^*x)^{-1}x^* \in A,$$

so  $x \in \text{GL}(A)$ . □

**Example 2.18.** Let  $X$  be a topological space and  $C_b(X)$  be the set of continuous bounded complex functions on  $X$ , endowed with the sup metric. Then  $C_b(X)$  is a  $C^*$ -algebra (where  $f^*(x) = \overline{f(x)}$ ). If  $X$  is compact, then  $C(X)$  is also an  $C^*$ -algebra.

**Example 2.19.** Define  $C_0(X)$  as the subset of  $f \in C_b(X)$  that vanish at infinity ( $\forall \varepsilon > 0 : \exists K \subseteq X$  compact, such that  $\|f|_{X \setminus K}\| < \varepsilon$ ). Then  $C_0(X)$  is a (possibly non-unital)  $C^*$ -subalgebra of  $C_b(X)$ .

**Example 2.20.** Let  $(X, \mu)$  be a measure space. Then  $L^\infty(X, \mu)$ , the set of essentially bounded functions on  $X$  endowed with the essential supremum norm, is a unital abelian  $C^*$ -algebra.

**Example 2.21.** For a Hilbert space  $\mathcal{H}$ ,  $\mathcal{B}(\mathcal{H})$  is a non-abelian  $C^*$ -algebra: for all  $x \in \mathcal{B}(\mathcal{H})$  we have  $\|x^*x\| = \|x\|^2$ .

**Example 2.22.** If  $\Gamma$  is a group, we define

$$\ell^1(\Gamma) = \{(\alpha_s)_{s \in \Gamma} \mid \alpha_s \in \mathbb{C}, \sum_{s \in \Gamma} |\alpha_s| < \infty\}.$$

We can then introduce the convolution multiplication on  $\ell^1(\Gamma)$ :

$$(\alpha * \beta)_s = \sum_{t \in \Gamma} \alpha_{st} \beta_{t^{-1}}.$$

This is a Banach algebra; it is even a Banach  $*$ -algebra with involution  $(\alpha^*)_s = \overline{\alpha_{s^{-1}}}$ . However, it is not a  $C^*$ -algebra if the group  $\Gamma$  has more than one element. In that case,

there exists  $z \in \Gamma$  such that  $z \neq 1$ . Define  $\alpha = (\alpha_s) \in \ell^1(G)$  such that

$$\alpha_s = \begin{cases} 1; & s = 1 \\ i; & s = z, z^{-1} \\ 0; & \text{otherwise} \end{cases}$$

If  $z \neq z^{-1}$ , we have

$$\begin{aligned} \|\alpha\alpha^*\| &= \sum_{s \in \Gamma} \left| \sum_{t \in \Gamma} \alpha_{st} \overline{\alpha_t} \right| \\ &= \sum_{s \in \Gamma} (3 \cdot \mathbf{1}_{s=1} + \mathbf{1}_{s=z^2} + \mathbf{1}_{s=z^{-2}}) \\ &< \sum_{s \in \Gamma} (3 \cdot \mathbf{1}_{s=1} + 2 \cdot \mathbf{1}_{s=z} + 2 \cdot \mathbf{1}_{s=z^{-1}} + \mathbf{1}_{s=z^2} + \mathbf{1}_{s=z^{-2}}) \\ &= \sum_{s \in \Gamma} \sum_{t \in \Gamma} |\alpha_{st} \alpha_t| = \sum_{t \in \Gamma} |\alpha_t| \cdot \sum_{s \in \Gamma} |\alpha_{st}| \\ &= \sum_{t \in \Gamma} |\alpha_t| \cdot \sum_{s \in \Gamma} |\alpha_s| = \|\alpha\|^2. \end{aligned}$$

Otherwise, we get

$$\begin{aligned} \|\alpha\alpha^*\| &= \sum_{s \in \Gamma} \left| \sum_{t \in \Gamma} \alpha_{st} \overline{\alpha_t} \right| \\ &= \sum_{s \in \Gamma} (2 \cdot \mathbf{1}_{s=1}) \\ &< \sum_{s \in \Gamma} (2 \cdot \mathbf{1}_{s=1} + 2 \cdot \mathbf{1}_{s=z}) \\ &= \sum_{s \in \Gamma} \sum_{t \in \Gamma} |\alpha_{st} \alpha_t| = \sum_{t \in \Gamma} |\alpha_t| \cdot \sum_{s \in \Gamma} |\alpha_{st}| \\ &= \sum_{t \in \Gamma} |\alpha_t| \cdot \sum_{s \in \Gamma} |\alpha_s| = \|\alpha\|^2. \end{aligned}$$

Therefore,  $\ell^1(\Gamma)$  is not a  $C^*$ -algebra if  $\Gamma$  has order greater than one.

## 2.3 Gelfand transform

**Definition 2.23.** Let  $A$  be an abelian Banach algebra. The *spectrum* of  $A$  is defined as

$$\sigma(A) := \{\varphi : A \rightarrow \mathbb{C} \mid \varphi \neq 0 \text{ continuous algebra homomorphism}\} \subseteq A^*$$

endowed with a weak-\* topology. Its elements are called *characters*.



If  $\varphi \in \sigma(A)$ , then  $\ker \varphi \cap \text{GL}(A) = \emptyset$ . For  $x \in A$ , we have

$$\begin{aligned}\varphi(x - \varphi(x)) &= \varphi(x) - \varphi(\varphi(x) \cdot 1) \\ &= \varphi(x) - \varphi(x)\varphi(1) \\ &= \varphi(x) - \varphi(x) = 0,\end{aligned}$$

which implies that  $\varphi(x) \in \sigma_A(x)$ . Consequently,  $|\varphi(x)| \leq r(x) \leq \|x\|$ , giving us the bound  $\|\varphi\| \leq 1$ . But since  $\varphi(1) = 1$ , we get  $\|\varphi\| = 1$ . We know that  $\sigma(A)$  is closed in  $(A^*)_1$ , making  $\sigma(A)$  a compact Hausdorff space by Banach-Alaoglu.

**Proposition 2.24.** *Let  $A$  be a  $C^*$ -algebra and  $h : A \rightarrow \mathbb{C}$  a non-zero homomorphism (not necessarily a  $*$ -homomorphism). Then the following statements hold:*

- (1.)  $h(a) \in \mathbb{R}$  for self-adjoint  $a$ ;
- (2.)  $h(a^*) = \overline{h(a)}$  for all  $a \in A$ ;
- (3.)  $h(aa^*) \geq 0$  for all  $a \in A$ ;
- (4.) if  $uu^* = 1$  or  $u^*u = 1$ , then  $|h(u)| = 1$ .

*Remark.* The first three items also hold for non-unital algebras.

*Proof.* (1.) Since  $h(a) \in \sigma_A(a)$  and self-adjoint elements have real spectrum, this is trivial.  
(2.) Let  $a = a_1 + ia_2$ , where  $a_1, a_2$  are self-adjoint. Then  $a^* = a_1 - ia_2$  and

$$h(a^*) = h(a_1 - ia_2) = h(a_1) - ih(a_2) = \overline{h(a_1) + ih(a_2)} = \overline{h(a)}.$$

(3.) Follows from (b).

(4.) If  $u$  is unitary, then  $|h(u)|^2 = h(u)h(u^*) = h(uu^*) = h(1) = 1$ . □

**Corollary 2.25.** *Every nonzero algebra homomorphism  $h : A \rightarrow \mathbb{C}$  is a character.*

**Proposition 2.26.** *Let  $A$  be an abelian Banach algebra. Then the map  $\varphi \mapsto \ker \varphi$  is a bijection from  $\sigma(A)$  to the set of all maximal ideals of  $A$ .*

*Proof.* If  $\varphi \in \sigma(A)$ , then  $\ker \varphi \triangleleft A$ . Suppose that  $\ker \varphi \subsetneq I \triangleleft A$ . Then there exists an element  $x \in I \setminus \ker \varphi$ . Thus,  $\varphi(x) \neq 0$  and from  $1 - \frac{x}{\varphi(x)} \in \ker \varphi$ . From there, it follows that

$$1 = \left(1 - \frac{x}{\varphi(x)}\right) + \frac{1}{\varphi(x)} \cdot x \in I.$$

Hence,  $\ker \varphi$  is a maximal ideal. Conversely, let  $I \triangleleft A$  be a maximal ideal. Then  $I \cap \text{GL}(A) = \emptyset$  and since  $\text{GL}(A)$  is open, we also have  $\overline{I} \cap \text{GL}(A) = \emptyset$ . Thus,  $\overline{I} \triangleleft A$  and  $1 \notin \overline{I}$ , so  $I \subseteq \overline{I} \subsetneq A$ . By maximality,  $\overline{I} = I$ . Then  $A/I$  is a Banach algebra and since  $I$  is maximal, every nonzero element in  $A/I$  is invertible. By Gelfand-Mazur,  $A/I \cong \mathbb{C}$ . The projection  $\pi : A \rightarrow A/I \cong \mathbb{C}$  is in  $\sigma(A)$  and  $\ker \pi = I$ . □

**Corollary 2.27.** *Let  $A$  be an abelian Banach algebra and  $x \in A \setminus \text{GL}(A)$ . Then there exists  $\varphi \in \sigma(A)$  such that  $\varphi(x) = 0$ . In particular,  $\sigma(A) \neq \emptyset$ .*

*Proof.* If  $x \notin \text{GL}(A)$ , then it generates an ideal  $\langle x \rangle \subsetneq A$ . By Zorn's lemma,  $\langle x \rangle$  has to be included in some maximal ideal  $I \triangleleft A$ . By the previous proposition, there exists a character  $\varphi : A \rightarrow \mathbb{C}$  in  $\sigma(A)$  such that  $x \in I = \ker \varphi$ .  $\square$

**Theorem 2.28** (Stone-Čech).

Let  $X$  be a topological space. For  $x \in X$ , let  $\beta_x : C_b(X) \rightarrow \mathbb{C}$  be the evaluation homomorphism  $f \mapsto f(x)$ . Then

$$\beta : X \rightarrow \sigma(C_b(X)), \quad x \mapsto \beta_x$$

is a continuous map whose image is dense in the codomain and has the following universal property: if  $\pi : X \rightarrow K^{T_2, \text{compact}}$  is continuous, then there exists a unique continuous mapping

$$\beta_\pi : \sigma(C_b(X)) \rightarrow K$$

such that  $\pi(x) = \beta_\pi(\beta_x)$  for all  $x \in X$ . In particular, if  $X$  is compact  $T_2$ , then  $\beta$  is a homeomorphism.

$$\begin{array}{ccc} X & \xrightarrow{\pi} & K^{T_2, \text{compact}} \\ \downarrow \beta & \nearrow \exists! \beta_\pi & \\ \sigma(C_b(X)) & & \end{array}$$

*Proof.* (1.) First, we prove that  $\beta$  is continuous. Let  $(x_i)_i$  be a net in  $X$  and  $x_i \rightarrow x$ , then for all  $f \in C_b(X)$  we have  $\beta_{x_i}(f) = f(x_i) \rightarrow f(x) = \beta_x(f)$ . Hence  $\beta_{x_i} \rightarrow \beta_x$  in the weak-\* topology.

(2.) Next, we prove that  $\text{im } \beta$  is dense. Assume otherwise and pick  $\varphi \in \sigma(C_b(X)) \setminus \overline{\beta(X)}$ . Define  $I := \ker \varphi$ . For all  $\psi \in \overline{\beta(X)}$ , there exists  $f_\psi \in I$  such that  $f_\psi \in \ker \varphi$ . Hence, there exists  $c_\psi$  and a neighborhood  $U_\psi$  of  $\psi$  such that  $|\tilde{\psi}(f)| > c_\psi$  for all  $\tilde{\psi} \in U_\psi$ . Thus,  $\overline{\beta(X)} \subseteq \bigcup_{\psi \in \overline{\beta(X)}} U_\psi$ . By compactness, there exists a finite subcovering of  $\overline{\beta(X)}$ , so  $\overline{\beta(X)} \subseteq \bigcup_{i=1}^n U_{\psi_i}$ . Then there exist  $f_{\psi_1}, \dots, f_{\psi_n} \in I$  and  $c > 0$  such that

$$\sum_{i=1}^n \psi(|f_{\psi_i}|^2) > c, \quad \forall \psi \in \overline{\beta(X)}.$$

Hence,

$$\sum_{i=1}^n |f_{\psi_i}|^2(x) = \sum_{i=1}^n \beta(x)(|f_{\psi_i}|^2) > c, \quad \forall x \in X.$$

It follows that  $\sum_{i=1}^n |f_{\psi_i}|^2 \in I$  and  $(\sum_{i=1}^n |f_{\psi_i}|^2)^{-1} \in C_b(X)$ . As a result,  $I = C_b(X)$ .

(3.) If  $X$  is compact and Hausdorff, then  $\beta$  is surjective since  $\beta(X)$  is dense and compact. Also,  $\beta$  is injective since  $C_b(X)$  separates points. In that case,  $\beta$  is a continuous bijection between compact Hausdorff spaces, and therefore a homeomorphism.

(4.) For the universal property: let  $\pi : X \rightarrow K$ , where  $K$  is compact Hausdorff. Then there exists a continuous map

$$\pi^* : C(K) \rightarrow C_b(X), \quad f \mapsto f \circ \pi.$$

This induces a continuous map

$$\tilde{\pi} : \sigma(C_b(X)) \rightarrow \sigma(C(K)), \quad \varphi \mapsto \varphi \circ \pi^*.$$

Since  $K$  is compact Hausdorff, the map  $\beta^K : K \rightarrow \sigma(C(K))$  is a homeomorphism. Define

$$\beta_\pi : \sigma(C_b(X)) \rightarrow K, \quad \beta_\pi = (\beta^K)^{-1} \circ \tilde{\pi}.$$

Then we have

$$\tilde{\pi}(\beta_x)(g) = \beta_x(\pi^*(g)) = \pi^*(g)(x) = g(\pi(x)) = \beta_{\pi(x)}^K(g).$$

By left multiplying by  $(\beta^K)^{-1}$ , we get  $\beta_\pi(\beta_x) = \pi(x)$ .  $\square$

**Definition 2.29.** Let  $A$  be an abelian Banach algebra. The Gelfand transform of  $A$  is the map

$$\Gamma : A \rightarrow C(\sigma(A)), \quad x \mapsto (\varphi \mapsto \varphi(x)).$$

**Theorem 2.30.**

*Let  $A$  be an abelian Banach algebra. Then  $\Gamma$  is a homomorphism, contraction and for  $x \in A$  we have*

$$\Gamma(x) \in \text{GL}(C(\sigma(A))) \Leftrightarrow x \in \text{GL}(A).$$

*Proof.* The homomorphism part is routine. We prove that  $\Gamma$  is a contraction as follows:

$$\|\Gamma(x)\| = \sup_{\varphi \in \sigma(A)} \|\Gamma(x)\varphi\| = \sup_{\varphi} |\varphi(x)| \leq \|x\|.$$

Next, we prove the equivalence. The right implication ( $\Rightarrow$ ) is trivial, since

$$\Gamma(x^{-1})\Gamma(x) = \Gamma(x^{-1}x) = \Gamma(1) = 1.$$

Now the converse ( $\Leftarrow$ ): if  $x \notin \text{GL}(A)$ , then by corollary 2.27 there exists  $\varphi \in \sigma(A)$  such that  $\varphi(x) = 0$ . Then  $\Gamma(x)(\varphi) = \varphi(x) = 0$ , so the continuous map  $\Gamma(x)$  is not invertible.  $\square$

**Corollary 2.31.** *Let  $A$  be an abelian Banach algebra. Then we have*

$$\sigma(\Gamma(x)) = \sigma(x)$$

and

$$\|\Gamma(x)\| = r(\Gamma(x)) = r(x).$$

**Theorem 2.32 (Gelfand).**

*Let  $A$  be an abelian  $C^*$ -algebra. Then  $\Gamma$  is an isometric  $*$ -isomorphism.*

*Proof.* For a self-adjoint  $x \in A$  we have  $\sigma(\Gamma(x)) = \sigma(x) \subseteq \mathbb{R}$ . Then  $\overline{\Gamma(x)} = \Gamma(x)$ . An arbitrary  $x \in A$  can be written as  $x = a + ib$  for self-adjoint  $a = \frac{x+x^*}{2}$  and  $b = \frac{i(x^*-x)}{2}$ . Then

$$\Gamma(x^*) = \Gamma(a - ib) = \Gamma(a) - i\Gamma(b) = \overline{\Gamma(a) + i\Gamma(b)} = \overline{\Gamma(x)}.$$

This implies that  $\Gamma$  is a  $*$ -homomorphism. Since  $A$  is abelian, each  $x \in A$  is normal so

$$\|x\| = r(x) = r(\Gamma(x)) = \|\Gamma(x)\|$$

and  $\Gamma$  is an isometry. In particular,  $\Gamma$  is injective. We know that  $\Gamma(A)$  is closed under  $*$ . Since  $\Gamma$  is isometric, the subalgebra  $\Gamma(A) \subseteq C(\sigma(A))$  is complete in the norm, so it is closed. It can be easily checked that  $\Gamma(A)$  separates points. By Stone-Weierstrass,  $\Gamma(A) = C(\sigma(A))$ .  $\square$

*Remark.* Let  $A$  be a  $C^*$ -algebra. If  $x \in A$  is normal, then it generates an abelian  $C^*$ -subalgebra of  $A$ :

$$C^*(x) = \overline{\{p(x, x^*) \mid p \in \mathbb{C}[x, y]\}}.$$

**Corollary 2.33.** *Let  $A$  be an abelian  $C^*$ -algebra, generated by  $x \in A$ . Then  $\sigma(A) \cong \sigma(x)$ .*

*Proof.* Let  $\Gamma : A \rightarrow C(\sigma(A))$  be the Gelfand transform. Define

$$\tau : \sigma(A) \rightarrow \sigma(x), \quad \varphi \mapsto \varphi(x) = \Gamma(x)(\varphi).$$

Clearly,  $\tau$  is well-defined since  $\varphi(x) \in \sigma(x)$  for all  $\varphi \in \sigma(A)$ . Next we show that  $\tau$  is onto. For  $\lambda \in \sigma(x)$  we have  $x - \lambda \notin \text{GL}(A)$ , so there exists  $\psi \in \sigma(A)$  such that  $\psi(x) - \psi(\lambda) = \psi(x - \lambda) = 0$ . We show that  $\tau$  is injective. Let  $\tau(\varphi_1) = \tau(\varphi_2)$ . Then  $\varphi_1(x) = \varphi_2(x)$ . Since

$$\varphi_j(x^*) = \Gamma(x^*)(\varphi_j) = \overline{\Gamma(x)(\varphi_j)} = \overline{\varphi_j(x)},$$

we have  $\varphi_1(x^*) = \varphi_2(x^*)$ . Hence  $\varphi_1(p(x, x^*)) = \varphi_2(p(x, x^*))$  for every polynomial  $p \in \mathbb{C}[x, y]$ . Since  $\{p(x, x^*) \mid p \text{ polynomial}\}$  is dense in  $A$ , we have  $\varphi_1 = \varphi_2$ . Finally, we prove the continuity of  $\tau$ . Let  $(\varphi_\alpha)_\alpha$  be a net in  $\sigma(A)$  such that  $\varphi_\alpha \rightarrow \varphi$ . Then  $\varphi_\alpha(y) \rightarrow \varphi(y)$  for all  $y \in A$ , so in particular  $\varphi_\alpha(x) \rightarrow \varphi(x)$ , which proves that  $\tau(\varphi_\alpha) \rightarrow \tau(\varphi)$ . Since  $\tau$  is a continuous bijection between compact Hausdorff spaces, it is a homeomorphism.  $\square$

*Remark.* Since  $\varphi \in \sigma(A)$  is an algebra homomorphism, we have  $\varphi(p(x, x^*)) = p(\varphi(x), \overline{\varphi(x)})$  for a complex polynomial  $p(z, \bar{z})$  in  $z$  and  $\bar{z}$ . Using the notation from above proof, we get  $\Gamma(p(x, x^*)) = p \circ \tau$ .

## 2.4 Continuous functional theorem

Now let  $A$  be any  $C^*$ -algebra and  $x \in A$  normal. Then  $C^*(x)$  is an abelian  $C^*$ -subalgebra of  $A$ . Since  $\sigma(x) = \sigma_{C^*(x)}$ , we have the map

$$\tau^\# : C(\sigma(x)) \rightarrow C(C^*(x)), \quad f \mapsto f \circ \tau,$$

which is a  $*$ -isomorphism and an isometry. Define a map  $\rho = \Gamma^{-1} \circ \tau^\# : C(\sigma(x)) \rightarrow C^*(x)$ .

$$\begin{array}{ccc} C^*(x) & \xrightarrow{\Gamma} & C(\sigma(A)) \\ & \swarrow \rho \quad \searrow \tau^\# & \\ & C(\sigma(x)) & \end{array}$$

We know that  $C^*(x) = \overline{\{p(x, x^*) \mid p(z, \bar{z}) \text{ polynomial}\}}$  and  $\Gamma(p(x, x^*)) = \tau^\#(p)$ , which means that  $\rho(p) = p(x, x^*)$  for any polynomial  $p \in \mathbb{C}[x, y]$ . This map  $\rho : C(\sigma(x)) \rightarrow C^*(x) \subseteq A$  is called the continuous functional calculus. We use the notation  $f(x) := \rho(f)$ .

**Theorem 2.34** (Continuous functional calculus).

Let  $A, B$  be  $C^*$ -algebras and let  $x \in A$  be normal.

(1.)  $f \mapsto f(x)$  is an isometric  $*$ -isomorphism  $C(\sigma(x)) \rightarrow A$  and if

$$f = \sum_{j,k=0}^n a_{jk} z^j \bar{z}^k$$

is a polynomial, then

$$f(x) = \sum_{j,k=0}^n a_{jk} x^j (x^*)^k.$$

In particular, if  $f(z) = z$  is the identity polynomial, then  $f(x) = x$ .

(2.) For  $f \in C(\sigma(x))$ , we have  $\sigma(f(x)) = f(\sigma(x))$ .

(3.) (**Spectral mapping theorem**) If  $\Phi : A \rightarrow B$  is a  $*$ -homomorphism, then  $\Phi(f(x)) = f(\Phi(x))$ .

(4.) Let  $(x_n)_n$  be a sequence of normal elements of  $A$  that converge to  $x$ ,  $\Omega$  a compact neighborhood of  $\sigma(x)$ , and  $f \in C(\Omega)$ . Then for any sufficiently large  $n$ , we have  $\sigma(x_n) \subseteq \Omega$  and  $\|f(x_n) - f(x)\| \rightarrow 0$ .

*Proof.* The items (1) and (2) follow directly from Gelfand theorem and properties of continuous functions on compact sets. The item (3) is obvious for polynomials  $f$  and the general case follows from Stone-Weierstrass. We prove the item (4). Let  $C = \sup_n \|x_n\| < \infty$ . First we need to show that  $\sigma(x_n) \subseteq \Omega$  for large enough  $n$ . If that wasn't the case, then for every  $n \in \mathbb{N}$  there would exist  $N_n > n$  such that there exists  $\lambda_n \in \sigma(x_{N_n}) \setminus \Omega \subseteq \overline{B_C(0)}$ . Thus there exists a convergent subsequence  $(\lambda_{n_k})_k$  such that  $\lambda_{n_k} \rightarrow \lambda \in U$ , where  $U$  is an open neighborhood of  $\sigma(x)$  and  $\lambda \notin \sigma(x)$ . But then

$$\underbrace{x_{n_k} - \lambda_{n_k}}_{\notin \text{GL}(A)} \rightarrow \underbrace{x - \lambda}_{\in \text{GL}(A)},$$

which contradicts the openness of  $\text{GL}(A)$ . For every  $\varepsilon > 0$  there exists a polynomial  $g : \Omega \rightarrow$

$\mathbb{C}$  such that  $\|f - g\|_\infty < \varepsilon$ . Now

$$\begin{aligned} \limsup_n \|f(x_n) - g(x_n)\| + \|g(x_n) - g(x)\| + \|g(x) - f(x)\| \\ \leq 2 \cdot C \cdot \varepsilon + \limsup_n \|g(x_n) - g(x)\| \\ = 2C\varepsilon. \end{aligned}$$

Since  $\varepsilon$  was arbitrary, we conclude that  $\lim_{n \rightarrow \infty} \|f(x_n) - f(x)\| = 0$ .  $\square$

We illustrate the use of continuous functional calculus to obtain the strengthening of corollary 2.13.

**Corollary 2.35.** *If  $A, B$  are  $C^*$ -algebras and  $\Phi : A \rightarrow B$  is a  $*$ -monomorphism, then it is an isometry.*

*Proof.* Let  $a \in A$  be self-adjoint. Then  $\Phi(a) \in B$  is self-adjoint as well. As in the proof of 2.13, we observe that  $\sigma_B(\Phi(a)) \subseteq \sigma_A(a)$ . Suppose that  $\sigma_B(\Phi(a)) \neq \sigma_A(a)$ . Since  $\sigma_B(\Phi(a))$  is compact, it is closed in  $\sigma_A(a)$ . This implies that  $U := \sigma_A(a) \setminus \sigma_B(\Phi(a))$  is a nonempty open set. It follows that there exists a function  $f$  which is zero on  $\sigma_B(\Phi(a))$ , but not identically zero on  $\sigma_A(a)$  (take for example any bump function on  $U$ ). Then  $f(\Phi(a)) = 0$ , but  $f(a) \neq 0$ . By Stone-Weierstrass, we can approximate  $f$  uniformly on  $\sigma_A(a)$  by polynomials  $\{p_n\}_{n \in \mathbb{N}}$ . Thus  $p_n(a) \rightarrow f(a)$  and  $p_n(\Phi(a)) \rightarrow f(\Phi(a)) = 0$ . On the other hand,  $p_n(\Phi(a)) = \Phi(p_n(a)) \rightarrow \Phi(f(a))$ , which implies that  $\Phi(f(a)) = f(\Phi(a)) = 0$ . But  $\Phi$  was assumed injective, so  $f(a) = 0$ , contradiction. Therefore,  $\sigma_B(\Phi(a)) = \sigma_A(a)$  for self-adjoint  $a$  and

$$\|a\| = r(a) = r(\Phi(a)) = \|\Phi(a)\|.$$

Now for a completely arbitrary  $a \in A$ , we have

$$\|a\|^2 = \|a^*a\| = \|\Phi(a^*a)\| = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a)\|^2,$$

concluding our proof.  $\square$

The argument in this proof is very common. We first approximate some function on the spectrum with polynomials using Stone-Weierstrass. Then we observe that the CFC of a polynomial has desired properties and deduce the same for the CFC of the original function.

## 2.5 Application of the continuous functional theorem

**Definition 2.36.** Let  $A$  be a  $C^*$ -algebra and  $x \in A$ .

- $x$  is *positive* if  $x = y^*y$  for some  $y \in A$  (i.e.,  $x$  is a hermitian square). The set of positive elements is denoted  $A_+$ .
- $x$  is a *projection* if  $x^2 = x^* = x$ .
- $x$  is *unitary* if  $xx^* = x^*x = 1$ . The set of positive elements is denoted  $U(A)$ .
- $x$  is an *isometry* if  $x^*x = 1$ .
- $x$  is a *partial isometry* if  $x^*x$  is a projection.

*Remark.* The first three are automatically normal (the first two are even self-adjoint).

The set of all positive operators (denoted as  $A_+$ ) induces a partial ordering on  $A_{\text{sa}}$ : for two elements  $a, b \in A_{\text{sa}}$  we define

$$a \leq b \Leftrightarrow b - a \in A_+.$$

We notice that  $x^*A_+x \subseteq A_+$  for every  $x \in A$ . For any  $a, b \in A_{\text{sa}}$  and  $x \in A$ , we have

$$a \leq b \Rightarrow x^*ax \leq x^*bx.$$

**Proposition 2.37.** *Let  $A$  be a  $C^*$ -algebra and  $x \in A$ . Then  $x$  is a linear combination of four unitaries.*

*Proof.* Since  $x = \operatorname{Re} x + i \operatorname{Im} x$ , where  $\operatorname{Re} x, \operatorname{Im} x \in A_{\text{sa}}$ , it's enough to show that every self-adjoint element is a linear combination of two unitaries. Without loss of generality, assume  $\|x\| \leq 1$ , so  $\sigma(x) \subseteq [-1, 1]$ . Consider the continuous function

$$f : [-1, 1] \rightarrow \mathbb{T}, \quad z \mapsto z + i(1 - z^2)^{\frac{1}{2}}.$$

Since  $f \cdot \bar{f} \equiv 1$  on  $[-1, 1]$ , it follows from continuous functional calculus that

$$f(x)f(x)^* = f(x)^*f(x) = 1.$$

Consequently,  $f(x) = u$  is unitary and  $x = \frac{1}{2}(f(x) + f(x)^*)$  is a linear combination of two unitaries.  $\square$

*Remark.* We use the notation  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ .

**Definition 2.38.** Let  $x \in A_{\text{sa}}$ . Then  $\sigma(x) \subseteq \mathbb{R}$  and we can define

$$x_+ = \max\{0, x\}, \quad x_- = -\min\{0, x\} \quad (x \in A).$$

Then  $\sigma(x_+), \sigma(x_-) \subseteq [0, \infty)$ ,  $x = x_+ - x_-$  and  $x_+x_- = x_-x_+ = 0$ .

**Lemma 2.39.** *Suppose  $x, y \in A_{\text{sa}}$  satisfy  $\sigma(x), \sigma(y) \subseteq [0, \infty)$ . Then  $\sigma(x + y) \subseteq [0, \infty)$ .*

*Proof.* Let  $a := \|x\|$  and  $b := \|y\|$ . Since  $x = x_+$  and  $\sigma(x) \subseteq [0, a]$ , we deduce that  $\sigma(a - x) \subseteq [0, a]$ , where  $\|a - x\| = r(a - x) \leq a$ . Likewise,  $\|b - y\| \leq b$ . Then

$$\begin{aligned} \sup_{\lambda \in \sigma(x+y)} \{a + b - \lambda\} &= r(a + b - (x + y)) \\ &= \|(a + b) - (x + y)\| \\ &\leq \|a - x\| + \|b - y\| \\ &\leq a + b. \end{aligned}$$

$\square$

**Theorem 2.40.**

*Let  $A$  be a  $C^*$ -algebra and  $x \in A$  normal. Then:*

ADD A PICTURE

- (1.)  $x \in A_{\text{sa}} \Leftrightarrow \sigma(x) \subseteq \mathbb{R};$
- (2.)  $x \in A_+ \Leftrightarrow \sigma(x) \subseteq [0, \infty);$
- (3.)  $x \in U(A) \Leftrightarrow \sigma(x) \subseteq \mathbb{T};$
- (4.)  $x^2 = x^* = x \Leftrightarrow \sigma(x) \subseteq \{0, 1\}.$

*Proof.* Throughout this proof, let  $f(z) = z$  denote the identity polynomial.

