FUNCTIONAL ANALYSIS - NOTES

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1 Convexity

1.1 Locally convex spaces

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ be a field.

Definition 1.1. A *topological vector space* (TVS) is a $\mathbb{F}$-vector space that is also a topological space and the two structures are compatible. This means that the usual operations on vector spaces

$$V \times V \to v,\ (x,y) \mapsto x+y, \qquad \mathbb{F} \times V \to V,\ (\lambda, x) \mapsto \lambda x$$

are continuous maps.

Example 1.2. *Normed spaces are TVS.*

Definition 1.3. Let V be a $\mathbb{F}$-space. Map $p : V \to \mathbb{R}$ is a *seminorm* if:
(1.) $p(x) \geq 0,\ \forall x \in V$ (positivity);
(2.) $p(\lambda x) = |\lambda| p(x),\ \forall x \in V,\ \forall \lambda \in \mathbb{F}$ (positive homogeneity);
(3.) $p(x+y) \leq p(x) + q(x),\ \forall x, y \in \mathbb{F}$ (triangle inequality).
A seminorm is therefore almost a norm, except that it's not necessarily positive definite.

Let V be a $\mathbb{F}$-vector space and $\mathcal{P}$ a family of seminorms in V. Let $\mathcal{T}$ be the topology in V with the following subbasis:

$$U(x_0, p, \varepsilon) = \{x \in V \mid p(x - x_0) < \varepsilon\};\ x_0 \in V,\ p \in \mathcal{P},\ \varepsilon > 0.$$

Basis of $\mathcal{T}$ are finite intersections of such sets. The set $U \subseteq V$ is open iff for every $x_0 \in U$ there exist seminorms $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$U \supset \bigcap_{j=1}^{n} U(x_0, p_j, \varepsilon_j).$$

The space $(V, \mathcal{T})$ is then a TVS. If $\mathcal{P}$ is a singleton and its element is a norm, then $(V, \mathcal{T})$ is a normed space.

Definition 1.4. A TVS X is a *locally-convex space* (LCS) if its topology is generated by a family of seminorms $\mathcal{P}$ satisfying

$$\bigcap_{p \in \mathcal{P}} \{x \in X \mid p(x) = 0\} = \{0\}.$$

Equivalently, for every $x \in X \setminus \{0\}$ there exists a seminorm $p \in \mathcal{P}$ such that $p(x) \neq 0$.

Corollary 1.5. *Let X be a space with a topology generated by a family of seminorms $\mathcal{P}$. Then X is a LCS iff it is Hausdorff.*

Proof. Start with ($\Rightarrow$). Let $x, y \in X$ be two distinct points. There exists a seminorm $p \in \mathcal{P}$ such that $p(x-y) = b \neq 0$. Define the sets

$$V = U\left(x, p, \frac{b}{2}\right), \quad W = U\left(y, p, \frac{b}{2}\right).$$

By the triangle inequality property of a seminorm, V and W separate the points x, y. Now the converse ($\Leftarrow$). Choose a point $X \ni x \neq 0$. Then there exist open sets $0 \in V, x \in W$ that separate 0 from x. There exists an open basis set $\bigcap_{j=1}^{n} U(0, p_j, \varepsilon_j) \subseteq V$, so $x \notin U(0, p_j, \varepsilon_j)$ for some index j. Hence, $p_j(x-0) = p_j(x) \geq \varepsilon > 0$. $\square$

LCS generally aren't first-countable, so we need to go beyond the usual sequences to describe the topology.

Definition 1.6. Partially ordered set $(I, \leq)$ is *upwards-directed* if

$$\forall i', i'' \in I : \ \exists i \in I : \ i \geq i', i \geq i''.$$

Example 1.7. *(1.) Every linearly ordered set is upwards-directed.*
(2.) Let $(X, \mathcal{T})$ be a topological space and $x_0 \in X$. Define a family of sets

$$\mathcal{U} = \{U^{open} \subseteq X \mid x_0 \in U\}$$

and a relation $U \geq V \Leftrightarrow U \subseteq V$. Then $(\mathcal{U}, \leq)$ is an upwards-directed set.
(3.) Let S be a set and $\mathcal{F}$ a family of all finite subsets of S. Define $F_1 \geq F_2$ in $\mathcal{F}$ if $F_1 \supseteq F_2$. Then $(\mathcal{F}, \leq)$ is again an upwards-directed set.

Definition 1.8. A *generalized sequence (net)* is $((I, \leq), x)$, where $(I, \leq)$ is upwards-directed and $x : I \to X$ is a function. We usually write $(x_i)_{i \in I}$ or $(x(i))_{i \in I}$.

Example 1.9. *(1.) Every sequence is a net.*
(2.) Let $(X, \mathcal{T})$ be a topological space, $x_0 \in X$ and $\mathcal{U}$ a collection of all open sets which contain x_0 (see example 1.7). For each $U \in \mathcal{U}$ pick a $x_U \in U$. Then $(x_U)_{U \in \mathcal{U}}$ is a net.

Definition 1.10. Let X be a topological space. A net $(x_i)_{i \in I}$ *converges* to an $x \in X$ if

$$\forall U^{\text{open}} \subseteq X, \ x \in U : \ \exists i_0 \in I : \ \forall i \geq i_0 : \ x_i \in U.$$

We write $\lim_{i \in I} x_i = x$, or alternatively, $x_i \xrightarrow[i \in I]{} x$. A point $x \in X$ is called a *cluster point* of a net $(x_i)_{i \in I}$ if

$$\forall U^{\text{open}} \subseteq X, \ x \in U : \ \forall i_0 \in I : \ \exists i \geq i_0 : \ x_i \in U.$$

Example 1.11. *Take the net $(x_U)_{U\in\mathcal{U}}$ from example 1.9. It follows from the definition that $x_U \xrightarrow[U\in\mathcal{U}]{} x_0$.*

Proposition 1.12. *(1.) Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ iff there exists a net $(a_i)_{i\in I}$ in A such that $a_i \to x$.*

(2.) Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous at $x_0 \in X$ iff $f(x_i) \to f(x_0)$ for every net $(x_i)_{i\in I}$ that converges to x_0.

Proof. (1.) We begin with the implication to the left ($\Leftarrow$). Take any $U^{\text{open}} \subseteq X$ such that $x \in U$. Since $a_i \to x$, there exists an index $i_0 \in I$, such that for every $i \geq i_0$ we have $a_i \in U$. Hence $a_i \in A \cap U \neq \emptyset$ and $x \in \overline{A}$. The converse ($\Rightarrow$) is similar. Define $\mathcal{U} = \{U^{\text{open}} \subseteq X \mid x \in U\}$. Since $x \in \overline{A}$, for each $U \in \mathcal{U}$, we have $A \cap U \neq \emptyset$. Pick $a_U \in A \cap U$. Then the net $(a_U)_{U\in\mathcal{U}}$ in A converges to x.

(2.) Start with the implication ($\Rightarrow$). Let f be a continuous function and let $(x_i)_{i\in I}$ converge to x_0. Let $f(x_0) \in U^{\text{open}} \subseteq Y$. Then $x_0 \in f^{-1}(U)^{\text{open}} \subseteq X$, which means there exists an $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, $x_i \in f^{-1}(U)$. But that implies that for every $i \geq i_0$, $f(x_i) \in U$, which is what we wanted. Now we prove the converse ($\Leftarrow$). Let's say that for every net $(x_i)_{i\in I}$ that converges to x_0, we have $f(x_i) \xrightarrow[i\in I]{} f(x_0)$. So for every set $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$ (using the first item), which proves that f is continuous. ☐

Proposition 1.13. *(a) A net $(x_i)_{i\in I}$ in a LCS converges to x_0 iff a net $(p(x_i - x_0))_{i\in I}$ converges to 0 for all $p \in \mathcal{P}$.*

(b) The topology in a LCS X is the coarsest (smallest) topology in which all the maps $x \mapsto p(x - x_0)$ are continuous for every $x_0 \in X$ and $p \in \mathcal{P}$.

Proof. (a) Start with the implication ($\Rightarrow$). Take any $p \in \mathcal{P}$. If we take $U = U(x_0, p, \varepsilon)$ in the definition of a limit of a net, we get

$$\forall \varepsilon > 0 : \ \exists i_0 \in I : \ \forall i \geq i_0 : \ p(x_i - x_0) \in (-\varepsilon, \varepsilon).$$

This proves our claim. Now for the opposite direction ($\Leftarrow$). For every $p \in \mathcal{P}$ and $\varepsilon_p > 0$ there exists an i_p such that for every $i \geq i_p$, $x_i \in U(x_0, p, \varepsilon_p)$. Now let U be an arbitrary basis set that includes the point x_0. That means U is the finite intersection of the sets $U(x_0, p, \varepsilon_p)$. Now let i_0 be greater than all indices i_p. By our assumption, for every $i \geq i_0$ we have $x_i \in U$.

(b) Pick any point $x_0 \in X$ and a seminorm $p \in \mathcal{P}$. Denote

$$f_{x_0,p} : X \to \mathbb{R}, \quad f_{x_0,p}(x) = p(x - x_0).$$

We essentially have to prove that the sets

$$f^{-1}_{x_0,p}(V), \ V^{\text{open}} \subseteq \mathbb{R}, \ x_0 \in X, \ p \in \mathcal{P}$$

generate a subbasis for the seminorm topology of a LCS space. Since f_{p,x_0} are continuous functions (by the first item and Proposition 1.12), these are all open sets in the seminorm topology. But on the other hand, all subbasis sets $U(x_0, p, \varepsilon)$ of the seminorm topology are of this type, so the above subbasis generates the seminorm topology, thus concluding our proof. $\square$

Example 1.14. *Let X be a topological space. For every $K^{compact} \subseteq X$ we define a seminorm*

$$p_K : C(X) \to \mathbb{R}, \quad f \mapsto \sup_{x \in K} |f(x)|.$$

We endow $C(X)$ with the topology induced by the family of seminorms $\{p_K \mid K^{compact} \subseteq X\}$. It's trivial to see that $C(X)$ is then a LCS. Moreover, we notice that the induced seminorm topology coincides with the topology of compact convergence on X. In the future, we will require X to be locally compact Hausdorff (this implies complete regularity) so that $C(X)$ has nice properties. There are examples of not completely regular spaces X such that the only elements of $C(X)$ are constant maps.

Example 1.15. *Let $D^{open} \subseteq \mathbb{C}$ and let $\mathcal{H}(D)$ be the set of all holomorphic functions on D. As in the example 1.14, we define $\mathcal{P} = \{p_K \mid K^{compact} \subseteq D\}$. This endows $\mathcal{H}(D)$ with a topology and makes $\mathcal{H}(D)$ into a LCS. Convergence in this topology concides with the uniform convergence on compacts in D.*

1.2 Weak topology

Let X be a normed space and let X^* be its dual. For every $f \in X^*$ we define a seminorm

$$p_f : X \to \mathbb{R}, \quad x \mapsto |f(x)|.$$

We claim that $\mathcal{P} = \{p_f \mid f \in X^*\}$ is a family of seminorms that induces a topology on X which makes X a LCS. Indeed, for any $x \in X \setminus \{0\}$ define a nonzero linear functional

$$f : \operatorname{span}(x) \to \mathbb{F}, \quad f(\lambda x) = \lambda$$

and extend it to $F : X \to \mathbb{F}$ using Hahn–Banach. Then $p_F(x) \neq 0$. The induced topology is the *weak topology* on X. We denote it as $\sigma(X, X^*)$.

Proposition 1.16. *A net $(x_i)_{i \in I}$ converges to $x_0 \in X$ with respect to the weak topology iff $f(x_i) \to f(x_0)$, $\forall f \in X^*$.*

Remark. We use the notation $x_i \xrightarrow{w} x_0$. Furthermore, a closure of a set $A \subseteq X$ in the weak topology will be denoted by $\overline{A}^w$.

Remark. The closure of a set $A \subseteq X$ in the weak topology will be denoted as $\overline{A}^w$.

Example 1.17. *Let $X = \mathbb{R}^n$. Then $X^* = \mathbb{R}^n$ and every linear functional f is of the form $f(x) = \langle x, y \rangle$ for some $y \in X$ (Riesz' representation theorem). The subbasis sets are*

$$U(0, p_y, \varepsilon) = \{x \in \mathbb{R}^n \mid |\langle x, y \rangle| < \varepsilon\}.$$

Weak topology in this case coincides with Euclidean topology.

Let X again be a normed space. To $x \in X$ we assign the seminorm

$$p_x : X^* \to \mathbb{R}, \quad f \mapsto |f(x)|.$$

The family $\{p_x \mid x \in X\}$ defines a topology in X^* in which X^* becomes a LCS. This topology is called the *weak-$*$ topology* and is denoted by $\sigma(X^*, X)$.

Proposition 1.18. *A net $(f_i)_{i \in I}$ converges to $f \in X^*$ with respect to the weak-$*$ topology iff $f_i(x) - f(x) \to 0, \forall x \in X$.*

Remark. We use the notation $f_i \xrightarrow{w^*} f$. Furthermore, a closure of a set $A \subseteq X^*$ in the weak-$*$ topology will be denoted by $\overline{A}^{w^*}$.

We can compare weak-$*$ topology on X^* with its weak topology. As a consequence of Hahn-Banach, we have for every $x \in X$

$$\|x\| = \sup\{|f(x)|| \ f \in X^*, \ \|f\| \leq 1\},$$

which implies that the map

$$\iota : X \hookrightarrow X^{**}, \quad x \mapsto (f \mapsto f(x))$$

is an isometry and therefore injective. This means that every seminorm in the weak-$*$ topology is also a seminorm in a weak topology on X^*, so the weak topology is finer (stronger) than the weak-$*$ topology on X^*.

Remark. Weak and weak-$*$ topology can be defined even if X is merely a LCS. In that case, X^* is of course defined as the space of continuous linear functionals on X.

1.3 Banach-Alaoglu theorem

Theorem 1.19 (Banach-Alaoglu)**.**

Let X be a normed space. Then the closed unit ball in X^* (denoted by $(X^*)_1$) is compact in the weak-$*$ topology in X^*.

Proof. To $x \in X$ we assign $D_x = \{z \in \mathbb{F} \mid |z| \leq \|x\|\}$ and endow D_x with the Euclidean topology. Then D_x is clearly compact. The set $P = \prod_{x \in X} D_x$ is compact in the product topology (Tychonoff theorem). Now we construct a map

$$\Phi : (X^*)_1 \to P, \quad f \mapsto (f(x))_{x \in X} \in P.$$

Clearly, Φ is well-defined and injective. We start by proving that Φ is continuous. Let $(f_i)_{i \in I}$ be a net in $(X^*)_1$ that converges to $f \in X^*$ in the weak-$*$ topology. Then $f_i(x) \to f(x)$ for each $x \in X$. By the definition of the product topology in P, this means that $\Phi(f_i) \mapsto \Phi(f)$ in P. Hence Φ is continuous. Since Φ is injective, it induces an inverse map

$$\Phi^{-1} : \operatorname{im}(\Phi) \to (X^*)_1$$

that is also continuous (we read the previous argument backwards).
Finally, we prove that $\operatorname{im}(\Phi)$ is closed in P. Suppose that $(\Phi(f_i))_{i \in I}$ converges to $p = (p_x)_{x \in X} \in P$. By definition of the product topology, this means that $f_i(x) \to p_x$ for all

$x \in X$. Define

$$f : X \to \mathbb{F}, \quad x \mapsto p_x.$$

Then f is linear and $f \in (X^*)_1$. Thus $p = \Phi(f) \in \operatorname{im}(\Phi)$. This in turn implies that $(\operatorname{im}\Phi)^{\text{closed}} \subseteq P^{\text{compact}}$. But we know that $(X^*)_1 \approx \operatorname{im}(\Phi)$, which implies that $(X^*)_1$ is also compact. $\square$

Corollary 1.20. *Every Banach space X is isometrically isomorphic to a closed subspace of $C(K)$ for some compact T_2 space K.*

Proof. Denote $K = (X^*)_1$ endowed with the weak-$*$ topology. By the Banach-Alaoglu theorem, K is compact and T_2. We now define the map

$$\Delta : X \to C(K), \quad x \mapsto (f \mapsto f(x)).$$

First, we prove that Δ is isometric. By Hahn-Banach, for every $x \in X \setminus \{0\}$ there exists an $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Then we have

$$\|\Delta(x)\|_\infty = \sup_{g \in K} |g(x)| = \|x\|.$$

Since Δ is an isometry, its image is complete and thus closed in $C(K)$. Obviously Δ is a linear map, so we are done. $\square$

1.4 Minkowski gauge

ADD MOTIVATION

Definition 1.21. Let X be a $\mathbb{F}$-vector space. A set $A \subseteq X$ is

- balanced if:
$$\forall x \in A : \ \forall \alpha \in \mathbb{F}, \ |\alpha| \leq 1 : \ \alpha x \in A.$$
- absorbing if:
$$\forall x \in X : \ \exists \varepsilon > 0 : \ \forall t \in (0, \varepsilon) : \ tx \in A.$$
- absorbing in $a \in A$ if $A - a = \{x - a \mid x \in A\}$ is absorbing.

Example 1.22. *Let X be a vector space and p a seminorm in X. Then*

$$V = \{x \in X \mid p(x) < 1\}$$

is convex, balanced, absorbing in each of its points.

Theorem 1.23.

Let X be a vector space and $V \subseteq X$ convex, balanced and absorbing in each of its points. Then there exists a unique seminorm p on X such that

$$V = \{x \in X \mid p(x) < 1\}.$$

Proof. To V we associate the Minkowski gauge:

$$p(x) = \inf\{t \geq 0 \mid x \in t \cdot V\},$$

where $t \cdot V = \{t \cdot v \mid v \in V\}$. First we prove that p is well defined. Since V is absorbing, we have $X = \bigcup_{n \in \mathbb{N}} n \cdot V$, so for every $x \in X$ the set $\{t \geq 0 \mid x \in t \cdot V\}$ is nonempty. It's also clear to see that $p(0) = 0$. Next we check for homogeneity. Suppose $\alpha \neq 0$. Then

$$\begin{aligned}
p(\alpha x) &= \inf\{t \geq 0 \mid \alpha x \in t \cdot V\} \\
&= \inf\left\{t \geq 0 \mid x \in \frac{t}{\alpha} \cdot V\right\} \\
&= \inf\left\{t \geq 0 \mid x \in \frac{t}{|\alpha|} \cdot V\right\} \\
&= \inf |\alpha| \left\{\frac{t}{|\alpha|} \geq 0 \mid x \in \frac{t}{|\alpha|} \cdot V\right\} \\
&= |\alpha| p(x).
\end{aligned}$$

Now we do the same for triangle inequality: let $\alpha, \beta \geq 0$ so that $\alpha + \beta > 0$. Let $a, b \in V$. Then

$$\alpha a + \beta b = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b\right) \in (\alpha + \beta) \cdot V.$$

This means that $\alpha \cdot V + \beta \cdot V \subseteq (\alpha + \beta) \cdot V$. Now let $x, y \in X$ and $p(x) = \alpha, p(y) = \beta$. Take $\delta > 0$. Then $x \in (\alpha + \delta) \cdot V, y \in (\beta + \delta) \cdot V$. Hence

$$x + y \in (\alpha + \delta) \cdot V + (\beta + \delta) \cdot V \subseteq (\alpha + \beta + 2\delta) \cdot V,$$

and by definition, $p(x + y) \leq \alpha + \beta + 2\delta$. Since $\delta > 0$ was arbitrary, we have $p(x + y) \leq \alpha + \beta = p(x) + p(y)$. Now that we have proved that p is a seminorm, we can show that

$$V = \{x \in X \mid p(x) < 1\}.$$

The inclusion $(\supseteq)$ is easy: if $p(x) < 1$, then $x \in (p(x) + \varepsilon) \cdot V$ for all $\varepsilon > 0$. By choosing $\varepsilon = 1 - p(x) > 0$, we get $x \in V$. Now we prove the other inclusion $(\subseteq)$. Let $x \in V$. Since V is absorbing in x, there exists an $\varepsilon > 0$ such that $y = x + tx \in V$ for all $t \in (0, \varepsilon)$. This means that $x = \frac{1}{t+1} y$, where $y \in V$. This implies that

$$p(x) = \frac{1}{t + 1} p(y) \leq \frac{1}{1 + t} \leq 1,$$

which proves the equality. Lastly, we prove the p is unique. Suppose there is some other seminorm q such that

$$\{x \in X \mid p(x) < 1\} = \{x \in X \mid q(x) < 1\}.$$

Suppose $p \neq q$. W.l.o.g. there exists an $x \neq 0$ such that $p(x) > q(x)$. By homogeneity, we can assume that $p(x) = 1 > q(x)$, contradicting our assumption. $\square$

Remark. If X is a TVS and V is an open subset, then V is absorbing at each of its points.

