

FUNCTIONAL ANALYSIS - NOTES

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1 Convexity

1.1 Locally convex spaces

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ be a field.

Definition 1.1. A *topological vector space* (TVS) is a \mathbb{F} -vector space that is also a topological space and the two structures are compatible. This means that the usual operations on vector spaces

$$V \times V \rightarrow V, (x, y) \mapsto x + y, \quad \mathbb{F} \times V \rightarrow V, (\lambda, x) \mapsto \lambda x$$

are continuous maps.

Example 1.2. Normed spaces are TVS.

Definition 1.3. Let V be a \mathbb{F} -space. Map $p : V \rightarrow \mathbb{R}$ is a *seminorm* if:

- (1.) $p(x) \geq 0, \forall x \in V$ (positivity);
- (2.) $p(\lambda x) = |\lambda|p(x), \forall x \in V, \forall \lambda \in \mathbb{F}$ (positive homogeneity);
- (3.) $p(x + y) \leq p(x) + p(y), \forall x, y \in V$ (triangle inequality).

A seminorm is therefore almost a norm, except that it's not necessarily positive definite.

Let V be a \mathbb{F} -vector space and \mathcal{P} a family of seminorms in V . Let \mathcal{T} be the topology in V with the following subbasis:

$$U(x_0, p, \varepsilon) = \{x \in V \mid p(x - x_0) < \varepsilon\}; \quad x_0 \in V, p \in \mathcal{P}, \varepsilon > 0.$$

Basis of \mathcal{T} are finite intersections of such sets. The set $U \subseteq V$ is open iff for every $x_0 \in U$ there exist seminorms $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon_1, \dots, \varepsilon_n > 0$ such that

$$U \supset \bigcap_{j=1}^n U(x_0, p_j, \varepsilon_j).$$

The space (V, \mathcal{T}) is then a TVS. If \mathcal{P} is a singleton and its element is a norm, then (V, \mathcal{T}) is a normed space.

Definition 1.4. A TVS X is a *locally-convex space* (LCS) if its topology is generated by a family of seminorms \mathcal{P} satisfying

$$\bigcap_{p \in \mathcal{P}} \{x \in X \mid p(x) = 0\} = \{0\}.$$

Equivalently, for every $x \in X \setminus \{0\}$ there exists a seminorm $p \in \mathcal{P}$ such that $p(x) \neq 0$.

Corollary 1.5. Let X be a space with a topology generated by a family of seminorms \mathcal{P} . Then X is a LCS iff it is Hausdorff.

Proof. Start with (\Rightarrow) . Let $x, y \in X$ be two distinct points. There exists a seminorm $p \in \mathcal{P}$ such that $p(x - y) = b \neq 0$. Define the sets

$$V = U\left(x, p, \frac{b}{2}\right), \quad W = U\left(y, p, \frac{b}{2}\right).$$

By the triangle inequality property of a seminorm, V and W separate the points x, y . Now the converse (\Leftarrow) . Choose a point $x \neq 0$. Then there exist open sets $0 \in V, x \in W$ that separate 0 from x . There exists an open basis set $\bigcap_{j=1}^n U(0, p_j, \varepsilon_j) \subseteq V$, so $x \notin U(0, p_j, \varepsilon_j)$ for some index j . Hence, $p_j(x - 0) = p_j(x) \geq \varepsilon > 0$. \square

LCS generally aren't first-countable, so we need to go beyond the usual sequences to describe the topology.

Definition 1.6. Partially ordered set (I, \leq) is *upwards-directed* if

$$\forall i', i'' \in I : \exists i \in I : i \geq i', i \geq i''.$$

Example 1.7. (1.) Every linearly ordered set is upwards-directed.

(2.) Let (X, \mathcal{T}) be a topological space and $x_0 \in X$. Define a family of sets

$$\mathcal{U} = \{U^{\text{open}} \subseteq X \mid x_0 \in U\}$$

and a relation $U \geq V \Leftrightarrow U \subseteq V$. Then (\mathcal{U}, \leq) is an upwards-directed set.

(3.) Let S be a set and \mathcal{F} a family of all finite subsets of S . Define $F_1 \geq F_2$ in \mathcal{F} if $F_1 \supseteq F_2$. Then (\mathcal{F}, \leq) is again an upwards-directed set.

Definition 1.8. A *generalized sequence (net)* is $((I, \leq), x)$, where (I, \leq) is upwards-directed and $x : I \rightarrow X$ is a function. We usually write $(x_i)_{i \in I}$ or $(x(i))_{i \in I}$.

Example 1.9. (1.) Every sequence is a net.

(2.) Let (X, \mathcal{T}) be a topological space, $x_0 \in X$ and \mathcal{U} a collection of all open sets which contain x_0 (see example 1.7). For each $U \in \mathcal{U}$ pick a $x_U \in U$. Then $(x_U)_{U \in \mathcal{U}}$ is a net.

Definition 1.10. Let X be a topological space. A net $(x_i)_{i \in I}$ *converges* to an $x \in X$ if

$$\forall U^{\text{open}} \subseteq X, x \in U : \exists i_0 \in I : \forall i \geq i_0 : x_i \in U.$$

We write $\lim_{i \in I} x_i = x$, or alternatively, $x_i \xrightarrow{i \in I} x$. A point $x \in X$ is called a *cluster point* of a net $(x_i)_{i \in I}$ if

$$\forall U^{\text{open}} \subseteq X, x \in U : \forall i_0 \in I : \exists i \geq i_0 : x_i \in U.$$

Example 1.11. Take the net $(x_U)_{U \in \mathcal{U}}$ from example 1.9. It follows from the definition that $x_U \xrightarrow{U \in \mathcal{U}} x_0$.

Proposition 1.12. (1.) Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ iff there exists a net $(a_i)_{i \in I}$ in A such that $a_i \rightarrow x$.
(2.) Let X, Y be topological spaces and $f : X \rightarrow Y$. Then f is continuous at $x_0 \in X$ iff $f(x_i) \rightarrow f(x_0)$ for every net $(x_i)_{i \in I}$ that converges to x_0 .

Proof. (1.) We begin with the implication to the left (\Leftarrow). Take any $U^{\text{open}} \subseteq X$ such that $x \in U$. Since $a_i \rightarrow x$, there exists an index $i_0 \in I$, such that for every $i \geq i_0$ we have $a_i \in U$. Hence $a_i \in A \cap U \neq \emptyset$ and $x \in \overline{A}$. The converse (\Rightarrow) is similar. Define $\mathcal{U} = \{U^{\text{open}} \subseteq X \mid x \in U\}$. Since $x \in \overline{A}$, for each $U \in \mathcal{U}$, we have $A \cap U \neq \emptyset$. Pick $a_U \in A \cap U$. Then the net $(a_U)_{U \in \mathcal{U}}$ in A converges to x .
(2.) Start with the implication (\Rightarrow). Let f be a continuous function and let $(x_i)_{i \in I}$ converge to x_0 . Let $f(x_0) \in U^{\text{open}} \subseteq Y$. Then $x_0 \in f^{-1}(U)^{\text{open}} \subseteq X$, which means there exists an $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, $x_i \in f^{-1}(U)$. But that implies that for every $i \geq i_0$, $f(x_i) \in U$, which is what we wanted. Now we prove the converse (\Leftarrow). Let's say that for every net $(x_i)_{i \in I}$ that converges to x_0 , we have $f(x_i) \xrightarrow{i \in I} f(x_0)$. So for every set $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$ (using the first item), which proves that f is continuous. \square

Proposition 1.13. (a) A net $(x_i)_{i \in I}$ in a LCS converges to x_0 iff a net $(p(x_i - x_0))_{i \in I}$ converges to 0 for all $p \in \mathcal{P}$.

(b) The topology in a LCS X is the coarsest (smallest) topology in which all the maps $x \mapsto p(x - x_0)$ are continuous for every $x_0 \in X$ and $p \in \mathcal{P}$.

Proof. (a) Start with the implication (\Rightarrow). Take any $p \in \mathcal{P}$. If we take $U = U(x_0, p, \varepsilon)$ in the definition of a limit of a net, we get

$$\forall \varepsilon > 0 : \exists i_0 \in I : \forall i \geq i_0 : p(x_i - x_0) \in (-\varepsilon, \varepsilon).$$

This proves our claim. Now for the opposite direction (\Leftarrow). For every $p \in \mathcal{P}$ and $\varepsilon_p > 0$ there exists an i_p such that for every $i \geq i_p$, $x_i \in U(x_0, p, \varepsilon_p)$. Now let U be an arbitrary basis set that includes the point x_0 . That means U is the finite intersection of the sets $U(x_0, p, \varepsilon_p)$. Now let i_0 be greater than all indices i_p . By our assumption, for every $i \geq i_0$ we have $x_i \in U$.

(b) Pick any point $x_0 \in X$ and a seminorm $p \in \mathcal{P}$. Denote

$$f_{x_0, p} : X \rightarrow \mathbb{R}, \quad f_{x_0, p}(x) = p(x - x_0).$$

We essentially have to prove that the sets

$$f_{x_0, p}^{-1}(V), \quad V^{\text{open}} \subseteq \mathbb{R}, \quad x_0 \in X, \quad p \in \mathcal{P}$$

generate a subbasis for the seminorm topology of a LCS space. Since f_{p, x_0} are continuous functions (by the first item and proposition 1.12), these are all open sets in the

seminorm topology. But on the other hand, all subbasis sets $U(x_0, p, \varepsilon)$ of the seminorm topology are of this type, so the above subbasis generates the seminorm topology, thus concluding our proof. \square

Example 1.14. Let X be a topological space. For every $K^{\text{compact}} \subseteq X$ we define a seminorm

$$p_K : C(X) \rightarrow \mathbb{R}, \quad f \mapsto \sup_{x \in K} |f(x)|.$$

We endow $C(X)$ with the topology induced by the family of seminorms $\{p_K \mid K^{\text{compact}} \subseteq X\}$. It's trivial to see that $C(X)$ is then a LCS. Moreover, we notice that the induced seminorm topology coincides with the topology of compact convergence on X . In the future, we will require X to be locally compact Hausdorff (this implies complete regularity) so that $C(X)$ has nice properties. There are examples of not completely regular spaces X such that the only elements of $C(X)$ are constant maps.

Example 1.15. Let $D^{\text{open}} \subseteq \mathbb{C}$ and let $\mathcal{H}(D)$ be the set of all holomorphic functions on D . As in the example 1.14, we define $\mathcal{P} = \{p_K \mid K^{\text{compact}} \subseteq D\}$. This endows $\mathcal{H}(D)$ with a topology and makes $\mathcal{H}(D)$ into a LCS. Convergence in this topology coincides with the uniform convergence on compacts in D .

1.2 Weak topology

Let X be a normed space and let X^* be its dual. For every $f \in X^*$ we define a seminorm

$$p_f : X \rightarrow \mathbb{R}, \quad x \mapsto |f(x)|.$$

We claim that $\mathcal{P} = \{p_f \mid f \in X^*\}$ is a family of seminorms that induces a topology on X which makes X a LCS. Indeed, for any $x \in X \setminus \{0\}$ define a nonzero linear functional

$$f : \text{span}(x) \rightarrow \mathbb{F}, \quad f(\lambda x) = \lambda$$

and extend it to $F : X \rightarrow \mathbb{F}$ using Hahn–Banach. Then $p_F(x) \neq 0$. The induced topology is the *weak topology* on X . We denote it as $\sigma(X, X^*)$.

Proposition 1.16. A net $(x_i)_{i \in I}$ converges to $x_0 \in X$ with respect to the weak topology iff $f(x_i) \rightarrow f(x_0)$, $\forall f \in X^*$.

Remark. We use the notation $x_i \xrightarrow{w} x_0$. Furthermore, a closure of a set $A \subseteq X$ in the weak topology will be denoted by \overline{A}^w .

Remark. The closure of a set $A \subseteq X$ in the weak topology will be denoted as \overline{A}^w .

Example 1.17. Let $X = \mathbb{R}^n$. Then $X^* = \mathbb{R}^n$ and every linear functional f is of the form $f(x) = \langle x, y \rangle$ for some $y \in X$ (Riesz' representation theorem). The subbasis sets are

$$U(0, p_y, \varepsilon) = \{x \in \mathbb{R}^n \mid |\langle x, y \rangle| < \varepsilon\}.$$

Weak topology in this case coincides with Euclidean topology.

Let X again be a normed space. To $x \in X$ we assign the seminorm

$$p_x : X^* \rightarrow \mathbb{R}, \quad f \mapsto |f(x)|.$$

The family $\{p_x \mid x \in X\}$ defines a topology in X^* in which X^* becomes a LCS. This topology is called the *weak-* topology* and is denoted by $\sigma(X^*, X)$.

Proposition 1.18. *A net $(f_i)_{i \in I}$ converges to $f \in X^*$ with respect to the weak-* topology iff $f_i(x) - f(x) \rightarrow 0, \forall x \in X$.*

Remark. We use the notation $f_i \xrightarrow{w^*} f$. Furthermore, a closure of a set $A \subseteq X^*$ in the weak-* topology will be denoted by \overline{A}^{w^*} .

We can compare weak-* topology on X^* with its weak topology. As a consequence of Hahn–Banach, we have for every $x \in X$

$$\|x\| = \sup\{|f(x)| \mid f \in X^*, \|f\| \leq 1\},$$

which implies that the map

$$\iota : X \hookrightarrow X^{**}, \quad x \mapsto (f \mapsto f(x))$$

is an isometry and therefore injective. This means that every seminorm in the weak-* topology is also a seminorm in a weak topology on X^* , so the weak topology is finer (stronger) than the weak-* topology on X^* .

Remark. Weak and weak-* topology can be defined even if X is merely a LCS. In that case, X^* is of course defined as the space of continuous linear functionals on X .

1.3 Banach–Alaoglu theorem

Theorem 1.19 (Banach–Alaoglu).

Let X be a normed space. Then the closed unit ball in X^ (denoted by $(X^*)_1$) is compact in the weak-* topology in X^* .*

Proof. To $x \in X$ we assign $D_x = \{z \in \mathbb{F} \mid |z| \leq \|x\|\}$ and endow D_x with the Euclidean topology. Then D_x is clearly compact. The set $P = \prod_{x \in X} D_x$ is compact in the product topology (Tychonoff theorem). Now we construct a map

$$\Phi : (X^*)_1 \rightarrow P, \quad f \mapsto (f(x))_{x \in X} \in P.$$

Clearly, Φ is well-defined and injective. We start by proving that Φ is continuous. Let $(f_i)_{i \in I}$ be a net in $(X^*)_1$ that converges to $f \in X^*$ in the weak-* topology. Then $f_i(x) \rightarrow f(x)$ for each $x \in X$. By the definition of the product topology in P , this means that $\Phi(f_i) \mapsto \Phi(f)$ in P . Hence Φ is continuous. Since Φ is injective, it induces an inverse map

$$\Phi^{-1} : \text{im}(\Phi) \rightarrow (X^*)_1$$

that is also continuous (we read the previous argument backwards).

Finally, we prove that $\text{im}(\Phi)$ is closed in P . Suppose that $(\Phi(f_i))_{i \in I}$ converges to $p = (p_x)_{x \in X} \in P$. By definition of the product topology, this means that $f_i(x) \rightarrow p_x$ for all

$x \in X$. Define

$$f : X \rightarrow \mathbb{F}, \quad x \mapsto p_x.$$

Then f is linear and $f \in (X^*)_1$. Thus $p = \Phi(f) \in \text{im}(\Phi)$. This in turn implies that $(\text{im } \Phi)^{\text{closed}} \subseteq P^{\text{compact}}$. But we know that $(X^*)_1 \approx \text{im}(\Phi)$, which implies that $(X^*)_1$ is also compact. \square

Corollary 1.20. *Every Banach space X is isometrically isomorphic to a closed subspace of $C(K)$ for some compact T_2 space K .*

Proof. Denote $K = (X^*)_1$ endowed with the weak-* topology. By the Banach–Alaoglu theorem, K is compact and T_2 . We now define the map

$$\Delta : X \rightarrow C(K), \quad x \mapsto (f \mapsto f(x)).$$

First, we prove that Δ is isometric. By Hahn–Banach, for every $x \in X \setminus \{0\}$ there exists an $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Then we have

$$\|\Delta(x)\|_\infty = \sup_{g \in K} |g(x)| = \|x\|.$$

Since Δ is an isometry, its image is complete and thus closed in $C(K)$. Obviously Δ is a linear map, so we are done. \square

1.4 Minkowski gauge

ADD MOTIVATION

Definition 1.21. Let X be a \mathbb{F} -vector space. A set $A \subseteq X$ is

- balanced if:

$$\forall x \in A : \forall \alpha \in \mathbb{F}, |\alpha| \leq 1 : \alpha x \in A.$$

- absorbing if:

$$\forall x \in X : \exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : tx \in A.$$

- absorbing in $a \in A$ if $A - a = \{x - a \mid x \in A\}$ is absorbing.

Example 1.22. Let X be a vector space and p a seminorm in X . Then

$$V = \{x \in X \mid p(x) < 1\}$$

is convex, balanced, absorbing in each of its points.

Theorem 1.23.

Let X be a vector space and $V \subseteq X$ convex, balanced and absorbing in each of its points. Then there exists a unique seminorm p on X such that

$$V = \{x \in X \mid p(x) < 1\}.$$

Proof. To V we associate the Minkowski gauge:

$$p(x) = \inf\{t \geq 0 \mid x \in t \cdot V\},$$

where $t \cdot V = \{t \cdot v \mid v \in V\}$. First we prove that p is well defined. Since V is absorbing, we have $X = \bigcup_{n \in \mathbb{N}} n \cdot V$, so for every $x \in X$ the set $\{t \geq 0 \mid x \in t \cdot V\}$ is nonempty. It's also clear to see that $p(0) = 0$. Next we check for homogeneity. Suppose $\alpha \neq 0$. Then

$$\begin{aligned} p(\alpha x) &= \inf\{t \geq 0 \mid \alpha x \in t \cdot V\} \\ &= \inf\left\{t \geq 0 \mid x \in \frac{t}{\alpha} \cdot V\right\} \\ &= \inf\left\{t \geq 0 \mid x \in \frac{t}{|\alpha|} \cdot V\right\} \\ &= \inf |\alpha| \left\{\frac{t}{|\alpha|} \geq 0 \mid x \in \frac{t}{|\alpha|} \cdot V\right\} \\ &= |\alpha| p(x). \end{aligned}$$

Now we do the same for triangle inequality: let $\alpha, \beta \geq 0$ so that $\alpha + \beta > 0$. Let $a, b \in V$. Then

$$\alpha a + \beta b = (\alpha + \beta) \left(\frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b \right) \in (\alpha + \beta) \cdot V.$$

This means that $\alpha \cdot V + \beta \cdot V \subseteq (\alpha + \beta) \cdot V$. Now let $x, y \in X$ and $p(x) = \alpha, p(y) = \beta$. Take $\delta > 0$. Then $x \in (\alpha + \delta) \cdot V, y \in (\beta + \delta) \cdot V$. Hence

$$x + y \in (\alpha + \delta) \cdot V + (\beta + \delta) \cdot V \subseteq (\alpha + \beta + 2\delta) \cdot V,$$

and by definition, $p(x + y) \leq \alpha + \beta + 2\delta$. Since $\delta > 0$ was arbitrary, we have $p(x + y) \leq \alpha + \beta = p(x) + p(y)$. Now that we have proved that p is a seminorm, we can show that

$$V = \{x \in X \mid p(x) < 1\}.$$

The inclusion (\supseteq) is easy: if $p(x) < 1$, then $x \in (p(x) + \varepsilon) \cdot V$ for all $\varepsilon > 0$. By choosing $\varepsilon = 1 - p(x) > 0$, we get $x \in V$. Now we prove the other inclusion (\subseteq) . Let $x \in V$. Since V is absorbing in x , there exists an $\varepsilon > 0$ such that $y = x + tx \in V$ for all $t \in (0, \varepsilon)$. This means that $x = \frac{1}{t+1}y$, where $y \in V$. This implies that

$$p(x) = \frac{1}{t+1} p(y) \leq \frac{1}{1+t} \leq 1,$$

which proves the equality. Lastly, we prove the p is unique. Suppose there is some other seminorm q such that

$$\{x \in X \mid p(x) < 1\} = \{x \in X \mid q(x) < 1\}.$$

Suppose $p \neq q$. W.l.o.g. there exists an $x \neq 0$ such that $p(x) > q(x)$. By homogeneity, we can assume that $p(x) = 1 > q(x)$, contradicting our assumption. \square

Remark. If X is a TVS and V is an open subset, then V is absorbing at each of its points.

Corollary 1.24. *Let X be a TVS and \mathcal{U} a collection of all open convex balanced subsets of X . Then X is locally convex iff \mathcal{U} is a basis for the neighborhood system at 0.*

1.5 Applications of Hahn–Banach

Recall: if X is a \mathbb{R} -vector space then $p : X \rightarrow \mathbb{R}$ is a sublinear functional if

$$p(x + y) \leq p(x) + p(y), \quad \forall x, y \in X$$

and

$$p(\alpha x) = \alpha x, \quad \forall x \in X, \alpha > 0.$$

Theorem 1.25 (Hahn–Banach theorem).

\mathbb{R} : Suppose X is a \mathbb{R} -vector space and $p : X \rightarrow \mathbb{R}$ is a sublinear functional. Given a linear functional f on $Y \leq X$ such that $f(y) \leq p(y)$ for every $y \in Y$, f extends to a linear functional $F : X \rightarrow \mathbb{R}$ such that $F(x) \leq p(x)$ for every $x \in X$.

\mathbb{C} : Suppose X is a \mathbb{C} -vector space and $p : X \rightarrow \mathbb{R}$ is a seminorm. Given a linear functional f on $Y \leq X$ such that $|f(y)| \leq p(y)$ for every $y \in Y$, f extends to a linear functional $F : X \rightarrow \mathbb{R}$ such that $|F(x)| \leq p(x)$ for every $x \in X$.

Corollary 1.26 (Hahn–Banach extension theorem). *Let X be a normed space, $f \in X^*$ and $Y \leq X$. Then there exists an $F \in X^*$ such that $F|_Y = f$ and $\|F\| = \|f\|$.*

Corollary 1.27 (Hahn–Banach separation theorem). *Suppose X is a LCS and $A, B \subseteq X$ are disjoint closed convex sets. If B is compact then there exists an $f \in X^*$ that separates A from B :*

$$\exists \alpha, \beta \in \mathbb{R} : \forall a, b \in B : \operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(b).$$

Theorem 1.28.

Let X be a LCS and $A \subseteq X$ convex. Then $\overline{A} = \overline{A}^w$.

Proof. Since the weak topology is weaker than the original topology, we have $\overline{A} \subseteq \overline{A}^w$. Let $x \notin \overline{A}$. We now separate \overline{A} and the compact set $\{x\}$: there exists $f \in X^*$ so that there exist $\alpha, \beta \in \mathbb{R}$ and we have

$$\operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(x)$$

for all $a \in \overline{A}$. This means that

$$\overline{A} \subseteq \{y \in X \mid \operatorname{Re} f(y) \leq \alpha\} = (\operatorname{Re} f)^{-1}(-\infty, \alpha] = C.$$

Since C is closed in the weak topology, it follows from $A \subseteq C$ that $\overline{A}^w \subseteq \overline{C}^w = C$. Since $x \notin C$, we have $x \notin \overline{A}^w$. \square

Corollary 1.29. *A convex set in a LCS is closed iff it is weakly closed.*

Proposition 1.30. *Let X be a TVS and $f : X \rightarrow \mathbb{F}$ a linear functional. The following are equivalent:*

- (1.) f is continuous;
- (2.) f is continuous in 0;
- (3.) f is continuous in some point;
- (4.) $\ker f$ is closed;
- (5.) $x \mapsto |f(x)|$ is a seminorm.

If X is a LCS, then these are also equivalent to

- (6.) $\exists \alpha_1, \dots, \alpha_n \in \mathbb{R}_{>0}$ and $\exists p_1, \dots, p_n \in \mathcal{P}$ such that

$$|f(x)| \leq \sum_{k=1}^n \alpha_k p_k(x), \quad \forall x \in X.$$

Proof. Equivalence of the first five statements is routine. Assume that X is a LCS. We prove the equivalence of (2) and (6). We start with (6) \Rightarrow (2). Let $(x_i)_{i \in I}$ be a net in X that converges to 0. Then we have

$$0 \leq |f(x_i)| \leq \sum_{k=1}^n \alpha_k p_k(x_i) \xrightarrow{i \in I} 0.$$

This implies that $f(x_i) \xrightarrow{i \in I} 0$, proving the implication. Now the opposite: (2) \Rightarrow (6). We know that $f^{-1}(B_1^\circ(0)) = \{x \in X \mid |f(x)| < 1\}$ is an open neighborhood of 0 in X . Then there exist $p_1, \dots, p_r \in \mathcal{P}$ and an $\varepsilon > 0$ such that

$$0 \in \bigcap_{i=1}^r U(0, p_i, \varepsilon) \subseteq f^{-1}(B_1^\circ(0)).$$

If $p_i(x) < \varepsilon$ for all $i \leq r$, then $|f(x)| < 1$. Pick any $\delta > 0$. Then

$$p_i \left(x \cdot \frac{\varepsilon}{\sum p_i(x) + \delta} \right) = \frac{\varepsilon}{\delta + \sum p_i(x)} \cdot p_i(x) < \varepsilon,$$

which implies

$$\left| f \left(x \cdot \frac{\varepsilon}{\sum p_i(x) + \delta} \right) \right| < 1.$$

From this we get $|f(x)| < \frac{1}{\varepsilon} (\sum p_i(x) + \delta)$. Since $\delta > 0$ was arbitrary, we get

$$|f(x)| \leq \sum_{i=1}^r \frac{1}{\varepsilon} p_i(x).$$

□

Recall the following theorem from measure theory (theorem 2.14 in [4]).

Theorem 1.31 (Riesz–Markoff).

Let X be a compact T_2 space, $\Phi \in C(X)^*$. Then there exists a unique regular Borel measure μ such that

$$\Phi(f) = \int_X f d\mu, \quad \forall f \in C(X).$$

Further, $\|\Phi\| = \|\mu\| = |\mu|(X)$.

Remark. The above also works if X is locally compact and $\Phi \in C_0(X)^*$.

As a corollary, we get the following proposition.

Proposition 1.32. Let X be completely regular. Endow $C(X)$ with a topology induced by its seminorms. If $L \in C(X)^*$ then there exists a compact $K \subseteq X$ and a regular Borel measure on K such that

$$L(f) = \int_K f d\mu, \quad \forall f \in C(X).$$

Conversely, every such pair (K, μ) defines $L \in C(X)^*$ with the above equation.

Proof. Begin with the implication (\Leftarrow) . Given (K, μ) , we just need to prove that the induced functional L is continuous on X . We have

$$|L(f)| = \left| \int_K f d\mu \right| \leq \|\mu\| \sup_K |f| = \|\mu\| p_K(f)$$

and L is continuous. Now the converse (\Rightarrow) . Let $L \in C(X)^*$. By the previous proposition, there exist compact sets $K_1, \dots, K_p \subseteq X$ and $\alpha_1, \dots, \alpha_p > 0$ such that

$$|L(f)| \leq \sum_{j=1}^p \alpha_j p_{K_j}(f).$$

Let $K = \bigcup_{j=1}^p K_j$ and $\alpha = \max\{\alpha_1, \dots, \alpha_p\}$. Then $\|f\| \leq \alpha p_K(f)$ for all $f \in C(X)$. Observe that if $f \in C(X)$ and $f|_K = 0$, then $L(f) = 0$. We now define a map $F : C(K) \rightarrow \mathbb{F}$. Since X is completely regular, we have a Tietze-like extension theorem: for any compact $K \subseteq X$ and a continuous function $g \in C(K)$, there exists an extension $\tilde{g} \in C(X)$. Define $F(g) := L(\tilde{g})$. First we need to check that F is well defined. Suppose we have two extensions \tilde{g} and $\tilde{\tilde{g}}$ of $g \in C(K)$. Since $\tilde{g} - \tilde{\tilde{g}}$ is evidently zero on K , we have

$$L(\tilde{g}) - L(\tilde{\tilde{g}}) = L(\tilde{g} - \tilde{\tilde{g}}) = 0$$

and F really is well defined. It is also clearly linear, so we just need to check continuity:

$$|F(g)| = |L(\tilde{g})| \leq \alpha \cdot p_K(\tilde{g}) = \alpha \cdot \|g\|_{\infty, K},$$

therefore $\|F\| \leq \alpha$ and F is continuous. Lastly we apply Riesz–Markoff: there exists a regular Borel measure μ on K so that $F(g) = \int_K g d\mu$. If $f \in C(X)$, then $g := f|_K \in C(K)$ and we have

$$L(f) = F(g) = \int_K g d\mu = \int_K f d\mu. \quad \square$$

1.6 Krein–Milman theorem

Definition 1.33. Let X be a vector space and $C \subseteq X$ a convex subset.

(a) A nonempty convex subset $F \subseteq C$ is a *face* if for any $x, y \in C$ we have

$$(\exists t \in (0, 1) : tx + (1 - t)y \in F) \Rightarrow x, y \in F.$$

(b) A point $x \in C$ is called an *extreme point* if $\{x\} \subseteq C$ is a face.

We use the notation $\text{ext}(C)$ for the set of all extreme points of C .

Example 1.34. If we consider spaces of real sequences, we have

- $\text{ext}((\ell^\infty)_1) = \{(\pm 1, \pm 1, \dots)\}$;
- $\text{ext}((\ell^1)_1) = \{(0, 0, \dots, \pm 1, \dots)\}$.

Example 1.35. We prove that for c_0 (the space of complex sequences that converge to 0) we have $\text{ext}(c_0)_1 = \emptyset$. Indeed, let $x = (x_n)_n \in (c_0)_1$. Since $\lim_n x_n = 0$, there exists $N \in \mathbb{N}$ such that $|x_n| < \frac{1}{2}$ for all $n > N$. Now define $y, z \in c_0$ by setting $y_n = z_n = x_n$ for $n \leq N$ and

$$y_n = x_n + \frac{1}{2^n}, \quad z_n = x_n - \frac{1}{2^n}$$

for $n > N$. Then $y, z \in (c_0)_1$ and $x = \frac{1}{2}(y + z)$, so $x \notin \text{ext}(c_0)_1$.

Example 1.36. Let us show that $\text{ext}(L^1[0, 1])_1 = \emptyset$. Take any $f \in (L^1[0, 1])_1$. Then $\int_0^1 |f(t)| dt = 1$, so there must exist an $x \in [0, 1]$ such that $\int_0^x |f(t)| dt = 1/2$. Now define $g := 2 \cdot f \cdot \chi_{[0, x]}$ and $h := 2 \cdot f \cdot \chi_{[x, 1]}$. Now we have $g, h \in (L^1[0, 1])_1$ and $f = \frac{1}{2}g + \frac{1}{2}h$, so f cannot be an extreme point.

Example 1.37. Finally, let us prove that $\text{ext}(C[0, 1])_1 = \{\pm 1\}$ for real valued functions. Take any $f \in (C[0, 1])_1$. Then define functions $g(t) = \min\{2f(t) + 1, 1\}$ and $h(t) = \max\{2f(t) - 1, -1\}$. Clearly $g, h \in (C[0, 1])_1$ and $f = \frac{1}{2}g + \frac{1}{2}h$. If f is an extreme point, then $g = h$, which happens only if $f = \pm 1$.

Definition 1.38. For a vector space X and $A \subseteq X$, define a *convex hull* $\text{co} A$ as the intersection of all convex sets in X that contain A . If X is a TVS, then define a *closed convex hull* $\overline{\text{co}} A$ as the intersection of all closed convex sets that contain A .

Convex hull of a set A can be given explicitly:

$$\text{co} A = \left\{ \sum_{i=0}^n \alpha_i x_i \mid n \in \mathbb{N}, \alpha_i \geq 0, \sum_{i=0}^n \alpha_i = 1, x_i \in A \right\}.$$

If X is a TVS, then $\overline{\text{co}} A = \overline{\text{co} A}$.

Lemma 1.39. If $C \subseteq X$ is a convex subset of a vector space and $a \in C$, then the following are equivalent.

- (a) $a \in \text{ext } C$.
- (b) If $x_1, x_2 \in C$ and $a = \frac{1}{2}(x_1 + x_2)$, then $x_1 = x_2 = a$.
- (c) If $x_1, x_2 \in C$, $t \in (0, 1)$ and $a = tx_1 + (1 - t)x_2$, then $x_1 = x_2 = a$.
- (d) $C \setminus \{a\}$ is a convex set.
- (e) If $x_1, \dots, x_n \in C$ and $a \in \text{co}\{x_1, \dots, x_n\}$, then $a = x_k$ for some index k .

Proof. Items (a) and (c) are equivalent by definition.

(b) \Rightarrow (c): Let $a = tx_1 + (1 - t)x_2$. Then

$$a = \frac{1}{2}(2tx_1 + (1 - 2t)x_2) + \frac{1}{2}x_2,$$

so we get $2tx_1 + (1 - 2t)x_2 = x_2$, which gives us $x_1 = x_2$.

(c) \Rightarrow (d): Take any $x_1, x_2 \in C \setminus \{a\}$. Since C is convex, $tx_1 + (1 - t)x_2 \in C$. Now if $a = tx_1 + (1 - t)x_2 \in \text{co}\{x_1, x_2\}$, then $a = x_1 = x_2$, which contradicts our assumption.

So $tx_1 + (1 - t)x_2 \in C \setminus \{a\}$ and $C \setminus \{a\}$ is convex.

(d) \Rightarrow (e) If $x_1, \dots, x_n \in C \setminus \{a\}$, then $\text{co}\{x_1, \dots, x_n\} \subseteq C \setminus \{a\}$ by convexity, contradiction.

(e) \Rightarrow (b): Suppose $a = \frac{1}{2}(x_1 + x_2)$. Then either $x_1 = a$ or $x_2 = a$ by our assumption. W.l.o.g. assume $x_1 = a$. Then $a = \frac{1}{2}(a + x_2)$, which implies $a = x_2$. \square

Lemma 1.40. Let X be a TVS and $C \subseteq X$ a nonempty compact convex set. Then for $\Phi \in X^*$ the set

$$F = \{x \in C \mid \text{Re } \Phi(x) = \min_C \text{Re } \Phi\}$$

is a closed face of C .

Proof. Since C is compact and $x \mapsto \text{Re } \Phi(x)$ is continuous, it attains its minimum on C . Hence F is nonempty. Since F is a continuous preimage of a point, it is also closed. By the linearity of Φ , F is convex. Now suppose that $t \in (0, 1)$ and $x, y \in C$ are such that $tx + (1 - t)y \in F$. Then

$$\begin{aligned} \min_C \text{Re } \Phi &= \text{Re } \Phi(tx + (1 - t)y) \\ &= t \cdot \text{Re } \Phi(x) + (1 - t) \text{Re } \Phi(y) \\ &\geq t \cdot \min_C \text{Re } \Phi + (1 - t) \min_C \text{Re } \Phi \\ &= \min_C \text{Re } \Phi. \end{aligned}$$

Since we have the equality in the second-to-last line, we have $\text{Re } \Phi(x) = \min_C \text{Re } \Phi$ and $\text{Re } \Phi(y) = \min_C \text{Re } \Phi$, meaning that $x, y \in F$. \square

Remark. Not all closed convex faces are of this form.

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Theorem 1.41 (Krein–Milman).

Let X be a LCS and $C \subseteq X$ a nonempty compact convex subset. Then $C = \overline{\text{co}}(\text{ext } C)$. In particular, $\text{ext } C \neq \emptyset$.

Proof. Let $\mathcal{F} = \{\text{closed faces in } C\}$ be ordered with \supseteq . Since $C \in \mathcal{F}$, it is nonempty. The set \mathcal{F} is then partially ordered. Since any increasing chain in \mathcal{F} has the finite intersection property, \mathcal{F} has a nonempty intersection due to C being compact. As a result, any increasing chain in \mathcal{F} has an upper bound. This tells us that we can apply Zorn's lemma to obtain a maximal element $F_0 \in \mathcal{F}$.

We prove that $F_0 = \{p\}$ for some $p \in X$. Assume for a contradiction that there are distinct $x, y \in F_0$. By Hahn–Banach, there exists a $\Phi \in X^*$ such that $\Phi(x) \neq \Phi(y)$. W.l.o.g. we assume that $\text{Re } \Phi(x) < \text{Re } \Phi(y)$. Define a set

$$F_1 = \{z \in F_0 \mid \text{Re } \Phi(z) = \min_{F_0} \text{Re } \Phi\}.$$

Then $F_1 \subsetneq F_0$, since $y \notin F_1$. By the previous lemma, F_1 is a closed face in F_0 , so it is a closed face in C , contradicting maximality of F_0 . As a result, $F_0 = \{p\}$, which implies that $p \in \text{ext}(C)$ and the set of extreme points of C is non-empty.

Since we have $C \supseteq \text{ext } C$, we also have $C = \overline{\text{co}}(C) \supseteq \overline{\text{co}}(\text{ext } C)$. Suppose $x \in C \setminus \overline{\text{co}}(\text{ext } C)$. By Hahn–Banach, there exists a $\Psi \in X^*$ such that $\text{Re } \Psi(x) < \min_{\overline{\text{co}}(\text{ext } C)} \text{Re } \Psi$. So the set

$$F = \{z \in C \mid \text{Re } \Psi(z) = \min_C \text{Re } \Psi\}$$

is a closed face in C . By the first part of this proof, there exists a $z \in \text{ext } F \subseteq \text{ext } C$. Hence

$$\min_C \text{Re } \Psi = \text{Re } \Psi(z) = \min_{\overline{\text{co}}(\text{ext } C)} \text{Re } \Psi > \text{Re } \Psi(x) \geq \min_C \text{Re } \Psi,$$

which leads to a contradiction. Therefore $\overline{\text{co}}(\text{ext } C) = C$. □

Example 1.42. Let \mathcal{H} be a Hilbert space. Then

$$\text{ext}(\mathcal{H})_1 = \{v \in \mathcal{H} \mid \|v\| = 1\}.$$

First we prove the inclusion (\supseteq). Suppose that $\|v\| = 1$ and $v = tx + (1-t)y$, where $t \in (0, 1)$ and $x, y \in (\mathcal{H})_1$. We have

$$\begin{aligned} 1 &= \|v\|^2 \\ &= \|tx + (1-t)y\|^2 \\ &= \langle tx + (1-t)y, tx + (1-t)y \rangle \\ &= t^2\|x\|^2 + (1-t)^2\|y\|^2 + 2t(1-t)\text{Re}\langle x, y \rangle \\ &\leq t^2 + (1-t)^2 + 2t(1-t) = 1. \end{aligned}$$

We get equality in the Cauchy–Schwartz inequality, so x, y are linearly dependent and there-

fore equal. For the reverse inclusion, let $v \in \text{ext}(\mathcal{H})_1$. If $\|v\| < 1$, then

$$v = \frac{1}{2} \cdot \frac{v}{\|v\|} + \frac{1}{2} \cdot (2\|v\| - 1) \frac{v}{\|v\|},$$

so v cannot be an extreme point of $(\mathcal{H})_1$.