(1.)

$$\begin{aligned}
 x = x^* &\Leftrightarrow f(x) = \bar{f}(x) \\
 &\Leftrightarrow f \equiv \bar{f} \text{ on } \sigma(x) \\
 &\Leftrightarrow z = \bar{z} \text{ for all } z \in \sigma(x) \\
 &\Leftrightarrow \sigma(x) \subseteq \mathbb{R}.
 \end{aligned}$$

(2.) ( $\Rightarrow$ ) Let  $x = y^*y$  for some  $y \in A$ . Write  $x = x_+ - x_-$  and let  $z := y \cdot x_-$ . Then

$$z^*z = x_-y^*yx_- = x_-xx_- = -x_-^3.$$

From there we get

$$\sigma(zz^*) \subseteq \sigma(z^*z) \cup \{0\} \subseteq (-\infty, 0].$$

Let  $z = a + ib$  for  $a, b \in A_{\text{sa}}$ . Then  $zz^* + z^*z = 2a^2 + 2b^2$ , which implies that  $\sigma(zz^* + z^*z) \subseteq [0, \infty)$ . It follows that

$$\sigma(z^*z) = \sigma((2a^2 + 2b^2) - zz^*) \subseteq [0, \infty).$$

As a result,

$$\sigma(-x_-^3) = \sigma(z^*z) \subseteq \{0\},$$

so  $x_-^3 = 0$  and  $x_- = 0$ . This proves that  $x = x_+$  has nonnegative spectrum. For the converse implication ( $\Leftarrow$ ), apply the function  $\sqrt{\cdot} : [0, \infty) \rightarrow \mathbb{R}$ . Then

$$x = (\sqrt{x})^2 = (\sqrt{x})^* \cdot \sqrt{x} \in A_+.$$

(3.)

$$\begin{aligned}
 xx^* = 1 &\Leftrightarrow f(x) \cdot \bar{f}(x) = 1 \\
 &\Leftrightarrow f \cdot \bar{f} \equiv 1 \text{ on } \sigma(x) \\
 &\Leftrightarrow |z|^2 = 1 \text{ for all } z \in \sigma(x) \\
 &\Leftrightarrow \sigma(x) \subseteq \mathbb{T}.
 \end{aligned}$$

(4.)

$$\begin{aligned}
 x^2 = x^* = x &\Leftrightarrow f(x) \cdot \bar{f}(x) = \bar{f}(x) = f(x) \\
 &\Leftrightarrow f \cdot \bar{f} \equiv \bar{f} \equiv f \text{ on } \sigma(x) \\
 &\Leftrightarrow |z|^2 = \bar{z} = z \text{ for all } z \in \sigma(x) \\
 &\Leftrightarrow \sigma(x) \subseteq \{0, 1\}.
 \end{aligned}$$

□



**Corollary 2.41.** *Let  $A$  be a  $C^*$ -algebra and  $x \in A$ . Then  $x$  is a partial isometry iff  $x^*$  is a partial isometry.*

*Proof.*

$$\begin{aligned}
x \text{ partial isometry} &\Leftrightarrow x^*x \text{ projection} \\
&\Leftrightarrow \sigma(x^*x) \subseteq \{0, 1\} \\
&\Leftrightarrow \sigma(xx^*) \subseteq \{0, 1\} \\
&\Leftrightarrow xx^* \text{ projection} \\
&\Leftrightarrow x^* \text{ partial isometry.}
\end{aligned}$$

□

**Corollary 2.42.** *Let  $A$  be a  $C^*$ -algebra.*

- (1.)  $A_+$  is a closed convex cone ( $\lambda A_+ \subseteq A_+$  for  $\lambda \in \mathbb{R}_{\geq 0}$ ).
- (2.) If  $a \in A_{\text{sa}}$ , then  $a \leq \|a\|$ .

**Proposition 2.43.** *Let  $A$  be a  $C^*$ -algebra and  $x, y \in A_+$ .*

- (1.) If  $x \leq y$ , then  $\sqrt{x} \leq \sqrt{y}$ .
- (2.) If  $x, y \in \text{GL}(A)$  and  $x \leq y$ , then  $y^{-1} \leq x^{-1}$ .

*Proof.* Let us prove the second point first. Suppose  $x, y \in \text{GL}(A)$ . Then we have  $y^{-\frac{1}{2}}xy^{-\frac{1}{2}} \leq 1$  and

$$\begin{aligned}
x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}} &\leq \|x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}}\| \\
&= r(x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}}) \\
&= r(y^{-\frac{1}{2}}xy^{-\frac{1}{2}}) \\
&\leq 1.
\end{aligned}$$

Multiplying on both sides by  $x^{-\frac{1}{2}}$ , we get  $y^{-1} \leq x^{-1}$ . Now we prove the first point. For invertible  $x \leq y$ , we have

$$\begin{aligned}
\|y^{-\frac{1}{2}}x^{\frac{1}{2}}\|^2 &= \|(y^{-\frac{1}{2}}x^{\frac{1}{2}})(y^{-\frac{1}{2}}x^{\frac{1}{2}})^*\| \\
&= \|y^{-\frac{1}{2}}xy^{-\frac{1}{2}}\| \\
&\leq 1,
\end{aligned}$$

which implies

$$\begin{aligned}
y^{-\frac{1}{4}}x^{\frac{1}{2}}y^{-\frac{1}{4}} &\leq \|y^{-\frac{1}{4}}x^{\frac{1}{2}}y^{-\frac{1}{4}}\| \\
&= r(y^{-\frac{1}{4}}x^{\frac{1}{2}}y^{-\frac{1}{4}}) \\
&= r(y^{-\frac{1}{2}}x^{\frac{1}{2}}) \\
&= \|y^{-\frac{1}{2}}x^{\frac{1}{2}}\| \leq 1.
\end{aligned}$$

Multiplying on both sides by  $y^{\frac{1}{4}}$ , we get  $y^{\frac{1}{2}} \leq x^{\frac{1}{2}}$ . For general non-invertible  $x \leq y$ , pick  $\varepsilon > 0$  and notice that

$$0 \leq x + \varepsilon \leq y + \varepsilon.$$

However, since  $x, y$  are positive, we also have  $x + \varepsilon, y + \varepsilon \in \text{GL}(A)$ . We use the above calculation to obtain  $(x + \varepsilon)^{\frac{1}{2}} \leq (y + \varepsilon)^{\frac{1}{2}}$ . If we send  $\varepsilon \rightarrow 0$ , we get  $x^{\frac{1}{2}} \leq y^{\frac{1}{2}}$ .  $\square$

*Remark.* Let  $I \subseteq \mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be continuous. Then the function  $f$  is operator monotone if for every  $C^*$ -algebra  $A$  and  $a, b \in A_{\text{sa}}$  with  $a \leq b$  and  $\sigma(a), \sigma(b) \subseteq I$ , we have  $f(a) \leq f(b)$ . By the above proposition,  $z \mapsto \sqrt{z}$  and  $z \mapsto \frac{1}{z}$  are operator monotone on  $[0, \infty)$ . Actually, this is also true for functions  $z \mapsto z^r$  for  $r \in [0, 1]$ , but not for  $r > 1$ .

**Definition 2.44.** Absolute value of  $x \in A$  is defined as

$$|x| = (x^*x)^{\frac{1}{2}} \in A_+.$$

**Corollary 2.45.** For  $x, y \in A$ , we have  $|xy| \leq \|x\||y|$ .

*Proof.* Notice that

$$|xy|^2 = y^*x^*xy \leq y^*\|x^*x\|y = \|x\|^2(y^*y)$$

and now apply the operator-monotone  $\sqrt{\cdot}$  and the previous proposition.  $\square$

**Theorem 2.46.**

Let  $A$  be a  $C^*$ -algebra.

- (1.)  $\text{ext}(A_+)_1 = \{\text{projections in } A\}$ .
- (2.)  $\text{ext}(A)_1 \subseteq \{\text{partial isometries in } A\}$ .
- (3.)  $\text{ext}(A_{\text{sa}})_1 = U(A) \cap A_{\text{sa}}$ .

*Proof.* (1.) Let  $x \in (A_+)_1$ . Then  $x^2 \leq 2x$ , since  $z^2 - 2z \leq 0$  on  $[0, 1] \supseteq \sigma(x)$ . So  $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$ . If  $x$  is an extreme point, then  $x = x^2$  and  $x \in A_+ \subseteq A_{\text{sa}}$ , so  $x$  is a projection. For the converse, assume  $A$  is abelian, meaning  $A = C(K)$  for some compact Hausdorff space  $K$  (by Gelfand). If  $x \in A = C(K)$  is a projection, then  $x = \chi_E$  for some clopen  $E \subseteq K$ . Since  $\text{ext}([0, 1]) = \{0, 1\}$ ,  $\chi_E$  is an extreme point. Let  $A$  now be a general  $C^*$ -algebra and  $p \in A^*$  a projection. Suppose  $p = \frac{1}{2}(a + b)$  for some  $a, b \in (A_+)_1$ . Then  $\frac{1}{2}a = p - \frac{1}{2}b \leq p$ . Hence

$$0 \leq (1 - p)a(1 - p) \leq (1 - p)2p(1 - p) = 0,$$

so

$$(\sqrt{a}(1 - p))^*(\sqrt{a}(1 - p)) = (1 - p)a(1 - p) = 0.$$

This implies that  $\sqrt{a}(1 - p) = 0$  and  $a(1 - p) = 0$ . It follows that

$$ap = a = a^* = (ap)^* = p^*a^* = pa.$$

Similarly, we can show that  $a, b, p$  all commute, so the  $C^*$ -subalgebra  $C^*(a, b, p)$  is abelian and we can just use the previous observation.

- (2.) Suppose  $x \in (A)_1$  is not a partial isometry (alternatively,  $x^*x$  is not a projection). First, we notice that  $\|x^*x\| = \|x\|^2 \leq 1$ . Since  $x$  is not a projection,  $\sigma(x^*x) \cap (0, 1) \neq \emptyset$ . Then we apply the continuous functional calculus to obtain a function  $f : \sigma(x^*x) \rightarrow [0, 1]$  such that  $|t(1 \pm f(t))^2| \leq 1$  for  $t \in \sigma(x^*x)$  (for example,  $f$  can be a small bump function on an interval  $[a, b] \subseteq (0, 1)$ , where  $[a, b] \cap \sigma(x^*x) \neq \emptyset$ ). Then  $y := f(x^*x) \in A_+$  gives us  $yx^*x = x^*xy \neq 0$  and  $\|x^*x(1 \pm y)^2\| \leq 1$ . Hence,  $\|x(1 \pm y)\|^2 \leq 1$  and

$$x = \frac{1}{2}((x + xy) + (x - xy)) \notin \text{ext}(A)_1.$$

- (3.) If  $u \in U(A) \cap A_{\text{sa}}$ , then  $x \mapsto ux$  is an isometry. As in the case of  $\mathcal{B}(\mathcal{H})$ ,  $u$  is an extreme point, so  $A_{\text{sa}} \cap U(A) \subseteq \text{ext}(A_{\text{sa}})_1$ . For the converse, assume  $x \in \text{ext}(A_{\text{sa}})_1$  and  $x_+ = \frac{1}{2}(a + b)$  for  $a, b \in (A_+)_1$ . Then

$$0 = x_-x_+x_- = \frac{1}{2}(x_-ax_- + x_-bx_-) \geq 0.$$

From  $x_-ax_- = 0$ , we get  $(\sqrt{a}x_-)^*(\sqrt{a}x_-) = 0$ , which implies that  $\sqrt{a}x_- = 0$  and  $ax_- = 0$ . Likewise,  $x_-a = bx_- = x_-b = 0$ . By Gelfand, the commutative  $C^*$ -algebra  $C^*(a, b, x_-)$  is isometrically  $*$ -isomorphic to  $C(K)$  for some compact  $K$ . This means that  $a$  and  $x_-$  are functions such that for every point in  $K$ , at least one of them is zero. Thus,  $a - x_-$  is bounded above by 1, and we have  $a - x_- \in (A_{\text{sa}})_1$ . Similarly,  $b - x_- \in (A_{\text{sa}})_1$ , so

$$x = \frac{1}{2}((a - x_-) + (b - x_-)) \in (A_{\text{sa}})_1.$$

But since  $x$  is an extreme point, we have  $a - x_- = b - x_-$  and  $a = b = x_+$ . Thus,  $x_+ \in \text{ext}(A_+)_1$  is a projection by (1.), and by symmetry, so is  $x_-$ . Now we prove that  $x$  is unitary:

$$x^*x = x^2 = (x_+ - x_-)^2 = x_+^2 + x_-^2 = x_+ + x_- = |x|.$$

This implies that  $|x|$  is a projection. Now set  $q := 1 - |x|$ . Then  $x + q$  and  $x - q$  are both in  $(A_{\text{sa}})_1$ . But since

$$x = \frac{1}{2}((x + q) + (x - q)),$$

we obtain  $q = 0$ , which further implies  $|x| = 1$  and  $x^*x = xx^* = 1$ .  $\square$

### 3 Representations of $C^*$ -algebras and states

#### 3.1 States

Let  $A$  be a  $C^*$ -algebra, then  $A^*$  can be given an  $A$ -bimodule structure: if  $\psi \in A^*$  and  $a, b \in A$ , then

$$(a \cdot \psi \cdot b)(x) = \psi(bxa), \quad \forall x \in A.$$

We have

$$\|a \cdot \psi \cdot b\| = \sup_{x \in (A)_1} \|\psi(bxa)\| \leq \sup_{x \in (A)_1} \|\psi\| \|bxa\| \leq \|\psi\| \|a\| \|b\|.$$

**Definition 3.1.** Let  $A$  be a  $C^*$ -algebra and  $\varphi \in A^*$ .

- We say that  $\varphi$  is *positive* if  $\varphi(x) \geq 0$ ,  $\forall x \in A_+$ . If  $\varphi$  is positive and  $a \in A$ , then  $a\varphi a^*$  is also positive.
- A positive element  $\varphi \in A^*$  is *faithful* if  $\varphi(x) \neq 0$ ,  $\forall x \in A_+ \setminus \{0\}$ .
- An element  $\varphi \in A^*$  is a *state* if it is *positive* and  $\|\varphi\| = 1$ . The set of states is denoted  $S(A) \subseteq (A^*)_1$ .

*Remark.* The set  $S(A)$  is compact Hausdorff in the weak-\* topology.

We notice that if  $\varphi \in A^*$  is positive and  $x \in A_{sa}$ , then

$$\varphi(x) = \varphi(x_+ - x_-) = \varphi(x_+) - \varphi(x_-) \in \mathbb{R}.$$

If  $y \in A$ , then  $y = y_1 + iy_2$ , where  $y_1, y_2$  are self-adjoint. Then

$$\begin{aligned} \varphi(y^*) &= \varphi((y_1 + iy_2)^*) = \varphi(y_1 - iy_2) \\ &= \varphi(y_1) - i\varphi(y_2) = \overline{\varphi(y_1) + i\varphi(y_2)} \\ &= \overline{\varphi(y_1 + iy_2)} = \overline{\varphi(y)} \end{aligned}$$

Such a functional  $\varphi \in A^*$  is called *hermitian*. For any  $\varphi \in A^*$ ,  $\varphi^*(y) = \overline{\varphi(y^*)}$ . Then  $\varphi + \varphi^*$  and  $i(\varphi - \varphi^*)$  are hermitian. One can, of course, define these notions also for unbounded linear functionals. However, positivity implies continuity: for every  $a \in A_{sa}$  we have  $-\|a\| \cdot 1 \leq a \leq \|a\| \cdot 1$ , which implies

$$-\|a\|\varphi(1) \leq \varphi(a) \leq \|a\|\varphi(1)$$

and  $\varphi$  is bounded. For  $a \in A$ , we can of course write  $a = b + ic$  for  $b, c \in A_{sa}$ . Here,

$$\|b\| = \left\| \frac{a + a^*}{2} \right\| \leq \frac{\|a\|}{2} + \frac{\|a^*\|}{2} = \|a\|$$

and likewise  $\|c\| \leq \|a\|$ . Let  $\varphi(1) = C$ . Then

$$|\varphi(a)|^2 = |\varphi(b) + i\varphi(c)|^2 = \varphi(b)^2 + \varphi(c)^2 \leq C^2(\|b\|^2 + \|c\|^2) \leq 2C^2\|a\|^2.$$

**Lemma 3.2.** Let  $\varphi \in A^*$  be positive. Then  $\forall x, y \in A$ :

$$|\varphi(y^*x)|^2 \leq \varphi(y^*y) \cdot \varphi(x^*x).$$

*Proof.* Consider the sesquilinear form  $\langle x, y \rangle = \varphi(y^*x)$ . Since  $\varphi$  is positive, this is a positive sesquilinear form and we can apply Cauchy-Schwartz.  $\square$

**Theorem 3.3.**

An element  $\varphi \in A^*$  is positive iff  $\|\varphi\| = \varphi(1)$ .

*Remark.* This implies that the set of states  $S(A)$  is convex.

*Proof.* First we prove the right implication ( $\Rightarrow$ ). We know that  $x^*x \leq \|x^*x\|$ , so

$$\begin{aligned} |\varphi(x)|^2 &\leq \varphi(1)\varphi(x^*x) \\ &\leq \varphi(1)\varphi(\|x^*x\|) \\ &= \varphi(1)^2\|x^*x\| \\ &= \varphi(1)^2\|x\|^2, \end{aligned}$$

so  $|\varphi(x)| \leq \varphi(1)\|x\|$ . From there we get  $\|\varphi\| \leq \varphi(1) \leq \|\varphi\|$ , so  $\varphi(1) = \|\varphi\|$ . Now the converse ( $\Leftarrow$ ). Suppose  $x \in A_+$  and  $\varphi(x) = \alpha + i\beta$ . For each  $t \in \mathbb{R}$ , we have

$$\begin{aligned} \alpha^2 + (\beta + t\|\varphi\|)^2 &= |\alpha + i(\beta + t\varphi(1))|^2 \\ &= |\varphi(x + it)|^2 \\ &\leq \|x + it\|^2 \cdot \|\varphi\|^2 \\ &= (\|x\|^2 + t^2) \|\varphi\|^2. \end{aligned}$$

From this it directly follows  $2\beta t\|\varphi\| \leq \|x\|^2 \cdot \|\varphi\|^2$ . Since  $t \in \mathbb{R}$  was arbitrary, we have  $\beta = 0$  and  $\varphi(x) = \alpha \in \mathbb{R}$ . Lastly, we derive

$$\begin{aligned} \|x\| \cdot \|\varphi\| - \varphi(x) &= \varphi(\|x\| - x) \\ &\leq \|\|x\| - x\| \cdot \|\varphi\| \\ &\leq \|x\| \cdot \|\varphi\|, \end{aligned}$$

so  $\varphi(x) \geq 0$ .  $\square$

**Proposition 3.4.** Let  $A$  be a  $C^*$ -algebra and  $x \in A$ . Then  $\forall \lambda \in \sigma(x)$  there exists a  $\varphi \in S(A)$  such that  $\varphi(x) = \lambda$ .

*Proof.* We know that  $\mathbb{C}x + \mathbb{C} \cdot 1 \subseteq A$ . Define

$$\varphi_0 : \mathbb{C}x + \mathbb{C}1 \rightarrow \mathbb{C}, \quad \alpha x + \beta \mapsto \alpha \cdot \lambda + \beta.$$

Since  $\varphi_0(\alpha x + \beta) \in \sigma(\alpha x + \beta)$ , we have

$$\|\varphi_0\| \leq 1 = \varphi_0(1),$$

therefore  $\|\varphi_0\| = 1$ . Now we apply Hahn-Banach to get an extension  $\varphi \in A^*$  such that  $\varphi|_{\mathbb{C}x + \mathbb{C}1} = \varphi_0$  and  $\|\varphi\| = 1 = \varphi(1)$ , so  $\varphi \in S(A)$  by theorem 3.3.  $\square$

**Proposition 3.5.** *Let  $A$  be a  $C^*$ -algebra and  $x \in A$ .*

- (1.)  $x = 0$  iff  $\varphi(x) = 0$ ,  $\forall \varphi \in S(A)$ .
- (2.)  $x \in A_{\text{sa}}$  iff  $\varphi(x) \in \mathbb{R}$ ,  $\forall \varphi \in S(A)$ .
- (3.)  $x \in A_+$  iff  $\varphi(x) \geq 0$ ,  $\forall \varphi \in S(A)$ .

*Proof.* (1.) If  $\varphi(x) = 0$  for all  $\varphi \in S(A)$ , then writing  $x = x_1 + ix_2$  for self-adjoint  $x_1, x_2$  gives us

$$0 = \varphi(x) = \varphi(x_1) + i\varphi(x_2),$$

which implies  $\varphi(x_1) = \varphi(x_2) = 0$ . Now use the proposition 3.5 to get  $\sigma(x_1) = \sigma(x_2) = \{0\}$ , which can only imply  $x_1 = x_2 = 0$ , whence  $x = 0$ .

- (2.) If  $\varphi(x) \in \mathbb{R}$  for all  $\varphi \in S(A)$ , then

$$\varphi(x - x^*) = \varphi(x) - \varphi(x^*) = \varphi(x) - \overline{\varphi(x)} = 0$$

and we use the previous item to show that  $x - x^* = 0$ . The converse implication follows from the fact that every positive functional is hermitian.

- (3.) If  $\varphi(x) \geq 0$  for all  $\varphi \in S(A)$ , then  $x \in A_{\text{sa}}$  by previous item and  $\sigma(x) \subseteq [0, \infty)$ , so  $x \in A_+$ . The converse once again follows from positivity of  $\varphi$ .  $\square$

## 3.2 Gelfand-Naimark-Segal (GNS) construction

**Definition 3.6.** • A *representation* of a  $C^*$ -algebra  $A$  is a  $*$ -homomorphism  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ .

- If  $\mathcal{K}^{\text{closed}} \leq \mathcal{H}$  and  $\pi(x)\mathcal{K} \subseteq \mathcal{K}$ ,  $\forall x \in A$  (we say that  $\mathcal{K}$  is *invariant* under  $\pi$ ), then the restriction of  $\pi$  to  $\mathcal{K}$  is a *subrepresentation*.
- If a representation has no other subrepresentations besides  $\mathcal{K} = (0)$  and  $\mathcal{K} = \mathcal{H}$  (equivalently,  $\pi(A)$  only has  $(0)$  and  $\mathcal{H}$  as closed invariant subspaces), then  $\pi$  is called *irreducible*.
- Representations  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  and  $\rho : A \rightarrow \mathcal{B}(\mathcal{K})$  are *equivalent* if there exists a unitary  $U : \mathcal{H} \rightarrow \mathcal{K}$  such that

$$U\pi(x) = \rho(x)U, \quad \forall x \in A.$$

- Vector  $\zeta \in \mathcal{H}$  is *cyclic* for a representation  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  if

$$\pi(A)\zeta := \{\pi(a)\zeta \mid a \in A\}$$

is dense in  $\mathcal{H}$  (this means that  $\overline{\pi(A)\zeta} = \mathcal{H}$ ).

**Example 3.7.** Each  $w \in \mathcal{H}$  defines a subrepresentation on  $K := \overline{\pi(A)w}$ .

**Example 3.8.** Let  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  be a representation and  $\mu \in \mathcal{H}$ ,  $\|\mu\| = 1$ . Then

$$\varphi_\mu : A \rightarrow \mathbb{C}, \quad x \mapsto \langle \pi(x)\mu, \mu \rangle$$

is a state. Indeed,

$$\varphi_\mu(1) = \langle 1 \cdot \mu, \mu \rangle = \|\mu\|^2 = 1$$

and

$$\begin{aligned} \varphi_\mu(x^*x) &= \langle \pi(x^*x)\mu, \mu \rangle \\ &= \langle \pi(x^*)\pi(x)\mu, \mu \rangle \\ &= \langle \pi(x)^*\pi(x)\mu, \mu \rangle \\ &= \langle \pi(x)\mu, \pi(x)\mu \rangle \\ &= \|\pi(x)\mu\|^2 \geq 0. \end{aligned}$$

**Theorem 3.9** (Gelfand-Naimark-Segal construction).

Let  $A$  be a  $C^*$ -algebra and  $\rho \in S(A)$ . Then there exists a Hilbert space  $L^2(A, \varphi)$  and a unique (up to equivalence) representation  $\pi : A \rightarrow \mathcal{B}(L^2(A, \varphi))$  and a unit cyclic vector  $1_\varphi$  such that

$$\varphi(x) = \langle \pi(x)1_\varphi, 1_\varphi \rangle, \quad \forall x \in A.$$

*Proof.* (1.) We start by defining

$$N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$$

whose elements we call nullvectors of  $\varphi$ . By the Cauchy-Schwartz lemma, we have

$$N_\varphi = \{x \in A \mid \varphi(yx) = 0, \forall y \in A\}.$$

Thus  $N_\varphi$  is a closed subspace of  $A$ .

(2.) We prove that  $N_\varphi$  is a left ideal: for  $x \in N_\varphi$  and  $a \in A$ , we have  $ax \in N_\varphi$ . Indeed,

$$\varphi((ax)^*ax) = \varphi((x^*a^*a)x) = 0.$$

(3.) Now  $\mathcal{H}_0 = A/N_\varphi$  is a vector space and we can endow it with the dot product  $\langle [x], [y] \rangle := \varphi(y^*x)$  for  $x, y \in A$ . It can easily be checked that this is a well-defined dot product in  $\mathcal{H}_0$ . We denote the completion of  $\mathcal{H}_0$  by  $L^2(A, \varphi)$ .

(4.) To an arbitrary  $a \in A$ , we associate the map

$$\pi_0(a) : \mathcal{H}_0 \rightarrow \mathcal{H}_0, \quad [x] \mapsto [ax].$$

Since  $N_\varphi$  is a left ideal of  $A$ ,  $\pi_0(a)$  is a well-defined linear map. We have

$$\begin{aligned} \|\pi_0(a)[x]\|^2 &= \|[ax]\|^2 \\ &= \langle [ax], [ax] \rangle \\ &= \varphi((ax)^*ax) \\ &= \varphi(x^*a^*ax) \\ &\leq \|a\|^2 \cdot \varphi(x^*x) \leq \|a\|^2 \|x\|^2. \end{aligned}$$

Since  $\pi_0(a)$  is a bounded linear map, it extends uniquely to  $\pi(a) \in \mathcal{B}(L^2(A, \varphi))$  with  $\|\pi(a)\| \leq \|a\|$ . Then we get

$$\pi : A \rightarrow \mathcal{B}(L^2(A, \varphi)), \quad a \mapsto \pi(a),$$

which is a homomorphism and has the property

$$\begin{aligned}
\langle [x], \pi(a^*)[y] \rangle &= \langle [x], [a^*y] \rangle \\
&= \varphi((a^*y)^*x) \\
&= \varphi(y^*ax) \\
&= \langle [ax], [y] \rangle \\
&= \langle \pi(a)[x], [y] \rangle.
\end{aligned}$$

So  $\pi(a)^* = \pi(a^*)$  and  $\pi$  is a representation.

(5.) We define  $1_\varphi := [1] \in \mathcal{H}_0 \subseteq L^2(A, \varphi)$  and notice that

$$\langle \pi(a)1_\varphi, 1_\varphi \rangle = \langle \pi(a)[1], [1] \rangle = \langle [a], [1] \rangle = \varphi(a).$$

Since  $\{\pi(a)1_\varphi \mid a \in A\} = \mathcal{H}_0$  is dense in  $L^2(A, \varphi)$ , the vector  $1_\varphi$  is cyclic for  $\pi$ .

(6.) Next we prove uniqueness: let  $\rho : A \rightarrow \mathcal{B}(\mathcal{K})$  be a representation,  $\mu \in \mathcal{K}$  a unit cyclic vector and assume  $\varphi(a) = \langle \rho(a)\mu, \mu \rangle$ ,  $\forall a \in A$ . We will prove that  $\rho$  is equivalent to  $\pi$ . Define

$$U_0 : \mathcal{H}_0 \rightarrow \mathcal{K}, \quad [x] \mapsto \rho(x)\mu.$$

Then we have

$$\begin{aligned}
\langle U_0[x], U_0[y] \rangle_{\mathcal{K}} &= \langle \rho(x)\mu, \rho(y)\mu \rangle \\
&= \langle \rho(y)^* \rho(x)\mu, \mu \rangle \\
&= \langle \rho(y^*x)\mu, \mu \rangle = \varphi(y^*x) = \langle [x], [y] \rangle_{L^2(A, \varphi)},
\end{aligned}$$

so  $U_0$  really is a well-defined isometry. For all  $a, x \in A$ :

$$U_0(\pi(a)[x]) = U_0([ax]) = \rho(ax)\mu = \rho(a)\rho(x)\mu = \rho(a)U_0[x].$$

Therefore,  $U_0$  induces an isometry  $U : L^2(A, \varphi) \rightarrow \mathcal{K}$  such that  $U\pi(a) = \rho(a)U$  for all  $a \in A$ . Since  $\mu$  is cyclic and  $\rho(a)\mu \subseteq \text{im } U$ , the range of  $U$  is dense in  $\mathcal{K}$ . It is also closed since  $U$  is isometric. We just proved that  $U$  is isometric and onto, so it is unitary.  $\square$

**Corollary 3.10.** *Every  $C^*$ -algebra has a faithful (i.e. injective) representation. In particular, every  $C^*$ -algebra is isometrically  $*$ -isomorphic to a closed subalgebra of  $\mathcal{B}(H)$  for some Hilbert space  $\mathcal{H}$ .*

*Proof.* Let  $\pi$  be a direct sum of all representations from GNS construction over all states. Then the Proposition 3.5 tells us that  $\pi$  is injective. An injective  $*$ -monomorphism is isometric and we are done.  $\square$

The preceding corollary enables us to view abstract  $C^*$ -algebras as concrete algebras of operators on some Hilbert space.

**Definition 3.11.** If  $S \subseteq A$ , then

$$S' := \{x \in A \mid \forall s \in S : xs = sx\}$$



is its commutant.