Corollary 1.24. *Let X be a TVS and $\mathcal{U}$ a collection of all open convex balanced subsets of X. Then X is locally convex iff $\mathcal{U}$ is a basis for the neighborhood system at 0.*

1.5 Applications of Hahn-Banach

Recall: if X is a $\mathbb{R}$-vector space then $p : X \to \mathbb{R}$ is a sublinear functional if

$$p(x + y) \leq p(x) + p(y), \ \forall x, y \in X$$

and

$$p(\alpha x) = \alpha x, \ \forall x \in X, \ \alpha > 0.$$

Theorem 1.25 (Hahn-Banach theorem)**.**

- $\mathbb{R}$: *Suppose X is a $\mathbb{R}$-vector space and $p : X \to \mathbb{R}$ is a sublinear functional. Given a linear functional f on $Y \leq X$ such that $f(y) \leq p(y)$ for every $y \in Y$, f extends to a linear functional $F : X \to \mathbb{R}$ such that $F(x) \leq p(x)$ for every $x \in X$.*
- $\mathbb{C}$: *Suppose X is a $\mathbb{C}$-vector space and $p : X \to \mathbb{R}$ is a seminorm. Given a linear functional f on $Y \leq X$ such that $|f(y)| \leq p(y)$ for every $y \in Y$, f extends to a linear functional $F : X \to \mathbb{R}$ such that $|F(x)| \leq p(x)$ for every $x \in X$.*

Corollary 1.26 (Hahn-Banach extension theorem)**.** *Let X be a normed space, $f \in X^*$ and $Y \leq X$. Then there exists an $F \in X^*$ such that $F\big|_Y = f$ and $\|F\| = \|f\|$.*

Corollary 1.27 (Hahn-Banach separation theorem)**.** *Suppose X is a LCS and $A, B \subseteq X$ are disjoint closed convex sets. If B is compact then there exists an $f \in X^*$ that seperates A from B:*

$$\exists \alpha, \beta \in \mathbb{R} : \ \forall a, b \in B : \ \operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(b).$$

Theorem 1.28.

Let X be a LCS and $A \subseteq X$ convex. Then $\overline{A} = \overline{A}^w$.

Proof. Since the weak topology is weaker than the original topology, we have $\overline{A} \subseteq \overline{A}^w$. Let $x \notin \overline{A}$. We now separate $\overline{A}$ and the compact set $\{x\}$: there exists $f \in X^*$ so that there exist $\alpha, \beta \in \mathbb{R}$ and we have

$$\operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(x)$$

for all $a \in \overline{A}$. This means that

$$\overline{A} \subseteq \{y \in X \mid \operatorname{Re} f(y) \leq \alpha\} = (\operatorname{Re} f)^{-1}(-\infty, \alpha] = C.$$

Since C is closed in the weak topology, it follows from $A \subseteq C$ that $\overline{A}^w \subseteq \overline{C}^w = C$. Since $x \notin C$, we have $x \notin \overline{A}^w$. $\square$

Corollary 1.29. *A convex set in a LCS is closed iff it is weakly closed.*

Proposition 1.30. *Let X be a TVS and $f : X \to \mathbb{F}$ a linear functional. The following are equivalent:*

(1.) f *is continuous;*

(2.) f *is continuous in* 0*;*

(3.) f *is continuous in some point;*

(4.) $\ker f$ *is closed;*

(5.) $x \mapsto |f(x)|$ *is a seminorm.*

If X is a LCS, then these are also equivalent to

(6.) $\exists \alpha_1, \dots, \alpha_n \in \mathbb{R}_{>0}$ *and* $\exists p_1, \dots, p_n \in \mathcal{P}$ *such that*

$$|f(x)| \leq \sum_{k=1}^{n} \alpha_k p_k(x), \ \forall x \in X.$$

Proof. Equivalence of the first five statements is routine. Assume that X is a LCS. We prove the equivalence of (2) and (6). We start with (6) $\Rightarrow$ (2). Let $(x_i)_{i \in I}$ be a net in X that converges to 0. Then we have

$$0 \leq |f(x_i)| \leq \sum_{k=1}^{n} \alpha_k p_k(x_i) \xrightarrow[i \in I]{} 0.$$

This implies that $f(x_i) \xrightarrow[i \in I]{} 0$, proving the implication. Now the opposite: (2) $\Rightarrow$ (6). We know that $f^{-1}(B_1^\circ(0)) = \{x \in X \mid |f(x)| < 1\}$ is an open neighborhood of 0 in X. Then there exist $p_1, \dots, p_r \in \mathcal{P}$ and an $\varepsilon > 0$ such that

$$0 \in \bigcap_{i=1}^{r} U(0, p_i, \varepsilon) \subseteq f^{-1}(B_1^\circ(0)).$$

If $p_i(x) < \varepsilon$ for all $i \leq r$, then $|f(x)| < 1$. Pick any $\delta > 0$. Then

$$p_i\left(x \cdot \frac{\varepsilon}{\sum p_i(x) + \delta}\right) = \frac{\varepsilon}{\delta + \sum p_i(x)} \cdot p_i(x) < \varepsilon,$$

which implies

$$\left| f\left(x \cdot \frac{\varepsilon}{\sum p_i(x) + \delta}\right)\right| < 1.$$

From this we get $|f(x)| < \frac{1}{\varepsilon}\left(\sum p_i(x) + \delta\right)$. Since $\delta > 0$ was arbitrary, we get

$$|f(x)| \leq \sum_{i=1}^{r} \frac{1}{\varepsilon} p_i(x). \qquad \square$$

Recall the following theorem from measure theory.

Theorem 1.31 (Riesz-Markoff theorem).

Let X be a compact T_2 space, $\Phi \in C(X)^*$. Then there exists a regular Borel measure μ such that

$$\Phi(f) = \int_X f \, d\mu, \ \forall f \in C(X).$$

Further, $\|\Phi\| = \|\mu\| = |\mu|(X)$.

Remark. The above also works if X is locally compact and $\Phi \in C_0(X)^*$.

As a corollary, we get the following proposition.

Proposition 1.32. *Let X be completely regular. Endow $C(X)$ with a topology induced by its seminorms. If $L \in C(X)^*$ then there exists a compact $K \subseteq X$ and a regular Borel measure on K such that*

$$L(f) = \int_K f \, d\mu, \ \forall f \in C(X).$$

Conversely, every such pair (K, μ) defines $L \in C(K)^*$ with the above equation.

Proof. Begin with the implication ($\Leftarrow$). Given (K, μ), we just need to prove that the induced functional L is continuous on X. We have

$$|L(f)| = \left| \int_K f \, d\mu \right| \leq \|\mu\| \sup_K |f| = \|\mu\| p_K(f)$$

and L is continuous. Now the converse ($\Rightarrow$). Let $L \in C(X)$. By the previous proposition, there exist compact sets $K_1, \dots, K_p \subseteq X$ and $\alpha_1, \dots, \alpha_p > 0$ such that

$$|L(f)| \leq \sum_{j=1}^{p} \alpha_j p_{K_j}(f).$$

Let $K = \bigcup_{j=1}^{p} K_j$ and $\alpha = \max\{\alpha_1, \dots, \alpha_p\}$. Then $\|f\| \leq \alpha p_K(f)$ for all $f \in C(X)$. Observe that if $f \in C(X)$ and $f\big|_K = 0$, then $L(f) = 0$. We now define a map $F : C(K) \to \mathbb{F}$. Since X is completely regular, we have a Tietze-like extension theorem: for any compact $K \subseteq X$ and a continuous function $g \in C(K)$, there exists an extension $\widetilde{g} \in C(X)$. Define $F(g) := L(\widetilde{g})$. First we need to check that F is well defined. Suppose we have two extensions $\widetilde{g}$ and $\widetilde{\widetilde{g}}$ of $g \in C(K)$. Since $\widetilde{g} - \widetilde{\widetilde{g}}$ is evidently zero on K, we have

$$L(\widetilde{g}) - L(\widetilde{\widetilde{g}}) = L(\widetilde{g} - \widetilde{\widetilde{g}}) = 0$$

and F really is well defined. It is also clearly linear, so we just need to check continuity:

$$|F(g)| = |L(\widetilde{g})| \leq \alpha \cdot p_K(\widetilde{g}) = \alpha \cdot \|g\|_{\infty, K},$$

therefore $\|F\| \leq \alpha$ and F is continuous. Lastly we apply Riesz-Markoff: there exists a regular Borel measure μ on K so that $F(g) = \int_K g \, d\mu$. If $f \in C(X)$, then $g := f\big|_K \in C(K)$ and we have

$$L(f) = F(g) = \int_K g \, d\mu = \int_K f \, d\mu. \qquad \square$$

1.6 Krein-Milman theorem

Definition 1.33. Let X be a vector space and $C \subseteq X$ a convex subset.
(a) A nonempty convex subset $F \subseteq C$ is a *face* if for any $x, y \in C$ we have

$$(\exists t \in (0,1): \ tx + (1-t)y \in F) \Rightarrow x, y \in F.$$

(b) A point $x \in C$ is a called an *extreme point* if $\{x\} \subseteq C$ is a face.
We use the notation $\operatorname{ext}(C)$ for the set of all extreme points of C.

Example 1.34. *If we consider spaces of real sequences, we have*
- $\operatorname{ext}((\ell^\infty)_1) = \{(\pm 1, \pm 1, \dots)\}$*;*
- $\operatorname{ext}((\ell^1)_1) = \{(0, 0, \dots, \pm 1, \dots)\}$*.*

Example 1.35. *We prove that for c_0 (the space of complex sequences that converge to 0) we have $\operatorname{ext}(c_0)_1 = \emptyset$. Indeed, let $x = (x_n)_n \in (c_0)_1$. Since $\lim_n x_n = 0$, there exists $N \in \mathbb{N}$ such that $|x_n| < \frac{1}{2}$ for all $n > N$. Now define $y, z \in c_0$ by setting $y_n = z_n = x_n$ for $n \leq N$ and*

$$y_n = x_n + \frac{1}{2^n}, \quad z_n = x_n - \frac{1}{2^n}$$

for $n > N$. Then $y, z \in (c_0)_1$ and $x = \frac{1}{2}(y+z)$, so $x \notin \operatorname{ext}(c_0)_1$.

Example 1.36. *Let us show that $\operatorname{ext}(L^1[0,1])_1 = \emptyset$. Take any $f \in (L^1[0,1])_1$. Then $\int_0^1 |f(t)|\, dt = 1$, so there must exist an $x \in [0,1]$ such that $\int_0^x |f(t)|\, dt = 1/2$. Now define $g := 2 \cdot f \cdot \chi_{[0,x]}$ and $h := 2 \cdot f \cdot \chi_{[x,1]}$. Now we have $g, h \in (L^1[0,1])_1$ and $f = \frac{1}{2}g + \frac{1}{2}h$, so f cannot be an extreme point.*

Example 1.37. *Finally, let us prove that $\operatorname{ext}(C[0,1])_1 = \{\pm 1\}$ for real valued functions. Take any $f \in (C[0,1])_1$. Then define functions $g(t) = \min\{2f(t)+1, 1\}$ and $h(t) = \max\{2f(t)-1, -1\}$. Clearly $g, h \in (C[0,1])_1$ and $f = \frac{1}{2}g + \frac{1}{2}h$. If f is an extreme point, then $g = h$, which happens only if $f = \pm 1$.*

Definition 1.38. For a vector space X and $A \subseteq X$, define a *convex hull* $\operatorname{co} A$ as the intersection of all convex sets in X that contain A. If X is a TVS, then define a *closed convex hull* $\overline{\operatorname{co}} A$ as the intersection of all closed convex sets that contain A.

Convex hull of a set A can be given explicitly:

$$\operatorname{co} A = \left\{ \sum_{i=0}^n \alpha_i x_i \mid n \in \mathbb{N}, \alpha_i \geq 0, \sum_{i=0}^n \alpha_i = 1, x_i \in A \right\}.$$

If X is a TVS, then $\overline{\operatorname{co}} A = \overline{\operatorname{co} A}$.

Lemma 1.39. *If $C \subseteq X$ is a convex subset of a vector space and $a \in C$, then the following are equivalent.*
(a) $a \in \operatorname{ext} C$.
(b) *If* $x_1, x_2 \in C$ *and* $a = \frac{1}{2}(x_1 + x_2)$, *then* $x_1 = x_2 = a$.
(c) *If* $x_1, x_2 \in C$, $t \in (0, 1)$ *and* $a = tx_1 + (1 - t)x_2$, *then* $x_1 = x_2 = a$.
(d) $C \setminus \{a\}$ *is a convex set.*
(e) *If* $x_1, \dots, x_n \in C$ *and* $a \in \operatorname{co}\{x_1, \dots, x_n\}$, *then* $a = x_k$ *for some index* k.

Proof. Items (a) and (c) are equivalent by definition.
(b) $\Rightarrow$ (c): Let $a = tx_1 + (1 - t)x_2$. Then

$$a = \frac{1}{2}(2tx_1 + (1 - 2t)x_2) + \frac{1}{2}x_2,$$

so we get $2tx_1 + (1 - 2t)x_2 = x_2$, which gives us $x_1 = x_2$.
(c) $\Rightarrow$ (d): Take any $x_1, x_2 \in C \setminus \{a\}$. Since C is convex, $tx_1 + (1 - t)x_2 \in C$. Now if $a = tx_1 + (1 - t)x_2 \in \operatorname{co}\{x_1, x_2\}$, then $a = x_1 = x_2$, which contradicts our assumption. So $tx_1 + (1 - t)x_2 \in C \setminus \{a\}$ and $C \setminus \{a\}$ is convex.
(d) $\Rightarrow$ (e) If $x_1, \dots, x_n \in C \setminus \{a\}$, then $\operatorname{co}\{x_1, \dots, x_n\} \subseteq C \setminus \{a\}$ by convexity, contradiction.
(e) $\Rightarrow$ (b): Suppose $a = \frac{1}{2}(x_1 + x_2)$. Then either $x_1 = a$ or $x_2 = a$ by our assumption. W.l.o.g. assume $x_1 = a$. Then $a = \frac{1}{2}(a + x_2)$, which implies $a = x_2$. □

Lemma 1.40. *Let X be a TVS and $C \subseteq X$ a nonempty compact convex set. Then for $\Phi \in X^*$ the set*

$$F = \{x \in C \mid \operatorname{Re}\Phi(x) = \min_C \operatorname{Re}\Phi\}$$

is a closed face of C.

Proof. Since C is compact and $x \mapsto \operatorname{Re}\Phi(x)$ is continuous, it attains its minimum on C. Hence F is nonempty. Since F is a continuous preimage of a point, it is also closed. By the linearity of Φ, F is convex. Now suppose that $t \in (0, 1)$ and $x, y \in C$ are such that $tx + (1 - t)y \in F$. Then

$$\begin{aligned} \min_C \operatorname{Re}\Phi &= \operatorname{Re}\Phi(tx + (1 - t)y) \\ &= t \cdot \operatorname{Re}\Phi(x) + (1 - t)\operatorname{Re}\Phi(y) \\ &\geq t \cdot \min_C \operatorname{Re}\Phi + (1 - t)\min_C \operatorname{Re} \\ &= \min_C \operatorname{Re}\Phi. \end{aligned}$$

Since we have the equality in the second-to-last line, we have $\operatorname{Re}\Phi(x) = \min_C \operatorname{Re}\Phi$ and $\operatorname{Re}\Phi(y) = \min_C \operatorname{Re}\Phi$, meaning that $x, y \in F$. □

Remark. Not all closed convex faces are of this form.

ADD A PICTURE

Theorem 1.41 (Krein-Milman)**.**

Let X be a LCS and $C \subseteq X$ a nonempty compact convex subset. Then $C = \overline{\mathrm{co}}(\operatorname{ext} C)$. In particular, $\operatorname{ext} C \neq \emptyset$.

Proof. Let $\mathcal{F} = \{\text{closed faces in } C\}$ be ordered with $\supset$. Since $C \in \mathcal{F}$, it is nonempty. The set $\mathcal{F}$ is then partially ordered. Since any increasing chain in $\mathcal{F}$ has the finite intersection property, $\mathcal{F}$ has a nonempty intersection due to C being compact. As a result, any increasing chain in $\mathcal{F}$ has an upper bound. This tells us that we can apply Zorn's lemma to obtain a maximal element $F_0 \in \mathcal{F}$.

We prove that $F_0 = \{p\}$ for some $p \in X$. Assume for a contradiction that there are distinct $x, y \in F_0$. By Hahn-Banach, there exists a $\Phi \in X^*$ such that $\Phi(x) \neq \Phi(y)$. W.l.o.g. we assume that $\operatorname{Re} \Phi(x) < \operatorname{Re} \Phi(y)$. Define a set

$$F_1 = \{z \in F_0 \mid \operatorname{Re} \Phi(z) = \min_{F_0} \operatorname{Re} \Phi\}.$$

Then $F_1 \subsetneq F_0$, since $y \notin F_0$. By the previous lemma, F_1 is a closed face in F_0, so it is a closed face in C, contradicting maximality of F_0. As a result, $F_0 = \{p\}$, which implies that $p \in \operatorname{ext}(C)$ and the set of extreme points of C is non-empty.

Since we have $C \supset \operatorname{ext} C$, we also have $C = \overline{\mathrm{co}}(C) \supseteq \overline{\mathrm{co}}(\operatorname{ext} C)$. Suppose $x \in C \setminus \overline{\mathrm{co}}(\operatorname{ext} C)$. By Hahn-Banach, there exists a $\Psi \in X^*$ such that $\operatorname{Re} \Psi(x) < \min_{\overline{\mathrm{co}}(\operatorname{ext} C)} \operatorname{Re} \Psi$. So the set

$$F = \{z \in C \mid \operatorname{Re} \Psi(z) = \min_{C} \operatorname{Re} \Psi\}$$

is a closed face in C. By the first part of this proof, there exists a $z \in \operatorname{ext} F \subseteq \operatorname{ext} C$. Hence

$$\min_C \operatorname{Re} \Psi = \operatorname{Re} \Psi(z) = \min_{\overline{\mathrm{co}}(\operatorname{ext} C)} \operatorname{Re} \Psi > \operatorname{Re} \Psi(x) \geq \min_C \operatorname{Re} \Psi,$$

which leads to a contradiction. Therefore $\overline{\mathrm{co}}(\operatorname{ext} C) = C$. □

Example 1.42. *Let $\mathcal{H}$ be a Hilbert space. Then*

$$\operatorname{ext}(\mathcal{H})_1 = \{v \in \mathcal{H} \mid \|v\| = 1\}.$$

First we prove the inclusion ($\supseteq$). Suppose that $\|v\| = 1$ and $v = tx + (1-t)y$, where $t \in (0,1)$ and $x, y \in (\mathcal{H})_1$. We have

$$\begin{aligned} 1 &= \|v\|^2 \\ &= \|tx + (1-t)y\|^2 \\ &= \langle tx + (1-t)y, tx + (1-t)y \rangle \\ &= t^2\|x\|^2 + (1-t)^2\|y\|^2 + 2t(1-t)\operatorname{Re}\langle x, y\rangle \\ &\leq t^2 + (1-t)^2 + 2t(1-t) = 1. \end{aligned}$$

We get equality in the Cauchy-Schwartz inequality, so x, y are linearly dependent and there-

fore equal. For the reverse inclusion, let $v \in \operatorname{ext}(\mathcal{H})_1$. If $\|v\| < 1$, then

$$v = \frac{1}{2} \cdot \frac{v}{\|v\|} + \frac{1}{2} \cdot (2\|v\| - 1) \frac{v}{\|v\|},$$

so v cannot be an extreme point of $(\mathcal{H})_1$.

Example 1.43. *It holds that*

$$\operatorname{ext}(\mathcal{B}(\mathcal{H}))_1 = \{V \in \mathcal{B}(\mathcal{H}) \mid V \text{ or } V^* \text{ is an isometry}\}.$$

Here, we will just prove the inclusion ($\supseteq$). Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry and suppose $V = tS + (1-t)T$ for $t \in (0,1)$ and $S, T \in (\mathcal{B}(\mathcal{H}))_1$. For $x \in \mathcal{H}$ we have:

$$\begin{aligned} \|x\| &= \|Vx\| \\ &= \|tSx + (1-t)Tx\| \\ &\leq t\|Sx\| + (1-t)\|Tx\| \\ &\leq t\|S\|\|x\| + (1-t)\|T\|\|x\| \\ &\leq t\|x\| + (1-t)\|x\| = \|x\|. \end{aligned}$$

Since we have equality, we get $\|S\| = \|T\| = 1$ and $\|Sx\| = \|Tx\| = \|x\|$. So S, T are isometries. For every $x \in \partial(\mathcal{H})_1 = \operatorname{ext}(\mathcal{H})_1$, we have

$$Vx = t \cdot Sx + (1-t)Tx$$

and by the previous example that implies $Tx = Sx = Vx$, so we really have $S = T = V$. We use the same argument if V^* is an isometry. For now, we lack some tools to prove the reverse inclusion. We will prove the equality in Corollary 4.4.

Example 1.44. *If X be a Banach space, then $(X^*)_1$ is weak-$*$ compact (by Banach-Alaoglu), so Krein-Milman gives us $(X^*)_1 = \overline{\operatorname{co}}(\operatorname{ext}(X^*)_1)$. Hence $(X^*)_1$ has a lot of extreme points. As a corollary, c_0, $L^1[0,1]$ and $C[0,1]$ are not duals of Banach spaces.*

Theorem 1.45 (Milman)**.**

Let X be a LCS, $K \subseteq X$ compact and assume $\overline{\operatorname{co}}(K)$ is compact. Then $\operatorname{ext}(\overline{\operatorname{co}}(K)) \subseteq K$.