Example 1.43. It holds that

$$\text{ext}(\mathcal{B}(\mathcal{H}))_1 = \{V \in \mathcal{B}(\mathcal{H}) \mid V \text{ or } V^* \text{ is an isometry}\}.$$

Here, we will just prove the inclusion (\supseteq) . Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry and suppose $V = tS + (1-t)T$ for $t \in (0, 1)$ and $S, T \in (\mathcal{B}(\mathcal{H}))_1$. For $x \in \mathcal{H}$ we have:

$$\begin{aligned} \|x\| &= \|Vx\| \\ &= \|tSx + (1-t)Tx\| \\ &\leq t\|Sx\| + (1-t)\|Tx\| \\ &\leq t\|S\|\|x\| + (1-t)\|T\|\|x\| \\ &\leq t\|x\| + (1-t)\|x\| = \|x\|. \end{aligned}$$

Since we have equality, we get $\|S\| = \|T\| = 1$ and $\|Sx\| = \|Tx\| = \|x\|$. So S, T are isometries. For every $x \in \partial(\mathcal{H})_1 = \text{ext}(\mathcal{H})_1$, we have

$$Vx = t \cdot Sx + (1-t)Tx$$

and by the previous example that implies $Tx = Sx = Vx$, so we really have $S = T = V$. We use the same argument if V^* is an isometry. For now, we lack some tools to prove the reverse inclusion. We will prove the equality in corollary 4.4.

Example 1.44. If X be a Banach space, then $(X^*)_1$ is weak-* compact (by Banach-Alaoglu), so Krein-Milman gives us $(X^*)_1 = \overline{\text{co}}(\text{ext}(X^*)_1)$. Hence $(X^*)_1$ has a lot of extreme points. As a corollary, c_0 , $L^1[0, 1]$ and $C[0, 1]$ are not duals of Banach spaces.

Theorem 1.45 (Milman).

Let X be a LCS, $K \subseteq X$ compact and assume $\overline{\text{co}}(K)$ is compact. Then $\text{ext}(\overline{\text{co}}(K)) \subseteq K$.

Proof. Assume there exists $x_0 \in \text{ext}(\overline{\text{co}}(K)) \setminus K$. Then there exists a basis neighborhood V of 0 in X such that $(x_0 + \overline{V}) \cap K = \emptyset$, or equivalently, $x_0 \notin K + \overline{V}$. If we write $K \subseteq \bigcup_{x \in K} (x + V)$, we get

$$K \subseteq \bigcup_{j=1}^n (x_j + V).$$

Form $K_j = \overline{\text{co}}(K \cap (x_j + V))$. Then K_j is convex and compact since $K_j \subseteq \overline{\text{co}}(K)$. We also have $K_j \subseteq x_j + \overline{V} = x_j + \overline{V}$ since V is convex. Also, $K \subseteq K_1 \cup \dots \cup K_n$. Next we prove

that $\text{co}(K_1 \cup \dots \cup K_n)$ is compact. Define

$$\Sigma = \{(t_1, \dots, t_n) \in [0, 1]^n \mid \sum_{j=1}^n t_j = 1\}$$

and the function

$$f : \Sigma \times K_1 \times \dots \times K_n \rightarrow X, \quad (t, k_1, \dots, k_n) \mapsto \sum_{j=1}^n t_j k_j.$$

Denote $C := \text{im } f$. Obviously, $C \subseteq \text{co}(K_1 \cup \dots \cup K_n)$ and C is a convex compact set. Furthermore, $C \supset K_j$ for each j , so $C = \text{co}(K_1 \cup \dots \cup K_n)$. From there, we get

$$\overline{\text{co}}(K) \subseteq \overline{\text{co}}(K_1 \cup \dots \cup K_n) = \text{co}(K_1 \cup \dots \cup K_n).$$

But since $K_j \subseteq \overline{\text{co}}(K)$ for all j , we deduce $\overline{\text{co}}(K) = \overline{\text{co}}(K_1 \cup \dots \cup K_n)$. We know that $x_0 \in \overline{\text{co}}(K)$, so

$$x_0 = t_1 y_1 + \dots + t_n y_n$$

for some $t_i \in [0, 1]$, $\sum t_i = 1$ and $y_j \in K_j$. But $x_0 \in \text{ext}(\overline{\text{co}})(K)$, so $y_j = x_0$ for some j . So we get $x_0 \in K_j \subseteq x_j + \bar{V} \subseteq K + \bar{V}$, a contradiction. \square

Remark. (1.) In finite dimensions, the convex hull of a compact set is compact. In infinite dimensions this fails.

(2.) The set $\text{ext}(C)$ is not always closed, even if $C \subseteq \mathbb{R}^3$ is convex and compact.

ADD A PICTURE

2 C^* -algebras and continuous functional calculus

2.1 Spectrum

Remark. From here on, all algebras are over \mathbb{C} .

Let A be a complex algebra with a unit 1 and

$$\mathrm{GL}(A) = \{a \in A \mid a \text{ is invertible}\}.$$

If $x \in A$, we define the spectrum

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \cdot 1 \notin \mathrm{GL}(A)\}.$$

Proposition 2.1. *Let A be a complex algebra with unity 1 and $x, y \in A$. Then*

$$\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}.$$

Proof. Suppose $1 - xy \in \mathrm{GL}(A)$. Formally, we can write

$$(1 - xy)^{-1} = 1 + xy + (xy)^2 + \cdots$$

and

$$(1 - yx)^{-1} = 1 + yx + (yx)^2 + \cdots = 1 + y(1 - xy)^{-1}x.$$

From this, we claim that indeed $1 - yx \in \mathrm{GL}(A)$ and

$$(1 - yx)^{-1} = 1 + y(1 - xy)^{-1}x.$$

The proof is straightforward: we have

$$\begin{aligned} (1 + y(1 - xy)^{-1}x)(1 - yx) &= (1 - yx) + y(1 - xy)^{-1}(x - xyx) \\ &= (1 - yx) + y(1 - xy)^{-1}(1 - xy)x \\ &= (1 - yx) + yx = 1 \end{aligned}$$

and

$$\begin{aligned} (1 - yx)(1 + y(1 - xy)^{-1}x) &= (1 - yx) + (y - yxy)(1 - xy)^{-1}x \\ &= (1 - yx) + y(1 - xy)(1 - xy)^{-1}x \\ &= (1 - yx) + yx = 1. \end{aligned}$$

Now the proof of the statement is at hand: if $\lambda \in \sigma_A(xy) \setminus \{0\}$, then

$$\lambda - xy \notin \mathrm{GL}(A) \Rightarrow 1 - \frac{x}{\lambda}y \notin \mathrm{GL}(A) \Rightarrow 1 - y\frac{x}{\lambda} \notin \mathrm{GL}(A) \Rightarrow \lambda - yx \notin \mathrm{GL}(A).$$

Thus, $\lambda \in \sigma_A(yx)$. Similarly, if $\lambda \in \sigma_A(yx) \setminus \{0\}$, then $\lambda \in \sigma_A(xy)$. □

Example 2.2. Let $S, S^* \in \mathcal{B}(\ell^2)$ be the right and left shift operators, respectively. Then $SS^* = I$, but

$$SS^*(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

This implies that $0 \in \sigma(SS^*)$, but $0 \notin \sigma(S^*S)$.

2.2 Banach and C^* -algebras

Definition 2.3. • A Banach algebra is a Banach space A that is also an algebra, satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. If a Banach algebra has a unit 1 , we also demand $\|1\| = 1$.

- An involution on a Banach algebra A is a skew-linear map

$$*: A \rightarrow A, \quad a \mapsto a^*$$

satisfying

$$(xy)^* = y^*x^*, \quad (x^*)^* = x, \quad \|x^*\| = \|x\|.$$

A C^* -algebra is a Banach $*$ -algebra A that also satisfies $\|x^*x\| = \|x\|^2$ for all $x \in A$.

Unless otherwise mentioned, all algebras in this section will be unital.

Proposition 2.4. We collect some basic properties of Banach algebras.

(1.) If A is a Banach $*$ -algebra, then $(x^*)^{-1} = (x^{-1})^*$ and $\sigma_A(x^*) = (\sigma_A(x))^*$.

(2.) Let A be a Banach algebra. If $\|x\| < 1$, then $1 - x \in \text{GL}(A)$ and

$$(1 - x)^{-1} = 1 + x + x^2 + \dots$$

As a consequence, if $\|1 - x\| < 1$, then $x \in \text{GL}(A)$.

(3.) Let A be a Banach algebra. Then $\text{GL}(A) \subseteq A$ is open, and the map $x \mapsto x^{-1}$ is continuous on $\text{GL}(A)$.

(4.) If A is a Banach algebra and $x \in A$, then $\sigma_A(x)$ is a nonempty compact set.

Proof. (1.) Suppose that the inverse $(x^*)^{-1}$ exists. Then $(x^*)^{-1} \cdot (x^*) = 1$, so starring gives us $(x^*)^* \cdot ((x^*)^{-1})^* = 1$ and $x \cdot ((x^*)^{-1})^* = 1$. Similarly, we have $(x^*) \cdot (x^*)^{-1} = 1$, which implies $((x^*)^{-1})^* \cdot x = 1$. This means that x is invertible and $((x^*)^{-1})^* = x^{-1}$. Starring this equation now gives us $(x^*)^{-1} = (x^{-1})^*$. For the opposite direction, suppose that x is invertible. Then

$$(x^{-1})^* \cdot x^* = (x \cdot x^{-1})^* = 1^* = 1$$

and

$$x^* \cdot (x^{-1})^* = (x^{-1} \cdot x)^* = 1^* = 1,$$

which means that x^* is invertible and $(x^*)^{-1} = (x^{-1})^*$. The rest is a matter of simple

computation:

$$\begin{aligned}\lambda \in \sigma_A(x^*) &\Leftrightarrow x^* - \lambda \notin \text{GL}(A) \Leftrightarrow (x - \bar{\lambda})^* \notin \text{GL}(A) \\ &\Leftrightarrow (x - \bar{\lambda}) \notin \text{GL}(A) \Leftrightarrow \bar{\lambda} \in \sigma_A(x) \\ &\Leftrightarrow \lambda \in (\sigma_A(x))^*.\end{aligned}$$

If $\|x\| \leq 1$, then the series $\sum_{n=0}^{\infty} x^n$ converges in norm to some x' . Since multiplication between elements of a Banach algebra is norm-continuous, we get we get

$$(1-x)x' = (1-x) \cdot \lim_{k \rightarrow \infty} \sum_{n=1}^k x^n = \lim_{k \rightarrow \infty} (1-x) \cdot \sum_{n=1}^k x^n = \lim_{k \rightarrow \infty} 1 - x^{k+1} = 1$$

and similarly for $x'(1-x)$.

Let $y \in \text{GL}(A)$. If $\|x - y\| \leq \frac{1}{\|y^{-1}\|}$, then

$$\|1 - xy^{-1}\| = \|(y-x)y^{-1}\| \leq \|y-x\|\|y^{-1}\| \leq 1,$$

which implies that $xy^{-1} \in \text{GL}(A)$, and thus $x = xy^{-1} \cdot y \in \text{GL}(A)$. We have shown that $\text{GL}(A)$ is open. Using the same notation and noting that $(xy^{-1})^{-1} = (1 - (1 - xy^{-1}))^{-1}$, we get

$$\|(xy^{-1})^{-1}\| \leq \sum_{n=0}^{\infty} \|(1 - xy^{-1})\|^n \leq \sum_{n=0}^{\infty} \|y^{-1}\|^n \|x - y\|^n \leq \frac{1}{1 - \|y^{-1}\| \cdot \|x - y\|}.$$

Now,

$$\begin{aligned}\|x^{-1} - y^{-1}\| &= \|x^{-1}(y-x)y^{-1}\| \\ &\leq \|y^{-1}(xy^{-1})^{-1}\| \|y-x\| \|y^{-1}\| \\ &\leq \|(xy^{-1})^{-1}\| \|y-x\| \|y^{-1}\|^2 \\ &\leq \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\| \cdot \|x - y\|} \|y-x\|.\end{aligned}$$

Since the function $t \mapsto \frac{\|y^{-1}\|^2}{1 - \|y^{-1}\| \cdot t} t$ is continuous at $t = 0$, the map $x \mapsto x^{-1}$ is continuous.

First, we prove compactness by showing that $\sigma_A(x)$ is bounded and closed. Suppose there exists $\lambda \in \sigma_A(x)$, such that $|\lambda| > \|x\|$. Then $(1 - \frac{x}{\lambda})$ is invertible by (2.), so $(-\lambda) \cdot (1 - \frac{x}{\lambda}) = x - \lambda$ is invertible as well. But this contradicts the fact that $\lambda \in \sigma_A(x)$, so we have shown that $\sigma_A(x) \subseteq \overline{B(0, \|x\|)}$. Next, we prove that the spectrum is closed. Define a continuous map

$$\mathbb{C} \rightarrow A, \quad \lambda \mapsto x - \lambda$$

and notice that the inverse image of $\text{GL}(A)$ (which is open by (3.)) is exactly $\mathbb{C} \setminus \sigma_A(x)$. This means that $\mathbb{C} \setminus \sigma_A(x)$ is open and $\sigma_A(x)$ is closed. For non-emptiness, we have to employ some standard Banach algebra techniques. We say that a function f from a domain $\Omega \subseteq \mathbb{C}$ to a Banach space X is analytical if there exists a limit

$$f'(z_0) := \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

for every $z_0 \in \Omega$ and the function f' is continuous on Ω . A lot of theory for complex analytic functions also applies to Banach space-valued analytic functions; in particular, we have Cauchy's integral formula, Liouville's theorem and the fact that every vector valued analytic function can be locally expressed as a power series with coefficients in X . Now we can define the resolvent function

$$F : \mathbb{C} \setminus \sigma_A(x) \rightarrow A, \quad F(z) = (z - x)^{-1}.$$

It's routine to show that F is analytic and its derivative is $F'(z) = (z - x)^{-2}$. Now for $z \in \mathbb{C} \setminus \overline{B(0, \|x\|)}$, we have $F(z) = z^{-1} \cdot (1 - a/z)$, which goes to 0 as $z \rightarrow \infty$. Now if $\sigma_A(x) = \emptyset$, then F would be an entire function that vanishes at ∞ . By Liouville's theorem, F is constant and so $F' = 0$. This is a contradiction. \square

Theorem 2.5 (Gelfand–Mazur).

If A is Banach algebra that is also a division ring, then $A = \mathbb{C}$.

Proof. Let $x \in A$ and $\lambda \in \sigma_A(x)$. Then $x - \lambda \cdot 1 \notin \text{GL}(A)$, implying $x - \lambda = 0$, hence $x = \lambda \in \mathbb{C}$. \square

Definition 2.6. If $f(x) = \sum_{j=0}^n a_j x^j$ is a polynomial and $a \in A$, we define $f(a) = \sum_{j=0}^n a_j a^j \in A$.

Theorem 2.7 (Spectral mapping theorem for polynomials).

Let A be a complex unitary algebra and $f \in \mathbb{C}[x]$. Then $f(\sigma_A(a)) = \sigma_A(f(a))$ for all $a \in A$.

Proof. First, we prove the inclusion (\subseteq) . If $\lambda \in \sigma_A(a)$ and $f(x) = \sum_{j=0}^n a_j x^j$, then

$$f(x) - f(\lambda) = \sum_{j=1}^n a_j (x^j - \lambda^j) = (x - \lambda) \cdot \sum_{j=1}^n a_j \sum_{k=0}^{j-1} x^k \lambda^{j-1-k}.$$

Substituting $x = a$, we obtain

$$f(a) - f(\lambda) = (a - \lambda) \left(\sum_{j=1}^n a_j \sum_{k=0}^{j-1} a^k \lambda^{j-1-k} \right).$$

Since $a - \lambda$ commutes with the second factor, $f(a) - f(\lambda)$ is not invertible and $f(\lambda) \in \sigma_A(f(a))$. For the converse inclusion (\supseteq) , assume $\mu \notin f(\sigma_A(a))$. We factor

$$f(x) - \mu = a_n (x - \lambda_1) \cdots (x - \lambda_n).$$

Since $f(\lambda) - \mu \neq 0$ for any $\lambda \in \sigma_A(a)$, it follows that $\lambda_i \notin \sigma_A(a)$ for all i . Therefore, $f(a) - \mu \in \text{GL}(A)$. \square

Definition 2.8. Let A be a Banach algebra and $x \in A$. The spectral radius of x is

$$r(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|.$$

Remark. By proposition 2.1, we have $r(xy) = r(yx)$.

In the introductory course, we proved the following.

Theorem 2.9 (Spectral radius formula).

Let A be a Banach algebra and $x \in A$. Then $\lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}}$ exists and is equal to $r(x)$.

Definition 2.10. Let A be a Banach $*$ -algebra and $x \in A$.

- x is *normal* iff $xx^* = x^*x$.
- x is *self-adjoint* iff $x^* = x$.
- x is *skew self-adjoint* iff $x^* = -x$.

The set of all self-adjoint operators is denoted as A_{sa} .

Remark. Every $a \in A$ can be uniquely expressed as a sum of a self-adjoint and skew self-adjoint element:

$$a = \frac{a + a^*}{2} + \frac{a - a^*}{2}.$$

Alternatively, we can uniquely write it in the form of

$$a = \left(\frac{a + a^*}{2} \right) + i \cdot \left(\frac{a - a^*}{2i} \right)$$

where both terms in parentheses are self-adjoint.

Corollary 2.11. *Let A be a Banach $*$ -algebra and $x \in A$ normal. Then $r(x^*x) \leq r(x)^2$. If A is a C^* -algebra, then $r(x^*x) = r(x)^2$.*

Proof. We use the spectral radius formula:

$$\begin{aligned} r(x^*x) &= \lim_{n \rightarrow \infty} \|(x^*x)^n\|^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \|(x^*)^n x^n\|^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \|(x^n)^* x^n\|^{\frac{1}{n}} \\ &\leq \lim_{n \rightarrow \infty} \|x^n\|^{\frac{2}{n}} = r(x)^2. \end{aligned}$$

If A is a C^* -algebra, we have an equality in the last line of the above calculation. \square

Proposition 2.12. *Let A be a C^* -algebra and $x \in A$ normal. Then $r(x) = \|x\|$.*

Proof. First, assume x is self-adjoint. Then

$$\|x^2\| = \|xx^*\| = \|x\|^2.$$

By induction, we get $\|x^{2^n}\| = \|x\|^{2^n}$ for every $n \in \mathbb{N}$. Therefore,

$$r(x) = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|.$$

If x is only normal, then

$$\|x\|^2 = \|x^*x\| = r(x^*x) = r(x)^2,$$

which implies $\|x\| = r(x)$. □

Corollary 2.13. *Let A, B be C^* -algebras and $\Phi : A \rightarrow B$ a $*$ -homomorphism ($\Phi(x^*) = \Phi(x)^*$). Then Φ is a contraction. Furthermore, if Φ is a $*$ -isomorphism, then it is isometric.*

Proof. Clearly, Φ maps invertible elements to invertible elements, so $\Phi(\text{GL}(A)) \subseteq \text{GL}(B)$. This implies $\sigma_B(\Phi(x)) \subseteq \sigma_A(x)$, hence $r(\Phi(x)) \leq r(x)$. Then

$$\begin{aligned} \|\Phi(x)\|^2 &= \|\Phi(x)\Phi(x)^*\| = \|\Phi(x)\Phi(x^*)\| \\ &= \|\Phi(xx^*)\| = r(\Phi(xx^*)) \\ &\leq r(xx^*) = \|xx^*\| = \|x\|^2. \end{aligned}$$

If Φ is a $*$ -isomorphism, we apply the same reasoning to its inverse, which implies that Φ must be an isometry. □

Corollary 2.14. *If A is a $*$ -algebra, then there exists at most one norm on A that makes it into a C^* -algebra.*

Proof. Considering the identity map

$$(A, \|\cdot\|_1) \rightarrow (A, \|\cdot\|_2),$$

it is a $*$ -isomorphism, so it preserves the norm by the previous corollary. □

Lemma 2.15. *Let A be a C^* -algebra and $x \in A$ self-adjoint. Then $\sigma_A(x) \subseteq \mathbb{R}$.*

Proof. Suppose $\lambda = \alpha + i\beta \in \sigma_A(x)$ for some $\alpha, \beta \in \mathbb{R}$. Define $y = x - \alpha + it$ for $t \in \mathbb{R}$. Then $i(\beta + t) \in \sigma_A(y)$ and y is normal. Thus,

$$\begin{aligned} |i(\beta + t)|^2 &= (\beta + t)^2 \leq r(y) \\ &= \|y\|^2 = \|yy^*\| \\ &= \|(x - \alpha)^2 + t^2\| \leq \|x - \alpha\|^2 + t^2. \end{aligned}$$

Simplifying, we get $\beta^2 + 2\beta t \leq \|x - \alpha\|^2$, and since $t \in \mathbb{R}$ was arbitrary, we have $\beta = 0$. □

Lemma 2.16. *Let A be a Banach algebra and $x \notin \text{GL}(A)$. If $(x_n)_n \subseteq \text{GL}(A)$ satisfies $x_n \rightarrow x$, then $\|x_n^{-1}\| \rightarrow \infty$.*

Proof. If the sequence $\|x_n^{-1}\|$ is bounded, then

$$\|1 - xx_n^{-1}\| = \|(x_n - x)x_n^{-1}\| \leq \|x_n - x\| \cdot \|x_n^{-1}\| \rightarrow 0.$$

In particular, there exists some $n \in \mathbb{N}$ such that $\|1 - xx_n^{-1}\| < 1$, which implies $xx_n^{-1} \in \text{GL}(A)$ and therefore $x = (xx_n^{-1})x_n \in \text{GL}(A)$, a contradiction. \square

Proposition 2.17. *Let B be a C^* -algebra and $A \subseteq B$ a unital C^* -subalgebra. Then for all $x \in A$, we have $\sigma_A(x) = \sigma_B(x)$.*

Proof. Obviously, $\text{GL}(A) \subseteq \text{GL}(B)$. For a self adjoint $x \in A \setminus \text{GL}(A)$, we have $it \notin \sigma_A(x)$ for $t \in \mathbb{R}$. So there exists $(x - it)^{-1} \in A$. Clearly,

$$x - it \in \text{GL}(A) \xrightarrow{t \rightarrow 0} x \notin \text{GL}(A),$$

thus $\|(x - it)^{-1}\| \rightarrow \infty$. Since the inverse function is continuous, this immediately yields $x \notin \text{GL}(B)$. For general $x \in A$: if $x \in \text{GL}(B)$, then $x^*x \in \text{GL}(B)$ is self-adjoint. By the first part of the proof, $x^*x \in \text{GL}(A)$. It follows that

$$x^{-1} = (x^*x)^{-1}x^* \in A,$$

so $x \in \text{GL}(A)$. \square

Example 2.18. *Let X be a Hausdorff topological space and $C_b(X)$ be the set of continuous bounded complex functions on X , endowed with the sup metric. Then $C_b(X)$ is a unital abelian C^* -algebra (where $f^*(x) = \overline{f(x)}$).*

Example 2.19. *Let X be a locally compact Hausdorff space and*

$$C_0(X) = \{f \in C(X) \mid \forall \varepsilon > 0 : \exists K^{\text{compact}} \subseteq X : |f(x)| \leq \varepsilon, \forall x \in X \setminus K\}$$

the set of all complex continuous functions on X that vanish at infinity. Then $C_0(X)$ is an abelian C^ -algebra. In some sense, it is the natural abelian C^* -algebra. The algebra $C_0(X)$ is unital iff X is compact – in that case, $C_0(X) = C_b(X) = C(X)$.*

Example 2.20. *Let (X, μ) be a measure space. Then $L^\infty(X, \mu)$, the set of essentially bounded functions on X endowed with the essential supremum norm, is a unital abelian C^* -algebra.*

Example 2.21. For a Hilbert space \mathcal{H} , $\mathcal{B}(\mathcal{H})$ is a non-abelian C^* -algebra: for all $x \in \mathcal{B}(\mathcal{H})$ we have $\|x^*x\| = \|x\|^2$.

Example 2.22. If Γ is a group, we define

$$\ell^1(\Gamma) = \{(\alpha_s)_{s \in \Gamma} \mid \alpha_s \in \mathbb{C}, \sum_{s \in \Gamma} |\alpha_s| < \infty\}.$$

We can then introduce the convolution multiplication on $\ell^1(\Gamma)$:

$$(\alpha * \beta)_s = \sum_{t \in \Gamma} \alpha_{st} \beta_{t^{-1}}.$$

This is a Banach algebra; it is even a Banach $*$ -algebra with involution $(\alpha^*)_s = \overline{\alpha_{s^{-1}}}$. However, it is not a C^* -algebra if the group Γ has more than one element. In that case, there exists $z \in \Gamma$ such that $z \neq 1$. Define $\alpha = (\alpha_s) \in \ell^1(\Gamma)$ such that

$$\alpha_s = \begin{cases} 1; & s = 1 \\ i; & s = z, z^{-1} \\ 0; & \text{otherwise} \end{cases}.$$

If $z \neq z^{-1}$, we have

$$\begin{aligned} \|\alpha\alpha^*\| &= \sum_{s \in \Gamma} \left| \sum_{t \in \Gamma} \alpha_{st} \overline{\alpha_t} \right| \\ &= \sum_{s \in \Gamma} (3 \cdot \mathbf{1}_{s=1} + \mathbf{1}_{s=z^2} + \mathbf{1}_{s=z^{-2}}) \\ &< \sum_{s \in \Gamma} (3 \cdot \mathbf{1}_{s=1} + 2 \cdot \mathbf{1}_{s=z} + 2 \cdot \mathbf{1}_{s=z^{-1}} + \mathbf{1}_{s=z^2} + \mathbf{1}_{s=z^{-2}}) \\ &= \sum_{s \in \Gamma} \sum_{t \in \Gamma} |\alpha_{st} \alpha_t| = \sum_{t \in \Gamma} |\alpha_t| \cdot \sum_{s \in \Gamma} |\alpha_{st}| \\ &= \sum_{t \in \Gamma} |\alpha_t| \cdot \sum_{s \in \Gamma} |\alpha_s| = \|\alpha\|^2. \end{aligned}$$

Otherwise, we get

$$\begin{aligned} \|\alpha\alpha^*\| &= \sum_{s \in \Gamma} \left| \sum_{t \in \Gamma} \alpha_{st} \overline{\alpha_t} \right| \\ &= \sum_{s \in \Gamma} (2 \cdot \mathbf{1}_{s=1}) \\ &< \sum_{s \in \Gamma} (2 \cdot \mathbf{1}_{s=1} + 2 \cdot \mathbf{1}_{s=z}) \\ &= \sum_{s \in \Gamma} \sum_{t \in \Gamma} |\alpha_{st} \alpha_t| = \sum_{t \in \Gamma} |\alpha_t| \cdot \sum_{s \in \Gamma} |\alpha_{st}| \\ &= \sum_{t \in \Gamma} |\alpha_t| \cdot \sum_{s \in \Gamma} |\alpha_s| = \|\alpha\|^2. \end{aligned}$$

Therefore, $\ell^1(\Gamma)$ is not a C^* -algebra if Γ has order greater than one.

2.3 Gelfand transform

Definition 2.23. Let A be an abelian Banach algebra. The *spectrum* of A is defined as

$$\sigma(A) := \{\varphi : A \rightarrow \mathbb{C} \mid \varphi \neq 0 \text{ continuous algebra homomorphism}\} \subseteq A^*$$

endowed with a weak-* topology. Its elements are called *characters*.

If $\varphi \in \sigma(A)$, then $\ker \varphi \cap \text{GL}(A) = \emptyset$. For $x \in A$, we have

$$\begin{aligned} \varphi(x - \varphi(x)) &= \varphi(x) - \varphi(\varphi(x) \cdot 1) \\ &= \varphi(x) - \varphi(x)\varphi(1) \\ &= \varphi(x) - \varphi(x) = 0, \end{aligned}$$

which implies that $\varphi(x) \in \sigma_A(x)$. Consequently, $|\varphi(x)| \leq r(x) \leq \|x\|$, giving us the bound $\|\varphi\| \leq 1$. But since $\varphi(1) = 1$, we get $\|\varphi\| = 1$. We know that $\sigma(A)$ is closed in $(A^*)_1$, making $\sigma(A)$ a compact Hausdorff space by Banach–Alaoglu.

Proposition 2.24. Let A be a C^* -algebra and $h : A \rightarrow \mathbb{C}$ a non-zero homomorphism (not necessarily a $*$ -homomorphism). Then the following statements hold:

- (1.) $h(a) \in \mathbb{R}$ for self-adjoint a ;
- (2.) $h(a^*) = \overline{h(a)}$ for all $a \in A$;
- (3.) $h(aa^*) \geq 0$ for all $a \in A$;
- (4.) if $uu^* = 1$ or $u^*u = 1$, then $|h(u)| = 1$.

Remark. The first three items also hold for non-unital algebras.

Proof. (1.) Since $h(a) \in \sigma_A(a)$ and self-adjoint elements have real spectrum, this is trivial.

(2.) Let $a = a_1 + ia_2$, where a_1, a_2 are self-adjoint. Then $a^* = a_1 - ia_2$ and

$$h(a^*) = h(a_1 - ia_2) = h(a_1) - ih(a_2) = \overline{h(a_1) + ih(a_2)} = \overline{h(a)}.$$

(3.) Follows from (b).

(4.) If u is unitary, then $|h(u)|^2 = h(u)h(u^*) = h(uu^*) = h(1) = 1$. □

Corollary 2.25. Every nonzero algebra homomorphism $h : A \rightarrow \mathbb{C}$ is a character.

Proposition 2.26. Let A be an abelian Banach algebra. Then the map $\varphi \mapsto \ker \varphi$ is a bijection from $\sigma(A)$ to the set of all maximal ideals of A .

Proof. If $\varphi \in \sigma(A)$, then $\ker \varphi \triangleleft A$. Suppose that $\ker \varphi \subsetneq I \triangleleft A$. Then there exists an element $x \in I \setminus \ker \varphi$. Thus, $\varphi(x) \neq 0$ and from $1 - \frac{x}{\varphi(x)} \in \ker \varphi$. From there, it follows that

$$1 = \left(1 - \frac{x}{\varphi(x)}\right) + \frac{1}{\varphi(x)} \cdot x \in I.$$

Hence, $\ker \varphi$ is a maximal ideal. Conversely, let $I \triangleleft A$ be a maximal ideal. Then $I \cap \text{GL}(A) = \emptyset$

and since $\text{GL}(A)$ is open, we also have $\bar{I} \cap \text{GL}(A) = \emptyset$. Thus, $\bar{I} \triangleleft A$ and $1 \notin \bar{I}$, so $I \subseteq \bar{I} \subsetneq A$. By maximality, $\bar{I} = I$. Then A/I is a Banach algebra and since I is maximal, every nonzero element in A/I is invertible. By Gelfand–Mazur, $A/I \cong \mathbb{C}$. The projection $\pi : A \rightarrow A/I \cong \mathbb{C}$ is in $\sigma(A)$ and $\ker \pi = I$. \square

Corollary 2.27. *Let A be an abelian Banach algebra and $x \in A \setminus \text{GL}(A)$. Then there exists $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$. In particular, $\sigma(A) \neq \emptyset$.*

Proof. If $x \notin \text{GL}(A)$, then it generates an ideal $\langle x \rangle \subsetneq A$. By Zorn's lemma, $\langle x \rangle$ has to be included in some maximal ideal $I \triangleleft A$. By the previous proposition, there exists a character $\varphi : A \rightarrow \mathbb{C}$ in $\sigma(A)$ such that $x \in I = \ker \varphi$. \square

Theorem 2.28 (Stone–Čech compactification).

Let X be a topological space. For $x \in X$, let $\beta_x : C_b(X) \rightarrow \mathbb{C}$ be the evaluation homomorphism $f \mapsto f(x)$. Then

$$\beta : X \rightarrow \sigma(C_b(X)), \quad x \mapsto \beta_x$$

is a continuous map whose image is dense in the codomain and has the following universal property: if $\pi : X \rightarrow K^{T_2, \text{compact}}$ is continuous, then there exists a unique continuous mapping

$$\beta_\pi : \sigma(C_b(X)) \rightarrow K$$

such that $\pi(x) = \beta_\pi(\beta_x)$ for all $x \in X$. In particular, if X is compact T_2 , then β is a homeomorphism.

$$\begin{array}{ccc} X & \xrightarrow{\pi} & K^{T_2, \text{compact}} \\ \downarrow \beta & \nearrow \exists! \beta_\pi & \\ \sigma(C_b(X)) & & \end{array}$$

Proof. (1.) First, we prove that β is continuous. Let $(x_i)_i$ be a net in X and $x_i \rightarrow x$, then for all $f \in C_b(X)$ we have $\beta_{x_i} = f(x_i) \rightarrow f(x) = \beta_x(f)$. Hence $\beta_{x_i} \rightarrow \beta_x$ in the weak-* topology.

(2.) Next, we prove that $\text{im } \beta$ is dense. Assume otherwise and pick $\varphi \in \sigma(C_b(X)) \setminus \overline{\beta(X)}$. Define $I := \ker \varphi$. For all $\psi \in \overline{\beta(X)}$, there exists $f_\psi \in I$ such that $f_\psi \in \ker \psi$. Hence, there exists c_ψ and a neighborhood U_ψ of ψ such that $|\tilde{\psi}(f)| > c_\psi$ for all $\tilde{\psi} \in U_\psi$. Thus, $\overline{\beta(X)} \subseteq \bigcup_{\psi \in \overline{\beta(X)}} U_\psi$. By compactness, there exists a finite subcovering of $\overline{\beta(X)}$, so $\overline{\beta(X)} \subseteq \bigcup_{i=1}^n U_{\psi_i}$. Then there exist $f_{\psi_1}, \dots, f_{\psi_n} \in I$ and $c > 0$ such that

$$\sum_{i=1}^n \psi(|f_{\psi_i}|^2) > c, \quad \forall \psi \in \overline{\beta(X)}.$$

Hence,

$$\sum_{i=1}^n |f_{\psi_i}|^2(x) = \sum_{i=1}^n \beta(x)(|f_{\psi_i}|^2) > c, \quad \forall x \in X.$$

It follows that $\sum_{i=1}^n |f_{\psi_i}|^2 \in I$ and $(\sum |f_{\psi_i}|^2)^{-1} \in C_b(X)$. As a result, $I = C_b(X)$.

- (3.) If X is compact and Hausdorff, then β is surjective since $\beta(X)$ is dense and compact. Also, β is injective since $C_b(X)$ separates points. In that case, β is a continuous bijection between compact Hausdorff spaces, and therefore a homeomorphism.
- (4.) For the universal property: let $\pi : X \rightarrow K$, where K is compact Hausdorff. Then there exists a continuous map

$$\pi^* : C(K) \rightarrow C_b(X), \quad f \mapsto f \circ \pi.$$

This induces a continuous map

$$\tilde{\pi} : \sigma(C_b(X)) \rightarrow \sigma(C(K)), \quad \varphi \mapsto \varphi \circ \pi^*.$$

Since K is compact Hausdorff, the map $\beta^K : K \rightarrow \sigma(C(K))$ is a homeomorphism. Define

$$\beta_\pi : \sigma(C_b(X)) \rightarrow K, \quad \beta_\pi = (\beta^K)^{-1} \circ \tilde{\pi}.$$

Then we have

$$\tilde{\pi}(\beta_x)(g) = \beta_x(\pi^*(g)) = \pi^*(g)(x) = g(\pi(x)) = \beta_{\pi(x)}^K(g).$$

By left multiplying by $(\beta^K)^{-1}$, we get $\beta_\pi(\beta_x) = \pi(x)$. □

Definition 2.29. Let A be an abelian Banach algebra. The Gelfand transform of A is the map

$$\Gamma : A \rightarrow C(\sigma(A)), \quad x \mapsto (\varphi \mapsto \varphi(x)).$$

Theorem 2.30.

Let A be an abelian Banach algebra. Then Γ is a homomorphism, contraction and for $x \in A$ we have

$$\Gamma(x) \in \text{GL}(C(\sigma(A))) \Leftrightarrow x \in \text{GL}(A).$$

Proof. The homomorphism part is routine. We prove that Γ is a contraction as follows:

$$\|\Gamma(x)\| = \sup_{\varphi \in \sigma(A)} \|\Gamma(x)\varphi\| = \sup_{\varphi} |\varphi(x)| \leq \|x\|.$$

Next, we prove the equivalence. The right implication (\Rightarrow) is trivial, since

$$\Gamma(x^{-1})\Gamma(x) = \Gamma(x^{-1}x) = \Gamma(1) = 1.$$

Now the converse (\Leftarrow): if $x \notin \text{GL}(A)$, then by corollary 2.27 there exists $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$. Then $\Gamma(x)(\varphi) = \varphi(x) = 0$, so the continuous map $\Gamma(x)$ is not invertible. □

Corollary 2.31. Let A be an abelian Banach algebra. Then we have

$$\sigma(\Gamma(x)) = \sigma(x)$$

and

$$\|\Gamma(x)\| = r(\Gamma(x)) = r(x).$$

Theorem 2.32 (Gelfand).