**Proposition 3.12** (Radon-Nikodym for linear functionals). *Let  $\varphi, \psi$  be positive linear functionals on a  $C^*$ -algebra  $A$  and  $\varphi \in S(A)$ . Then  $\varphi \leq \psi$  iff there exists a unique  $y \in \pi_\psi(A)'$  such that  $0 \leq y \leq 1$  and*

$$\varphi(a) = \langle \pi_\psi(a)y1_\psi, 1_\psi \rangle, \quad \forall a \in A.$$

*Proof.* Start with ( $\Leftarrow$ ). For  $a \in A_+$  we have

$$\pi_\psi(a)y = \pi_\psi(a)^{\frac{1}{2}}y\pi_\psi(a)^{\frac{1}{2}} \leq \pi_\psi(a).$$

Then

$$\varphi(a) = \langle \pi_\psi(a)y1_\psi, 1_\psi \rangle \leq \langle \pi_\psi(a)1_\psi, 1_\psi \rangle = \psi(a).$$

Now the converse ( $\Rightarrow$ ). By Cauchy-Schwartz,

$$\begin{aligned} |\varphi(b^*a)|^2 &\leq \varphi(a^*a)\varphi(b^*b) \\ &\leq \psi(a^*a)\psi(b^*b) \\ &= \|\pi_\psi(a)1_\psi\|^2 \cdot \|\pi_\psi(b)1_\psi\|^2. \end{aligned}$$

This means that  $\langle \pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle_\varphi := \varphi(b^*a)$  is a nonnegative sesquilinear form on  $\pi_\psi(A)1_\psi^{\text{dense}} \subseteq L^2(A, \psi)$ , which is bounded by 1. This further implies that it is continuous and we can extend it to  $L^2(A, \psi)$ . By Riesz, there exists  $y \in \mathcal{B}(L^2(A, \psi))$  such that

$$\varphi(b^*a) = \langle y\pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle, \quad \forall a, b \in A$$

and  $0 \leq y \leq 1$ . For  $a, b, c \in A$  we have

$$\begin{aligned} \langle y\pi_\psi(a)\pi_\psi(b)1_\psi, \pi_\psi(c)1_\psi \rangle &= \langle y\pi_\psi(ab)1_\psi, \pi_\psi(c)1_\psi \rangle \\ &= \varphi(c^* \cdot ab) = \varphi((a^*c)^*b) \\ &= \langle y\pi_\psi(b)1_\psi, \pi_\psi(a^*)\pi_\psi(c)1_\psi \rangle \\ &= \langle \pi_\psi(a)y\pi_\psi(b)1_\psi, \pi_\psi(c)1_\psi \rangle, \end{aligned}$$

so  $y\pi_\psi(a) = \pi_\psi(a)y$  for all  $a \in A$  and  $y \in \pi_\psi(A)'$ . Finally, the uniqueness. Say that there exists a  $z \in \pi_\psi(A)'$  such that  $0 \leq z \leq 1$  and

$$\langle \pi_\psi(a)y1_\psi, 1_\psi \rangle = \langle \pi_\psi(a)z1_\psi, 1_\psi \rangle, \quad \forall a \in A.$$

Then

$$\begin{aligned} \langle \pi_\psi(b^*a)z1_\psi, 1_\psi \rangle &= \langle \pi_\psi(b^*a)y1_\psi, 1_\psi \rangle \\ &= \langle y\pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle \\ &= \langle z\pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle, \end{aligned}$$

which implies  $y = z$ . □

**Proposition 3.13.** *Suppose that  $A$  is a separable  $C^*$ -algebra. Then  $A$  has a faithful cyclic representation on a separable Hilbert space.*

*Proof.* If  $A$  is separable, then it has a dense subset  $\{a_i\}_{i=1}^\infty$ . We can embed  $S(A)$  into the space  $\prod_{i=1}^\infty \overline{B_1(0)}$ , where  $\overline{B_1(0)}$  is a closed unit ball in  $\mathbb{C}$ , by sending  $\varphi \in S(A)$  to  $(\varphi(a_i))_{i=1}^\infty$ . The latter topological space is metrizable by the metric  $\rho(x, y) = \sum_{i=1}^\infty \frac{\rho_i(x_i, y_i)}{2^i(\rho_i(x_i, y_i)+1)}$ , and so is  $S(A)$ . Therefore,  $S(A)$  with the weak-\* topology is a metrizable compact, therefore separable. Let  $\{f_i\}_i^\infty$  countable weak-\* dense subset of  $S(A)$ . Then

$$f(a) := \sum_{i=1}^\infty 2^{-i} f_i(a)$$

defines a faithful ( $f(a^*a) = 0$  iff  $a = 0$ ) state on  $A$ . Then the GNS construction  $\pi_f$  is faithful: if  $\pi_f(a) = 0$ , then

$$f(b^*a^*ab) = \langle \pi_f(a)[b], \pi_f(a)[b] \rangle = 0$$

for every  $b \in A$ . In particular for  $b = 1$ , we get  $f(a^*a) = 0$  and so  $a = 0$ . Since  $a \mapsto [a]$  is a continuous map of  $A$  onto a dense subspace of some Hilbert space  $\mathcal{H}_f$  (induced by  $\pi_f : A \rightarrow \mathcal{B}(\mathcal{H}_f)$ ), the latter space is separable.  $\square$

**Proposition 3.14.** *Every representation of a  $C^*$ -algebra is equivalent to a direct sum of cyclic representations.*

*Proof.* Let  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  be some representation of  $A$ . Let  $\mathcal{E}$  be the collection of all subsets  $E$  of nonzero vectors in  $\mathcal{H}$  such that  $\pi(A)e \perp \pi(A)f$  for any  $e, f \in E$ . If we order  $\mathcal{E}$  by inclusion, then Zorn's lemma tells us that  $\mathcal{E}$  has a maximal element  $E_0$ . Let  $\mathcal{H}_0 = \bigoplus_{e \in E_0} \pi(A)e$ . Take  $h \in \mathcal{H}_0^\perp$  in  $\mathcal{H}$ . Then for any  $a, b \in A$  and  $e \in E_0$  we have

$$\langle \pi(a)e, \pi(b)h \rangle = \langle \pi(b)^* \pi(a)e, h \rangle = \langle \pi(b^*a)e, h \rangle = 0,$$

so  $\pi(A)e \perp \pi(A)h$  for each  $e \in E_0$ . By maximality,  $h = 0$  and  $\mathcal{H} = \mathcal{H}_0$ . For  $e \in E_0$ , define  $\mathcal{H}_e := \pi(A)e$ . Obviously,  $\mathcal{H}_e$  is invariant for  $\pi$ , so  $\pi_e := \pi|_{\mathcal{H}_e}$  is a cyclic representation of  $A$ . Clearly,  $\pi = \bigoplus_{e \in E_0} \pi_e$ .  $\square$

### 3.3 Pure states and irregular representations

**Definition 3.15.** A state  $\varphi \in S(A)$  is called *pure* if it's an extreme point of  $S(A)$ .

**Proposition 3.16.** *A state  $\varphi \in S(A)$  is pure iff the representation GNS  $\pi_\varphi : A \rightarrow \mathcal{B}(L^2(A, \varphi))$  with cyclic vector  $1_\varphi$  is irreducible.*

*Proof.* ( $\Rightarrow$ ) Let  $\mathcal{K} \leq L^2(A, \varphi)$  be a closed invariant subspace. Then  $\mathcal{K}^\perp$  is also a closed invariant subspace: for  $a \in A$ ,  $x \in \mathcal{K}^\perp$  and  $k \in \mathcal{K}$  we have

$$\langle \pi_\varphi(a)x, k \rangle = \langle x, \pi_\varphi(a^*)k \rangle = 0.$$

Since  $L^2(A, \varphi) = \mathcal{K} \oplus \mathcal{K}^\perp$  we write  $1_\varphi = \underbrace{\mu_1}_{\in \mathcal{K}} + \underbrace{\mu_2}_{\in \mathcal{K}^\perp}$  and form

$$\varphi_j := \frac{\langle \pi_\varphi(x) \mu_j, \mu_j \rangle}{\|\mu_j\|^2}, \quad j = 1, 2.$$

These are states and so is

$$\varphi(x) = \|\mu_1\|^2 \varphi_1(x) + \|\mu_2\|^2 \varphi_2(x)$$

because  $1 = \|1_\varphi\|^2 = \|\mu_1\|^2 + \|\mu_2\|^2$ . Since  $\varphi \in \text{ext } S(A)$ , we either have  $\mu_1 = 0$  or  $\mu_2 = 0$ , which implies that  $\mathcal{K}$  is either  $(0)$  or  $L^2(A, \varphi)$ .

( $\Leftarrow$ ) Suppose  $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2)$  for  $\varphi_1, \varphi_2 \in S(A)$ . Define a linear map

$$U : L^2(A, \varphi) \rightarrow L^2(A, \varphi_1) \oplus L^2(A, \varphi_2), \quad \pi_\varphi(x)1_\varphi \mapsto \frac{1}{\sqrt{2}}\pi_{\varphi_1}(x)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(x)1_{\varphi_2}.$$

First we notice that  $U$  preserves the scalar product:

$$\begin{aligned} \langle \pi_\varphi(x)1_\varphi, \pi_\varphi(y)1_\varphi \rangle &= \varphi(x^*y) \\ &= \frac{1}{2}\varphi_1(x^*y) + \frac{1}{2}\varphi_2(x^*y) \\ &= \left\langle \frac{1}{\sqrt{2}}\pi_{\varphi_1}(x)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(x)1_{\varphi_2}, \frac{1}{\sqrt{2}}\pi_{\varphi_1}(y)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(y)1_{\varphi_2} \right\rangle \\ &= \langle U\pi_\varphi(x)1_\varphi, U\pi_\varphi(y)1_\varphi \rangle. \end{aligned}$$

Additionally,  $U$  intertwines: for all  $x \in A$ , we have

$$\begin{aligned} U\pi_\varphi(x)(\pi_\varphi(y)1_\varphi) &= U\pi_\varphi(xy)1_\varphi \\ &= \frac{1}{\sqrt{2}}\pi_{\varphi_1}(xy)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(xy)1_{\varphi_2} \\ &= (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))(\pi_{\varphi_1}(y)1_{\varphi_1} \oplus \pi_{\varphi_2}(y)1_{\varphi_2}) \\ &= (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))U(\pi_\varphi(y)1_\varphi). \end{aligned}$$

If we star the intertwining identity, we get

$$\pi_\varphi(x^*)U^* = U^*(\pi_{\varphi_1}(x^*) \oplus \pi_{\varphi_2}(x^*)), \quad \forall x^* \in A.$$

If we plug in  $x$  instead of  $x^*$ , we get

$$\pi_\varphi(x)U^* = U^*(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)), \quad \forall x \in A.$$

Now let

$$p_1 \in \mathcal{B}(L^2(A, \varphi_1) \oplus L^2(A, \varphi_2))$$

be orthogonal projection onto the first direct summand. Clearly, we have

$$p_1(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)) = (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))p_1$$

Putting it all together, we get

$$\begin{aligned}\pi_\varphi(x)U^*p_1U &= U^*(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))p_1U \\ &= U^*p_1(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))U \\ &= U^*p_1U(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)),\end{aligned}$$

so  $U^*p_1U$  commutes with  $\pi_\varphi(x)$  for all  $x \in A$ . If  $\sigma(U^*p_1U)$  has more than one element, then  $\exists t \in (0, 1]$  such that  $\sigma(U^*p_1U - t)$  has both positive and negative elements. By Definition 2.38, we can write  $U^*p_1U = a - b$  for positive  $0 \neq a, b$  such that  $ab = ba = 0$ . Then  $a, b$  commute with  $\pi_\varphi(A)$ , so  $\ker a \neq 0$  is a closed subspace of  $L^2(A, \varphi)$  that is invariant under  $\pi_\varphi(x)$ , which is a contradiction. So  $U^*p_1U$  has a single element spectrum  $\{\alpha\}$  and since  $U^*p_1U$  is normal (because it is positive), so is  $U^*p_1U - \alpha I$ . But now we can write

$$\|U^*p_1U - \alpha I\| = r(U^*p_1U) = 0,$$

which proves that  $U^*p_1U = \alpha I$ . Then

$$\begin{aligned}\alpha &= \alpha\varphi(1) = \varphi(\alpha) \\ &= \langle \alpha 1_\varphi, 1_\varphi \rangle \\ &= \langle U^*p_1U 1_\varphi, 1_\varphi \rangle \\ &= \left\langle \frac{1}{\sqrt{2}}1_{\varphi_1} \oplus 0, \frac{1}{\sqrt{2}}1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}1_{\varphi_2} \right\rangle \\ &= \left\langle \frac{1}{\sqrt{2}}1_{\varphi_1}, \frac{1}{\sqrt{2}}1_{\varphi_1} \right\rangle_{\varphi_1} = \frac{1}{2}.\end{aligned}$$

This means that we can write

$$\left(\sqrt{2}p_1U\right)^* \left(\sqrt{2}p_1U\right) = 1,$$

so

$$u_1 = \frac{1}{\sqrt{2}}p_1U : L^2(A, \varphi) \rightarrow L^2(A, \varphi_1)$$

is an isometry. We also have the identities

$$u_1 1_\varphi = 1_{\varphi_1}, \quad u_1 \pi_\varphi(x) = \pi_{\varphi_1}(x) u_1.$$

It follows that

$$\begin{aligned}\varphi(x) &= \langle \pi_\varphi(x) 1_\varphi, 1_\varphi \rangle \\ &= \langle u_1^* u_1 \pi_\varphi(x) 1_\varphi, 1_\varphi \rangle \\ &= \langle u_1^* \pi_{\varphi_1}(x) u_1 1_\varphi, 1_\varphi \rangle \\ &= \langle \pi_{\varphi_1}(x) u_1 1_\varphi, u_1 1_\varphi \rangle \\ &= \langle \pi_{\varphi_1}(x) 1_{\varphi_1}, 1_{\varphi_1} \rangle = \varphi_1(x)\end{aligned}$$

and we are done. □

**Theorem 3.17.**

A representation  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  is irreducible iff  $\pi(A)' = \mathbb{C} \cdot \text{id}$ .

*Proof.* Start with  $(\Leftarrow)$ . Suppose there exists a closed invariant subspace  $(0) \neq \mathcal{K} \subsetneq \mathcal{H}$ . Let  $p \in \mathcal{B}(\mathcal{H})$  be the orthogonal projection onto  $\mathcal{K}$ . Then  $p \notin \mathbb{C} \cdot \text{id}$ . Now we prove that  $p \in \pi(A)'$ . Let  $a \in A$ . For  $\mu \in \mathcal{K}$ , we have

$$(p\pi(a))\mu = p(\pi(a)\mu) = \pi(a)\mu = \pi(a)(p\mu) = (\pi(a)p)\mu.$$

Now for  $\mu \in \mathcal{K}^\perp$ , we get

$$(p\pi(a))\mu = p(\pi(a)\mu) = 0 = \pi(a)(0) = \pi(a)(p\mu) = (\pi(a)p)\mu.$$

For the converse  $(\Rightarrow)$ , suppose there exists a non-scalar self-adjoint  $h \in \pi(A)'$ . Then  $\sigma(h)$  has at least two elements. We can define two bump functions  $f, g \in C(\sigma(h))$  in the respective neighborhoods of these two elements of  $\sigma(h)$  such that  $fg = 0$ . Then  $f(h) \neq 0$  since  $f \neq 0$ . Then also  $\mathcal{K} := \text{im } f(h) \leq \mathcal{H}$  is nonzero. Also,  $g(h) \neq 0$  and  $g(h)|_{\mathcal{K}} = 0$  since  $g(h) \cdot f(h) = 0$ . In particular,  $\mathcal{K} \subsetneq \mathcal{H}$ . Take any  $x \in \pi(A)$ . By Stone-Weierstrass, we can approximate  $f(h)$  in norm by  $(p_i(h))_{i=1}^\infty$ , where  $(p_i)_{i=1}^\infty$  are complex polynomials. Notice that since  $h \in \pi(A)'$ , we also have  $p_i(h) \in \pi(A)'$ . Therefore, we have  $p_i(h)x = xp_i(h)$ . By sending  $i \rightarrow \infty$ , the left side converges in norm to  $f(h)x$ , while the right one converges in norm to  $af(h)$ . As a result,  $f(h)x = xf(h)$  and  $f(h) \in \pi(A)'$ . We claim that  $\mathcal{K}$  is invariant; it's enough to show that  $\text{im } f(h)$  is invariant. For  $a \in A, \mu \in \mathcal{H}$  we have

$$\pi(a)(f(h)\mu) = \pi(a)f(h)\mu = f(h)\pi(a)\mu \in \text{im } f(h).$$

In general, if  $q \in \pi(A)'$ , then  $q^* \in \pi(A)'$  and we can reduce the problem to the self-adjoint case handled above.  $\square$

**Corollary 3.18.** Irreducible representations of abelian  $C^*$ -algebras are 1-dimensional.

*Proof.* Let  $A$  be an abelian  $C^*$ -algebra and  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  an irrep. Then by Theorem 3.17,  $\pi(A)' = \mathbb{C}$ . Moreover,

$$\pi(A) = Z(\pi(A)) = \pi(A)' \cap \pi(A) = \mathbb{C} \cdot \text{id}.$$

$\square$

**Corollary 3.19.** If  $A$  is an abelian  $C^*$ -algebra, then  $\text{ext } S(A) = \sigma(A)$ .

*Proof.* Let  $\varphi \in \sigma(A)$ . Then  $\varphi$  is 1-dimensional (therefore irreducible) representation and so  $\varphi \in \text{ext } S(A)$ . For the converse, take  $\varphi \in \text{ext } S(A)$ . Then the GNS construction  $\pi_\varphi$  is irreducible, therefore 1-dimensional. So  $L^2(A, \varphi) = \mathbb{C}$  with the standard scalar product and  $\varphi(x) = \langle \pi_\varphi(x)1_\varphi, 1_\varphi \rangle = \pi_\varphi(x)$ .  $\square$

**Proposition 3.20.** Let  $A$  be a  $C^*$ -algebra. Then  $\text{co ext } S(A)$  is weak-\* dense in  $S(A)$ .

*Proof.* We know that  $S(A)$  is compact Hausdorff with respect to the weak-\* topology. The conclusion follows from Krein-Milman.  $\square$

**Corollary 3.21.** *Let  $A$  be a  $C^*$ -algebra and  $x \in A \setminus \{0\}$ . Then there exist an irrep  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  such that  $\pi(x) \neq 0$ .*

*Proof.* By Proposition 3.5, there exists  $\varphi \in S(A)$  such that  $\varphi(x) \neq 0$ . By the previous proposition (Krein-Milman), there exists a  $\tau \in \text{ext } S(A)$  such that  $\tau(x) \neq 0$ . Then apply GNS:  $\pi_\tau$  is irreducible and  $\pi_\tau(x) \neq 0$ .  $\square$

**Theorem 3.22** (Jordan decomposition for linear functionals).

*Let  $A$  be a  $C^*$ -algebra and  $\varphi \in A^*$  hermitian. Then there exist (unique - without proof) positive linear functionals  $\varphi_+, \varphi_- \in A^*$  such that  $\varphi = \varphi_+ - \varphi_-$  and  $\|\varphi\| = \|\varphi_+\| = \|\varphi_-\|$ .*

*Proof.* W.l.o.g.  $\|\varphi\| = 1$ . Let  $\Sigma$  denote the set of positive linear functionals with norm  $\leq 1$ . By Banach-Alaoglu,  $\Sigma$  is weak-\* compact and Hausdorff. Consider

$$\gamma : A \rightarrow C(\Sigma), \quad a \mapsto (\psi \mapsto \psi(a)).$$

This is an isometry and  $\gamma(A_+) \subseteq C(\Sigma)_+$ . By Hahn-Banach, there exists a  $\tilde{\varphi} : C(\Sigma) \rightarrow \mathbb{C}$  such that  $\|\tilde{\varphi}\| = \|\varphi\|$  and  $\varphi = \tilde{\varphi} \circ \gamma$ . Assume  $\tilde{\varphi}$  is hermitian (otherwise, we can replace it by  $\frac{\tilde{\varphi} + \tilde{\varphi}^*}{2}$ ). By Riesz-Markoff, there exists a regular Radon Measure  $\mu$  on  $\Sigma$  such that  $\tilde{\varphi}(f) = \int f d\mu$  for all  $f \in C(\Sigma)$ . Then we use Jordan decomposition for measures to obtain  $\mu_+, \mu_-$  such that  $\mu = \mu_+ - \mu_-$  and  $\|\mu\| = \|\mu_+\| = \|\mu_-\|$ . Now we just define  $\varphi_\pm(a) := \int a d\mu_\pm$ .  $\square$

**Corollary 3.23.** *For a  $C^*$ -algebra  $A$ ,  $A^*$  is the span of positive linear functionals on  $A$ .*

**Corollary 3.24.** *Let  $A$  be a  $C^*$ -algebra and  $\varphi \in A^*$ . Then there exists a representation  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  and  $\mu, \theta \in \mathcal{H}$  such that  $\varphi(a) = \langle \pi(a)\theta, \mu \rangle$ .*

*Proof.* Write  $\varphi = \sum_{i=1}^n \alpha_i \psi_i$  for some  $\psi_i \in S(A)$ . Let  $\pi_i$  be the GNS representation of  $\psi_i$ . Define  $\pi := \bigoplus_i \pi_i$ ,  $\theta := \bigoplus_i \alpha_i 1_{\psi_i}$  and  $\mu = \bigoplus_i 1_{\psi_i}$ . The result then follows immediately.  $\square$

### 3.4 Examples of $C^*$ -algebras

**Example 3.25.** *The most canonical example of a  $C^*$ -algebra is  $\mathcal{B}(\mathcal{H})$ . Similarly, the algebra of compact operators  $\mathcal{K}(\mathcal{H})$  is a  $C^*$ -algebra (if  $\dim \mathcal{H} = \infty$ , it is non-unital).*

**Example 3.26.** *If  $\dim \mathcal{H} = \infty$ , then  $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$  is a so-called Calkin algebra. Calkin algebra is simple and it does not have a separable representation. We will prove this later, when we have more tools at our disposal.*

**Example 3.27.** *The algebra of matrices  $M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$  is a  $C^*$ -algebra.*

**Proposition 3.28** (Artin-Wedderburn for finite-dimensional  $C^*$ -algebras). *Every finite-dimensional*

$C^*$ -algebra  $A$  is

$$A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$$

for uniquely determined  $n_1, \dots, n_r$ .

*Proof.* (TO DO)

□

### 3.4.1 Group $C^*$ -algebras

Let  $G$  be a group. Then the (complex) group algebra  $\mathbb{C}[G]$  is the algebra with basis  $\{u_g \mid g \in G\}$  and multiplication given by  $u_g \cdot u_h = u_{gh}$ . Then  $\mathbb{C}[G]$  has the involution

$$\left( \sum_{g \in G}^{\text{finite}} a_g u_g \right)^* = \sum_g \overline{a_g} u_{g^{-1}}.$$

Multiplication is convolutive:

$$\begin{aligned} \left( \sum_g a_g u_g \right) \left( \sum_h b_h u_h \right) &= \sum_{g,h} a_g b_h u_g u_h \\ &= \sum_{g,h} a_g b_h u_{gh} \\ &= \sum_k \left( \sum_g a_g b_{g^{-1}k} \right) u_k. \end{aligned}$$

To introduce norms on  $\mathbb{C}[G]$  we use representations  $\pi : \mathbb{C}[G] \rightarrow \mathcal{B}(\mathcal{H})$ . Given such a  $\pi$ , we define the  $C^*$ -algebra

$$C_\pi^*(G) := \overline{\pi(\mathbb{C}[G])} \subseteq \mathcal{B}(\mathcal{H}).$$

For  $g \in G$ , we get

$$\begin{aligned} \pi(u_g) \pi(u_g)^* &= \pi(u_g) \cdot \pi(u_g^*) \\ &= \pi(u_g) \cdot \pi(u_{g^{-1}}) \\ &= \pi(u_g \cdot u_{g^{-1}}) = \pi(u_e) = 1. \end{aligned}$$

Similarly,  $\pi(u_g)^* \pi(u_g) = 1$ . Under any representation of  $\mathbb{C}[G]$ , each  $u_g$  is mapped to a unitary.

**Example 3.29.** Take  $\mathcal{H} = \ell^2(G)$  (this is a Hilbert space with ONB  $\{\delta_g \mid g \in G\}$ ). Then

$$\lambda : \mathbb{C}[G] \rightarrow \mathcal{B}(\ell^2(G)), \quad u_g \mapsto (\delta_h \mapsto \delta_{gh})$$

is a faithful representation. We call it the left regular representation of  $G$ . The closure of its image is called the reduced group  $C^*$ -algebra  $C_r^*(G) := \overline{\lambda(\mathbb{C}[G])} \subseteq \mathcal{B}(\ell^2(G))$ .

**Definition 3.30.** Universal (or full) group  $C^*$ -algebra is the completion  $\mathbb{C}[G]$ , where the norm of an element  $a \in \mathbb{C}[G]$  is  $\|a\|_u = \sup\{\|\pi(a)\| \mid \pi \text{ representation of } \mathbb{C}[G]\}$ .

**Lemma 3.31.** If  $\pi$  is a representation of  $\mathbb{C}[G]$  and  $a = \sum_{g \in G}^{\text{finite}} a_g u_g \in \mathbb{C}[G]$ , then  $\|\pi(a)\| \leq \sum |a_g|$ .

*Proof.* Then  $\pi(a) = \sum a_g \cdot \pi(u_g)$ . Then

$$\|\pi(a)\| = \left\| \sum a_g \pi(u_g) \right\| \leq \sum |a_g| \cdot \|\pi(u_g)\| = \sum |a_g|.$$

□

This implies that  $\|\cdot\|_u$  is indeed a norm on  $\mathbb{C}[G]$ , making the universal  $C^*$ -algebra  $C^*(G)$  of  $G$  in Definition 3.30 well-defined.

*Remark.* The algebraic group algebra  $\mathbb{C}[G]$  is dense in both  $C_\lambda^*(G)$  and  $C^*(G)$ .

**Theorem 3.32** (Universal property).

For each representation  $\pi$  of  $\mathbb{C}[G]$  there exists a surjective  $*$ -homomorphism  $\hat{\pi} : C^*(G) \rightarrow C_\pi^*(G)$  such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{C}[G] & \xrightarrow{\pi} & C_\pi^*(G) \\ \downarrow & \nearrow \exists! \hat{\pi} & \\ C^*(G) & & \end{array}$$

*Proof.* Define first  $\hat{\pi}$  on  $\mathbb{C}[G] \subseteq C^*G$  by  $\hat{\pi}(a) := \pi(a) \in C_\pi^*(G)$ . Firstly,  $\hat{\pi}$  on  $\mathbb{C}[G]$  is contractive:

$$\|\hat{\pi}(a)\| = \|\pi(a)\| \leq \|a\|_u.$$

By density,  $\hat{\pi}$  uniquely extends to a continuous  $*$ -homomorphism  $\hat{\pi} : C^*(G) \rightarrow C_\pi^*(G)$ . This  $\hat{\pi}$  is also contractive and  $\text{im } \pi$  is dense, so  $\hat{\pi}$  is onto. □

**Example 3.33.** Let  $G$  be abelian and  $|G| = n$ . Then  $\mathbb{C}[G] \cong \mathbb{C}^{|G|}$  as a vector space. Hence  $C^*G = \mathbb{C}[G] = C_r^*(G)$ . Furthermore,  $\mathbb{C}[G]$  is commutative, so by the structure theorem we have

$$\mathbb{C}[G] \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ times}}$$

as a  $C^*$ -algebra. For instance,  $\mathbb{C}[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}] \cong \mathbb{C}[\mathbb{Z}/4\mathbb{Z}]$ .

**Example 3.34.** Let  $G = S_3$ . Then  $|G| = 6$  and once again  $C^*(G) = \mathbb{C}[G] = C_r^*(G)$ . By structure theorem and the dimension consideration,  $\mathbb{C}[G] \cong M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$  (otherwise it would be commutative).



**Example 3.35.** Let  $G = S_4$ . Again,  $C^*(G) = \mathbb{C}[G] = C_r^*(G)$ . By Maschke's theorem,  $\mathbb{C}[G]$  is semisimple, therefore it is a direct sum of matrix algebras over  $\mathbb{C}$ . Since  $S_4$  has five conjugacy classes, there are five factors<sup>a</sup>. Adding up all the dimensions, the only combination that works is  $9 + 9 + 4 + 1 + 1 = 24$ , therefore

$$\mathbb{C}[G] \cong M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}.$$

<sup>a</sup>Pierre Antoine Grillet, *Abstract algebra*, theorem IX.7.9.

**Example 3.36.** What is  $C^*(\mathbb{Z})$ ? Representations  $\pi(\mathbb{C}[Z]) \rightarrow \mathcal{B}(\mathcal{H})$  are determined by choice of unitary  $U \in \mathcal{B}(\mathcal{H})$  such that  $\pi(u_1) = U$ . By universal property, for every  $\mathcal{H}$  and  $U \in \mathcal{B}(\mathcal{H})$  there exists a unique  $*$ -homomorphism

$$\widehat{\pi} : C^*(\mathbb{Z}) \rightarrow C^*(\{U\}),$$

where the latter is a  $C^*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$ , generated by  $U$ . We call  $C^*(\mathbb{Z})$  the universal  $C^*$ -algebra, generated by a unitary.

### 3.5 Abelian group $C^*$ -algebras

If  $G$  is abelian, then  $\mathbb{C}[G]$  is commutative and  $C_r^*(G)$  is abelian. By Gelfand, there exists a compact hausdorff space  $\widehat{G}$  such that  $C_r^*(G) \cong C(\widehat{G})$  and  $\widehat{G} = \sigma(C_r^*(G))$ .

**Definition 3.37.** To each abelian group  $G$  we associate its Pontryagin dual

$$\widehat{G} = \{w : G \rightarrow \mathbb{T} \text{ group homomorphism}\}.$$

Then  $\widehat{G}$  is a group under pointwise multiplication. We endow  $\widehat{G}$  with the topology induced from  $\widehat{G} \subseteq \mathbb{T}^G$ . The basis sets for this topology are

$$B_{\varepsilon, F}(w) = \{\eta \in \widehat{G} \mid |\eta(h) - w(h)| < \varepsilon, \forall h \in F\}$$

for  $\varepsilon > 0$ ,  $w \in \widehat{G}$  and  $F \subseteq G$  finite. Notice that a net  $(w_i)_{i \in I} \subseteq \widehat{G}$  is Cauchy iff  $(w_i(g))_{i \in I} \subseteq \mathbb{T}$  is Cauchy for all  $g \in G$ .

#### Theorem 3.38.

The map

$$h : \widehat{G} \rightarrow \sigma(C_r^*(G)), \quad w \mapsto \left( \sum a_g u_g \mapsto \sum a_g w(g) \right)$$

is a homeomorphism.