Proof. Assume there exists $x_0 \in \operatorname{ext}(\overline{\operatorname{co}}(K)) \setminus K$. Then there exists a basis neighborhood V of 0 in X such that $(x_0 + \overline{V}) \cap K = \emptyset$, or equivalently, $x_0 \notin K + \overline{V}$. If we write $K \subseteq \bigcup_{x \in K}(x + V)$, we get

$$K \subseteq \bigcup_{j=1}^{n} (x_j + V).$$

Form $K_j = \overline{\operatorname{co}}(K \cap (x_j + V))$. Then K_j is convex and compact since $K_j \subseteq \overline{\operatorname{co}}(K)$. We also have $K_j \subseteq \overline{x_j + V} = x_j + \overline{V}$ since V is convex. Also, $K \subseteq K_1 \cup \cdots \cup K_n$. Next we prove that $\mathrm{co}(K_1 \cup \cdots \cup K_n)$ is compact. Define

$$\Sigma = \{(t_1, \ldots, t_n) \in [0,1]^n \mid \sum_{j=1}^{n} t_j = 1\}$$

and the function

$$f : \Sigma \times K_1 \times \cdots \times K_n \to X, \quad (t, k_1, \ldots, k_n) \mapsto \sum_{j=1}^{k} t_j k_j.$$

Denote $C := \mathrm{im}\, f$. Obviously, $C \subseteq \mathrm{co}(K_1 \cup \cdots \cup K_n)$ and C is a convex compact set. Furthermore, $C \supset K_j$ for each j, so $C = \mathrm{co}(K_1 \cup \cdots \cup K_n)$. From there, we get

$$\overline{\mathrm{co}}(K) \subseteq \overline{\mathrm{co}}(K_1 \cup \cdots \cup K_n) = \mathrm{co}(K_1 \cup \cdots \cup K_n).$$

But since $K_j \subseteq \overline{\mathrm{co}}(K)$ for all j, we deduce $\overline{\mathrm{co}}(K) = \overline{\mathrm{co}}(K_1 \cup \cdots \cup K_n)$. We know that $x_0 \in \overline{\mathrm{co}}(K)$, so

$$x_0 = t_1 y_1 + \cdots + t_n y_n$$

for some $t_i \in [0,1]$, $\sum t_i = 1$ and $y_j \in K_j$. But $x_0 \in \mathrm{ext}(\overline{\mathrm{co}})(K)$, so $y_j = x_0$ for some j. So we get $x_0 \in K_j \subseteq x_j + \overline{V} \subseteq K + \overline{V}$, a contradiction. □

Remark. (1.) In finite dimensions, the convex hull of a compact set is compact. In infinite dimensions this fails.

(2.) The set $\mathrm{ext}(C)$ is not always closed, even if $C \subseteq \mathbb{R}^3$ is convex and compact.

ADD A PICTURE

2 C^*-algebras and continuous functional calculus

2.1 Spectrum

Let A be a complex algebra with a unit 1 and

$$\mathrm{GL}(A) = \{a \in A \mid a \text{ is invertible}\}.$$

If $x \in A$, we define the spectrum

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \cdot 1 \notin \mathrm{GL}(A)\}.$$

Proposition 2.1. *Let A be a complex algebra with unity 1 and $x, y \in A$. Then*

$$\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}.$$

Proof. Suppose $1 - xy \in \mathrm{GL}(A)$. Formally, we can write

$$(1 - xy)^{-1} = 1 + xy + (xy)^2 + \cdots$$

and

$$(1 - yx)^{-1} = 1 + yx + (yx)^2 + \cdots = 1 + y(1 - xy)^{-1}x.$$

From this, we claim that indeed $1 - yx \in \mathrm{GL}(A)$ and

$$(1 - yx)^{-1} = 1 + y(1 - xy)^{-1}x.$$

The proof is straightforward: we have

$$\begin{aligned}(1 + y(1 - xy)^{-1}x)(1 - yx) &= (1 - yx) + y(1 - xy)^{-1}(x - xyx) \\ &= (1 - yx) + y(1 - xy)^{-1}(1 - xy)x \\ &= (1 - yx) + yx = 1\end{aligned}$$

and

$$\begin{aligned}(1 - yx)(1 + y(1 - xy)^{-1}x) &= (1 - yx) + (y - yxy)(1 - xy)^{-1}x \\ &= (1 - yx) + y(1 - xy)(1 - xy)^{-1}x \\ &= (1 - yx) + yx = 1.\end{aligned}$$

Now the proof of the statement is at hand: if $\lambda \in \sigma_A(xy) \setminus \{0\}$, then

$$\lambda - xy \notin \mathrm{GL}(A) \Rightarrow 1 - \frac{x}{\lambda}y \notin \mathrm{GL}(A) \Rightarrow 1 - y\frac{x}{\lambda} \notin \mathrm{GL}(A) \Rightarrow \lambda - yx \notin \mathrm{GL}(A).$$

Thus, $\lambda \in \sigma_A(yx)$. Similarly, if $\lambda \in \sigma_A(yx) \setminus \{0\}$, then $\lambda \in \sigma_A(xy)$. □

Example 2.2. *Let $S, S^* \in \mathcal{B}(\ell^2)$ be the right and left shift operators, respectively. Then $SS^* = I$, but*

$$SS^*(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

This implies that $0 \in \sigma(SS^*)$, but $0 \notin \sigma(S^*S)$.

2.2 Banach and C^*-algebras

Definition 2.3.
- A Banach algebra is a Banach space A that is also an algebra, satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. If a Banach algebra has a unit 1, we also demand $\|1\| = 1$.
- An involution on a Banach algebra A is a skew-linear map

  $$* : A \to A, \quad a \mapsto a^*$$

  satisfying

  $$(xy)^* = y^* x^*, \quad (x^*)^* = x, \quad \|x^*\| = \|x\|.$$

  A C^*-algebra is a Banach $*$-algebra A that also satisfies $\|x^* x\| = \|x\|^2$ for all $x \in A$.

Unless otherwise mentioned, all algebras in this section are unital.

Are all our Banach algebras complex? We probably need that for nonempty spectra.

Proposition 2.4. *We collect some basic properties of Banach algebras.*

(1.) If A is a Banach $*$-algebra, then $(x^*)^{-1} = (x^{-1})^*$ and $\sigma_A(x^*) = (\sigma_A(x))^*$.

(2.) Let A be a Banach algebra. If $\|x\| < 1$, then $1 - x \in \mathrm{GL}(A)$ and

$$(1-x)^{-1} = 1 + x + x^2 + \cdots$$

As a consequence, if $\|1 - x\| < 1$, then $x \in \mathrm{GL}(A)$.

(3.) Let A be a Banach algebra. Then $\mathrm{GL}(A) \subseteq A$ is open, and the map $x \mapsto x^{-1}$ is continuous on $\mathrm{GL}(A)$.

(4.) If A is a Banach algebra and $x \in A$, then $\sigma_A(x)$ is a nonempty compact set.

Proof. (1.) Suppose that the inverse $(x^*)^{-1}$ exists. Then $(x^*)^{-1} \cdot (x^*) = 1$, so starring gives us $(x^*)^* \cdot ((x^*)^{-1})^* = 1$ and $x \cdot ((x^*)^{-1})^* = 1$. Similarly, we have $(x^*) \cdot (x^*)^{-1} = 1$, which implies $((x^*)^{-1})^* \cdot x = 1$. This means that x is invertible and $((x^*)^{-1})^* = x^{-1}$. Starring this equation now gives us $(x^*)^{-1} = (x^{-1})^*$. For the opposite direction, suppose that x is invertible. Then

$$(x^{-1})^* \cdot x^* = (x \cdot x^{-1})^* = 1^* = 1$$

and

$$x^* \cdot (x^{-1})^* = (x^{-1} \cdot x)^* = 1^* = 1,$$

which means that x^* is invertible and $(x^*)^{-1} = (x^{-1})^*$. The rest is a matter of simple computation:

$$\begin{aligned}
\lambda \in \sigma_A(x^*) &\Leftrightarrow x^* - \lambda \notin \mathrm{GL}(A) \Leftrightarrow (x - \overline{\lambda})^* \notin \mathrm{GL}(A) \\
&\Leftrightarrow (x - \overline{\lambda}) \notin \mathrm{GL}(A) \Leftrightarrow \overline{\lambda} \in \sigma_A(x) \\
&\Leftrightarrow \lambda \in (\sigma_A(x))^*.
\end{aligned}$$

If $\|x\| \leq 1$, then the series $\sum_{n=0}^{\infty} x^n$ converges in norm to some x'. Since multiplication between elements of a Banach algebra is norm-continuous, we get we get

$$(1-x)x' = (1-x) \cdot \lim_{k\to\infty} \sum_{n=1}^{k} x^n = \lim_{k\to\infty} (1-x) \cdot \sum_{n=1}^{k} x^n = \lim_{k\to\infty} 1 - x^{k+1} = 1$$

and similarly for $x'(1-x)$.
Let $y \in \mathrm{GL}(A)$. If $\|x-y\| \leq \frac{1}{\|y^{-1}\|}$, then

$$\|1 - xy^{-1}\| = \|(y-x)y^{-1}\| \leq \|y-x\|\|y^{-1}\| \leq 1,$$

which implies that $xy^{-1} \in \mathrm{GL}(A)$, and thus $x = xy^{-1} \cdot y \in \mathrm{GL}(A)$. We have shown that $\mathrm{GL}(A)$ is open. Using the same notation and noting that $(xy^{-1})^{-1} = (1-(1-xy^{-1}))^{-1}$, we get

$$\|(xy^{-1})^{-1}\| \leq \sum_{n=0}^{\infty} \|(1-xy^{-1})\|^n \leq \sum_{n=0}^{\infty} \|y^{-1}\|^n \|x-y\|^n \leq \frac{1}{1-\|y^{-1}\| \cdot \|x-y\|}.$$

Now,

$$\begin{aligned}
\|x^{-1} - y^{-1}\| &= \|x^{-1}(y-x)y^{-1}\| \\
&\leq \|y^{-1}(xy^{-1})^{-1}\|\|y-x\|\|y^{-1}\| \\
&\leq \|(xy^{-1})^{-1}\|\|y-x\|\|y^{-1}\|^2 \\
&\leq \frac{\|y^{-1}\|^2}{1-\|y^{-1}\| \cdot \|x-y\|}\|y-x\|.
\end{aligned}$$

Since the function $t \mapsto \frac{\|y^{-1}\|^2}{1-\|y^{-1}\|\cdot t} t$ is continuous at $t = 0$, the map $x \mapsto x^{-1}$ is continuous.
First, we prove compactness by showing that $\sigma_A(x)$ is bounded and closed. Suppose there exists $\lambda \in \sigma_A(x)$, such that $|\lambda| > \|x\|$. Then $\left(1 - \frac{x}{\lambda}\right)$ is invertible by (2.), so $(-\lambda) \cdot \left(1 - \frac{x}{\lambda}\right) = x - \lambda$ is invertible as well. But this contradicts the fact that $\lambda \in \sigma_A(x)$, so we have shown that $\sigma_A(x) \subseteq \overline{B(0, \|x\|)}$. Next, we prove that the spectrum is closed. Define a continuous map

$$\mathbb{C} \to A, \quad \lambda \mapsto x - \lambda$$

and notice that the inverse image of $\mathrm{GL}(A)$ (which is open by (3.)) is exactly $\mathbb{C} \setminus \sigma_A(x)$. This means that $\mathbb{C} \setminus \sigma_A(x)$ is open and $\sigma_A(x)$ is closed. For non-emptyness, we have to employ some standard Banach algebra techniques. We say that a function from f from a domain $\Omega \subseteq \mathbb{C}$ to a Banach space X is analytical if there exists a limit

$$f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

for every $z_0 \in \Omega$ and the function f' is continuous on Ω. A lot of theory for complex analytic functions also applies to Banach space-valued analytic functions; in particular, we have Cauchy's integral formula, Liouville's theorem and the fact that every vector valued analytic function can be locally expressed as a power series with coefficients in X. Now we can define the resolvent function

$$F : \mathbb{C} \setminus \sigma_A(x) \to A, \quad F(z) = (z-x)^{-1}.$$

It's routine to show that F is analytic and its derivative is $F'(z) = (z-x)^{-2}$. Now for $z \in \mathbb{C} \setminus \overline{B(0, \|x\|)}$, we have $F(z) = z^{-1} \cdot (1 - a/z)$, which goes to 0 as $z \to \infty$. Now if $\sigma_A(x) = \emptyset$, then F would be an entire function that vanishes at ∞. By Liouville's theorem, F is constant and so $F' = 0$. This is a contradiction. $\square$

Theorem 2.5 (Gelfand-Mazur)**.**

If A is Banach algebra that is also a division ring, then $A = \mathbb{C}$.

Proof. Let $x \in A$ and $\lambda \in \sigma_A(x)$. Then $x - \lambda \cdot 1 \notin \mathrm{GL}(A)$, implying $x - \lambda = 0$, hence $x = \lambda \in \mathbb{C}$. $\square$

Definition 2.6. If $f(x) = \sum_{j=0}^n a_j x^j$ is a polynomial and $a \in A$, we define $f(a) = \sum_{j=0}^n a_j a^j \in A$.

Theorem 2.7 (Spectral mapping theorem for polynomials)**.**

Let A be a complex unitary algebra and $f \in \mathbb{C}[x]$. Then $f(\sigma_A(a)) = \sigma_A(f(a))$ for all $a \in A$.

Proof. First, we prove the inclusion ($\subseteq$). If $\lambda \in \sigma_A(a)$ and $f(x) = \sum_{j=0}^n a_j x^j$, then

$$f(x) - f(\lambda) = \sum_{j=1}^n a_j (x^j - \lambda^j) = (x - \lambda) \cdot \sum_{j=1}^n a_j \sum_{k=0}^{j-1} x^k \lambda^{j-1-k}.$$

Substituting $x = a$, we obtain

$$f(a) - f(\lambda) = (a - \lambda) \left(\sum_{j=1}^n a_j \sum_{k=0}^{j-1} a^k \lambda^{j-1-k} \right).$$

Since $a - \lambda$ commutes with the second factor, $f(a) - f(\lambda)$ is not invertible and $f(\lambda) \in \sigma_A(f(a))$. For the converse inclusion ($\supseteq$), assume $\mu \notin f(\sigma_A(a))$. We factor

$$f(x) - \mu = a_n (x - \lambda_1) \cdots (x - \lambda_n).$$

Since $f(\lambda) - \mu \neq 0$ for any $\lambda \in \sigma_A(a)$, it follows that $\lambda_i \notin \sigma_A(a)$ for all i. Therefore, $f(a) - \mu \in \mathrm{GL}(A)$. $\square$

Definition 2.8. Let A be a Banach algebra and $x \in A$. The spectral radius of x is

$$r(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|.$$

Remark. By proposition 2.1, we have $r(xy) = r(yx)$.

In the introductory course, we proved the following.

Theorem 2.9 (Spectral radius formula)**.**

Let A be a Banach algebra and $x \in A$. Then $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and is equal to $r(x)$.

Definition 2.10. Let A be a Banach $*$-algebra and $x \in A$.

- x is *normal* iff $xx^* = x^*x$.
- x is *self-adjoint* iff $x^* = x$.
- x is *skew self-adjoint* iff $x^* = -x$.

The set of all self-adjoint operators is denoted as A_{sa}.

Remark. Every $a \in A$ can be uniquely expressed as a sum of a self-adjoint and skew self-adjoint element:

$$a = \frac{a + a^*}{2} + \frac{a - a^*}{2}.$$

Alternatively, we can uniquely write it in the form of

$$a = \left(\frac{a + a^*}{2}\right) + i \cdot \left(\frac{a - a^*}{2i}\right)$$

where both terms in parentheses are self-adjoint.

Corollary 2.11. *Let A be a Banach $*$-algebra and $x \in A$ normal. Then $r(x^*x) \le r(x)^2$. If A is a C^*-algebra, then $r(x^*x) = r(x)^2$.*

Proof. We use the spectral radius formula:

$$\begin{aligned} r(x^*x) &= \lim_{n\to\infty} \|(x^*x)^n\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|(x^*)^n x^n\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|(x^n)^* x^n\|^{\frac{1}{n}} \\ &\le \lim_{n\to\infty} \|x^n\|^{\frac{2}{n}} = r(x)^2. \end{aligned}$$

If A is a C^*-algebra, we have an equality in the last line of the above calculation. □

Proposition 2.12. *Let A be a C^*-algebra and $x \in A$ normal. Then $r(x) = \|x\|$.*

Proof. First, assume x is self-adjoint. Then

$$\|x^2\| = \|xx^*\| = \|x\|^2.$$

By induction, we get $\|x^{2^n}\| = \|x\|^{2^n}$ for every $n \in \mathbb{N}$. Therefore,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|.$$

If x is only normal, then

$$\|x\|^2 = \|x^*x\| = r(x^*x) = r(x)^2,$$

which implies $\|x\| = r(x)$. □

Corollary 2.13. *Let A, B be C^*-algebras and $\Phi : A \to B$ a $*$-homomorphism ($\Phi(x^*) = \Phi(x)^*$). Then Φ is a contraction. Furthermore, if Φ is a $*$-isomorphism, then it is isometric.*

Proof. Clearly, Φ maps invertible elements to invertible elements, so $\Phi(\mathrm{GL}(A)) \subseteq \mathrm{GL}(B)$. This implies $\sigma_B(\Phi(x)) \subseteq \sigma_A(x)$, hence $r(\Phi(x)) \leq r(x)$. Then

$$\begin{aligned}\|\Phi(x)\|^2 &= \|\Phi(x)\Phi(x)^*\| = \|\Phi(x)\Phi(x^*)\| \\ &= \|\Phi(xx^*)\| = r(\Phi(xx^*)) \\ &\leq r(xx^*) = \|xx^*\| = \|x\|^2.\end{aligned}$$

If Φ is a $*$-isomorphism, we apply the same reasoning to its inverse, which implies that Φ must be an isometry. □

Corollary 2.14. *If A is a $*$-algebra, then there exists at most one norm on A that makes it into a C^*-algebra.*

Proof. Considering the identity map

$$(A, \|\|_1) \to (A, \|\|_2),$$

it is a $*$-isomorphism, so it preserves the norm by the previous corollary. □

Lemma 2.15. *Let A be a C^*-algebra and $x \in A$ self-adjoint. Then $\sigma_A(x) \subseteq \mathbb{R}$.*

Proof. Suppose $\lambda = \alpha + i\beta \in \sigma_A(x)$ for some $\alpha, \beta \in \mathbb{R}$. Define $y = x - \alpha + it$ for $t \in \mathbb{R}$. Then $i(\beta + t) \in \sigma_A(y)$ and y is normal. Thus,

$$\begin{aligned}|i(\beta + t)|^2 = (\beta + t)^2 &\leq r(y) \\ &= \|y\|^2 = \|yy^*\| \\ &= \|(x - \alpha)^2 + t^2\| \leq \|x - \alpha\|^2 + t^2.\end{aligned}$$

Simplifying, we get $\beta^2 + 2\beta t \leq \|x - \alpha\|^2$, and since $t \in \mathbb{R}$ was arbitrary, we have $\beta = 0$. □

Lemma 2.16. *Let A be a Banach algebra and $x \notin \mathrm{GL}(A)$. If $(x_n)_n \subseteq \mathrm{GL}(A)$ satisfies $x_n \to x$, then $\|x_n^{-1}\| \to \infty$.*

Proof. If the sequence $\|x_n^{-1}\|$ is bounded, then

$$\|1 - xx_n^{-1}\| = \|(x_n - x)x_n^{-1}\| \leq \|x_n - x\| \cdot \|x_n^{-1}\| \to 0.$$

In particular, there exists some $n \in \mathbb{N}$ such that $\|1 - xx_n^{-1}\| < 1$, which implies $xx_n^{-1} \in \mathrm{GL}(A)$ and therefore $x = (xx_n^{-1})x_n \in \mathrm{GL}(A)$, a contradiction. □

Proposition 2.17. *Let B be a C^*-algebra and $A \subseteq B$ a unital C^*-subalgebra. Then for all*

$x \in A$, *we have* $\sigma_A(x) = \sigma_B(x)$.