Let A be an abelian C^ -algebra. Then Γ is an isometric $*$ -isomorphism.*

Proof. For a self-adjoint $x \in A$ we have $\sigma(\Gamma(x)) = \sigma(x) \subseteq \mathbb{R}$. Then $\overline{\Gamma(x)} = \Gamma(x)$. An arbitrary $x \in A$ can be written as $x = a + ib$ for self-adjoint $a = \frac{x+x^*}{2}$ and $b = \frac{i(x^*-x)}{2}$. Then

$$\Gamma(x^*) = \Gamma(a - ib) = \Gamma(a) - i\Gamma(b) = \overline{\Gamma(a) + i\Gamma(b)} = \overline{\Gamma(x)}.$$

This implies that Γ is a $*$ -homomorphism. Since A is abelian, each $x \in A$ is normal so

$$\|x\| = r(x) = r(\Gamma(x)) = \|\Gamma(x)\|$$

and Γ is an isometry. In particular, Γ is injective. We know that $\Gamma(A)$ is closed under $*$. Since Γ is isometric, the subalgebra $\Gamma(A) \subseteq C(\sigma(A))$ is complete in the norm, so it is closed. It can be easily checked that $\Gamma(A)$ separates points. By Stone–Weierstrass, $\Gamma(A) = C(\sigma(A))$. \square

Remark. Let A be a C^* -algebra. If $x \in A$ is normal, then it generates an abelian C^* -subalgebra of A :

$$C^*(x) = \overline{\{p(x, x^*) \mid p \in \mathbb{C}[x, y]\}}.$$

Corollary 2.33. *Let A be an abelian C^* -algebra, generated by $x \in A$. Then $\sigma(A) \cong \sigma(x)$.*

Proof. Let $\Gamma : A \rightarrow C(\sigma(A))$ be the Gelfand transform. Define

$$\tau : \sigma(A) \rightarrow \sigma(x), \quad \varphi \mapsto \varphi(x) = \Gamma(x)(\varphi).$$

Clearly, τ is well-defined since $\varphi(x) \in \sigma(x)$ for all $\varphi \in \sigma(A)$. Next we show that τ is onto. For $\lambda \in \sigma(x)$ we have $x - \lambda \notin \text{GL}(A)$, so there exists $\psi \in \sigma(A)$ such that $\psi(x) - \psi(\lambda) = \psi(x - \lambda) = 0$. We show that τ is injective. Let $\tau(\varphi_1) = \tau(\varphi_2)$. Then $\varphi_1(x) = \varphi_2(x)$. Since

$$\varphi_j(x^*) = \Gamma(x^*)(\varphi_j) = \overline{\Gamma(x)(\varphi_j)} = \overline{\varphi_j(x)},$$

we have $\varphi_1(x^*) = \varphi_2(x^*)$. Hence $\varphi_1(p(x, x^*)) = \varphi_2(p(x, x^*))$ for every polynomial $p \in \mathbb{C}[x, y]$. Since $\{p(x, x^*) \mid p \text{ polynomial}\}$ is dense in A , we have $\varphi_1 = \varphi_2$. Finally, we prove the continuity of τ . Let $(\varphi_\alpha)_\alpha$ be a net in $\sigma(A)$ such that $\varphi_\alpha \rightarrow \varphi$. Then $\varphi_\alpha(y) \rightarrow \varphi(y)$ for all $y \in A$, so in particular $\varphi_\alpha(x) \rightarrow \varphi(x)$, which proves that $\tau(\varphi_\alpha) \rightarrow \tau(\varphi)$. Since τ is a continuous bijection between compact Hausdorff spaces, it is a homeomorphism. \square

Remark. Since $\varphi \in \sigma(A)$ is an algebra homomorphism, we have $\varphi(p(x, x^*)) = p(\varphi(x), \overline{\varphi(x)})$ for a complex polynomial $p(z, \bar{z})$ in z and \bar{z} . Using the notation from above proof, we get $\Gamma(p(x, x^*)) = p \circ \tau$.

2.4 Continuous functional calculus

Now let A be any C^* -algebra and $x \in A$ normal. Then $C^*(x)$ is an abelian C^* -subalgebra of A . Since $\sigma(x) = \sigma_{C^*(x)}$, we have the map

$$\tau^\# : C(\sigma(x)) \rightarrow C(C^*(x)), \quad f \mapsto f \circ \tau,$$

which is a $*$ -isomorphism and an isometry. Define a map $\rho = \Gamma^{-1} \circ \tau^\# : C(\sigma(x)) \rightarrow C^*(x)$.

$$\begin{array}{ccc} C^*(x) & \xrightarrow{\Gamma} & C(\sigma(A)) \\ & \swarrow \rho \quad \searrow \tau^\# & \\ & C(\sigma(x)) & \end{array}$$

We know that $C^*(x) = \overline{\{p(x, x^*) \mid p(z, \bar{z}) \text{ polynomial}\}}$ and $\Gamma(p(x, x^*)) = \tau^\#(p)$, which means that $\rho(p) = p(x, x^*)$ for any polynomial $p \in \mathbb{C}[x, y]$. This map $\rho : C(\sigma(x)) \rightarrow C^*(x) \subseteq A$ is called the *continuous functional calculus*. We use the notation $f(x) := \rho(f)$.

Theorem 2.34 (Continuous functional calculus).

Let A, B be C^* -algebras and let $x \in A$ be normal.

(1.) $f \mapsto f(x)$ is an isometric $*$ -isomorphism $C(\sigma(x)) \rightarrow A$ and if

$$f = \sum_{j,k=0}^n a_{jk} z^j \bar{z}^k$$

is a polynomial, then

$$f(x) = \sum_{j,k=0}^n a_{jk} x^j (x^*)^k.$$

In particular, if $f(z) = z$ is the identity polynomial, then $f(x) = x$.

(2.) For $f \in C(\sigma(x))$, we have $\sigma(f(x)) = f(\sigma(x))$.

(3.) (**Spectral mapping theorem**) If $\Phi : A \rightarrow B$ is a $*$ -homomorphism, then $\Phi(f(x)) = f(\Phi(x))$.

(4.) Let $(x_n)_n$ be a sequence of normal elements of A that converge to x , Ω a compact neighborhood of $\sigma(x)$, and $f \in C(\Omega)$. Then for any sufficiently large n , we have $\sigma(x_n) \subseteq \Omega$ and $\|f(x_n) - f(x)\| \rightarrow 0$.

Proof. The items (1) and (2) follow directly from Gelfand's theorem and properties of continuous functions on compact sets. The item (3) is obvious for polynomials f and the general case follows from Stone–Weierstrass. We prove the item (4). Let $C = \sup_n \|x_n\| < \infty$. First we need to show that $\sigma(x_n) \subseteq \Omega$ for large enough n . If that wasn't the case, then for every $n \in \mathbb{N}$ there would exist $N_n > n$ such that there exists $\lambda_n \in \sigma(x_{N_n}) \setminus \Omega \subseteq \overline{B_C(0)}$. Thus there exists a convergent subsequence $(\lambda_{n_k})_k$ such that $\lambda_{n_k} \rightarrow \lambda \in U$, where U is an open neighborhood of $\sigma(x)$ and $\lambda \notin \sigma(x)$. But then

$$\underbrace{x_{n_k} - \lambda_{n_k}}_{\notin \text{GL}(A)} \rightarrow \underbrace{x - \lambda}_{\in \text{GL}(A)},$$

which contradicts the openness of $\text{GL}(A)$. For every $\varepsilon > 0$ there exists a polynomial $g : \Omega \rightarrow$

\mathbb{C} such that $\|f - g\|_\infty < \varepsilon$. Now

$$\begin{aligned} \limsup_n \|f(x_n) - g(x_n)\| + \|g(x_n) - g(x)\| + \|g(x) - f(x)\| \\ \leq 2 \cdot C \cdot \varepsilon + \limsup_n \|g(x_n) - g(x)\| \\ = 2C\varepsilon. \end{aligned}$$

Since ε was arbitrary, we conclude that $\lim_{n \rightarrow \infty} \|f(x_n) - f(x)\| = 0$. \square

We illustrate the use of continuous functional calculus to obtain the strengthening of corollary 2.13.

Corollary 2.35. *If A, B are C^* -algebras and $\Phi : A \rightarrow B$ is a $*$ -monomorphism, then it is an isometry.*

Proof. Let $a \in A$ be self-adjoint. Then $\Phi(a) \in B$ is self-adjoint as well. As in the proof of 2.13, we observe that $\sigma_B(\Phi(a)) \subseteq \sigma_A(a)$. Suppose that $\sigma_B(\Phi(a)) \neq \sigma_A(a)$. Since $\sigma_B(\Phi(a))$ is compact, it is closed in $\sigma_A(a)$. This implies that $U := \sigma_A(a) \setminus \sigma_B(\Phi(a))$ is a nonempty open set. It follows that there exists a function f which is zero on $\sigma_B(\Phi(a))$, but not identically zero on $\sigma_A(a)$ (take for example any bump function on U). Then $f(\Phi(a)) = 0$, but $f(a) \neq 0$. By Stone–Weierstrass, we can approximate f uniformly on $\sigma_A(a)$ by polynomials $\{p_n\}_{n \in \mathbb{N}}$. Thus $p_n(a) \rightarrow f(a)$ and $p_n(\Phi(a)) \rightarrow f(\Phi(a)) = 0$. On the other hand, $p_n(\Phi(a)) = \Phi(p_n(a)) \rightarrow \Phi(f(a))$, which implies that $\Phi(f(a)) = f(\Phi(a)) = 0$. But Φ was assumed injective, so $f(a) = 0$, contradiction. Therefore, $\sigma_B(\Phi(a)) = \sigma_A(a)$ for self-adjoint a and

$$\|a\| = r(a) = r(\Phi(a)) = \|\Phi(a)\|.$$

Now for a completely arbitrary $a \in A$, we have

$$\|a\|^2 = \|a^*a\| = \|\Phi(a^*a)\| = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a)\|^2,$$

concluding our proof. \square

The argument in this proof is very common. We first approximate some function on the spectrum with polynomials using Stone–Weierstrass. Then we observe that the continuous functional calculus of a polynomial has desired properties and deduce the same for the continuous functional calculus of the original function.

2.5 Applications of the continuous functional calculus

Definition 2.36. Let A be a C^* -algebra and $x \in A$.

- x is *positive* if $x = y^*y$ for some $y \in A$ (i.e., x is a hermitian square). The set of positive elements is denoted A_+ .
- x is a *projection* if $x^2 = x^* = x$.
- x is *unitary* if $xx^* = x^*x = 1$. The set of positive elements is denoted $U(A)$.
- x is an *isometry* if $x^*x = 1$.
- x is a *partial isometry* if x^*x is a projection.

Remark. The first three are automatically normal (the first two are even self-adjoint).

The set of all positive operators (denoted as A_+) induces a partial ordering on A_{sa} : for two elements $a, b \in A_{\text{sa}}$ we define

$$a \leq b \Leftrightarrow b - a \in A_+.$$

We notice that $x^*A_+x \subseteq A_+$ for every $x \in A$. For any $a, b \in A_{\text{sa}}$ and $x \in A$, we have

$$a \leq b \Rightarrow x^*ax \leq x^*bx.$$

Proposition 2.37. *Let A be a C^* -algebra and $x \in A$. Then x is a linear combination of four unitaries.*

Proof. Since $x = \operatorname{Re} x + i \operatorname{Im} x$, where $\operatorname{Re} x, \operatorname{Im} x \in A_{\text{sa}}$, it's enough to show that every self-adjoint element is a linear combination of two unitaries. Without loss of generality, assume $\|x\| \leq 1$, so $\sigma(x) \subseteq [-1, 1]$. Consider the continuous function

$$f : [-1, 1] \rightarrow \mathbb{T}, \quad z \mapsto z + i(1 - z^2)^{\frac{1}{2}}.$$

Since $f \cdot \bar{f} \equiv 1$ on $[-1, 1]$, it follows from continuous functional calculus that

$$f(x)f(x)^* = f(x)^*f(x) = 1.$$

Consequently, $f(x) = u$ is unitary and $x = \frac{1}{2}(f(x) + f(x)^*)$ is a linear combination of two unitaries. \square

Remark. We use the notation $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition 2.38. Let $x \in A_{\text{sa}}$. Then $\sigma(x) \subseteq \mathbb{R}$ and we can define

$$x_+ = \max\{0, x\}, \quad x_- = -\min\{0, x\} \quad (x \in A).$$

Then $\sigma(x_+), \sigma(x_-) \subseteq [0, \infty)$, $x = x_+ - x_-$ and $x_+x_- = x_-x_+ = 0$.

Lemma 2.39. *Suppose $x, y \in A_{\text{sa}}$ satisfy $\sigma(x), \sigma(y) \subseteq [0, \infty)$. Then $\sigma(x + y) \subseteq [0, \infty)$.*

Proof. Let $a := \|x\|$ and $b := \|y\|$. Since $x = x^*$ and $\sigma(x) \subseteq [0, a]$, we deduce that $\sigma(a - x) \subseteq [0, a]$, where $\|a - x\| = r(a - x) \leq a$. Likewise, $\|b - y\| \leq b$. Then

$$\begin{aligned} \sup_{\lambda \in \sigma(x+y)} \{a + b - \lambda\} &= r(a + b - (x + y)) \\ &= \|(a + b) - (x + y)\| \\ &\leq \|a - x\| + \|b - y\| \\ &\leq a + b. \end{aligned}$$

\square

ADD A PICTURE

Theorem 2.40.

Let A be a C^* -algebra and $x \in A$ normal. Then:

- (1.) $x \in A_{\text{sa}} \Leftrightarrow \sigma(x) \subseteq \mathbb{R}$;
- (2.) $x \in A_+ \Leftrightarrow \sigma(x) \subseteq [0, \infty)$;
- (3.) $x \in U(A) \Leftrightarrow \sigma(x) \subseteq \mathbb{T}$;
- (4.) $x^2 = x^* = x \Leftrightarrow \sigma(x) \subseteq \{0, 1\}$.

Proof. Throughout this proof, let $f(z) = z$ denote the identity polynomial.

(1.)

$$\begin{aligned}
 x = x^* &\Leftrightarrow f(x) = \bar{f}(x) \\
 &\Leftrightarrow f \equiv \bar{f} \text{ on } \sigma(x) \\
 &\Leftrightarrow z = \bar{z} \text{ for all } z \in \sigma(x) \\
 &\Leftrightarrow \sigma(x) \subseteq \mathbb{R}.
 \end{aligned}$$

(2.) (\Rightarrow) Let $x = y^*y$ for some $y \in A$. Write $x = x_+ - x_-$ and let $z := y \cdot x_-$. Then

$$z^*z = x_-y^*yx_- = x_-xx_- = -x_-^3.$$

From there we get

$$\sigma(zz^*) \subseteq \sigma(z^*z) \cup \{0\} \subseteq (-\infty, 0].$$

Let $z = a + ib$ for $a, b \in A_{\text{sa}}$. Then $zz^* + z^*z = 2a^2 + 2b^2$, which implies that $\sigma(zz^* + z^*z) \subseteq [0, \infty)$. It follows that

$$\sigma(z^*z) = \sigma((2a^2 + 2b^2) - zz^*) \subseteq [0, \infty).$$

As a result,

$$\sigma(-x_-^3) = \sigma(z^*z) \subseteq \{0\},$$

so $x_-^3 = 0$ and $x_- = 0$. This proves that $x = x_+$ has nonnegative spectrum. For the converse implication (\Leftarrow), apply the function $\sqrt{\cdot} : [0, \infty) \rightarrow \mathbb{R}$. Then

$$x = (\sqrt{x})^2 = (\sqrt{x})^* \cdot \sqrt{x} \in A_+.$$

(3.)

$$\begin{aligned}
 xx^* = 1 &\Leftrightarrow f(x) \cdot \bar{f}(x) = 1 \\
 &\Leftrightarrow f \cdot \bar{f} \equiv 1 \text{ on } \sigma(x) \\
 &\Leftrightarrow |z|^2 = 1 \text{ for all } z \in \sigma(x) \\
 &\Leftrightarrow \sigma(x) \subseteq \mathbb{T}.
 \end{aligned}$$

(4.)

$$\begin{aligned}
 x^2 = x^* = x &\Leftrightarrow f(x) \cdot \bar{f}(x) = \bar{f}(x) = f(x) \\
 &\Leftrightarrow f \cdot \bar{f} \equiv \bar{f} \equiv f \text{ on } \sigma(x) \\
 &\Leftrightarrow |z|^2 = \bar{z} = z \text{ for all } z \in \sigma(x) \\
 &\Leftrightarrow \sigma(x) \subseteq \{0, 1\}.
 \end{aligned}$$

□

Corollary 2.41. *Let A be a C^* -algebra and $x \in A$. Then x is a partial isometry iff x^* is a partial isometry.*

Proof.

$$\begin{aligned}
x \text{ partial isometry} &\Leftrightarrow x^*x \text{ projection} \\
&\Leftrightarrow \sigma(x^*x) \subseteq \{0, 1\} \\
&\Leftrightarrow \sigma(xx^*) \subseteq \{0, 1\} \\
&\Leftrightarrow xx^* \text{ projection} \\
&\Leftrightarrow x^* \text{ partial isometry.}
\end{aligned}$$

□

Corollary 2.42. *Let A be a C^* -algebra.*

- (1.) A_+ is a closed convex cone ($\lambda A_+ \subseteq A_+$ for $\lambda \in \mathbb{R}_{\geq 0}$).
- (2.) If $a \in A_{\text{sa}}$, then $a \leq \|a\|$.

Proposition 2.43. *Let A be a C^* -algebra and $x, y \in A_+$.*

- (1.) If $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.
- (2.) If $x, y \in \text{GL}(A)$ and $x \leq y$, then $y^{-1} \leq x^{-1}$.

Proof. Let us prove the second point first. Suppose $x, y \in \text{GL}(A)$. Then we have $y^{-\frac{1}{2}}xy^{-\frac{1}{2}} \leq 1$ and

$$\begin{aligned}
x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}} &\leq \|x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}}\| \\
&= r(x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}}) \\
&= r(y^{-\frac{1}{2}}xy^{-\frac{1}{2}}) \\
&\leq 1.
\end{aligned}$$

Multiplying on both sides by $x^{-\frac{1}{2}}$, we get $y^{-1} \leq x^{-1}$. Now we prove the first point. For invertible $x \leq y$, we have

$$\begin{aligned}
\|y^{-\frac{1}{2}}x^{\frac{1}{2}}\|^2 &= \|(y^{-\frac{1}{2}}x^{\frac{1}{2}})(y^{-\frac{1}{2}}x^{\frac{1}{2}})^*\| \\
&= \|y^{-\frac{1}{2}}xy^{-\frac{1}{2}}\| \\
&\leq 1,
\end{aligned}$$

which implies

$$\begin{aligned}
y^{-\frac{1}{4}}x^{\frac{1}{2}}y^{-\frac{1}{4}} &\leq \|y^{-\frac{1}{4}}x^{\frac{1}{2}}y^{-\frac{1}{4}}\| \\
&= r(y^{-\frac{1}{4}}x^{\frac{1}{2}}y^{-\frac{1}{4}}) \\
&= r(y^{-\frac{1}{2}}x^{\frac{1}{2}}) \\
&= \|y^{-\frac{1}{2}}x^{\frac{1}{2}}\| \leq 1.
\end{aligned}$$

Multiplying on both sides by $y^{\frac{1}{4}}$, we get $y^{\frac{1}{2}} \leq x^{\frac{1}{2}}$. For general non-invertible $x \leq y$, pick $\varepsilon > 0$ and notice that

$$0 \leq x + \varepsilon \leq y + \varepsilon.$$

However, since x, y are positive, we also have $x + \varepsilon, y + \varepsilon \in \text{GL}(A)$. We use the above calculation to obtain $(x + \varepsilon)^{\frac{1}{2}} \leq (y + \varepsilon)^{\frac{1}{2}}$. If we send $\varepsilon \rightarrow 0$, we get $x^{\frac{1}{2}} \leq y^{\frac{1}{2}}$. \square

Remark. Let $I \subseteq \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ be continuous. Then the function f is operator monotone if for every C^* -algebra A and $a, b \in A_{\text{sa}}$ with $a \leq b$ and $\sigma(a), \sigma(b) \subseteq I$, we have $f(a) \leq f(b)$. By the above proposition, $z \mapsto \sqrt{z}$ and $z \mapsto \frac{1}{z}$ are operator monotone on $[0, \infty)$. Actually, this is also true for functions $z \mapsto z^r$ for $r \in [0, 1]$, but not for $r > 1$.

Definition 2.44. Absolute value of $x \in A$ is defined as

$$|x| = (x^*x)^{\frac{1}{2}} \in A_+.$$

Corollary 2.45. For $x, y \in A$, we have $|xy| \leq \|x\||y|$.

Proof. Notice that

$$|xy|^2 = y^*x^*xy \leq y^*\|x^*x\|y = \|x\|^2(y^*y)$$

and now apply the operator-monotone $\sqrt{\cdot}$ and the previous proposition. \square

Theorem 2.46.

Let A be a C^* -algebra.

- (1.) $\text{ext}(A_+)_1 = \{\text{projections in } A\}$.
- (2.) $\text{ext}(A)_1 \subseteq \{\text{partial isometries in } A\}$.
- (3.) $\text{ext}(A_{\text{sa}})_1 = U(A) \cap A_{\text{sa}}$.

Proof. (1.) Let $x \in (A_+)_1$. Then $x^2 \leq 2x$, since $z^2 - 2z \leq 0$ on $[0, 1] \supseteq \sigma(x)$. So $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$. If x is an extreme point, then $x = x^2$ and $x \in A_+ \subseteq A_{\text{sa}}$, so x is a projection. For the converse, assume A is abelian, meaning $A = C(K)$ for some compact Hausdorff space K (by Gelfand). If $x \in A = C(K)$ is a projection, then $x = \chi_E$ for some clopen $E \subseteq K$. Since $\text{ext}([0, 1]) = \{0, 1\}$, χ_E is an extreme point. Let A now be a general C^* -algebra and $p \in A^*$ a projection. Suppose $p = \frac{1}{2}(a + b)$ for some $a, b \in (A_+)_1$. Then $\frac{1}{2}a = p - \frac{1}{2}b \leq p$. Hence

$$0 \leq (1 - p)a(1 - p) \leq (1 - p)2p(1 - p) = 0,$$

so

$$(\sqrt{a}(1 - p))^*(\sqrt{a}(1 - p)) = (1 - p)a(1 - p) = 0.$$

This implies that $\sqrt{a}(1 - p) = 0$ and $a(1 - p) = 0$. It follows that

$$ap = a = a^* = (ap)^* = p^*a^* = pa.$$

Similarly, we can show that a, b, p all commute, so the C^* -subalgebra $C^*(a, b, p)$ is abelian and we can just use the previous observation.

- (2.) Suppose $x \in (A)_1$ is not a partial isometry (alternatively, x^*x is not a projection). First, we notice that $\|x^*x\| = \|x\|^2 \leq 1$. Since x is not a projection, $\sigma(x^*x) \cap (0, 1) \neq \emptyset$. Then we apply the continuous functional calculus to obtain a function $f : \sigma(x^*x) \rightarrow [0, 1]$ such that $|t(1 \pm f(t))^2| \leq 1$ for $t \in \sigma(x^*x)$ (for example, f can be a small bump function on an interval $[a, b] \subseteq (0, 1)$, where $[a, b] \cap \sigma(x^*x) \neq \emptyset$). Then $y := f(x^*x) \in A_+$ gives us $yx^*x = x^*xy \neq 0$ and $\|x^*x(1 \pm y)^2\| \leq 1$. Hence, $\|x(1 \pm y)\|^2 \leq 1$ and

$$x = \frac{1}{2}((x + xy) + (x - xy)) \notin \text{ext}(A)_1.$$

- (3.) If $u \in U(A) \cap A_{\text{sa}}$, then $x \mapsto ux$ is an isometry. As in the case of $\mathcal{B}(\mathcal{H})$, u is an extreme point, so $A_{\text{sa}} \cap U(A) \subseteq \text{ext}(A_{\text{sa}})_1$. For the converse, assume $x \in \text{ext}(A_{\text{sa}})_1$ and $x_+ = \frac{1}{2}(a + b)$ for $a, b \in (A_+)_1$. Then

$$0 = x_-x_+x_- = \frac{1}{2}(x_-ax_- + x_-bx_-) \geq 0.$$

From $x_-ax_- = 0$, we get $(\sqrt{a}x_-)^*(\sqrt{a}x_-) = 0$, which implies that $\sqrt{a}x_- = 0$ and $ax_- = 0$. Likewise, $x_-a = bx_- = x_-b = 0$. By Gelfand, the commutative C^* -algebra $C^*(a, b, x_-)$ is isometrically $*$ -isomorphic to $C(K)$ for some compact K . This means that a and x_- are functions such that for every point in K , at least one of them is zero. Thus, $a - x_-$ is bounded above by 1, and we have $a - x_- \in (A_{\text{sa}})_1$. Similarly, $b - x_- \in (A_{\text{sa}})_1$, so

$$x = \frac{1}{2}((a - x_-) + (b - x_-)) \in (A_{\text{sa}})_1.$$

But since x is an extreme point, we have $a - x_- = b - x_-$ and $a = b = x_+$. Thus, $x_+ \in \text{ext}(A_+)_1$ is a projection by (1.), and by symmetry, so is x_- . Now we prove that x is unitary:

$$x^*x = x^2 = (x_+ - x_-)^2 = x_+^2 + x_-^2 = x_+ + x_- = |x|.$$

This implies that $|x|$ is a projection. Now set $q := 1 - |x|$. Then $x + q$ and $x - q$ are both in $(A_{\text{sa}})_1$. But since

$$x = \frac{1}{2}((x + q) + (x - q)),$$

we obtain $q = 0$, which further implies $|x| = 1$ and $x^*x = xx^* = 1$. \square

3 Representations of C^* -algebras and states

3.1 States

Let A be a C^* -algebra, then A^* can be given an A -bimodule structure: if $\psi \in A^*$ and $a, b \in A$, then

$$(a \cdot \psi \cdot b)(x) = \psi(bxa), \quad \forall x \in A.$$

We have

$$\|a \cdot \psi \cdot b\| = \sup_{x \in (A)_1} \|\psi(bxa)\| \leq \sup_{x \in (A)_1} \|\psi\| \|bxa\| \leq \|\psi\| \|a\| \|b\|.$$

Definition 3.1. Let A be a C^* -algebra and $\varphi \in A^*$.

- We say that φ is *positive* if $\varphi(x) \geq 0$, $\forall x \in A_+$. If φ is positive and $a \in A$, then $a\varphi a^*$ is also positive.
- A positive element $\varphi \in A^*$ is *faithful* if $\varphi(x) \neq 0$, $\forall x \in A_+ \setminus \{0\}$.
- An element $\varphi \in A^*$ is a *state* if it is *positive* and $\|\varphi\| = 1$. The set of states is denoted $S(A) \subseteq (A^*)_1$.

Remark. The set $S(A)$ is compact Hausdorff in the weak-* topology.

We notice that if $\varphi \in A^*$ is positive and $x \in A_{sa}$, then

$$\varphi(x) = \varphi(x_+ - x_-) = \varphi(x_+) - \varphi(x_-) \in \mathbb{R}.$$

If $y \in A$, then $y = y_1 + iy_2$, where y_1, y_2 are self-adjoint. Then

$$\begin{aligned} \varphi(y^*) &= \varphi((y_1 + iy_2)^*) = \varphi(y_1 - iy_2) \\ &= \varphi(y_1) - i\varphi(y_2) = \overline{\varphi(y_1) + i\varphi(y_2)} \\ &= \overline{\varphi(y_1 + iy_2)} = \overline{\varphi(y)} \end{aligned}$$

Such a functional $\varphi \in A^*$ is called *hermitian*. For any $\varphi \in A^*$, $\varphi^*(y) = \overline{\varphi(y^*)}$. Then $\varphi + \varphi^*$ and $i(\varphi - \varphi^*)$ are hermitian. One can, of course, define these notions also for unbounded linear functionals. However, positivity implies continuity: for every $a \in A_{sa}$ we have $-\|a\| \cdot 1 \leq a \leq \|a\| \cdot 1$, which implies

$$-\|a\|\varphi(1) \leq \varphi(a) \leq \|a\|\varphi(1)$$

and φ is bounded. For $a \in A$, we can of course write $a = b + ic$ for $b, c \in A_{sa}$. Here,

$$\|b\| = \left\| \frac{a + a^*}{2} \right\| \leq \frac{\|a\|}{2} + \frac{\|a^*\|}{2} = \|a\|$$

and likewise $\|c\| \leq \|a\|$. Let $\varphi(1) = C$. Then

$$|\varphi(a)|^2 = |\varphi(b) + i\varphi(c)|^2 = \varphi(b)^2 + \varphi(c)^2 \leq C^2(\|b\|^2 + \|c\|^2) \leq 2C^2\|a\|^2.$$

Lemma 3.2. Let $\varphi \in A^*$ be positive. Then $\forall x, y \in A$:

$$|\varphi(y^*x)|^2 \leq \varphi(y^*y) \cdot \varphi(x^*x).$$

Proof. Consider the sesquilinear form $\langle x, y \rangle = \varphi(y^*x)$. Since φ is positive, this is a positive sesquilinear form and we can apply Cauchy-Schwartz. \square

Theorem 3.3.

An element $\varphi \in A^*$ is positive iff $\|\varphi\| = \varphi(1)$.

Remark. This implies that the set of states $S(A)$ is convex.

Proof. First we prove the right implication (\Rightarrow). We know that $x^*x \leq \|x^*x\|$, so

$$\begin{aligned} |\varphi(x)|^2 &\leq \varphi(1)\varphi(x^*x) \\ &\leq \varphi(1)\varphi(\|x^*x\|) \\ &= \varphi(1)^2\|x^*x\| \\ &= \varphi(1)^2\|x\|^2, \end{aligned}$$

so $|\varphi(x)| \leq \varphi(1)\|x\|$. From there we get $\|\varphi\| \leq \varphi(1) \leq \|\varphi\|$, so $\varphi(1) = \|\varphi\|$. Now the converse (\Leftarrow). Suppose $x \in A_+$ and $\varphi(x) = \alpha + i\beta$. For each $t \in \mathbb{R}$, we have

$$\begin{aligned} \alpha^2 + (\beta + t\|\varphi\|)^2 &= |\alpha + i(\beta + t\varphi(1))|^2 \\ &= |\varphi(x + it)|^2 \\ &\leq \|x + it\|^2 \cdot \|\varphi\|^2 \\ &= (\|x\|^2 + t^2) \|\varphi\|^2. \end{aligned}$$

From this it directly follows $2\beta t\|\varphi\| \leq \|x\|^2 \cdot \|\varphi\|^2$. Since $t \in \mathbb{R}$ was arbitrary, we have $\beta = 0$ and $\varphi(x) = \alpha \in \mathbb{R}$. Lastly, we derive

$$\begin{aligned} \|x\| \cdot \|\varphi\| - \varphi(x) &= \varphi(\|x\| - x) \\ &\leq \|\|x\| - x\| \cdot \|\varphi\| \\ &\leq \|x\| \cdot \|\varphi\|, \end{aligned}$$

so $\varphi(x) \geq 0$. \square

Proposition 3.4. Let A be a C^* -algebra and $x \in A$. Then $\forall \lambda \in \sigma(x)$ there exists a $\varphi \in S(A)$ such that $\varphi(x) = \lambda$.

Proof. We know that $\mathbb{C}x + \mathbb{C} \cdot 1 \subseteq A$. Define

$$\varphi_0 : \mathbb{C}x + \mathbb{C}1 \rightarrow \mathbb{C}, \quad \alpha x + \beta \mapsto \alpha \cdot \lambda + \beta.$$

Since $\varphi_0(\alpha x + \beta) \in \sigma(\alpha x + \beta)$, we have

$$\|\varphi_0\| \leq 1 = \varphi_0(1),$$

therefore $\|\varphi_0\| = 1$. Now we apply Hahn–Banach to get an extension $\varphi \in A^*$ such that $\varphi|_{\mathbb{C}x + \mathbb{C}1} = \varphi_0$ and $\|\varphi\| = 1 = \varphi(1)$, so $\varphi \in S(A)$ by theorem 3.3. \square

Proposition 3.5. *Let A be a C^* -algebra and $x \in A$.*

- (1.) $x = 0$ iff $\varphi(x) = 0$, $\forall \varphi \in S(A)$.
- (2.) $x \in A_{\text{sa}}$ iff $\varphi(x) \in \mathbb{R}$, $\forall \varphi \in S(A)$.
- (3.) $x \in A_+$ iff $\varphi(x) \geq 0$, $\forall \varphi \in S(A)$.

Proof. (1.) If $\varphi(x) = 0$ for all $\varphi \in S(A)$, then writing $x = x_1 + ix_2$ for self-adjoint x_1, x_2 gives us

$$0 = \varphi(x) = \varphi(x_1) + i\varphi(x_2),$$

which implies $\varphi(x_1) = \varphi(x_2) = 0$. Now use the proposition 3.5 to get $\sigma(x_1) = \sigma(x_2) = \{0\}$, which can only imply $x_1 = x_2 = 0$, whence $x = 0$.

- (2.) If $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(A)$, then

$$\varphi(x - x^*) = \varphi(x) - \varphi(x^*) = \varphi(x) - \overline{\varphi(x)} = 0$$

and we use the previous item to show that $x - x^* = 0$. The converse implication follows from the fact that every positive functional is hermitian.

- (3.) If $\varphi(x) \geq 0$ for all $\varphi \in S(A)$, then $x \in A_{\text{sa}}$ by previous item and $\sigma(x) \subseteq [0, \infty)$, so $x \in A_+$. The converse once again follows from positivity of φ . \square

3.2 Gelfand-Naimark-Segal construction

Definition 3.6. • A *representation* of a C^* -algebra A is a $*$ -homomorphism $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} .

- If $\mathcal{K}^{\text{closed}} \leq \mathcal{H}$ and $\pi(x)\mathcal{K} \subseteq \mathcal{K}$, $\forall x \in A$ (we say that \mathcal{K} is *invariant* under π), then the restriction of π to \mathcal{K} is a *subrepresentation*.
- If a representation has no other subrepresentations besides $\mathcal{K} = (0)$ and $\mathcal{K} = \mathcal{H}$ (equivalently, $\pi(A)$ only has (0) and \mathcal{H} as closed invariant subspaces), then π is called *irreducible*.
- Representations $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ and $\rho : A \rightarrow \mathcal{B}(\mathcal{K})$ are *equivalent* if there exists a unitary $U : \mathcal{H} \rightarrow \mathcal{K}$ such that

$$U\pi(x) = \rho(x)U, \quad \forall x \in A.$$

- Vector $\zeta \in \mathcal{H}$ is *cyclic* for a representation $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ if

$$\pi(A)\zeta := \{\pi(a)\zeta \mid a \in A\}$$

is dense in \mathcal{H} (this means that $\overline{\pi(A)\zeta} = \mathcal{H}$).

Example 3.7. Each $w \in \mathcal{H}$ defines a subrepresentation on $K := \overline{\pi(A)w}$.

Example 3.8. Let $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ be a representation and $\mu \in \mathcal{H}$, $\|\mu\| = 1$. Then

$$\varphi_\mu : A \rightarrow \mathbb{C}, \quad x \mapsto \langle \pi(x)\mu, \mu \rangle$$

is a state. Indeed,

$$\varphi_\mu(1) = \langle 1 \cdot \mu, \mu \rangle = \|\mu\|^2 = 1$$

and

$$\begin{aligned} \varphi_\mu(x^*x) &= \langle \pi(x^*x)\mu, \mu \rangle \\ &= \langle \pi(x^*)\pi(x)\mu, \mu \rangle \\ &= \langle \pi(x)^*\pi(x)\mu, \mu \rangle \\ &= \langle \pi(x)\mu, \pi(x)\mu \rangle \\ &= \|\pi(x)\mu\|^2 \geq 0. \end{aligned}$$

Theorem 3.9 (Gelfand-Naimark-Segal construction).

Let A be a C^* -algebra and $\rho \in S(A)$. Then there exists a Hilbert space $L^2(A, \varphi)$ and a unique (up to equivalence) representation $\pi : A \rightarrow \mathcal{B}(L^2(A, \varphi))$ and a unit cyclic vector 1_φ such that

$$\varphi(x) = \langle \pi(x)1_\varphi, 1_\varphi \rangle, \quad \forall x \in A.$$

Proof. (1.) We start by defining

$$N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$$

whose elements we call nullvectors of φ . By the Cauchy-Schwartz lemma, we have

$$N_\varphi = \{x \in A \mid \varphi(yx) = 0, \forall y \in A\}.$$

Thus N_φ is a closed subspace of A .

(2.) We prove that N_φ is a left ideal: for $x \in N_\varphi$ and $a \in A$, we have $ax \in N_\varphi$. Indeed,

$$\varphi((ax)^*ax) = \varphi((x^*a^*a)x) = 0.$$

(3.) Now $\mathcal{H}_0 = A/N_\varphi$ is a vector space and we can endow it with the dot product $\langle [x], [y] \rangle := \varphi(y^*x)$ for $x, y \in A$. It can easily be checked that this is a well-defined dot product in \mathcal{H}_0 . We denote the completion of \mathcal{H}_0 by $L^2(A, \varphi)$.

(4.) To an arbitrary $a \in A$, we associate the map

$$\pi_0(a) : \mathcal{H}_0 \rightarrow \mathcal{H}_0, \quad [x] \mapsto [ax].$$

Since N_φ is a left ideal of A , $\pi_0(a)$ is a well-defined linear map. We have

$$\begin{aligned} \|\pi_0(a)[x]\|^2 &= \|[ax]\|^2 \\ &= \langle [ax], [ax] \rangle \\ &= \varphi((ax)^*ax) \\ &= \varphi(x^*a^*ax) \\ &\leq \|a\|^2 \cdot \varphi(x^*x) \leq \|a\|^2 \|x\|^2. \end{aligned}$$

Since $\pi_0(a)$ is a bounded linear map, it extends uniquely to $\pi(a) \in \mathcal{B}(L^2(A, \varphi))$ with $\|\pi(a)\| \leq \|a\|$. Then we get

$$\pi : A \rightarrow \mathcal{B}(L^2(A, \varphi)), \quad a \mapsto \pi(a),$$

which is a homomorphism and has the property

$$\begin{aligned}
\langle [x], \pi(a^*)[y] \rangle &= \langle [x], [a^*y] \rangle \\
&= \varphi((a^*y)^*x) \\
&= \varphi(y^*ax) \\
&= \langle [ax], [y] \rangle \\
&= \langle \pi(a)[x], [y] \rangle.
\end{aligned}$$

So $\pi(a)^* = \pi(a^*)$ and π is a representation.

(5.) We define $1_\varphi := [1] \in \mathcal{H}_0 \subseteq L^2(A, \varphi)$ and notice that

$$\langle \pi(a)1_\varphi, 1_\varphi \rangle = \langle \pi(a)[1], [1] \rangle = \langle [a], [1] \rangle = \varphi(a).$$

Since $\{\pi(a)1_\varphi \mid a \in A\} = \mathcal{H}_0$ is dense in $L^2(A, \varphi)$, the vector 1_φ is cyclic for π .

(6.) Next we prove uniqueness: let $\rho : A \rightarrow \mathcal{B}(\mathcal{K})$ be a representation, $\mu \in \mathcal{K}$ a unit cyclic vector and assume $\varphi(a) = \langle \rho(a)\mu, \mu \rangle$, $\forall a \in A$. We will prove that ρ is equivalent to π . Define

$$U_0 : \mathcal{H}_0 \rightarrow \mathcal{K}, \quad [x] \mapsto \rho(x)\mu.$$

Then we have

$$\begin{aligned}
\langle U_0[x], U_0[y] \rangle_{\mathcal{K}} &= \langle \rho(x)\mu, \rho(y)\mu \rangle \\
&= \langle \rho(y)^* \rho(x)\mu, \mu \rangle \\
&= \langle \rho(y^*x)\mu, \mu \rangle = \varphi(y^*x) = \langle [x], [y] \rangle_{L^2(A, \varphi)},
\end{aligned}$$

so U_0 really is a well-defined isometry. For all $a, x \in A$:

$$U_0(\pi(a)[x]) = U_0([ax]) = \rho(ax)\mu = \rho(a)\rho(x)\mu = \rho(a)U_0[x].$$

Therefore, U_0 induces an isometry $U : L^2(A, \varphi) \rightarrow \mathcal{K}$ such that $U\pi(a) = \rho(a)U$ for all $a \in A$. Since μ is cyclic and $\rho(a)\mu \subseteq \text{im } U$, the range of U is dense in \mathcal{K} . It is also closed since U is isometric. We just proved that U is isometric and onto, so it is unitary. \square

Corollary 3.10. *Every C^* -algebra has a faithful (i.e. injective) representation. In particular, every C^* -algebra is isometrically $*$ -isomorphic to a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space \mathcal{H} .*

Proof. Let π be a direct sum of all representations from GNS construction over all states. Then the proposition 3.5 tells us that π is injective. An injective $*$ -monomorphism is isometric and we are done. \square

The preceding corollary enables us to view abstract C^* -algebras as concrete algebras of operators on some Hilbert space.