*Proof.* First, we prove that  $h(w) \in \sigma(C_r^*(G))$  for all  $w \in \widehat{G}$ . We begin by showing  $h(w) : \mathbb{C}[G] \rightarrow \mathbb{C}$  is a homomorphism. Take  $b = \sum b_k u_k \in \mathbb{C}[G]$ . Then

$$h(w)(a \cdot b) = \sum_g \left( \sum_h a_h b_{h^{-1}g} \right) \cdot w(g)$$

and

$$h(w)(a) \cdot h(w)(b) = \left( \sum_g a_g w(g) \right) \cdot \left( \sum_h b_h w(h) \right) = \sum_k \left( \sum_h a_{kh^{-1}} b_h \right) w(k),$$

so  $h(w)$  is multiplicative. To extend it to  $C_r^*$ , we must prove that  $|h(w)a| \leq \|a\|_r$  for all  $a \in \mathbb{C}[G]$ . To  $\chi \in \sigma(C_r^*(G))$  and  $a \in \mathbb{C}[G]$  we associate

$$\tilde{a} = \sum a_g w(g) \cdot \overline{\chi(u_g)} u_g,$$

so  $h(w)a = \chi(\tilde{a})$ . By Gelfand,

$$\|\tilde{a}\|_r = \sup\{|\mu(\tilde{a})| \mid \mu \in \sigma(C_r^*(G))\} \geq |\chi(\tilde{a})| = |h(w)a|.$$

Next, we show that  $\|\tilde{a}\|_r = \|a\|_r$ : to  $\theta \in \ell^2(G)$  assign  $\tilde{\theta}$  by  $\tilde{\theta}_h := \chi(u_{h^{-1}}) \overline{w(h)} \theta_h$ . Then  $\|\theta\|_2 = \|\tilde{\theta}\|_2$ . Further,  $\|\lambda(\tilde{a})\tilde{\theta}\|_2 = \|\lambda(a)\theta\|_2$  (short calculation), so

$$\|\tilde{a}\|_r = \sup\{\|\lambda(\tilde{a})\tilde{\theta}\| \mid \|\tilde{\theta}\|_2 = 1\} = \sup\{\|\lambda(a)\theta\| \mid \|\theta\|_2 = 1\} = \|a\|_r.$$

Next, we prove that  $h$  is continuous. Suppose the net  $(w_i)_{i \in I} \subseteq \widehat{G}$  is Cauchy. We prove that for every  $a \in C_r^*(G)$  the net  $(h(w_i)(a))_{i \in I}$  is Cauchy. Pick  $\varepsilon > 0$ . There exists  $J$  such that for every  $i, j \geq J$ , we have

$$|w_i(g) - w_j(g)| < \frac{\varepsilon}{|\{g \mid a_g \neq 0\}|}, \quad \forall g \in G.$$

Then for all  $i, j \geq J$  we get  $|h(w_i)(g) - h(w_j)(g)| < \varepsilon$ . For the general case  $a \in C_r^*(G)$ , we can take  $a$  as a limit of a sequence  $(a_n)_n \subseteq \mathbb{C}[G]$ , approximate  $a$  with  $a_n$  and use the triangle inequality to establish that  $(h(w_i)(a))_i$  is Cauchy. Now on to bijectivity of  $h$ . It's enough to check that it is surjective: take  $\phi \in \sigma(C_r^*(G))$ . Define

$$w_\phi : G \rightarrow \mathbb{C}, \quad g \mapsto \phi(u_g).$$

Since  $\phi$  is a \*-homomorphism,  $\text{im } w_\phi \subseteq \mathbb{T}$ . We have to prove that  $w_\phi \in \widehat{G}$ . We just check the multiplicativity:

$$w_\phi(g) \cdot w_\phi(h) = \phi(u_g)\phi(u_h) = \phi(u_g u_h) = \phi(u_{gh}) = w_\phi(gh).$$

For every  $w \in \widehat{G}$ , we get  $w_{h(w)} = w$ . So  $w_\phi = w_{h(w_\phi)}$ , which gives us  $h(w_\phi) = \phi$ . Now since  $h$  is a bijective continuous map between compact Hausdorff spaces, it is a homeomorphism.  $\square$

**Example 3.39.**  $\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z}$ ,  $\widehat{\mathbb{R}} = \mathbb{R}$ ,  $\widehat{\mathbb{Z}} = \mathbb{T}$ ,  $\widehat{\mathbb{T}} = \mathbb{Z}$ .

**Theorem 3.40** (Pontryagin).

*If  $G$  is a locally compact abelian group (its underlying topological space is locally compact Hausdorff), then  $G \cong \widehat{\widehat{G}}$ .*

## 4 Bounded operators on Hilbert spaces

Let  $\mathcal{H}$  be a complex Hilbert space. Then  $\mathcal{B}(\mathcal{H})$  is a  $\mathbb{C}$ -algebra with norm  $\|A\| = \sup_{\mu \in \mathcal{H}, \|\mu\|=1} \|A\mu\|$ .

*Remark.* Recall that  $A \in \mathcal{B}(\mathcal{H})$  is:

- (1.) normal  $\Leftrightarrow A^*A = AA^* \Leftrightarrow \|A\mu\| = \|A^*\mu\|, \forall \mu \in \mathcal{H}$
- (2.) self-adjoint  $\Leftrightarrow A^* = A \Leftrightarrow \langle A\mu, \mu \rangle \in \mathbb{R}, \forall \mu \in \mathcal{H}$
- (3.) positive  $\Leftrightarrow A = B^*B$  for some  $B \in \mathcal{B}(\mathcal{H}) \Leftrightarrow \langle A\mu, \mu \rangle \geq 0, \forall \mu \in \mathcal{H}$
- (4.) isometry  $\Leftrightarrow A^*A = I \Leftrightarrow \|A\mu\| = \|\mu\|, \forall \mu \in \mathcal{H}$
- (5.) projection  $\Leftrightarrow A^2 = A = A^* \Leftrightarrow A$  is an orthogonal projection onto some closed subspace of  $\mathcal{H}$

**Lemma 4.1.** *An operator  $A \in \mathcal{B}(\mathcal{H})$  is a partial isometry iff there exists a closed subspace  $\mathcal{K} \leq \mathcal{H}$  such that  $A|_{\mathcal{K}}$  is an isometry and  $A|_{\mathcal{K}^\perp} = 0$ .*

*Proof.* We first prove  $(\Leftarrow)$ . Obviously,  $\mathcal{K}^\perp \subseteq \ker A$ . From  $Ax = 0$ , where  $x = y + z$  and  $y \in \mathcal{K}, z \in \mathcal{K}^\perp$ , we have

$$0 = Ax = A(y + z) = Ay + Az = Ay.$$

But since  $A|_{\mathcal{K}}$  is an isometry,  $\|Ay\| = \|y\| = 0$ , so  $y = 0$  and  $x \in \mathcal{K}^\perp$ . Now we prove that  $P = A^*A$  is a standard projection onto  $\mathcal{K}$ . For  $x \in \mathcal{K}$ , we have

$$\langle Px, x \rangle = \langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 = \|x\|^2,$$

so

$$\|P\| = \|A^*A\| \leq \|A\|\|A^*\| = \|A\|^2 = 1.$$

From Cauchy-Schwartz:

$$\langle Px, x \rangle \leq \|Px\|\|x\| \leq \|P\|\|x\|^2 \leq \|x\|^2.$$

Since we have equality in Cauchy-Schwartz, there exists a  $\lambda \in \mathbb{C}$  such that  $Px = \lambda x$ . But from  $\langle Px, x \rangle = \|x\|^2$ , it follows that  $\lambda = 1$ . So  $P|_{\mathcal{K}} = \text{id}$  and for  $x \in \mathcal{K}^\perp$ ,  $Px = A^*Ax = 0$ . Therefore,  $P = A^*A$  is indeed a projection. Now onto the opposite direction  $(\Rightarrow)$ . Suppose  $P = A^*A$  is a projection and denote  $\mathcal{K} = \text{im } P$ . Since  $\mathcal{K} = \ker(I - P)$ , it is a closed subspace of  $\mathcal{H}$ . For  $x \in \mathcal{K}$ , we have

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle Px, x \rangle = \langle x, x \rangle = \|x\|^2.$$

But for  $x \in \mathcal{K}^\perp$ , we use the identity

$$(\text{im } P)^\perp = \ker P^* = \ker P$$

to get  $Px = 0$ , so  $\|Ax\|^2 = \langle Px, x \rangle = 0$  and  $\|Ax\| = 0$ . □

**Theorem 4.2.**

Let  $\mathcal{H}$  be a Hilbert space and  $x \in \mathcal{B}(\mathcal{H})$ . Then there exists a partial isometry  $v$  such that  $x = v \cdot |x|$  and  $\ker v = \ker |x| = \ker x$ . This decomposition is unique if  $x = wy$  for  $y \geq 0$  and partial isometry  $w$  such that  $\ker y = \ker w$ . Then  $w = v$  and  $y = |x|$ .

*Proof.* First we prove the existence. Define

$$v_x : \operatorname{im} |x| \rightarrow \operatorname{im} x, \quad |x|y \mapsto xy.$$

Since

$$\begin{aligned} \| |x|y \|^2 &= \langle |x|y, |x|y \rangle \\ &= \langle |x|^2 y, y \rangle \\ &= \langle x^* xy, y \rangle \\ &= \langle xy, xy \rangle \\ &= \| xy \|^2. \end{aligned}$$

The above  $v_x$  is well defined. It is also linear and isometric. By continuity, extend  $v_x$  to a map  $\overline{\operatorname{im} |x|} \rightarrow \overline{\operatorname{im} x}$ . Now  $v_0$  can be extended to  $v : \mathcal{H} \rightarrow \mathcal{H}$  by setting  $v|_{(\operatorname{im} |x|)^\perp} = 0$ . By previous lemma,  $v$  is a partial isometry. By definition,  $x = v \cdot |x|$  and  $\ker v = (\operatorname{im} |x|)^\perp = \ker |x| = \ker x$ . Next, we prove uniqueness. If  $x = wy$  as in the statement, then  $\ker w = \ker y = (\operatorname{im} y)^\perp$ , so  $w$  is a partial isometry on  $\overline{\operatorname{im} y}$ . From there, we get

$$|x|^2 = (wy)^*(wy) = y^* w^* w y = y^* y = y^2,$$

which implies

$$|x| = (|x|^2)^{\frac{1}{2}} = (y^2)^{\frac{1}{2}} = y.$$

Now

$$w|x|\mu = wy\mu = x\mu$$

together with

$$\ker w = (\operatorname{im} y)^\perp = (\operatorname{im} |x|)^\perp$$

implies  $w = v$ . □

Now we can also prove the statement in the Example 1.41.

**Proposition 4.3.** *The extreme points of the unit ball of  $\mathcal{B}(\mathcal{H})$  are exactly the elements  $V \in \mathcal{B}(\mathcal{H})$  such that*

$$(1 - VV^*)\mathcal{B}(\mathcal{H})(1 - V^*V) = 0.$$

*In particular,  $V^*V$  and  $VV^*$  are projections.*

*Proof.* Let  $V \in A$  be an extreme point of the unit ball of  $\mathcal{B}(\mathcal{H})$ , so  $\sigma(V) \subseteq [-1, 1]$ . Write

$$V = \frac{1}{2}V(2 - |V|) + \frac{1}{2}V|V|.$$

Since the functions  $z \mapsto z(2 - |z|)$  and  $z \mapsto |z|(2 - |z|)$  coincide and are both bounded above

by 1 on  $\sigma(x)$ , we have  $\|V(2 - |V|)\| = \||V|(2 - |V|)\| \leq 1$  by continuous functional calculus. This implies that  $V(2 - |V|)$  is in the unit ball of  $\mathcal{B}(\mathcal{H})$ . The same can be said about  $V|V|$  by the same argument. Now since  $V$  is an extreme point, we must have  $V = V|V|$ . Multiplying on the left with  $V^*$ , we get  $|V|^2 = |V|^3$ . This means that the functions  $z \mapsto z^2$  and  $z \mapsto z^3$  coincide on  $\sigma(|V|)$ , which implies that  $\sigma(|V|) \subseteq \{0, 1\}$ . As a result,  $|V|$  is a projection, so  $P := |V| = |V|^2 = V^*V$ . The same can be said about  $Q := VV^*$ , since we know that  $\sigma(V^*V) \setminus \{0\} = \sigma(VV^*) \setminus \{0\}$ . This means that  $V$  is a partial isometry. By the previous lemma,  $P$  is a projection onto the initial space of  $V$ , so  $QV = VP = V$ . Now suppose  $W := (1 - Q)Z(1 - P) \neq 0$  for some  $Z$  in the unit ball. Then

$$\begin{aligned} \|V + W\|^2 &= \|QVP + (1 - Q)Z(1 - P)\|^2 \\ &= \|(QVP + (1 - Q)Z(1 - P))^*(QVP + (1 - Q)Z(1 - P))\| \\ &= \|(PV^*Q + (1 - P)Z^*(1 - Q))(QVP + (1 - Q)Z(1 - P))\| \\ &= \|PV^*QVP + (1 - P)Z^*(1 - Q)Z(1 - P)\| \\ &= \|PV^*VP + (1 - P)W^*W(1 - P)\| \\ &= \max\{\|V^*V\|, \|W^*W\|\} \\ &= \max\{\|V\|^2, \|W\|^2\} = 1 \end{aligned}$$

and similarly  $\|V - W\|^2 = 1$ . Therefore we have a decomposition

$$V = \frac{1}{2}(V + W) + \frac{1}{2}(V - W)$$

and  $V$  is not an extreme point, leading to a contradiction. Conversely, suppose that  $(1 - VV^*)\mathcal{B}(\mathcal{H})(1 - V^*V) = 0$ . Then we have

$$0 = V^*(1 - VV^*)V(1 - V^*V) = V^*V(1 - V^*V)^2.$$

This implies that the function  $z \mapsto z(1 - z)^2$  must be zero on  $\sigma(V^*V)$ , so  $\sigma(V^*V) \subseteq \{0, 1\}$  and  $P := V^*V$  is a projection. By the same argument,

$$0 = (1 - VV^*)V(1 - V^*V)V^* = (1 - VV^*)^2VV^*$$

and  $Q := VV^*$  is a projection as well. Assume that  $V = \frac{1}{2}U + \frac{1}{2}W$  for  $U, W$  in the unit ball. Again we have  $V = VP = QV$ , so

$$V = \frac{1}{2}UP + \frac{1}{2}WP$$

and

$$\begin{aligned} 4P &= 4V^*V = PU^*UP + PW^*WP + PU^*WP + PW^*UP \\ &= 2(PU^*UP + PW^*WP) - P(U - W)^*(U - W)P \\ &\leq 4P - P(U - W)^*(U - W)P. \end{aligned}$$

This immediately implies that  $(U - W)P = 0$ . Similarly, we have  $Q(U - W) = 0$ . Now

$$\begin{aligned} U - W &= Q(U - W)P + (1 - Q)(U - W)P \\ &\quad + Q(U - W)(1 - P) + (1 - Q)(U - W)(1 - P) = 0. \end{aligned}$$

□

*Remark.* The above theorem holds for any  $C^*$ -algebra, not just  $\mathcal{B}(\mathcal{H})$ . We can identify any general  $C^*$ -algebra with an algebra of operators on some Hilbert space and then the above proof carries over verbatim.

**Corollary 4.4.**

$$\text{ext}(\mathcal{B}(\mathcal{H}))_1 = \{V \in \mathcal{B}(\mathcal{H}) \mid V \text{ or } V^* \text{ is an isometry}\}.$$

*Proof.* We need to prove the inclusion ( $\subseteq$ ). If  $V$  is an extreme point, then  $V^*V$  and  $VV^*$  are projections. Therefore,  $V$  is a partial isometry with the initial space  $(\ker V)^\perp$  and  $V^*$  is a partial isometry with the initial space  $(\ker V^*)^\perp$ . Assume neither  $V$  nor  $V^*$  are full isometries, so their initial spaces are proper subspaces of  $\mathcal{H}$ . This means that there exist vectors  $0 \neq x \in \ker V$  and  $0 \neq y \in \ker V^*$ . Define  $P$  as a rank-one projection from  $x$  to  $y$ . Then

$$(1 - VV^*)P(1 - V^*V)x = y \neq 0,$$

so  $(1 - VV^*)P(1 - V^*V) \neq 0$ , contradiction.  $\square$

#### 4.1 Trace class operators

Recall that  $A \in \mathcal{B}(\mathcal{H})$  has finite rank if  $\text{rank } A = \dim \overline{\text{im } A} < \infty$ . We know that

$$\text{im } A^* = \text{im}(A^*|_{(\ker A^*)^\perp}) = \text{im}(A^*|_{\overline{\text{im } A}}).$$

So  $\text{rank } A < \infty$  iff  $\text{rank } A^* < \infty$ . Define a set

$$\mathcal{F}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid \text{rank } A < \infty\}$$

of finite rank operators. Now if  $\alpha, \beta \in \mathcal{H}$ , then we can define

$$\alpha \otimes \bar{\beta} : \mathcal{H} \rightarrow \mathcal{H}, \quad y \mapsto \langle y, \beta \rangle \cdot \alpha.$$

It is trivial to see that  $\text{rank}(\alpha \otimes \bar{\beta}) \leq 1$  and  $(\alpha \otimes \bar{\beta})^* = \beta \otimes \bar{\alpha}$ . If  $\|\alpha\| = \|\beta\| = 1$ , then  $\alpha \otimes \bar{\beta}$  is a partial isometry with initial space  $\mathbb{C}\beta$  and image  $\mathbb{C}\alpha$ . Then

$$\mathcal{F}(\mathcal{H}) = \text{span } \{\alpha \otimes \bar{\beta} \mid \alpha, \beta \in \mathcal{H}\}.$$

For  $x, y \in \mathcal{B}(\mathcal{H})$  we have

$$x(\alpha \otimes \bar{\beta})y = (x\alpha) \otimes \overline{(y^*\beta)}.$$

**Lemma 4.5.** *Let  $x \in \mathcal{B}(\mathcal{H})$  have the polar decomposition  $x = v \cdot |x|$ . Then for all  $y \in \mathcal{H}$ , we have*

$$2|\langle xy, y \rangle| \leq \langle |x|y, y \rangle + \langle |x|v^*y, v^*y \rangle.$$

*Proof.* Let  $\lambda \in \mathbb{T}$ . Then

$$\begin{aligned} 0 &\leq \|(|x|^{\frac{1}{2}} - \lambda|x|^{\frac{1}{2}}v^*)y\|^2 \\ &= \| |x|^{\frac{1}{2}}y \|^2 - 2\text{Re } \bar{\lambda} \langle |x|^{\frac{1}{2}}y, |x|^{\frac{1}{2}}v^*y \rangle + \| |x|^{\frac{1}{2}}v^*y \|^2. \end{aligned}$$

Now pick  $\lambda$  such that  $\bar{\lambda} \langle |x|^{\frac{1}{2}}y, |x|^{\frac{1}{2}}v^*y \rangle \geq 0$  and we are done.  $\square$

**Definition 4.6.** Let  $(e_i)_{i \in I}$  be an orthonormal system for  $\mathcal{H}$ . For  $x \in \mathcal{B}(\mathcal{H})_+$ , define the trace

$$\text{Tr}(x) = \sum_{i \in I} \langle x e_i, e_i \rangle \in [0, \infty].$$

We call  $x \in \mathcal{B}(\mathcal{H})$  trace class if

$$\|x\| = \text{Tr}(|x|) < \infty.$$

The set of trace class operators on  $\mathcal{H}$  will be denoted by  $L^1(\mathcal{B}(\mathcal{H}), \text{Tr})$ .

*Remark.* By  $\sum_{i \in I} \langle x e_i, e_i \rangle$ , we mean the limit of the net  $a_F = \sum_{i \in F} \langle x e_i, e_i \rangle$ , where  $F \subseteq I$  is a finite subset. Indeed, one can check that finite subsets  $F \subseteq I$  together with the inclusion relation form a directed set. Furthermore, the above net is monotone, therefore it either converges (if it is above bounded) or it goes to infinity. Therefore, it has a generalized limit. If the sum converges, then  $\langle x e_i, e_i \rangle$  is non-zero for at most countable many  $e_i$  and our definition of the sum coincides with the usual sum of the series.

**Lemma 4.7.** For all  $x \in \mathcal{B}(\mathcal{H})$  we have  $\text{Tr}(x^* x) = \text{Tr}(x x^*)$ .

*Proof.*

$$\begin{aligned} \text{Tr}(x^* x) &= \sum_i \langle x^* x e_i, e_i \rangle = \sum_i \langle x e_i, x e_i \rangle \\ &= \sum_i \|x e_i\|^2 = \sum_i \sum_j \langle x e_i, e_j \rangle \overline{\langle x e_i, e_j \rangle} \\ &= \sum_j \sum_i \langle e_i, x^* e_j \rangle \overline{\langle e_i, x^* e_j \rangle} = \sum_j \sum_i \langle x^* e_j, e_i \rangle \overline{\langle x^* e_j, e_i \rangle} \\ &= \sum_j \|x^* e_j\|^2 = \sum_j \langle x^* e_j, x^* e_j \rangle \\ &= \sum_j \langle x x^* e_j, e_j \rangle = \text{Tr}(x x^*) \end{aligned}$$

□

**Corollary 4.8.** If  $x \in \mathcal{B}(\mathcal{H})_+$  and  $u \in \mathcal{U}(\mathcal{H})$ , then

$$\text{Tr}(u^* x u) = \text{Tr}(x).$$

In particular, the trace of a positive operator is independent of the choice of the orthonormal basis for  $\mathcal{H}$ .

*Proof.* Since  $x \in \mathcal{B}(\mathcal{H})_+$ , there exists a  $y \in \mathcal{B}(\mathcal{H})$  such that  $x = y^* y$ . By lemma, we have

$$\begin{aligned} \text{Tr}(x) &= \text{Tr}(y^* y) = \text{Tr}(y y^*) \\ &= \text{Tr}(u^* y^* y u) = \text{Tr}(u^* x u). \end{aligned}$$

If  $(f_i)$  is another ONB for  $\mathcal{H}$ , then there exists  $u \in \mathcal{U}(\mathcal{H})$  such that  $ue_i = f_i$  for all indices  $i$ :

$$\begin{aligned}\sum_i \langle xf_i, f_i \rangle &= \sum_i \langle xue_i, ue_i \rangle \\ &= \sum_i \langle u^* xue_i, e_i \rangle \\ &= \text{Tr}(u^* xu) = \text{Tr}(x).\end{aligned}$$

□

**Definition 4.9.** If  $(e_i)$  is ONB for  $\mathcal{H}$  and  $x \in L^1(\mathcal{B}(\mathcal{H}))$ , then its trace is

$$\text{Tr}(x) := \sum_{i \in I} \langle xe_i, e_i \rangle.$$

By one of the lemmas above, this series is absolutely convergent:

$$2|\text{Tr}(x)| \leq \text{Tr}(|x|) + \text{Tr}(v|x|v^*) \leq \|x\|_1 + \|x\|_1 = 2\|x\|_1.$$

This is the result of the following proof.

**Theorem 4.10.**

- (1.)  $L^1(\mathcal{B}(\mathcal{H}))$  is a two-sided ideal in  $\mathcal{B}(\mathcal{H})$  that is closed under involution.
- (2.)  $L^1(\mathcal{B}(\mathcal{H}))$  is a linear span of all positive operators of finite trace.
- (3.) Trace is independent of the ONB and  $\|\cdot\|_1$  is a norm on  $L^1(\mathcal{B}(\mathcal{H}))$ .

*Proof.* Let  $A, B \in L^1(\mathcal{B}(\mathcal{H}))$  and satisfy the polar decompositions:

$$A + B = U|A + B|, \quad A = V|A|, \quad B = W|B|.$$

Let  $(e_i)$  be an ONB. Then

$$\begin{aligned}\sum_{i=1}^N \langle |A + B|e_i, e_i \rangle &= \sum_{n=1}^N |\langle U^*(A + B)e_n, e_n \rangle| \\ &\leq \sum_{n=1}^N |\langle U^*Ae_n, e_n \rangle| + \sum_{n=1}^N |\langle U^*Be_n, e_n \rangle| \\ &= \sum_{n=1}^N |\langle U^*V|A|e_n, e_n \rangle| + \sum_{n=1}^N |\langle U^*W|B|e_n, e_n \rangle|.\end{aligned}$$



We can bound the first term:

$$\begin{aligned}
\sum_{n=1}^N |\langle U^* V | A | e_n, e_n \rangle| &= \sum_{n=1}^N |\langle |A|^{\frac{1}{2}} e_n, |A|^{\frac{1}{2}} V^* U e_n \rangle| \\
&\leq \sum_{n=1}^N \| |A|^{\frac{1}{2}} e_n \| \| |A|^{\frac{1}{2}} V^* U e_n \| \\
&\leq \left( \sum_{n=1}^N \| |A|^{\frac{1}{2}} e_n \|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^N \| |A|^{\frac{1}{2}} V^* U e_n \|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Since  $\| |A|^{\frac{1}{2}} e_n \|^2 = \langle |A|^{\frac{1}{2}} e_n, |A|^{\frac{1}{2}} e_n \rangle = \langle |A| e_n, e_n \rangle$ , the expression in the first bracket goes to  $\text{Tr } |A|$ . Next, we prove that the expression in the second bracket is less or equal to  $\text{Tr } |A|$ :

$$\sum_{n=1}^N \langle |A|^{\frac{1}{2}} V^* U e_n, |A|^{\frac{1}{2}} V^* U e_n \rangle = \sum_{n=1}^N \langle U^* V | A | V^* U e_n, e_n \rangle \xrightarrow{N \rightarrow \infty} \text{Tr } |A|.$$

Pick an ONB for  $\mathcal{H}$  as follows: each  $f_j$  should be in  $\ker U$  or  $(\ker U)^\perp$ . Then

$$\text{Tr}(U^* V | A | V^* U) \leq \text{Tr}(V | A | V^*).$$

By similar argument,

$$\text{Tr}(V | A | V^*) \leq \text{Tr}(|A|)$$

and we are done:

$$\sum_{n=1}^N |\langle U^* V | A | e_n, e_n \rangle| \leq \text{Tr } |A|.$$

Similarly,

$$\sum_{n=1}^N |\langle U^* W | B | e_n, e_n \rangle| \leq \text{Tr } |B|,$$

which implies  $\text{Tr } |A + B| \leq \text{Tr } |A| + \text{Tr } |B|$ . We have proved that  $L^1(\mathcal{B}(\mathcal{H}))$  is a vector space and  $\| \cdot \|_1$  is a norm. Clearly,  $L^1(\mathcal{B}(\mathcal{H}))$  contains all positive operators with finite trace, so also their linear span. Next we prove that it is a two-sided ideal of  $\mathcal{B}(\mathcal{H})$ . Let  $A \in L^1(\mathcal{B}(\mathcal{H}))$  and  $B \in \mathcal{B}(\mathcal{H})$ . Since every operator is a linear combination of four unitaries, we can assume WLOG that  $B = U$  is a unitary. Then

$$|UA| = (A^* U^* U A)^{\frac{1}{2}} = (A^* A)^{\frac{1}{2}} = |A|,$$

so  $BA = UA \in L^1(\mathcal{B}(\mathcal{H}))$ . Furthermore,

$$|AU| = (U^* A^* A U)^{\frac{1}{2}} = U^* |A| U,$$

which implies

$$\text{Tr } |AU| = \text{Tr}(U^* |A| U) = \text{Tr } |A|$$

and  $AB = AU \in L^1(\mathcal{B}(\mathcal{H}))$ . Now we prove that  $L^1(\mathcal{B}(\mathcal{H}))$  is closed under involution. Let  $A = U|A|$  and  $A^* = V|A^*|$  be polar decompositions. Then

$$|A^*| = V^* A^* = V^* (U|A|)^* = V^* |A| U^*.$$

If  $A \in L^1(\mathcal{B}(\mathcal{H}))$ , then  $|A| \in L^1(\mathcal{B}(\mathcal{H}))$ , so

$$|A^*| = V^*|A|U^* \in L^1(\mathcal{B}(\mathcal{H})).$$

This gives us  $A^* \in L^1(\mathcal{B}(\mathcal{H}))$ . Finally, we prove that  $L^1(\mathcal{B}(\mathcal{H}))$  is a linear span of all positive operators of finite trace. Let  $x \in L^1(\mathcal{B}(\mathcal{H}))$  and  $a \in \mathcal{B}(\mathcal{H})$ . The following glorization identity holds:

$$4a|x| = \sum_{k=0}^3 i^k \underbrace{(a + i^k)|x|(a + i^k)^*}_{\text{positive and finite trace}}.$$

If  $a = v$  partial isometry from the polar decomposition theorem, then

$$x = v|x| = \sum_{k=0}^3 \frac{i^k}{4} (v + i^k)|x|(v + i^k)^*.$$

is a linear combination of four positive operators with finite trace. □

As an obvious corollary, the definition of trace is independent of the choice of an ONB of  $\mathcal{H}$ .

**Proposition 4.11.** *Let  $x \in L^1(\mathcal{B}(\mathcal{H}))$  and  $a, b \in \mathcal{B}(\mathcal{H})$ . Then*

- $\|x\| \leq \|x\|_1$ ;
- $\|axb\|_1 \leq \|a\|\|b\|\|x\|_1$ ;
- $\text{Tr}(ax) = \text{Tr}(xa)$ .

*Proof.* (1.)

$$\begin{aligned} \|x\| &= \||x|\| = \||x|^{\frac{1}{2}}\|^2 \\ &= \sup_{\|\alpha\|=1} \langle |x|^{\frac{1}{2}}\alpha, |x|^{\frac{1}{2}}\alpha \rangle = \sup_{\|\alpha\|=1} \langle |x|\alpha, \alpha \rangle \\ &\leq \text{Tr } |x| = \|x\|_1. \end{aligned}$$

(2.) We begin with

$$|ax|^2 = x^*a^*ax \leq \|a^*a\|x^*x = \|a^*a\| \cdot |x|^2 = \|a\|^2 \cdot |x|^2$$

and since  $|ax| \leq \|a\| \cdot |x|$  we get  $\|ax\|_1 \leq \|a\| \cdot \|x\|_1$ . But  $\|x\|_1 = \|x^*\|_1$ , we also get  $\|xb\|_1 \leq \|b\| \cdot \|x\|_1$ .

(3.) WLOG  $a = u \in \mathcal{U}(\mathcal{H})$ . Then

$$\begin{aligned} \text{Tr}(xu) &= \sum_i \langle xue_i, e_i \rangle = \sum_i \langle xue_i, u^*ue_i \rangle \\ &= \sum_i \langle uxue_i, ue_i \rangle = \text{Tr}(ux). \end{aligned} \quad \square$$

*Remark.* We have the following identities:

- (1.)  $\text{Tr}(\alpha \otimes \bar{\beta}) = \langle \alpha, \beta \rangle$ ;
- (2.)  $\mathcal{F}(\mathcal{H})$  is dense in  $(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$ . Of course, we know that  $\mathcal{F}(\mathcal{H})$  is dense in  $(\mathcal{K}(\mathcal{H}), \|\cdot\|)$ .