Proof. Obviously, $\mathrm{GL}(A) \subseteq \mathrm{GL}(B)$. For a self adjoint $x \in A \setminus \mathrm{GL}(A)$, we have $it \notin \sigma_A(x)$ for $t \in \mathbb{R}$. So there exists $(x - it)^{-1} \in A$. Clearly,

$$x - it \in \mathrm{GL}(A) \xrightarrow[t \to 0]{} x \notin \mathrm{GL}(A),$$

thus $\|(x - it)^{-1}\| \to \infty$. Since the inverse function is continuous, this immediately yields $x \notin \mathrm{GL}(B)$. For general $x \in A$: if $x \in \mathrm{GL}(B)$, then $x^*x \in \mathrm{GL}(B)$ is self-adjoint. By the first part of the proof, $x^*x \in \mathrm{GL}(A)$. It follows that

$$x^{-1} = (x^*x)^{-1}x^* \in A,$$

so $x \in \mathrm{GL}(A)$. □

Example 2.18. *Let* X *be a Hausdorff topological space and* $C_b(X)$ *be the set of continuous bounded complex functions on* X, *endowed with the sup metric. Then* $C_b(X)$ *is a unital abelian* C^**-algebra (where* $f^*(x) = \overline{f(x)}$*).*

Example 2.19. *Let* X *be a locally compact Hausdorff space and*

$$C_0(X) = \{f \in C(X) \mid \forall \varepsilon > 0 : \ \exists K^{compact} \subseteq X : \ |f(x)| \leq \varepsilon, \ \forall x \in X \setminus K\}$$

the set of all complex continuous functions on X *that vanish at infinity. Then* $C_0(X)$ *is an abelian* C^**-algebra. In some sense, it is the natural abelian* C^**-algebra. The algebra* $C_0(X)$ *is unital iff* X *is compact – in that case,* $C_0(X) = C_b(X) = C(X)$.

Example 2.20. *Let* (X, μ) *be a measure space. Then* $L^\infty(X, \mu)$*, the set of essentially bounded functions on* X *endowed with the essential supremum norm, is a unital abelian* C^**-algebra.*

Example 2.21. *For a Hilbert space* $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ *is a non-abelian* C^**-algebra: for all* $x \in \mathcal{B}(\mathcal{H})$ *we have* $\|x^*x\| = \|x\|^2$.

Example 2.22. *If* Γ *is a group, we define*

$$\ell^1(\Gamma) = \{(\alpha_s)_{s\in\Gamma} \mid \alpha_s \in \mathbb{C}, \sum_{s\in\Gamma} |\alpha_s| < \infty\}.$$

We can then introduce the convolution multiplication on $\ell^1(\Gamma)$*:*

$$(\alpha * \beta)_s = \sum_{t\in\Gamma} \alpha_{st}\beta_{t^{-1}}.$$

This is a Banach algebra; it is even a Banach $*$*-algebra with involution* $(\alpha^*)_s = \overline{\alpha_{s^{-1}}}$*. However, it is not a* C^**-algebra if the group* Γ *has more than one element. In that case,* *there exists $z \in \Gamma$ such that $z \neq 1$. Define $\alpha = (\alpha_s) \in \ell^1(G)$ such that*

$$\alpha_s = \begin{cases} 1; & s = 1 \\ i; & s = z, z^{-1} \\ 0; & \textit{otherwise} \end{cases}.$$

If $z \neq z^{-1}$, we have

$$\begin{aligned}
\|\alpha\alpha^*\| &= \sum_{s\in\Gamma} \left| \sum_{t\in\Gamma} \alpha_{st}\overline{\alpha_t} \right| \\
&= \sum_{s\in\Gamma} (3 \cdot \mathbf{1}_{s=1} + \mathbf{1}_{s=z^2} + \mathbf{1}_{s=z^{-2}}) \\
&< \sum_{s\in\Gamma} (3 \cdot \mathbf{1}_{s=1} + 2 \cdot \mathbf{1}_{s=z} + 2 \cdot \mathbf{1}_{s=z^{-1}} + \mathbf{1}_{s=z^2} + \mathbf{1}_{s=z^{-2}}) \\
&= \sum_{s\in\Gamma} \sum_{t\in\Gamma} |\alpha_{st}\alpha_t| = \sum_{t\in\Gamma} |\alpha_t| \cdot \sum_{s\in\Gamma} |\alpha_{st}| \\
&= \sum_{t\in\Gamma} |\alpha_t| \cdot \sum_{s\in\Gamma} |\alpha_s| = \|\alpha\|^2.
\end{aligned}$$

Otherwise, we get

$$\begin{aligned}
\|\alpha\alpha^*\| &= \sum_{s\in\Gamma} \left| \sum_{t\in\Gamma} \alpha_{st}\overline{\alpha_t} \right| \\
&= \sum_{s\in\Gamma} (2 \cdot \mathbf{1}_{s=1}) \\
&< \sum_{s\in\Gamma} (2 \cdot \mathbf{1}_{s=1} + 2 \cdot \mathbf{1}_{s=z}) \\
&= \sum_{s\in\Gamma} \sum_{t\in\Gamma} |\alpha_{st}\alpha_t| = \sum_{t\in\Gamma} |\alpha_t| \cdot \sum_{s\in\Gamma} |\alpha_{st}| \\
&= \sum_{t\in\Gamma} |\alpha_t| \cdot \sum_{s\in\Gamma} |\alpha_s| = \|\alpha\|^2.
\end{aligned}$$

Therefore, $\ell^1(\Gamma)$ is not a C^*-algebra if Γ has order greater than one.

2.3 Gelfand transform

Definition 2.23. Let A be an abelian Banach algebra. The *spectrum* of A is defined as

$$\sigma(A) := \{\varphi : A \to \mathbb{C} \mid \varphi \neq 0 \text{ continuous algebra homomorphism}\} \subseteq A^*$$

endowed with a weak-$*$ topology. Its elements are called *characters.*

If $\varphi \in \sigma(A)$, then $\ker\varphi \cap \mathrm{GL}(A) = \emptyset$. For $x \in A$, we have

$$\begin{aligned}\varphi(x - \varphi(x)) &= \varphi(x) - \varphi(\varphi(x) \cdot 1)\\ &= \varphi(x) - \varphi(x)\varphi(1)\\ &= \varphi(x) - \varphi(x) = 0,\end{aligned}$$

which implies that $\varphi(x) \in \sigma_A(x)$. Consequently, $|\varphi(x)| \leq r(x) \leq \|x\|$, giving us the bound $\|\varphi\| \leq 1$. But since $\varphi(1) = 1$, we get $\|\varphi\| = 1$. We know that $\sigma(A)$ is closed in $(A^*)_1$, making $\sigma(A)$ is a compact Hausdorff space by Banach-Alaoglu.

Proposition 2.24. *Let A be a C^*-algebra and $h : A \to \mathbb{C}$ a non-zero homomorphism (not necessarily a $*$-homomorphism). Then the following statements hold:*

(1.) $h(a) \in \mathbb{R}$ for self-adjoint a;

(2.) $h(a^*) = \overline{h(a)}$ for all $a \in A$;

(3.) $h(aa^*) \geq 0$ for all $a \in A$;

(4.) if $uu^* = 1$ or u^*u, then $|h(u)| = 1$.

Remark. The first three item also hold for non-unital algebras.

Proof. (1.) Since $h(a) \in \sigma_A(a)$ and self-adjoint elements have real spectrum, this is trivial.
(2.) Let $a = a_1 + ia_2$, where a_1, a_2 are self-adjoint. Then $a^* = a_1 - ia_2$ and

$$h(a^*) = h(a_1 - ia_2) = h(a_1) - ih(a_2) = \overline{h(a_1) + ih(a_2)} = \overline{h(a)}.$$

(3.) Follows from (b).
(4.) If u is unitary, then $|h(u)|^2 = h(u)h(u^*) = h(uu^*) = h(1) = 1.$ ☐

Corollary 2.25. *Every nonzero algebra homomorphism $h : A \to \mathbb{C}$ is a character.*

Proposition 2.26. *Let A be an abelian Banach algebra. Then the map $\varphi \mapsto \ker\varphi$ is a bijection from $\sigma(A)$ to the set of all maximal ideals of A.*

Proof. If $\varphi \in \sigma(A)$, then $\ker\varphi \lhd A$. Suppose that $\ker\varphi \subsetneq I \lhd A$. Then there exists an element $x \in I \setminus \ker\varphi$. Thus, $\varphi(x) \neq 0$ and from $1 - \frac{x}{\varphi(x)} \in \ker\varphi$. From there, it follows that

$$1 = \left(1 - \frac{x}{\varphi(x)}\right) + \frac{1}{\varphi(x)} \cdot x \in I.$$

Hence, $\ker\varphi$ is a maximal ideal. Conversely, let $I \lhd A$ be a maximal ideal. Then $I \cap \mathrm{GL}(A) = \emptyset$ and since $\mathrm{GL}(A)$ is open, we also have $\overline{I} \cap \mathrm{GL}(A) = \emptyset$. Thus, $\overline{I} \lhd A$ and $1 \notin \overline{I}$, so $I \subseteq \overline{I} \subsetneq A$. By maximality, $\overline{I} = I$. Then A/I is a Banach algebra and since I is maximal, every nonzero element in A/I is invertible. By Gelfand-Mazur, $A/I \cong \mathbb{C}$. The projection $\pi : A \to A/I \cong \mathbb{C}$ is in $\sigma(A)$ and $\ker\pi = I$. ☐

Corollary 2.27. *Let A be an abelian Banach algebra and $x \in A \setminus \mathrm{GL}(A)$. Then there exists $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$. In particular, $\sigma(A) \neq 0$.*

Proof. If $x \notin \mathrm{GL}(A)$, then it generates an ideal $\langle x \rangle \subsetneq A$. By Zorn's lemma, $\langle x \rangle$ has to be included in some maximal ideal $I \lhd A$. By the previous proposition, there exists a character $\varphi : A \to \mathbb{C}$ in $\sigma(A)$ such that $x \in I = \ker \varphi$. $\square$

Theorem 2.28 (Stone-Čech).

Let X be a topological space. For $x \in X$, let $\beta_x : C_b(X) \to \mathbb{C}$ be the evaluation homomorphism $f \mapsto f(x)$. Then

$$\beta : X \to \sigma(C_b(X)), \quad x \mapsto \beta_x$$

is a continuous map whose image is dense in the codomain and has the following universal property: if $\pi : X \to K^{T_2,\ compact}$ is continuous, then there exists a unique continuous mapping

$$\beta_\pi : \sigma(C_b(X)) \to K$$

such that $\pi(x) = \beta_\pi(\beta_x)$ for all $x \in X$. In particular, if X is compact T_2, then β is a homeomorphism.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\beta"] & K^{T_2,\ compact} \\ \sigma(C_b(X)) \arrow[ur, dashed, "\exists ! \beta_\pi"'] & \end{tikzcd}$$

Proof. (1.) First, we prove that β is continuous. Let $(x_i)_i$ ibes a net in X and $x_i \to x$, then for all $f \in C_b(X)$ we have $\beta_{x_i} = f(x_i) \to f(x) = \beta_x(f)$. Hence $\beta_{x_i} \to \beta_x$ in the weak-$*$ topology.

(2.) Next, we prove that $\mathrm{im}\, \beta$ is dense. Assume otherwise and pick $\varphi \in \sigma(C_b(X)) \setminus \overline{\beta(X)}$. Define $I := \ker \varphi$. For all $\psi \in \overline{\beta(X)}$, there exists $f_\psi \in I$ such that $f_\psi \in \ker \psi$. Hence, there exists c_ψ and a neighborhood U_ψ of ψ such that $|\widetilde{\psi}(f)| > c_\psi$ for all $\widetilde{\psi} \in U_\psi$. Thus, $\overline{\beta(X)} \subseteq U_{\psi \in \overline{\beta(X)}} U_\psi$. By compactness, there exists a finite subcovering of $\overline{\beta(X)}$, so $\overline{\beta(X)} \subseteq \bigcup_{i=1}^n U_{\psi_j}$. Then there exist $f_{\psi_1}, \ldots, f_{\psi_n} \in I$ and $c > 0$ such that

$$\sum_{i=1}^n \psi(|f_{\psi_i}|^2) > c, \quad \forall \psi \in \overline{\beta(X)}.$$

Hence,

$$\sum_{i=1}^n |f_{\psi_i}|^2(x) = \sum_{i=1}^n \beta(x)(|f_{\psi_i}|^2) > c, \quad \forall x \in X.$$

It follows that $\sum_{i=1}^n |f_{\psi_i}|^2 \in I$ and $(\sum |f_{\psi_i}|^2)^{-1} \in C_b(X)$. As a result, $I = C_b(X)$.

(3.) If X is compact and Hausdorff, then β is surjective since $\beta(X)$ is dense and compact. Also, β is injective since $C_b(X)$ separates points. In that case, β is a continuous bijection between compact Hausdorff spaces, and therefore a homeomorphism.

(4.) For the universal property: let $\pi : X \to K$, where K is compact Hausdorff. Then there exists a continuous map

$$\pi^* : C(K) \to C_b(X), \quad f \mapsto f \circ \pi.$$

This induces a continuous map

$$\widetilde{\pi} : \sigma(C_b(X)) \to \sigma(C(K)), \quad \varphi \mapsto \varphi \circ \pi^*.$$

Since K is compact Hausdorff, the map $\beta^K : K \to \sigma(C(K))$ is a homeomorphism. Define

$$\beta_\pi : \sigma(C_b(X)) \to K, \quad \beta_\pi = (\beta^K)^{-1} \circ \widetilde{\pi}.$$

Then we have

$$\widetilde{\pi}(\beta_x)(g) = \beta_x(\pi^*(g)) = \pi^*(g)(x) = g(\pi(x)) = \beta^K_{\pi(x)}(g).$$

By left multiplying by $(\beta^K)^{-1}$, we get $\beta_\pi(\beta_x) = \pi(x)$. $\square$

Definition 2.29. Let A be an abelian Banach algebra. The Gelfand transform of A is the map

$$\Gamma : A \to C(\sigma(A)), \quad x \mapsto (\varphi \mapsto \varphi(x)).$$

Theorem 2.30.

Let A be an abelian Banach algebra. Then Γ is a homomorphism, contraction and for $x \in A$ we have

$$\Gamma(x) \in \mathrm{GL}(C(\sigma(A))) \Leftrightarrow x \in \mathrm{GL}(A).$$

Proof. The homomorphism part is routine. We prove that Γ is a contraction as follows:

$$\|\Gamma(x)\| = \sup_{\varphi \in \sigma(A)} \|\Gamma(x)\varphi\| = \sup_{\varphi} |\varphi(x)| \le \|x\|.$$

Next, we prove the equivalence. The right implication ($\Rightarrow$) is trivial, since

$$\Gamma(x^{-1})\Gamma(x) = \Gamma(x^{-1}x) = \Gamma(1) = 1.$$

Now the converse ($\Leftarrow$): if $x \notin \mathrm{GL}(A)$, then by corollary 2.27 there exists $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$. Then $\Gamma(x)(\varphi) = \varphi(x) = 0$, so the continuous map $\Gamma(x)$ is not invertible. $\square$

Corollary 2.31. *Let A be an abelian Banach algebra. Then we have*

$$\sigma(\Gamma(x)) = \sigma(x)$$

and

$$\|\Gamma(x)\| = r(\Gamma(x)) = r(x).$$

Theorem 2.32 (Gelfand)**.**

Let A be an abelian C^*-algebra. Then Γ is an isometric $*$-isomorphism.

Proof. For a self-adjoint $x \in A$ we have $\sigma(\Gamma(x)) = \sigma(x) \subseteq \mathbb{R}$. Then $\overline{\Gamma(x)} = \Gamma(x)$. An arbitrary $x \in A$ can be written as $x = a + ib$ for self-adjoint $a = \frac{x+x^*}{2}$ and $b = \frac{i(x^*-x)}{2}$. Then

$$\Gamma(x^*) = \Gamma(a - ib) = \Gamma(a) - i\Gamma(b) = \overline{\Gamma(a) + i\Gamma(b)} = \overline{\Gamma(x)}.$$

This implies that Γ is a $*$-homomorphism. Since A is abelian, each $x \in A$ is normal so

$$\|x\| = r(x) = r(\Gamma(x)) = \|\Gamma(x)\|$$

and Γ is an isometry. In particular, Γ is injective. We know that $\Gamma(A)$ is closed under $*$. Since Γ is isometric, the subalgebra $\Gamma(A) \subseteq C(\sigma(A))$ is complete in the norm, so it is closed. It can be easily checked that $\Gamma(A)$ separates points. By Stone-Weierstrass, $\Gamma(A) = C(\sigma(A))$. ☐

Remark. Let A be a C^*-algebra. If $x \in A$ is normal, then it generates an abelian C^*-subalgebra of A:

$$C^*(x) = \overline{\{p(x, x^*) \mid p \in \mathbb{C}[x, y]\}}.$$

Corollary 2.33. *Let A be an abelian C^*-algebra, generated by $x \in A$. Then $\sigma(A) \cong \sigma(x)$.*

Proof. Let $\Gamma : A \to C(\sigma(A))$ be the Gelfand transform. Define

$$\tau : \sigma(A) \to \sigma(x), \quad \varphi \mapsto \varphi(x) = \Gamma(x)(\varphi).$$

Clearly, τ is well-defined since $\varphi(x) \in \sigma(x)$ for all $\varphi \in \sigma(A)$. Next we show that τ is onto. For $\lambda \in \sigma(x)$ we have $x - \lambda \notin \mathrm{GL}(A)$, so there exists $\psi \in \sigma(A)$ such that $\psi(x) - \psi(\lambda) = \psi(x - \lambda) = 0$. We show that τ is injective. Let $\tau(\varphi_1) = \tau(\varphi_2)$. Then $\varphi_1(x) = \varphi_2(x)$. Since

$$\varphi_j(x^*) = \Gamma(x^*)(\varphi_j) = \overline{\Gamma(x)(\varphi_j)} = \overline{\varphi_j(x)},$$

we have $\varphi_1(x^*) = \varphi_2(x^*)$. Hence $\varphi_1(p(x, x^*)) = \varphi_2(p(x, x^*))$ for every polynomial $p \in \mathbb{C}[x, y]$. Since $\{p(x, x^*) \mid p \text{ polynomial}\}$ is dense in A, we have $\varphi_1 = \varphi_2$. Finally, we prove the continuity of τ. Let $(\varphi_\alpha)_\alpha$ be a net in $\sigma(A)$ such that $\varphi_\alpha \to \varphi$. Then $\varphi_\alpha(y) \to \varphi(y)$ for all $y \in A$, so in particular $\varphi_\alpha(x) \to \varphi(x)$, which proves that $\tau(\varphi_\alpha) \to \tau(\varphi)$. Since τ is a continuous bijection between compact Hausdorff spaces, it is a homeomorphism. ☐

Remark. Since $\varphi \in \sigma(A)$ is an algebra homomorphism, we have $\varphi(p(x, x^*)) = p(\varphi(x), \overline{\varphi(x)})$ for a complex polynomial $p(z, \overline{z})$ in z and $\overline{z}$. Using the notation from above proof, we get $\Gamma(p(x, x^*)) = p \circ \tau$.

2.4 Continuous functional theorem

Now let A be any C^*-algebra and $x \in A$ normal. Then $C^*(x)$ is an abelian C^*-subalgebra of A. Since $\sigma(x) = \sigma_{C^*(x)}$, we have the map

$$\tau^{\#} : C(\sigma(x)) \to C(C^*(x)), \quad f \mapsto f \circ \tau,$$

which is a $*$-isomorphism and an isometry. Define a map $\rho = \Gamma^{-1} \circ \tau^{\#} : C(\sigma(x)) \to C^*(x)$.

$$\begin{tikzcd} C^*(x) \arrow[rr, "\Gamma"] & & C(\sigma(A)) \\ & C(\sigma(x)) \arrow[ul, "\rho"] \arrow[ur, "\tau^{\#}"'] & \end{tikzcd}$$

We know that $C^*(x) = \overline{\{p(x, x^*) \mid p(z, \overline{z}) \text{ polynomial}\}}$ and $\Gamma(p(x, x^*)) = \tau^{\#}(p)$, which means that $\rho(p) = p(x, x^*)$ for any polynomial $p \in \mathbb{C}[x, y]$. This map $\rho : C(\sigma(x)) \to C^*(x) \subseteq A$ is called the continuous functional calculus. We use the notation $f(x) := \rho(f)$.

Theorem 2.34 (Continuous functional calculus)**.**

Let A, B be C^*-algebras and let $x \in A$ be normal.

(1.) *$f \mapsto f(x)$ is an isometric $*$-isomorphism $C(\sigma(x)) \to A$ and if*

$$f = \sum_{j,k=0}^{n} a_{jk} z^j \overline{z}^k$$

is a polynomial, then

$$f(x) = \sum_{j,k=0}^{n} a_{jk} x^j (x^*)^k.$$

In particular, if $f(z) = z$ is the identity polynomial, then $f(x) = x$.