Definition 3.11. If $S \subseteq A$, then

$$S' := \{x \in A \mid \forall s \in S : xs = sx\}$$

is its commutant.

Proposition 3.12 (Radon-Nikodym for linear functionals). *Let φ, ψ be positive linear functionals on a C^* -algebra A and $\varphi \in S(A)$. Then $\varphi \leq \psi$ iff there exists a unique $y \in \pi_\psi(A)'$ such that $0 \leq y \leq 1$ and*

$$\varphi(a) = \langle \pi_\psi(a)y1_\psi, 1_\psi \rangle, \quad \forall a \in A.$$

Proof. Start with (\Leftarrow). For $a \in A_+$ we have

$$\pi_\psi(a)y = \pi_\psi(a)^{\frac{1}{2}}y\pi_\psi(a)^{\frac{1}{2}} \leq \pi_\psi(a).$$

Then

$$\varphi(a) = \langle \pi_\psi(a)y1_\psi, 1_\psi \rangle \leq \langle \pi_\psi(a)1_\psi, 1_\psi \rangle = \psi(a).$$

Now the converse (\Rightarrow). By Cauchy-Schwartz,

$$\begin{aligned} |\varphi(b^*a)|^2 &\leq \varphi(a^*a)\varphi(b^*b) \\ &\leq \psi(a^*a)\psi(b^*b) \\ &= \|\pi_\psi(a)1_\psi\|^2 \cdot \|\pi_\psi(b)1_\psi\|^2. \end{aligned}$$

This means that $\langle \pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle_\varphi := \varphi(b^*a)$ is a nonnegative sesquilinear form on $\pi_\psi(A)1_\psi^{\text{dense}} \subseteq L^2(A, \psi)$, which is bounded by 1. This further implies that it is continuous and we can extend it to $L^2(A, \psi)$. By Riesz, there exists $y \in \mathcal{B}(L^2(A, \psi))$ such that

$$\varphi(b^*a) = \langle y\pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle, \quad \forall a, b \in A$$

and $0 \leq y \leq 1$. For $a, b, c \in A$ we have

$$\begin{aligned} \langle y\pi_\psi(a)\pi_\psi(b)1_\psi, \pi_\psi(c)1_\psi \rangle &= \langle y\pi_\psi(ab)1_\psi, \pi_\psi(c)1_\psi \rangle \\ &= \varphi(c^* \cdot ab) = \varphi((a^*c)^*b) \\ &= \langle y\pi_\psi(b)1_\psi, \pi_\psi(a^*)\pi_\psi(c)1_\psi \rangle \\ &= \langle \pi_\psi(a)y\pi_\psi(b)1_\psi, \pi_\psi(c)1_\psi \rangle, \end{aligned}$$

so $y\pi_\psi(a) = \pi_\psi(a)y$ for all $a \in A$ and $y \in \pi_\psi(A)'$. Finally, the uniqueness. Say that there exists a $z \in \pi_\psi(A)'$ such that $0 \leq z \leq 1$ and

$$\langle \pi_\psi(a)y1_\psi, 1_\psi \rangle = \langle \pi_\psi(a)z1_\psi, 1_\psi \rangle, \quad \forall a \in A.$$

Then

$$\begin{aligned} \langle \pi_\psi(b^*a)z1_\psi, 1_\psi \rangle &= \langle \pi_\psi(b^*a)y1_\psi, 1_\psi \rangle \\ &= \langle y\pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle \\ &= \langle z\pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle, \end{aligned}$$

which implies $y = z$. □

Proposition 3.13. *Suppose that A is a separable C^* -algebra. Then A has a faithful cyclic representation on a separable Hilbert space.*

Proof. If A is separable, then it has a dense subset $\{a_i\}_{i=1}^\infty$. We can embed $S(A)$ into the space $\prod_{i=1}^\infty \overline{B_1(0)}$, where $\overline{B_1(0)}$ is a closed unit ball in \mathbb{C} , by sending $\varphi \in S(A)$ to $(\varphi(a_i))_{i=1}^\infty$. The latter topological space is metrizable by the metric $\rho(x, y) = \sum_{i=1}^\infty \frac{\rho_i(x_i, y_i)}{2^i(\rho_i(x_i, y_i)+1)}$, and so is $S(A)$. Therefore, $S(A)$ with the weak-* topology is a metrizable compact, therefore separable. Let $\{f_i\}_i^\infty$ countable weak-* dense subset of $S(A)$. Then

$$f(a) := \sum_{i=1}^\infty 2^{-i} f_i(a)$$

defines a faithful ($f(a^*a) = 0$ iff $a = 0$) state on A . Then the GNS construction π_f is faithful: if $\pi_f(a) = 0$, then

$$f(b^*a^*ab) = \langle \pi_f(a)[b], \pi_f(a)[b] \rangle = 0$$

for every $b \in A$. In particular for $b = 1$, we get $f(a^*a) = 0$ and so $a = 0$. Since $a \mapsto [a]$ is a continuous map of A onto a dense subspace of some Hilbert space \mathcal{H}_f (induced by $\pi_f : A \rightarrow \mathcal{B}(\mathcal{H}_f)$), the latter space is separable. \square

Proposition 3.14. *Every representation of a C^* -algebra is equivalent to a direct sum of cyclic representations.*

Proof. Let $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ be some representation of A . Let \mathcal{E} be the collection of all subsets E of nonzero vectors in \mathcal{H} such that $\pi(A)e \perp \pi(A)f$ for any $e, f \in E$. If we order \mathcal{E} by inclusion, then Zorn's lemma tells us that \mathcal{E} has a maximal element E_0 . Let $\mathcal{H}_0 = \bigoplus_{e \in E_0} \pi(A)e$. Take $h \in \mathcal{H}_0^\perp$ in \mathcal{H} . Then for any $a, b \in A$ and $e \in E_0$ we have

$$\langle \pi(a)e, \pi(b)h \rangle = \langle \pi(b)^* \pi(a)e, h \rangle = \langle \pi(b^*a)e, h \rangle = 0,$$

so $\pi(A)e \perp \pi(A)h$ for each $e \in E_0$. By maximality, $h = 0$ and $\mathcal{H} = \mathcal{H}_0$. For $e \in E_0$, define $\mathcal{H}_e := \pi(A)e$. Obviously, \mathcal{H}_e is invariant for π , so $\pi_e := \pi|_{\mathcal{H}_e}$ is a cyclic representation of A . Clearly, $\pi = \bigoplus_{e \in E_0} \pi_e$. \square

3.3 Pure states and irregular representations

Definition 3.15. A state $\varphi \in S(A)$ is called *pure* if it's an extreme point of $S(A)$.

Proposition 3.16. *A state $\varphi \in S(A)$ is pure iff the representation GNS $\pi_\varphi : A \rightarrow \mathcal{B}(L^2(A, \varphi))$ with cyclic vector 1_φ is irreducible.*

Proof. (\Rightarrow) Let $\mathcal{K} \leq L^2(A, \varphi)$ be a closed invariant subspace. Then \mathcal{K}^\perp is also a closed invariant subspace: for $a \in A$, $x \in \mathcal{K}^\perp$ and $k \in \mathcal{K}$ we have

$$\langle \pi_\varphi(a)x, k \rangle = \langle x, \pi_\varphi(a^*)k \rangle = 0.$$

Since $L^2(A, \varphi) = \mathcal{K} \oplus \mathcal{K}^\perp$ we write $1_\varphi = \underbrace{\mu_1}_{\in \mathcal{K}} + \underbrace{\mu_2}_{\in \mathcal{K}^\perp}$ and form

$$\varphi_j := \frac{\langle \pi_\varphi(x) \mu_j, \mu_j \rangle}{\|\mu_j\|^2}, \quad j = 1, 2.$$

These are states and so is

$$\varphi(x) = \|\mu_1\|^2 \varphi_1(x) + \|\mu_2\|^2 \varphi_2(x)$$

because $1 = \|1_\varphi\|^2 = \|\mu_1\|^2 + \|\mu_2\|^2$. Since $\varphi \in \text{ext } S(A)$, we either have $\mu_1 = 0$ or $\mu_2 = 0$, which implies that \mathcal{K} is either (0) or $L^2(A, \varphi)$.

(\Leftarrow) Suppose $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2)$ for $\varphi_1, \varphi_2 \in S(A)$. Define a linear map

$$U : L^2(A, \varphi) \rightarrow L^2(A, \varphi_1) \oplus L^2(A, \varphi_2), \quad \pi_\varphi(x)1_\varphi \mapsto \frac{1}{\sqrt{2}}\pi_{\varphi_1}(x)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(x)1_{\varphi_2}.$$

First we notice that U preserves the scalar product:

$$\begin{aligned} \langle \pi_\varphi(x)1_\varphi, \pi_\varphi(y)1_\varphi \rangle &= \varphi(x^*y) \\ &= \frac{1}{2}\varphi_1(x^*y) + \frac{1}{2}\varphi_2(x^*y) \\ &= \left\langle \frac{1}{\sqrt{2}}\pi_{\varphi_1}(x)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(x)1_{\varphi_2}, \frac{1}{\sqrt{2}}\pi_{\varphi_1}(y)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(y)1_{\varphi_2} \right\rangle \\ &= \langle U\pi_\varphi(x)1_\varphi, U\pi_\varphi(y)1_\varphi \rangle. \end{aligned}$$

Additionally, U intertwines: for all $x \in A$, we have

$$\begin{aligned} U\pi_\varphi(x)(\pi_\varphi(y)1_\varphi) &= U\pi_\varphi(xy)1_\varphi \\ &= \frac{1}{\sqrt{2}}\pi_{\varphi_1}(xy)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(xy)1_{\varphi_2} \\ &= (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))(\pi_{\varphi_1}(y)1_{\varphi_1} \oplus \pi_{\varphi_2}(y)1_{\varphi_2}) \\ &= (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))U(\pi_\varphi(y)1_\varphi). \end{aligned}$$

If we star the intertwining identity, we get

$$\pi_\varphi(x^*)U^* = U^*(\pi_{\varphi_1}(x^*) \oplus \pi_{\varphi_2}(x^*)), \quad \forall x^* \in A.$$

If we plug in x instead of x^* , we get

$$\pi_\varphi(x)U^* = U^*(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)), \quad \forall x \in A.$$

Now let

$$p_1 \in \mathcal{B}(L^2(A, \varphi_1) \oplus L^2(A, \varphi_2))$$

be orthogonal projection onto the first direct summand. Clearly, we have

$$p_1(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)) = (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))p_1$$

Putting it all together, we get

$$\begin{aligned}\pi_\varphi(x)U^*p_1U &= U^*(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))p_1U \\ &= U^*p_1(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))U \\ &= U^*p_1U(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)),\end{aligned}$$

so U^*p_1U commutes with $\pi_\varphi(x)$ for all $x \in A$. If $\sigma(U^*p_1U)$ has more than one element, then $\exists t \in (0, 1]$ such that $\sigma(U^*p_1U - t)$ has both positive and negative elements. By definition 2.38, we can write $U^*p_1U = a - b$ for positive $0 \neq a, b$ such that $ab = ba = 0$. Then a, b commute with $\pi_\varphi(A)$, so $\ker a \neq 0$ is a closed subspace of $L^2(A, \varphi)$ that is invariant under $\pi_\varphi(x)$, which is a contradiction. So U^*p_1U has a single element spectrum $\{\alpha\}$ and since U^*p_1U is normal (because it is positive), so is $U^*p_1U - \alpha I$. But now we can write

$$\|U^*p_1U - \alpha I\| = r(U^*p_1U) = 0,$$

which proves that $U^*p_1U = \alpha I$. Then

$$\begin{aligned}\alpha &= \alpha\varphi(1) = \varphi(\alpha) \\ &= \langle \alpha 1_\varphi, 1_\varphi \rangle \\ &= \langle U^*p_1U 1_\varphi, 1_\varphi \rangle \\ &= \left\langle \frac{1}{\sqrt{2}}1_{\varphi_1} \oplus 0, \frac{1}{\sqrt{2}}1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}1_{\varphi_2} \right\rangle \\ &= \left\langle \frac{1}{\sqrt{2}}1_{\varphi_1}, \frac{1}{\sqrt{2}}1_{\varphi_1} \right\rangle_{\varphi_1} = \frac{1}{2}.\end{aligned}$$

This means that we can write

$$\left(\sqrt{2}p_1U\right)^* \left(\sqrt{2}p_1U\right) = 1,$$

so

$$u_1 = \frac{1}{\sqrt{2}}p_1U : L^2(A, \varphi) \rightarrow L^2(A, \varphi_1)$$

is an isometry. We also have the identities

$$u_1 1_\varphi = 1_{\varphi_1}, \quad u_1 \pi_\varphi(x) = \pi_{\varphi_1}(x) u_1.$$

It follows that

$$\begin{aligned}\varphi(x) &= \langle \pi_\varphi(x) 1_\varphi, 1_\varphi \rangle \\ &= \langle u_1^* u_1 \pi_\varphi(x) 1_\varphi, 1_\varphi \rangle \\ &= \langle u_1^* \pi_{\varphi_1}(x) u_1 1_\varphi, 1_\varphi \rangle \\ &= \langle \pi_{\varphi_1}(x) u_1 1_\varphi, u_1 1_\varphi \rangle \\ &= \langle \pi_{\varphi_1}(x) 1_{\varphi_1}, 1_{\varphi_1} \rangle = \varphi_1(x)\end{aligned}$$

and we are done. □

Theorem 3.17.

A representation $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ is irreducible iff $\pi(A)' = \mathbb{C} \cdot \text{id}$.

Proof. Start with (\Leftarrow) . Suppose there exists a closed invariant subspace $(0) \neq \mathcal{K} \subsetneq \mathcal{H}$. Let $p \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection onto \mathcal{K} . Then $p \notin \mathbb{C} \cdot \text{id}$. Now we prove that $p \in \pi(A)'$. Let $a \in A$. For $\mu \in \mathcal{K}$, we have

$$(p\pi(a))\mu = p(\pi(a)\mu) = \pi(a)\mu = \pi(a)(p\mu) = (\pi(a)p)\mu.$$

Now for $\mu \in \mathcal{K}^\perp$, we get

$$(p\pi(a))\mu = p(\pi(a)\mu) = 0 = \pi(a)(0) = \pi(a)(p\mu) = (\pi(a)p)\mu.$$

For the converse (\Rightarrow) , suppose there exists a non-scalar self-adjoint $h \in \pi(A)'$. Then $\sigma(h)$ has at least two elements. We can define two bump functions $f, g \in C(\sigma(h))$ in the respective neighborhoods of these two elements of $\sigma(h)$ such that $fg = 0$. Then $f(h) \neq 0$ since $f \neq 0$. Then also $\mathcal{K} := \text{im } f(h) \leq \mathcal{H}$ is nonzero. Also, $g(h) \neq 0$ and $g(h)|_{\mathcal{K}} = 0$ since $g(h) \cdot f(h) = 0$. In particular, $\mathcal{K} \subsetneq \mathcal{H}$. Take any $x \in \pi(A)$. By Stone-Weierstrass, we can approximate $f(h)$ in norm by $(p_i(h))_{i=1}^\infty$, where $(p_i)_{i=1}^\infty$ are complex polynomials. Notice that since $h \in \pi(A)'$, we also have $p_i(h) \in \pi(A)'$. Therefore, we have $p_i(h)x = xp_i(h)$. By sending $i \rightarrow \infty$, the left side converges in norm to $f(h)x$, while the right one converges in norm to $af(h)$. As a result, $f(h)x = xf(h)$ and $f(h) \in \pi(A)'$. We claim that \mathcal{K} is invariant; it's enough to show that $\text{im } f(h)$ is invariant. For $a \in A, \mu \in \mathcal{H}$ we have

$$\pi(a)(f(h)\mu) = \pi(a)f(h)\mu = f(h)\pi(a)\mu \in \text{im } f(h).$$

In general, if $q \in \pi(A)'$, then $q^* \in \pi(A)'$ and we can reduce the problem to the self-adjoint case handled above. \square

Corollary 3.18. Irreducible representations of abelian C^* -algebras are 1-dimensional.

Proof. Let A be an abelian C^* -algebra and $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ an irrep. Then by theorem 3.17, $\pi(A)' = \mathbb{C}$. Moreover,

$$\pi(A) = Z(\pi(A)) = \pi(A)' \cap \pi(A) = \mathbb{C} \cdot \text{id}.$$

\square

Corollary 3.19. If A is an abelian C^* -algebra, then $\text{ext } S(A) = \sigma(A)$.

Proof. Let $\varphi \in \sigma(A)$. Then φ is 1-dimensional (therefore irreducible) representation and so $\varphi \in \text{ext } S(A)$. For the converse, take $\varphi \in \text{ext } S(A)$. Then the GNS construction π_φ is irreducible, therefore 1-dimensional. So $L^2(A, \varphi) = \mathbb{C}$ with the standard scalar product and $\varphi(x) = \langle \pi_\varphi(x)1_\varphi, 1_\varphi \rangle = \pi_\varphi(x)$. \square

Proposition 3.20. Let A be a C^* -algebra. Then $\text{co ext } S(A)$ is weak-* dense in $S(A)$.

Proof. We know that $S(A)$ is compact Hausdorff with respect to the weak-* topology. The conclusion follows from Krein–Milman. \square

Corollary 3.21. *Let A be a C^* -algebra and $x \in A \setminus \{0\}$. Then there exist an irrep $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ such that $\pi(x) \neq 0$.*

Proof. By proposition 3.5, there exists $\varphi \in S(A)$ such that $\varphi(x) \neq 0$. By the previous proposition (Krein–Milman), there exists a $\tau \in \text{ext } S(A)$ such that $\tau(x) \neq 0$. Then apply GNS: π_τ is irreducible and $\pi_\tau(x) \neq 0$. \square

Theorem 3.22 (Jordan decomposition for linear functionals).

Let A be a C^ -algebra and $\varphi \in A^*$ hermitian. Then there exist (unique - without proof) positive linear functionals $\varphi_+, \varphi_- \in A^*$ such that $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| = \|\varphi_-\|$.*

Proof. W.l.o.g. $\|\varphi\| = 1$. Let Σ denote the set of positive linear functionals with norm ≤ 1 . By Banach–Alaoglu, Σ is weak-* compact and Hausdorff. Consider

$$\gamma : A \rightarrow C(\Sigma), \quad a \mapsto (\psi \mapsto \psi(a)).$$

This is an isometry and $\gamma(A_+) \subseteq C(\Sigma)_+$. By Hahn–Banach, there exists a $\tilde{\varphi} : C(\Sigma) \rightarrow \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$ and $\varphi = \tilde{\varphi} \circ \gamma$. Assume $\tilde{\varphi}$ is hermitian (otherwise, we can replace it by $\frac{\tilde{\varphi} + \tilde{\varphi}^*}{2}$). By Riesz–Markoff, there exists a regular Radon Measure μ on Σ such that $\tilde{\varphi}(f) = \int f d\mu$ for all $f \in C(\Sigma)$. Then we use Jordan decomposition for measures to obtain μ_+, μ_- such that $\mu = \mu_+ - \mu_-$ and $\|\mu\| = \|\mu_+\| = \|\mu_-\|$. Now we just define $\varphi_\pm(a) := \int a d\mu_\pm$. \square

Corollary 3.23. *For a C^* -algebra A , A^* is the span of positive linear functionals on A .*

Corollary 3.24. *Let A be a C^* -algebra and $\varphi \in A^*$. Then there exists a representation $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ and $\mu, \theta \in \mathcal{H}$ such that $\varphi(a) = \langle \pi(a)\theta, \mu \rangle$.*

Proof. Write $\varphi = \sum_{i=1}^n \alpha_i \psi_i$ for some $\psi_j \in S(A)$. Let π_i be the GNS representation of ψ_i . Define $\pi := \bigoplus_i \pi_i$, $\theta := \bigoplus_i \alpha_i 1_{\psi_i}$ and $\mu = \bigoplus_i 1_{\psi_i}$. The result then follows immediately. \square

3.4 Examples of C^* -algebras

Example 3.25. *The most canonical example of a C^* -algebra is $\mathcal{B}(\mathcal{H})$. Similarly, the algebra of compact operators $\mathcal{K}(\mathcal{H})$ is a C^* -algebra (if $\dim \mathcal{H} = \infty$, it is non-unital).*

Definition 3.26. A C^* -algebra A is *simple* if it has no closed two-sided ideals.

The following results require some tools from nonunital C^* -algebras (namely, the approximate identity), so we state them without proofs. The reader can find additional information in the section VIII.4 of [1].

Lemma 3.27. *Any closed ideal I of a C^* -algebra A is closed for involution: if $a \in I$, then $a^* \in I$.*

Proposition 3.28. *If I is a closed ideal of a C^* -algebra A , then A/I is a C^* -algebra for involution*

$$(a + I)^* := a^* + I$$

and the usual quotient norm.

Example 3.29. *From the introductory course, we know that the set of compact operators $\mathcal{K}(\mathcal{H})$ on a Hilbert space forms a closed ideal in $\mathcal{B}(\mathcal{H})$. If $\dim \mathcal{H} = \infty$, then $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a Calkin algebra. If $\dim \mathcal{H} = |\mathbb{N}|$, then $\mathcal{K}(\mathcal{H})$ is the only proper nontrivial closed ideal in $\mathcal{B}(\mathcal{H})$ (a fact, later proved in corollary 6.14). In that case, Calkin algebra is simple and it does not admit a separable representation.*

Example 3.30. *The algebra of matrices $M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$ is a C^* -algebra.*

3.4.1 Structure theorem for finite-dimensional C^* -algebras

For finite-dimensional C^* -algebras, we have the Artin-Wedderburn type theorem.

Proposition 3.31 (Structure theorem for finite-dimensional C^* -algebras). *Every finite-dimensional C^* -algebra A is*

$$A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$$

for uniquely determined n_1, \dots, n_r .

To prove this, we need a few preliminary lemmas.

Lemma 3.32. *Every finite-dimensional C^* -algebra is unital.*

This fact, which we will not prove here, follows essentially from Krein–Milman theorem. The reader should consult theorem I.10.2 of [5].

Corollary 3.33. *If A is a finite-dimensional C^* -algebra, then every ideal in A is of the form $I = Ap$ for some central projection $p \in A$.*

Proof. If $I \triangleleft A$, then it is itself a (finite-dimensional) C^* -algebra, so it must have a unit p . As a result, we have $I = Ip \subseteq Ap$, but since I is an ideal we also have $Ap \subseteq I$, so $Ap = I$. Since p is a unit in I , we have $p^2 = p = p^*$, so p is a projection. For any $x \in A$, we get $xp \in I$ and so $p \cdot (xp) = xp$. By starring this equation, we get $px^*p = px^*$. Combining the last two equations gives us

$$x^*p = p \cdot (x^*p) = px^*p = px^*,$$

so p commutes with x^* . As a result, p commutes with the entire A . □

Lemma 3.34. *If A is a finite-dimensional abelian C^* -algebra, then its spectrum is finite.*

Proof. We know that by Gelfand, $A \cong C(\sigma(A))$, where $\sigma(A)$ is a compact Hausdorff (and therefore normal) space. Suppose that $\sigma(A)$ is infinite.

- (1.) First, we will inductively construct an infinite subset $X = \{x_n\}_{n \in \mathbb{N}} \subseteq \sigma(A)$ that does not contain any of its accumulation points. Pick any point $x \in \sigma(A)$. If x is not an accumulation point of $\sigma(A)$ (so it is an isolated point), then take $x_1 := x$ and choose any new point in $\sigma(A) \setminus \{x_1\}$ to repeat this process. However, if x is an accumulation point of $\sigma(A)$, then take any $x_1 \in \sigma(A) \setminus \{x\}$. Since $\sigma(A)$ is Hausdorff, there exist disjoint open neighborhoods $V_1 \ni x, U_1 \ni x_1$. Now since x is an accumulation point of $\sigma(A)$, there must exist some $x_2 \in V_1$, such that $x_2 \neq x$. By the Hausdorff property, there must exist open disjoint neighborhoods $V_2 \ni x, U_2 \ni x_2$ inside V_1 . Now repeat this process indefinitely to obtain a set $\{x_n\}_{n \in \mathbb{N}}$ which does not contain its accumulation points.
- (2.) Notice that from the previous item, every point x_n has an open neighborhood U_n , where $U_n \cap U_m = \emptyset$ for any $n \neq m$. By Urysohn's lemma, there exists a continuous function $f_n : \sigma(A) \rightarrow [0, 1]$ for every $n \in \mathbb{N}$ such that $f_n(x_n) = 1$ and $f_n = 0$ on $\sigma(A) \setminus U_n$.
- (3.) Finally, we have an infinite linearly independent set $\{f_n\}_{n \in \mathbb{N}}$ in $C(\sigma(A))$, so the latter algebra must be infinite-dimensional. \square

Let A be any finite-dimensional C^* -algebra. Then its center, say C , is a finite-dimensional abelian C^* -algebra with spectrum $\{\omega_1, \dots, \omega_n\}$. Let $p_i \in C$ be an element that corresponds to the characteristic function $\chi_{\{\omega_i\}} \in C(\sigma(C))$. It follows from Gelfand that $C \cong \mathbb{C}p_1 \oplus \dots \oplus \mathbb{C}p_n$, so $p_1 + \dots + p_n = 1$. As a result,

$$A \cong Ap_1 \oplus \dots \oplus Ap_n,$$

where each of Ap_i has trivial center. By the previous lemma, Ap_i is a simple finite-dimensional C^* -algebra. Therefore, it suffices to prove the structure theorem for simple finite-dimensional C^* -algebras.

Proof of the structure theorem. Assume A is simple and finite-dimensional. We first note that $aAb \neq \{0\}$ for any nonzero $a, b \in A$. This follows from the observation that the set AaA is an ideal of A which must be the entire algebra A , since it is nonzero. Let B be a maximal abelian $*$ -subalgebra of A and let $\sigma(B) = \{\omega_1, \omega_2, \dots, \omega_n\}$ be its spectrum. Let $e_i \in B$ denote the projection, corresponding to the characteristic function $\chi_{\{\omega_i\}}$. By our previous arguments, e_i are orthogonal and $\sum_{i=1}^n e_i = 1$. Furthermore, $B \cong \mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_n$. It follows that $e_i A e_i$ commutes with every e_j . Next, we prove that the C^* -algebra $e_i A e_i$ has dimension one. Take any normal element $x \in e_i A e_i$. By Gelfand, $\sigma(x)$ is homeomorphic to the spectrum of a C^* -algebra, generated by x . This C^* -algebra is finite-dimensional (since it lives inside the finite-dimensional C^* -algebra $e_i A e_i$), so its spectrum must be finite and as a result, $\sigma(x)$ is finite (and discrete). Suppose that there exists a normal element $x \in e_i A e_i$ such that $\sigma(x)$ has at least two distinct elements λ_1, λ_2 . Then functional calculus gives us orthogonal projections $p = \chi_{\{\lambda_1\}}(x)$ and $q = \chi_{\{\lambda_2\}}(x)$ in $e_i A e_i$. Then $C^*(p, q, e_2, \dots, e_n)$ is an abelian C^* -algebra in A of dimension $n + 1$, which is in contradiction with assumption that B is maximal abelian. This means that every normal element $x \in e_i A e_i$ has singleton spectrum $\sigma(x) = \lambda$, so it is of the form $x = \lambda \cdot e_i$ by functional calculus. Since $e_i A e_i$ is the span of self-adjoint (which are normal) elements, we have $e_i A e_i = \mathbb{C}e_i \subseteq B$. For fixed i, j , we know that $e_i A e_j \neq \{0\}$. Choose a nonzero $x \in e_i A e_j$ and notice that $x = e_i x e_j$. This

implies that $x^*x = e_j x^* x e_j = \lambda e_j$ and $xx^* = e_i x x^* e_i = \mu e_i$ for some $\lambda, \mu \neq 0$. But from

$$\lambda = \|x^*x\| = \|x\|^2 = \|x^*\|^2 = \|xx^*\| = \mu$$

we get $\lambda = \mu > 0$. If we define $u = \lambda^{-\frac{1}{2}}x$, we get $u^*u = e_j$ and $uu^* = e_i$. For each i , let $u_i \in A$ be an element such that $u_i^*u_i = e_1 = u_i u_i^* = e_i$. Then, define $u_{i,j} = u_i u_j^*$. From our arguments, the following has to be true:

$$u_{i,j}^* = u_{j,i}, \quad \sum_{i=1}^n u_{i,i} = 1, \quad u_{i,j} u_{k,l} = \delta_{j,k} u_{i,l}$$

We claim that $e_i A e_j = \mathbb{C} u_{i,j}$. Indeed, if $x \in e_i A e_j$, then $x u_{i,j} \in e_i A e_i$, so $x u_{i,j} = \lambda e_i$ for some $\lambda \in \mathbb{C}$. Hence we get

$$x = x e_j = x u_{j,i} u_{i,j} \lambda e_i u_{i,j} \lambda u_{i,j}.$$

For each $x \in A$, let $\lambda_{i,j}(x)$ be a scalar such that $e_i x e_j = \lambda_{i,j}(x) u_{i,j}$. It follows that

$$x = \sum_{i,j=1}^n e_i x e_j = \sum_{i,j=1}^n \lambda_{i,j}(x) u_{i,j}.$$

Now the map

$$A \rightarrow M_n(\mathbb{C}), \quad x \mapsto (\lambda_{i,j}(x))_{i,j}$$

is a $*$ -isomorphism of A onto the algebra $M_n(\mathbb{C})$, where $\dim A = n^2$. □

3.4.2 Group C^* -algebras

Let G be a group. Then the (complex) group algebra $\mathbb{C}[G]$ is defined as the algebra with basis $\{u_g \mid g \in G\}$ and multiplication given by $u_g \cdot u_h = u_{gh}$. Multiplication is convolutive:

$$\begin{aligned} \left(\sum_g^{\text{finite}} a_g u_g \right) \left(\sum_h^{\text{finite}} b_h u_h \right) &= \sum_{g,h} a_g b_h u_g u_h \\ &= \sum_{g,h} a_g b_h u_{gh} \\ &= \sum_k \left(\sum_g a_g b_{g^{-1}k} \right) u_k. \end{aligned}$$

We can equip $\mathbb{C}[G]$ with an involution

$$\left(\sum_{g \in G}^{\text{finite}} a_g u_g \right)^* = \sum_g \overline{a_g} u_{g^{-1}}.$$

Given a representation (homomorphism of $*$ -algebras) $\pi : \mathbb{C}[G] \rightarrow \mathcal{B}(\mathcal{H})$, we define the C^* -algebra

$$C_\pi^*(G) := \overline{\pi(\mathbb{C}[G])} \subseteq \mathcal{B}(\mathcal{H}).$$

For $g \in G$, we get

$$\begin{aligned}\pi(u_g)\pi(u_g)^* &= \pi(u_g) \cdot \pi(u_g^*) \\ &= \pi(u_g) \cdot \pi(u_{g^{-1}}) \\ &= \pi(u_g \cdot u_{g^{-1}}) = \pi(u_e) = 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\pi(u_g)^*\pi(u_g) &= \pi(u_g^*) \cdot \pi(u_g) \\ &= \pi(u_{g^{-1}}) \cdot \pi(u_g) \\ &= \pi(u_{g^{-1}} \cdot u_g) = \pi(u_e) = 1.\end{aligned}$$

We have thus proved that under any representation of $\mathbb{C}[G]$, each u_g is mapped to a unitary.

Example 3.35. Take $\mathcal{H} = \ell^2(G)$ (this is a Hilbert space with ONB $\{\delta_g \mid g \in G\}$). Then

$$\lambda : \mathbb{C}[G] \rightarrow \mathcal{B}(\ell^2(G)), \quad u_g \mapsto (\delta_h \mapsto \delta_{gh})$$

is a faithful representation. We call it the left regular representation of G . The closure of its image is called the reduced group C^* -algebra $C_r^*(G) := \overline{\lambda(\mathbb{C}[G])} \subseteq \mathcal{B}(\ell^2(G))$.

Definition 3.36. Universal (or full) group C^* -algebra is the completion $\mathbb{C}[G]$, where the norm of an element $a \in \mathbb{C}[G]$ is $\|a\|_u = \sup\{\|\pi(a)\| \mid \pi \text{ representation of } \mathbb{C}[G]\}$.

Lemma 3.37. If π is a representation of $\mathbb{C}[G]$ and $a = \sum_{g \in G}^{\text{finite}} a_g u_g \in \mathbb{C}[G]$, then $\|\pi(a)\| \leq \sum |a_g|$.

Proof. Then $\pi(a) = \sum a_g \cdot \pi(u_g)$. Then

$$\|\pi(a)\| = \left\| \sum a_g \pi(u_g) \right\| \leq \sum |a_g| \cdot \|\pi(u_g)\| = \sum |a_g|. \quad \square$$

This implies that $\|\cdot\|_u$ is indeed a norm on $\mathbb{C}[G]$, making the universal C^* -algebra $C^*(G)$ of G in definition 3.36 well-defined.

Remark. The group algebra $\mathbb{C}[G]$ is dense in both $C_\pi^*(G)$ and $C^*(G)$.

Theorem 3.38 (Universal property).

For each representation π of $\mathbb{C}[G]$ there exists a surjective $*$ -homomorphism $\hat{\pi} : C^*(G) \rightarrow C_\pi^*(G)$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{C}[G] & \xrightarrow{\pi} & C_\pi^*(G) \\ \downarrow & \nearrow \hat{\pi} & \\ C^*(G) & & \end{array} \quad \exists! \hat{\pi}$$

Proof. Define first $\hat{\pi}$ on $\mathbb{C}[G] \subseteq C^*G$ by $\hat{\pi}(a) := \pi(a) \in C_\pi^*(G)$. Firstly, $\hat{\pi}$ on $\mathbb{C}[G]$ is contractive:

$$\|\hat{\pi}(a)\| = \|\pi(a)\| \leq \|a\|_u.$$

By density, $\hat{\pi}$ uniquely extends to a continuous $*$ -homomorphism $\hat{\pi} : C^*(G) \rightarrow C_\pi^*(G)$. This $\hat{\pi}$ is also contractive and $\text{im } \pi$ is dense, so $\hat{\pi}$ is onto. \square

Example 3.39. Let G be abelian and $|G| = n$. Then $\mathbb{C}[G] \cong \mathbb{C}^{|G|}$ as a vector space. Hence $C^*G = \mathbb{C}[G] = C_r^*(G)$. Furthermore, $\mathbb{C}[G]$ is commutative, so by the structure theorem we have

$$\mathbb{C}[G] \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ times}}$$

as a C^* -algebra. For instance, $\mathbb{C}[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}] \cong \mathbb{C}[\mathbb{Z}/4\mathbb{Z}]$.

Example 3.40. Let $G = S_3$. Then $|G| = 6$ and once again $C^*(G) = \mathbb{C}[G] = C_r^*(G)$. By structure theorem and the dimension consideration, $\mathbb{C}[G] \cong M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ (otherwise it would be commutative).

Example 3.41. Let $G = S_4$. Again, $C^*(G) = \mathbb{C}[G] = C_r^*(G)$. By Maschke's theorem, $\mathbb{C}[G]$ is semisimple, therefore it is isomorphic (as an \mathbb{C} -algebra) to a direct sum of matrix algebras over \mathbb{C} . This decomposition is unique. Since S_4 has five conjugacy classes, there are five factors (see, for example, theorem IX.7.9. in [2]). Adding up all the dimensions, the only combination that works is $9 + 9 + 4 + 1 + 1 = 24$, therefore

$$\mathbb{C}[G] \cong M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$$

as a \mathbb{C} -algebra. By the structure theorem, it is also isomorphic to this direct sum as a C^* -algebra.

Example 3.42. What is $C^*(\mathbb{Z})$? Representations $\pi(\mathbb{C}[\mathbb{Z}]) \rightarrow \mathcal{B}(\mathcal{H})$ are determined by choice of unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\pi(u_1) = U$. By universal property, for every \mathcal{H} and $U \in \mathcal{B}(\mathcal{H})$ there exists a unique $*$ -homomorphism

$$\hat{\pi} : C^*(\mathbb{Z}) \rightarrow C^*(\{U\}),$$

where the latter is a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$, generated by U . We call $C^*(\mathbb{Z})$ the universal C^* -algebra, generated by a unitary.

3.5 Abelian group C^* -algebras

If G is abelian, then $\mathbb{C}[G]$ is commutative and $C_r^*(G)$ is abelian. By Gelfand, there exists a compact Hausdorff space Σ such that $C_r^*(G) \cong C(\Sigma)$ and $\Sigma = \sigma(C_r^*(G))$.

Definition 3.43. To each abelian group G we associate its *Pontryagin dual*

$$\widehat{G} = \{w : G \rightarrow \mathbb{T} \text{ group homomorphism}\}.$$

Then \widehat{G} is a group under pointwise multiplication. We endow \widehat{G} with the compact-open topology induced from $\widehat{G} \subseteq \mathbb{T}^G$. Recall that the basis sets for this topology are

$$B_{\varepsilon, F}(w) = \{\eta \in \widehat{G} \mid |\eta(h) - w(h)| < \varepsilon, \forall h \in F\}$$

for $\varepsilon > 0$, $w \in \widehat{G}$ and $F \subseteq G$ finite.

Remark. A net $(w_i)_{i \in I} \subseteq \widehat{G}$ is Cauchy iff $(w_i(g))_{i \in I} \subseteq \mathbb{T}$ is Cauchy for all $g \in G$.

Let us calculate the Pontryagin dual of some very simple abelian groups.

Example 3.44. Let us prove that $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ as a topological group. Take any $k \in \mathbb{Z}/n\mathbb{Z}$ and define a group homomorphism

$$\varphi_k : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{T}, \quad l \mapsto e^{\frac{2\pi i k l}{n}}.$$

Then

$$\Phi : \mathbb{Z}/n\mathbb{Z} \rightarrow \widehat{\mathbb{Z}/n\mathbb{Z}}, \quad k \mapsto \varphi_k$$

is easily seen to be a group isomorphism. Since both groups are obviously endowed with discrete topology, the above map is also a homeomorphism, hence $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ as topological groups.