**Theorem 4.12.**

$(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$  is a Banach space.

*Proof.* We only have to prove completeness. Let  $(x_n)_n$  be a Cauchy sequence in  $(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$ . Since  $\|\cdot\| \leq \|\cdot\|_1$ ,  $(x_n)$  is a Cauchy sequence in  $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ . But  $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$  is a Banach space, so there exists  $x \in \mathcal{B}(\mathcal{H})$  such that  $x_n \rightarrow x$  in  $\|\cdot\|$ -topology. Notice that

$$x^*x - x_n^*x_n = x^*(x - x_n) + (x - x_n)^*x_n.$$

By continuity of continuous functional calculus, this implies  $|x_n| \rightarrow |x|$ , meaning  $\||x_n| - |x|\| \rightarrow 0$ . Next we prove that  $x \in L^1(\mathcal{B}(\mathcal{H}))$ . For any ONB  $(e_i)_i$ , we have

$$\sum_{i=1}^k \langle |x|e_i, e_i \rangle = \lim_{n \rightarrow \infty} \sum_{i=1}^k \langle |x_n|e_i, e_i \rangle \leq \lim_{n \rightarrow \infty} \text{Tr } |x_n| = \lim_{n \rightarrow \infty} \|x_n\|_1 < \infty.$$

Here, we used the fact that  $\|x_n - x_k\|_1 \geq \|x_n\|_1 - \|x_k\|_1$ , so the sequence  $(\|x_n\|_1)_n$  is Cauchy and therefore has a limit. This proves that  $x \in L^1(\mathcal{B}(\mathcal{H}))$  and  $\|x\|_1 \leq \lim_{n \rightarrow \infty} \|x_n\|_1$ . Finally, we have to show that  $\|x_n - x\|_1 \rightarrow 0$ . Let  $\varepsilon > 0$ . Pick  $N \in \mathbb{N}$  such that for every  $n > N$ , we get  $\|x_n - x_N\|_1 < \frac{\varepsilon}{3}$ . Let  $\mathcal{H}_0 \subseteq \mathcal{H}$  be a finite dimensional subspace such that

$$\|x_N P_{\mathcal{H}_0^\perp}\|_1, \|x P_{\mathcal{H}_0^\perp}\|_1 < \frac{\varepsilon}{3}.$$

Then for every  $n > N$ , we have

$$\begin{aligned} \|x - x_n\|_1 &\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|(x - x_n)P_{\mathcal{H}_0^\perp}\|_1 \\ &\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|x P_{\mathcal{H}_0^\perp} - x_N P_{\mathcal{H}_0^\perp}\|_1 + \|x_N P_{\mathcal{H}_0^\perp} - x_n P_{\mathcal{H}_0^\perp}\|_1 \\ &\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|x_N - x_n\|_1 \|P_{\mathcal{H}_0^\perp}\| \\ &= \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \varepsilon \\ &\leq \|(x - x_n)\| \|P_{\mathcal{H}_0}\|_1 + \varepsilon \xrightarrow{n \rightarrow \infty} \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, this shows  $x_n \xrightarrow{\|\cdot\|_1} x$ . □

**Theorem 4.13.**

The map

$$\Psi : \mathcal{B}(\mathcal{H}) \rightarrow L^1(\mathcal{B}(\mathcal{H}))^*, \quad a \mapsto (\psi_a : x \mapsto \text{Tr}(ax))$$

is an isometric isomorphism of Banach spaces.

**Corollary 4.14.** The map

$$\Phi : L^1(\mathcal{B}(\mathcal{H})) \rightarrow \mathcal{K}(\mathcal{H})^*, \quad x \mapsto (\varphi_x : a \mapsto \text{Tr}(ax))$$

is an isometric isomorphism of Banach spaces.

*Proof of the theorem.* We notice that  $\Psi$  is linear and a contraction because the norms  $\|\cdot\|$  and  $\|\cdot\|_1$  are comparable. We will first show that  $\Psi$  is surjective. Let  $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$ . Notice that

$$(\alpha, \beta) \mapsto \varphi(\alpha \otimes \bar{\beta})$$

is a bounded sesquilinear form in  $\mathcal{H}$ . By the introductory course, there exists an  $a \in \mathcal{B}(\mathcal{H})$  such that

$$\varphi(\alpha \otimes \bar{\beta}) = \langle a\alpha, \beta \rangle = \text{Tr}(a\alpha \otimes \bar{\beta}) = \text{Tr}(a(\alpha \otimes \bar{\beta})) = \psi_a(\alpha \otimes \bar{\beta}).$$

So  $\varphi$  and  $\psi_a$  agree on  $\mathcal{F}(\mathcal{H})$ , so by bounded density  $\varphi = \psi_a$ . Finally,

$$\|a\| = \sup_{\alpha, \beta \in (\mathcal{H})_1} |\langle a\alpha, \beta \rangle| = \sup_{\alpha, \beta \in (\mathcal{H})_1} |\text{Tr}(a(\alpha \otimes \bar{\beta}))| \leq \|\psi_a\|_1.$$

But since

$$\|\psi_a\|_1 = \sup_{x \in (L^1(\mathcal{B}(\mathcal{H})))_1} |\text{Tr}(ax)| = \sup_{x \in (L^1(\mathcal{B}(\mathcal{H})))_1} \|ax\|_1 \leq \sup_{x \in (L^1(\mathcal{B}(\mathcal{H})))_1} \|a\| \|x\|_1 = \|a\|,$$

we have  $\|a\| = \|\psi_a\|_1$  and  $\psi$  is isometric.  $\square$

Recall that  $T \in \mathcal{B}(\mathcal{H})$  is compact if  $T((\mathcal{H})_1)$  is relatively compact. Equivalently, image under  $T$  of a bounded sequence in  $\mathcal{H}$  has a convergent subsequence. For Hilbert spaces, this is also equivalent to  $T \in \overline{\mathcal{F}(\mathcal{H})}$  in the norm  $\|\cdot\|$ . For compact operators in Hilbert space, we have the singular value decomposition. For  $K \in \mathcal{K}(\mathcal{H})$ , then there exists an orthonormal basis  $(e_i)_i$  and  $(f_j)_j$  for  $\mathcal{H}$  and  $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$  such that

$$Kx = \sum_{n=1}^{\infty} \sigma_n \langle x, e_n \rangle f_n.$$

This implies

$$|K|x = \sum \sigma_n \langle x, e_n \rangle e_n.$$

#### Theorem 4.15.

- (1.)  $L^1(\mathcal{B}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$ .
- (2.)  $K \in \mathcal{H}$  is a  $L^1(\mathcal{B}(\mathcal{H}))$  iff  $\sum_{k=1}^{\infty} \sigma_k < \infty$ .

*Proof.* (1.) If  $x \in L^1(\mathcal{B}(\mathcal{H})) = \mathcal{F}$  then there exists  $(x_n)_n$  in  $\mathcal{F}(\mathcal{H})$  such that  $\|x_n - x\|_1 \rightarrow 0$ .

Since  $\|\cdot\| \leq \|\cdot\|_1$ , we get  $\|x_n - x\| \rightarrow 0$  and  $x \in \overline{\mathcal{F}}^{\|\cdot\|} = \mathcal{K}(\mathcal{H})$ .

(2.) This follows from  $\text{Tr } |K| = \sum \sigma_n$ .  $\square$

## 4.2 Hilbert-Schmidt operators

**Definition 4.16.** An element  $x \in \mathcal{B}(\mathcal{H})$  is a Hilbert-Schmidt operator if

$$|x|^2 = x^*x \in L^1(\mathcal{B}(\mathcal{H})).$$

The set of all such elements is denoted by  $L^2(\mathcal{B}(\mathcal{H}), \text{Tr})$ .

**Proposition 4.17.** (1.)  $L^2(\mathcal{B}(\mathcal{H})) \triangleleft \mathcal{B}(\mathcal{H})$  and is closed under  $*$ .

(2.) If  $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ , then  $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$  and  $\text{Tr}(xy) = \text{Tr}(yx)$ .

*Remark.* Beware:  $\exists a, b \in \mathcal{B}(\mathcal{H})$  such that  $ab \in L^1(\mathcal{B}(\mathcal{H}))$  and  $ba \notin L^1(\mathcal{B}(\mathcal{H}))$ . However, if  $ab, ba \in L^1(\mathcal{B}(\mathcal{H}))$ , then  $\text{Tr}(ab) = \text{Tr}(ba)$ .

*Proof.* For  $\alpha \in \mathbb{C}$  and  $x \in \mathcal{B}(\mathcal{H})$ , we have  $|\alpha x|^2 = |\alpha|^2 |x|^2$ . Similarly,  $|x + y|^2 \leq |x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$ , so  $L^2(\mathcal{B}(\mathcal{H}))$  is a complex vector space. Since  $|ax|^2 \leq \|a\|^2 \cdot |x|^2$ , we have  $L^2(\mathcal{B}(\mathcal{H}))$  is a left ideal of  $\mathcal{B}(\mathcal{H})$ . From

$$\text{Tr}|x|^2 = \text{Tr}(x^*x) = \text{Tr}(xx^*) = \text{Tr}|x^*|^2,$$

we deduce that  $L^2(\mathcal{B}(\mathcal{H}))$  is closed under involution. If  $x \in L^2(\mathcal{B}(\mathcal{H}))$  and  $b \in \mathcal{B}(\mathcal{H})$ , then  $x^* \in L^2$ , which implies  $b^*x^* \in L^2$  and finally  $xb = (b^*x^*)^* \in L^2(\mathcal{B}(\mathcal{H}))$ , so  $L^2(\mathcal{B}(\mathcal{H})) \triangleleft \mathcal{B}(\mathcal{H})$ . Next, we use the polarization identity

$$4y^*x = \sum_{k=0}^3 i^k |x + i^k y|^2.$$

If  $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ , then this shows  $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$  and

$$\begin{aligned} 4 \text{Tr}(y^*x) &= \sum_{k=0}^3 i^k \text{Tr}((x + i^k y)^*(x + i^k y)) \\ &= \sum_{k=0}^3 i^k \text{Tr}((x + i^k y)(x + i^k y)^*) \\ &= 4 \text{Tr}(xy^*). \end{aligned}$$

On  $L^2(\mathcal{B}(\mathcal{H}))$  we have the sesquilinear form  $\langle x, y \rangle_2 := \text{Tr}(y^*x)$ . It is well-defined and positive definite, so it is a scalar product. The induced norm is denoted as  $\|\cdot\|_2$ . For every  $y \in L^2(\mathcal{B}(\mathcal{H}))$ , we have

$$\|y\| = \|y^*y\|^{\frac{1}{2}} \leq \|y^*y\|_1^{\frac{1}{2}} = \|y\|_2.$$

Similarly, we have

$$\|axb\|_2 = \|a\| \cdot \|x\|_2 \cdot \|b\|$$

for all  $x \in L^2(\mathcal{B}(\mathcal{H}))$  and  $a, b \in \mathcal{B}(\mathcal{H})$ . As before,  $\mathcal{F}(\mathcal{H})$  are dense in  $L^2(\mathcal{B}(\mathcal{H}))$  with regards to  $\|\cdot\|_2$  and  $L^2(\mathcal{B}(\mathcal{H}))$ . Using singular values  $(\sigma_n)_n$  of compact  $K \in \mathcal{K}(\mathcal{H})$ , we have  $K \in L^2(\mathcal{B}(\mathcal{H}))$  iff  $\sum_{j=0}^{\infty} \sigma_j^2 < \infty$ . For every  $x \in L^1(\mathcal{B}(\mathcal{H}))$ , we have

$$\|x\|_2 = \sup_{y \in L^2(\mathcal{B}(\mathcal{H})), \|y\|_2=1} |\text{Tr}(y^*x)| \leq \sup_{y \in L^2(\mathcal{B}(\mathcal{H})), \|y\|_2=1} \|y\| \cdot \|x\|_1 \leq \|x\|_1.$$

As a result  $(L^2(\mathcal{B}(\mathcal{H})), \langle \cdot, \cdot \rangle_2)$  is a Hilbert space.  $\square$

**Theorem 4.18** (Hölder inequality).

For all  $x, y \in L^2(\mathcal{B}(\mathcal{H}))$  we have

$$\|xy\|_1 \leq \|x\|_2 \|y\|_2.$$

*Proof.*

$$\begin{aligned} \|xy\|_1 &= \operatorname{Tr} |xy| = |\operatorname{Tr}(v^* xy)| \\ &= |\langle y, x^* v \rangle_2| \leq \|x^* v\|_2 \|y\|_2 \\ &\leq \|x^*\|_2 \|v\| \|y\|_2 \leq \|x\|_2 \cdot \|y\|_2. \end{aligned}$$

□

### 4.3 Hilbert-Schmidt integral operators

Let  $(X, \mu)$  be a  $\sigma$ -finite measure space. This means that  $X$  is a countable union of finite-measure sets:

$$X = \bigcup_{j=1}^{\infty} A_j, \quad \mu(A_j) < \infty.$$

For  $K \in L^2(X \times X, \mu \times \mu)$ , then we can define a Hilbert-Schmidt integral operator with kernel  $K$ :

$$T_K : L^2(X, \mu) \rightarrow L^2(X, \mu), \quad f \mapsto \left( y \mapsto \int_X K(x, y) f(x) d\mu(x) \right).$$

Suppose  $(\varphi_\alpha)_\alpha$  is an ONB for  $L^2(K, \mu)$ . By Fubini,  $(\overline{\varphi_\alpha(x)} \varphi_\beta(y))_{\alpha, \beta}$  is an orthonormal basis for  $L^2(X \times X, \mu \times \mu)$ . Since  $K \in L^2(X \times X, \mu \times \mu)$ , there exist  $c_{ij} \in \mathbb{C}$  such that

$$K(x, y) = \sum_{i, j} c_{ij} \overline{\varphi_i(x)} \varphi_j(y), \quad \|K\|_{L^2(X \times X)}^2 = \sum |c_{ij}|^2 < \infty.$$

We show that  $T_K$  is well-defined: for  $f \in L^2(X, \mu)$ , we have  $T_K f \in L^2(X, \mu)$ . Indeed,

$$T_K f(y) = \sum_{i, j} c_{ij} \langle f, \varphi_i \rangle \varphi_j(y),$$

which implies

$$\begin{aligned} \|T_K f\|_{L^2(X)}^2 &\leq \sum_{i, j} |c_{ij}|^2 |\langle f, \varphi_j \rangle|^2 \|\varphi_j\|_{L^2(X)}^2 \\ &\leq \|f\|_{L^2}^2 \sum_{i, j} |c_{ij}|^2 \|\varphi_i\|_{L^2}^2 \|\varphi_j\|_{L^2}^2 \\ &= \|f\|_{L^2}^2 \sum |c_{ij}|^2 \\ &= \|f\|_{L^2}^2 \|K\|_{L^2(X \times X)}^2 \end{aligned}$$

and finally  $\|T_K\| \leq \|K\|_{L^2}$ . Suppose that  $T_K^* : L^2(X, \mu) \rightarrow L^2(X, \mu)$  is the integral operator with kernel

$$K^*(y, x) := \overline{K(x, y)}.$$

Now

$$\begin{aligned}
\langle T_K f, g \rangle &= \int_Y \left( \int_X K(x, y) f(x) d\mu(x) \right) \cdot \overline{g(y)} d\mu(y) \\
&= \int_X f(x) \cdot \left( \int_Y \overline{K(x, y)} g(y) d\mu(y) \right) d\mu(x) \\
&= \langle f, T_{K^*} g \rangle.
\end{aligned}$$

*Remark* (Fubini's theorem). If  $X, Y$  are measure spaces and  $\int_{X \times Y} |f| d(x, y) < \infty$ , then

$$\int_{X \times Y} f = \int_Y \left( \int_X f dx \right) dy = \int_X \left( \int_Y f dy \right) dx.$$

#### Theorem 4.19.

- (1.) For  $K \in L^2(X \times X, \mu \times \mu)$  we have  $T_K \in L^2(\mathcal{B}(L^2(X, \mu)))$ .  
(2.) The mapping  $\Phi : K \mapsto T_K$  is a unitary

$$L^2(X \times X, \mu \times \mu) \mapsto L^2(\mathcal{B}(L^2(X, \mu))).$$

*Proof.* (1.) We will prove that  $\|T_K\|_2 = \|K\|_{L^2}$ . We want to approximate  $T_K$  with finite rank operators, so we first approximate  $K$ :

$$K(x, y) = \sum_{i,j=1}^{\infty} c_{ij} \overline{\varphi_i(x)} \varphi_j(y)$$

for orthonormal basis  $(\varphi_\alpha)_\alpha$  for  $L^2(X, \mu)$ . For  $N \in \mathbb{N}$  let  $K_N(x, y) = \sum_{i,j=1}^N c_{ij} \overline{\varphi_i(x)} \varphi_j(y)$ . Then

$$T_{K_N} f = \sum_{i,j=1}^N c_{ij} \langle f, \varphi_i \rangle \varphi_j \in \mathcal{F}(L^2(X, \mu)).$$

By the above inequality,

$$\|T_K - T_{K_N}\| \leq \|K - K_N\|_{L^2} \rightarrow 0,$$

so  $T_K \in \overline{\mathcal{F}}^{\|\cdot\|} = \mathcal{K}(\mathcal{H})$ . Then

$$\|T_K\|_2^2 = \sum_i \|T_K \varphi_i\|_{L^2}^2 = \sum_{i,j,k} \|c_{jk} \varphi_j(x) \delta_{ik}\|^2 = \sum |c_{ij}|^2 = \|K\|_{L^2}^2.$$

- (2.) It remains to prove surjectivity. Since  $\Phi$  is isometric,  $\text{im } \Phi$  is closed. So it suffices to show that  $\text{im } \Phi$  is dense. In particular, we will show that  $\text{im } \Phi \supseteq \mathcal{F}(L^2(X, \mu))$ . Let  $A \in \mathcal{F}(L^2(X, \mu))$ , so  $\text{rank } A < \infty$ . Let  $(\psi_1, \dots, \psi_m)$  be an orthonormal basis for  $\text{im } A$ . Then  $A\varphi = c_1(\varphi)\psi_1 + \dots + c_m(\varphi)\psi_m$  for some bounded linear functionals  $c_j$  on

$L^2(X, \mu)$ . By Riesz, there exist  $\mu_j \in L^2(X, \mu)$  such that  $c_j(\varphi) = \langle \varphi, \mu_j \rangle$ . Hence

$$A\varphi(x) = \int_X \left( \sum_{j=1}^m \psi_j(x) \cdot \overline{\mu_j(y)} \cdot \varphi(y) \right) d\mu(y) = T_{\sum_{j=1}^m \psi_j(x) \overline{\mu_j(y)}} \in \text{im } \Phi. \quad \square$$

Now let  $\mathcal{H}, \mathcal{K}$  be Hilbert spaces and  $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ . We associate to  $A$  the map

$$\tilde{A} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}, \mathcal{H} \oplus \mathcal{K}), \quad \alpha \oplus \beta \mapsto 0 \oplus A\alpha$$

or in matrix form

$$\tilde{A} = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}.$$

We denote the set of Hilbert-Schmidt operators  $\mathcal{H} \mapsto \mathcal{K}$  as

$$HS(\mathcal{H}, \mathbb{C}) = \{A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \mid \tilde{A} \in L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))\}.$$

Using this notation,  $HS(\mathcal{H}, \mathbb{C})$  is just the dual of  $\mathcal{H}$ , so  $\mathcal{H}^*$ . By Riesz, we have a natural anti-isomorphism

$$\mathcal{H} \rightarrow \mathcal{H}^*, \quad \alpha \mapsto \langle \cdot, \alpha \rangle =: \bar{\alpha}$$

and a natural anti-linear bijection

$$\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}^*), \quad A \mapsto (\bar{A} : \bar{\alpha} \mapsto \overline{A\alpha}).$$

We denote the Hilbert space  $\mathcal{H} \bar{\otimes} \mathcal{K} := HS(\mathcal{K}^*, \mathcal{H})$ . This is called the tensor product of Hilbert spaces. If  $(e_\alpha)_\alpha$  is an ONB for  $\mathcal{H}$  and  $(f_\beta)_\beta$  for  $\mathcal{K}$ , then  $(e_\alpha \otimes \bar{f}_\beta)_{\alpha, \beta}$  is an ONB for  $\mathcal{H} \bar{\otimes} \mathcal{K}$ . The algebraic tensor product  $\mathcal{H} \otimes \mathcal{K}$  are exactly the operators in  $HS(\mathcal{K}^*, \mathcal{H})$  that are of finite rank, which are  $\text{span}\{\varphi \otimes \bar{\psi} \mid \varphi \in \mathcal{H}, \psi \in \mathcal{K}\}$ .

#### 4.4 Locally convex topologies on $\mathcal{B}(\mathcal{H})$

If  $\mathcal{H}$  is a Hilbert space, then  $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$  is a Banach algebra with its norm topology.

**Definition 4.20.** (1.) The weak operator topology (WOT) is given by the seminorms

$$T \mapsto |\langle T\alpha, \beta \rangle|, \quad \forall \alpha, \beta \in \mathcal{H}.$$

(2.) The strong operator topology (SOT) is given by the seminorms

$$T \mapsto \|T\alpha\|, \quad \forall \alpha \in \mathcal{H}.$$

These topologies are comparable:  $\text{WOT} \subseteq \text{SOT} \subseteq \text{uniform}$ .

- Uniform topology has the subbasis

$$\{S \in \mathcal{B}(\mathcal{H}) \mid \|S - T\| < \varepsilon\}$$

for  $T \in \mathcal{B}(\mathcal{H})$  and  $\varepsilon > 0$ . The net  $T_i$  converges to  $T$  iff  $\|T_i - T\|$  converges to 0.



- WOT topology has the subbasis

$$\{S \in \mathcal{B}(\mathcal{H}) \mid \langle (S - T)\alpha, \beta \rangle < \varepsilon\}$$

for  $\alpha, \beta \in \mathcal{H}$ ,  $T \in \mathcal{B}(\mathcal{H})$  and  $\varepsilon > 0$ . The net  $T_i$  converges to  $T$  iff  $\langle T_i \alpha, \beta \rangle$  converges to  $\langle T \alpha, \beta \rangle$  for all  $\alpha, \beta$ .

- SOT topology has the subbasis

$$\{S \in \mathcal{B}(\mathcal{H}) \mid \|(S - T)\alpha\| < \varepsilon\}$$

for  $\alpha \in \mathcal{H}$ ,  $T \in \mathcal{B}(\mathcal{H})$  and  $\varepsilon > 0$ . The net  $T_i$  converges to  $T$  iff  $\|(T_i - T)\alpha\|$  converges to 0 for all  $\alpha$ .

**Example 4.21.** Let  $\mathcal{H} = \ell^2(\mathbb{N})$  and denote  $T_n = \frac{1}{n} \circ \text{id}$ . Then  $T_n \rightarrow 0$  in the norm topology. Now if we introduce the operator

$$S(x_1, x_2, \dots) = (0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots),$$

then  $S_n \rightarrow 0$  in SOT, but not in norm topology, since  $\|S_n\| = 1$ . Lastly, we define

$$W_n(x_1, x_2, \dots) = (0, 0, \dots, x_1, x_2, \dots).$$

We get that  $W_n \rightarrow 0$  in WOT, but not in SOT and norm topology.

**Example 4.22.** Let  $(y_n)_n$  be a countable dense subset of  $\mathcal{H} = \ell^2$ . Consider the following metrics on  $(\mathcal{B}(\mathcal{H}))_1$ :

$$d_S(A, B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|(A - B)y_n\|, \quad d_W(A, B) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle (A - B)y_n, y_n \rangle|.$$

Then  $d_S$  induces SOT and  $d_W$  induces WOT on  $(\mathcal{B}(\mathcal{H}))_1$ .

**Example 4.23.** The multiplication

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}), \quad (A, B) \mapsto A \cdot B$$

is not jointly continuous with regards to SOT or WOT. Indeed, if  $S : \ell^2 \rightarrow \ell^2$  is right shift (and  $S^*$  left shift), then  $S^n \rightarrow 0$  and  $(S^*)^n \rightarrow 0$  in WOT, but  $(S^*)^n S^n = I$ . However, multiplication is WOT- and SOT- continuous in each factor separately. Suppose that  $(x_\alpha)_\alpha \rightarrow x$  in WOT and  $y \in \mathcal{B}(\mathcal{H})$ . Then for each  $v, w \in \mathcal{B}(\mathcal{H})$ , we have

$$|\langle x_\alpha y v - x y v, w \rangle| \rightarrow 0,$$

since  $x_\alpha \rightarrow x$  in WOT. Similarly,

$$|\langle y x_\alpha v - y x v, w \rangle| = |\langle x_\alpha v - x v, y^* w \rangle| \rightarrow 0,$$

which implies  $x_\alpha y \rightarrow xy$  and  $y x_\alpha \rightarrow yx$  in WOT.

**Example 4.24.** The adjoint is isometric in uniform topology. It is also continuous in WOT:

$$|\langle x^*v - y^*v, w \rangle| < \varepsilon \Leftrightarrow |\langle xw - yw, v \rangle| < \varepsilon.$$

However, it is not continuous with regards to SOT. If  $(e_n)_n$  is an ONB for  $\mathcal{H}$ , consider  $e_1 \otimes \overline{e_n}$ . Then for every  $x \in \mathcal{H}$ , we have

$$\|(e_1 \otimes \overline{e_n})x\| = |\langle x, e_n \rangle| \xrightarrow{n \rightarrow \infty} 0,$$

so  $e_1 \otimes \overline{e_n} \rightarrow 0$  in SOT. However,

$$\|(e_1 \otimes \overline{e_n})^*x\| = \|(e_n \otimes \overline{e_1})x\| = |\langle x, e_1 \rangle|$$

does not go to 0 for all  $x \in \mathcal{H}$ , which proves our statement.

*Remark.* If  $T : X \rightarrow Y$  is continuous, then  $T$  remains continuous if  $X$  is given a finer topology or  $Y$  is given a coarser topology. But if both topologies are made coarser or both finer, nothing can be said in general. In particular, if  $T : X \rightarrow X$  is continuous with respect to a given topology on  $X$  in both domain and codomain, you cannot generally conclude anything about continuity of  $T$  when  $X$  is given a finer or coarser topology on both domain and codomain. The previous example illustrates this.

**Lemma 4.25.** Let  $\varphi : \mathcal{B} \rightarrow \mathbb{C}$  be linear. The following is equivalent.

(1.) There exist  $v_1, \dots, v_n \in \mathcal{H}$  and  $w_1, \dots, w_n \in \mathcal{H}$  such that

$$\varphi(T) = \sum_{i=1}^n \langle Tv_i, w_i \rangle.$$

(2.)  $\varphi$  is WOT-continuous.

(3.)  $\varphi$  is SOT-continuous.

*Proof.* The implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) is obvious. Let us prove (3)  $\Rightarrow$  (1). By a proposition in chapter 1 (the application of Hahn-Banach), there exists a  $K > 0$  and  $v_1, \dots, v_n \in \mathcal{H}$  such that

$$|\varphi(T)|^2 \leq K \cdot \sum_{i=1}^n \|Tv_i\|^2.$$

Define

$$\mathcal{H}_0 := \overline{\left\{ \bigoplus_{i=1}^n Tv_i \mid T \in \mathcal{B}(\mathcal{H}) \right\}} \leq \mathcal{H}^{\oplus n}.$$

The map

$$\mathcal{H}_0 \ni \bigoplus_{i=1}^n Tv_i \mapsto \varphi(T) \in \mathbb{C}$$

is a well-defined and bounded linear functional, which by continuity extends to  $\mathcal{H}_0 \rightarrow \mathbb{C}$ . By

Riesz, there exist  $w_1, \dots, w_n \in \mathcal{H}$  such that

$$\varphi(T) = \sum_{i=1}^n \langle Tv_i, w_i \rangle.$$

Recall that  $v \otimes \bar{w} \in \mathcal{FH}$  and  $\text{Tr}(v \otimes \bar{w}) = \langle v, w \rangle$ , so

$$\text{Tr}(T(v \otimes \bar{w})) = \langle Tv, w \rangle.$$

The previous identity is really

$$\varphi(T) = \sum_{i=1}^n \text{Tr}(T(v \otimes \bar{w})) = \text{Tr}(T \cdot \sum_{i=1}^n v_i \otimes \bar{w}_i).$$

This means that  $\varphi(T) = \text{Tr}(T \cdot A)$  for  $A \in \mathcal{F}(\mathcal{H})$ . □

**Corollary 4.26.** *If  $K \subseteq \mathcal{B}(\mathcal{H})$  is convex, then*

$$\overline{K}^{\text{WOT}} = \overline{K}^{\text{SOT}}.$$

*Proof.*

$$\overline{K}^{\text{WOT}} = \overline{K}^{\text{weak, WOT}} = \overline{K}^{\text{weak, SOT}} = \overline{K}^{\text{SOT}} \quad \square$$

**Definition 4.27.** The  $\sigma$ -weak operator topology ( $\sigma$ -WOT or ultra-weak) is the topology in  $\mathcal{B}(\mathcal{H})$  given by the seminorms

$$x \mapsto \left| \sum_{i=1}^{\infty} \langle x\alpha_i, \alpha_i \rangle \right|$$

for  $\alpha_j \in \mathcal{H}$  with  $\sum_{j=1}^{\infty} \|\alpha_j\|^2 < \infty$ . A subbasis of open sets is thus

$$\left\{ x \in \mathcal{B}(\mathcal{H}) \mid \left| \sum \langle (x - x_0)\alpha_i, \alpha_i \rangle \right| < \varepsilon \right\}$$

for  $\alpha_j \in \mathcal{H}$  with  $\varepsilon > 0$ ,  $x_0 \in \mathcal{B}(\mathcal{H})$  and  $\sum \|\alpha_j\|^2 < \infty$ .

**Definition 4.28.** The  $\sigma$ -strong operator topology ( $\sigma$ -SOT or ultra-strong) is the topology in  $\mathcal{B}(\mathcal{H})$  given by the seminorms

$$x \mapsto \left( \sum_{i=1}^{\infty} \|x\alpha_i\|^2 \right)^{\frac{1}{2}}$$

for  $\alpha_j \in \mathcal{H}$  with  $\sum_{j=1}^{\infty} \|\alpha_j\|^2 < \infty$ . A subbasis of open sets is thus

$$\left\{ x \in \mathcal{B}(\mathcal{H}) \mid \left( \sum \|(x - x_0)\alpha_i\|^2 \right)^{\frac{1}{2}} < \varepsilon \right\}$$

for  $\alpha_j \in \mathcal{H}$  with  $\varepsilon > 0$ ,  $x_0 \in \mathcal{B}(\mathcal{H})$  and  $\sum \|\alpha_j\|^2 < \infty$ .