(2.) *For $f \in C(\sigma(x))$, we have $\sigma(f(x)) = f(\sigma(x))$.*

(3.) ***(Spectral mapping theorem)** If $\Phi : A \to B$ is a $*$-homomorphism, then $\Phi(f(x)) = f(\Phi(x))$.*

(4.) *Let $(x_n)_n$ be a sequence of normal elements of A that converge to x, Ω a compact neighborhood of $\sigma(x)$, and $f \in C(\Omega)$. Then for any sufficiently large n, we have $\sigma(x_n) \subseteq \Omega$ and $\|f(x_n) - f(x)\| \to 0$.*

Proof. The items (1) and (2) follow directly from Gelfand theorem and properties of continuous functions on compact sets. The item (3) is obvious for polynomials f and the general case follows from Stone-Weierstrass. We prove the item (4). Let $C = \sup_n \|x_n\| < \infty$. First we need to show that $\sigma(x_n) \subseteq \Omega$ for large enough n. If that wasn't the case, then for every $n \in \mathbb{N}$ there would exist $N_n > n$ such that there exists $\lambda_n \in \sigma(x_{N_n}) \setminus \Omega \subseteq \overline{B_C(0)}$. Thus there exists a convergent subsequence $(\lambda_{n_k})_k$ such that $\lambda_{n_k} \to \lambda \in U$, where U is an open neighborhood of $\sigma(x)$ and $\lambda \notin \sigma(x)$. But then

$$\underbrace{x_{n_k} - \lambda_{n_k}}_{\notin \mathrm{GL}(A)} \to \underbrace{x - \lambda}_{\in \mathrm{GL}(A)},$$

which contradicts the openness of $\mathrm{GL}(A)$. For every $\varepsilon > 0$ there exists a polynomial $g : \Omega \to$

$\mathbb{C}$ such that $\|f - g\|_\infty < \varepsilon$. Now

$$\begin{aligned}\limsup_n \|f(x_n) - g(x_n)\| + \|g(x_n) - g(x)\| + \|g(x) - f(x)\| \\ &\leq 2 \cdot C \cdot \varepsilon + \limsup_n \|g(x_n) - g(x)\| \\ &= 2C\varepsilon.\end{aligned}$$

Since ε was arbitrary, we conclude that $\lim_{n\to\infty} \|f(x_n) - f(x)\| = 0$. □

We illustrate the use of continuous functional calculus to obtain the strengthening of corollary 2.13.

Corollary 2.35. *If A, B are C^*-algebras and $\Phi : A \to B$ is a $*$-monomorphism, then it is an isometry.*

Proof. Let $a \in A$ be self-adjoint. Then $\Phi(a) \in B$ is self-adjoint as well. As in the proof of 2.13, we observe that $\sigma_B(\Phi(a)) \subseteq \sigma_A(a)$. Suppose that $\sigma_B(\Phi(a)) \neq \sigma_A(a)$. Since $\sigma_B(\Phi(a))$ is compact, it is closed in $\sigma_A(a)$. This implies that $U := \sigma_A(a) \setminus \sigma_B(\Phi(a))$ is a nonempty open set. It follows that there exists a function f which is zero on $\sigma_B(\Phi(a))$, but not identically zero on $\sigma_A(a)$ (take for example any bump function on U). Then $f(\Phi(a)) = 0$, but $f(a) \neq 0$. By Stone-Weierstrass, we can approximate f uniformly on $\sigma_A(a)$ by polynomials $\{p_n\}_{n\in\mathbb{N}}$. Thus $p_n(a) \to f(a)$ and $p_n(\Phi(a)) \to f(\Phi(a)) = 0$. On the other hand, $p_n(\Phi(a)) = \Phi(p_n(a)) \to \Phi(f(a))$, which implies that $\Phi(f(a)) = f(\Phi(a)) = 0$. But Φ was assumed injective, so $f(a) = 0$, contradiction. Therefore, $\sigma_B(\Phi(a)) = \sigma_A(a)$ for self-adjoint a and

$$\|a\| = r(a) = r(\Phi(a)) = \|\Phi(a)\|.$$

Now for a completely arbitrary $a \in A$, we have

$$\|a\|^2 = \|a^*a\| = \|\Phi(a^*a)\| = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a)\|^2,$$

concluding our proof. □

The argument in this proof is very common. We first approximate some function on the spectrum with polynomials using Stone-Weierstrass. Then we observe that the CFC of a polynomial has desired properties and deduce the same for the CFC of the original function.

2.5 Application of the continuous functional theorem

Definition 2.36. Let A be a C^*-algebra and $x \in A$.

- x is *positive* if $x = y^*y$ for some $y \in A$ (i.e., x is a hernitian square). The set of positive elements is denoted A_+.
- x is a *projection* if $x^2 = x^* = x$.
- x is *unitary* if $xx^* = x^*x = 1$. The set of positive elements is denoted $U(A)$.
- x is an *isometry* if $x^*x = 1$.
- x is a *partial isometry* if x^*x is a projection.

Remark. The first three are automatically normal (the first two are even self-adjoint).

The set of all positive operators (denoted as A_+) induces a partial ordering on A_{sa}: for two elements $a, b \in A_{\text{sa}}$ we define

$$a \leq b \Leftrightarrow b - a \in A_+.$$

We notice that $x^* A_+ x \subseteq A_+$ for every $x \in A$. For any $a, b \in A_{\text{sa}}$ and $x \in A$, we have

$$a \leq b \Rightarrow x^* a x \leq x^* b x.$$

Proposition 2.37. *Let A be a C^*-algebra and $x \in A$. Then x is a linear combination of four unitaries.*

Proof. Since $x = \operatorname{Re} x + i \operatorname{Im} x$, where $\operatorname{Re} x, \operatorname{Im} x \in A_{\text{sa}}$, it's enough to show that every self-adjoint element is a linear combination of two unitaries. Without loss of generality, assume $\|x\| \leq 1$, so $\sigma(x) \subseteq [-1, 1]$. Consider the continuous function

$$f : [-1, 1] \to \mathbb{T}, \quad z \mapsto z + i(1 - z^2)^{\frac{1}{2}}.$$

Since $f \cdot \overline{f} \equiv 1$ on $[-1, 1]$, it follows from continuous functional calculus that

$$f(x) f(x)^* = f(x)^* f(x) = 1.$$

Consequently, $f(x) = u$ is unitary and $x = \frac{1}{2}(f(x) + f(x)^*)$ is a linear combination of two unitaries. $\square$

Remark. We use the notation $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition 2.38. Let $x \in A_{\text{sa}}$. Then $\sigma(x) \subseteq \mathbb{R}$ and we can define

$$x_+ = \max\{0, z\}(x) \in A, \quad x_- = -\min\{0, z\}(x) \in A.$$

Then $\sigma(x_+), \sigma(x_-) \subseteq [0, \infty)$, $x = x_+ - x_-$ and $x_+ x_- = x_- x_+ = 0$.

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Lemma 2.39. *Suppose $x, y \in A_{\text{sa}}$ satisfy $\sigma(x), \sigma(y) \subseteq [0, \infty)$. Then $\sigma(x + y) \subseteq [0, \infty)$.*

Proof. Let $a := \|x\|$ and $b := \|y\|$. Since $x = x^*$ and $\sigma(x) \subseteq [0, a]$, we deduce that $\sigma(a - x) \subseteq [0, a]$, where $\|a - x\| = r(a - x) \leq a$. Likewise, $\|b - y\| \leq b$. Then

$$\begin{aligned}
\sup_{\lambda \in \sigma(x+y)} \{a + b - \lambda\} &= r(a + b - (x + y)) \\
&= \|(a + b) - (x + y)\| \\
&\leq \|a - x\| + \|b - y\| \\
&\leq a + b.
\end{aligned}$$

$\square$

Theorem 2.40.

Let A be a C^*-algebra and $x \in A$ normal. Then:

(1.) $x \in A_{\text{sa}} \Leftrightarrow \sigma(x) \subseteq \mathbb{R}$;

(2.) $x \in A_+ \Leftrightarrow \sigma(x) \subseteq [0, \infty)$;

(3.) $x \in U(A) \Leftrightarrow \sigma(x) \subseteq \mathbb{T}$;

(4.) $x^2 = x^* = x \Leftrightarrow \sigma(x) \subseteq \{0, 1\}$.

Proof. Throughout this proof, let $f(z) = z$ denote the identity polynomial.

(1.)

$$\begin{aligned} x = x^* &\Leftrightarrow f(x) = \overline{f}(x) \\ &\Leftrightarrow f \equiv \overline{f} \text{ on } \sigma(x) \\ &\Leftrightarrow z = \overline{z} \text{ for all } z \in \sigma(x) \\ &\Leftrightarrow \sigma(x) \subseteq \mathbb{R}. \end{aligned}$$

(2.) ($\Rightarrow$) Let $x = y^* y$ for some $y \in A$. Write $x = x_+ - x_-$ and let $z := y \cdot x_-$. Then

$$z^* z = x_- y^* y x_- = x_- x x_- = -x_-^3.$$

From there we get

$$\sigma(zz^*) \subseteq \sigma(z^* z) \cup \{0\} \subseteq (-\infty, 0].$$

Let $z = a + ib$ for $a, b \in A_{\text{sa}}$. Then $zz^* + z^* z = 2a^2 + 2b^2$, which implies that $\sigma(zz^* + z^* z) \subseteq [0, \infty)$. It follows that

$$\sigma(z^* z) = \sigma((2a^2 + 2b^2) - zz^*) \subseteq [0, \infty).$$

As a result,

$$\sigma(-x_-^3) = \sigma(z^* z) \subseteq \{0\},$$

so $x_-^3 = 0$ and $x_- = 0$. This proves that $x = x_+$ has nonnegative spectrum. For the converse implication ($\Leftarrow$), apply the function $\sqrt{\cdot} : [0, \infty) \to \mathbb{R}$. Then

$$x = (\sqrt{x})^2 = (\sqrt{x})^* \cdot \sqrt{x} \in A_+.$$

(3.)

$$\begin{aligned} xx^* = 1 &\Leftrightarrow f(x) \cdot \overline{f}(x) = 1 \\ &\Leftrightarrow f \cdot \overline{f} \equiv 1 \text{ on } \sigma(x) \\ &\Leftrightarrow |z|^2 = 1 \text{ for all } z \in \sigma(x) \\ &\Leftrightarrow \sigma(x) \subseteq \mathbb{T}. \end{aligned}$$

(4.)

$$\begin{aligned} x^2 = x^* = x &\Leftrightarrow f(x) \cdot \overline{f}(x) = \overline{f}(x) = f(x) \\ &\Leftrightarrow f \cdot \overline{f} \equiv \overline{f} \equiv f \text{ on } \sigma(x) \\ &\Leftrightarrow |z|^2 = \overline{z} = z \text{ for all } z \in \sigma(x) \\ &\Leftrightarrow \sigma(x) \subseteq \{0, 1\}. \end{aligned}$$

□

Corollary 2.41. *Let A be a C^*-algebra and $x \in A$. Then x is a partial isometry iff x^* is a partial isometry.*

Proof.

$$\begin{aligned} x \text{ partial isometry} &\Leftrightarrow x^*x \text{ projection} \\ &\Leftrightarrow \sigma(x^*x) \subseteq \{0,1\} \\ &\Leftrightarrow \sigma(xx^*) \subseteq \{0,1\} \\ &\Leftrightarrow xx^* \text{ projection} \\ &\Leftrightarrow x^* \text{ partial isometry.} \end{aligned}$$

□

Corollary 2.42. *Let A be a C^*-algebra.*

(1.) A_+ is a closed convex cone ($\lambda A_+ \subseteq A_+$ for $\lambda \in \mathbb{R}_{\geq 0}$).

(2.) If $a \in A_{\mathrm{sa}}$, then $a \leq \|a\|$.

Proposition 2.43. *Let A be a C^*-algebra and $x, y \in A_+$.*

(1.) If $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.

(2.) If $x, y \in \mathrm{GL}(A)$ and $x \leq y$, then $y^{-1} \leq x^{-1}$.

Proof. Let us prove the second point first. Suppose $x, y \in \mathrm{GL}(A)$. Then we have $y^{-\frac{1}{2}}xy^{-\frac{1}{2}} \leq 1$ and

$$\begin{aligned} x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}} &\leq \|x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}}\| \\ &= r(x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}}) \\ &= r(y^{-\frac{1}{2}}xy^{-\frac{1}{2}}) \\ &\leq 1. \end{aligned}$$

Multiplying on both sides by $x^{-\frac{1}{2}}$, we get $y^{-1} \leq x^{-1}$. Now we prove the first point. For invertible $x \leq y$, we have

$$\begin{aligned} \|y^{-\frac{1}{2}}x^{\frac{1}{2}}\|^2 &= \|(y^{-\frac{1}{2}}x^{\frac{1}{2}})(y^{-\frac{1}{2}}x^{\frac{1}{2}})^*\| \\ &= \|y^{-\frac{1}{2}}xy^{-\frac{1}{2}}\| \\ &\leq 1, \end{aligned}$$

which implies

$$\begin{aligned} y^{-\frac{1}{4}}x^{\frac{1}{2}}y^{-\frac{1}{4}} &\leq \|y^{-\frac{1}{4}}x^{\frac{1}{2}}y^{-\frac{1}{4}}\| \\ &= r(y^{-\frac{1}{4}}x^{\frac{1}{2}}y^{-\frac{1}{4}}) \\ &= r(y^{-\frac{1}{2}}x^{\frac{1}{2}}) \\ &= \|y^{-\frac{1}{2}}x^{\frac{1}{2}}\| \leq 1. \end{aligned}$$

Multiplying on both sides by $y^{\frac{1}{4}}$, we get $y^{\frac{1}{2}} \leq x^{\frac{1}{2}}$. For general non-invertible $x \leq y$, pick $\varepsilon > 0$ and notice that

$$0 \leq x + \varepsilon \leq y + \varepsilon.$$

However, since x, y are positive, we also have $x + \varepsilon, y + \varepsilon \in \mathrm{GL}(A)$. We use the above calculation to obtain $(x+\varepsilon)^{\frac{1}{2}} \leq (y+\varepsilon)^{\frac{1}{2}}$. If we send $\varepsilon \to 0$, we get $x^{\frac{1}{2}} \leq y^{\frac{1}{2}}$. □

Remark. Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ be continuous. Then the function f is operator monotone if for every C^*-algebra A and $a, b \in A_{\mathrm{sa}}$ with $a \leq b$ and $\sigma(a), \sigma(b) \subseteq I$, we have $f(a) \leq f(b)$. By the above proposition, $z \mapsto \sqrt{z}$ and $z \mapsto \frac{1}{z}$ are operator monotone on $[0, \infty)$. Actually, this is also true for functions $z \mapsto z^r$ for $r \in [0, 1]$, but not for $r > 1$.

Definition 2.44. Absolute value of $x \in A$ is defined as

$$|x| = (x^* x)^{\frac{1}{2}} \in A_+.$$

Corollary 2.45. *For $x, y \in A$, we have $|xy| \leq \|x\||y|$.*

Proof. Notice that

$$|xy|^2 = y^* x^* x y \leq y^* \|x^* x\| y = \|x\|^2 (y^* y)$$

and now apply the operator-monotone $\sqrt{\cdot}$ and the previous proposition. □

Theorem 2.46.

Let A be a C^*-algebra.

(1.) $\mathrm{ext}(A_+)_1 = \{$*projections in* $A\}$.

(2.) $\mathrm{ext}(A)_1 \subseteq \{$*partial isometries in* $A\}$.

(3.) $\mathrm{ext}(A_{\mathrm{sa}})_1 = U(A) \cap A_{\mathrm{sa}}$.

Proof. (1.) Let $x \in (A_+)_1$. Then $x^2 \leq 2x$, since $z^2 - 2z \leq 0$ on $[0, 1] \supseteq \sigma(x)$. So $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$. If x is an extreme point, then $x = x^2$ and $x \in A_+ \subseteq A_{\mathrm{sa}}$, so x is a projection. For the converse, assume A is abelian, meaning $A = C(K)$ for some compact Hausdorff space K (by Gelfand). If $x \in A = C(K)$ is a projection, then $x = \chi_E$ for some clopen $E \subseteq K$. Since $\mathrm{ext}([0, 1]) = \{0, 1\}$, χ_E is an extreme point. Let A now be a general C^*-algebra and $p \in A^*$ a projection. Suppose $p = \frac{1}{2}(a + b)$ for some $a, b \in (A_+)_1$. Then $\frac{1}{2}a = p - \frac{1}{2}b \leq p$. Hence

$$0 \leq (1 - p)a(1 - p) \leq (1 - p)2p(1 - p) = 0,$$

so

$$(\sqrt{a}(1 - p))^*(\sqrt{a}(1 - p)) = (1 - p)a(1 - p) = 0.$$

This implies that $\sqrt{a}(1 - p) = 0$ and $a(1 - p) = 0$. It follows that

$$ap = a = a^* = (ap)^* = p^* a^* = pa.$$

Similarly, we can show that a, b, p all commute, so the C^*-subalgebra $C^*(a, b, p)$ is abelian and we can just use the previous observation.

(2.) Suppose $x \in (A)_1$ is not a partial isometry (alternatively, x^*x is not a projection). First, we notice that $\|x^*x\| = \|x\|^2 \leq 1$. Since x is not a projection, $\sigma(x^*x) \cap (0,1) \neq \emptyset$. Then we apply the continuous functional calculus to obtain a function $f : \sigma(x^*x) \to [0,1]$ such that $|t(1 \pm f(t))^2| \leq 1$ for $t \in \sigma(x^*x)$ (for example, f can be a small bump function on an interval $[a,b] \subseteq (0,1)$, where $[a,b] \cap \sigma(x^*x) \neq \emptyset$). Then $y := f(x^*x) \in A_+$ gives us $yx^*x = x^*xy \neq 0$ and $\|x^*x(1 \pm y)^2\| \leq 1$. Hence, $\|x(1 \pm y)\|^2 \leq 1$ and

$$x = \frac{1}{2}((x + xy) + (x - xy)) \notin \operatorname{ext}(A)_1.$$

(3.) If $u \in U(A) \cap A_{\text{sa}}$, then $x \mapsto ux$ is an isometry. As in the case of $\mathcal{B}(\mathcal{H})$, u is an extreme point, so $A_{\text{sa}} \cap U(A) \subseteq \operatorname{ext}(A_{\text{sa}})_1$. For the converse, assume $x \in \operatorname{ext}(A_{\text{sa}})_1$ and $x_+ = \frac{1}{2}(a + b)$ for $a, b \in (A_+)_1$. Then

$$0 = x_- x_+ x_- = \frac{1}{2}(x_- a x_- + x_- b x_-) \geq 0.$$

From $x_- a x_- = 0$, we get $(\sqrt{a} x_-)^* (\sqrt{a} x_-) = 0$, which implies that $\sqrt{a} x_- = 0$ and $a x_- = 0$. Likewise, $x_- a = b x_- = x_- b = 0$. By Gelfand, the commutative C^*-algebra $C^*(a, b, x_-)$ is isometrically $*$-isomorphic to $C(K)$ for some compact K. This means that a and x_- are functions such that for every point in K, at least one of them is zero. Thus, $a - x_-$ is bounded above by 1, and we have $a - x_- \in (A_{\text{sa}})_1$. Similarly, $b - x_- \in (A_{\text{sa}})_1$, so

$$x = \frac{1}{2}((a - x_-) + (b - x_-)) \in (A_{\text{sa}})_1.$$

But since x is an extreme point, we have $a - x_- = b - x_-$ and $a = b = x_+$. Thus, $x_+ \in \operatorname{ext}(A_+)_1$ is a projection by (1.), and by symmetry, so is x_-. Now we prove that x is unitary:

$$x^*x = x^2 = (x_+ - x_-)^2 = x_+^2 + x_-^2 = x_+ + x_- = |x|.$$

This implies that $|x|$ is a projection. Now set $q := 1 - |x|$. Then $x + q$ and $x - q$ are both in $(A_{\text{sa}})_1$. But since

$$x = \frac{1}{2}((x + q) + (x - q)),$$

we obtain $q = 0$, which further implies $|x| = 1$ and $x^*x = xx^* = 1$. $\square$

3 Representations of C^*-algebras and states

3.1 States

Let A be a C^*-algebra, then A^* can be given an A-bimodule structure: if $\psi \in A^*$ and $a, b \in A$, then

$$(a \cdot \psi \cdot b)(x) = \psi(bxa), \quad \forall x \in A.$$

We have

$$\|a \cdot \psi \cdot b\| = \sup_{x \in (A)_1} \|\psi(bxa)\| \leq \sup_{x \in (A)_1} \|\psi\|\|bxa\| \leq \|\psi\|\|a\|\|b\|.$$

Definition 3.1. Let A be a C^*-algebra and $\varphi \in A^*$.

- We say that φ is *positive* if $\varphi(x) \geq 0$, $\forall x \in A_+$. If φ is positive and $a \in A$, then $a\varphi a^*$ is also positive.
- A positive element $\varphi \in A^*$ is *faithful* if $\varphi(x) \neq 0$, $\forall x \in A_+ \setminus \{0\}$.
- An element $\varphi \in A^*$ is a state if it is *positive* and $\|\varphi\| = 1$. The set of states is denoted $S(A) \subseteq (A^*)_1$.

Remark. The set $S(A)$ is compact Hausdorff in the weak-$*$ topology.