Example 3.45. Next, we prove that $\widehat{\mathbb{Z}} \cong \mathbb{T}$. Take any $\xi \in \mathbb{T}$ and define

$$\varphi_\xi : \mathbb{Z} \rightarrow \mathbb{T}, \quad k \mapsto \xi^k.$$

Then

$$\Phi : \mathbb{T} \rightarrow \widehat{\mathbb{Z}}, \quad \xi \mapsto \varphi_\xi$$

is a group isomorphism. If $(\xi_i)_{i \in I}$ is a net in \mathbb{T} such that $\xi_i \rightarrow \xi$, then $\Phi \xi_i = \varphi_{\xi_i} \rightarrow \varphi_\xi = \Phi \xi$. As a result, Φ is continuous. But since Φ maps from a compact to a Hausdorff space, it is also closed and thus a homeomorphism.

Theorem 3.46.

The map

$$h : \widehat{G} \rightarrow \sigma(C_r^*(G)), \quad w \mapsto \left(\sum a_g u_g \mapsto \sum a_g w(g) \right)$$

is a homeomorphism.

Proof. First, we prove that $h(w) \in \sigma(C_r^*(G))$ for all $w \in \widehat{G}$. We begin by showing $h(w) :$

$\mathbb{C}[G] \rightarrow \mathbb{C}$ is a homomorphism. Take $b = \sum b_k u_k \in \mathbb{C}[G]$. Then

$$h(w)(a \cdot b) = \sum_g \left(\sum_h a_h b_{h^{-1}g} \right) \cdot w(g)$$

and

$$h(w)(a) \cdot h(w)(b) = \left(\sum_g a_g w(g) \right) \cdot \left(\sum_h b_h w(h) \right) = \sum_k \left(\sum_h a_{kh^{-1}} b_h \right) w(k),$$

so $h(w)$ is multiplicative. To extend it to C_r^* , we must prove that $|h(w)a| \leq \|a\|_r$ for all $a \in \mathbb{C}[G]$. To $\chi \in \sigma(C_r^*(G))$ and $a \in \mathbb{C}[G]$ we associate

$$\tilde{a} = \sum a_g w(g) \cdot \overline{\chi(u_g)} u_g,$$

so $h(w)a = \chi(\tilde{a})$. By Gelfand,

$$\|\tilde{a}\|_r = \sup\{|\mu(\tilde{a})| \mid \mu \in \sigma(C_r^*(G))\} \geq |\chi(\tilde{a})| = |h(w)a|.$$

Next, we show that $\|\tilde{a}\|_r = \|a\|_r$: to $\theta \in \ell^2(G)$ assign $\tilde{\theta}$ by $\tilde{\theta}_h := \chi(u_{h^{-1}}) \overline{w(h)} \theta_h$. Then $\|\theta\|_2 = \|\tilde{\theta}\|_2$. Further, $\|\lambda(\tilde{a})\tilde{\theta}\|_2 = \|\lambda(a)\theta\|_2$ (short calculation), so

$$\|\tilde{a}\|_r = \sup\{\|\lambda(\tilde{a})\tilde{\theta}\| \mid \|\tilde{\theta}\|_2 = 1\} = \sup\{\|\lambda(a)\theta\| \mid \|\theta\|_2 = 1\} = \|a\|_r.$$

Next, we prove that h is continuous. Suppose the net $(w_i)_{i \in I} \subseteq \widehat{G}$ is Cauchy. We prove that for every $a \in C_r^*(G)$ the net $(h(w_i)(a))_{i \in I}$ is Cauchy. Pick $\varepsilon > 0$. There exists J such that for every $i, j \geq J$, we have

$$|w_i(g) - w_j(g)| < \frac{\varepsilon}{|\{g \mid a_g \neq 0\}|}, \quad \forall g \in G.$$

Then for all $i, j \geq J$ we get $|h(w_i)(g) - h(w_j)(g)| < \varepsilon$. For the general case $a \in C_r^*(G)$, we can take a as a limit of a sequence $(a_n)_n \subseteq \mathbb{C}[G]$, approximate a with a_n and use the triangle inequality to establish that $(h(w_i)(a))_i$ is Cauchy. Now on to bijectivity of h . It's enough to check that it is surjective: take $\phi \in \sigma(C_r^*(G))$. Define

$$w_\phi : G \rightarrow \mathbb{C}, \quad g \mapsto \phi(u_g).$$

Since ϕ is a $*$ -homomorphism, $\text{im } w_\phi \subseteq \mathbb{T}$. We have to prove that $w_\phi \in \widehat{G}$. We just check the multiplicativity:

$$w_\phi(g) \cdot w_\phi(h) = \phi(u_g)\phi(u_h) = \phi(u_g u_h) = \phi(u_{gh}) = w_\phi(gh).$$

For every $w \in \widehat{G}$, we get $w_{h(w)} = w$. So $w_\phi = w_{h(w_\phi)}$, which gives us $h(w_\phi) = \phi$. Now since h is a bijective continuous map between compact Hausdorff spaces, it is a homeomorphism. \square

Example 3.47. We can now explicitly calculate Gelfand transform of a reduced group

C^* -algebra $C_r^*(\mathbb{Z}/n\mathbb{Z})$:

$$C_r^*(\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\Gamma} C(\sigma(C_r^*(\mathbb{Z}/n\mathbb{Z}))) \rightarrow C(\widehat{\mathbb{Z}/n\mathbb{Z}}) \rightarrow C(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{C}^n,$$

which maps an element $\sum_{l=0}^{n-1} a_l u_l \in C_r^*(\mathbb{Z}/n\mathbb{Z})$ to

$$\begin{bmatrix} \sum_{l=0}^{n-1} a_l \\ \sum_{l=0}^{n-1} a_l e^{\frac{2\pi i l}{n}} \\ \vdots \\ \sum_{l=0}^{n-1} a_l e^{\frac{2\pi i (n-1) l}{n}} \end{bmatrix}.$$

Example 3.48. Similarly, we can explicitly calculate Gelfand transform of a reduced group algebra $C_r^*(\mathbb{Z})$:

$$C_r^*(\mathbb{Z}) \xrightarrow{\Gamma} C(\sigma(C_r^*(\mathbb{Z}))) \rightarrow C(\widehat{\mathbb{Z}}) \rightarrow C(\mathbb{T}),$$

which maps $\sum_{l \in \mathbb{Z}} a_l u_l$ into a function

$$f \in C(\mathbb{T}), \quad f(\xi) = \sum_{l \in \mathbb{Z}} a_l \xi^l.$$

Therefore, the inverse Gelfand transform maps a function $f \in C(\mathbb{T})$ into its Fourier series.

In operator algebras, we usually consider topological groups G that are locally compact and Hausdorff. For such groups, there exists a so-called *Haar measure* μ on G . This measure allows us to consider the C^* -algebras $L^2(G)$, $C_r^*(G)$ and $C^*(G)$ for general locally compact and Hausdorff topological groups. If a group G is equipped with discrete topology, then these notions coincide with the ones from the previous subsection.

Example 3.49. It turns out that the Pontryagin dual of a locally compact Hausdorff abelian group is itself a locally compact Hausdorff abelian group. As in the above examples, we can prove the following:

- $\widehat{\widehat{\mathbb{T}}} = \mathbb{Z}$;
- $\widehat{\widehat{\mathbb{R}}} = \mathbb{R}$.

We notice that for each of the groups that we have seen so far in the examples, the dual of a dual is the original group. This is not a coincidence.

Theorem 3.50 (Pontryagin).

If G is a locally compact Hausdorff abelian group, then $G \cong \widehat{\widehat{G}}$.

4 Bounded operators on Hilbert spaces

4.1 Polar decomposition

Let \mathcal{H} be a complex Hilbert space. Then $\mathcal{B}(\mathcal{H})$ is a C^* -algebra with the operator norm

$$\|A\| = \sup_{\mu \in \mathcal{H}, \mu \neq 0} \frac{\|A\mu\|}{\|\mu\|} = \sup_{\mu \in \mathcal{H}, \|\mu\|=1} \|A\mu\| = \sup_{\mu \in \mathcal{H}, \|\mu\| \leq 1} \|A\mu\|$$

Remark. Recall that $A \in \mathcal{B}(\mathcal{H})$ is:

- (1.) normal $\Leftrightarrow A^*A = AA^* \Leftrightarrow \|A\mu\| = \|A^*\mu\|, \forall \mu \in \mathcal{H}$;
- (2.) self-adjoint $\Leftrightarrow A^* = A \Leftrightarrow \langle A\mu, \mu \rangle \in \mathbb{R}, \forall \mu \in \mathcal{H}$;
- (3.) positive $\Leftrightarrow A = B^*B$ for some $B \in \mathcal{B}(\mathcal{H}) \Leftrightarrow \langle A\mu, \mu \rangle \geq 0, \forall \mu \in \mathcal{H}$;
- (4.) isometry $\Leftrightarrow A^*A = I \Leftrightarrow \|A\mu\| = \|\mu\|, \forall \mu \in \mathcal{H}$;
- (5.) projection $\Leftrightarrow A^2 = A = A^* \Leftrightarrow A$ is an orthogonal projection onto some closed subspace of \mathcal{H} .

Lemma 4.1. *An operator $A \in \mathcal{B}(\mathcal{H})$ is a partial isometry iff there exists a closed subspace $\mathcal{K} \subseteq \mathcal{H}$ such that $A|_{\mathcal{K}}$ is an isometry and $A|_{\mathcal{K}^\perp} = 0$.*

Proof. We first prove (\Leftarrow) . Obviously, $\mathcal{K}^\perp \subseteq \ker A$. From $Ax = 0$, where $x = y + z$ and $y \in \mathcal{K}, z \in \mathcal{K}^\perp$, we have

$$0 = Ax = A(y + z) = Ay + Az = Ay.$$

But since $A|_{\mathcal{K}}$ is an isometry, $\|Ay\| = \|y\| = 0$, so $y = 0$ and $x \in \mathcal{K}^\perp$. Now we prove that $P = A^*A$ is the projection onto \mathcal{K} . For $x \in \mathcal{K}$, we have

$$\langle Px, x \rangle = \langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 = \|x\|^2,$$

so

$$\|P\| = \|A^*A\| \leq \|A\|\|A^*\| = \|A\|^2 = 1.$$

From Cauchy-Schwartz:

$$\langle Px, x \rangle \leq \|Px\|\|x\| \leq \|P\|\|x\|^2 \leq \|x\|^2.$$

Since we have equality in Cauchy-Schwartz, there exists a $\lambda \in \mathbb{C}$ such that $Px = \lambda x$. But from $\langle Px, x \rangle = \|x\|^2$, it follows that $\lambda = 1$. So $P|_{\mathcal{K}} = \text{id}$ and for $x \in \mathcal{K}^\perp$, $Px = A^*Ax = 0$. Therefore, $P = A^*A$ is indeed a projection. Now onto the opposite direction (\Rightarrow) . Suppose $P = A^*A$ is a projection and denote $\mathcal{K} = \text{im } P$. Since $\mathcal{K} = \ker(I - P)$, it is a closed subspace of \mathcal{H} . For $x \in \mathcal{K}$, we have

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle Px, x \rangle = \langle x, x \rangle = \|x\|^2.$$

But for $x \in \mathcal{K}^\perp$, we use the identity

$$(\text{im } P)^\perp = \ker P^* = \ker P$$

to get $Px = 0$, so $\|Ax\|^2 = \langle Px, x \rangle = 0$ and $\|Ax\| = 0$. □

Theorem 4.2 (Polar decomposition).

Let \mathcal{H} be a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$. Then there exists a partial isometry v such that $x = v \cdot |x|$ and $\ker v = \ker |x| = \ker x$. This decomposition is unique: if $x = wy$ for $y \geq 0$ and partial isometry w such that $\ker y = \ker w$, then $w = v$ and $y = |x|$.

Proof. First we prove the existence. Define

$$v_0 : \operatorname{im} |x| \rightarrow \operatorname{im} x, \quad |x|y \mapsto xy.$$

Since

$$\begin{aligned} \| |x|y \|^2 &= \langle |x|y, |x|y \rangle \\ &= \langle |x|^2 y, y \rangle \\ &= \langle x^* x y, y \rangle \\ &= \langle x y, x y \rangle \\ &= \| x y \|^2. \end{aligned}$$

The above v_0 is well defined. It is also linear and isometric. By continuity, extend v_0 to a map $\overline{\operatorname{im} |x|} \rightarrow \overline{\operatorname{im} x}$. Now v_0 can be extended to $v : \mathcal{H} \rightarrow \mathcal{H}$ by setting $v|_{(\operatorname{im} |x|)^\perp} = 0$. By previous lemma, v is a partial isometry. By definition, $x = v \cdot |x|$ and $\ker v = (\operatorname{im} |x|)^\perp = \ker |x| = \ker x$. Next, we prove uniqueness. If $x = wy$ as in the statement, then $\ker w = \ker y = (\operatorname{im} y)^\perp$, so w is a partial isometry on $\overline{\operatorname{im} y}$. From there, we get

$$|x|^2 = (wy)^*(wy) = y^* w^* w y = y^* y = y^2,$$

which implies

$$|x| = (|x|^2)^{\frac{1}{2}} = (y^2)^{\frac{1}{2}} = y.$$

Now

$$w|x|\mu = wy\mu = x\mu$$

together with

$$\ker w = (\operatorname{im} y)^\perp = (\operatorname{im} |x|)^\perp$$

implies $w = v$. □

Now we can also prove the statement in the example 1.43.

Proposition 4.3. The extreme points of the unit ball of $\mathcal{B}(\mathcal{H})$ are exactly the elements $V \in \mathcal{B}(\mathcal{H})$ such that

$$(1 - VV^*) \mathcal{B}(\mathcal{H})(1 - V^*V) = 0.$$

In particular, V^*V and VV^* are projections.

Proof. Let $V \in A$ be an extreme point of the unit ball of $\mathcal{B}(\mathcal{H})$, so $\sigma(V) \subseteq [-1, 1]$. Write

$$V = \frac{1}{2}V(2 - |V|) + \frac{1}{2}V|V|.$$

Since the functions $z \mapsto z(2 - |z|)$ and $z \mapsto |z|(2 - |z|)$ coincide and are both bounded above by 1 on $\sigma(x)$, we have $\|V(2 - |V|)\| = \| |V|(2 - |V|)\| \leq 1$ by continuous functional calculus. This implies that $V(2 - |V|)$ is in the unit ball of $\mathcal{B}(\mathcal{H})$. The same can be said about $V|V|$ by the same argument. Now since V is an extreme point, we must have $V = V|V|$. Multiplying on the left with V^* , we get $|V|^2 = |V|^3$. This means that the functions $z \mapsto z^2$ and $z \mapsto z^3$ coincide on $\sigma(|V|)$, which implies that $\sigma(|V|) \subseteq \{0, 1\}$. As a result, $|V|$ is a projection, so $P := |V| = |V|^2 = V^*V$. The same can be said about $Q := VV^*$, since we know that $\sigma(V^*V) \setminus \{0\} = \sigma(VV^*) \setminus \{0\}$. This means that V is a partial isometry. By the previous lemma, P is a projection onto the initial space of V , so $QV = VP = V$. Now suppose $W := (1 - Q)Z(1 - P) \neq 0$ for some Z in the unit ball. Then

$$\begin{aligned} \|V + W\|^2 &= \|QVP + (1 - Q)Z(1 - P)\|^2 \\ &= \|(QVP + (1 - Q)Z(1 - P))^*(QVP + (1 - Q)Z(1 - P))\| \\ &= \|(PV^*Q + (1 - P)Z^*(1 - Q))(QVP + (1 - Q)Z(1 - P))\| \\ &= \|PV^*QVP + (1 - P)Z^*(1 - Q)Z(1 - P)\| \\ &= \|PV^*VP + (1 - P)W^*W(1 - P)\| \\ &= \max\{\|V^*V\|, \|W^*W\|\} \\ &= \max\{\|V\|^2, \|W\|^2\} = 1 \end{aligned}$$

and similarly $\|V - W\|^2 = 1$. Therefore we have a decomposition

$$V = \frac{1}{2}(V + W) + \frac{1}{2}(V - W)$$

and V is not an extreme point, leading to a contradiction. Conversely, suppose that $(1 - VV^*)\mathcal{B}(\mathcal{H})(1 - V^*V) = 0$. Then we have

$$0 = V^*(1 - VV^*)V(1 - V^*V) = V^*V(1 - V^*V)^2.$$

This implies that the function $z \mapsto z(1 - z)^2$ must be zero on $\sigma(V^*V)$, so $\sigma(V^*V) \subseteq \{0, 1\}$ and $P := V^*V$ is a projection. By the same argument,

$$0 = (1 - VV^*)V(1 - V^*V)V^* = (1 - VV^*)^2VV^*$$

and $Q := VV^*$ is a projection as well. Assume that $V = \frac{1}{2}U + \frac{1}{2}W$ for U, W in the unit ball. Again we have $V = VP = QV$, so

$$V = \frac{1}{2}UP + \frac{1}{2}WP$$

and

$$\begin{aligned} 4P &= 4V^*V = PU^*UP + PW^*WP + PU^*WP + PW^*UP \\ &= 2(PU^*UP + PW^*WP) - P(U - W)^*(U - W)P \\ &\leq 4P - P(U - W)^*(U - W)P. \end{aligned}$$

This immediately implies that $(U - W)P = 0$. Similarly, we have $Q(U - W) = 0$. Now

$$\begin{aligned} U - W &= Q(U - W)P + (1 - Q)(U - W)P \\ &\quad + Q(U - W)(1 - P) + (1 - Q)(U - W)(1 - P) = 0. \end{aligned}$$

□

Remark. The above theorem holds for any C^* -algebra, not just $\mathcal{B}(\mathcal{H})$. We can identify any general C^* -algebra with an algebra of operators on some Hilbert space and then the above proof carries over verbatim.

Corollary 4.4.

$$\text{ext}(\mathcal{B}(\mathcal{H}))_1 = \{V \in \mathcal{B}(\mathcal{H}) \mid V \text{ or } V^* \text{ is an isometry}\}.$$

Proof. We need to prove the inclusion (\subseteq). If V is an extreme point, then V^*V and VV^* are projections. Therefore, V is a partial isometry with the initial space $(\ker V)^\perp$ and V^* is a partial isometry with the initial space $(\ker V^*)^\perp$. Assume neither V nor V^* are full isometries, so their initial spaces are proper subspaces of \mathcal{H} . This means that there exist vectors $0 \neq x \in \ker V$ and $0 \neq y \in \ker V^*$. Define P as a rank-one projection from x to y . Then

$$(1 - VV^*)P(1 - V^*V)x = y \neq 0,$$

so $(1 - VV^*)P(1 - V^*V) \neq 0$, contradiction. \square

4.2 Trace class operators

Definition 4.5. Let X, Y be Banach spaces. An operator $A \in \mathcal{B}(X, Y)$ has *finite rank* if $\text{rank } A := \dim \overline{\text{im } A} < \infty$. The set of finite rank operators is denoted by $\mathcal{F}(X, Y)$. We also denote $\mathcal{F}(X) := \mathcal{F}(X, X)$.

Remark. Let $A \in \mathcal{B}(\mathcal{H})$, where \mathcal{H} is a Hilbert space. We know that

$$\text{im } A^* = \text{im}(A^*|_{(\ker A)^{\perp}}) = \text{im}(A^*|_{\overline{\text{im } A}}).$$

From there, we can conclude that $\text{rank } A < \infty$ iff $\text{rank } A^* < \infty$.

If $\alpha, \beta \in \mathcal{H}$, then we can define the operator

$$\alpha \otimes \bar{\beta} : \mathcal{H} \rightarrow \mathcal{H}, \quad y \mapsto \langle y, \beta \rangle \cdot \alpha.$$

It is trivial to see that $\text{rank}(\alpha \otimes \bar{\beta}) \leq 1$ and $(\alpha \otimes \bar{\beta})^* = \beta \otimes \bar{\alpha}$. By Riesz's representation theorem, we also know that every rank-one operator on \mathcal{H} is of this form. If $\|\alpha\| = \|\beta\| = 1$, then $\alpha \otimes \bar{\beta}$ is a partial isometry with initial space $\mathbb{C}\beta$ and image $\mathbb{C}\alpha$. Then

$$\mathcal{F}(\mathcal{H}) = \text{span } \{\alpha \otimes \bar{\beta} \mid \alpha, \beta \in \mathcal{H}\}.$$

For $x, y \in \mathcal{B}(\mathcal{H})$ we have

$$x(\alpha \otimes \bar{\beta})y = (x\alpha) \otimes \overline{(y^*\beta)}.$$

Lemma 4.6. Let $x \in \mathcal{B}(\mathcal{H})$ have the polar decomposition $x = v \cdot |x|$. Then for all $y \in \mathcal{H}$, we have

$$2|\langle xy, y \rangle| \leq \langle |x|y, y \rangle + \langle |x|v^*y, v^*y \rangle.$$

Proof. Let $\lambda \in \mathbb{T}$. Then

$$\begin{aligned} 0 &\leq \|(|x|^{\frac{1}{2}} - \lambda|x|^{\frac{1}{2}}v^*)y\|^2 \\ &= \| |x|^{\frac{1}{2}}y \|^2 - 2 \operatorname{Re} \bar{\lambda} \langle |x|^{\frac{1}{2}}y, |x|^{\frac{1}{2}}v^*y \rangle + \| |x|^{\frac{1}{2}}v^*y \|^2. \end{aligned}$$

Now pick λ such that $\bar{\lambda} \langle |x|^{\frac{1}{2}}y, |x|^{\frac{1}{2}}v^*y \rangle \geq 0$ and we are done. \square

Definition 4.7. Let $(e_i)_{i \in I}$ be an orthonormal basis for \mathcal{H} . For $x \in \mathcal{B}(\mathcal{H})_+$, define the *trace*

$$\operatorname{Tr}(x) = \sum_{i \in I} \langle x e_i, e_i \rangle \in [0, \infty].$$

We call $x \in \mathcal{B}(\mathcal{H})$ *trace class* if

$$\|x\|_1 := \operatorname{Tr}(|x|) < \infty.$$

The set of trace class operators on \mathcal{H} will be denoted by $L^1(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

Remark. Let $\{h_i \mid i \in I\} \subseteq \mathcal{H}$ be a set of vectors in a Hilbert space. We already know that the collection of finite sets $F \subseteq I$ forms a directed set. Then vectors $h_F := \sum_{i \in F} h_i$ form a net in \mathcal{H} . We define $\sum_{i \in I} h_i$ as the limit of the net (h_F) , if it exists. Note that if I is countable, this definition of a convergent sum does not necessarily coincide with the usual one. In other words, for a set $\{h_n \mid n \in \mathbb{N}\}$ in a Hilbert space \mathcal{H} , the convergence of a sum $\sum_{n \in \mathbb{N}} h_n$ is not equivalent to the convergence of a sum $\sum_{n=1}^{\infty} h_n$ – in fact, the convergence of a former sum implies the convergence of the latter one (with the sums being equal). The converse holds if $\sum_{n=1}^{\infty} \|h_n\| < \infty$.

Remark. If $x \in \mathcal{B}(\mathcal{H})_+$ and $\operatorname{Tr}(x) = \sum_{i \in I} \langle x e_i, e_i \rangle < \infty$, then $\langle x e_i, e_i \rangle > 0$ holds for at most countably many e_i . Let $(e_n)_{n \in \mathbb{N}}$ be a set of such basis vectors. Then $\sum_{i \in I} \langle x e_i, e_i \rangle = \sum_{n=1}^{\infty} \langle x e_n, e_n \rangle$.

Lemma 4.8. For all $x \in \mathcal{B}(\mathcal{H})$ we have $\operatorname{Tr}(x^*x) = \operatorname{Tr}(xx^*)$.

Proof.

$$\begin{aligned} \operatorname{Tr}(x^*x) &= \sum_i \langle x^*x e_i, e_i \rangle = \sum_i \langle x e_i, x e_i \rangle \\ &= \sum_i \|x e_i\|^2 = \sum_i \sum_j \langle x e_i, e_j \rangle \overline{\langle x e_i, e_j \rangle} \\ &= \sum_j \sum_i \langle e_i, x^* e_j \rangle \overline{\langle e_i, x^* e_j \rangle} = \sum_j \sum_i \langle x^* e_j, e_i \rangle \overline{\langle x^* e_j, e_i \rangle} \\ &= \sum_j \|x^* e_j\|^2 = \sum_j \langle x^* e_j, x^* e_j \rangle \\ &= \sum_j \langle x x^* e_j, e_j \rangle = \operatorname{Tr}(x x^*) \end{aligned}$$

\square

Corollary 4.9. *If $x \in \mathcal{B}(\mathcal{H})_+$ and $u \in \mathcal{U}(\mathcal{H})$, then*

$$\mathrm{Tr}(u^*xu) = \mathrm{Tr}(x).$$

In particular, the trace of a positive operator is independent of the choice of the orthonormal basis for \mathcal{H} .

Proof. Since $x \in \mathcal{B}(\mathcal{H})_+$, there exists a $y \in \mathcal{B}(\mathcal{H})$ such that $x = y^*y$. By lemma 4.8, we have

$$\begin{aligned} \mathrm{Tr}(x) &= \mathrm{Tr}(y^*y) = \mathrm{Tr}(yy^*) \\ &= \mathrm{Tr}(u^*y^*yu) = \mathrm{Tr}(u^*xu). \end{aligned}$$

If (f_i) is another ONB for \mathcal{H} , then there exists $u \in \mathcal{U}(\mathcal{H})$ such that $ue_i = f_i$ for all indices i :

$$\begin{aligned} \sum_i \langle xf_i, f_i \rangle &= \sum_i \langle xue_i, ue_i \rangle \\ &= \sum_i \langle u^*xue_i, e_i \rangle \\ &= \mathrm{Tr}(u^*xu) = \mathrm{Tr}(x). \end{aligned} \quad \square$$

Definition 4.10. If (e_i) is ONB for \mathcal{H} and $x \in L^1(\mathcal{B}(\mathcal{H}))$, then its trace is

$$\mathrm{Tr}(x) := \sum_{i \in I} \langle xe_i, e_i \rangle.$$

By lemma 4.6 and the proof below, we get

$$\begin{aligned} 2|\mathrm{Tr}(x)| &\leq \sum_{i \in I} 2|\langle xe_i, e_i \rangle| \\ &\leq \sum_{i \in I} \langle |x|e_i, e_i \rangle + \langle |x|v^*e_i, v^*e_i \rangle \\ &= \mathrm{Tr}(|x|) + \mathrm{Tr}(v|x|v^*) \\ &\leq \|x\|_1 + \|x\|_1 \\ &= 2\|x\|_1. \end{aligned}$$

Theorem 4.11.

- (1.) $L^1(\mathcal{B}(\mathcal{H}))$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$ that is closed under involution.
- (2.) $L^1(\mathcal{B}(\mathcal{H}))$ is a linear span of all positive operators of finite trace.
- (3.) Trace is independent of the ONB and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.

Proof. Let $A, B \in L^1(\mathcal{B}(\mathcal{H}))$ and satisfy the polar decompositions:

$$A + B = U|A + B|, \quad A = V|A|, \quad B = W|B|.$$

Let (e_i) be an ONB. Then

$$\begin{aligned} \sum_{i=1}^N \langle |A + B| e_i, e_i \rangle &= \sum_{n=1}^N |\langle U^*(A + B) e_n, e_n \rangle| \\ &\leq \sum_{n=1}^N |\langle U^* A e_n, e_n \rangle| + \sum_{n=1}^N |\langle U^* B e_n, e_n \rangle| \\ &= \sum_{n=1}^N |\langle U^* V |A| e_n, e_n \rangle| + \sum_{n=1}^N |\langle U^* W |B| e_n, e_n \rangle|. \end{aligned}$$

We can bound the first term:

$$\begin{aligned} \sum_{n=1}^N |\langle U^* V |A| e_n, e_n \rangle| &= \sum_{n=1}^N |\langle |A|^{\frac{1}{2}} e_n, |A|^{\frac{1}{2}} V^* U e_n \rangle| \\ &\leq \sum_{n=1}^N \| |A|^{\frac{1}{2}} e_n \| \| |A|^{\frac{1}{2}} V^* U e_n \| \\ &\leq \left(\sum_{n=1}^N \| |A|^{\frac{1}{2}} e_n \|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^N \| |A|^{\frac{1}{2}} V^* U e_n \|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since $\| |A|^{\frac{1}{2}} e_n \|^2 = \langle |A|^{\frac{1}{2}} e_n, |A|^{\frac{1}{2}} e_n \rangle = \langle |A| e_n, e_n \rangle$, the expression in the first bracket goes to $\text{Tr } |A|$. Next, we prove that the expression in the second bracket is less or equal to $\text{Tr } |A|$:

$$\sum_{n=1}^N \langle |A|^{\frac{1}{2}} V^* U e_n, |A|^{\frac{1}{2}} V^* U e_n \rangle = \sum_{n=1}^N \langle U^* V |A| V^* U e_n, e_n \rangle \xrightarrow{N \rightarrow \infty} \text{Tr } |A|.$$

Pick an ONB for \mathcal{H} as follows: each f_j should be in $\ker U$ or $(\ker U)^\perp$. Then

$$\text{Tr}(U^* V |A| V^* U) \leq \text{Tr}(V |A| V^*).$$

By similar argument,

$$\text{Tr}(V |A| V^*) \leq \text{Tr}(|A|)$$

and we are done:

$$\sum_{n=1}^N |\langle U^* V |A| e_n, e_n \rangle| \leq \text{Tr } |A|.$$

Similarly,

$$\sum_{n=1}^N |\langle U^* W |B| e_n, e_n \rangle| \leq \text{Tr } |B|,$$

which implies $\text{Tr } |A + B| \leq \text{Tr } |A| + \text{Tr } |B|$. We have proved that $L^1(\mathcal{B}(\mathcal{H}))$ is a vector space and $\| \cdot \|_1$ is a norm. Clearly, $L^1(\mathcal{B}(\mathcal{H}))$ contains all positive operators with finite trace, so

also their linear span. Next we prove that it is a two-sided ideal of $\mathcal{B}(\mathcal{H})$. Let $A \in L^1(\mathcal{B}(\mathcal{H}))$ and $B \in \mathcal{B}(\mathcal{H})$. Since every operator is a linear combination of four unitaries, we can assume w.l.o.g. that $B = U$ is a unitary. Then

$$|UA| = (A^*U^*UA)^{\frac{1}{2}} = (A^*A)^{\frac{1}{2}} = |A|,$$

so $BA = UA \in L^1(\mathcal{B}(\mathcal{H}))$. Furthermore,

$$|AU| = (U^*A^*AU)^{\frac{1}{2}} = U^*|A|U,$$

which implies

$$\text{Tr } |AU| = \text{Tr}(U^*|A|U) = \text{Tr } |A|$$

and $AB = AU \in L^1(\mathcal{B}(\mathcal{H}))$. Now we prove that $L^1(\mathcal{B}(\mathcal{H}))$ is closed under involution. Let $A = U|A|$ and $A^* = V|A^*|$ be polar decompositions. Then

$$|A^*| = V^*A^* = V^*(U|A|)^* = V^*|A|U^*.$$

If $A \in L^1(\mathcal{B}(\mathcal{H}))$, then $|A| \in L^1(\mathcal{B}(\mathcal{H}))$, so

$$|A^*| = V^*|A|U^* \in L^1(\mathcal{B}(\mathcal{H})).$$

This gives us $A^* \in L^1(\mathcal{B}(\mathcal{H}))$. Finally, we prove that $L^1(\mathcal{B}(\mathcal{H}))$ is the linear span of all positive operators of finite trace. Let $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $a \in \mathcal{B}(\mathcal{H})$. The following polarization identity holds:

$$4a|x| = \sum_{k=0}^3 i^k \underbrace{(a + i^k)|x|(a + i^k)^*}_{\text{positive and finite trace}}.$$

If $a = v$ partial isometry from the polar decomposition theorem, then

$$x = v|x| = \sum_{k=0}^3 \frac{i^k}{4} (v + i^k)|x|(v + i^k)^*.$$

is a linear combination of four positive operators with finite trace. □

Proposition 4.12. *Let $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $a, b \in \mathcal{B}(\mathcal{H})$. Then*

- $\|x\| \leq \|x\|_1$;
- $\|axb\|_1 \leq \|a\| \|b\| \|x\|_1$;
- $\text{Tr}(ax) = \text{Tr}(xa)$.

Proof. (1.)

$$\begin{aligned} \|x\| &= \| |x| \| = \| |x|^{\frac{1}{2}} \|^2 \\ &= \sup_{\|\alpha\|=1} \langle |x|^{\frac{1}{2}} \alpha, |x|^{\frac{1}{2}} \alpha \rangle = \sup_{\|\alpha\|=1} \langle |x| \alpha, \alpha \rangle \\ &\leq \text{Tr } |x| = \|x\|_1. \end{aligned}$$

(2.) We begin with

$$|ax|^2 = x^* a^* a x \leq \|a^* a\| x^* x = \|a^* a\| \cdot |x|^2 = \|a\|^2 \cdot |x|^2$$

and since $|ax| \leq \|a\| \cdot |x|$ we get $\|ax\|_1 \leq \|a\| \cdot \|x\|_1$. But $\|x\|_1 = \|x^*\|_1$, so we also get $\|xb\|_1 \leq \|b\| \cdot \|x\|_1$.

(3.) Since every element of $\mathcal{B}(\mathcal{H})$ is a linear combination of 4 unitaries, we can w.l.o.g. assume $a = u \in \mathcal{U}(\mathcal{H})$. Then

$$\begin{aligned} \text{Tr}(xu) &= \sum_i \langle xue_i, e_i \rangle = \sum_i \langle xue_i, u^* u e_i \rangle \\ &= \sum_i \langle u x u e_i, u e_i \rangle = \text{Tr}(ux). \end{aligned} \quad \square$$

Remark. We have the following identities:

- (1.) $\text{Tr}(\alpha \otimes \bar{\beta}) = \langle \alpha, \beta \rangle$;
- (2.) $\mathcal{F}(\mathcal{H})$ is dense in $(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$.

Theorem 4.13.

$(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$ is a Banach space.

Proof. We only have to prove completeness. Let $(x_n)_n$ be a Cauchy sequence in $(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$. Since $\|\cdot\| \leq \|\cdot\|_1$, (x_n) is a Cauchy sequence in $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$. But $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ is a Banach space, so there exists $x \in \mathcal{B}(\mathcal{H})$ such that $x_n \rightarrow x$ in norm-topology. Notice that

$$x^* x - x_n^* x_n = x^* (x - x_n) + (x - x_n)^* x_n.$$

By continuity of the continuous functional calculus, this implies $|x_n| \rightarrow |x|$, meaning that $\||x_n| - |x|\| \rightarrow 0$. Next we prove that $x \in L^1(\mathcal{B}(\mathcal{H}))$. For any ONB $(e_i)_i$, we have

$$\sum_{i=1}^k \langle |x| e_i, e_i \rangle = \lim_{n \rightarrow \infty} \sum_{i=1}^k \langle |x_n| e_i, e_i \rangle \leq \lim_{n \rightarrow \infty} \text{Tr} |x_n| = \lim_{n \rightarrow \infty} \|x_n\|_1 < \infty.$$

Here, we used the fact that $\|x_n - x_k\|_1 \geq \|x_n\|_1 - \|x_k\|_1$, so the sequence $(\|x_n\|_1)_n$ is Cauchy and therefore has a limit. This proves that $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $\|x\|_1 \leq \lim_{n \rightarrow \infty} \|x_n\|_1$. Finally, we have to show that $\|x_n - x\|_1 \rightarrow 0$. Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that for every $n > N$, we get $\|x_n - x_N\|_1 < \frac{\varepsilon}{3}$. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a finite dimensional subspace such that

$$\|x_N P_{\mathcal{H}_0^\perp}\|_1, \|x P_{\mathcal{H}_0^\perp}\|_1 < \frac{\varepsilon}{3}.$$

Then for every $n > N$, we get that

$$\begin{aligned}
\|x - x_n\|_1 &\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|(x - x_n)P_{\mathcal{H}_0^\perp}\|_1 \\
&\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|xP_{\mathcal{H}_0^\perp} - x_N P_{\mathcal{H}_0^\perp}\|_1 + \|x_N P_{\mathcal{H}_0^\perp} - x_n P_{\mathcal{H}_0^\perp}\|_1 \\
&\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|x_N - x_n\|_1 \|P_{\mathcal{H}_0^\perp}\| \\
&< \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \varepsilon \\
&\leq \|(x - x_n)\| \|P_{\mathcal{H}_0}\|_1 + \varepsilon \xrightarrow[n \rightarrow \infty]{} \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows $x_n \xrightarrow{\|\cdot\|_1} x$. \square

Theorem 4.14.

The map

$$\Psi : \mathcal{B}(\mathcal{H}) \rightarrow L^1(\mathcal{B}(\mathcal{H}))^*, \quad a \mapsto (\psi_a : x \mapsto \text{Tr}(ax))$$

is an isometric isomorphism of Banach spaces.

Proof. We notice that Ψ is linear and a contraction because the norms $\|\cdot\|$ and $\|\cdot\|_1$ are comparable. We will first show that Ψ is surjective. Let $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$. Notice that

$$(\alpha, \beta) \mapsto \varphi(\alpha \otimes \bar{\beta})$$

is a bounded sesquilinear form in \mathcal{H} . By the introductory course, there exists an $a \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(\alpha \otimes \bar{\beta}) = \langle a\alpha, \beta \rangle = \text{Tr}(a\alpha \otimes \bar{\beta}) = \text{Tr}(a(\alpha \otimes \bar{\beta})) = \psi_a(\alpha \otimes \bar{\beta}).$$

So φ and ψ_a agree on $\mathcal{F}(\mathcal{H})$, so by bounded density $\varphi = \psi_a$. Finally,

$$\|a\| = \sup_{\alpha, \beta \in (\mathcal{H})_1} |\langle a\alpha, \beta \rangle| = \sup_{\alpha, \beta \in (\mathcal{H})_1} |\text{Tr}(a(\alpha \otimes \bar{\beta}))| \leq \|\psi_a\|_1.$$

But since

$$\|\psi_a\|_1 = \sup_{x \in (L^1(\mathcal{B}(\mathcal{H})))_1} |\text{Tr}(ax)| = \sup_{x \in (L^1(\mathcal{B}(\mathcal{H})))_1} \|ax\|_1 \leq \sup_{x \in (L^1(\mathcal{B}(\mathcal{H})))_1} \|a\| \|x\|_1 = \|a\|,$$

we have $\|a\| = \|\psi_a\|_1$ and ψ is isometric. \square

Corollary 4.15. The map

$$\Phi : L^1(\mathcal{B}(\mathcal{H})) \rightarrow \mathcal{K}(\mathcal{H})^*, \quad x \mapsto (\varphi_x : a \mapsto \text{Tr}(ax))$$

is an isometric isomorphism of Banach spaces.