*Remark.*  $\sigma$ -WOT can also be given by seminorms

$$x \mapsto |\operatorname{Tr}(xa)|$$

for  $a \in L^1(\mathcal{B}(\mathcal{H}))$  positive. Let  $(f_i)_i$  be an ONB for  $\mathcal{H}$  and define

$$b : \mathcal{H} \rightarrow \mathcal{H}, \quad f_i \mapsto \alpha_i.$$

Since  $\sum \|\alpha_j\|^2 < \infty$ , we can conclude  $b \in L^2(\mathcal{B}(\mathcal{H}))$ . Then:

$$\begin{aligned} \sum_i \langle x\alpha_i, \alpha_i \rangle &= \sum_i \langle xbf_i, bf_i \rangle \\ &= \sum_i \langle b^* xbf_i, f_i \rangle \\ &= \operatorname{Tr}(b^* xb) \\ &= \operatorname{Tr}(xbb^*), \end{aligned}$$

where  $a := bb^* \in L^1(\mathcal{B}(\mathcal{H}))$ . Since  $\mathcal{B}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))^*$ , the  $\sigma$ -WOT is just the weak-\* topology (with regards to this pairing).

*Remark.* The map

$$\operatorname{id} \otimes 1 : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2), \quad x \mapsto x \otimes 1$$

is an isometric \*-isomorphism of  $C^*$ -algebras. It is neither SOT- nor WOT-continuous. Despite that,  $\sigma$ -WOT on  $\mathcal{B}(\mathcal{H})$  is induced by WOT on  $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2)$  and the  $\sigma$ -SOT on  $\mathcal{B}(\mathcal{H})$  is induced by SOT on  $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2)$ . Indeed, if  $(e_i)_{i \in \mathbb{N}}$  is an ONB for  $\ell^2$ , define  $\alpha := \sum_{i=1}^{\infty} \alpha_i \otimes e_i \in \mathcal{H} \overline{\otimes} \ell^2$ . Then

$$\sum_{i \in \mathbb{N}} \langle x\alpha_i, \alpha_i \rangle_{\mathcal{H}} = \langle (\operatorname{id} \otimes 1)(x)\alpha, \alpha \rangle_{\mathcal{H} \overline{\otimes} \ell^2}$$

and similarly

$$\left( \sum_{i \in \mathbb{N}} \|x\alpha_i\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} = \|(\operatorname{id} \otimes 1)(x)\alpha\|_{\mathcal{H} \overline{\otimes} \ell^2}$$

**Lemma 4.29.** *Let  $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$  be a linear functional operator. Then the following is equivalent.*

- (1.)  $\exists a \in L^1(\mathcal{B}(\mathcal{H}))$  such that  $\varphi(x) = \operatorname{Tr}(ax)$ ,  $\forall x \in \mathcal{B}(\mathcal{H})$ ;
- (2.)  $\varphi$  is  $\sigma$ -WOT continuous;
- (3.)  $\varphi$  is  $\sigma$ -SOT continuous.

*Proof.* As previously, the implication (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) is obvious. Let us prove (3)  $\Rightarrow$  (1). Assume  $\varphi$  is  $\sigma$ -SOT continuous. By identifying  $\mathcal{B}(\mathcal{H})$  via  $\operatorname{id} \otimes 1$  with a subspace in  $\mathcal{B}(\mathcal{H} \otimes \ell^2)$ ,  $\varphi$  is SOT-continuous on this subspace. By Hahn-Banach,  $\varphi$  extends to a SOT-continuous linear functional on  $\mathcal{B}(\mathcal{H} \otimes \ell^2)$ . By the previous lemma,  $\exists \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathcal{H} \overline{\otimes} \ell^2$ .

$$\varphi(x) = \sum_{i=1}^n \langle (\operatorname{id} \otimes 1)(x)\alpha_i, \beta_i \rangle.$$

With

$$\alpha_i \sum_{j=1}^{\infty} \alpha_{ij} \otimes e_j, \quad \sum_j \|\alpha_{ij}\|^2 < \infty$$

and

$$\beta_i \sum_{j=1}^{\infty} \beta_{ij} \otimes e_j, \quad \sum_j \|\beta_{ij}\|^2 < \infty.$$

Then

$$\begin{aligned} \varphi(x) &= \sum_{i=1}^n \langle (x \otimes 1) \sum_{j=1}^{\infty} \alpha_{ij} \otimes e_j, \sum_{k=1}^{\infty} \beta_{ik} \otimes e_k \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle x \alpha_{ij}, \beta_{ik} \rangle \langle e_j, e_k \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^{\infty} \langle x \alpha_{ij}, \beta_{ij} \rangle. \end{aligned}$$

Define

$$A_i : \mathcal{H} \rightarrow \mathcal{H}, \quad A_i f_k = \alpha_{ik}$$

and

$$B_i : \mathcal{H} \rightarrow \mathcal{H}, \quad B_i f_k = \beta_{ik}$$

for an orthonormal basis  $(f_k)_{k \in \mathbb{N}}$ . By assumption,  $A_i, B_i \in L^2(\mathcal{B}(\mathcal{H}))$ . As before, this gives  $\varphi(x) = \sum_i \text{Tr}(B_i^* x A_i) = \text{Tr}(x A_i B_i^*)$ .  $\square$

**Corollary 4.30.** *The unit disk  $(\mathcal{B}(\mathcal{H}))_1$  is compact with respect to  $\sigma$ -WOT topology.*

*Proof.*  $\sigma$ -WOT on  $\mathcal{B}(\mathcal{H})$  is the weak-\* topology from  $L^1(\mathcal{B}(\mathcal{H}))^* = \mathcal{B}(\mathcal{H})$ . The statement now follows from Banach-Alaoglu.  $\square$

**Corollary 4.31.** *WOT and  $\sigma$ -WOT agree on bounded subsets  $B \subseteq \mathcal{B}(\mathcal{H})$ .*

*Proof.* WLOG  $B$  is closed. Then the identity  $(B, \sigma\text{-WOT}) \rightarrow (B, \text{WOT})$  is a continuous map from a  $T_2$  compact (previous corollary) to a  $T_2$  space. Therefore an identity map is a closed continuous bijection, so a homeomorphism.  $\square$

We use the following notation: if  $A$  is vector space and  $B \subseteq \mathcal{L}(A, \mathbb{C})$  is a set of linear functionals, then  $\sigma(A, B)$  is the weakest topology in  $A$  such that linear functionals in  $B$  are continuous.

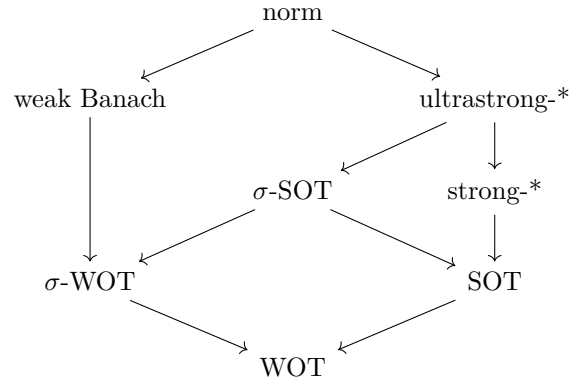
**Definition 4.32.** Let  $A$  be a vector space and  $B \subseteq \mathcal{L}(A, \mathbb{C})$  a set of some of its linear functionals. Then we define  $\sigma(A, B)$  as the weakest topology such that functionals in  $B$  are continuous.

*Remark.*  $\sigma$ -WOT is  $\sigma(\mathcal{B}(\mathcal{H}), L^1(\mathcal{B}(\mathcal{H})))$ .

**Definition 4.33.** Let us define the following topologies on  $\mathcal{B}(\mathcal{H})$ .

- (1.) Weak Banach topology is  $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^*)$ .
- (2.) Ultrastrong-\* topology is the weakest topology stronger than  $\sigma$ -SOT such that \* is continuous.
- (3.) Strong-\* topology is generated by seminorms  $x \mapsto \|x\alpha\|$  and  $x \mapsto \|x^*\alpha\|$  for  $\alpha \in \mathcal{H}$ .

In the end, we get the following diagram which demonstrates which topologies are comparable.



## 5 von Neumann algebras

### 5.1 Bicommutant theorem

**Definition 5.1.** A von Neumann algebra (on Hilbert space  $\mathcal{H}$ ) is a  $*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$  that is WOT-closed. Equivalently, a  $*$ -subalgebra that is SOT-closed.

**Definition 5.2.** If  $A \subseteq \mathcal{B}(\mathcal{H})$ , then  $W^*(A)$  denotes the vNa generated by  $A$ . This is equivalent to the smallest vNa in  $\mathcal{B}(\mathcal{H})$  that contains  $A$ , which is equivalent to

$$\bigcap \{W \mid A \subseteq W, W \subseteq \mathcal{B}(\mathcal{H}) \text{ is vNa}\}.$$

**Lemma 5.3.** Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a vNa. Then  $(A)_1$  is WOT-compact.

*Proof.* Follows from Banach-Alaoglu and  $\sigma$ -WOT and WOT topologies being equivalent on bounded sets.  $\square$

**Corollary 5.4.** Let  $A \subseteq \mathcal{B}(\mathcal{H})$  vNa. Then  $(A)_1$  and  $A_{\text{sa}}$  are SOT-closed and WOT-closed.

*Proof.* By the lemma above,  $*$  is continuous in WOT, so  $A_{\text{sa}}$  is closed in WOT. Since  $A_{\text{sa}}$  is convex, we use Hahn-Banach to see that it is also SOT-closed. The same exact argument applies for  $(A)_1$ .  $\square$

**Definition 5.5.** Commutant of  $B \subseteq \mathcal{B}(\mathcal{H})$  is

$$B' = \{T \in \mathcal{B}(\mathcal{H}) \mid \forall S \in B : ST = TS\}.$$

Bicommutant is  $B'' := (B')'$ . By definition,  $B'' \supseteq B$ .

**Theorem 5.6.**

Suppose  $A \subseteq \mathcal{B}(\mathcal{H})$  is closed under  $*$ . Then  $A'$  is vNa.

*Proof.* Obviously,  $A'$  is a subalgebra of  $\mathcal{B}(\mathcal{H})$  closed under  $*$ . We prove that it is WOT-closed. Let  $(x_\alpha)_\alpha$  be a net in  $A'$  that WOT-converges to  $x \in \mathcal{B}(\mathcal{H})$ . Pick  $a \in A$  and  $\varphi, \mu \in \mathcal{H}$ . Then

$$\begin{aligned} \langle [x, a]\varphi, \mu \rangle &= \langle (xa - ax)\varphi, \mu \rangle \\ &= \langle xa\varphi, \mu \rangle - \langle ax\varphi, \mu \rangle \\ &= \langle xa\varphi, \mu \rangle - \langle x\varphi, a^*\mu \rangle \\ &= \lim_\alpha \langle x_\alpha a\varphi, \mu \rangle - \langle x_\alpha \varphi, a^*\mu \rangle \\ &= \lim_\alpha \langle (x_\alpha a - ax_\alpha)\varphi, \mu \rangle \\ &= \lim_\alpha \langle [x_\alpha, a]\varphi, \mu \rangle = 0, \end{aligned}$$

so  $x \in A'$  and we're done.  $\square$

**Corollary 5.7.** *Every vNa is unital.*

**Example 5.8.** *For an infinitely-dimensional Hilbert space  $\mathcal{H}$ , the set of all compact operators  $\mathcal{K}(\mathcal{H})$  is not a vNa, since it doesn't include the identity (the latter is a consequence of Riesz lemma). Another way: as we'll see later, the finite-rank projections converge strongly to identity,  $\mathcal{K}(\mathcal{H})$  is not SOT-closed, so it is also not WOT-closed.*

**Corollary 5.9.** *Suppose  $A \subseteq \mathcal{B}(\mathcal{H})$  is a maximal commutative subalgebra and is closed under  $*$ . Then  $A$  is vNa.*

*Proof.* Since  $A$  is commutative,  $A' \supseteq A$ . Take  $b \in A' \subseteq A$  and consider the subalgebra, generated by  $A$  and  $b$ . This is an abelian algebra, so by maximality we have  $b \in A$  and  $A = A'$ . Then by theorem,  $A$  is a vNa.  $\square$

**Lemma 5.10.** *Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a  $*$ -subalgebra. Then  $\forall \mu \in \mathcal{H}$  and  $\forall x \in A''$  there exists a net  $(x_\alpha)_\alpha$  in  $A$  such that  $\lim_\alpha \|(x_\alpha - x)\mu\| = 0$ .*

*Proof.* Define  $\mathcal{K} := \overline{A\mu} \leq \mathcal{H}$ . Let  $p : \mathcal{H} \rightarrow \mathcal{K}$  be the orthogonal projection onto  $\mathcal{K}$ . By definition,  $a\mathcal{K} \subseteq \mathcal{K}$ ,  $\forall a \in A$ . Equivalently,  $pap = ap$ . Then

$$pa = (a^*p)^* = (pa^*p)^* = pap = ap,$$

so  $p \in A'$ . But  $x \in A''$ , so

$$xp = xp^2 = pxp$$

and  $x\mathcal{K} \subseteq \mathcal{K}$ . In particular, since  $\mu \in \mathcal{K}$ , we have  $x\mu \in \mathcal{K} = \overline{A\mu}$ . So there must exist some net in  $A\mu$  that converges to  $x\mu$ .  $\square$

**Theorem 5.11** (von Neumann's bicommutant theorem).

*Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a  $*$ -subalgebra. Then  $\overline{A}^{\text{WOT}} = A''$ .*

*Proof.* By the previous theorem,  $A''$  is a vNa. In particular, it is WOT-closed. Since  $A \subseteq A''$ , it suffices to show that  $A$  is WOT-dense in  $A''$ . Because  $A$  is convex, it is enough to show that  $A$  is SOT-dense in  $A''$ . Let  $x \in A''$  and  $\mu_1, \dots, \mu_n \in \mathcal{H}$ . Form the following subalgebra of  $\mathcal{B}(\mathcal{H}^n) \cong M_n(\mathcal{B}(\mathcal{H}))$ . Define

$$\tilde{A} = \left\{ \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \in M_n(\mathcal{B}(\mathcal{H})) \mid a \in A \right\}.$$



Then  $\widetilde{A}' = M_n(A')$ . Hence we get

$$\widetilde{A}'' \subseteq M_n(A')' = \widetilde{A}''.$$

This implies that

$$\begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} \in \widetilde{A}'' \subseteq \widetilde{A}''.$$

Now we apply lemma to  $\widetilde{A}$  to get a net  $(a_i)_i$  in  $A$  such that

$$\lim_i \|(x - a_i)\mu_j\| = 0, \quad \forall j = 1, \dots, n.$$

□

**Corollary 5.12.** *Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a  $*$ -subalgebra. Then  $A$  is a vNa iff  $A = A''$ .*

*Remark.* WOT-closed implies norm-closed. In particular, every vNa is a  $C^*$ -algebra. However, the converse is not always true:  $\mathcal{C}([0, 1])$  is a  $C^*$ -algebra that is not vNa.

**Corollary 5.13** (Polar decomposition in vNa). *Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a vNa and  $x \in A$ . Then for its polar decomposition  $x = v|x|$  we have  $v \in A$ .*

*Proof.* We know that

$$\ker v = (\operatorname{im} |x|)^\perp = \ker |x| = \ker x.$$

For  $a \in A'$  and  $\mu \in \ker x$  we have  $a\mu \in \ker x$ :

$$x(a\mu) = ax\mu = 0,$$

which implies  $a \ker |x| \subseteq \ker |x|$ . We know that  $\mathcal{H} = \ker |x| \oplus \overline{\operatorname{im} |x|}$ . Suppose that  $|x|\mu \in \operatorname{im} |x|$ . Then

$$\begin{aligned} [a, v]|x|\mu &= (av - va)|x|\mu = av|x|\mu - va|x|\mu \\ &= ax\mu - v|x|a\mu = ax\mu - xa\mu \\ &= [a, x]\mu = 0. \end{aligned}$$

But for  $\beta \in \ker |x| = \ker v$ , we have

$$[a, v]\beta = (av - va)\beta = av\beta - va\beta = 0.$$

Since  $av$  and  $va$  agree on  $\ker |x| \oplus \overline{\operatorname{im} |x|} = \mathcal{H}$ , we have  $v \in A'' = A$ .

□

**Example 5.14** (Commutative vNa - IMPORTANT). *Let  $(X, \mu)$  be a  $\sigma$ -finite measure space and*

$$M : L^\infty(X, \mu) \rightarrow B(L^p(X, \mu)), \quad g \mapsto M_g,$$

*where we define*

$$(M_g f)(x) = g(x)f(x).$$

*Then  $M$  is an isometric  $*$ -isomorphism  $L^\infty(X, \mu) \rightarrow M(L^\infty(X, \mu))$  and  $M(L^\infty(X, \mu))$  is*

a maximal commutative  $vNa$  in  $\mathcal{B}(L^2(X, \mu))$ .

*Remark.* A measurable function  $f : X \rightarrow \mathbb{C}$  is essentially bounded if there exists a real number  $M$  (called an essential bound) such that

$$\mu(\{x \in X \mid |f(x)| > M\}) = 0.$$

We define  $\|f\|_\infty$  to be an essential supremum of  $f$ , which is the infimum of its essential bounds. If  $\mathcal{L}^\infty(X, \mu)$  are essentially-bounded functions and  $\mathcal{N} = \{f \mid \|f\|_\infty = 0\}$ . Then  $\|\cdot\|_\infty$  is a norm on the space  $L^\infty(X, \mu) = \mathcal{L}^\infty(X, \mu) / \mathcal{N}$ .

*Proof of the example.* Clearly,  $M$  is injective, additive and multiplicative. First, we prove that  $M$  is a  $*$ -homomorphism. This follows from the next calculation:

$$\begin{aligned} \langle M_{\bar{g}}\mu, \varphi \rangle &= \int_X M_{\bar{g}}\mu \cdot \bar{\varphi} \, d\mu \\ &= \int_X \bar{g}\mu \bar{\varphi} \\ &= \int_X \mu \bar{g}\bar{\varphi} \, d\mu \\ &= \langle \mu, M_g\varphi \rangle = \langle M_g^*\mu, \varphi \rangle, \end{aligned}$$

so  $M_{\bar{g}} = M_g^*$ . Next, we prove that  $M$  is isometric. For  $g \in L^\infty(X, \mu)$ , there exists a sequence  $E_n \subseteq X$  such that  $0 < \mu(E_n) < \infty$  and  $|g|_{E_n} \geq \|g\|_\infty - \frac{1}{n}$  for all  $n \in \mathbb{N}$ . Then

$$\|M_g\| \geq \frac{\|M_g 1_{E_n}\|_2}{\|1_{E_n}\|_2} \geq \|g\|_\infty - \frac{1}{n}, \quad \forall n \in \mathbb{N},$$

which implies  $\|M_g\| \geq \|g\|_\infty$ . For the reverse, notice that

$$\begin{aligned} \|M_g 1_{E_n}\|^2 &= \int_X |g \cdot 1_{E_n}|^2 \, d\mu \\ &= \int_{E_n} |g|^2 \, d\mu \\ &\geq \int_{E_n} (\|g\|_\infty - \frac{1}{n})^2 \, d\mu \\ &= (\|g\|_\infty - \frac{1}{n})^2 \cdot \mu(E_n) \end{aligned}$$

and

$$\begin{aligned} \|M_g\|^2 &= \sup_{\|\mu\|_2=1} \|M_g\mu\|_2^2 = \sup_{\|\mu\|_2=1} \int_X |g\mu|^2 \, d\mu \\ &\leq \|g\|_\infty^2 \cdot \sup_{\|\mu\|_2=1} \int_X |\mu|^2 \, d\mu = \|g\|_\infty^2. \end{aligned}$$

We've just shown that  $\|Mg\| = \|g\|_\infty$ . Lastly, we prove that  $M(L^\infty(X, \mu))$  is a maximal commutative subalgebra of  $\mathcal{B}(L^2(X, \mu))$ . Take  $T \in \mathcal{B}(L^2(X, \mu))$  and assume it commutes with all  $M_g$ 's. Now pick a measurable sequence  $E_n \subseteq X$  such that  $0 < \mu(E_n) < \infty$ ,

$E_n \subseteq E_{n+1}$  and  $X = \bigcup_{n \in \mathbb{N}} E_n$ . Define  $f_n := T(1_{E_n}) \in (X, \mu)$ . First we prove that  $f_n \in L^\infty(X, \mu)$ . If  $A$  is measurable and  $0 < \mu(A) < \infty$ , then

$$\begin{aligned} \frac{1}{\mu(A)} \int_X |f_n \cdot 1_A|^2 d\mu &= \frac{1}{\mu(A)} \cdot \|M_{1_A} T(1_{E_n})\|^2 \\ &= \frac{1}{\mu(A)} \cdot \|T(1_{A \cap E_n})\|^2 \\ &\leq \frac{1}{\mu(A)} \cdot \|T\|^2 \cdot \|1_A\|^2 = \|T\|^2. \end{aligned}$$

If  $f \notin L^\infty$ , then for all  $M \in \mathbb{R}$  we have

$$0 < \mu(\underbrace{\{x \in X \mid |f_n(x)| > M\}}_{A_{n,M}}) < \infty,$$

since  $f_n \in L^2$ . By above calculation,

$$M^2 \leq \frac{1}{\mu(A_{n,M})} \cdot \int_X |f \cdot 1_{A_{n,M}}|^2 d\mu \leq \|T\|^2,$$

which is of course a contradiction. This proves that  $f_n \in L^\infty(X, \mu)$  and  $\|f_n\|_\infty \leq \|T\|$ . For  $n \leq m$  we have

$$\begin{aligned} 1_{E_n} \cdot f_m &= 1_{E_n} \cdot T(1_{E_m}) \\ &= M_{1_{E_n}}(T(1_{E_m})) \\ &= T(M_{1_{E_n}} 1_{E_m}) \\ &= T(1_{E_n} 1_{E_m}) = f_n. \end{aligned}$$

Therefore,  $f_m|_{E_n} = f_n$ . Sequence  $(f_n)_n$  converges to a measurable  $f : X \rightarrow \mathbb{C}$ . From  $\|f_n\|_\infty \leq \|T\|$  for all  $n \in \mathbb{N}$  we also deduce  $\|f\|_\infty \leq \|T\|$ , so  $f \in L^\infty(X, \mu)$ . Lastly, we prove  $T = M_f$ . Note that simple functions  $\sum_{j=1}^r \alpha_j 1_{A_j}$  are  $L^2$ -dense. Let  $A \subseteq X$  be measurable with  $\mu(A) < \infty$ . Then  $\|1_{A \cap E_n} - 1_A\|_2 \xrightarrow{n \rightarrow \infty} 0$ . Hence

$$\|(T - M_f)1_A\|_2 = \lim_{n \rightarrow \infty} \|(T - M_f)1_{A \cap E_n}\|_2 = 0,$$

as we shall prove.

$$\begin{aligned} T(1_{A \cap E_n}) &= T(1_A \cdot 1_{E_n}) = T(M_{1_A} 1_{E_n}) \\ &= M_{1_A}(T(1_{E_n})) = M_{1_A}(f_n) \\ &= 1_A \cdot f_n. \end{aligned}$$

On the other hand,

$$M_f(1_{A \cap E_n}) = f \cdot 1_{A \cap E_n} = f \cdot 1_{E_n} \cdot 1_A = 1_A \cdot f_n$$

and we are done. □

Another possible characterization of vNa's is given by the following.

**Theorem 5.15 (Sakai).**

Let  $A$  be a  $C^*$ -algebra such that for a Banach space there exists an isometric  $*$ -isomorphism  $A \rightarrow E^*$ . Then there exists a vNa  $B \subseteq \mathcal{B}(\mathcal{H})$  such that  $A \cong B$  as a  $C^*$ -algebra.

For the proof, see R.V.Kadison's *The von Neumann algebra characterization theorems* (1985).

## 5.2 Kaplansky's density theorem

**Lemma 5.16.** Multiplication  $(A, B) \mapsto A \cdot B$  is SOT-continuous on bounded sets.

*Proof.* Let  $(A_i)_i$  and  $(B_i)_i$  be nets with  $\sup \|A_i\|, \sup \|B_i\| < M$  for some  $M \in \mathbb{R}$ . Suppose  $A_i \rightarrow A$  and  $B_i \rightarrow B$  in SOT. For any  $x$ , we get

$$\begin{aligned} \|ABx - A_i B_i x\| &= \|ABx - A_i Bx + A_i Bx - A_i B_i x\| \\ &\leq \|ABx - A_i Bx\| + \|A_i Bx - A_i B_i x\| \\ &\leq \|A(Bx) - A_i(Bx)\| + \|A_i\| \cdot \|Bx - B_i x\| \\ &\leq \|A(Bx) - A_i(Bx)\| + M \cdot \|Bx - B_i x\| \rightarrow 0, \end{aligned}$$

so  $A_i B_i \xrightarrow{\text{SOT}} AB$ . □

**Proposition 5.17.** Let  $f \in C(\mathbb{C})$ . Then  $x \mapsto f(x)$  is SOT-continuous on each bounded set of normal operators in  $\mathcal{B}(\mathcal{H})$ .

*Proof.* By Stone-Weierstrass, we can uniformly approximate  $f$  by polynomials on a bounded subset  $B_R(0) \subseteq \mathbb{C}$ . By the previous lemma, multiplication is SOT-continuous on this bounded set of normal operators. But for normal operator  $A$ , we have  $\|Ax\| = \|A^*x\|$  for every  $x \in \mathcal{H}$ , so  $*$  is also SOT-continuous on normal operators and we're done. □

**Theorem 5.18 (Cayley transform).**

Mapping  $x \mapsto (x - i)(x + i)^{-1}$  is SOT-continuous  $\mathcal{B}(\mathcal{H})_{\text{sa}} \rightarrow \mathcal{U}(\mathcal{H})$ .

*Proof.* If  $x \in \mathcal{B}(\mathcal{H})_{\text{sa}}$ , then  $\sigma(x) \subseteq \mathbb{R}$  and  $(x + i) \in \mathcal{B}(\mathcal{H})$  is invertible. We notice that  $z \mapsto \frac{z-i}{z+i} : \mathbb{R} \rightarrow \mathbb{C}$  has its range in  $\mathbb{T}$ , so the Cayley transform does in fact map into the unitaries. Now onto the SOT-continuity: let  $(x_k)_k$  be a net in  $\mathcal{B}(\mathcal{H})_{\text{sa}}$  with  $x_k \rightarrow x$  in SOT.

By spectral mapping theorem,  $\|(x_k + i)^{-1}\| \leq 1$ . For each  $\alpha \in \mathcal{H}$ , we have

$$\begin{aligned}
\|(x - i)(x + i)^{-1}\alpha - (x_k - i)(x_k + i)^{-1}\alpha\| &= \|(x_k + i)^{-1}((x_k + i)(x - i)(x + i)^{-1} - (x_k - i))\alpha\| \\
&= \|(x_k + i)^{-1}((x_k + i)(x - i) - (x_k - i)(x + i))(x + i)^{-1}\alpha\| \\
&= \|(x_k + i)^{-1}2i(x - x_k)(x + i)^{-1}\alpha\| \\
&\leq 2\|(x_k + i)^{-1}\|\underbrace{\|(x - x_k)(x + i)^{-1}\alpha\|}_{\beta} \\
&\leq 2\|(x - x_k)\beta\| \rightarrow 0.
\end{aligned}$$

□

**Corollary 5.19.** *If  $f \in C_0(\mathbb{R})$ , then  $x \mapsto f(x)$  is SOT-continuous on  $\mathcal{B}(\mathcal{H})_{\text{sa}}$ .*

*Proof.* Suppose

$$g(t) = \begin{cases} f\left(i\frac{1+t}{1-t}\right); & t \neq 1 \\ 0; & t = 1 \end{cases}$$

which maps  $\mathbb{T} \rightarrow \mathbb{C}$ . By previous proposition,  $x \mapsto g(x)$  is SOT-continuous on unitaries. Letting  $U(z) = \frac{z-i}{z+i}$ , denote the Cayley transform, we have that  $f = g \circ U$  is a composite of two SOT-continuous maps, which is a SOT-continuous map of itself. □

**Theorem 5.20** (Kaplansky's density theorem).

Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a  $*$ -subalgebra and  $B = \overline{A}^{\text{SOT}}$ , then

- (1.)  $\overline{A_{\text{sa}}}^{\text{SOT}} = B_{\text{sa}}$ ;
- (2.)  $\overline{(A)_1}^{\text{SOT}} = (B)_1$ .

*Proof.* WLOG  $A$  is a  $C^*$ -algebra, so norm-closed.