We notice that if $\varphi \in A^*$ is positive and $x \in A_{\text{sa}}$, then

$$\varphi(x) = \varphi(x_+ - x_-) = \varphi(x_+) - \varphi(x_-) \in \mathbb{R}.$$

If $y \in A$, then $y = y_1 + iy_2$, where y_1, y_2 are self-adjoint. Then

$$\begin{aligned} \varphi(y^*) &= \varphi((y_1 + iy_2)^*) = \varphi(y_1 - iy_2) \\ &= \varphi(y_1) - i\varphi(y_2) = \overline{\varphi(y_1) + i\varphi(y_2)} \\ &= \overline{\varphi(y_1 + iy_2)} = \overline{\varphi(y)} \end{aligned}$$

Such a functional $\varphi \in A^*$ is called *hermitian*. For any $\varphi \in A^*$, $\varphi^*(y) = \overline{\varphi(y^*)}$. Then $\varphi + \varphi^*$ and $i(\varphi - \varphi^*)$ are hermitian. One can, of course, define these notions also for unbounded linear functionals. However, positivity implies continuity: for every $a \in A_{\text{sa}}$ we have $-\|a\| \cdot 1 \leq a \leq \|a\| \cdot 1$, which implies

$$-\|a\|\varphi(1) \leq \varphi(a) \leq \|a\|\varphi(1)$$

and φ is bounded. For $a \in A$, we can of course write $a = b + ic$ for $b, c \in A_{\text{sa}}$. Here,

$$\|b\| = \left\| \frac{a + a^*}{2} \right\| \leq \frac{\|a\|}{2} + \frac{\|a^*\|}{2} = \|a\|$$

and likewise $\|c\| \leq \|a\|$. Let $\varphi(1) = C$. Then

$$|\varphi(a)|^2 = |\varphi(b) + i\varphi(c)|^2 = \varphi(b)^2 + \varphi(c)^2 \leq C^2(\|b\|^2 + \|c\|^2) \leq 2C^2\|a\|^2.$$

Lemma 3.2. *Let $\varphi \in A^*$ be positive. Then $\forall x, y \in A$:*

$$|\varphi(y^*x)|^2 \leq \varphi(y^*y) \cdot \varphi(x^*x).$$

Proof. Consider the sesquilinear form $\langle x, y\rangle = \varphi(y^*x)$. Since φ is positive, this is a positive sesquilinear form and we can apply Cauchy-Schwartz. $\square$

Theorem 3.3.

An element $\varphi \in A^*$ is positive iff $\|\varphi\| = \varphi(1)$.

Remark. This implies that the set of states $S(A)$ is convex.

Proof. First we prove the right implication ($\Rightarrow$). We know that $x^*x \leq \|x^*x\|$, so

$$\begin{aligned}|\varphi(x)|^2 &\leq \varphi(1)\varphi(x^*x)\\ &\leq \varphi(1)\varphi(\|x^*x\|)\\ &= \varphi(1)^2\|x^*x\|\\ &= \varphi(1)^2\|x\|^2,\end{aligned}$$

so $|\varphi(x)| \leq \varphi(1)\|x\|$. From there we get $\|\varphi\| \leq \varphi(1) \leq \|\varphi\|$, so $\varphi(1) = \|\varphi\|$. Now the converse ($\Leftarrow$). Suppose $x \in A_+$ and $\varphi(x) = \alpha + i\beta$. For each $t \in \mathbb{R}$, we have

$$\begin{aligned}\alpha^2 + (\beta + t\|\varphi\|)^2 &= |\alpha + i(\beta + t\varphi(1))|^2\\ &= |\varphi(x + it)|^2\\ &\leq \|x + it\|^2 \cdot \|\varphi\|^2\\ &= \left(\|x\|^2 + t^2\right)\|\varphi\|^2.\end{aligned}$$

From this it directly follows $2\beta t\|\varphi\| \leq \|x\|^2 \cdot \|\varphi\|^2$. Since $t \in \mathbb{R}$ was arbitrary, we have $\beta = 0$ and $\varphi(x) = \alpha \in \mathbb{R}$. Lastly, we derive

$$\begin{aligned}\|x\| \cdot \|\varphi\| - \varphi(x) &= \varphi(\|x\| - x)\\ &\leq \|\|x\| - x\| \cdot \|\varphi\|\\ &\leq \|x\| \cdot \|\varphi\|,\end{aligned}$$

so $\varphi(x) \geq 0$. $\square$

Proposition 3.4. *Let A be a C^*-algebra and $x \in A$. Then $\forall \lambda \in \sigma(x)$ there exists a $\varphi \in S(A)$ such that $\varphi(x) = \lambda$.*

Proof. We know that $\mathbb{C}x + \mathbb{C} \cdot 1 \subseteq A$. Define

$$\varphi_0 : \mathbb{C}x + \mathbb{C}1 \to \mathbb{C}, \quad \alpha x + \beta \mapsto \alpha \cdot \lambda + \beta.$$

Since $\varphi_0(\alpha x + \beta) \in \sigma(\alpha x + \beta)$, we have

$$\|\varphi_0\| \leq 1 = \varphi_0(1),$$

therefore $\|\varphi_0\| = 1$. Now we apply Hahn-Banach to get an extension $\varphi \in A^*$ such that $\varphi\big|_{\mathbb{C}x+\mathbb{C}1} = \varphi_0$ and $\|\varphi\| = 1 = \varphi(1)$, so $\varphi \in S(A)$ by theorem 3.3. $\square$

Proposition 3.5. *Let A be a C^*-algebra and $x \in A$.*

(1.) $x = 0$ *iff* $\varphi(x) = 0,\ \forall \varphi \in S(A)$.

(2.) $x \in A_{\mathrm{sa}}$ *iff* $\varphi(x) \in \mathbb{R},\ \forall \varphi \in S(A)$.

(3.) $x \in A_+$ *iff* $\varphi(x) \geq 0,\ \forall \varphi \in S(A)$.

Proof. (1.) If $\varphi(x) = 0$ for all $\varphi \in S(A)$, then writing $x = x_1 + ix_2$ for self-adjoint x_1, x_2 gives us

$$0 = \varphi(x) = \varphi(x_1) + i\varphi(x_2),$$

which implies $\varphi(x_1) = \varphi(x_2) = 0$. Now use the proposition 3.5 to get $\sigma(x_1) = \sigma(x_2) = \{0\}$, which can only imply $x_1 = x_2 = 0$, whence $x = 0$.

(2.) If $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(A)$, then

$$\varphi(x - x^*) = \varphi(x) - \varphi(x^*) = \varphi(x) - \overline{\varphi(x)} = 0$$

and we use the previous item to show that $x - x^*$. The converse implication follows from the fact that every positive functional is hermitian.

(3.) If $\varphi(x) \geq 0$ for all $\varphi \in S(A)$, then $x \in A_{\mathrm{sa}}$ by previous item and $\sigma(x) \subseteq [0, \infty)$, so $x \in A_+$. The converse once again follows from positivity of φ. $\square$

3.2 Gelfand-Naimark-Segal (GNS) construction

Definition 3.6.

- A *representation* of a C^*-algebra A is a *-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.
- If $\mathcal{K}^{\text{closed}} \leq \mathcal{H}$ and $\pi(x)\mathcal{K} \subseteq \mathcal{K},\ \forall x \in A$ (we say that $\mathcal{K}$ is *invariant* under π), then the restriction of π to $\mathcal{K}$ is a *subrepresentation*.
- If a representation has no other subrepresentations besides $\mathcal{K} = (0)$ and $\mathcal{K} = \mathcal{H}$ (equivalently, $\pi(A)$ only has (0) and $\mathcal{H}$ as closed invariant subspaces), then π is called irreducible.
- Representations $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\rho : A \to \mathcal{B}(\mathcal{K})$ are *equivalent* if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that

$$U\pi(x) = \rho(x)U, \quad \forall x \in A.$$

- Vector $\zeta \in \mathcal{H}$ is *cyclic* for a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ if

$$\pi(A)\zeta := \{\pi(a)\zeta \mid a \in A\}$$

is dense in $\mathcal{H}$ (this means that $\overline{\pi(A)\zeta} = \mathcal{H}$).

Example 3.7. *Each $w \in \mathcal{H}$ defines a subrepresentation on $K := \overline{\pi(A)w}$.*

Example 3.8. *Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation and $\mu \in \mathcal{H}$, $\|\mu\| = 1$. Then*

$$\varphi_\mu : A \to \mathbb{C}, \quad x \mapsto \langle \pi(x)\mu, \mu \rangle$$

is a state. Indeed,

$$\varphi_\mu(1) = \langle 1 \cdot \mu, \mu\rangle = \|\mu\|^2 = 1$$

and

$$\begin{aligned}\varphi_\mu(x^*x) &= \langle \pi(x^*x)\mu, \mu\rangle \\ &= \langle \pi(x^*)\pi(x)\mu, \mu\rangle \\ &= \langle \pi(x)^*\pi(x)\mu, \mu\rangle \\ &= \langle \pi(x)\mu, \pi(x)\mu\rangle \\ &= \|\pi(x)\mu\|^2 \geq 0.\end{aligned}$$

Theorem 3.9 (Gelfand-Naimark-Segal construction)**.**

Let A be a C^*-algebra and $\rho \in S(A)$. Then there exists a Hilbert space $L^2(A, \varphi)$ and a unique (up to equivalence) representation $\pi : A \to \mathcal{B}(L^2(A, \varphi))$ and a unit cyclic vector 1_φ such that

$$\varphi(x) = \langle \pi(x)1_\varphi, 1_\varphi\rangle, \quad \forall x \in A.$$

Proof. (1.) We start by defining

$$N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$$

whose elements we call nullvectors of φ. By the Cauchy-Schwartz lemma, we have

$$N_\varphi = \{x \in A \mid \varphi(yx) = 0, \ \forall y \in A\}.$$

Thus N_φ is a closed subspace of A.

(2.) We prove that N_φ is a left ideal: for $x \in N_\varphi$ and $a \in A$, we have $ax \in N_\varphi$. Indeed,

$$\varphi((ax)^*ax) = \varphi((x^*a^*a)x) = 0.$$

(3.) Now $\mathcal{H}_0 = A/N_\varphi$ is a vector space and we can endow it with the dot product $\langle [x], [y]\rangle := \varphi(y^*x)$ for $x, y \in A$. It can easily be checked that this is a well-defined dot product in $\mathcal{H}_0$. We denote the completion of $\mathcal{H}_0$ by $L^2(A, \varphi)$.

(4.) To an arbitrary $a \in A$, we associate the map

$$\pi_0(a) : \mathcal{H}_0 \to \mathcal{H}_0, \quad [x] \mapsto [ax].$$

Since N_φ is a left ideal of A, $\pi_0(a)$ is a well-defined linear map. We have

$$\begin{aligned}\|\pi_0(a)[x]\|^2 &= \|[ax]\|^2 \\ &= \langle [ax], [ax]\rangle \\ &= \varphi((ax)^*ax) \\ &= \varphi(x^*a^*ax) \\ &\leq \|a\|^2 \cdot \varphi(x^*x) \leq \|a\|^2\|x\|^2.\end{aligned}$$

Since $\pi_0(a)$ is a bounded linear map, it exdends uniquely to $\pi(a) \in \mathcal{B}(L^2(A, \varphi))$ with $\|\pi(a)\| \leq \|a\|$. Then we get

$$\pi : A \to \mathcal{B}(L^2(A, \varphi)), \quad a \mapsto \pi(a),$$

which is a homomorphism and has the property

$$\begin{aligned}\langle [x], \pi(a^*)[y]\rangle &= \langle [x], [a^*y]\rangle \\ &= \varphi((a^*y)^*x) \\ &= \varphi(y^*ax) \\ &= \langle [ax], [y]\rangle \\ &= \langle \pi(a)[x], [y]\rangle.\end{aligned}$$

So $\pi(a)^* = \pi(a^*)$ and π is a representation.

(5.) We define $1_\varphi := [1] \in \mathcal{H}_0 \subseteq L^2(A, \varphi)$ and notice that

$$\langle \pi(a)1_\varphi, 1_\varphi\rangle = \langle \pi(a)[1], [1]\rangle = \langle [a], [1]\rangle = \varphi(a).$$

Since $\{\pi(a)1_\varphi \mid a \in A\} = \mathcal{H}_0$ is dense in $L^2(A, \varphi)$, the vector 1_φ is cyclic for π.

(6.) Next we prove uniqueness: let $\rho : A \to \mathcal{B}(\mathcal{K})$ be a representation, $\mu \in \mathcal{K}$ a unit cyclic vector and assume $\varphi(a) = \langle \rho(a)\mu, \mu\rangle$, $\forall a \in A$. We will prove that ρ is equivalent to π. Define

$$U_0 : \mathcal{H}_0 \to \mathcal{K}, \quad [x] \mapsto \rho(x)\mu.$$

Then we have

$$\begin{aligned}\langle U_0[x], U_0[y]\rangle_{\mathcal{K}} &= \langle \rho(x)\mu, \rho(y)\mu\rangle \\ &= \langle \rho(y)^*\rho(x)\mu, \mu\rangle \\ &= \langle \rho(y^*x)\mu, \mu\rangle = \varphi(y^*x) = \langle [x], [y]\rangle_{L^2(A,\varphi)},\end{aligned}$$

so U_0 really is a well-defined isometry. For all $a, x \in A$:

$$U_0(\pi(a)[x]) = U_0([ax]) = \rho(ax)\mu = \rho(a)\rho(x)\mu = \rho(a)U_0[x].$$

Therefore, U_0 induces an isometry $U : L^2(A, \varphi) \to \mathcal{K}$ such that $U\pi(a) = \rho(a)U$ for all $a \in A$. Since μ is cyclic and $\rho(a)\mu \subseteq \operatorname{im} U$, the range of U is dense in $\mathcal{K}$. It is also closed since U is isometric. We just proved that U is isometric and onto, so it is unitary. ☐

Corollary 3.10. *Every C^*-algebra has a faithful (i.e. injective) representation. In particular, every C^*-algebra is isometrically *-isomorphic to a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space $\mathcal{H}$.*

Proof. Let π be a direct sum of all representations from GNS construction over all states. Then the Proposition 3.5 tells us that π is injective. An injective *-monomorphism is isometric and we are done. ☐

The preceding corollary enables us to view abstract C^*-algebras as concrete algebras of operators on some Hilbert space.

Definition 3.11. If $S \subseteq A$, then

$$S' := \{x \in A \mid \forall s \in S : \; xs = sx\}$$

is its *commutant*.

Proposition 3.12 (Radon-Nikodym for linear functionals)**.** *Let φ, ψ be positive linear functionals on a C^*-algebra A and $\varphi \in S(A)$. Then $\varphi \leq \psi$ iff there exists a unique $y \in \pi_\psi(A)'$ such that $0 \leq y \leq 1$ and*

$$\varphi(a) = \langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle, \quad \forall a \in A.$$

Proof. Start with ($\Leftarrow$). For $a \in A_+$ we have

$$\pi_\psi(a) y = \pi_\psi(a)^{\frac{1}{2}} y \pi_\psi(a)^{\frac{1}{2}} \leq \pi_\psi(a).$$

Then

$$\varphi(a) = \langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle \leq \langle \pi_\psi(a) 1_\psi, 1_\psi \rangle = \psi(a).$$

Now the converse ($\Rightarrow$). By Cauchy-Schwartz,

$$\begin{aligned} |\varphi(b^*a)|^2 &\leq \varphi(a^*a)\varphi(b^*b) \\ &\leq \psi(a^*a)\psi(b^*b) \\ &= \|\pi_\psi(a)1_\psi\|^2 \cdot \|\pi_\psi(b)1_\psi\|^2. \end{aligned}$$

This means that $\langle \pi_\psi(a)1_\psi, \pi_\psi(b)1_\psi \rangle_\varphi := \varphi(b^*a)$ is a nonnegative sesquilinear form on $\pi_\varphi(A)1_\psi^{\text{dense}} \subseteq L^2(A, \psi)$, which is bounded by 1. This further implies that it is continuous and we can extend it to $L^2(A, \psi)$. By Riesz, there exists $y \in \mathcal{B}(L^2(A, \psi))$ such that

$$\varphi(b^*a) = \langle y \pi_\psi(a) 1_\psi, \pi_\psi(b) 1_\psi \rangle, \quad \forall a, b \in A$$

and $0 \leq y \leq 1$. For $a, b, c \in A$ we have

$$\begin{aligned} \langle y \pi_\psi(a) \pi_\psi(b) 1_\psi, \pi_\psi(c) 1_\psi \rangle &= \langle y \pi_\psi(ab) 1_\psi, \pi_\psi(c) 1_\psi \rangle \\ &= \varphi(c^* \cdot ab) = \varphi((a^*c)^* b) \\ &= \langle y \pi_\psi(b) 1_\psi, \pi_\psi(a^*) \pi_\psi(c) 1_\psi \rangle \\ &= \langle \pi_\psi(a) y \pi_\psi(b) 1_\psi, \pi_\psi(c) 1_\psi \rangle, \end{aligned}$$

so $y \pi_\psi(a) = \pi_\psi(a) y$ for all $a \in A$ and $y \in \pi_\psi(A)'$. Finally, the uniqueness. Say that there exists a $z \in \pi_\psi(A)'$ such that $0 \leq z \leq 1$ and

$$\langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle = \langle \pi_\psi(a) z 1_\psi, 1_\psi \rangle, \quad \forall a \in A.$$

Then

$$\begin{aligned} \langle \pi_\psi(b^*a) z 1_\psi, 1_\psi \rangle &= \langle \pi_\psi(b^*a) y 1_\psi, 1_\psi \rangle \\ &= \langle y \pi_\psi(a) 1_\psi, \pi_\psi(b) 1_\psi \rangle \\ &= \langle z \pi_\psi(a) 1_\psi, \pi_\psi(b) 1_\psi \rangle, \end{aligned}$$

which implies $y = z$. □

Proposition 3.13. *Suppose that A is a separable C^*-algebra. Then A has a faithful cyclic representation on a separable Hilbert space.*

Proof. If A is separable, then it has a dense subset $\{a_i\}_{i=1}^{\infty}$. We can embed $S(A)$ into the space $\prod_{i=1}^{\infty} \overline{B_1(0)}$, where $\overline{B_1(0)}$ is a closed unit ball in $\mathbb{C}$, by sending $\varphi \in S(A)$ to $(\varphi(a_i))_{i=1}^{\infty}$. The latter topological space is metrizable by the metric $\rho(x, y) = \sum_{i=1}^{\infty} \frac{\rho_i(x_i, y_i)}{2^i(\rho_i(x_i, y_i)+1)}$, and so is $S(A)$. Therefore, $S(A)$ with the weak-$*$ topology is a metrizable compact, therefore separable. Let $\{f_i\}_i^{\infty}$ countable weak-$*$ dense subset of $S(A)$. Then

$$f(a) := \sum_{i=1}^{\infty} 2^{-i} f_i(a)$$

defines a faithful ($f(a^*a) = 0$ iff $a = 0$) state on A. Then the GNS construction π_f is faithful: if $\pi_f(a) = 0$, then

$$f(b^*a^*ab) = \langle \pi_f(a)[b], \pi_f(a)[b] \rangle = 0$$

for every $b \in A$. In particular for $b = 1$, we get $f(a^*a) = 0$ and so $a = 0$. Since $a \mapsto [a]$ is a continuous map of A onto a dense subspace of some Hilbert space $\mathcal{H}_f$ (induced by $\pi_f : A \to \mathcal{B}(\mathcal{H}_f)$), the latter space is separable. $\square$

Proposition 3.14. *Every representation of a C^*-algebra is equivalent to a direct sum of cyclic representations.*

Proof. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be some representation of A. Let $\mathcal{E}$ be the collection of all subsets E of nonzero vectors in $\mathcal{H}$ such that $\pi(A)e \perp \pi(A)f$ for any $e, f \in E$. If we order $\mathcal{E}$ by inclusion, then Zorn's lemma tells us that $\mathcal{E}$ has a maximal element E_0. Let $\mathcal{H}_0 = \bigoplus_{e \in E_0} \overline{\pi(A)e}$. Take $h \in \mathcal{H}_0^{\perp}$ in $\mathcal{H}$. Then for any $a, b \in A$ and $e \in E_0$ we have

$$\langle \pi(a)e, \pi(b)h \rangle = \langle \pi(b)^*\pi(a)e, h \rangle = \langle \pi(b^*a)e, h \rangle = 0,$$

so $\pi(A)e \perp \pi(A)h$ for each $e \in E_0$. By maximality, $h = 0$ and $\mathcal{H} = \mathcal{H}_0$. For $e \in E_0$, define $\mathcal{H}_e := \overline{\pi(A)e}$. Obviously, $\mathcal{H}_e$ is invariant for π, so $\pi_e := \pi\big|_{\mathcal{H}_e}$ is a cyclic representation of A. Clearly, $\pi = \bigoplus_{e \in E_0} \pi_e$. $\square$

3.3 Pure states and irregular representations

Definition 3.15. A state $\varphi \in S(A)$ is called *pure* if it's an extreme point of $S(A)$.