Proof. Same as that of theorem 4.14. \square

Definition 4.16. Let X, Y be Banach spaces. An operator $T \in \mathcal{B}(X, Y)$ is said to be *compact* if $\overline{T((X)_1)}$ is compact. The space of compact operators is $\mathcal{K}(X, Y)$. We also write $\mathcal{K}(X) := \mathcal{K}(X, X)$.

From the introductory course, we know the following statements about compact operators.

Proposition 4.17. Let $T \in \mathcal{B}(X, Y)$. The following are equivalent.

- (1.) T is compact;
- (2.) T maps bounded maps in X into relatively compact maps in Y ;
- (3.) T maps bounded sequences in X into sequences in Y that have an accumulation point.

If X, Y are Hilbert spaces, then this is also equivalent to the following.

- (4.) $T \in \overline{\mathcal{F}(X, Y)}$.

Remark. $\mathcal{K}(\mathcal{H})$ is a closed ideal in $\mathcal{B}(\mathcal{H})$.

Theorem 4.18 (Singular value decomposition).

For $K \in \mathcal{K}(\mathcal{H})$, there exists orthonormal bases $(e_i)_i$ and $(f_j)_j$ for \mathcal{H} and $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$ such that

$$Kx = \sum_{n=1}^{\infty} \sigma_n \langle x, e_n \rangle f_n. \quad (4.1)$$

As a result,

$$|K|x = \sum \sigma_n \langle x, e_n \rangle e_n.$$

Proposition 4.19. Let X be a Banach space. Then the following statements are equivalent:

- (1.) $\text{id} : X \rightarrow X$ is compact;
- (2.) $(X)_1$ is compact;
- (3.) $\dim X < \infty$.

The equivalence of the last two items is also known as the Riesz lemma.

Theorem 4.20.

- (1.) $L^1(\mathcal{B}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$.
- (2.) $K \in \mathcal{H}$ is a $L^1(\mathcal{B}(\mathcal{H}))$ iff $\sum_{k=1}^{\infty} \sigma_k < \infty$.

Proof. (1.) If $x \in L^1(\mathcal{B}(\mathcal{H}))$ then there exists $(x_n)_n$ in $\mathcal{F}(\mathcal{H})$ such that $\|x_n - x\|_1 \rightarrow 0$. Since $\|\cdot\| \leq \|\cdot\|_1$, we get $\|x_n - x\| \rightarrow 0$ and $x \in (\mathcal{F}, \|\cdot\|) = \mathcal{K}(\mathcal{H})$.

(2.) This follows from $\text{Tr}|K| = \sum \sigma_n$ for K as in (4.1). \square

4.3 Hilbert–Schmidt operators

Definition 4.21. An element $x \in \mathcal{B}(\mathcal{H})$ is a *Hilbert–Schmidt operator* if

$$|x|^2 = x^*x \in L^1(\mathcal{B}(\mathcal{H})).$$

The set of all such elements is denoted by $L^2(\mathcal{B}(\mathcal{H}), \text{Tr})$.

Proposition 4.22. (1.) $L^2(\mathcal{B}(\mathcal{H})) \triangleleft \mathcal{B}(\mathcal{H})$ and is closed under $*$.

(2.) If $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$ and $\text{Tr}(xy) = \text{Tr}(yx)$.

Remark. Beware: there exist $a, b \in \mathcal{B}(\mathcal{H})$ such that $ab \in L^1(\mathcal{B}(\mathcal{H}))$ and $ba \notin L^1(\mathcal{B}(\mathcal{H}))$. However, if $ab, ba \in L^1(\mathcal{B}(\mathcal{H}))$, then $\text{Tr}(ab) = \text{Tr}(ba)$.

Proof. For $\alpha \in \mathbb{C}$ and $x, y \in \mathcal{B}(\mathcal{H})$, we have $|\alpha x|^2 = |\alpha|^2 |x|^2$. Similarly, $|x + y|^2 \leq |x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$, so $L^2(\mathcal{B}(\mathcal{H}))$ is a complex vector space. Since $|ax|^2 \leq \|a\|^2 \cdot |x|^2$, we have $L^2(\mathcal{B}(\mathcal{H}))$ is a left ideal of $\mathcal{B}(\mathcal{H})$. From

$$\text{Tr}|x|^2 = \text{Tr}(x^*x) = \text{Tr}(xx^*) = \text{Tr}|x^*|^2,$$

we deduce that $L^2(\mathcal{B}(\mathcal{H}))$ is closed under involution. If $x \in L^2(\mathcal{B}(\mathcal{H}))$ and $b \in \mathcal{B}(\mathcal{H})$, then $x^* \in L^2(\mathcal{B}(\mathcal{H}))$, which implies $b^*x^* \in L^2(\mathcal{B}(\mathcal{H}))$ and finally $xb = (b^*x^*)^* \in L^2(\mathcal{B}(\mathcal{H}))$, so $L^2(\mathcal{B}(\mathcal{H})) \triangleleft \mathcal{B}(\mathcal{H})$. Next, we use the polarization identity

$$4y^*x = \sum_{k=0}^3 i^k |x + i^k y|^2.$$

If $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then this shows $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$ and

$$\begin{aligned} 4 \text{Tr}(y^*x) &= \sum_{k=0}^3 i^k \text{Tr}((x + i^k y)^*(x + i^k y)) \\ &= \sum_{k=0}^3 i^k \text{Tr}((x + i^k y)(x + i^k y)^*) \\ &= 4 \text{Tr}(xy^*). \end{aligned}$$

\square

On $L^2(\mathcal{B}(\mathcal{H}))$ we have the sesquilinear form $\langle x, y \rangle_2 := \text{Tr}(y^*x)$. It is well-defined and positive definite, so it is a scalar product. The induced norm is denoted by $\|\cdot\|_2$. For every $y \in L^2(\mathcal{B}(\mathcal{H}))$, we have

$$\|y\| = \|y^*y\|_1^{\frac{1}{2}} \leq \|y^*y\|_1^{\frac{1}{2}} = \|y\|_2.$$

Similarly, we have

$$\|axb\|_2 = \|a\| \cdot \|x\|_2 \cdot \|b\|$$

for all $x \in L^2(\mathcal{B}(\mathcal{H}))$ and $a, b \in \mathcal{B}(\mathcal{H})$. As before, $\mathcal{F}(\mathcal{H})$ are dense in $L^2(\mathcal{B}(\mathcal{H}))$ with respect to $\|\cdot\|_2$ and $L^2(\mathcal{B}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$. Using singular values $(\sigma_n)_n$ of a compact $K \in \mathcal{K}(\mathcal{H})$, we have $K \in L^2(\mathcal{B}(\mathcal{H}))$ iff $\sum_{k=0}^{\infty} \sigma_j^2 < \infty$. For every $x \in L^1(\mathcal{B}(\mathcal{H}))$, we have

$$\|x\|_2 = \sup_{y \in L^2(\mathcal{B}(\mathcal{H})), \|y\|_2=1} |\operatorname{Tr}(y^*x)| \leq \sup_{y \in L^2(\mathcal{B}(\mathcal{H})), \|y\|_2=1} \|y\| \cdot \|x\|_1 \leq \|x\|_1.$$

As a result, $(L^2(\mathcal{B}(\mathcal{H})), \langle \cdot, \cdot \rangle_2)$ is a Hilbert space.

Theorem 4.23 (Hölder's inequality).

For all $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ we have

$$\|xy\|_1 \leq \|x\|_2 \|y\|_2.$$

Proof. Let $xy = v|xy|$ be the polar decomposition of xy . Then

$$\begin{aligned} \|xy\|_1 &= \operatorname{Tr} |xy| = \operatorname{Tr}(v^*xy) \\ &= |\langle y, x^*v \rangle_2| \leq \|x^*v\|_2 \|y\|_2 \\ &\leq \|x^*\|_2 \|v\| \|y\|_2 \leq \|x\|_2 \cdot \|y\|_2. \end{aligned}$$

□

4.4 Hilbert–Schmidt integral operators

Let (X, μ) be a σ -finite measure space. This means that X is a countable union of finite-measure sets:

$$X = \bigcup_{j=1}^{\infty} A_j, \quad \mu(A_j) < \infty.$$

For $K \in L^2(X \times X, \mu \times \mu)$, we can define a Hilbert–Schmidt integral operator with kernel K :

$$T_K : L^2(X, \mu) \rightarrow L^2(X, \mu), \quad f \mapsto \left(y \mapsto \int_X K(x, y) f(x) d\mu(x) \right).$$

Suppose $(\varphi_\alpha)_\alpha$ is an ONB for $L^2(K, \mu)$. By Fubini, $(\overline{\varphi_\alpha(x)} \varphi_\beta(y))_{\alpha, \beta}$ is an orthonormal basis for $L^2(X \times X, \mu \times \mu)$. Since $K \in L^2(X \times X, \mu \times \mu)$, there exist $c_{ij} \in \mathbb{C}$ such that

$$K(x, y) = \sum_{i, j} c_{ij} \overline{\varphi_i(x)} \varphi_j(y), \quad \|K\|_{L^2(X \times X)}^2 = \sum |c_{ij}|^2 < \infty.$$

We show that T_K is well-defined: for $f \in L^2(X, \mu)$, we have $T_K f \in L^2(X, \mu)$. Indeed,

$$T_K f(y) = \sum_{i, j} c_{ij} \langle f, \varphi_i \rangle \varphi_j(y),$$

which implies

$$\begin{aligned}
\|T_K f\|_{L^2(X)}^2 &\leq \sum_{i,j} |c_{ij}|^2 |\langle f, \varphi_j \rangle|^2 \|\varphi_j\|_{L^2(X)}^2 \\
&\leq \|f\|_{L^2}^2 \sum_{i,j} |c_{ij}|^2 \|\varphi_j\|_{L^2}^2 \\
&= \|f\|_{L^2}^2 \sum_{i,j} |c_{ij}|^2 \\
&= \|f\|_{L^2}^2 \|K\|_{L^2(X \times X)}^2
\end{aligned}$$

and finally $\|T_K\| \leq \|K\|_{L^2}$. We claim that $T_K^* : L^2(X, \mu) \rightarrow L^2(X, \mu)$ is the integral operator with kernel

$$K^*(y, x) := \overline{K(x, y)}.$$

Indeed,

$$\begin{aligned}
\langle T_K f, g \rangle &= \int_Y \left(\int_X K(x, y) f(x) d\mu(x) \right) \cdot \overline{g(y)} d\mu(y) \\
&= \int_X f(x) \cdot \left(\int_Y \overline{K(x, y)} g(y) d\mu(y) \right) d\mu(x) \\
&= \langle f, T_{K^*} g \rangle.
\end{aligned}$$

Remark (Fubini's theorem). If $(X, \mu), (Y, \lambda)$ are σ -finite measure spaces and $\int_{X \times Y} |f| d(\mu \times \lambda)(x, y) < \infty$, then

$$\int_{X \times Y} f d(\mu \times \lambda)(x, y) = \int_Y \left(\int_X f d\mu(x) \right) d\lambda(y) = \int_X \left(\int_Y f d\lambda(y) \right) d\mu(x).$$

Theorem 4.24.

- (1.) For $K \in L^2(X \times X, \mu \times \mu)$ we have $T_K \in L^2(\mathcal{B}(L^2(X, \mu)))$.
- (2.) The mapping $\Phi : K \mapsto T_K$ is a unitary $L^2(X \times X, \mu \times \mu) \rightarrow L^2(\mathcal{B}(L^2(X, \mu)))$.

Proof. (1.) We will prove that $\|T_K\|_2 = \|K\|_{L^2}$. We want to approximate T_K with finite rank operators, so we first approximate K :

$$K(x, y) = \sum_{i,j=1}^{\infty} c_{ij} \overline{\varphi_i(x)} \varphi_j(x)$$

for an orthonormal basis $(\varphi_\alpha)_\alpha$ for $L^2(X, \mu)$. For $N \in \mathbb{N}$ let $K_N(x, y) = \sum_{i,j=1}^N c_{ij} \overline{\varphi_i(x)} \varphi_j(x)$. Then

$$T_{K_N} f = \sum_{i,j=1}^N c_{ij} \langle f, \varphi_i \rangle \varphi_j \in \mathcal{F}(L^2(X, \mu)).$$

By the above inequality,

$$\|T_K - T_{K_N}\| \leq \|K - K_N\|_{L^2} \rightarrow 0,$$

so $T_K \in \overline{(\mathcal{F}, \|\cdot\|)} = \mathcal{K}(\mathcal{H})$. Then

$$\|T_K\|_2^2 = \sum_i \|T_K \varphi_i\|_{L^2}^2 = \sum_{i,j,k} \|c_{jk} \varphi_j(x) \delta_{ik}\|^2 = \sum |c_{ij}|^2 = \|K\|_{L^2}^2.$$

- (2.) It remains to prove surjectivity. Since Φ is isometric, $\text{im } \Phi$ is closed. So it suffices to show that $\text{im } \Phi$ is dense. In particular, we will show that $\text{im } \Phi \supseteq \mathcal{F}(L^2(X, \mu))$. Let $A \in \mathcal{F}(L^2(X, \mu))$, so $\text{rank } A < \infty$. Let (ψ_1, \dots, ψ_m) be an orthonormal basis for $\text{im } A$. Then $A\varphi = c_1(\varphi)\psi_1 + \dots + c_m(\varphi)\psi_m$ for some bounded linear functionals c_j on $L^2(X, \mu)$. By Riesz, there exist $\mu_j \in L^2(X, \mu)$ such that $c_j(\varphi) = \langle \varphi, \mu_j \rangle$. Hence

$$A\varphi(x) = \int_X \left(\sum_{j=1}^m \psi_j(x) \cdot \overline{\mu_j(y)} \cdot \varphi(y) \right) d\mu(y) = T_{\sum_{j=1}^m \psi_j(x) \overline{\mu_j(y)}} \in \text{im } \Phi. \quad \square$$

4.5 Hilbert–Schmidt operators $\mathcal{H} \rightarrow \mathcal{K}$ and tensor products

Let \mathcal{H}, \mathcal{K} be Hilbert spaces and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. We associate to A the map

$$\tilde{A} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}, \mathcal{H} \oplus \mathcal{K}), \quad \alpha \oplus \beta \mapsto 0 \oplus A\alpha,$$

or in matrix form,

$$\tilde{A} = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}.$$

We denote the set of Hilbert–Schmidt operators $\mathcal{H} \rightarrow \mathcal{K}$ as

$$HS(\mathcal{H}, \mathcal{K}) := \{A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \mid \tilde{A} \in L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))\}.$$

It is trivial to show that this coincides with

$$\{A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \mid A^*A \in L^1(\mathcal{B}(\mathcal{H}))\}.$$

With the usual scalar product $\langle A, B \rangle_2 = \text{Tr}(B^*A)$, this becomes a Hilbert space.

Remark. By Riesz’s representation theorem, every functional in \mathcal{H}^* is of the form $\bar{\alpha} : x \mapsto \langle x, \alpha \rangle$, where $\alpha \in \mathcal{H}$. This means that we can introduce a scalar product on \mathcal{H}^* by $\langle \bar{\alpha}, \bar{\beta} \rangle_{\mathcal{H}^*} := \langle \beta, \alpha \rangle_{\mathcal{H}}$. This scalar product induces the usual operator norm on \mathcal{H}^* , so it makes \mathcal{H}^* into a Hilbert space.

Example 4.25. The dual \mathcal{H}^* is isomorphic as a Hilbert space to $HS(\mathcal{H}, \mathbb{C})$. To prove

this, it's enough to compare the scalar products. For any $\bar{\alpha}, \bar{\beta} \in \mathcal{H}^*$, we have

$$\begin{aligned}
\langle \bar{\alpha}, \bar{\beta} \rangle &= \text{Tr}(\bar{\beta}^* \bar{\alpha}) \\
&= \sum_{i \in I} \langle \bar{\alpha} e_i, \bar{\beta} e_i \rangle \\
&= \sum_{i \in I} \langle e_i, \alpha \rangle \overline{\langle e_i, \beta \rangle} \\
&= \sum_{i \in I} \langle e_i, \alpha \rangle \langle \beta, e_i \rangle \\
&= \langle \beta, \alpha \rangle = \langle \bar{\alpha}, \bar{\beta} \rangle_{\mathcal{H}^*}.
\end{aligned}$$

Definition 4.26. We define the *tensor product of Hilbert spaces* as $\mathcal{H} \bar{\otimes} \mathcal{K} := HS(\mathcal{K}^*, \mathcal{H})$.

Remark. For any $\alpha \in \mathcal{H}$ and $\beta \in \mathcal{K}$, we define the elementary tensors as

$$\alpha \otimes \beta : \mathcal{H}^* \rightarrow \mathcal{K}, \quad f \mapsto f(\alpha)\beta.$$

The span of these operators is the usual algebraic tensor product of vector spaces $\mathcal{H} \otimes \mathcal{K}$ and consists of all the finite-rank operators in $\mathcal{H} \bar{\otimes} \mathcal{K}$.

4.6 Locally convex topologies on $\mathcal{B}(\mathcal{H})$

If \mathcal{H} is a Hilbert space, then $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ is a Banach algebra with its norm topology.

Definition 4.27. (1.) The *weak operator topology* (WOT) is given by the seminorms

$$T \mapsto |\langle T\alpha, \beta \rangle|, \quad \forall \alpha, \beta \in \mathcal{H}.$$

(2.) The *strong operator topology* (SOT) is given by the seminorms

$$T \mapsto \|T\alpha\|, \quad \forall \alpha \in \mathcal{H}.$$

These topologies are comparable: $\text{WOT} \subseteq \text{SOT} \subseteq \text{norm topology}$.

- Norm topology has the subbasis

$$\{S \in \mathcal{B}(\mathcal{H}) \mid \|S - T\| < \varepsilon\}$$

for $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. The net T_i converges to T iff $\|T_i - T\|$ converges to 0.

- WOT topology has the subbasis

$$\{S \in \mathcal{B}(\mathcal{H}) \mid \langle (S - T)\alpha, \beta \rangle < \varepsilon\}$$

for $\alpha, \beta \in \mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. The net T_i converges to T iff $\langle T_i \alpha, \beta \rangle$ converges to $\langle T \alpha, \beta \rangle$ for all α, β .

- SOT topology has the subbasis

$$\{S \in \mathcal{B}(\mathcal{H}) \mid \|(S - T)\alpha\| < \varepsilon\}$$

for $\alpha \in \mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. The net T_i converges to T iff $\|(T_i - T)\alpha\|$ converges to 0 for all α .

Example 4.28. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and denote $T_n = \frac{1}{n} \cdot \text{id}$. Then $T_n \rightarrow 0$ in the norm topology. Now if we introduce the operator

$$S(x_1, x_2, \dots) = (0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots),$$

then $S_n \rightarrow 0$ in SOT, but not in norm topology, since $\|S_n\| = 1$. Lastly, we define

$$W_n(x_1, x_2, \dots) = (0, 0, \dots, x_1, x_2, \dots).$$

We get that $W_n \rightarrow 0$ in WOT, but not in SOT or norm topology.

Example 4.29. Let $(y_n)_n$ be a countable dense subset of $\mathcal{H} = \ell^2$. Consider the following two metrics on $(\mathcal{B}(\mathcal{H}))_1$:

$$d_S(A, B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|(A - B)y_n\|, \quad d_W(A, B) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle (A - B)y_n, y_n \rangle|.$$

Then d_S induces SOT and d_W induces WOT on $(\mathcal{B}(\mathcal{H}))_1$.

Example 4.30. The multiplication

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}), \quad (A, B) \mapsto A \cdot B$$

is not jointly continuous with regards to SOT or WOT. Indeed, if $S : \ell^2 \rightarrow \ell^2$ is the right shift (and S^* the left shift), then $S^n \rightarrow 0$ and $(S^*)^n \rightarrow 0$ in SOT and WOT, but $(S^*)^n S^n = I$. However, multiplication is WOT- and SOT-continuous in each factor separately. Suppose that $(x_\alpha)_\alpha \rightarrow x$ in WOT and $y \in \mathcal{B}(\mathcal{H})$. Then for each $v, w \in \mathcal{H}$, we have

$$|\langle x_\alpha y v - x y v, w \rangle| \rightarrow 0,$$

since $x_\alpha \rightarrow x$ in WOT. Similarly,

$$|\langle y x_\alpha v - y x v, w \rangle| = |\langle x_\alpha v - x v, y^* w \rangle| \rightarrow 0,$$

which implies $x_\alpha y \rightarrow xy$ and $y x_\alpha \rightarrow yx$ in WOT. Similarly, if $(x_\alpha)_\alpha \rightarrow x$ in SOT and $y \in \mathcal{B}(\mathcal{H})$, then for each $v \in \mathcal{H}$ we have

$$\|(x_\alpha - x)y v\| \rightarrow 0, \quad \|y(x_\alpha - x)v\| \rightarrow 0,$$

so $x_\alpha y \rightarrow xy$ and $y x_\alpha \rightarrow yx$ in SOT.

Example 4.31. The adjoint is isometric in the norm topology. It is also continuous in WOT:

$$|\langle x^*v - y^*v, w \rangle| < \varepsilon \Leftrightarrow |\langle xw - yw, v \rangle| < \varepsilon.$$

However, it is not continuous with respect to SOT. If $(e_n)_n$ is an ONB for \mathcal{H} , consider $e_1 \otimes \overline{e_n}$. Then for every $x \in \mathcal{H}$, we have

$$\|(e_1 \otimes \overline{e_n})x\| = |\langle x, e_n \rangle| \xrightarrow{n \rightarrow \infty} 0,$$

so $e_1 \otimes \overline{e_n} \rightarrow 0$ in SOT. However,

$$\|(e_1 \otimes \overline{e_n})^*x\| = \|(e_n \otimes \overline{e_1})x\| = |\langle x, e_1 \rangle|$$

does not go to 0 for all $x \in \mathcal{H}$, which proves our statement.

Remark. If $T : X \rightarrow Y$ is continuous, then T remains continuous if X is given a finer topology or Y is given a coarser topology. But if both topologies are made coarser or both finer, nothing can be said in general. In particular, if $T : X \rightarrow X$ is continuous with respect to a given topology on X in both domain and codomain, you cannot generally conclude anything about continuity of T when X is given a finer or coarser topology on both domain and codomain. The previous example illustrates this.

Lemma 4.32. Let $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ be linear. The following are equivalent.

(1.) There exist $v_1, \dots, v_n \in \mathcal{H}$ and $w_1, \dots, w_n \in \mathcal{H}$ such that

$$\varphi(T) = \sum_{i=1}^n \langle Tv_i, w_i \rangle.$$

(2.) φ is WOT-continuous.

(3.) φ is SOT-continuous.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) are obvious. Let us prove (3) \Rightarrow (1). By proposition 1.30, there exists a $K > 0$ and $v_1, \dots, v_n \in \mathcal{H}$ such that

$$|\varphi(T)|^2 \leq K \cdot \sum_{i=1}^n \|Tv_i\|^2.$$

Define

$$\mathcal{H}_0 := \overline{\left\{ \bigoplus_{i=1}^n Tv_i \mid T \in \mathcal{B}(\mathcal{H}) \right\}} \leq \mathcal{H} \oplus^n.$$

The map

$$\mathcal{H}_0 \ni \bigoplus_{i=1}^n Tv_i \mapsto \varphi(T) \in \mathbb{C}$$

is a well-defined and bounded linear functional, which by continuity extends to $\mathcal{H}_0 \rightarrow \mathbb{C}$. By

Riesz, there exist $w_1, \dots, w_n \in \mathcal{H}$ such that

$$\varphi(T) = \sum_{i=1}^n \langle T v_i, w_i \rangle.$$

Recall that $v \otimes \bar{w} \in \mathcal{F}(\mathcal{H})$ and $\text{Tr}(v \otimes \bar{w}) = \langle v, w \rangle$, so

$$\text{Tr}(T(v \otimes \bar{w})) = \langle T v, w \rangle.$$

The previous identity is really

$$\varphi(T) = \sum_{i=1}^n \text{Tr}(T(v \otimes \bar{w}_i)) = \text{Tr}(T \cdot \sum_{i=1}^n v_i \otimes \bar{w}_i).$$

This means that $\varphi(T) = \text{Tr}(T \cdot A)$ for $A \in \mathcal{F}(\mathcal{H})$. □

Corollary 4.33. *If $K \subseteq \mathcal{B}(\mathcal{H})$ is convex, then*

$$\overline{K}^{\text{WOT}} = \overline{K}^{\text{SOT}}.$$

Proof. Consider $\mathcal{B}(\mathcal{H})$, equipped with WOT topology. This is a LCS, so $\overline{K}^{w, \text{WOT}} = \overline{K}^{\text{WOT}}$ by theorem 1.28. Similarly, we have that $\mathcal{B}(\mathcal{H})$ is a LCS when equipped with SOT topology, so $\overline{K}^{w, \text{SOT}} = \overline{K}^{\text{SOT}}$. Now

$$\begin{aligned} x \in \overline{K}^{w, \text{WOT}} &\Leftrightarrow \exists \text{ a net } (x_\alpha)_\alpha \subseteq K, \text{ such that } x_\alpha \rightarrow x \text{ WOT-weakly} \\ &\Leftrightarrow f(x_\alpha) \rightarrow f(x) \text{ for all WOT-continuous functionals } f : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C} \\ &\Leftrightarrow f(x_\alpha) \rightarrow f(x) \text{ for all SOT-continuous functionals } f : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C} \\ &\Leftrightarrow \exists \text{ a net } (x_\alpha)_\alpha \subseteq K, \text{ such that } x_\alpha \rightarrow x \text{ SOT-weakly} \\ &\Leftrightarrow x \in \overline{K}^{w, \text{SOT}}. \end{aligned}$$

Therefore, $\overline{K}^{w, \text{WOT}} = \overline{K}^{w, \text{SOT}}$ and we are done. □

Definition 4.34. The σ -weak operator topology (σ -WOT or ultra-weak) is the topology in $\mathcal{B}(\mathcal{H})$ given by the seminorms

$$x \mapsto \left| \sum_{i=1}^{\infty} \langle x \alpha_i, \alpha_i \rangle \right|$$

for $\alpha_i \in \mathcal{H}$ with $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$. A subbasis of open sets is thus

$$\left\{ x \in \mathcal{B}(\mathcal{H}) \mid \left| \sum_{i=1}^{\infty} \langle (x - x_0) \alpha_i, \alpha_i \rangle \right| < \varepsilon \right\}$$

for $\alpha_i \in \mathcal{H}$ with $\varepsilon > 0$, $x_0 \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$.

Definition 4.35. The σ -strong operator topology (σ -SOT or ultra-strong) is the topology in $\mathcal{B}(\mathcal{H})$ given by the seminorms

$$x \mapsto \left(\sum_{i=1}^{\infty} \|x\alpha_i\|^2 \right)^{\frac{1}{2}}$$

for $\alpha_i \in \mathcal{H}$ with $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$. A subbasis of open sets is thus

$$\left\{ x \in \mathcal{B}(\mathcal{H}) \mid \left(\sum_{i=1}^{\infty} \|(x - x_0)\alpha_i\|^2 \right)^{\frac{1}{2}} < \varepsilon \right\}$$

for $\alpha_i \in \mathcal{H}$ with $\varepsilon > 0$, $x_0 \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$.

Remark. σ -WOT can also be given by seminorms

$$x \mapsto |\operatorname{Tr}(xa)|$$

for $a \in L^1(\mathcal{B}(\mathcal{H}))$ positive. Let $(f_i)_i$ be an ONB for \mathcal{H} and define

$$b : \mathcal{H} \rightarrow \mathcal{H}, \quad f_i \mapsto \alpha_i.$$

Since $\sum_i \|\alpha_i\|^2 < \infty$, we can conclude $b \in L^2(\mathcal{B}(\mathcal{H}))$. Then:

$$\begin{aligned} \sum_i \langle x\alpha_i, \alpha_i \rangle &= \sum_i \langle xbf_i, bf_i \rangle \\ &= \sum_i \langle b^* xbf_i, f_i \rangle \\ &= \operatorname{Tr}(b^* xb) \\ &= \operatorname{Tr}(xbb^*), \end{aligned}$$

where $a := bb^* \in L^1(\mathcal{B}(\mathcal{H}))$. Since $\mathcal{B}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))^*$, the σ -WOT is just the weak-* topology (with respect to this pairing).

Remark. The map

$$\operatorname{id} \otimes 1 : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2), \quad x \mapsto x \otimes 1$$

is an isometric *-isomorphism of C^* -algebras. It is neither SOT- nor WOT-continuous. Despite that, σ -WOT on $\mathcal{B}(\mathcal{H})$ is induced by WOT on $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2)$ and the σ -SOT on $\mathcal{B}(\mathcal{H})$ is induced by SOT on $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2)$. Indeed, if $(e_i)_{i \in \mathbb{N}}$ is an ONB for ℓ^2 , define $\alpha := \sum_{i=1}^{\infty} \alpha_i \otimes e_i \in \mathcal{H} \overline{\otimes} \ell^2$. Then

$$\sum_{i \in \mathbb{N}} \langle x\alpha_i, \alpha_i \rangle_{\mathcal{H}} = \langle (\operatorname{id} \otimes 1)(x)\alpha, \alpha \rangle_{\mathcal{H} \overline{\otimes} \ell^2}$$

and similarly

$$\left(\sum_{i \in \mathbb{N}} \|x\alpha_i\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} = \|(\operatorname{id} \otimes 1)(x)\alpha\|_{\mathcal{H} \overline{\otimes} \ell^2}$$

Lemma 4.36. Let $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ be a linear functional operator. Then the following are equivalent.

- (1.) $\exists a \in L^1(\mathcal{B}(\mathcal{H}))$ such that $\varphi(x) = \text{Tr}(ax)$, $\forall x \in \mathcal{B}(\mathcal{H})$;
- (2.) φ is σ -WOT continuous;
- (3.) φ is σ -SOT continuous.

Proof. As previously, the implication (1) \Rightarrow (2) \Rightarrow (3) is obvious. Let us prove (3) \Rightarrow (1). Assume φ is σ -SOT continuous. By identifying $\mathcal{B}(\mathcal{H})$ via $\text{id} \otimes 1$ with a subspace in $\mathcal{B}(\mathcal{H} \otimes \ell^2)$, φ is SOT-continuous on this subspace. By Hahn–Banach, φ extends to a SOT-continuous linear functional on $\mathcal{B}(\mathcal{H} \otimes \ell^2)$. By the previous lemma, $\exists \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathcal{H} \otimes \ell^2$.

$$\varphi(x) = \sum_{i=1}^n \langle (\text{id} \otimes 1)(x) \alpha_i, \beta_i \rangle.$$

With

$$\alpha_i \sum_{j=1}^{\infty} \alpha_{ij} \otimes e_j, \quad \sum_j \|\alpha_{ij}\|^2 < \infty$$

and

$$\beta_i \sum_{j=1}^{\infty} \beta_{ij} \otimes e_j, \quad \sum_j \|\beta_{ij}\|^2 < \infty.$$

Then

$$\begin{aligned} \varphi(x) &= \sum_{i=1}^n \langle (x \otimes 1) \sum_{j=1}^{\infty} \alpha_{ij} \otimes e_j, \sum_{k=1}^{\infty} \beta_{ik} \otimes e_k \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle x \alpha_{ij}, \beta_{ik} \rangle \langle e_j, e_k \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^{\infty} \langle x \alpha_{ij}, \beta_{ij} \rangle. \end{aligned}$$

Define

$$A_i : \mathcal{H} \rightarrow \mathcal{H}, \quad A_i f_k = \alpha_{ik}$$

and

$$B_i : \mathcal{H} \rightarrow \mathcal{H}, \quad B_i f_k = \beta_{ik}$$

for an orthonormal basis $(f_k)_{k \in \mathbb{N}}$. By assumption, $A_i, B_i \in L^2(\mathcal{B}(\mathcal{H}))$. As before, this gives $\varphi(x) = \sum_i \text{Tr}(B_i^* x A_i) = \text{Tr}(x A_i B_i^*)$. \square

Corollary 4.37. The unit disk $(\mathcal{B}(\mathcal{H}))_1$ is compact with respect to the σ -WOT topology.

Proof. σ -WOT on $\mathcal{B}(\mathcal{H})$ is the weak-* topology from $L^1(\mathcal{B}(\mathcal{H}))^* = \mathcal{B}(\mathcal{H})$. The statement now follows from Banach–Alaoglu. \square

Corollary 4.38. WOT and σ -WOT topologies agree on bounded subsets $B \subseteq \mathcal{B}(\mathcal{H})$.

Proof. W.l.o.g. $B = M \cdot (\mathcal{B}(\mathcal{H}))_1$ for some $M > 0$. Then the identity $(B, \sigma\text{-WOT}) \rightarrow (B, \text{WOT})$ is a continuous map from a Hausdorff compact space (previous corollary) to a Hausdorff space. Therefore the identity map is a closed continuous bijection, so it's a homeomorphism. \square

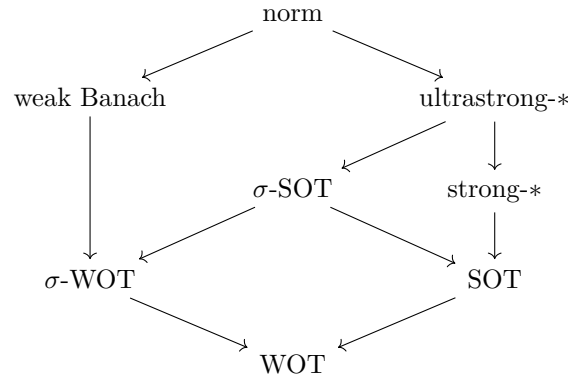
Definition 4.39. Let A be a vector space and $B \subseteq \mathcal{L}(A, \mathbb{C})$ a set of some of its linear functionals. Then we define $\sigma(A, B)$ as the weakest topology in A such that functionals in B are continuous.

Remark. $\sigma\text{-WOT}$ topology is $\sigma(\mathcal{B}(\mathcal{H}), L^1(\mathcal{B}(\mathcal{H})))$.

Remark. Let us define the following topologies on $\mathcal{B}(\mathcal{H})$.

- (1.) Weak Banach topology is $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^*)$.
- (2.) Ultrastrong-* topology is the weakest topology stronger than $\sigma\text{-SOT}$ such that $*$ is continuous.
- (3.) Strong-* topology is generated by seminorms $x \mapsto \|x\alpha\|$ and $x \mapsto \|x^*\alpha\|$ for $\alpha \in \mathcal{H}$.

In the end, we get the following diagram that demonstrates which topologies are comparable.



5 von Neumann algebras

5.1 Bicommutant theorem

Definition 5.1. A *von Neumann algebra* (on Hilbert space \mathcal{H}) is a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ that is WOT-closed. Equivalently, it is a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ that is SOT-closed.

Remark. To shorten the notation, we will abbreviate “von Neumann algebra” to vNa.

If $A \subseteq \mathcal{B}(\mathcal{H})$, then $W^*(A)$ denotes the vNa generated by A , or the smallest vNa in $\mathcal{B}(\mathcal{H})$ that contains A . This is well defined, since

$$W^*(A) = \bigcap \{W \mid A \subseteq W, W \subseteq \mathcal{B}(\mathcal{H}) \text{ is vNa}\}.$$

Lemma 5.2. If $A \subseteq \mathcal{B}(\mathcal{H})$ is a vNa, then $(A)_1$ is WOT-compact.

Proof. By corollary 4.37, $(\mathcal{B}(\mathcal{H}))_1$ is compact in σ -WOT topology. By corollary 4.38, the WOT and σ -WOT topologies on $(\mathcal{B}(\mathcal{H}))_1$ are equivalent, so $(\mathcal{B}(\mathcal{H}))_1$ is also compact in WOT topology. Next, we prove that $(A)_1$ is WOT-closed in $(\mathcal{B}(\mathcal{H}))_1$. Suppose that the net $(x_i)_i$ in $(A)_1$ converges to some x . Since A is WOT-closed, we must have $x \in A$. Assume that $x \notin (A)_1$, so $\|x\| > 1$. Since $\|x\| = \sup_{\alpha, \beta \in (\mathcal{H})_1} |\langle x\alpha, \beta \rangle|$, there must exist some $\alpha, \beta \in (\mathcal{H})_1$ such that $|\langle x\alpha, \beta \rangle| > 1$. However, for every x_i we have $|\langle x_i\alpha, \beta \rangle| \leq \|x_i\| \cdot \|\alpha\| \cdot \|\beta\| \leq 1$, contradicting the fact that $\langle x_i\alpha, \beta \rangle \rightarrow \langle x\alpha, \beta \rangle$. Therefore, $x \in (A)_1$ and $(A)_1$ is WOT-closed in $(\mathcal{B}(\mathcal{H}))_1$, so it is compact. \square

Corollary 5.3. Let $A \subseteq \mathcal{B}(\mathcal{H})$ vNa. Then $(A)_1$ and A_{sa} are SOT-closed and WOT-closed.

Proof. We already know that the adjoint is continuous in WOT, so A_{sa} is closed in WOT. Since A_{sa} is convex, it is also SOT-closed. The same exact argument applies for $(A)_1$. \square

Definition 5.4. The *commutant* of a set $B \subseteq \mathcal{B}(\mathcal{H})$ is

$$B' := \{T \in \mathcal{B}(\mathcal{H}) \mid \forall S \in B : ST = TS\}$$

and its *bicommutant* is $B'' := (B')'$.

Remark. By definition, $B'' \supseteq B$.

Theorem 5.5.

Suppose $A \subseteq \mathcal{B}(\mathcal{H})$ is closed under $*$. Then A' is vNa.

Proof. Obviously, A' is also a subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed under $*$. We prove that it is WOT-closed. Let $(x_\alpha)_\alpha$ be a net in A' that WOT-converges to $x \in \mathcal{B}(\mathcal{H})$. Pick any $a \in A$

and $\varphi, \mu \in \mathcal{H}$. Then

$$\begin{aligned}
\langle [x, a]\varphi, \mu \rangle &= \langle (xa - ax)\varphi, \mu \rangle \\
&= \langle xa\varphi, \mu \rangle - \langle ax\varphi, \mu \rangle \\
&= \langle xa\varphi, \mu \rangle - \langle x\varphi, a^*\mu \rangle \\
&= \lim_{\alpha} \langle x_{\alpha}a\varphi, \mu \rangle - \langle x_{\alpha}\varphi, a^*\mu \rangle \\
&= \lim_{\alpha} \langle (x_{\alpha}a - ax_{\alpha})\varphi, \mu \rangle \\
&= \lim_{\alpha} \langle [x_{\alpha}, a]\varphi, \mu \rangle = 0,
\end{aligned}$$

so $x \in A'$ and we are done. \square

Corollary 5.6. *Every vNa is unital.*

Example 5.7. *For an infinitely-dimensional Hilbert space \mathcal{H} , the set of all compact operators $\mathcal{K}(\mathcal{H})$ is not a vNa, since it doesn't include the identity (by the Riesz lemma). In particular, $\mathcal{K}(\mathcal{H})$ is neither SOT- nor WOT-closed.*

Remark. As we will see later, the finite-rank projections on a Hilbert space converge strongly to identity.