- (1.) First we prove that  $\overline{A_{\text{sa}}}^{\text{SOT}} \subseteq B_{\text{sa}}$ . Since  $\overline{A_{\text{sa}}}^{\text{SOT}} = \overline{A_{\text{sa}}}^{\text{WOT}}$ , take  $x \in \overline{A_{\text{sa}}}^{\text{SOT}}$  and the net  $(x_k)_k \subseteq A_{\text{sa}}$  converges to  $x$ . Since  $*$  is WOT continuous,  $(x_k^*)_k = (x_k)_k$  converge to  $x^*$ , so  $x = x^*$ . Now the converse inclusion: suppose the net  $(x_k)_k$  SOT-converges to  $x \in B_{\text{sa}}$ . Then  $\frac{x_k + x_k^*}{2} \rightarrow x$  in WOT-topology, which implies

$$B_{\text{sa}} \subseteq \overline{A_{\text{sa}}}^{\text{WOT}} = \overline{A_{\text{sa}}}^{\text{SOT}}.$$

- (2.) Suppose the net  $(y_i)_i$  in  $A_{\text{sa}}$  SOT-converges to  $x \in B_{\text{sa}}$ . Take  $f \in C_0(\mathbb{R})$  such that for all we have  $f(t) = t$ ,  $\forall |t| \leq \|x\|$  and  $|f(t)| \leq \|x\|$ ,  $\forall t \in \mathbb{R}$ . By functional calculus,  $\|f(y_k)\| \leq \|x\|$ . By the previous corollary,  $(f(y_i))_i \xrightarrow{\text{SOT}} f(x) = x$ . This proves that  $(A)_1 \cap A_{\text{sa}}$  is SOT-dense in  $(B)_1 \cap B_{\text{sa}}$ . Pass over to  $M_2(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ . Then  $M_2(A)$  is SOT-dense in  $M_2(B)$  by assumption. For  $x \in (B)_1$ , we have

$$\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (M_2(B))_1 \cap (M_2(B))_{\text{sa}}.$$

That means there exists a net

$$\tilde{x}_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in (M_2(A))_1$$

such that  $\tilde{x}_i \rightarrow \tilde{x}$  and therefore  $b_i \in (A)_1$  converge to  $x$ .  $\square$

**Corollary 5.21.** *Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a  $*$ -algebra. Then  $A$  is a vNa iff  $(A)_1$  is SOT-closed.*

### 5.3 Examples of vNa's

**Definition 5.22.** A vNa  $M$  is called a factor if  $Z(M) = M \cap M' = \mathbb{C} \cdot 1$ .

**Example 5.23.** *Clearly,  $\mathcal{B}(\mathcal{H})$  is a factor. If  $n = \dim \mathcal{H} < \infty$ , then  $M_n(\mathbb{C})$  is a factor.*

Let  $\Gamma$  be a group and  $\mathcal{H} = \ell^2(\Gamma)$ . Consider the left regular representation

$$\lambda : \Gamma \rightarrow \mathcal{B}(\ell^2(\Gamma)), \quad g \mapsto (\delta_h \mapsto \delta_{gh})$$

and extend linearly to  $\lambda : \mathbb{C}[\Gamma] \rightarrow \mathcal{B}(\ell^2(\Gamma))$ . The group vNa of  $\Gamma$  is  $VN(\Gamma) := \lambda(\mathbb{C}[\Gamma])''$  in  $\mathcal{B}(\ell^2(\Gamma))$ . It has a trace, which is the linear functional

$$\tau : VN(\Gamma) \rightarrow \mathbb{C}, \quad x \mapsto \langle x\delta_e, \delta_e \rangle.$$

For  $g \in \Gamma$ ,  $\tau(\lambda(g)) = 1$  if  $g = e$ , otherwise zero. For  $g_1, \dots, g_r \in \Gamma$ , we have

$$g_1 \dots g_r = e \Leftrightarrow \tau(\lambda(g_1) \dots \lambda(g_r)) = 1.$$

Since  $\tau$  is a positive linear functional and  $\tau(1) = 1$ ,  $\tau$  is a state. For any two elements  $g, h \in \Gamma$  we have  $gh = e \Leftrightarrow hg = e$ , which together with the above line implies

$$\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g)).$$

By linearity,  $\tau$  has a cyclic property on  $\lambda(\mathbb{C}[\Gamma])$ . But since  $\tau$  is, by definition, WOT-continuous and  $VN(\Gamma) = (\lambda(\mathbb{C}[\Gamma]))'' = \overline{\lambda(\mathbb{C}[\Gamma])}^{\text{WOT}}$ ,  $\tau$  is cyclic on the entire  $VN(\Gamma)$ . Now if  $|\Gamma| = \infty$ , then  $VN(\Gamma) \neq \mathcal{B}(\mathcal{H})$ , since the latter does not have a trace if  $\dim \mathcal{H} = \infty$ . If  $\Gamma$  is Abelian, then  $VN(\Gamma)$  is commutative.

**Definition 5.24.** Group  $\Gamma$  has icc (infinite conjugacy classes) if for all  $g \in \Gamma \setminus \{e\}$  the set  $\{f^{-1}gf \mid f \in \Gamma\}$  is infinite.

**Example 5.25.** *The group*

$$S_\infty = \{\text{bijections } \mathbb{N} \rightarrow \mathbb{N} \text{ that only permute finitely many elements}\}$$

*has icc.*

**Example 5.26.** Free groups  $\mathbb{F}_n$  for  $n > 1$  have *icc*.

**Theorem 5.27.**

If  $\Gamma$  has *icc*, then  $VN(\Gamma)$  is a factor.

**Definition 5.28.**  $VN(S_\infty) =: R$  is the hyperfinite  $II_1$ -factor.

Open problem: does  $VN(\mathbb{F}_2) \cong VN(\mathbb{F}_3)$  hold?

## 5.4 Operations with vNa's

### Direct sums

Let  $M_i \subseteq \mathcal{B}(\mathcal{H}_i)$  be vNa's. Define the isometric embedding

$$\iota_j : \mathcal{B}(\mathcal{H}_j) \rightarrow \mathcal{B}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n), \quad x \mapsto ((\alpha_1, \dots, \alpha_n) \mapsto (0, \dots, 0, x\alpha_j, 0, \dots, 0)).$$

This map is the  $n \times n$  bounded matrix where the  $(j, j)$ -th element is  $x$  and the rest are zero. Then

$$M_1 \oplus \cdots \oplus M_n := \text{span}\{\iota_j(x) \mid j = 1, \dots, n, x \in M_j\}$$

is the direct sum of vNa's. If  $n \geq 2$ , then from

$$Z(M_1 \oplus \cdots \oplus M_n) = Z(M_1) \oplus \cdots \oplus Z(M_n),$$

we deduce that  $M_1 \oplus \cdots \oplus M_n$  is not a factor.

### Tensor products

The algebraic tensor product  $\mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n)$  acts on  $\mathcal{H}_1 \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_n$  by

$$(x_1 \otimes \cdots \otimes x_n)(\alpha_1 \otimes \cdots \otimes \alpha_n) = (x_1 \alpha_1) \otimes \cdots \otimes (x_n \alpha_n)$$

for  $x_j \in \mathcal{B}(\mathcal{H}_j)$  and  $\alpha_j \in \mathcal{H}_j$ , which implies

$$\mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n) \subseteq \mathcal{B}(\mathcal{H}_1 \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_n).$$

Finally, we define the tensor product of vNa's as

$$M_1 \bar{\otimes} \cdots \bar{\otimes} M_n = (M_1 \otimes \cdots \otimes M_n)'' \cap \mathcal{B}(\mathcal{H}_1 \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_n).$$

### Compressions

**Definition 5.29.** Let  $M \subseteq \mathcal{B}(\mathcal{H})$  be a vNa and  $p \in \mathcal{B}(\mathcal{H})$  a projection. A compression of  $M$  is  $pMp = \{p x p \mid x \in M\}$ . When  $p \in M$ , it is also called a corner.

If  $\mathcal{H} = \text{im } p \oplus (\text{im } p)^\perp = \text{im } p \oplus (1 - p)$ . In this basis, elements of  $pMp$  has the matrix form

$$\begin{bmatrix} p x p & 0 \\ 0 & 0 \end{bmatrix}.$$

If  $M \ni p \neq 1$ , then  $pMp$  is a  $*$ -algebra and  $pMp \subseteq M$  but it is not a subalgebra since  $1_M = 1_{\mathcal{B}(\mathcal{H})} \notin pMp$ . However,  $pMp$  is a subalgebra of  $\mathcal{B}(p\mathcal{H})$  with identity element  $p$ .

**Definition 5.30.** Let  $\mathcal{K} \subseteq \mathcal{H}$  and  $x \in \mathcal{B}(\mathcal{H})$ .

- (1.)  $\mathcal{K}$  is invariant for  $x$  if  $x\mathcal{K} \subseteq \mathcal{K}$ ;
- (2.)  $\mathcal{K}$  is reducing for  $x$  if  $\mathcal{K}$  is invariant for both  $x$  and  $x^*$ .

Now if  $S \subseteq \mathcal{B}(\mathcal{H})$ , then

- (1.)  $\mathcal{K}$  is invariant for  $S$  if  $x\mathcal{K} \subseteq \mathcal{K}$  for all  $x \in S$ ;
- (2.)  $\mathcal{K}$  is reducing for  $S$  if  $\mathcal{K}$  is reducing for all  $x \in S$ .

If  $S \subseteq \mathcal{B}(\mathcal{H})$  is closed under  $*$ , then  $\mathcal{K}$  is invariant for  $S$  iff it is reducing for  $S$ .

**Lemma 5.31.** Let  $\mathcal{K}^{\text{closed}} \leq \mathcal{H}$  and  $M \subseteq \mathcal{B}(\mathcal{H})$  an  $*$ -algebra. Let  $p : \mathcal{H} \rightarrow \mathcal{K}$  be an orthogonal projection. Then  $\mathcal{K}$  is reducing for  $M$  iff  $p \in M'$ .

**Theorem 5.32.**

Let  $M \subseteq \mathcal{B}(\mathcal{H})$  be a vNa and  $p \in M$  a projection. Then  $pMp$  and  $M'p$  are vNa's in  $\mathcal{B}(p\mathcal{H})$ .

*Proof.* We will show that

$$(M'p)' \cap \mathcal{B}(p\mathcal{H}) = pMp, \quad (pMp)' \cap \mathcal{B}(p\mathcal{H}) = M'p.$$

Then the bicommutant theorem will take care of the rest. It is obvious that  $(M'p)' \cap \mathcal{B}(p\mathcal{H}) \supseteq pMp$ . For the converse, pick  $x \in (M'p)' \cap \mathcal{B}(p\mathcal{H})$ . Define  $\tilde{x} = xp = px \in \mathcal{B}(\mathcal{H})$ . For  $y \in M'$ , we have

$$y\tilde{x} = ypx = xyp = xpy = \tilde{x}y,$$

which implies  $\tilde{x} \in M'' = M$ . Then  $x = pxp = p\tilde{x}p \in pMp$ . As before  $(pMp)' \cap \mathcal{B}(p\mathcal{H}) \supseteq M'p$  is trivial and we just prove the converse. Take  $y \in (pMp)' \cap \mathcal{B}(p\mathcal{H})$ . Using CFC, we can write  $y$  as a linear combinations of 4 unitaries. Since  $pMp$  is closed under  $*$ ,  $(pMp)'$  is a vNa (and therefore a  $C^*$ -algebra). So we can assume WLOG that  $y = u$  a unitary. Set  $\mathcal{K} := \overline{Mp\mathcal{H}}$  and  $q : \mathcal{H} \rightarrow \mathcal{K}$  an orthogonal projection. Since  $\mathcal{K}$  is reducing for  $M$  and  $M'$ , which implies

$$q \in M' \cap M'' = M' \cap M = Z(M).$$

Next, we extend  $u$  to  $\mathcal{K}$ :

$$\tilde{u}\left(\sum_i \underbrace{x_i}_{\in M} p \underbrace{\alpha_i}_{\in \mathcal{H}}\right) = \sum_i x_i u p \alpha_i.$$

We shall show that this is a well-defined isometry in  $Mp\mathcal{H}$ :

$$\begin{aligned} \|\tilde{u} \sum_i x_i p \alpha_i\|^2 &= \sum_{i,j} \langle x_i u p \alpha_i, x_j u p \alpha_j \rangle \\ &= \sum_{i,j} \langle (p x_j^* x_i p) u \alpha_i, u \alpha_j \rangle \\ &= \sum_{i,j} \langle u p x_j^* x_i p \alpha_i, u \alpha_j \rangle \\ &= \sum_{i,j} \langle p x_j^* x_i p \alpha_i, \alpha_j \rangle = \left\| \sum_i x_i p \alpha_i \right\|^2. \end{aligned}$$



So  $\tilde{u}$  extends to an isometry on  $\mathcal{K} = \overline{Mp\mathcal{H}}$ . By definition,  $\tilde{u}$  commutes with  $M$  on  $Mp\mathcal{H}$ , so also on  $\mathcal{K}$ . Thus for every  $x \in M$  and  $\alpha \in \mathcal{H}$ , we have

$$x(\tilde{u}q)\alpha = \tilde{u}xq\alpha = (\tilde{u}q)x\alpha,$$

which implies  $\tilde{u}q \in M' \cap \mathcal{B}(\mathcal{H})$ . Then

$$\tilde{u}qp\alpha = \tilde{u}1p\alpha = 1up\alpha,$$

which implies  $u = \tilde{u}qp \in \mathcal{B}(\mathcal{H})$  and  $u \in M'p$ . □

**Corollary 5.33.** *Suppose the vNa  $M \subseteq \mathcal{B}(\mathcal{H})$  is a factor and let  $p \in M$  be a projection. Then  $pMp$  and  $M'p$  are factor (in  $\mathcal{B}(p\mathcal{H})$ ).*

*Proof.* Let  $\mathcal{K} = \overline{Mp\mathcal{H}}$  and  $q : \mathcal{H} \rightarrow \mathcal{K}$  a projection. From the previous proof,  $q \in Z(M) = \mathbb{C}$ . Then  $q \in \{0, 1\}$ . WLOG  $p \neq 0$ , so  $q = 1$ . Thus  $\mathcal{K} = \mathcal{H}$ , so  $Mp\mathcal{H}$  is dense in  $\mathcal{H}$ . Consider

$$\psi : M' \rightarrow M'p, \quad y \mapsto yp.$$

We will prove that  $\psi$  is an isomorphism of algebras. Obviously, it is additive. Since

$$\psi(xy) = xyp = xyp^2 = xpy p = \psi(x)\psi(y),$$

it is also multiplicative. Same calculation shows  $\psi(y^*) = \psi(y)^*$ . Obviously,  $\psi$  is surjective. Finally, we prove injectivity. Suppose  $y \in M'$  satisfies  $yp = 0$ . Then for every  $x \in M$  and  $\alpha \in \mathcal{H}$ , we get  $xyp\alpha = x(yp)\alpha = 0$ . Hence  $y|_{Mp\mathcal{H}} = 0$ , so by continuity,  $y|_{\overline{Mp\mathcal{H}}} = y|_{\mathcal{K}} = 0$ . But because  $\mathcal{K} = \mathcal{H}$ , this yields  $y|_{\mathcal{H}} = 0$ . As a result, we get

$$Z(M'p) = Z(M')p = \mathbb{C} \cdot p,$$

so  $M'p$  is a factor. Similarly,

$$Z(pMp) = (pMp) \cap (pMp)' = (M'p)' \cap M'p = Z(M'p) = \mathbb{C}p,$$

so  $pMp$  is a factor. □

## 6 Spectral theorem and Borel functional calculus

### 6.1 Spectral theorem

Recall the spectral theorem for  $\mathcal{K}(\mathcal{H})$ . Let  $T \in \mathcal{K}(\mathcal{H})$  be self-adjoint and for  $\lambda \in \sigma_p(T)$ , define  $E(\lambda)$  as an orthogonal projection onto the eigenspace  $\ker(T - \lambda I)$ . For  $\mu \neq \lambda$ , we get  $E(\lambda)E(\mu) = 0$  and

$$T = \sum_{\lambda \in \sigma_p(T) \setminus \{0\}} \lambda E(\lambda).$$

Our first goal will be to generalize this to non-compact self-adjoint operator.

**Theorem 6.1 (Vigier).**

*Let  $(u_\lambda)$  be a net of increasing (decreasing) and above (below) bounded self-adjoint operators on  $\mathcal{H}$ . Then  $(u_\lambda)$  converges.*

*Proof.* We prove the statement for above-bounded increasing net. We can assume  $(u_\lambda)$  has a lower bound  $m$  by considering a truncated net. WLOG we can assume  $u_\lambda$  are positive (otherwise we can consider  $u_\lambda - m$ ). There exists  $M \geq 0$  such that  $\|u_\lambda\| \leq M$  for indices  $\lambda$ . So the net  $\langle u_\lambda x, x \rangle$  is real and increasing and bounded above by  $M\|x\|^2$ . Using the polarization identity

$$\langle u_\lambda x, x \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \langle u_\lambda (x + i^k y), x + i^k y \rangle,$$

we see that  $\langle u_\lambda x, y \rangle$  is a convergent net for all  $x, y \in \mathcal{H}$ . Letting  $\sigma(x, y)$  denote its limit, we can easily check that  $\sigma$  is a bounded sesquilinear form ( $|\sigma(x, y)| \leq M\|x\|\|y\|$ ). By Riesz, there exists an operator  $u \in \mathcal{B}(\mathcal{H})$  such that  $\langle ux, y \rangle = \sigma(x, y)$ . Then  $u$  is self-adjoint,  $\|u\| \leq M$  and  $u_\lambda \leq u$ . Note that

$$\begin{aligned} \|(u - u_\lambda)x\|^2 &\leq \|(u - u_\lambda)^{\frac{1}{2}}(u - u_\lambda)^{\frac{1}{2}}x\|^2 \\ &\leq \|(u - u_\lambda)\| \|(u - u_\lambda)^{\frac{1}{2}}x\|^2 \\ &\leq 2M \langle (u - u_\lambda)x, x \rangle \rightarrow 0, \end{aligned}$$

so  $u_\lambda$  converge strongly to  $u$ . □

*Remark.* If  $(p_\lambda)$  is a net of projections converging strongly to some operator  $u$ , then  $u$  is also a projection. Clearly,  $u$  is self-adjoint and

$$\begin{aligned} \langle ux, y \rangle &= \lim_\lambda \langle p_\lambda x, y \rangle = \lim_\lambda \langle p_\lambda x, p_\lambda y \rangle \\ &= \langle ux, uy \rangle = \langle u^2 x, y \rangle, \end{aligned}$$

therefore  $u^2 = u$ .

**Corollary 6.2.** *If  $(p_n)_{n \in \mathbb{N}}$  is a sequence of pairwise orthogonal orthogonal projections in  $\mathcal{B}(\mathcal{H})$ , then  $(\sum_{n=1}^N p_n)$  SOT-converges for  $N \rightarrow \infty$  (we denote the limit by  $\sum_{n=1}^\infty p_n$ ).*

**Definition 6.3.** Let  $X$  be a set,  $\Omega$  a  $\sigma$ -algebra in  $X$  and  $\mathcal{H}$  a Hilbert space. Then we define a projection-valued measure (PVM) for  $(X, \Omega, \mathcal{H})$  is a map  $E : \Omega \rightarrow \mathcal{B}(\mathcal{H})$  such that

- (1.)  $E(S)$  is a projection for all  $S \in \Omega$ ;
- (2.)  $E(\emptyset) = 0$  and  $E(X) = 1$ ;
- (3.)  $E(S \cap T) = E(S)E(T)$  for all  $S, T \in \Omega$ ;
- (4.) If  $(S_n)_{n \in \mathbb{N}} \subseteq \Omega$  is a sequence of pairwise disjoint sets, then

$$E\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} E(S_n).$$

*Remark.* Projections  $E(S)$  commute with each other (follows directly from the third point of the definition).

**Example 6.4.** Let  $(X, \Omega, \mu)$  be a  $\sigma$ -finite measure space. Let  $\mathcal{H} = L^2(X, \mu)$  and  $S \in \Omega$ , then  $\chi_S \in L^\infty(X, \mu) \subseteq \mathcal{B}(L^2(X, \mu))$  is a projection (with pointwise multiplication in  $\mathcal{B}(\mathcal{H})$ ). Letting  $E(S) := \chi_S \in \mathcal{B}(L^2(X, \mu))$ , we get a PVM  $E : \Omega \rightarrow \mathcal{B}(L^2(X, \mu))$ .

**Lemma 6.5.** Let  $E$  be a PVM for  $(X, \Omega, \mathcal{H})$ . Then for all  $\alpha, \beta \in \mathcal{H}$  the mapping

$$E_{\alpha, \beta} : \Omega \rightarrow \mathbb{C}, \quad S \mapsto \langle E(S)\alpha, \beta \rangle$$

is a complex measure in  $\Omega$  with total variation  $\leq \|\alpha\| \|\beta\|$ .

*Proof.* Let  $\alpha, \beta \in \mathcal{H}$ . Then  $E_{\alpha, \beta}$  is  $\sigma$ -additive (countably-additive for disjoint sets) since  $E$  is  $\sigma$ -additive by (4). Total variation of a complex measure is

$$\|E_{\alpha, \beta}\| := \sup\left\{\sum_{S \in \pi} |E_{\alpha, \beta}(S)|\right\},$$

where the sum is over all partitions of  $X$  into finitely many pieces of measurable sets. Let  $\pi = \{S_1, \dots, S_n\}$  be a partition of  $X$  with  $S_j \in \Omega$ . For each  $j$  pick  $\alpha_j \in \mathbb{C}$  such that  $|\alpha_j| = 1$  and

$$\alpha_j \cdot E_{\alpha, \beta}(S_j) = \alpha_j \langle E(S_j)\alpha, \beta \rangle = |\langle E(S_j)\alpha, \beta \rangle| = |E_{\alpha, \beta}(S_j)|.$$

Then

$$\sum_{j=1}^n |E_{\alpha, \beta}(S_j)| = \left\langle \sum_{j=1}^n \alpha_j E(S_j)\alpha, \beta \right\rangle \leq \left\| \sum_{j=1}^n \alpha_j E(S_j)\alpha \right\| \cdot \|\beta\|.$$

For  $i \neq j$  we have

$$E(S_i)E(S_j) = E(S_i \cap S_j) = E(\emptyset) = 0,$$

so  $E(S_i)$  are pairwise orthogonal. Finally, we can use Pythagoras to get

$$\begin{aligned}
\left\| \sum_{j=1}^n \alpha_j E(S_j) \alpha \right\|^2 &= \sum_{j=1}^n \|E(S_j) \alpha\|^2 \\
&= \left\| \sum_{j=1}^n E(S_j) \alpha \right\|^2 \\
&= \left\| E \left( \bigcup_{j=1}^n S_j \right) \alpha \right\|^2 \\
&= \|E(X) \alpha\|^2 = \|\alpha\|^2.
\end{aligned}$$

□

*Remark.* Let  $E$  be a PVM for  $(X, \Omega, \mathcal{H})$ ,  $\alpha \in \mathcal{H}$  and  $S \in \Omega$ . Then

$$\begin{aligned}
E_{\alpha, \alpha}(S) &= \langle E(S) \alpha, \alpha \rangle \\
&= \langle E(S)^2 \alpha, \alpha \rangle \\
&= \langle E(S) \alpha, E(S) \alpha \rangle \geq 0,
\end{aligned}$$

so  $E_{\alpha, \alpha}$  is a positive measure on  $X$ . Furthermore, if  $\|\alpha\| = 1$ , then  $E_{\alpha, \alpha}$  is a probability measure.

Let

$$(\alpha, \beta) \mapsto \int_X 1 dE_{\alpha, \beta}.$$

Since  $E_{\alpha + \lambda \alpha', \beta} = E_{\alpha, \beta} + \lambda E_{\alpha', \beta}$  and  $E_{\alpha, \beta + \lambda \beta'} = E_{\alpha, \beta} + \bar{\lambda} E_{\alpha, \beta'}$ , the above is a sesquilinear form on  $\mathcal{H}$ . In particular, it is bounded

$$\left\| \int_X dE_{\alpha, \beta} \right\| \leq \|E_{\alpha, \beta}\| \leq \|\alpha\| \|\beta\|.$$

Suppose  $f : X \rightarrow \mathbb{C}$  is a bounded  $\Omega$ -measurable function. Then

$$(\alpha, \beta) \mapsto \int_X f dE_{\alpha, \beta}$$

is a bounded sesquilinear form:

$$\left\| \int_X f dE_{\alpha, \beta} \right\| \leq \|f\|_{\infty} \|E_{\alpha, \beta}\| \leq \|f\|_{\infty} \|\alpha\| \|\beta\|.$$

So there exists an  $x \in \mathcal{B}(\mathcal{H})$  such that  $\|x\| \leq \|f\|_{\infty}$  and  $\langle x \alpha, \beta \rangle = \int_X f dE_{\alpha, \beta}$ . If  $f = \chi_S$  for  $S \in \Omega$ , then  $x = E(S)$ :

$$\int_X \chi_S dE_{\alpha, \beta} = E_{\alpha, \beta}(S) = \langle E(S) \alpha, \beta \rangle.$$

**Definition 6.6.** Let  $E$  be a PVM for  $(X, \Omega, \mathcal{H})$  and  $f : X \rightarrow \mathbb{C}$  be a bounded  $\Omega$ -measurable

function and  $x \in \mathcal{B}(\mathcal{H})$ . We call  $x$  the integral of  $f$  with regards to  $E$  if

$$\langle x\alpha, \beta \rangle = \int_X f dE_{\alpha, \beta}, \quad \forall \alpha, \beta \in \mathcal{H}.$$

We denote it by  $x := \int_X f dE$ .

*Remark.* Define  $B(X, \Omega)$  as the set of all bounded  $\Omega$ -measurable complex functions on  $X$  and endow it with the sup norm. If  $X$  is a topological space and  $\Omega = \mathcal{B}_X$  is the Borel  $\sigma$ -algebra on  $X$ , then  $B(X) = B(X, \mathcal{B}_X)$ .

**Proposition 6.7.** *Let  $E$  be a PVM for  $(X, \Omega, \mathcal{H})$ . Then*

$$\rho : B(X, \Omega) \rightarrow \mathcal{B}(\mathcal{H}), \quad f \mapsto \int_X f dE.$$

*is a  $*$ -homomorphism and contractive. Furthermore:*

- (1.) *If  $(f_n)_n \subseteq B(X, \Omega)$  is an increasing sequence of nonnegative functions and  $f = \sup_n f_n \in B(X, \Omega)$ , then  $\int_X f_n dE \rightarrow \int_X f dE$  in SOT.*
- (2.) *If  $X$  is compact and  $T_2$ , then  $\rho(B(X)) \subseteq \rho(C(X))''$ .*

*Proof.* We already saw that  $\|\rho(f)\| \leq \|f\|_\infty$ , hence  $\rho$  is contractive. It is also clear that  $\rho$  is linear and  $\rho(f)^* = \rho(\bar{f})$ . Next, we prove multiplicativity:  $\rho(\chi_S) = E(S)$  for  $S \in \Omega$ . Then

$$\rho(\chi_S) \cdot \rho(\chi_T) = E(S) \cdot E(T) = E(S \cap T) = \rho(\chi_{S \cap T}) = \rho(\chi_S \cdot \chi_T).$$

Since  $\rho$  is linear, it is also multiplicative on simple functions (these are finite linear combinations of characteristic functions). Since each  $f \in B(X, \Omega)$  is a uniform limit of a uniformly bounded sequence of simple functions, we deduce that  $\rho(fg) = \rho(f)\rho(g)$  for all  $f, g \in B(X, \Omega)$ .

- (1.) Let  $f, f_n$  be as in the statement. Since  $\rho$  is a  $*$ -homomorphism,  $(\rho(f_n))_n$  is an increasing sequence of positive operators and  $\sup_n \|\rho(f_n)\| \leq \sup_n \|f_n\|_\infty = \|f\|$ . By Vigier, there exists  $x \in \mathcal{B}(\mathcal{H})$  such that  $\rho(f_n) \xrightarrow{\text{SOT}} x$ . This  $x$  is a natural candidate for  $\rho(f)$ . Indeed, for  $\alpha, \beta \in \mathcal{H}$ , we have

$$\begin{aligned} \langle \rho(f)\alpha, \beta \rangle &= \int_X f dE_{\alpha, \beta} \\ &= \lim_{n \rightarrow \infty} \int_X f_n dE_{\alpha, \beta} \\ &= \lim_{n \rightarrow \infty} \langle \rho(f_n)\alpha, \beta \rangle, \end{aligned}$$

so  $\rho(f_n) \xrightarrow{\text{WOT}} \rho(f)$  and therefore  $\rho(f) = x$ .

- (2.) Let  $X$  be compact Hausdorff and  $a \in \rho(C(X))'$ . Take  $\alpha, \beta \in \mathcal{H}$ . Then for all  $f \in C(X)$ ,

we have

$$\begin{aligned} 0 &= \langle (a\rho(f) - \rho(f)a)\alpha, \beta \rangle \\ &= \langle \rho(f)\alpha, a^*\beta \rangle - \langle \rho(f)(a\alpha), \beta \rangle \\ &= \int_X f dE_{\alpha, a^*\beta} - \int_X f dE_{a\alpha, \beta}, \end{aligned}$$

so by uniqueness from Riesz-Markoff we get  $E_{\alpha, a^*\beta} = E_{a\alpha, \beta}$ . This same calculation backwards tells us that  $a$  commutes with all  $\rho(g) = \int_X g dE$  for  $g \in B(X)$ , so  $\rho(B(X)) \subseteq \rho(C(X))''$   $\square$

*Remark.* The map  $\rho$  is not necessarily isometric. However, we can define  $E$ -null sets

$$\{S \in \Omega \mid E(S) = 0\},$$

which gives us an equivalence relation on  $B(X, \Omega)$  as follows:  $f \sim_E g$  if  $f(x) = g(x)$  except possibly on some  $E$ -null set. Then we have

$$\ker \rho = \{f \in B(X, \Omega) \mid f \sim_E 0\}$$

and an essential supremum

$$\|\rho(f)\| = \|f\|_\infty := \inf\{t > 0 \mid E(\{x \in X \mid |f(x)| \geq t\}) = 0\}.$$

Define  $L^\infty(X, E) = \frac{B(X, \Omega)}{\sim_E}$  with norm induced by an essential supremum above. Then the map  $\rho$  from the above proposition induces an isometric  $*$ -isomorphism  $\tilde{\rho}: L^\infty(X, E) \rightarrow \mathcal{B}(\mathcal{H})$ .

Recall that for a commutative  $C^*$ -algebra  $A$  the Gelfand transform

$$\Gamma: A \rightarrow C(\sigma(A))$$

is an isometric  $*$ -isomorphism.

**Theorem 6.8 (Spectral theorem).**

Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a commutative  $C^*$ -algebra and  $\mathcal{B}_{\sigma(A)}$  be the Borel  $\sigma$ -algebra on  $\sigma(A)$ . Then there exists a PVM  $E$  for  $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$  such that

$$x = \int_{\sigma(A)} \Gamma(x) dE$$

for all  $x \in A$ .