Proposition 3.16. *A state $\varphi \in S(A)$ is pure iff the representation GNS $\pi_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$ with cyclic vector 1_φ is irreducible.*

Proof. ($\Rightarrow$) Let $\mathcal{K} \leq L^2(A, \varphi)$ be a closed invariant subspace. Then $\mathcal{K}^{\perp}$ is also a closed invariant subspace: for $a \in A$, $x \in \mathcal{K}^{\perp}$ and $k \in \mathcal{K}$ we have

$$\langle \pi_\varphi(a)x, k \rangle = \langle x, \pi_\varphi(a^*)k \rangle = 0.$$

Since $L^2(A,\varphi) = \mathcal{K} \oplus \mathcal{K}^\perp$ we write $1_\varphi = \underbrace{\mu_1}_{\in \mathcal{K}} + \underbrace{\mu_2}_{\in \mathcal{K}^\perp}$ and form

$$\varphi_j := \frac{\langle \pi_\varphi(x)\mu_j, \mu_j\rangle}{\|\mu_j\|^2}, \quad j = 1, 2.$$

These are states and so is

$$\varphi(x) = \|\mu_1\|^2 \varphi_1(x) + \|\mu_2\|^2 \varphi_2(x)$$

because $1 = \|1_\varphi\|^2 = \|\mu_1\|^2 + \|\mu_2\|^2$. Since $\varphi \in \operatorname{ext} S(A)$, we either have $\mu_1 = 0$ or $\mu_2 = 0$, which implies that $\mathcal{K}$ is either (0) or $L^2(A,\varphi)$.

($\Leftarrow$) Suppose $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2)$ for $\varphi_1, \varphi_2 \in S(A)$. Define a linear map

$$U : L^2(A,\varphi) \to L^2(A,\varphi_1) \oplus L^2(A,\varphi_2), \quad \pi_\varphi(x)1_\varphi \mapsto \frac{1}{\sqrt{2}}\pi_{\varphi_1}(x)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(x)1_{\varphi_2}.$$

First we notice that U preserves the scalar product:

$$\begin{aligned}
\langle \pi_\varphi(x)1_\varphi, \pi_\varphi(y)1_\varphi\rangle &= \varphi(x^*y) \\
&= \frac{1}{2}\varphi_1(x^*y) + \frac{1}{2}\varphi_2(x^*y) \\
&= \langle \frac{1}{\sqrt{2}}\pi_{\varphi_1}(x)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(x)1_{\varphi_2}, \frac{1}{\sqrt{2}}\pi_{\varphi_1}(y)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(y)1_{\varphi_2}\rangle \\
&= \langle U\pi_\varphi(x)1_\varphi, U\pi_\varphi(y)1_\varphi\rangle.
\end{aligned}$$

Additionally, U intertwines: for all $x \in A$, we have

$$\begin{aligned}
U\pi_\varphi(x)(\pi_\varphi(y)1_\varphi) &= U\pi_\varphi(xy)1_\varphi \\
&= \frac{1}{\sqrt{2}}\pi_{\varphi_1}(xy)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(xy)1_{\varphi_2} \\
&= (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))\,(\pi_{\varphi_1}(y)1_{\varphi_1} \oplus \pi_{\varphi_2}(y)1_{\varphi_2}) \\
&= (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))\,U(\pi_\varphi(y)1_\varphi).
\end{aligned}$$

If we star the intertwining identity, we get

$$\pi_\varphi(x^*)U^* = U^*\left(\pi_{\varphi_1}(x^*) \oplus \pi_{\varphi_2}(x^*)\right), \quad \forall x^* \in A.$$

If we plug in x instead of x^*, we get

$$\pi_\varphi(x)U^* = U^*\left(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)\right), \quad \forall x \in A.$$

Now let

$$p_1 \in \mathcal{B}(L^2(A,\varphi_1) \oplus L^2(A,\varphi_2))$$

be orthogonal projection onto the first direct summand. Clearly, we have

$$p_1\left(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)\right) = \left(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)\right)p_1$$

Putting it all together, we get

$$\begin{aligned}\pi_\varphi(x)U^*p_1U &= U^*(\pi_{\varphi_1}(x)\oplus\pi_{\varphi_2}(x))p_1U\\ &= U^*p_1(\pi_{\varphi_1}(x)\oplus\pi_{\varphi_2}(x))U\\ &= U^*p_1U(\pi_{\varphi_1}(x)\oplus\pi_{\varphi_2}(x)),\end{aligned}$$

so U^*p_1U commutes with $\pi_\varphi(x)$ for all $x \in A$. If $\sigma(U^*p_1U)$ has more than one element, then $\exists t \in (0,1]$ such that $\sigma(U^*p_1U - t)$ has both positive and negative elements. By Definition 2.38, we can write $U^*p_1U = a - b$ for positive $0 \neq a, b$ such that $ab = ba = 0$. Then a, b commute with $\pi_\varphi(A)$, so $\ker a \neq 0$ is an closed subspace of $L^2(A,\varphi)$ that is invariant under $\pi_\varphi(x)$, which is a contradiction. So U^*p_1U has a single element spectrum $\{\alpha\}$ and since U^*p_1U is normal (because it is positive), so is $U^*p_1U - \alpha I$. But now we can write

$$\|U^*p_1U - \alpha I\| = r(U^*p_1U) = 0,$$

which proves that $U^*p_1U = \alpha I$. Then

$$\begin{aligned}\alpha &= \alpha\varphi(1) = \varphi(\alpha)\\ &= \langle \alpha 1_\varphi, 1_\varphi\rangle\\ &= \langle U^*p_1U1_\varphi, 1_\varphi\rangle\\ &= \left\langle \frac{1}{\sqrt{2}}1_{\varphi_1}\oplus 0, \frac{1}{\sqrt{2}}1_{\varphi_1}\oplus\frac{1}{\sqrt{2}}1_{\varphi_2}\right\rangle\\ &= \left\langle \frac{1}{\sqrt{2}}1_{\varphi_1}, \frac{1}{\sqrt{2}}1_{\varphi_1}\right\rangle_{\varphi_1} = \frac{1}{2}.\end{aligned}$$

This means that we can write

$$\left(\sqrt{2}p_1U\right)^*\left(\sqrt{2}p_1U\right) = 1,$$

so

$$u_1 = \frac{1}{\sqrt{2}}p_1U : L^2(A,\varphi) \to L^2(A,\varphi_1)$$

is an isometry. We also have the identities

$$u_1 1_\varphi = 1_{\varphi_1}, \quad u_1\pi_\varphi(x) = \pi_{\varphi_1}(x)u_1.$$

It follows that

$$\begin{aligned}\varphi(x) &= \langle \pi_\varphi(x)1_\varphi, 1_\varphi\rangle\\ &= \langle u_1^*u_1\pi_\varphi(x)1_\varphi, 1_\varphi\rangle\\ &= \langle u_1^*\pi_{\varphi_1}(x)u_1 1_\varphi, 1_\varphi\rangle\\ &= \langle \pi_{\varphi_1}(x)u_1 1_\varphi, u_1 1_\varphi\rangle\\ &= \langle \pi_{\varphi_1}(x)1_{\varphi_1}, 1_{\varphi_1}\rangle = \varphi_1(x)\end{aligned}$$

and we are done. □

Theorem 3.17.

A representation $\pi : A \to \mathcal{B}(\mathcal{H})$ if irreducible iff $\pi(A)' = \mathbb{C} \cdot \mathrm{id}$.

Proof. Start with ($\Leftarrow$). Suppose there exists a closed invariant subspace $(0) \neq \mathcal{K} \subsetneq \mathcal{H}$. Let $p \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection onto $\mathcal{K}$. Then $p \notin \mathbb{C}\cdot\mathrm{id}$. Now we prove that $p \in \pi(A)'$. Let $a \in A$. For $\mu \in \mathcal{K}$, we have

$$(p\pi(a))\mu = p(\pi(a)\mu) = \pi(a)\mu = \pi(a)(p\mu) = (\pi(a)p)\mu.$$

Now for $\mu \in \mathcal{K}^\perp$, we get

$$(p\pi(a))\mu = p(\pi(a)\mu) = 0 = \pi(a)(0) = \pi(a)(p\mu) = (\pi(a)p)\mu.$$

For the converse ($\Rightarrow$), suppose there exists a non-scalar self-adjoint $h \in \pi(A)'$. Then $\sigma(h)$ has at least two elements. We can define two bump functions $f, g \in C(\sigma(h))$ in the respective neighborhoods of these two elements of $\sigma(h)$ such that $fg = 0$. Then $f(h) \neq 0$ since $f \neq 0$. Then also $\mathcal{K} := \overline{\mathrm{im}\, f(h)} \leq \mathcal{H}$ is nonzero. Also, $g(h) \neq 0$ and $g(h)\big|_{\mathcal{K}} = 0$ since $g(h) \cdot f(h) = 0$. In particular, $\mathcal{K} \subsetneq \mathcal{H}$. Take any $x \in \pi(A)$. By Stone-Weierstrass, we can approximate $f(h)$ in norm by $(p_i(h))_{i=1}^\infty$, where $(p_i)_{i=1}^\infty$ are complex polynomials. Notice that since $h \in \pi(A)'$, we also have $p_i(h) \in \pi(A)'$. Therefore, we have $p_i(h)x = xp_i(h)$. By sending $i \to \infty$, the left side converges in norm to $f(h)a$, while the right one converges in norm to $af(h)$. As a result, $f(h)x = xf(h)$ and $f(h) \in \pi(A)'$. We claim that $\mathcal{K}$ is invariant; it's enough to show that $\mathrm{im}\, f(h)$ is invariant. For $a \in A, \mu \in \mathcal{H}$ we have

$$\pi(a)(f(h)\mu) = \pi(a)f(h)\mu = f(h)\pi(a)\mu \in \mathrm{im}\, f(h).$$

In general, if $q \in \pi(A)'$, then $q^* \in \pi(A)'$ and we can reduce the problem to the self-adjoint case handled above. □

Corollary 3.18. *Irreducible representations of abelian C^*-algebras are 1-dimensional.*

Proof. Let A be an abelian C^*-algebra and $\pi : A \to \mathcal{B}(\mathcal{H})$ an irrep. Then by Theorem 3.17, $\pi(A)' = \mathbb{C}$. Moreover,

$$\pi(A) = Z(\pi(A)) = \pi(A)' \cap \pi(A) = \mathbb{C} \cdot \mathrm{id}\,.$$ □

Corollary 3.19. *If A is an abelian C^*-algebra, then* $\mathrm{ext}\, S(A) = \sigma(A)$.

Proof. Let $\varphi \in \sigma(A)$. Then φ is 1-dimensional (therefore irreducible) representation and so $\varphi \in \mathrm{ext}\, S(A)$. For the converse, take $\varphi \in \mathrm{ext}\, S(A)$. Then the GNS construction π_φ is irreducible, therefore 1-dimensional. So $L^2(A, \varphi) = \mathbb{C}$ with the standard scalar product and $\varphi(x) = \langle \pi_\varphi(x)1_\varphi, 1_\varphi\rangle = \pi_\varphi(x)$. □

Proposition 3.20. *Let A be a C^*-algebra. Then* $\mathrm{co}\,\mathrm{ext}\, S(A)$ *is weak-$*$ dense in* $S(A)$.

Proof. We know that $S(A)$ is compact Hausdorff with respect to the weak-$*$ topology. The conclusion follows from Krein-Milman. □

Corollary 3.21. *Let A be a C^*-algebra and $x \in A \setminus (0)$. Then there exist an irrep $\pi : A \to \mathcal{B}(\mathcal{H})$ such that $\pi(x) \neq 0$.*

Proof. By Proposition 3.5, there exists $\varphi \in S(A)$ such that $\varphi(x) \neq 0$. By the previous proposition (Krein-Milman), there exists a $\tau \in \operatorname{ext} S(A)$ such that $\tau(x) \neq 0$. Then apply GNS: π_τ is irreducible and $\pi_\tau(x) \neq 0$. ☐

Theorem 3.22 (Jordan decomposition for linear functionals)**.**

Let A be a C^*-algebra and $\varphi \in A^*$ hermitian. Then there exist (unique - without proof) positive linear functionals $\varphi_+, \varphi_- \in A^*$ such that $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| = \|\varphi_-\|$.

Proof. W.l.o.g. $\|\varphi\| = 1$. Let Σ denote the set of positive linear functionals with norm ≤ 1. By Banach-Alaoglu, Σ is weak-$*$ compact and Hausdorff. Consider

$$\gamma : A \to C(\Sigma), \quad a \mapsto (\psi \mapsto \psi(a)).$$

This is an isometry and $\gamma(A_+) \subseteq C(\Sigma)_+$. By Hahn-Banach, there exists a $\widetilde{\varphi} : C(\Sigma) \to \mathbb{C}$ such that $\|\widetilde{\varphi}\| = \|\varphi\|$ and $\varphi = \widetilde{\varphi} \circ \gamma$. Assume $\widetilde{\varphi}$ is hermitian (otherwise, we can replace it by $\frac{\widetilde{\varphi}+\widetilde{\varphi}^*}{2}$). By Riesz-Markoff, there exists a regular Radon Measure μ on Σ such that $\widetilde{\varphi}(f) = \int f \, d\mu$ for all $f \in C(\Sigma)$. Then we use Jordan decomposition for measures to obtain μ_+, μ_- such that $\mu = \mu_+ - \mu_-$ and $\|\mu\| = \|\mu_+\| = \|\mu_-\|$. Now we just define $\varphi_\pm(a) := \int a \, d\mu_\pm$. ☐

Corollary 3.23. *For a C^*-algebra A, A^* is the span of positive linear functionals on A.*

Corollary 3.24. *Let A be a C^*-algebra and $\varphi \in A^*$. Then there exists a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\mu, \theta \in \mathcal{H}$ such that $\varphi(a) = \langle \pi(a)\theta, \mu \rangle$.*

Proof. Write $\varphi = \sum_{i=1}^n \alpha_i \psi_i$ for some $\psi_j \in S(A)$. Let π_i be the GNS representation of ψ_i. Define $\pi := \bigoplus_i \pi_i$, $\theta := \bigoplus_i \alpha_i 1_{\psi_i}$ and $\mu = \bigoplus_i 1_{\psi_i}$. The result then follows immediately. ☐

3.4 Examples of C^*-algebras

Example 3.25. *The most canonical example of a C^*-algebra is $\mathcal{B}(\mathcal{H})$. Similarly, the algebra of compact operators $\mathcal{K}(\mathcal{H})$ is a C^*-algebra (if $\dim \mathcal{H} = \infty$, it is non-unital).*

Example 3.26. *If $\dim \mathcal{H} = \infty$, then $\mathcal{B}(\mathcal{H}) / \mathcal{K}(\mathcal{H})$ is a so-called* Calkin algebra. *Calkin algebra is simple and it does not have a separable representation. We will prove this later, when we have more tools at our disposal.*

Example 3.27. *The algebra of matrices $M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$ is a C^*-algebra.*

3.4.1 Structure theorem for finite-dimensional C^*-algebras

For finite-dimensional C^*-algebras, we have the Artin-Wedderburn type theorem.

Proposition 3.28 (Structure theorem for finite-dimensional C^*-algebras)**.** *Every finite-dimensional C^*-algebra A is*

$$A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$$

for uniquely determined $n_1, \ldots, n_r$.

To prove this, we need a few preliminary lemmas.

Lemma 3.29. *Every finite-dimensional C^*-algebra is unital.*

This fact, which we will not prove here, follows essentially from Krein-Milman theorem. [1]

Corollary 3.30. *If A is a finite-dimensional C^*-algebra, then every ideal in A is of the form $I = Ap$ for some central projection $p \in A$.*

Proof. If $I \lhd A$, then it is itself a (finite-dimensional) C^*-algebra, so it must have a unit p. As a result, we have $I = Ip \subseteq Ap$, but since I is an ideal we also have $Ap \subseteq I$, so $Ap = I$. Since p is a unit in I, we have $p^2 = p = p^*$, so p is a projection. For any $x \in A$, we get $xp \in I$ and so $p \cdot (xp) = xp$. By starring this equation, we get $px^*p = px^*$. Combining the last two equations gives us

$$x^*p = p \cdot (x^*p) = px^*p = px^*,$$

so p commutes with x^*. As a result, p commutes with the entire A. □

Lemma 3.31. *If A is a finite-dimensional abelian C^*-algebra, then its spectrum is finite.*

Proof. We know that by Gelfand, $A \cong C(\sigma(A))$, where $\sigma(A)$ is a compact Hausdorff (and therefore normal) space. Suppose that $\sigma(A)$ is infinite.

(1.) First, we will inductively construct an infinite subset $X = \{x_n\}_{n\in\mathbb{N}} \subseteq \sigma(A)$ that does not contain any of its accumulation points. Pick any point $x \in \sigma(A)$. If x is not an accumulation point of $\sigma(A)$ (so it is an isolated point), then take $x_1 := x$ and choose any new point in $\sigma(A) \setminus \{x_1\}$ to repeat this process. Howevery, if x is an accumulation point of $\sigma(A)$, then take any $x_1 \in \sigma(A) \setminus \{x\}$. Since $\sigma(A)$ is Hausdorff, there exist disjoint open neighborhoods $V_1 \ni x, U_1 \ni x_1$. Now since x is an accumulation point of $\sigma(A)$, there must exist some $x_2 \in V_1$, such that $x_2 \neq x$. By Hausdorff property, there must exist open disjoint neighborhoods $V_2 \ni x, U_2 \ni x_2$ inside V_1. Now repeat this process indefinitely to obtain a set $\{x_n\}_{n\in\mathbb{N}}$ which does not contain its accumulation points.

(2.) Notice that from the previous item, every point x_n has an open neighborhood U_n, where $U_n \cap U_m = \emptyset$ for any $n \neq m$. By Urysohn's lemma, there exists a continuous function $f_n : \sigma(A) \to [0,1]$ for every $n \in \mathbb{N}$ such that $f_n(x_n) = 1$ and $f_n = 0$ on

[1] The reader should consult Misamichi Takesaki, *Theory of Operator Algebras I*, Theorem I.10.2

$\sigma(A) \setminus U_n$.

(3.) Finally, we have an infinite linearly independent set $\{f_n\}_{n\in\mathbb{N}}$ in $C(\sigma(A))$, so the latter algebra must be infinite-dimensional. $\square$

Let A be any finite-dimensional C^*-algebra. Then its center, say C, is a finite-dimensional abelian C^*-algebra with spectrum $\{\omega_1, \dots, \omega_n\}$. Let $p_i \in C$ be an element that corresponds to characteristic function $\chi_{\{\omega_i\}} \in C(\sigma(C))$. It follows from Gelfand that $C \cong \mathbb{C}p_1 \oplus \cdots \oplus \mathbb{C}p_n$, so $p_1 + dots + p_n = 1$. As a result,

$$A \cong Ap_1 \oplus \cdots \oplus Ap_n,$$

where each of Ap_i has trivial center. By previous lemma, Ap_i is a simple finite-dimensional C^*-algebra. So it suffices to prove the structure theorem for a simple finite-dimensional C^*-algebras.