Corollary 5.8. *Suppose that $A \subseteq \mathcal{B}(\mathcal{H})$ is a maximal commutative subalgebra. If A is closed under $*$, then it is a vNa.*

Proof. Since A is commutative, $A' \supseteq A$. Take $b \in A' \subseteq A$ and consider the subalgebra, generated by A and b . This is an abelian algebra, so by maximality we have $b \in A$ and $A = A'$. Then by theorem 5.5, A is a vNa. \square

Lemma 5.9. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$ -subalgebra. For any $\mu \in \mathcal{H}$ and $x \in A''$ there exists a net $(x_{\alpha})_{\alpha}$ in A such that $\lim_{\alpha} \|(x_{\alpha} - x)\mu\| = 0$.*

Proof. Define $\mathcal{K} := \overline{A\mu} \leq \mathcal{H}$. Let $p : \mathcal{H} \rightarrow \mathcal{K}$ be the orthogonal projection onto \mathcal{K} . By definition, $a\mathcal{K} \subseteq \mathcal{K}$ for any $a \in A$. Equivalently, $pap = ap$ for any $a \in A$. Then

$$pa = (a^*p)^* = (pa^*p)^* = pap = ap,$$

so $p \in A'$. But $x \in A''$, so

$$xp = xp^2 = pxp$$

and $x\mathcal{K} \subseteq \mathcal{K}$. In particular, since $\mu \in \mathcal{K}$, we have $x\mu \in \mathcal{K} = \overline{A\mu}$. So there must exist some net in $A\mu$ that converges to $x\mu$. \square

Theorem 5.10 (von Neumann's bicommutant theorem).

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$ -subalgebra. Then $\overline{A}^{\text{WOT}} = A''$.

Proof. By the previous theorem, A'' is a vNa. In particular, it is WOT-closed. Since $A \subseteq A''$, it suffices to show that A is WOT-dense in A'' . Because A is convex, it is enough to show that A is SOT-dense in A'' . Let $x \in A''$ and $\mu_1, \dots, \mu_n \in \mathcal{H}$. Consider the matrix $*$ -algebra $M_n(\mathcal{B}(\mathcal{H}))$ with the usual matrix involution. There exists a canonical $*$ -isomorphism $M_n(\mathcal{B}(\mathcal{H})) \rightarrow \mathcal{B}(\mathcal{H}^n)$, which allows us to introduce a (unique) norm on $M_n(\mathcal{B}(\mathcal{H}))$, making it a C^* -algebra. Define

$$\tilde{A} = \left\{ \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \in M_n(\mathcal{B}(\mathcal{H})) \mid a \in A \right\}.$$

Then $\tilde{A}' = M_n(A')$. Hence we get

$$\tilde{A}'' \subseteq M_n(A')' = \tilde{A}''.$$

This implies that

$$\begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} \in \tilde{A}'' \subseteq \tilde{A}''.$$

Now we apply lemma 5.9 to \tilde{A} to get a net $(a_i)_i$ in A such that

$$\lim_i \|(x - a_i)\mu_j\| = 0, \quad \forall j = 1, \dots, n.$$

Finally, we have to show that this implies that x is in the SOT-closure of A . Let U be an open neighborhood around x . Then U must contain some finite intersection of subbasis sets that generate the SOT topology. This means that there exists $\varepsilon > 0$ and $\mu_1, \dots, \mu_n \in \mathcal{H}$ such that

$$\bigcap_{j=1}^n \{y \in \mathcal{B}(\mathcal{H}) \mid \|(x - y)\mu_j\| < \varepsilon\} \subseteq U.$$

Now we can conclude that $U \cap A \neq \emptyset$ and x is in the SOT-closure of A . □

Corollary 5.11. Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$ -subalgebra. Then A is a vNa iff $A = A''$.

Remark. WOT-closed implies norm-closed. In particular, every vNa is a C^* -algebra. However, the converse is not always true: $\mathcal{C}([0, 1])$ is a C^* -algebra that is not vNa. As we will see, this is because it does not contain nontrivial projections.

Corollary 5.12 (Polar decomposition in vNa). Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $x \in A$. Suppose that $x = v|x|$ is the polar decomposition of x in $\mathcal{B}(\mathcal{H})$. Then $v \in A$.

Proof. We know that

$$\ker v = (\operatorname{im} |x|)^\perp = \ker |x| = \ker x.$$

For $a \in A'$ and $\mu \in \ker x$ we have $a\mu \in \ker x$:

$$x(a\mu) = ax\mu = 0,$$

which implies $a \ker |x| \subseteq \ker |x|$. We know that $\mathcal{H} = \ker |x| \oplus \overline{\operatorname{im} |x|}$. Suppose that $|x|\mu \in \operatorname{im} |x|$. Then

$$\begin{aligned} [a, v]|x|\mu &= (av - va)|x|\mu = av|x|\mu - va|x|\mu \\ &= ax\mu - v|x|a\mu = ax\mu - xa\mu \\ &= [a, x]\mu = 0. \end{aligned}$$

But for $\beta \in \ker |x| = \ker v$, we have

$$[a, v]\beta = (av - va)\beta = av\beta - va\beta = 0.$$

Since av and va agree on $\ker |x| \oplus \overline{\operatorname{im} |x|} = \mathcal{H}$, we have $v \in A'' = A$. □

The next example is of fundamental importance.

Example 5.13 (Commutative vNa). *Let (X, μ) be a σ -finite measure space and*

$$M : L^\infty(X, \mu) \rightarrow \mathcal{B}(L^2(X, \mu)), \quad g \mapsto M_g,$$

where we define

$$(M_g f)(x) = g(x)f(x).$$

Then M is an isometric $$ -isomorphism onto its image and $M(L^\infty(X, \mu))$ is a maximal commutative vNa in $\mathcal{B}(L^2(X, \mu))$.*

Proof of the example. Clearly, M is injective, additive and multiplicative. First, we prove that M is a $*$ -homomorphism. This follows from the next calculation:

$$\begin{aligned} \langle M_{\bar{g}}\mu, \varphi \rangle &= \int_X M_{\bar{g}}\mu \cdot \bar{\varphi} d\mu \\ &= \int_X \bar{g}\mu \bar{\varphi} \\ &= \int_X \mu \bar{g}\bar{\varphi} d\mu \\ &= \langle \mu, M_g \varphi \rangle = \langle M_g^* \mu, \varphi \rangle, \end{aligned}$$

so $M_{\bar{g}} = M_g^*$. Next, we prove that M is isometric. For $g \in L^\infty(X, \mu)$, there exists a sequence $E_n \subseteq X$ such that $0 < \mu(E_n) < \infty$ and $|g|_{E_n} \geq \|g\|_\infty - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\|M_g\| \geq \frac{\|M_g 1_{E_n}\|_2}{\|1_{E_n}\|_2} \geq \|g\|_\infty - \frac{1}{n}, \quad \forall n \in \mathbb{N},$$

which implies $\|M_g\| \geq \|g\|_\infty$. For the reverse, notice that

$$\begin{aligned}\|M_g 1_{E_n}\|^2 &= \int_X |g \cdot 1_{E_n}|^2 d\mu \\ &= \int_{E_n} |g|^2 d\mu \\ &\geq \int_{E_n} (\|g\|_\infty - \frac{1}{n})^2 d\mu \\ &= (\|g\|_\infty - \frac{1}{n})^2 \cdot \mu(E_n)\end{aligned}$$

and

$$\begin{aligned}\|M_g\|^2 &= \sup_{\|\mu\|_2=1} \|M_g \mu\|_2^2 = \sup_{\|\mu\|_2=1} \int_X |g\mu|^2 d\mu \\ &\leq \|g\|_\infty^2 \cdot \sup_{\|\mu\|_2=1} \int_X |\mu|^2 d\mu = \|g\|_\infty^2.\end{aligned}$$

We've just shown that $\|M_g\| = \|g\|_\infty$. Lastly, we prove that $M(L^\infty(X, \mu))$ is a maximal commutative subalgebra of $\mathcal{B}(L^2(X, \mu))$. Take $T \in \mathcal{B}(L^2(X, \mu))$ and assume it commutes with all M_g 's. Now pick a measurable sequence $E_n \subseteq X$ such that $0 < \mu(E_n) < \infty$, $E_n \subseteq E_{n+1}$ and $X = \bigcup_{n \in \mathbb{N}} E_n$. Define $f_n := T(1_{E_n}) \in (X, \mu)$. First we prove that $f_n \in L^\infty(X, \mu)$. If A is measurable and $0 < \mu(A) < \infty$, then

$$\begin{aligned}\frac{1}{\mu(A)} \int_X |f_n \cdot 1_A|^2 d\mu &= \frac{1}{\mu(A)} \cdot \|M_{1_A} T(1_{E_n})\|^2 \\ &= \frac{1}{\mu(A)} \cdot \|T(1_{A \cap E_n})\|^2 \\ &\leq \frac{1}{\mu(A)} \cdot \|T\|^2 \cdot \|1_A\|^2 = \|T\|^2.\end{aligned}$$

If $f \notin L^\infty(X, \mu)$, then for all $M \in \mathbb{R}$ we have

$$0 < \mu(\underbrace{\{x \in X \mid |f_n(x)| > M\}}_{A_{n,M}}) < \infty,$$

since $f_n \in L^2(X, \mu)$. By above calculation,

$$M^2 \leq \frac{1}{\mu(A_{n,M})} \cdot \int_X |f \cdot 1_{A_{n,M}}|^2 d\mu \leq \|T\|^2,$$

which is of course a contradiction. This proves that $f_n \in L^\infty(X, \mu)$ and $\|f_n\|_\infty \leq \|T\|$. For $n \leq m$ we have

$$\begin{aligned}1_{E_n} \cdot f_m &= 1_{E_n} \cdot T(1_{E_m}) \\ &= M_{1_{E_n}}(T(1_{E_m})) \\ &= T(M_{1_{E_n}} 1_{E_m}) \\ &= T(1_{E_n} 1_{E_m}) = f_n.\end{aligned}$$

Therefore, $f_m|_{E_n} = f_n$. The sequence $(f_n)_n$ converges to a measurable $f : X \rightarrow \mathbb{C}$. From $\|f_n\|_\infty \leq \|T\|$ for all $n \in \mathbb{N}$ we also deduce $\|f\|_\infty \leq T$, so $f \in L^\infty(X, \mu)$. Lastly, we prove $T = M_f$. Note that simple functions $\sum_{j=i}^r \alpha_j 1_{A_j}$ are $L^2(X, \mu)$ -dense. Let $A \subseteq X$ be measurable with $\mu(A) < \infty$. Then $\|1_{A \cap E_n} - 1_A\|_2 \xrightarrow{n \rightarrow \infty} 0$. Hence

$$\|(T - M_f)1_A\|_2 = \lim_{n \rightarrow \infty} \|(T - M_f)1_{A \cap E_n}\|_2 = 0,$$

as we shall prove.

$$\begin{aligned} T(1_{A \cap E_n}) &= T(1_A \cdot 1_{E_n}) = T(M_{1_A} 1_{E_n}) \\ &= M_{1_A}(T(1_{E_n})) = M_{1_A}(f_n) \\ &= 1_A \cdot f_n. \end{aligned}$$

On the other hand,

$$M_f(1_{A \cap E_n}) = f \cdot 1_{A \cap E_n} = f \cdot 1_{E_n} \cdot 1_A = 1_A \cdot f_n$$

and we are done. \square

Another possible characterization of vNa's is given by the following.

Theorem 5.14 (Sakai).

Let A be a C^ -algebra such that for a Banach space E there exists an isometric isomorphism $A \rightarrow E^*$. Then there exists a vNa $B \subseteq \mathcal{B}(\mathcal{H})$ such that $A \cong B$ as a C^* -algebra.*

For the proof, see the expository article [3].

5.2 Kaplansky's density theorem

Lemma 5.15. *The multiplication $(A, B) \mapsto A \cdot B$ is SOT-continuous on bounded sets.*

Proof. Let $(A_i)_i$ and $(B_i)_i$ be nets with $\sup \|A_i\|, \sup \|B_i\| < M$ for some $M \in \mathbb{R}$. Suppose $A_i \rightarrow A$ and $B_i \rightarrow B$ in SOT. For any x , we get

$$\begin{aligned} \|ABx - A_i B_i x\| &= \|ABx - A_i Bx + A_i Bx - A_i B_i x\| \\ &\leq \|ABx - A_i Bx\| + \|A_i Bx - A_i B_i x\| \\ &\leq \|A(Bx) - A_i(Bx)\| + \|A_i\| \cdot \|Bx - B_i x\| \\ &\leq \|A(Bx) - A_i(Bx)\| + M \cdot \|Bx - B_i x\| \rightarrow 0, \end{aligned}$$

so $A_i B_i \xrightarrow{\text{SOT}} AB$. \square

Proposition 5.16. *Let $f \in C(\mathbb{C})$. Then $x \mapsto f(x)$ is SOT-continuous on each bounded set of normal operators in $\mathcal{B}(\mathcal{H})$.*

Proof. By Stone–Weierstrass, we can uniformly approximate f by polynomials on a bounded subset $B_R(0) \subseteq \mathbb{C}$. By the previous lemma, multiplication is SOT-continuous on this bounded set of normal operators. But for a normal operator A , we have $\|Ax\| = \|A^*x\|$ for every $x \in \mathcal{H}$, so $*$ is also SOT-continuous on normal operators and we’re done. \square

Theorem 5.17 (Cayley transform).

Mapping $x \mapsto (x - i)(x + i)^{-1}$ is SOT-continuous $\mathcal{B}(\mathcal{H})_{\text{sa}} \rightarrow \mathcal{U}(\mathcal{H})$.

Proof. If $x \in \mathcal{B}(\mathcal{H})_{\text{sa}}$, then $\sigma(x) \subseteq \mathbb{R}$ and $(x + i) \in \mathcal{B}(\mathcal{H})$ is invertible. We notice that $z \mapsto \frac{z-i}{z+i} : \mathbb{R} \rightarrow \mathbb{C}$ has its range in \mathbb{T} , so the Cayley transform does in fact map into the unitaries. Now onto the SOT-continuity: let $(x_k)_k$ be a net in $\mathcal{B}(\mathcal{H})_{\text{sa}}$ with $x_k \rightarrow x$ in SOT. By the spectral mapping theorem, $\|(x_k + i)^{-1}\| \leq 1$. For each $\alpha \in \mathcal{H}$, we have

$$\begin{aligned} \|(x - i)(x + i)^{-1}\alpha - (x_k - i)(x_k + i)^{-1}\alpha\| &= \|(x_k + i)^{-1}((x_k + i)(x - i)(x + i)^{-1} - (x_k - i))\alpha\| \\ &= \|(x_k + i)^{-1}((x_k + i)(x - i) - (x_k - i)(x + i))(x + i)^{-1}\alpha\| \\ &= \|(x_k + i)^{-1}2i(x - x_k)(x + i)^{-1}\alpha\| \\ &\leq 2\|(x_k + i)^{-1}\|\underbrace{\|(x - x_k)(x + i)^{-1}\alpha\|}_{\beta} \\ &\leq 2\|(x - x_k)\beta\| \rightarrow 0. \end{aligned} \quad \square$$

Corollary 5.18. *If $f \in C_0(\mathbb{R})$, then $x \mapsto f(x)$ is SOT-continuous on $\mathcal{B}(\mathcal{H})_{\text{sa}}$.*

Proof. Consider the continuous function

$$g(t) = \begin{cases} f\left(i\frac{1+t}{1-t}\right); & t \neq 1 \\ 0; & t = 1 \end{cases}$$

which maps $\mathbb{T} \rightarrow \mathbb{C}$. By the previous proposition, $x \mapsto g(x)$ is SOT-continuous on unitaries. Letting $U(z) = \frac{z-i}{z+i}$, denote the Cayley transform, we have that $f = g \circ U$ is a composite of two SOT-continuous maps, which is a SOT-continuous map of itself. \square

Theorem 5.19 (Kaplansky’s density theorem).

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $$ -subalgebra and $B = \overline{A}^{\text{SOT}}$, then*

- (1.) $\overline{A_{\text{sa}}}^{\text{SOT}} = B_{\text{sa}};$
- (2.) $\overline{(A)_1}^{\text{SOT}} = (B)_1.$

Proof. W.l.o.g. A is a C^* -algebra, so norm-closed.

(1.) First we prove that $\overline{A_{\text{sa}}}^{\text{SOT}} \subseteq B_{\text{sa}}$. Since $\overline{A_{\text{sa}}}^{\text{SOT}} = \overline{A_{\text{sa}}}^{\text{WOT}}$, take $x \in \overline{A_{\text{sa}}}^{\text{SOT}}$ and a net

$(x_k)_k \subseteq A_{\text{sa}}$ that converges to x . Since $*$ is WOT continuous, $(x_k^*)_k = (x_k)_k$ converge to x^* , so $x = x^*$. Now the converse inclusion: suppose the net $(x_k)_k$ SOT-converges to $x \in B_{\text{sa}}$. Then $\frac{x_k + x_k^*}{2} \rightarrow x$ in the WOT-topology, which implies

$$B_{\text{sa}} \subseteq \overline{A_{\text{sa}}}^{\text{WOT}} = \overline{A_{\text{sa}}}^{\text{SOT}}.$$

- (2.) Suppose the net $(y_i)_i$ in A_{sa} SOT-converges to $x \in B_{\text{sa}}$. Take $f \in C_0(\mathbb{R})$ such that we have $f(t) = t$, $\forall |t| \leq \|x\|$ and $|f(t)| \leq \|x\|$, $\forall t \in \mathbb{R}$. By functional calculus, $\|f(y_k)\| \leq \|x\|$. By the previous corollary, $(f(y_i))_i \xrightarrow{\text{SOT}} f(x) = x$. This proves that $(A)_1 \cap A_{\text{sa}}$ is SOT-dense in $(B)_1 \cap B_{\text{sa}}$. Pass over to $M_2(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then $M_2(A)$ is SOT-dense in $M_2(B)$ by the first part of the proof. For $x \in (B)_1$, we have

$$\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (M_2(B))_1 \cap (M_2(B))_{\text{sa}}.$$

That means there exists a net

$$\tilde{x}_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in (M_2(A))_1$$

such that $\tilde{x}_i \rightarrow \tilde{x}$ and therefore $b_i \in (A)_1$ SOT-converge to x . □

Corollary 5.20. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$ -algebra. Then A is a vNa iff $(A)_1$ is SOT-closed.*

5.3 Examples of vNa's

Definition 5.21. A vNa M is called a *factor* if $Z(M) = M \cap M' = \mathbb{C} \cdot 1$.

Example 5.22. *Clearly, $\mathcal{B}(\mathcal{H})$ is a factor. In particular, $M_n(\mathbb{C})$ is a factor.*

Let Γ be a group and $\mathcal{H} = \ell^2(\Gamma)$. Consider the left regular representation

$$\lambda : \Gamma \rightarrow \mathcal{B}(\ell^2(\Gamma)), \quad g \mapsto (\delta_h \mapsto \delta_{gh})$$

and extend it linearly to $\lambda : \mathbb{C}[\Gamma] \rightarrow \mathcal{B}(\ell^2(\Gamma))$. The group vNa of Γ is $VN(\Gamma) := \lambda(\mathbb{C}[\Gamma])''$ in $\mathcal{B}(\ell^2(\Gamma))$. It has a *trace*, which is defined as the linear functional

$$\tau : VN(\Gamma) \rightarrow \mathbb{C}, \quad x \mapsto \langle x\delta_e, \delta_e \rangle.$$

For $g \in \Gamma$, $\tau(\lambda(g)) = 1$ if $g = e$, otherwise zero. For $g_1, \dots, g_r \in \Gamma$, we have

$$g_1 \dots g_r = e \Leftrightarrow \tau(\lambda(g_1) \dots \lambda(g_r)) = 1.$$

Since τ is a positive linear functional and $\tau(1) = 1$, τ is a state. For any two elements $g, h \in \Gamma$ we have $gh = e \Leftrightarrow hg = e$, which together with the above line implies

$$\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g)).$$

By linearity, τ has the same cyclic property on $\lambda(\mathbb{C}[\Gamma])$. But since τ is, by definition, WOT-continuous and $VN(\Gamma) = (\lambda(\mathbb{C}[\Gamma]))'' = \overline{\lambda(\mathbb{C}[\Gamma])}^{\text{WOT}}$, τ is cyclic on the entire $VN(\Gamma)$. Now if

$|\Gamma| = \infty$, then $VN(\Gamma) \neq \mathcal{B}(\mathcal{H})$, since the latter does not have a trace if $\dim \mathcal{H} = \infty$. If Γ is an abelian group, then $VN(\Gamma)$ is commutative.

Definition 5.23. Group Γ has *icc* (infinite conjugacy classes) if for all $g \in \Gamma \setminus \{e\}$ the set $\{f^{-1}gf \mid f \in \Gamma\}$ is infinite.

Example 5.24. The group

$$S_\infty = \{\text{bijections } \mathbb{N} \rightarrow \mathbb{N} \text{ that only permute finitely many elements}\}$$

has *icc*.

Example 5.25. Free groups \mathbb{F}_n for $n > 1$ have *icc*.

Theorem 5.26.

If Γ has *icc*, then $VN(\Gamma)$ is a factor.

Definition 5.27. $VN(S_\infty) =: R$ is the *hyperfinite II_1 -factor*.

Open problem: does $VN(\mathbb{F}_2) \cong VN(\mathbb{F}_3)$ hold?

5.4 Operations with vNa's

5.4.1 Direct sums

Let $M_i \subseteq \mathcal{B}(\mathcal{H}_i)$ be vNa's. Define the isometric embedding

$$\iota_j : \mathcal{B}(\mathcal{H}_j) \rightarrow \mathcal{B}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n), \quad x \mapsto ((\alpha_1, \dots, \alpha_n) \mapsto (0, \dots, 0, x\alpha_j, 0, \dots, 0)).$$

This map is the $n \times n$ bounded matrix where the (j, j) -th element is x and the rest are zero. Then

$$M_1 \oplus \cdots \oplus M_n := \text{span}\{\iota_j(x) \mid j = 1, \dots, n, x \in M_j\}$$

is the direct sum of vNa's. If $n \geq 2$, then from

$$Z(M_1 \oplus \cdots \oplus M_n) = Z(M_1) \oplus \cdots \oplus Z(M_n),$$

we deduce that $M_1 \oplus \cdots \oplus M_n$ is not a factor.

5.4.2 Tensor products

The algebraic tensor product $\mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n)$ acts on $\mathcal{H}_1 \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_n$ by

$$(x_1 \otimes \cdots \otimes x_n)(\alpha_1 \otimes \cdots \otimes \alpha_n) = (x_1\alpha_1) \otimes \cdots \otimes (x_n\alpha_n)$$

for $x_j \in \mathcal{B}(\mathcal{H}_j)$ and $\alpha_j \in \mathcal{H}_j$, which implies

$$\mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n) \subseteq \mathcal{B}(\mathcal{H}_1 \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_n).$$

Finally, we define the tensor product of vNa's as

$$M_1 \bar{\otimes} \cdots \bar{\otimes} M_n = (M_1 \otimes \cdots \otimes M_n)'' \cap \mathcal{B}(\mathcal{H}_1 \bar{\otimes} \cdots \bar{\otimes} \mathcal{H}_n).$$

5.4.3 Compressions

Definition 5.28. Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $p \in \mathcal{B}(\mathcal{H})$ a projection. A compression of M is $pMp = \{pxp \mid x \in M\}$. When $p \in M$, it is also called a corner.

If $\mathcal{H} = \text{im } p \oplus (\text{im } p)^\perp = \text{im } p \oplus \text{im}(1 - p)$. In this basis, elements of pMp have the matrix form

$$\begin{bmatrix} pxp & 0 \\ 0 & 0 \end{bmatrix}.$$

If $M \ni p \neq 1$, then pMp is a $*$ -algebra and $pMp \subseteq M$ but it is not a subalgebra since $1_M = 1_{\mathcal{B}(\mathcal{H})} \notin pMp$. However, pMp is a subalgebra of $\mathcal{B}(p\mathcal{H})$ with identity element p .

Definition 5.29. Let $\mathcal{K} \subseteq \mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$.

- (1.) \mathcal{K} is invariant under x if $x\mathcal{K} \subseteq \mathcal{K}$;
- (2.) \mathcal{K} is reducing under x if \mathcal{K} is invariant under both x and x^* .

Now if $S \subseteq \mathcal{B}(\mathcal{H})$, then

- (1.) \mathcal{K} is invariant under S if $x\mathcal{K} \subseteq \mathcal{K}$ under all $x \in S$;
- (2.) \mathcal{K} is reducing under S if \mathcal{K} is reducing under all $x \in S$.

If $S \subseteq \mathcal{B}(\mathcal{H})$ is closed under $*$, then \mathcal{K} is invariant under S iff it is reducing under S . The following lemma was proved in the introductory course.

Lemma 5.30. Let $\mathcal{K}^{\text{closed}} \leq \mathcal{H}$ and $M \subseteq \mathcal{B}(\mathcal{H})$ an $*$ -algebra. Let $p : \mathcal{H} \rightarrow \mathcal{K}$ be the orthogonal projection. Then \mathcal{K} is reducing under M iff $p \in M'$.

Theorem 5.31.

Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $p \in M$ a projection. Then pMp and $M'p$ are vNa's in $\mathcal{B}(p\mathcal{H})$.

Proof. We will show that

$$(M'p)' \cap \mathcal{B}(p\mathcal{H}) = pMp, \quad (pMp)' \cap \mathcal{B}(p\mathcal{H}) = M'p.$$

Then the bicommutant theorem will take care of the rest. It is obvious that $(M'p)' \cap \mathcal{B}(p\mathcal{H}) \supseteq pMp$. For the converse, pick $x \in (M'p)' \cap \mathcal{B}(p\mathcal{H})$. Define $\tilde{x} = xp = px \in \mathcal{B}(\mathcal{H})$. For $y \in M'$, we have

$$y\tilde{x} = ypx = xyp = xpy = \tilde{x}y,$$

which implies $\tilde{x} \in M'' = M$. Then $x = pxp = p\tilde{x}p \in pMp$. As before, $(pMp)' \cap \mathcal{B}(p\mathcal{H}) \supseteq M'p$ is trivial and we just prove the converse. Take $y \in (pMp)' \cap \mathcal{B}(p\mathcal{H})$. Using continuous functional calculus, we can write y as a linear combinations of 4 unitaries. Since pMp is closed under $*$, $(pMp)'$ is a vNa (and therefore a C^* -algebra). So we can assume w.l.o.g. that $y = u$ a unitary. Set $\mathcal{K} := \overline{Mp\mathcal{H}}$ and let $q : \mathcal{H} \rightarrow \mathcal{K}$ be the orthogonal projection. Since \mathcal{K} is reducing under M and M' , which implies

$$q \in M' \cap M'' = M' \cap M = Z(M).$$

Next, we extend u to \mathcal{K} :

$$\tilde{u}\left(\sum_i x_i \underbrace{p}_{\in M} \underbrace{\alpha_i}_{\in \mathcal{H}}\right) = \sum_i x_i u p \alpha_i.$$

We shall show that this is a well-defined isometry in $Mp\mathcal{H}$:

$$\begin{aligned} \|\tilde{u} \sum_i x_i p \alpha_i\|^2 &= \sum_{i,j} \langle x_i u p \alpha_i, x_j u p \alpha_j \rangle \\ &= \sum_{i,j} \langle (p x_j^* x_i p) u \alpha_i, u \alpha_j \rangle \\ &= \sum_{i,j} \langle u p x_j^* x_i p \alpha_i, u \alpha_j \rangle \\ &= \sum_{i,j} \langle p x_j^* x_i p \alpha_i, \alpha_j \rangle = \|\sum_i x_i p \alpha_i\|^2. \end{aligned}$$

So \tilde{u} extends to an isometry on $\mathcal{K} = \overline{Mp\mathcal{H}}$. By definition, \tilde{u} commutes with M on $Mp\mathcal{H}$, so also on \mathcal{K} . Thus for every $x \in M$ and $\alpha \in \mathcal{H}$, we have

$$x(\tilde{u}q)\alpha = \tilde{u}xq\alpha = (\tilde{u}q)x\alpha,$$

which implies $\tilde{u}q \in M' \cap \mathcal{B}(\mathcal{H})$. Then

$$\tilde{u}q p \alpha = \tilde{u}1 p \alpha = 1 u p \alpha,$$

which implies $u = \tilde{u}q p \in \mathcal{B}(\mathcal{H})$ and $u \in M'p$. □

Corollary 5.32. *Suppose the vNa $M \subseteq \mathcal{B}(\mathcal{H})$ is a factor and let $p \in M$ be a projection. Then pMp and $M'p$ are factors (in $\mathcal{B}(p\mathcal{H})$).*

Proof. Let $\mathcal{K} = \overline{Mp\mathcal{H}}$ and $q : \mathcal{H} \rightarrow \mathcal{K}$ the projection. From the previous proof, $q \in Z(M) = \mathbb{C}$. Then $q \in \{0, 1\}$. w.l.o.g. $p \neq 0$, so $q = 1$. Thus $\mathcal{K} = \mathcal{H}$, so $Mp\mathcal{H}$ is dense in \mathcal{H} . Consider

$$\psi : M' \rightarrow M'p, \quad y \mapsto yp.$$

We will prove that ψ is an isomorphism of algebras. Obviously, it is additive. Since

$$\psi(xy) = xyp = xyp^2 = xpyp = \psi(x)\psi(y),$$

it is also multiplicative. Same calculation shows $\psi(y^*) = \psi(y)^*$. Obviously, ψ is surjective. Finally, we prove injectivity. Suppose $y \in M'$ satisfies $yp = 0$. Then for every $x \in M$ and $\alpha \in \mathcal{H}$, we get $yx p \alpha = x(yp)\alpha = 0$. Hence $y|_{Mp\mathcal{H}} = 0$, so by continuity, $y|_{\overline{Mp\mathcal{H}}} = y|_{\mathcal{K}} = 0$. But because $\mathcal{K} = \mathcal{H}$, this yields $y|_{\mathcal{H}} = 0$. As a result, we get

$$Z(M'p) = Z(M')p = \mathbb{C} \cdot p,$$

so $M'p$ is a factor. Similarly,

$$Z(pMp) = (pMp) \cap (pMp)' = (M'p)' \cap M'p = Z(M'p) = \mathbb{C}p,$$

so pMp is a factor. □

6 Spectral theorem and Borel functional calculus

6.1 Spectral theorem

Recall the spectral theorem for compact operators.

Theorem 6.1 (Spectral theorem for compact operators).

If $T \in \mathcal{K}(\mathcal{H})$ is self-adjoint, then T has only a countable number of distinct eigenvalues, where each nonzero eigenvalue has finite multiplicity. If $\{\lambda_1, \lambda_2, \dots\}$ are the distinct eigenvalues of T , and P_n is the projection of \mathcal{H} onto $\ker(T - \lambda_n)$, then $P_n P_m = 0$ for $n \neq m$ and

$$T = \sum_{n=1}^{\infty} \lambda_n P_n.$$

Our first goal is to generalize this result to non-compact self-adjoint operators.

Theorem 6.2 (Vigier).

Let (u_λ) be a net of increasing (decreasing) and bounded above (below) self-adjoint operators on \mathcal{H} . Then (u_λ) converges.

Proof. We prove the statement for an increasing net that is bounded above. We can assume (u_λ) has a lower bound m by considering a truncated net. Without loss of generality, we assume u_λ is positive (otherwise, consider $u_\lambda - m$). There exists $M \geq 0$ such that $\|u_\lambda\| \leq M$ for all indices λ . So the net $\langle u_\lambda x, x \rangle$ is real, increasing, and bounded above by $M\|x\|^2$. Using the polarization identity,

$$\langle u_\lambda x, x \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \langle u_\lambda (x + i^k y), x + i^k y \rangle,$$

we see that $\langle u_\lambda x, y \rangle$ is a convergent net for all $x, y \in \mathcal{H}$. Letting $\sigma(x, y)$ denote its limit, we can easily check that σ is a bounded sesquilinear form ($|\sigma(x, y)| \leq M\|x\|\|y\|$). By Riesz's representation theorem, there exists an operator $u \in \mathcal{B}(\mathcal{H})$ such that $\langle ux, y \rangle = \sigma(x, y)$. Then u is self-adjoint, $\|u\| \leq M$, and $u_\lambda \leq u$. Note that

$$\begin{aligned} \|(u - u_\lambda)x\|^2 &\leq \|(u - u_\lambda)^{\frac{1}{2}}(u - u_\lambda)^{\frac{1}{2}}x\|^2 \\ &\leq \|(u - u_\lambda)\| \|(u - u_\lambda)^{\frac{1}{2}}x\|^2 \\ &\leq 2M \langle (u - u_\lambda)x, x \rangle \rightarrow 0, \end{aligned}$$

so u_λ converges strongly to u . □

Remark. If (p_λ) is a net of projections converging strongly to some operator u , then u is also a projection. Clearly, u is self-adjoint, and

$$\begin{aligned} \langle ux, y \rangle &= \lim_{\lambda} \langle p_\lambda x, y \rangle = \lim_{\lambda} \langle p_\lambda x, p_\lambda y \rangle \\ &= \langle ux, uy \rangle = \langle u^2 x, y \rangle, \end{aligned}$$

therefore, $u^2 = u$.

Corollary 6.3. *If $(p_n)_{n \in \mathbb{N}}$ is a sequence of pairwise orthogonal projections in $\mathcal{B}(\mathcal{H})$, then $(\sum_{n=1}^N p_n)$ SOT-converges as $N \rightarrow \infty$ (we denote the limit by $\sum_{n=1}^{\infty} p_n$).*

Definition 6.4. Let X be a set, Ω a σ -algebra in X , and \mathcal{H} a Hilbert space. A *projection-valued measure* (PVM) for (X, Ω, \mathcal{H}) is a map $E : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ such that

- (1.) $E(S)$ is a projection for all $S \in \Omega$;
- (2.) $E(\emptyset) = 0$ and $E(X) = 1$;
- (3.) $E(S \cap T) = E(S)E(T)$ for all $S, T \in \Omega$;
- (4.) If $(S_n)_{n \in \mathbb{N}} \subseteq \Omega$ is a sequence of pairwise disjoint sets, then

$$E\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} E(S_n).$$

Remark. The projections $E(S)$ commute with each other, which follows directly from the third point of the definition.

Example 6.5. Let (X, Ω, μ) be a σ -finite measure space. Let $\mathcal{H} = L^2(X, \mu)$, and for $S \in \Omega$, define $E(S) := \chi_S \in \mathcal{B}(L^2(X, \mu))$. Then $E : \Omega \rightarrow \mathcal{B}(L^2(X, \mu))$ is a PVM.

Lemma 6.6. Let E be a PVM for (X, Ω, \mathcal{H}) . Then, for all $\alpha, \beta \in \mathcal{H}$, the mapping

$$E_{\alpha, \beta} : \Omega \rightarrow \mathbb{C}, \quad S \mapsto \langle E(S)\alpha, \beta \rangle$$

is a complex measure on Ω with total variation $\leq \|\alpha\| \|\beta\|$.

Proof. Let $\alpha, \beta \in \mathcal{H}$. Then $E_{\alpha, \beta}$ is σ -additive (i.e., countably additive for disjoint sets) since E is σ -additive by (4). The total variation of a complex measure is given by

$$\|E_{\alpha, \beta}\| := \sup \left\{ \sum_{S \in \pi} |E_{\alpha, \beta}(S)| \right\},$$

where the sum is taken over all partitions of X into finitely many measurable sets. Let $\pi = \{S_1, \dots, S_n\}$ be a partition of X with $S_j \in \Omega$. For each j , pick $\alpha_j \in \mathbb{C}$ such that $|\alpha_j| = 1$ and

$$\alpha_j \cdot E_{\alpha, \beta}(S_j) = \alpha_j \langle E(S_j)\alpha, \beta \rangle = |\langle E(S_j)\alpha, \beta \rangle| = |E_{\alpha, \beta}(S_j)|.$$

Then,

$$\sum_{j=1}^n |E_{\alpha, \beta}(S_j)| = \left| \left\langle \sum_{j=1}^n \alpha_j E(S_j)\alpha, \beta \right\rangle \right| \leq \left\| \sum_{j=1}^n \alpha_j E(S_j)\alpha \right\| \cdot \|\beta\|.$$

For $i \neq j$, we have

$$E(S_i)E(S_j) = E(S_i \cap S_j) = E(\emptyset) = 0,$$

so $E(S_i)$ and $E(S_j)$ are pairwise orthogonal. Finally, applying the Pythagorean theorem, we get

$$\begin{aligned}
\left\| \sum_{j=1}^n \alpha_j E(S_j) \alpha \right\|^2 &= \sum_{j=1}^n \|E(S_j) \alpha\|^2 \\
&= \left\| \sum_{j=1}^n E(S_j) \alpha \right\|^2 \\
&= \left\| E \left(\bigcup_{j=1}^n S_j \right) \alpha \right\|^2 \\
&= \|E(X) \alpha\|^2 = \|\alpha\|^2.
\end{aligned}$$

□

Remark. Let E be a PVM for (X, Ω, \mathcal{H}) , and let $\alpha \in \mathcal{H}$ and $S \in \Omega$. Then,

$$\begin{aligned}
E_{\alpha, \alpha}(S) &= \langle E(S) \alpha, \alpha \rangle \\
&= \langle E(S)^2 \alpha, \alpha \rangle \\
&= \langle E(S) \alpha, E(S) \alpha \rangle \geq 0,
\end{aligned}$$

so $E_{\alpha, \alpha}$ is a positive measure on X . Furthermore, if $\|\alpha\| = 1$, then $E_{\alpha, \alpha}$ is a probability measure.