*Proof.* For all  $\alpha, \beta \in \mathcal{H}$  and

$$\varphi: C(\sigma(A)) \rightarrow \mathbb{C}, \quad f \mapsto \langle \Gamma^{-1}(f)\alpha, \beta \rangle$$

is a bounded linear functional. Indeed, since  $\Gamma$  is an isometry we get

$$\langle \Gamma^{-1}(f)\alpha, \beta \rangle \leq \|f\|_\infty \|\alpha\| \|\beta\|.$$

By Riesz-Markoff, there exists a unique regular Borel measure  $\mu_{\alpha,\beta}$  such that

$$\langle \Gamma^{-1}(f)\alpha, \beta \rangle = \int_{\sigma(A)} f d\mu_{\alpha,\beta}.$$

We will show that  $\mu_{\alpha,\beta} = E_{\alpha,\beta}$  for a PVM  $E$ . For  $f, g \in C(\sigma(A))$  we have

$$\int_{\sigma(A)} fg d\mu_{\alpha,\beta} = \langle \Gamma^{-1}(fg)\alpha, \beta \rangle = \langle \Gamma^{-1}(f)\Gamma(g)\alpha, \beta \rangle = \int_{\sigma(A)} f d\mu_{\Gamma^{-1}(g)\alpha,\beta}.$$

Now we notice that this is equal to

$$\langle \Gamma^{-1}(f)\alpha, \Gamma^{-1}(\bar{g})\beta \rangle = \int_{\sigma(A)} f d\mu_{\alpha,\Gamma^{-1}(\bar{g})\beta}.$$

From the uniqueness of Riesz-Markoff, we get

$$g d\mu_{\alpha,\beta} = d\mu_{\Gamma^{-1}(g)\alpha,\beta} = d\mu_{\alpha,\Gamma^{-1}(\bar{g})\beta}.$$

Finally, we have

$$\begin{aligned} \int_{\sigma(A)} f d\overline{\mu_{\alpha,\beta}} &= \overline{\int_{\sigma(A)} \bar{f} d\mu_{\alpha,\beta}} \\ &= \overline{\langle \Gamma^{-1}(\bar{f})\alpha, \beta \rangle} \\ &= \overline{\langle \alpha, \Gamma^{-1}(f)\beta \rangle} \\ &= \langle \Gamma^{-1}(f)\beta, \alpha \rangle \\ &= \int_{\sigma(A)} f d\mu_{\beta,\alpha} \end{aligned}$$

for all  $f \in C(\sigma(A))$ , which implies  $\overline{\mu_{\alpha,\beta}} = \mu_{\beta,\alpha}$ . To each  $S \in \mathcal{B}_{\sigma(A)}$  we assign the sesquilinear form

$$\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_{\sigma(A)} \chi_S d\mu_{\alpha,\beta}.$$

This form is bounded by  $\|\alpha\|\|\beta\| = \|\mu_{\alpha,\beta}\|$ . Thus there exists  $E(S) \in \mathcal{B}(\mathcal{H})$  such that

$$\int_{\sigma(A)} \chi_S d\mu_{\alpha,\beta} = \langle E(S)\alpha, \beta \rangle.$$

Now notice that

$$\begin{aligned} \langle E(S)^*\alpha, \beta \rangle &= \langle \alpha, E(S)\beta \rangle \\ &= \overline{\langle E(S)\beta, \alpha \rangle} \\ &= \overline{\int_{\sigma(A)} \chi_S d\mu_{\beta,\alpha}} \\ &= \int_{\sigma(A)} \chi_S d\overline{\mu_{\beta,\alpha}} \\ &= \int_{\sigma(A)} \chi_S d\mu_{\alpha,\beta} \\ &= \langle E(S)\alpha, \beta \rangle, \end{aligned}$$

so  $E(S) = E(S)^*$ . For any  $f \in C(\sigma(A))$ , we get

$$\begin{aligned}\langle \Gamma^{-1}(f)E(S)\alpha, \beta \rangle &= \langle E(S)\alpha, \Gamma^{-1}(\bar{f})\beta \rangle \\ &= \int \chi_S d\mu_{\alpha, \Gamma^{-1}(\bar{f})} \\ &= \int \chi_S f d\mu_{\alpha, \beta}.\end{aligned}$$

By the lemma below,  $C(\sigma(A))$  is weak-\* dense in  $C(\sigma(A))^{**}$ . Furthermore, the latter set contains  $B(\sigma(A))$ : indeed, given any  $\psi \in C(\sigma(A))^*$ , we have that  $\psi(\cdot) = \int \cdot d\mu$  for some measure  $\mu$ . For any  $r \in B(\sigma(A))$ , we have  $r(\psi) = \int_{\sigma(A)} r d\mu$ . Hence,  $i : C \rightarrow C^{**}$  extends to  $\widehat{i} : B \rightarrow C^{**}$ . In particular, for  $T \in \mathcal{B}_{\sigma(A)}$ , there exists a net  $(f_i)_i \subseteq C(\sigma(A))$  such that  $f_i \xrightarrow{\text{weak-}^*} \chi_T$ . As a result,  $\int_{\sigma(A)} f_i d\mu_{\alpha, \beta} \rightarrow \int_{\sigma(A)} \chi_T d\mu_{\alpha, \beta}$  for all  $\alpha, \beta \in \mathcal{H}$ , so  $\Gamma^{-1}(f_i) \xrightarrow{\text{WOT}} E(T)$ . Now

$$\begin{aligned}\langle E(T) \cdot E(S)\alpha, \beta \rangle &= \lim_i \langle \Gamma^{-1}(f_i)E(S)\alpha, \beta \rangle \\ &= \lim_i \langle E(S)\alpha, \Gamma^{-1}(\bar{f}_i)\beta \rangle \\ &= \lim_i \int_{\sigma(A)} \chi_S f_i d\mu_{\alpha, \beta} \\ &= \int_{\sigma(A)} \chi_S \cdot \chi_T d\mu_{\alpha, \beta} \\ &= \int_{\sigma(A)} \chi_{S \cap T} d\mu_{\alpha, \beta} = \langle E(S \cap T)\alpha, \beta \rangle.\end{aligned}$$

Since  $\alpha, \beta$  were arbitrary, we get  $E(S) \cdot E(T) = E(S \cap T)$  for all  $S, T \in \mathcal{B}_{\sigma(A)}$ . As a consequence,  $E(S)^2 = E(S)$ , so  $E(S)$  is a projection. Obviously,  $E(\emptyset) = 0$ . Further,

$$\langle E(\sigma(A))\alpha, \beta \rangle = \int_{\sigma(A)} 1 d\mu_{\alpha, \beta} = \langle \Gamma^{-1}(1)\alpha, \beta \rangle = \langle 1\alpha, \beta \rangle,$$

so  $E(\sigma(A)) = 1$ . Since all  $\mu_{\alpha, \beta}$  are  $\sigma$ -additive, we have

$$\begin{aligned}\langle E\left(\bigcup_{i=1}^{\infty} S_i\right)\alpha, \beta \rangle &= \mu_{\alpha, \beta}\left(\bigcup_{i=1}^{\infty} S_i\right) \\ &= \sum_{i=1}^{\infty} \mu_{\alpha, \beta}(S_i) \\ &= \left\langle \sum_{i=1}^{\infty} E(S_i)\alpha, \beta \right\rangle\end{aligned}$$

for each sequence  $(S_i)_i \subseteq \mathcal{B}_{\sigma(A)}$  of pairwise disjoint sets, so  $E(\bigcup_{i=1}^{\infty} S_i) = \sum_{i=1}^{\infty} E(S_i)$ . We have proved that  $E$  is a PVM for  $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$ . For any  $\alpha, \beta \in \mathcal{H}$ , we get

$$E_{\alpha, \beta}(S) = \langle E(S)\alpha, \beta \rangle = \int \chi_S d\mu_{\alpha, \beta} = \mu_{\alpha, \beta}(S),$$



so  $E_{\alpha,\beta} = \mu_{\alpha,\beta}$  and therefore  $\int f dE_{\alpha,\beta} = \langle \Gamma^{-1}\alpha, \beta \rangle$ . But this proves that  $x = \int_{\sigma(A)} \Gamma(x) dE$  for all  $x \in A$ . Uniqueness: suppose that  $E'$  is another such PVM. For  $\alpha, \beta \in \mathcal{H}$  and  $f \in C(\sigma(A))$ , we have

$$\int_{\sigma(A)} f dE_{\alpha,\beta} = \langle \Gamma^{-1}(f)\alpha, \beta \rangle = \int_{\sigma(A)} f dE'_{\alpha,\beta},$$

which proves that  $E_{\alpha,\beta} = E'_{\alpha,\beta}$  as elements in  $C(\sigma(A))^*$ . Thus

$$\langle E(S)\alpha, \beta \rangle = E_{\alpha,\beta}(S) = E'_{\alpha,\beta}(S) = \langle E'(S)\alpha, \beta \rangle.$$

Since  $\alpha, \beta$  were arbitrary,  $E(S) = E'(S)$ . But since  $S$  was also arbitrary,  $E = E'$ .  $\square$

**Lemma 6.9** (Goldstine's theorem). *Let  $X$  be a Banach space. Then the image of*

$$\iota : X \rightarrow X^{**}, \quad x \mapsto (f \mapsto f(x))$$

*is dense in weak-\* topology.*

*Proof.* Let  $\beta \in X^{**}$  and  $f_1, \dots, f_r \in X^*$  (WLOG linearly independent). Then

$$U = \{\alpha \in X^{**} \mid |(\alpha - \beta)(f_j)| < \varepsilon, j = 1, \dots, r\}.$$

is a basic open set in weak-\* topology in  $X^{**}$ . WLOG assume  $X$  is infinitely-dimensional. Consider the linear map

$$\Phi : X \rightarrow \mathbb{C}^r, \quad x \mapsto (f_1(x), \dots, f_r(x)).$$

This map is surjective. In particular, there exists  $x_0 \in X$  such that

$$\Phi(x_0) = (f_1(x_0), \dots, f_r(x_0)) = (\rho_0(f_1), \dots, \rho_0(f_r)),$$

hence  $\iota(x_0) \in U \cap \iota(X)$ .  $\square$

## 6.2 Borel functional calculus

Let  $x \in \mathcal{B}(\mathcal{H})$  be normal ( $x^*x = xx^*$ ) and  $A = C^*(x) \subseteq \mathcal{B}(\mathcal{H})$ . Then  $\sigma(A) \cong \sigma(x)$ . By the spectral theorem, there exists a PVM  $E$  for  $(\sigma(x), \mathcal{B}_{\sigma(x)}, \mathcal{H})$  and

$$B(\sigma(x)) \rightarrow \mathcal{B}(\mathcal{H}), \quad f \mapsto \int_{\sigma(x)} f dE$$

is a \*-homomorphism and a contraction (proposition above). Furthermore,  $f \in C(\sigma(x))$  maps into  $\int_{\sigma(x)} f dE = \Gamma^{-1}(f)$ , so the above map, when restricted to  $C(\sigma(x))$ , coincides with  $\Gamma^{-1}$ . Furthermore, If  $f = \text{id} \in B(\sigma(x))$ , meaning  $f(z) = z$  for  $z \in \sigma(x)$ , then

$$\int_{\sigma(x)} \text{id} dE = \Gamma^{-1}(\text{id}) = x.$$

For  $f \in B(\sigma(x))$ , define

$$f(x) := \int_{\sigma(x)} f dE \in A'' = W^*(x),$$

which is a vNa generated by  $x$ .

**Definition 6.10.** Let  $x \in \mathcal{B}(\mathcal{H})$  be normal. The mapping

$$B(\sigma(x)) \rightarrow W^*(x), \quad f \mapsto f(x)$$

is the Borel functional calculus.

**Theorem 6.11** (Spectral mapping theorem).

*Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a vNa and let  $x \in A$  be normal. Then the Borel functional calculus has the following properties:*

(1.) *The map*

$$B(\sigma(x)) \rightarrow A, \quad f \mapsto f(x)$$

*is a bounded \*-homomorphism.*

(2.) *If  $f \in C(\sigma(x))$ , then this  $f(x)$  is the same  $f(x)$  as in continuous functional calculus.*

(3.) *For  $f \in B(\sigma(x))$ , we have  $\sigma(f(x)) \subseteq f(\sigma(x))$ .*

*Proof.* (1.) This is the above proposition.

(2.) Obvious.

(3.) For  $\lambda \notin f(\sigma(x))$ , then  $f - \lambda \in B(\sigma(x))$  is invertible in  $B(\sigma(x))$ , so there exists  $g \in B(\sigma^{-1})$  such that  $(f - \lambda)g = \text{id}$ . By Borel functional calculus,  $(f(x) - \lambda I) \cdot g(x) = I$ , so  $\lambda \notin \sigma(f(x))$ .  $\square$

**Corollary 6.12.** *Every vNa is the norm-closure of the linear span of the projections.*

*Proof.* Let  $M \subseteq \mathcal{B}(\mathcal{H})$  be a vNa and  $x \in M$ . By using  $\text{Re } x, \text{Im } x \in M_{\text{sa}}$ , we may WLOG assume  $x \in A_{\text{sa}}$ . Hence  $x$  is normal and for all  $f \in B(\sigma(x))$  we have  $f(x) \in M$ . For  $S \in \mathcal{B}_{\sigma(x)}$ ,  $\chi_S(x) \in M$  is a projection. Now we can uniformly approximate  $\text{id}$  on  $\sigma(x)$  by using simple functions. By BFC,  $x$  is uniformly approximated by linear combinations of projections.  $\square$

*Remark.* There exist  $C^*$ -algebras without nontrivial projections. For example, for a compact Hausdorff connected  $X$ , the algebra  $C(X)$  only has trivial projections 0 and 1. There exist noncommutative examples, too.

### 6.3 Commutative von Neumann algebras

**Definition 6.13.** Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a subalgebra. Vector  $\alpha \in \mathcal{H}$  is:

- (1.) cyclic for  $A$  if  $A\alpha$  is dense in  $\mathcal{H}$ .
- (2.) separating for  $A$  if  $x\alpha = 0$  for  $x \in A$  implies  $x = 0$ .

**Proposition 6.14.** Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a subalgebra.

- (1.) If  $\alpha \in \mathcal{H}$  is cyclic for  $A$ , then it is separating for  $A'$ .
- (2.) Assume  $A$  is a  $*$ -subalgebra. Then if  $\alpha$  is separating for  $A'$ , it is cyclic for  $A$ .
- (3.) Suppose  $W \subseteq \mathcal{B}(\mathcal{H})$  is a vNa. Then  $\alpha$  is cyclic for  $M$  iff it is separating for  $M'$  and separating for  $M$  iff it is cyclic for  $M'$ .

*Proof.* (1.) Let  $\alpha$  be cyclic for  $A$ . Let  $y \in A'$  satisfy  $y\alpha = 0$ . Pick any  $\beta \in \mathcal{H}$ . there exists a sequence  $(x_n)_n \subseteq A$  such that  $\|x_n\alpha - \beta\| \rightarrow 0$ . Hence

$$y\beta = \lim_{n \rightarrow \infty} yx_n\alpha = \lim_{n \rightarrow \infty} x_n(y\alpha) = 0$$

and  $\alpha$  is separating for  $A'$ .

- (2.) Define  $\mathcal{K} := (A\alpha)^\perp \subseteq \mathcal{H}$ . Let  $p : \mathcal{H} \rightarrow \mathcal{K}$  be the orthogonal projection. For  $x_1, x_2 \in A$  and  $\beta \in \mathcal{K}$  we have

$$\langle x_1\beta, x_2\alpha \rangle = \langle \beta, x_1^*x_2\alpha \rangle = 0,$$

so  $x_1\beta \in \mathcal{K}$  and  $\mathcal{K}$  is an invariant subspace for  $A$ . But since  $A$  is  $*$ -closed,  $\mathcal{K}$  is reducing and by lemma 5.31  $p \in A'$ . Of course,  $\alpha \in A\alpha$  and  $p\alpha = 0$ . Now we use the fact that  $\alpha$  is separating for  $A'$  and therefore  $p = 0$ . This implies  $\mathcal{K} = (0)$ .

- (3.) This follows immediately from  $M = M''$  and the previous two points. □

**Example 6.15.** Recall  $VN(\Gamma) := \lambda(\mathbb{C}[\Gamma])'' \subseteq \mathcal{B}(\ell^2(\Gamma))$ . Similarly, we can use the right regular map

$$\rho : \Gamma \rightarrow \mathcal{B}(\ell^2(\Gamma)), \quad g \mapsto (\rho_g : \delta_k \mapsto \delta_{kg^{-1}})$$

to define  $VN_{\text{right}}(\Gamma) = \rho(\Gamma)'' \subseteq \mathcal{B}(\ell^2(\Gamma))$ . Notice that  $\delta_e \in \ell^2(\Gamma)$  is cyclic for  $\lambda(\mathbb{C}[\Gamma])$  as well as  $\rho(\mathbb{C}[\Gamma])$ . This means that it is cyclic for both  $VN(\Gamma)$  and  $VN_{\text{right}}(\Gamma)$ . It's easy to see that  $VN(\Gamma)' = VN_{\text{right}}(\Gamma)$ , so  $\delta_e$  is separating for  $VN(\Gamma)$  and  $VN_{\text{right}}(\Gamma)$ .

**Corollary 6.16.** If  $A \subseteq \mathcal{B}(\mathcal{H})$  is commutative, then each cyclic vector for  $A$  is also separating for  $A$ .

*Proof.* If  $\alpha \in \mathcal{H}$  is cyclic for  $A$ , then it is separating for  $A'$ , but since  $A \subseteq A'$  it is also separating for  $A$ . □

**Theorem 6.17** (Classification of commutative vNa's).

Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a commutative vNa with a cyclic vector  $\alpha \in \mathcal{H}$ . Suppose  $A_0 \subseteq A$  is a  $C^*$ -algebra that is SOT-dense. Then there exists a finite regular positive Borel measure  $\mu$  on  $\sigma(A_0)$  and an isomorphism

$$\tilde{\Gamma} : A \rightarrow L^\infty(\sigma(A_0), \mu) \subseteq \mathcal{B}(L^2(\sigma(A_0), \mu))$$

that extends the Gelfand transform  $\Gamma : A_0 \rightarrow C(\sigma(A_0))$ . Furthermore,  $\tilde{\Gamma}$  is spacial, that is induced by conjugation with a unitary  $U : \mathcal{H} \rightarrow L^2(\sigma(A_0), \mu)$ .

*Proof.* Since  $A_0$  is a commutative  $C^*$ -algebra, the Gelfand transform  $\Gamma : A_0 \rightarrow C(\sigma(A_0))$  is an isometric  $*$ -isomorphism. Define  $\varphi_0 : A \rightarrow \mathbb{C}$  by  $x \mapsto \langle x\alpha_0, \alpha_0 \rangle$ . Then  $\varphi_0\Gamma^{-1} : C(\sigma(A_0)) \rightarrow \mathbb{C}$  is a bounded linear functional, so by Riesz-Markoff there exists a regular Borel measure  $\mu$  on  $\sigma(A_0)$  such that

$$\varphi_0\Gamma^{-1}(f) = \int_{\sigma(A_0)} f d\mu.$$

For every positive function  $f \in C(\sigma(A_0))$  we have

$$\begin{aligned} \int_{\sigma(A_0)} f d\mu &= \int \sqrt{f}^2 d\mu = \varphi_0\Gamma^{-1}(\sqrt{f}^2) = \langle \Gamma^{-1}(\sqrt{f}^2)\alpha_0, \alpha_0 \rangle \\ &= \langle \Gamma^{-1}(\sqrt{f})^2\alpha_0, \alpha_0 \rangle = \langle \Gamma^{-1}(\sqrt{f})\alpha_0, \Gamma^{-1}(\sqrt{f})\alpha_0 \rangle \\ &= \|\Gamma^{-1}(\sqrt{f})\alpha_0\|^2 \geq 0 \end{aligned}$$

and  $\mu$  is a positive measure. Furthermore,  $\mu$  is finite, since

$$\mu(\sigma(A_0)) = \varphi_0(1) = \|\alpha_0\|^2 < \infty.$$

Now we prove that  $\text{supp } \mu = \sigma(A_0)$ . If  $\text{supp } \mu \subsetneq \sigma(A_0)$ , then there exists  $\emptyset \neq S^{\text{open}} \subseteq \sigma(A_0)$  with  $\mu(S) = 0$ . Consider a nonnegative  $f \in C(\sigma(A_0)) \setminus (0)$  with  $f|_{S^c} = 0$ . Then

$$\|\Gamma^{-1}(\sqrt{f})\alpha_0\|^2 = \int_{\sigma(A_0)} f d\mu = \int_S f d\mu = 0.$$

We get  $\Gamma^{-1}(\sqrt{f})\alpha_0 = 0$ , which by cyclicity of  $\alpha_0$  implies  $\Gamma^{-1}(\sqrt{f}) = 0$ ,  $\sqrt{f} = 0$  and  $f = 0$ , a contradiction. Define

$$U_0 : A_0\alpha_0 \rightarrow C(\sigma(A_0)) \subseteq L^2(\sigma(A_0), \mu), \quad x\alpha_0 \mapsto \Gamma(x).$$

Since  $\alpha_0$  is separating for  $A_0$ , this  $U_0$  is a well-defined linear map. For  $x, y \in A_0$ , we have

$$\begin{aligned} \langle U_0(x\alpha_0), U_0(y\alpha_0) \rangle &= \langle \Gamma(x), \Gamma(y) \rangle_2 \\ &= \int_{\sigma(A_0)} \overline{\Gamma(y)}\Gamma(x) d\mu \\ &= \int_{\sigma(A_0)} \Gamma(y^*x) d\mu \\ &= \varphi_0(y^*x) = \langle y^*x\alpha_0, \alpha_0 \rangle = \langle x\alpha_0, y\alpha_0 \rangle \end{aligned}$$

and so  $U_0$  is an isometry! Since  $\alpha_0$  is cyclic for  $A$  and  $A_0$  is SOT-dense in  $A$ ,  $\alpha_0$  is cyclic for  $A_0$ . So  $A_0\alpha_0$  is dense in  $\mathcal{H}$  and the image of  $U_0$  is the entire  $C(\sigma(A_0))$ . By continuity,  $U_0$  extends to a surjective isometry

$$U : \mathcal{H} \rightarrow L^2(\sigma(A_0), \mu) = \overline{C(\sigma(A_0), \mu)}^{\langle \cdot, \cdot \rangle^2},$$

where  $U$  is unitary. Next, define

$$\tilde{\Gamma} : A \rightarrow \mathcal{B}(L^2(\sigma(A_0), \mu)), \quad x \mapsto UxU^*.$$

We claim that  $\tilde{\Gamma}$  is an isometric \*-homomorphism. Since  $U$  is unitary, the isometric part is obvious and the homomorphism soon follows. Now we claim that  $\tilde{\Gamma}(A) = M(L^\infty(\sigma(A_0), \mu))$ . For  $x \in A_0$  and  $g \in C(\sigma(A_0))$ , we have

$$\begin{aligned} \tilde{\Gamma}(x)g &= UxU^*g = UxU^{-1}(\Gamma(\Gamma^{-1}(g))) \\ &= Ux(\Gamma^{-1}(g)\alpha_0) = \Gamma(x\Gamma^{-1}(g)) \\ &= \Gamma(x)g = M_{\Gamma(x)}g \end{aligned}$$

and since  $C(\sigma(A_0))$  is dense in  $L^2(\sigma(A_0), \mu)$ , we get  $\tilde{\Gamma}(x) = M_{\Gamma(x)}$ . It follows that

$$\tilde{\Gamma}(A_0) = M(C(\sigma(A_0))) \subseteq M(L^\infty(\sigma(A_0), \mu)).$$

Then we use the fact that  $\tilde{\Gamma}$  is SOT-continuous (by definition) and  $M(L^\infty)$  is a vNa (example 5.14) to get

$$\tilde{\Gamma}(A) = \tilde{\Gamma}(\overline{A_0}^{\text{SOT}}) \subseteq \overline{\tilde{\Gamma}(A_0)}^{\text{SOT}} \subseteq \overline{M(L^\infty(\sigma(A_0), \mu))}^{\text{SOT}} = M(L^\infty(\sigma(A_0), \mu)).$$

The reverse inclusion is done by nets. Suppose  $(\tilde{\Gamma}(x_i))_i \subseteq \tilde{\Gamma}(A_0)$  WOT-converges to  $T \in \mathcal{B}(L^2(\sigma(A_0), \mu))$ . Then for all  $\beta \in \mathcal{H}$  we have

$$\begin{aligned} \langle TU\beta, U\mu \rangle &= \lim_i \langle \tilde{\Gamma}(x_i)U\beta, U\mu \rangle \\ &= \lim_i \langle Ux_iU^*U\beta, U\mu \rangle \\ &= \lim_i \langle x_i\beta, \mu \rangle \end{aligned}$$

and  $(x_i)_i \xrightarrow{\text{WOT}} U^*TU \in \mathcal{B}(\mathcal{H})$ . Since  $\overline{A_0}^{\text{WOT}} = A$ , we get  $x = U^*TU \in A$  and  $\tilde{\Gamma}(x) = T$ , so  $\overline{\tilde{\Gamma}(A_0)}^{\text{WOT}} \subseteq \tilde{\Gamma}(A)$ . Finally, we ask what is  $\overline{M(C(\sigma(A_0)))}^{\text{WOT}}$ ? WOT on that set is generated by seminorms

$$|\langle M_g\alpha, \beta \rangle| = \left| \int_{\sigma(A_0)} g\alpha\beta d\mu \right|,$$

where  $g \in C(\sigma(A_0))$  and  $\alpha, \beta \in L^2(\sigma(A_0), \mu)$ . Accordingly, weak-\* topology on  $L^\infty(\sigma(A_0), \mu)$  is generated by seminorms

$$\left| \int_{\sigma(A_0)} g\gamma d\mu \right|,$$

where  $\mu \in L^1(\sigma(A_0), \mu)$  (this is because for a  $\sigma$ -finite  $X$ ,  $(L^1(X))^* = L^\infty(X)$ ). By Hölder,  $L^1$  function is a product of two  $L^2$  functions, so these two topologies coincide. By Goldstine's theorem,  $C(\sigma(A_0))$  are weak-\* dense in  $L^\infty(\sigma(A_0), \mu)$ . Then we have

$$M(L^\infty(\sigma(A_0), \mu)) = \overline{M(C(\sigma(A_0)))}^{\text{WOT}} = \overline{\tilde{\Gamma}(A_0)}^{\text{WOT}} \subseteq \tilde{\Gamma}(A)$$

and finally  $\tilde{\Gamma}(A) = M(L^\infty(\sigma(A_0), \mu))$ .  $\square$

*Remark.* Applying this theorem to  $A_0 = A$  we get

$$M(L^\infty(\sigma(A), \mu)) = \tilde{\Gamma}(A) = M(C(\sigma(A))).$$

The following statement from the above proof is important in its own right.

**Lemma 6.18.**  *$C(\sigma(A_0))$  are weak-\* dense in  $L^\infty(\sigma(A_0), \mu)$ .*

*Proof.* For any bounded measurable function  $f \in B(\sigma(A_0))$ , there exists a net  $(f_i)_i \subseteq C(\sigma(A_0))$  such that  $f_i \xrightarrow{\text{weak-*}} f$  by Goldstine (see the proof for theorem 6.8).  $\square$

In the proof, we used the following theorem.

**Theorem 6.19.**

*Suppose  $1 \leq p < \infty$  and  $\mu$  is a  $\sigma$ -finite positive measure on  $X$ , and  $\Phi$  is a bounded linear functional on  $L^p(X, \mu)$ . Then there is a unique  $g \in L^q(X, \mu)$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ , such that*

$$\Phi(f) = \int_X fg d\mu.$$

*Moreover,  $\|\Phi\| = \|g\|_q$ .*

The theorem tells us that under these conditions,  $L^q(X, \mu)$  is isometrically isomorphic to the dual space of  $L^p(X, \mu)$ . In particular, we used the fact that  $(L^1(X, \mu))^* = L^\infty(X, \mu)$ . This is theorem 6.16 in W.Rudin's *Real and complex analysis*.

How crucial is the cyclicity assumption? Let  $A \subseteq \mathcal{B}(\mathcal{H})$  be a commutative vNa. Pick  $0 \neq \alpha \in \mathcal{H}$ . Define  $\mathcal{K} := \overline{A\alpha}$  and let  $p : \mathcal{H} \rightarrow \mathcal{K}$  be an orthogonal projection, so by reducibility of  $\mathcal{K}$  we get  $p \in A'$ . Therefore  $pAp = Ap \subseteq \mathcal{B}(\mathcal{H})$  is a commutative vNa with a cyclic vector  $\alpha \in \mathcal{K}$ . Then, again by theorem,  $Ap \cong L^\infty(X, \mu)$  for some  $(X, \mu)$ . By Zorn's lemma,  $A \cong L^\infty(Y, \nu)$  for some disjoint union of measure spaces  $(Y, \nu)$ .

**Proposition 6.20.** *Let  $\mathcal{H}$  be a separable Hilbert space and  $A \subseteq \mathcal{B}(\mathcal{H})$  a commutative vNa. Then there exists a separating vector for  $A$ .*

*Proof.* By Zorn, there exists a maximal set of unit vectors  $(\alpha_k)_k$  such that  $A\alpha_k \perp A\alpha_l$  for  $k \neq l$ . By maximality,  $\sum_k A\alpha_k$  is dense in  $\mathcal{H}$ . Define  $\alpha = \sum_{n=1}^\infty \frac{1}{2^n} \alpha_n$ . We claim that

$\alpha$  is separating for  $A$ . Indeed, let  $x \in A$  such that  $x\alpha = 0$ . Then  $\sum_{n=1}^{\infty} \frac{1}{2^n} x\alpha_n = 0$ . By orthogonality,  $x\alpha_n = 0$  for all indices  $n$ . For all  $y \in A$ , we get  $xy\alpha_n = yx\alpha_n = 0$ , so  $x|_{A\alpha_n} = 0$  for all  $n$ . But since  $\sum_n A\alpha_n$  is dense in  $A$ , we get  $x = 0$ .  $\square$

**Corollary 6.21.** *Let  $\mathcal{H}$  be a separable Hilbert space and  $A \subseteq \mathcal{B}(\mathcal{H})$  is a maximal commutative  $vNa$ . Then there exists a cyclic vector for  $A$ .*

*Proof.* By the proposition, there exists a separating vector  $\alpha$  for  $A$ , which is then cyclic for  $A'$ . But since  $A$  is maximal, we get  $A = A'$ .  $\square$

**Theorem 6.22.**

*Let  $\mathcal{H}$  be a separable Hilbert space and  $A \subseteq \mathcal{B}(\mathcal{H})$  a commutative  $vNa$ . Then there exists a compact Hausdorff space  $X$  and a finite regular Borel measure  $\mu$  on  $X$  such that  $A \cong L^\infty(X, \mu)$ .*

*Proof.* By proposition, there exists a separating vector  $\alpha \in \mathcal{H}$  for  $A$ . Form  $\mathcal{K} := \overline{A\alpha}$ . Then the algebra  $\{x|_{\mathcal{K}} \mid x \in A\} \subseteq \mathcal{B}(\mathcal{K})$  is  $*$ -isomorphic to  $A$ , has cyclic vector  $\alpha$  and the above theorem applies.  $\square$

**Example 6.23.** *Let  $\Gamma = \mathbb{Z}/n\mathbb{Z}$ . Then the spectrum is  $\sigma(VN(\Gamma)) = \{e^{\frac{2k\pi i}{n}} \mid 0 \leq k < n\}$  and  $\mu(e^{\frac{2k\pi i}{n}}) = \frac{1}{n}$ . Then*

$$VN(\Gamma) = L^\infty(\sigma(VN(\Gamma)), \mu) \cong \mathbb{C}^n$$

*as an algebra. The generator for  $VN(\Gamma)$  is the matrix*

$$\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

**Example 6.24.** *Let  $\Gamma = \mathbb{Z}$ . Then  $\mathbb{T}$  is the Pontryagin dual of  $\Gamma$ , so  $C^*(\Gamma) = C(\mathbb{T})$  and  $VN(\Gamma) = L^\infty(\mathbb{T}, m)$ , where  $m$  is the normalized Lebesgue measure.*