Define

$$(\alpha, \beta) \mapsto \int_X 1 dE_{\alpha, \beta}.$$

Since

$$E_{\alpha + \lambda \alpha', \beta} = E_{\alpha, \beta} + \lambda E_{\alpha', \beta}$$

and

$$E_{\alpha, \beta + \lambda \beta'} = E_{\alpha, \beta} + \bar{\lambda} E_{\alpha, \beta'},$$

the above defines a sesquilinear form on \mathcal{H} . In particular, it is bounded:

$$\left\| \int_X dE_{\alpha, \beta} \right\| \leq \|E_{\alpha, \beta}\| \leq \|\alpha\| \|\beta\|.$$

Suppose $f : X \rightarrow \mathbb{C}$ is a bounded Ω -measurable function. Then

$$(\alpha, \beta) \mapsto \int_X f dE_{\alpha, \beta}$$

defines a bounded sesquilinear form:

$$\left\| \int_X f dE_{\alpha, \beta} \right\| \leq \|f\|_\infty \|E_{\alpha, \beta}\| \leq \|f\|_\infty \|\alpha\| \|\beta\|.$$

So there exists an $x \in \mathcal{B}(\mathcal{H})$ such that $\|x\| \leq \|f\|_\infty$ and

$$\langle x \alpha, \beta \rangle = \int_X f dE_{\alpha, \beta}.$$

If $f = \chi_S$ for $S \in \Omega$, then $x = E(S)$, i.e.,

$$\int_X \chi_S dE_{\alpha, \beta} = E_{\alpha, \beta}(S) = \langle E(S) \alpha, \beta \rangle.$$

Definition 6.7. Let E be a PVM for (X, Ω, \mathcal{H}) , and let $f : X \rightarrow \mathbb{C}$ be a bounded Ω -measurable function. We call $x \in \mathcal{B}(\mathcal{H})$ the *integral of f with respect to E* if

$$\langle x\alpha, \beta \rangle = \int_X f dE_{\alpha, \beta}, \quad \forall \alpha, \beta \in \mathcal{H}.$$

We denote it by

$$x := \int_X f dE.$$

Remark. Define $B(X, \Omega)$ as the set of all bounded Ω -measurable complex functions on X , endowed with the supremum norm. If X is a topological space and $\Omega = \mathcal{B}_X$ is the Borel σ -algebra on X , then $B(X) = B(X, \mathcal{B}_X)$.

Proposition 6.8. Let E be a PVM for (X, Ω, \mathcal{H}) . Then, the mapping

$$\Phi : B(X, \Omega) \rightarrow \mathcal{B}(\mathcal{H}), \quad f \mapsto \int_X f dE$$

is a $*$ -homomorphism and contractive. Furthermore:

- (1.) If $(f_n)_n \subseteq B(X, \Omega)$ is an increasing sequence of nonnegative functions and $f = \sup_n f_n \in B(X, \Omega)$, then $\int_X f_n dE \rightarrow \int_X f dE$ in SOT.
- (2.) If X is compact and T_2 , then $\Phi(B(X)) \subseteq \Phi(C(X))''$.

Proof. We already saw that $\|\Phi(f)\| \leq \|f\|_\infty$; hence, Φ is contractive. It is also clear that Φ is linear and that $\Phi(f)^* = \Phi(\bar{f})$. Next, we prove multiplicativity: $\Phi(\chi_S) = E(S)$ for $S \in \Omega$. Then,

$$\Phi(\chi_S) \cdot \Phi(\chi_T) = E(S) \cdot E(T) = E(S \cap T) = \Phi(\chi_{S \cap T}) = \Phi(\chi_S \cdot \chi_T).$$

Since Φ is linear, it is also multiplicative on simple functions (which are finite linear combinations of characteristic functions). Since each $f \in B(X, \Omega)$ is a uniform limit of a uniformly bounded sequence of simple functions, we deduce that $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in B(X, \Omega)$.

- (1.) Let f, f_n be as in the statement. Since Φ is a $*$ -homomorphism, $(\Phi(f_n))_n$ is an increasing sequence of positive operators, and $\sup_n \|\Phi(f_n)\| \leq \sup_n \|f_n\|_\infty = \|f\|$. By Vigier, there exists $x \in \mathcal{B}(\mathcal{H})$ such that $\Phi(f_n) \xrightarrow{\text{SOT}} x$. This x is a natural candidate for $\Phi(f)$. Indeed, for $\alpha, \beta \in \mathcal{H}$, we have

$$\begin{aligned} \langle \Phi(f)\alpha, \beta \rangle &= \int_X f dE_{\alpha, \beta} \\ &= \lim_{n \rightarrow \infty} \int_X f_n dE_{\alpha, \beta} \\ &= \lim_{n \rightarrow \infty} \langle \Phi(f_n)\alpha, \beta \rangle, \end{aligned}$$

so $\Phi(f_n) \xrightarrow{\text{WOT}} \Phi(f)$, and therefore $\Phi(f) = x$.

- (2.) Let X be compact Hausdorff and $a \in \Phi(C(X))'$. Take $\alpha, \beta \in \mathcal{H}$. Then, for all

$f \in C(X)$, we have

$$\begin{aligned} 0 &= \langle (a\Phi(f) - \Phi(f)a)\alpha, \beta \rangle \\ &= \langle \Phi(f)\alpha, a^*\beta \rangle - \langle \Phi(f)(a\alpha), \beta \rangle \\ &= \int_X f dE_{\alpha, a^*\beta} - \int_X f dE_{a\alpha, \beta}, \end{aligned}$$

so by uniqueness from Riesz–Markoff, we get $E_{\alpha, a^*\beta} = E_{a\alpha, \beta}$. Reversing this calculation shows that a commutes with all $\Phi(g) = \int_X g dE$ for $g \in B(X)$, so $\Phi(B(X)) \subseteq \Phi(C(X))''$. \square

Remark. The map Φ is not necessarily isometric. However, it induces an isometric $*$ -isomorphism $\tilde{\Phi} : L^\infty(X, \mu) \rightarrow \mathcal{B}(\mathcal{H})$.

Recall that for an abelian C^* -algebra A , the Gelfand transform

$$\Gamma : A \rightarrow C(\sigma(A))$$

is an isometric $*$ -isomorphism.

Theorem 6.9 (Spectral theorem).

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be an abelian C^* -algebra, and let $\mathcal{B}_{\sigma(A)}$ be the Borel σ -algebra on $\sigma(A)$. Then, there exists a PVM E for $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$ such that

$$x = \int_{\sigma(A)} \Gamma(x) dE$$

for all $x \in A$.

Proof. For all $\alpha, \beta \in \mathcal{H}$, define

$$\varphi : C(\sigma(A)) \rightarrow \mathbb{C}, \quad f \mapsto \langle \Gamma^{-1}(f)\alpha, \beta \rangle.$$

This is a bounded linear functional. Indeed, since Γ is an isometry, we get

$$\langle \Gamma^{-1}(f)\alpha, \beta \rangle \leq \|f\|_\infty \|\alpha\| \|\beta\|.$$

By the Riesz–Markoff theorem, there exists a unique regular Borel measure $\mu_{\alpha, \beta}$ such that

$$\langle \Gamma^{-1}(f)\alpha, \beta \rangle = \int_{\sigma(A)} f d\mu_{\alpha, \beta}.$$

We will show that $\mu_{\alpha, \beta} = E_{\alpha, \beta}$ for a PVM E . For $f, g \in C(\sigma(A))$, we have

$$\int_{\sigma(A)} fg d\mu_{\alpha, \beta} = \langle \Gamma^{-1}(fg)\alpha, \beta \rangle = \langle \Gamma^{-1}(f)\Gamma(g)\alpha, \beta \rangle = \int_{\sigma(A)} f d\mu_{\Gamma^{-1}(g)\alpha, \beta}.$$

This is also equal to

$$\langle \Gamma^{-1}(f)\alpha, \Gamma^{-1}(\bar{g})\beta \rangle = \int_{\sigma(A)} f d\mu_{\alpha, \Gamma^{-1}(\bar{g})\beta}.$$

By the uniqueness in Riesz–Markoff, we obtain

$$g d\mu_{\alpha,\beta} = d\mu_{\Gamma^{-1}(g)\alpha,\beta} = d\mu_{\alpha,\Gamma^{-1}(\bar{g})\beta}.$$

Finally, we have

$$\begin{aligned} \int_{\sigma(A)} f d\overline{\mu_{\alpha,\beta}} &= \overline{\int_{\sigma(A)} \bar{f} d\mu_{\alpha,\beta}} \\ &= \overline{\langle \Gamma^{-1}(\bar{f})\alpha, \beta \rangle} \\ &= \overline{\langle \alpha, \Gamma^{-1}(f)\beta \rangle} \\ &= \langle \Gamma^{-1}(f)\beta, \alpha \rangle \\ &= \int_{\sigma(A)} f d\mu_{\beta,\alpha} \end{aligned}$$

for all $f \in C(\sigma(A))$, which implies $\overline{\mu_{\alpha,\beta}} = \mu_{\beta,\alpha}$. To each $S \in \mathcal{B}_{\sigma(A)}$, we assign the sesquilinear form

$$\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_{\sigma(A)} \chi_S d\mu_{\alpha,\beta}.$$

This form is bounded by $\|\alpha\|\|\beta\| = \|\mu_{\alpha,\beta}\|$. Thus, there exists $E(S) \in \mathcal{B}(\mathcal{H})$ such that

$$\int_{\sigma(A)} \chi_S d\mu_{\alpha,\beta} = \langle E(S)\alpha, \beta \rangle.$$

Now notice that

$$\begin{aligned} \langle E(S)^*\alpha, \beta \rangle &= \langle \alpha, E(S)\beta \rangle \\ &= \overline{\langle E(S)\beta, \alpha \rangle} \\ &= \overline{\int_{\sigma(A)} \chi_S d\mu_{\beta,\alpha}} \\ &= \int_{\sigma(A)} \chi_S d\overline{\mu_{\beta,\alpha}} \\ &= \int_{\sigma(A)} \chi_S d\mu_{\alpha,\beta} \\ &= \langle E(S)\alpha, \beta \rangle, \end{aligned}$$

so $E(S) = E(S)^*$. We now show that E is a projection-valued measure. Using the weak-* density of $C(\sigma(A))$ in $B(\sigma(A))$, we obtain

$$E(S)E(T) = E(S \cap T).$$

This shows that $E(S)$ is a projection. Further,

$$E(\sigma(A)) = 1,$$

and since $\mu_{\alpha,\beta}$ is σ -additive, we conclude that

$$E\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} E(S_i)$$

for pairwise disjoint sets $(S_i)_i$. This proves that E is a PVM. Finally, for all $\alpha, \beta \in \mathcal{H}$, we get

$$E_{\alpha, \beta}(S) = \langle E(S)\alpha, \beta \rangle = \mu_{\alpha, \beta}(S),$$

so $\mu_{\alpha, \beta} = E_{\alpha, \beta}$, and

$$\int f dE_{\alpha, \beta} = \langle \Gamma^{-1}(f)\alpha, \beta \rangle.$$

This proves that

$$x = \int_{\sigma(A)} \Gamma(x) dE \quad \forall x \in A.$$

Uniqueness: Suppose E' is another such PVM. Then, for all $f \in C(\sigma(A))$,

$$\int_{\sigma(A)} f dE_{\alpha, \beta} = \langle \Gamma^{-1}(f)\alpha, \beta \rangle = \int_{\sigma(A)} f dE'_{\alpha, \beta}.$$

Thus, $E_{\alpha, \beta} = E'_{\alpha, \beta}$ in $C(\sigma(A))^*$, which implies $E = E'$. \square

Lemma 6.10 (Goldstine's theorem). *Let X be a Banach space. Then the image of*

$$\iota : X \rightarrow X^{**}, \quad x \mapsto (f \mapsto f(x))$$

is dense in the weak- topology.*

Proof. Let $\beta \in X^{**}$ and $f_1, \dots, f_r \in X^*$ (w.l.o.g. linearly independent). The set

$$U = \{\alpha \in X^{**} \mid |(\alpha - \beta)(f_j)| < \varepsilon, j = 1, \dots, r\}$$

is a basic weak-* open set in X^{**} . Assume X is infinite-dimensional. Consider the surjective linear map

$$\Phi : X \rightarrow \mathbb{C}^r, \quad x \mapsto (f_1(x), \dots, f_r(x)).$$

Then, there exists $x_0 \in X$ such that

$$\Phi(x_0) = (\beta(f_1), \dots, \beta(f_r)),$$

hence $\iota(x_0) \in U \cap \iota(X)$. \square

6.2 Borel functional calculus

Let $x \in \mathcal{B}(\mathcal{H})$ be normal and $A := C^*(x) \subseteq \mathcal{B}(\mathcal{H})$ an abelian C^* -algebra, generated by x . By the spectral theorem, there exists a PVM E for $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$ and

$$\Phi : B(\sigma(A)) \rightarrow \mathcal{B}(\mathcal{H}), \quad f \mapsto \int_{\sigma(A)} f dE$$

is a *-homomorphism and a contraction by proposition 6.8. Since $\sigma(x) = \sigma_{C^*(x)}$, we have the homeomorphism $\tau : \sigma(C^*(x)) \rightarrow \sigma(x)$, which induces the isometric *-isomorphism

$$\tau^\# : B(\sigma(x)) \rightarrow B(C^*(x)), \quad f \mapsto f \circ \tau.$$

We can now define a map

$$\rho = \Phi \circ \tau^\# : B(\sigma(x)) \rightarrow \mathcal{B}(\mathcal{H}),$$

called the *Borel functional calculus*.

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H}) & \xleftarrow{\Phi} & C(\sigma(A)) \\ & \nwarrow \rho \quad \nearrow \tau^\# & \\ & C(\sigma(x)) & \end{array}$$

As with the continuous functional calculus, we will employ the notation $f(x) := \rho(f)$. This notation makes sense because the Borel functional calculus actually extends the continuous functional calculus. Indeed, for $f \in C(\sigma(x))$, we have

$$\rho(f) = \int_{\sigma(x)} f dE = \int_{\sigma(A)} \tau^\#(f) dE = \Gamma^{-1}(\tau^\#(f)) = (\Gamma^{-1} \circ \tau^\#)(f),$$

which implies that ρ , when restricted to $C(\sigma(x))$, coincides with the continuous functional calculus. As a corollary, if $f = \text{id} \in B(\sigma(x))$, then

$$\rho(\text{id}) = \int_{\sigma(x)} \text{id} dE = x.$$

Since ρ extends continuous functional calculus, we have by the second item of proposition 6.8

$$\rho(B(\sigma(x))) = \Phi(B(\sigma(A))) \subseteq \Phi(C(\sigma(A)))'' = \rho(C(\sigma(x)))'' = A''.$$

By the Bicommutant theorem, we have $A'' = W^*(x)$, where $W^*(x)$ is the vNa, generated by x .

Theorem 6.11 (Spectral mapping theorem).

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and let $x \in A$ be normal. Then the Borel functional calculus has the following properties:

(1.) The map

$$B(\sigma(x)) \rightarrow W^*(x), \quad f \mapsto f(x)$$

is a contractive $*$ -homomorphism.

(2.) If $f \in C(\sigma(x))$, then this $f(x)$ coincides with the $f(x)$ from the continuous functional calculus.

(3.) For $f \in B(\sigma(x))$, we have $\sigma(f(x)) \subseteq f(\sigma(x))$.

Proof. The first and second item follow directly from the above discussion. Let us prove the third item. Suppose $\lambda \notin f(\sigma(x))$. Then $f - \lambda \in \mathcal{B}(\sigma(x))$ is invertible in $\mathcal{B}(\sigma(x))$, so there exists $g \in \mathcal{B}(\sigma(x))$ such that $(f - \lambda)g = \text{id}$. By the Borel functional calculus, $(f(x) - \lambda I) \cdot g(x) = I$, so $\lambda \notin \sigma(f(x))$. \square

The spectral theorem is a powerful result that enables us to answer many questions about normal operators. Let us mention some of its consequences.

Corollary 6.12. Every vNa is the norm-closure of the linear span of its projections.

Proof. Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $x \in M$. Using that $\operatorname{Re} x, \operatorname{Im} x \in M_{\text{sa}}$, we may assume without loss of generality that $x \in M_{\text{sa}}$. Hence x is normal, and for all $f \in B(\sigma(x))$, we have $f(x) \in M$. For $S \in \mathcal{B}_{\sigma(x)}$, the characteristic function $\chi_S(x) \in M$ is a projection. Now we can uniformly approximate id on $\sigma(x)$ using simple functions. By the Borel functional calculus, x is uniformly approximated by linear combinations of projections. \square

Remark. There exist C^* -algebras without nontrivial projections. For example, for a compact Hausdorff connected space X , the algebra $C(X)$ only has trivial projections 0 and 1. There also exist nonabelian examples.

We can now prove the statement in example 3.29. We begin with a lemma.

Lemma 6.13. *If $x \in \mathcal{B}(\mathcal{H})$ is a normal operator and $x = \int_{\sigma(x)} z dE$, then N is compact iff for every $\varepsilon > 0$, $E(\{z \mid |z| > \varepsilon\})$ has finite rank.*

Proof. If $\varepsilon > 0$, then define $\Delta_\varepsilon := \{z \mid |z| > \varepsilon\}$ and $e_\varepsilon := E(\Delta_\varepsilon)$. Then

$$\begin{aligned} x - xe_\varepsilon &= \int_{\sigma(x)} z dE - \int_{\sigma(x)} z \chi_{\Delta_\varepsilon}(z) dE \\ &= \int_{\sigma(x)} z \chi_{\mathbb{C} \setminus \Delta_\varepsilon}(z) dE = \phi(x), \end{aligned}$$

where $\phi(z) = z \chi_{\mathbb{C} \setminus \Delta_\varepsilon}$. Since the Borel functional calculus is a contraction, this implies that

$$\|x - xe_\varepsilon\| = \|\phi(x)\| \leq \|\phi\|_\infty \leq \varepsilon.$$

Therefore, we can approximate x with operators xe_ε . If every e_ε has finite rank, then every xe_ε has finite rank and x can be approximated using finite rank operators, so it must be compact. Conversely, let x be compact and take an arbitrary $\varepsilon > 0$. Define the function $\psi(z) = z^{-1} \chi_{\Delta_\varepsilon}$. Then

$$x\psi(x) = \int_{\sigma(x)} z z^{-1} \chi_{\Delta_\varepsilon} dE = \int_{\sigma(x)} \chi_{\Delta_\varepsilon} dE = E_\varepsilon$$

is compact. Define a map $\iota_\varepsilon : \operatorname{im} e_\varepsilon \hookrightarrow \mathcal{H}$. Of course, $\operatorname{im} e_\varepsilon$ is closed (and therefore Hilbert) in \mathcal{H} since e_ε is a projection. Now

$$E_\varepsilon \circ \iota_\varepsilon = \operatorname{id} : \operatorname{im} e_\varepsilon \rightarrow \operatorname{im} e_\varepsilon$$

must be compact, which can happen only if $\operatorname{im} e_\varepsilon$ has finite dimension. As a result, e_ε is a finite rank operator. \square

Corollary 6.14. *If \mathcal{H} is separable and I is an ideal of $\mathcal{B}(\mathcal{H})$ that contains a noncompact operator, then $I = \mathcal{B}(\mathcal{H})$.*

Proof. Suppose that $x \in I$ is noncompact and let

$$x^*x = \int_{\sigma(x^*x)} z dE$$

by the spectral theorem. By the previous lemma, there exists such an $\varepsilon > 0$ such that $e_\varepsilon = E(\varepsilon, \infty)$ has infinite rank. Now notice that

$$e_\varepsilon = \int_{\sigma(x)} \chi_{(\varepsilon, \infty)} dE = \int_{\sigma(x)} z^{-1} \chi_{(\varepsilon, \infty)} z dE = \left(\int_{\sigma(x)} z^{-1} \chi_{(\varepsilon, \infty)} dE \right) (x^*x) \in I.$$

Let $U : \mathcal{H} \rightarrow e_\varepsilon \mathcal{H}$ be a unitary (such an U exists because $\dim \mathcal{H} = \dim e_\varepsilon \mathcal{H} = |\mathbb{N}|$). Then $U^* e_\varepsilon U = \text{id}_{\mathcal{H}} \in I$, hence $I = \mathcal{B}(\mathcal{H})$. \square

Remark. The statement is not true for \mathcal{H} with dimension of higher cardinality. Notice that $\mathcal{K}(\mathcal{H})$ is a closed ideal, generated by the operators of finite rank. If \mathcal{H} is not separable, then operators with separable image (or any cardinality less than the dimension of \mathcal{H}) also generate a closed proper ideal in $\mathcal{B}(\mathcal{H})$.

6.3 Abelian vNa's

Definition 6.15. Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a subalgebra. A vector $\alpha \in \mathcal{H}$ is:

- (1.) *Cyclic* for A if $A\alpha$ is dense in \mathcal{H} .
- (2.) *Separating* for A if $x\alpha = 0$ for $x \in A$ implies $x = 0$.

Proposition 6.16. Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a subalgebra.

- (1.) If $\alpha \in \mathcal{H}$ is cyclic for A , then it is separating for A' .
- (2.) Assume A is a $*$ -subalgebra. Then, if α is separating for A' , it is cyclic for A .
- (3.) Suppose $W \subseteq \mathcal{B}(\mathcal{H})$ is a vNa. Then α is cyclic for M iff it is separating for M' , and separating for M iff it is cyclic for M' .

Proof. (1.) Let α be cyclic for A . Let $y \in A'$ satisfy $y\alpha = 0$. Pick any $\beta \in \mathcal{H}$. There exists a sequence $(x_n)_n \subseteq A$ such that $\|x_n\alpha - \beta\| \rightarrow 0$. Hence,

$$y\beta = \lim_{n \rightarrow \infty} yx_n\alpha = \lim_{n \rightarrow \infty} x_n(y\alpha) = 0,$$

and α is separating for A' .

- (2.) Define $\mathcal{K} := (A\alpha)^\perp \leq \mathcal{H}$. Let $p : \mathcal{H} \rightarrow \mathcal{K}$ be the orthogonal projection. For $x_1, x_2 \in A$ and $\beta \in \mathcal{K}$, we have

$$\langle x_1\beta, x_2\alpha \rangle = \langle \beta, x_1^*x_2\alpha \rangle = 0,$$

so $x_1\beta \in \mathcal{K}$, and \mathcal{K} is an invariant subspace for A . But since A is $*$ -closed, \mathcal{K} is reducing, and by lemma 5.30, $p \in A'$. Of course, $\alpha \in A\alpha$ and $p\alpha = 0$. Now we use the fact that α is separating for A' , and therefore $p = 0$. This implies $\mathcal{K} = (0)$.

- (3.) This follows immediately from $M = M''$ and the previous two points. \square

Example 6.17. Recall that $VN(\Gamma) := \lambda(\mathbb{C}[\Gamma])'' \subseteq \mathcal{B}(\ell^2(\Gamma))$. Similarly, we can use the right regular map

$$\rho : \Gamma \rightarrow \mathcal{B}(\ell^2(\Gamma)), \quad g \mapsto (\rho_g : \delta_k \mapsto \delta_{kg^{-1}})$$

to define $VN_{\text{right}}(\Gamma) = \rho(\Gamma)'' \subseteq \mathcal{B}(\ell^2(\Gamma))$. Notice that $\delta_e \in \ell^2(\Gamma)$ is cyclic for both $\lambda(\mathbb{C}[\Gamma])$ and $\rho(\mathbb{C}[\Gamma])$. This means that it is cyclic for both $VN(\Gamma)$ and $VN_{\text{right}}(\Gamma)$. It's easy to see that $VN(\Gamma)' = VN_{\text{right}}(\Gamma)$, so δ_e is separating for both $VN(\Gamma)$ and $VN_{\text{right}}(\Gamma)$.

Corollary 6.18. If $A \subseteq \mathcal{B}(\mathcal{H})$ is abelian, then each cyclic vector for A is also separating for A .

Proof. If $\alpha \in \mathcal{H}$ is cyclic for A , then it is separating for A' , but since $A \subseteq A'$, it is also separating for A . \square

Recall from example 5.13 that for a σ -finite measure on X , the map

$$M : L^\infty(X, \mu) \rightarrow \mathcal{B}(L^2(X, \mu))$$

is an isometric $*$ -isomorphism onto its image. In the remainder of this chapter, we will identify $L^\infty(X, \mu)$ as the image in $\mathcal{B}(L^2(X, \mu))$.

Theorem 6.19 (Classification of abelian vNa's).

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be an abelian vNa with a cyclic vector $\alpha_0 \in \mathcal{H}$. Suppose $A_0 \subseteq A$ is a C^* -algebra that is SOT-dense. Then, there exists a finite regular positive Borel measure μ on $\sigma(A_0)$ and an isomorphism

$$\tilde{\Gamma} : A \rightarrow L^\infty(\sigma(A_0), \mu) \subseteq \mathcal{B}(L^2(\sigma(A_0), \mu))$$

that extends the Gelfand transform $\Gamma : A_0 \rightarrow C(\sigma(A_0))$. Furthermore, $\tilde{\Gamma}$ is spatial, that is, it is induced by conjugation with a unitary $U : \mathcal{H} \rightarrow L^2(\sigma(A_0), \mu)$.

Remark. Applying this theorem to $A_0 = A$, we get

$$L^\infty(\sigma(A), \mu) = \tilde{\Gamma}(A) = C(\sigma(A)).$$

We begin with the following lemma.

Lemma 6.20. $C(\sigma(A_0))$ is weak- $*$ dense in $L^\infty(\sigma(A_0), \mu)$.

Proof. For any bounded measurable function $f \in B(\sigma(A_0))$, there exists a net $(f_i)_i \subseteq C(\sigma(A_0))$ such that $f_i \xrightarrow{\text{weak-}^*} f$ by Goldstine (see the proof of theorem 6.9). \square

Proof. Since A_0 is an abelian C^* -algebra, the Gelfand transform $\Gamma : A_0 \rightarrow C(\sigma(A_0))$ is an isometric $*$ -isomorphism. Define $\varphi_0 : A \rightarrow \mathbb{C}$ by $x \mapsto \langle x\alpha_0, \alpha_0 \rangle$. Then $\varphi_0\Gamma^{-1} : C(\sigma(A_0)) \rightarrow \mathbb{C}$ is a bounded linear functional, so by the Riesz–Markoff theorem, there exists a regular Borel

measure μ on $\sigma(A_0)$ such that

$$\varphi_0 \Gamma^{-1}(f) = \int_{\sigma(A_0)} f d\mu.$$

For every positive function $f \in C(\sigma(A_0))$, we have

$$\begin{aligned} \int_{\sigma(A_0)} f d\mu &= \int \sqrt{f}^2 d\mu = \varphi_0 \Gamma^{-1}(\sqrt{f}^2) = \langle \Gamma^{-1}(\sqrt{f}^2) \alpha_0, \alpha_0 \rangle \\ &= \langle \Gamma^{-1}(\sqrt{f})^2 \alpha_0, \alpha_0 \rangle = \langle \Gamma^{-1}(\sqrt{f}) \alpha_0, \Gamma^{-1}(\sqrt{f}) \alpha_0 \rangle \\ &= \|\Gamma^{-1}(\sqrt{f}) \alpha_0\|^2 \geq 0 \end{aligned}$$

and μ is a positive measure. Furthermore, μ is finite, since

$$\mu(\sigma(A_0)) = \varphi_0(1) = \|\alpha_0\|^2 < \infty.$$

Now we prove that $\text{supp } \mu = \sigma(A_0)$. If $\text{supp } \mu \subsetneq \sigma(A_0)$, then there exists a non-empty open set $S \subseteq \sigma(A_0)$ with $\mu(S) = 0$. Consider a nonnegative function $f \in C(\sigma(A_0)) \setminus \{0\}$ with $f|_{\sigma(A_0) \setminus S} = 0$. Then

$$\|\Gamma^{-1}(\sqrt{f}) \alpha_0\|^2 = \int_{\sigma(A_0)} f d\mu = \int_S f d\mu = 0.$$

We get $\Gamma^{-1}(\sqrt{f}) \alpha_0 = 0$, which, by the cyclicity of α_0 , implies $\Gamma^{-1}(\sqrt{f}) = 0$, $\sqrt{f} = 0$, and $f = 0$, a contradiction. Define

$$U_0 : A_0 \alpha_0 \rightarrow C(\sigma(A_0)) \subseteq L^2(\sigma(A_0), \mu), \quad x \alpha_0 \mapsto \Gamma(x).$$

Since α_0 is separating for A_0 , this U_0 is a well-defined linear map. For $x, y \in A_0$, we have

$$\begin{aligned} \langle U_0(x \alpha_0), U_0(y \alpha_0) \rangle &= \langle \Gamma(x), \Gamma(y) \rangle_2 \\ &= \int_{\sigma(A_0)} \overline{\Gamma(y)} \Gamma(x) d\mu \\ &= \int_{\sigma(A_0)} \Gamma(y^* x) d\mu \\ &= \varphi(y^* x) = \langle y^* x \alpha_0, \alpha_0 \rangle = \langle x \alpha_0, y \alpha_0 \rangle \end{aligned}$$

and so U_0 is an isometry! Since α_0 is cyclic for A and A_0 is SOT-dense in A , α_0 is cyclic for A_0 . Thus, $A_0 \alpha_0$ is dense in \mathcal{H} and the image of U_0 is the entire $C(\sigma(A_0))$. By continuity, U_0 extends to a surjective isometry

$$U : \mathcal{H} \rightarrow L^2(\sigma(A_0), \mu) = \overline{C(\sigma(A_0), \mu)}^{\langle \cdot, \cdot \rangle^2},$$

where U is unitary. Next, define

$$\tilde{\Gamma} : A \rightarrow \mathcal{B}(L^2(\sigma(A_0), \mu)), \quad x \mapsto UxU^*.$$

We claim that $\tilde{\Gamma}$ is an isometric $*$ -homomorphism. Since U is unitary, the isometric part is obvious, and the homomorphism property soon follows. Now we claim that $\tilde{\Gamma}(A) =$

$L^\infty(\sigma(A_0), \mu)$. For $x \in A_0$ and $g \in C(\sigma(A_0))$, we have

$$\begin{aligned}\tilde{\Gamma}(x)g &= UxU^*g = UxU^{-1}(\Gamma(\Gamma^{-1}(g))) \\ &= Ux(\Gamma^{-1}(g)\alpha_0) = \Gamma(x\Gamma^{-1}(g)) \\ &= \Gamma(x)g\end{aligned}$$

and since $C(\sigma(A_0))$ is dense in $L^2(\sigma(A_0), \mu)$, we get $\tilde{\Gamma}(x) = M_{\Gamma(x)}$. It follows that

$$\tilde{\Gamma}(A_0) = C(\sigma(A_0)) \subseteq L^\infty(\sigma(A_0), \mu).$$

Then we use the fact that $\tilde{\Gamma}$ is SOT-continuous (by definition) and that L^∞ is a vNa to get

$$\tilde{\Gamma}(A) = \tilde{\Gamma}(\overline{A_0}^{\text{SOT}}) \subseteq \overline{\tilde{\Gamma}(A_0)}^{\text{SOT}} \subseteq \overline{L^\infty(\sigma(A_0), \mu)}^{\text{SOT}} = L^\infty(\sigma(A_0), \mu).$$

The reverse inclusion is proved by nets. Suppose $(\tilde{\Gamma}(x_i))_i \subseteq \tilde{\Gamma}(A_0)$ WOT-converges to $T \in B(L^2(\sigma(A_0), \mu))$. Then for all $\beta \mu \in \mathcal{H}$ we have

$$\begin{aligned}\langle TU\beta, U\mu \rangle &= \lim_i \langle \tilde{\Gamma}(x_i)U\beta, U\mu \rangle \\ &= \lim_i \langle Ux_iU^*U\beta, U\mu \rangle \\ &= \lim_i \langle x_i\beta, \mu \rangle\end{aligned}$$

and $(x_i)_i \xrightarrow{\text{WOT}} U^*TU \in \mathcal{B}(\mathcal{H})$. Since $\overline{A_0}^{\text{WOT}} = A$, we get $x = U^*TU \in A$ and $\tilde{\Gamma}(x) = T$, so $\overline{\tilde{\Gamma}(A_0)}^{\text{WOT}} \subseteq \tilde{\Gamma}(A)$. Finally, we ask: what is $\overline{C(\sigma(A_0))}^{\text{WOT}}$? The WOT on that set is generated by seminorms

$$|\langle M_g\alpha, \beta \rangle| = \left| \int_{\sigma(A_0)} g\alpha\beta d\mu \right|,$$

where $g \in C(\sigma(A_0))$ and $\alpha, \beta \in L^2(\sigma(A_0), \mu)$. By theorem 6.21, the weak-* topology on $L^\infty(\sigma(A_0), \mu)$ is generated by seminorms

$$\left| \int_{\sigma(A_0)} g\gamma d\mu \right|,$$

where $\mu \in L^1(\sigma(A_0), \mu)$. By Hölder's inequality, an L^1 function is a product of two L^2 functions, so these two topologies coincide. By the lemma above, $C(\sigma(A_0))$ is weak-* dense in $L^\infty(\sigma(A_0), \mu)$. Therefore, we have

$$L^\infty(\sigma(A_0), \mu) = \overline{C(\sigma(A_0))}^{\text{WOT}} = \overline{\tilde{\Gamma}(A_0)}^{\text{WOT}} \subseteq \tilde{\Gamma}(A)$$

and finally $\tilde{\Gamma}(A) = L^\infty(\sigma(A_0), \mu)$. □

In the proof, we used the well known theorem from measure theory (theorem 6.16 in [4]).

Theorem 6.21.

Suppose $1 \leq p < \infty$ and μ is a σ -finite positive measure on X , and Φ is a bounded linear functional on $L^p(X, \mu)$. Then there is a unique $g \in L^q(X, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\Phi(f) = \int_X fg \, d\mu.$$

Moreover, $\|\Phi\| = \|g\|_q$.

How crucial is the cyclicity assumption? Let $A \subseteq \mathcal{B}(\mathcal{H})$ be an abelian vNa. Pick $0 \neq \alpha \in \mathcal{H}$. Define $\mathcal{K} := \overline{A\alpha}$ and let $p : \mathcal{H} \rightarrow \mathcal{K}$ be an orthogonal projection. By the reducibility of \mathcal{K} , we get $p \in A'$. Therefore, $pAp = Ap \subseteq \mathcal{B}(\mathcal{H})$ is an abelian vNa with a cyclic vector $\alpha \in \mathcal{K}$. Then, again by the theorem, $Ap \cong L^\infty(X, \mu)$ for some (X, μ) . By Zorn's lemma, $A \cong L^\infty(Y, \nu)$ for some disjoint union of measure spaces (Y, ν) .

Proposition 6.22. Let \mathcal{H} be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ be an abelian vNa. Then there exists a separating vector for A .

Proof. By Zorn's lemma, there exists a maximal set of unit vectors $(\alpha_k)_k$ such that $A\alpha_k \perp A\alpha_l$ for $k \neq l$. By maximality, $\sum_k A\alpha_k$ is dense in \mathcal{H} . Define $\alpha = \sum_{n=1}^\infty \frac{1}{2^n} \alpha_n$. We claim that α is separating for A . Indeed, let $x \in A$ such that $x\alpha = 0$. Then $\sum_{n=1}^\infty \frac{1}{2^n} x\alpha_n = 0$. By orthogonality, $x\alpha_n = 0$ for all indices n . For all $y \in A$, we get $xy\alpha_n = yx\alpha_n = 0$, so $x|_{A\alpha_n} = 0$ for all n . But since $\sum_n A\alpha_n$ is dense in A , we get $x = 0$. \square

Corollary 6.23. Let \mathcal{H} be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ be a maximal abelian vNa. Then there exists a cyclic vector for A .

Proof. By proposition 6.22, there exists a separating vector α for A , which is then cyclic for A' . But since A is maximal, we get $A = A'$. \square

Theorem 6.24.

Let \mathcal{H} be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ an abelian vNa. Then there exists a compact Hausdorff space X and a finite regular Borel measure μ on X such that $A \cong L^\infty(X, \mu)$.

Proof. By proposition 6.22, there exists a separating vector $\alpha \in \mathcal{H}$ for A . Form $\mathcal{K} := \overline{A\alpha}$. Then the algebra $\{x|_{\mathcal{K}} \mid x \in A\} \subseteq \mathcal{B}(\mathcal{K})$ is $*$ -isomorphic to A , has cyclic vector α , and the above theorem applies. \square

We can use theorem 6.19 to find $VN(G)$ of an abelian group G . Recall that by definition, $C_r^*(G)$ is a SOT-dense C^* -subalgebra in $VN(G)$. Moreover, $VN(G)$ has a cyclic vector $\delta_0 \in \ell^2(G)$. Therefore, there exists a positive Borel measure μ on $\sigma_r^*(G)$ such that

$$VN(G) \cong L^\infty(\sigma(C_r^*(G)), \mu) \cong L^\infty(\widehat{G}, \mu).$$

Example 6.25. Let us find $VN(\mathbb{Z}/n\mathbb{Z})$. Following the proof of theorem 6.19, all we need to do is find a positive Borel measure μ on $\mathbb{Z}/n\mathbb{Z}$, such that for any $x \in C_r^*(G)$, we have

$$\langle x\delta_0, \delta_0 \rangle = \int_{\mathbb{Z}/n\mathbb{Z}} \Gamma(x) d\mu.$$

Now if $x = \sum_{l=0}^{n-1} a_l u_l$, the LHS of this equation equals a_0 . On the other hand, example 3.47 tells us that the RHS is equal to

$$\sum_{k=0}^{n-1} \mu(\{k\}) \cdot \left(\sum_{l=0}^{n-1} a_l e^{\frac{2\pi i k l}{n}} \right) = \sum_{l=0}^{n-1} a_l \cdot \left(\sum_{k=0}^{n-1} \mu(\{k\}) e^{\frac{2\pi i k l}{n}} \right).$$

The above expression equals a_0 if we set $\mu(\{k\}) = \frac{1}{n}$ for any $k \in \mathbb{Z}/n\mathbb{Z}$. Therefore, we can take μ to be the normalized counting measure on $\mathbb{Z}/n\mathbb{Z}$ and

$$VN(\mathbb{Z}/n\mathbb{Z}) \cong L^\infty(\mathbb{Z}/n\mathbb{Z}, \mu)$$

as a C^* -algebra. In particular, $VN(\mathbb{Z}/n\mathbb{Z}) \cong C(\mathbb{Z}/n\mathbb{Z})$, so we have $VN(\mathbb{Z}/n\mathbb{Z}) = C_r^*(\mathbb{Z}/n\mathbb{Z})$ and $VN(\mathbb{Z}/n\mathbb{Z})$ is generated by

$$u_1 = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \ell^2(\mathbb{Z}/n\mathbb{Z}).$$

Example 6.26. Similar to the previous example, we can find $VN(\mathbb{Z})$. Again, it suffices to find a positive Borel measure μ on \mathbb{T} such that

$$\langle x\delta_0, \delta_0 \rangle = \int_{\mathbb{T}} \Gamma(x) d\mu$$

for any $x = \sum_{l \in \mathbb{Z}} a_l u_l \in C_r^*(\mathbb{Z})$. Now the LHS of the above equation equals a_0 , while the RHS is equal, by example 3.48, to

$$\int_{\mathbb{T}} \sum_{l \in \mathbb{Z}} a_l \xi^l d\mu = \sum_{l \in \mathbb{Z}} a_l \int_{\mathbb{T}} \xi^l d\mu$$

(here, we used Lebesgue's dominated convergence theorem). If we set μ to be the normalized Lebesgue measure m on \mathbb{T} , then the above sum equals a_0 . Therefore, we have

$$VN(\mathbb{Z}) \cong L^\infty(\mathbb{T}, m)$$

as a C^* -algebra by theorem 6.19.

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