Proof of the structure theorem. Assume A is simple and finite-dimensional. We first note that $aAb \neq \{0\}$ for any nonzero $a, b \in A$. This follows from the observation that the set AaA is an ideal of A which must be the entire algebra A, since it is nonzero. Let B be a maximal abelian $*$-subalgebra of A and let $\sigma(B) = \{\omega_1, \omega_2, \dots, \omega_n\}$ be its spectrum. Let $e_i \in B$ denote the projection, corresponding to the characteristic function $\chi_{\{\omega_i\}}$. By our previous arguments, e_i are orthogonal and $\sum_{i=1}^n e_i = 1$. Furthermore, $B \cong \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$. It follows that e_iAe_i commutes with every e_j. Next, we prove that the C^*-algebra e_iAe_i has dimension one. Take any normal element $x \in e_iAe_i$. By Gelfand, $\sigma(x)$ is homeomorphic to the spectrum of a C^*-algebra, generated by x. This C^*-algebra is finite-dimensional (since it lives inside the finite-dimensional C^*-algebra e_iAe_i), so its spectrum must be finite and as a result, $\sigma(x)$ is finite (and discrete). Suppose that there exists a normal element $x \in e_iAe_i$ such that $\sigma(x)$ has at least two distinct elements λ_1, λ_2. Then functional calculus gives us orthogonal projections $p = \chi_{\{\lambda_1\}}(x)$ and $q = \chi_{\{\lambda_2\}}(x)$ in e_iAe_i. Then $C^*(p, q, e_2, \dots, e_n)$ is an abelian C^*-algebra in A of dimension $n+1$, which is in contradiction with assumption that B is maximal abelian. This means that every normal element $x \in e_iAe_i$ has singleton spectrum $\sigma(x) = \lambda$, so it is of the form $x = \lambda \cdot e_i$ by functional calculus. Since e_iAe_i is the span of self-adjoint (which are normal) elements, we have $e_iAe_i = \mathbb{C}e_i \subseteq B$. For fixed i, j, we know that $e_iAe_j \neq \{0\}$. Choose a nonzero $x \in e_iAe_j$ and notice that $x = e_ixe_j$. This implies that $x^*x = e_jx^*xe_j = \lambda e_j$ and $xx^* = e_ixx^*e_i = \mu e_j$ for some $\lambda, \mu \neq 0$. But from

$$\lambda = \|x^*x\| = \|x\|^2 = \|x^*\|^2 = \|xx^*\| = \mu$$

we get $\lambda = \mu > 0$. If we define $u = \lambda^{-\frac{1}{2}}x$, we get $u^*u = e_j$ and $uu^* = e_i$. For each i, let $u_i \in A$ be an element such that $u_i^*u_i = e_1 = u_iu_i^* = e_i$. Then, define $u_{i,j} = u_iu_j^*$. From our arguments, the following has to be true:

$$u_{i,j}^* = u_{j,i}, \quad \sum_{i=1}^n u_{i,i} = 1, \quad u_{i,j}u_{k,l} = \delta_{j,k}u_{i,l}$$

We claim that $e_iAe_j = \mathbb{C}u_{i,j}$. Indeed, if $x \in e_iAe_j$, then $xu_{i,j} \in e_iAe_i$, so $xu_{i,j} = \lambda e_i$ for some $\lambda \in \mathbb{C}$. Hence we get

$$x = xe_j = xu_{j,i}u_{i,j}\lambda e_iu_{i,j}\lambda u_{i,j}.$$

For each $x \in A$, let $\lambda_{i,j}(x)$ be a scalar such that $e_ixe_j = \lambda_{i,j}(x)u_{i,j}$. It follows that

$$x = \sum_{i,j=1}^n e_ixe_j = \sum_{i,j=1}^n \lambda_{i,j}(x)u_{i,j}.$$

Now the map

$$A \to M_n(\mathbb{C}), \quad x \mapsto (\lambda_{i,j}(x))_{i,j}$$

is a $*$-isomorphism of A onto the algebra $M_n(\mathbb{C})$, where $\dim A = n^2$. $\square$

3.4.2 Group C^*-algebras

Let G be a group. Then the (complex) group algebra $\mathbb{C}[G]$ is defined as the algebra with basis $\{u_g \mid g \in G\}$ and multiplication given by $u_g \cdot u_h = u_{gh}$. Multiplication is convolutive:

$$\begin{aligned}\left(\sum_g^{\text{finite}} a_g u_g\right)\left(\sum_h^{\text{finite}} b_h u_h\right) &= \sum_{g,h} a_g b_h u_g u_h \\ &= \sum_{g,h} a_g b_h u_{gh} \\ &= \sum_k \left(\sum_g a_g b_{g^{-1}k}\right) u_k.\end{aligned}$$

We can equip $\mathbb{C}[G]$ with an involution

$$\left(\sum_{g\in G}^{\text{finite}} a_g u_g\right)^* = \sum_g \overline{a_g} u_{g^{-1}}.$$

Given a representation (homomorphism of $*$-algebras) $\pi : \mathbb{C}[G] \to \mathcal{B}(\mathcal{H})$, we define the C^*-algebra

$$C^*_\pi(G) := \overline{\pi(\mathbb{C}[G])} \subseteq \mathcal{B}(\mathcal{H})\,.$$

For $g \in G$, we get

$$\begin{aligned}\pi(u_g)\pi(u_g)^* &= \pi(u_g) \cdot \pi(u_g^*) \\ &= \pi(u_g) \cdot \pi(u_{g^{-1}}) \\ &= \pi(u_g \cdot u_{g^{-1}}) = \pi(u_e) = 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\pi(u_g)^*\pi(u_g) &= \pi(u_g^*) \cdot \pi(u_g) \\ &= \pi(u_{g^{-1}}) \cdot \pi(u_g) \\ &= \pi(u_{g^{-1}} \cdot u_g) = \pi(u_e) = 1.\end{aligned}$$

We have thus proved that under any representation of $\mathbb{C}[G]$, each u_g is mapped to a unitary.

Example 3.32. *Take $\mathcal{H} = \ell^2(G)$ (this is a Hilbert space with ONB $\{\delta_g \mid g \in G\}$). Then*

$$\lambda : \mathbb{C}[G] \to \mathcal{B}(\ell^2(G)), \quad u_g \mapsto (\delta_h \mapsto \delta_{gh})$$

is a faithful representation. We call it the left regular representation of G. The closure of its image is called the reduced group C^*-algebra $C^*_r(G) := \overline{\lambda(\mathbb{C}[G])} \subseteq \mathcal{B}(\ell^2(G))$.

Definition 3.33. *Universal* (or *full*) group C^*-algebra is the completion $\mathbb{C}[G]$, where the norm of an element $a \in \mathbb{C}[G]$ is $\|a\|_u = \sup\{\|\pi(a)\| \mid \pi \text{ representation of } \mathbb{C}[G]\}$.

Lemma 3.34. *If π is a representation of $\mathbb{C}[G]$ and $a = \sum_{g\in G}^{\text{finite}} a_g u_g \in \mathbb{C}[G]$, then $\|\pi(a)\| \leq \sum |a_g|$.*

Proof. Then $\pi(a) = \sum a_g \cdot \pi(u_g)$. Then

$$\|\pi(a)\| = \left\|\sum a_g \pi(u_g)\right\| \leq \sum |a_g| \cdot \|\pi(u_g)\| = \sum |a_g|. \qquad \square$$

This implies that $\|\cdot\|_u$ is indeed a norm on $\mathbb{C}[G]$, making the universal C^*-algebra $C(G)$ of G in Definition 3.33 well-defined.

Remark. The group algebra $\mathbb{C}[G]$ is dense in both $C^*_\pi(G)$ and $C^*(G)$.

Theorem 3.35 (Universal property)**.**

*For each representation π of $\mathbb{C}[G]$ there exists a surjective *-homomorphism $\widehat{\pi} : C^*(G) \to C^*_\pi(G)$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathbb{C}[G] \arrow[r, "\pi"] \arrow[d, hook] & C^*_\pi(G) \\ C^*(G) \arrow[ur, dashed, "\exists! \widehat{\pi}"'] & \end{tikzcd}$$

Proof. Define first $\widehat{\pi}$ on $\mathbb{C}[G] \subseteq C^*G$ by $\widehat{\pi}(a) := \pi(a) \in C^*_\pi(G)$. Firstly, $\widehat{\pi}$ on $\mathbb{C}[G]$ is contractive:

$$\|\widehat{\pi}(a)\| = \|\pi(a)\| \leq \|a\|_u.$$

By density, $\widehat{\pi}$ uniquely extends to a continuous *-homomorphism $\widehat{\pi} : C^*(G) \to C^*_\pi(G)$. This $\widehat{\pi}$ is also contractive and $\operatorname{im}\pi$ is dense, so $\widehat{\pi}$ is onto. $\square$

Example 3.36. *Let G be abelian and $|G| = n$. Then $\mathbb{C}[G] \cong \mathbb{C}^{|G|}$ as a vector space. Hence $C^*G = \mathbb{C}[G] = C^*_r(G)$. Furthermore, $\mathbb{C}[G]$ is commutative, so by the structure theorem we have*

$$\mathbb{C}[G] \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ times}}$$

as a C^*-algebra. For instance, $\mathbb{C}\left[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\right] \cong \mathbb{C}\left[\mathbb{Z}/4\mathbb{Z}\right]$.

Example 3.37. *Let $G = S_3$. Then $|G| = 6$ and once again $C^*(G) = \mathbb{C}[G] = C^*_r(G)$. By structure theorem and the dimension consideration, $\mathbb{C}[G] \cong M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ (otherwise it would be commutative).*

Example 3.38. *Let $G = S_4$. Again, $C^*(G) = \mathbb{C}[G] = C^*_r(G)$. By Maschke's theorem, $\mathbb{C}[G]$ is semisimple, therefore it is a direct sum of matrix algebras over $\mathbb{C}$. Since S_4 has five conjugacy classes, there are five factors[a]. Adding up all the dimensions, the only combination that works is $9+9+4+1+1=24$, therefore*

$$\mathbb{C}[G] \cong M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}.$$

[a] Pierre Antoine Grillet, *Abstract algebra*, theorem IX.7.9.

Example 3.39. *What is $C^*(\mathbb{Z})$? Representations $\pi(\mathbb{C}[Z]) \to \mathcal{B}(\mathcal{H})$ are determined by choice of unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\pi(u_1) = U$. By universal property, for every $\mathcal{H}$ and $U \in \mathcal{B}(\mathcal{H})$ there exists a unique *-homomorphism*

$$\widehat{\pi} : C^*(\mathbb{Z}) \to C^*(\{U\}),$$

where the latter is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, generated by U. We call $C^*(\mathbb{Z})$ the universal C^*-algebra, generated by a unitary.

3.5 Abelian group C^*-algebras

If G is abelian, then $\mathbb{C}[G]$ is commutative and $C^*_r(G)$ is abelian. By Gelfand, there exists a compact Hausdorff space Σ such that $C^*_r(G) \cong C(\Sigma)$ and $\Sigma = \sigma(C^*_r(G))$.

Definition 3.40. To each abelian group G we associate its *Pontryagin dual*

$$\widehat{G} = \{w : G \to \mathbb{T} \text{ group homomorphism}\}.$$

Then $\widehat{G}$ is a group under pointwise multiplication. We endow $\widehat{G}$ with the compact-open topology induced from $\widehat{G} \subseteq \mathbb{T}^G$. Recall that the basis sets for this topology are

$$B_{\varepsilon,F}(w) = \{\eta \in \widehat{G} \mid |\eta(h) - w(h)| < \varepsilon,\ \forall h \in F\}$$

for $\varepsilon > 0$, $w \in \widehat{G}$ and $F \subseteq G$ finite.

Remark. A net $(w_i)_{i\in I} \subseteq \widehat{G}$ is Cauchy iff $(w_i(g))_{i\in I} \subseteq \mathbb{T}$ is Cauchy for all $g \in G$.

Theorem 3.41.

The map

$$h : \widehat{G} \to \sigma(C^*_r(G)), \quad w \mapsto \left(\sum a_g u_g \mapsto \sum a_g w(g)\right)$$

is a homeomorphism.

Proof. First, we prove that $h(w) \in \sigma(C^*_r(G))$ for all $w \in \widehat{G}$. We begin by showing $h(w) : \mathbb{C}[G] \to \mathbb{C}$ is a homomorphism. Take $b = \sum b_k u_k \in \mathbb{C}[G]$. Then

$$h(w)(a \cdot b) = \sum_g \left(\sum_h a_h b_{h^{-1}g}\right) \cdot w(g)$$

and

$$h(w)(a) \cdot h(w)(b) = \left(\sum_g a_g w(g)\right) \cdot \left(\sum_h b_h w(h)\right) = \sum_k \left(\sum_h a_{kh^{-1}} b_h\right) w(k),$$

so $h(w)$ is multiplicative. To extend it to C_r^*, we must prove that $|h(w)a| \leq \|a\|_r$ for all $a \in \mathbb{C}[G]$. To $\chi \in \sigma(C_r^*(G))$ and $a \in \mathbb{C}[G]$ we associate

$$\widetilde{a} = \sum a_g w(g) \cdot \overline{\chi(u_g)} u_g,$$

so $h(w)a = \chi(\widetilde{a})$. By Gelfand,

$$\|\widetilde{a}\|_r = \sup\{|\mu(\widetilde{a})| \mid \mu \in \sigma(C_r^*(G))\} \geq |\chi(\widetilde{a})| = |h(w)a|.$$

Next, we show that $\|\widetilde{a}\|_r = \|a\|_r$: to $\theta \in \ell^2(G)$ assign $\widetilde{\theta}$ by $\widetilde{\theta}_h := \chi(u_{h^{-1}})\overline{w(h)}\theta_h$. Then $\|\theta\|_2 = \|\widetilde{\theta}\|_2$. Further, $\|\lambda(\widetilde{a})\widetilde{\theta}\|_2 = \|\lambda(a)\theta\|_2$ (short calculation), so

$$\|\widetilde{a}\|_r = \sup\{\|\lambda(\widetilde{a})\widetilde{\theta}\| \mid \|\widetilde{\theta}\|_2 = 1\} = \sup\{\|\lambda(\widetilde{a})\theta\| \mid \|\theta\|_2 = 1\} = \|a\|_r.$$

Next, we prove that h is continuous. Suppose the net $(w_i)_{i \in I} \subseteq \widehat{G}$ is Cauchy. We prove that for every $a \in C_r^*(G)$ the net $(h(w_i)(a))_{i \in I}$ is Cauchy. Pick $\varepsilon > 0$. There exists J such that for every $i, j \geq J$, we have

$$|w_i(g) - w_j(g)| < \frac{\varepsilon}{|\{g \mid a_g \neq 0\}|}, \quad \forall g \in G.$$

Then for all $i, j \geq J$ we get $|h(w_i)(g) - h(w_j)(g)| < \varepsilon$. For the general case $a \in C_r^*(G)$, we can take a as a limit of a sequence $(a_n)_n \subseteq \mathbb{C}[G]$, approximate a with a_n and use the triangle inequality to establish that $(h(w_i)(a))_i$ is Cauchy. Now on to bijectivity of h. It's enough to check that it is surjective: take $\phi \in \sigma(C_r^*(G))$. Define

$$w_\phi : G \to \mathbb{C}, \quad g \mapsto \phi(u_g).$$

Since ϕ is a *-homomorphism, $\operatorname{im} w_\phi \subseteq \mathbb{T}$. We have to prove that $w_\phi \in \widehat{G}$. We just check the multiplicativity:

$$w_\phi(g) \cdot w_\phi(h) = \phi(u_g)\phi(u_h) = \phi(u_g u_h) = \phi(u_{gh}) = w_\phi(gh).$$

For every $w \in \widehat{G}$, we get $w_{h(w)} = w$. So $w_\phi = w_{h(w_\phi)}$, which gives us $h(w_\phi) = \phi$. Now since h is a bijective continuous map between compact Hausdorff spaces, it is a homeomorphism. □

In operator algebras, we usually consider topological groups that are locally compact and Hausdorff. For such groups, there exists a so-called *Haar measure* μ. This measure allows us to consider the C^*-algebras $L^2(G)$, $C_r^*(G)$ and $C^*(G)$ for general locally compact and Hausdorff topological groups. If a group G is equipped with a discrete topology, then these notions coincide with the ones from the previous subsection.

Example 3.42. *Let us present the Pontryagin duals of some locally compact Hausdorff abelian groups.*

- $\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z}$;
- $\widehat{\mathbb{R}} = \mathbb{R}$;
- $\widehat{\mathbb{Z}} = \mathbb{T}$;
- $\widehat{\mathbb{T}} = \mathbb{Z}$.

We notice that for each of these groups, the dual of a dual is the original group. This is not a coincidence.

Theorem 3.43 (Pontryagin)**.**

If G is a locally compact Hausdorff abelian group, then $G \approx \widehat{\widehat{G}}$.

4 Bounded operators on Hilbert spaces

4.1 Polar decomposition

Let $\mathcal{H}$ be a complex Hilbert space. Then $\mathcal{B}(\mathcal{H})$ is a C^*-algebra with the operator norm

$$\|A\| = \sup_{\mu\in\mathcal{H},\mu\neq 0} \frac{\|A\mu\|}{\|\mu\|} = \sup_{\mu\in\mathcal{H},\|\mu\|=1} \|A\mu\| = \sup_{\mu\in\mathcal{H},\|\mu\|\leq 1} \|A\mu\|$$

Remark. Recall that $A \in \mathcal{B}(\mathcal{H})$ is:

(1.) normal $\Leftrightarrow A^*A = AA^* \Leftrightarrow \|A\mu\| = \|A^*\mu\|,\ \forall \mu \in \mathcal{H}$;

(2.) self-adjoint $\Leftrightarrow A^* = A \Leftrightarrow \langle A\mu, \mu\rangle \in \mathbb{R},\ \forall \mu \in \mathcal{H}$;

(3.) positive $\Leftrightarrow A = B^*B$ for some $B \in \mathcal{B}(\mathcal{H}) \Leftrightarrow \langle A\mu, \mu\rangle \geq 0,\ \forall \mu \in \mathcal{H}$;

(4.) isometry $\Leftrightarrow A^*A = I \Leftrightarrow \|A\mu\| = \|\mu\|,\ \forall \mu \in \mathcal{H}$;

(5.) projection $\Leftrightarrow A^2 = A = A^* \Leftrightarrow A$ is an orthogonal projection onto some closed subspace of $\mathcal{H}$.

Lemma 4.1. *An operator $A \in \mathcal{B}(\mathcal{H})$ is a partial isometry iff there exists a closed subspace $\mathcal{K} \leq \mathcal{H}$ such that $A\big|_{\mathcal{K}}$ is an isometry and $A\big|_{\mathcal{K}^\perp} = 0$.*

Proof. We first prove ($\Leftarrow$). Obviously, $\mathcal{K}^\perp \subseteq \ker A$. From $Ax = 0$, where $x = y + z$ and $y \in \mathcal{K}, z \in \mathcal{K}^\perp$, we have

$$0 = Ax = A(y + z) = Ay + Az = Ay.$$

But since $A\big|_{\mathcal{K}}$ is an isometry, $\|Ay\| = \|y\| = 0$, so $y = 0$ and $x \in \mathcal{K}^\perp$. Now we prove that $P = A^*A$ is the projection onto $\mathcal{K}$. For $x \in \mathcal{K}$, we have

$$\langle Px, x\rangle = \langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 = \|x\|^2,$$

so

$$\|P\| = \|A^*A\| \leq \|A\|\|A^*\| = \|A\|^2 = 1.$$

From Cauchy-Schwartz:

$$\langle Px, x\rangle \leq \|Px\|\|x\| \leq \|P\|\|x\|^2 \leq \|x\|^2.$$

Since we have equality in Cauchy-Schwartz, there exists a $\lambda \in \mathbb{C}$ such that $Px = \lambda x$. But from $\langle Px, x\rangle = \|x\|^2$, it follows that $\lambda = 1$. So $P\big|_{\mathcal{K}} = \mathrm{id}$ and for $x \in \mathcal{K}^\perp$, $Px = A^*Ax = 0$. Therefore, $P = A^*A$ is indeed a projection. Now onto the opposite direction ($\Rightarrow$). Suppose $P = A^*A$ is a projection and denote $\mathcal{K} = \operatorname{im} P$. Since $\mathcal{K} = \ker(I - P)$, it is a closed subspace of $\mathcal{H}$. For $x \in \mathcal{K}$, we have

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle Px, x\rangle = \langle x, x\rangle = \|x\|^2.$$

But for $x \in \mathcal{K}^\perp$, we use the identity

$$(\operatorname{im} P)^\perp = \ker P^* = \ker P$$

to get $Px = 0$, so $\|Ax\|^2 = \langle Px, x\rangle = 0$ and $\|Ax\| = 0$. $\square$

Theorem 4.2 (Polar decomposition)**.**

Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$. Then there exists a partial isometry v such that $x = v \cdot |x|$ and $\ker v = \ker |x| = \ker x$. This decomposition is unique: if $x = wy$ for $y \geq 0$ and partial isometry w such that $\ker y = \ker w$, then $w = v$ and $y = |x|$.

Proof. First we prove the existence. Define

$$v_0 : \operatorname{im}|x| \to \operatorname{im} x, \quad |x|y \mapsto xy.$$

Since

$$\begin{aligned} \||x|y\|^2 &= \langle |x|y, |x|y\rangle \\ &= \langle |x|^2 y, y\rangle \\ &= \langle x^* x y, y\rangle \\ &= \langle xy, xy\rangle \\ &= \|xy\|^2. \end{aligned}$$

The above v_0 is well defined. It is also linear and isometric. By continuity, extend v_0 to a map $\overline{\operatorname{im}|x|} \to \overline{\operatorname{im} x}$. Now v_0 can be extended to $v : \mathcal{H} \to \mathcal{H}$ by setting $v\big|_{(\operatorname{im}|x|)^\perp} = 0$. By previous lemma, v is a partial isometry. By definition, $x = v \cdot |x|$ and $\ker v = (\operatorname{im}|x|)^\perp = \ker |x| = \ker x$. Next, we prove uniqueness. If $x = wy$ as in the statement, then $\ker w = \ker y = (\operatorname{im} y)^\perp$, so w is a partial isometry on $\overline{\operatorname{im} y}$. From there, we get

$$|x|^2 = (wy)^*(wy) = y^* w^* w y = y^* y = y^2,$$

which implies

$$|x| = (|x|^2)^{\frac{1}{2}} = (y^2)^{\frac{1}{2}} = y.$$

Now

$$w|x|\mu = wy\mu = x\mu$$

together with

$$\ker w = (\operatorname{im} y)^\perp = (\operatorname{im}|x|)^\perp$$

implies $w = v$. □

Now we can also prove the statement in the Example 1.43.

Proposition 4.3. *The extreme points of the unit ball of $\mathcal{B}(\mathcal{H})$ are exactly the elements $V \in \mathcal{B}(\mathcal{H})$ such that*

$$(1 - VV^*)\,\mathcal{B}(\mathcal{H})(1 - V^*V) = 0.$$

In particular, V^*V and VV^* are projections.

Proof. Let $V \in A$ be an extreme point of the unit ball of $\mathcal{B}(\mathcal{H})$, so $\sigma(V) \subseteq [-1, 1]$. Write

$$V = \frac{1}{2}V(2 - |V|) + \frac{1}{2}V|V|.$$

Since the functions $z \mapsto z(2-|z|)$ and $z \mapsto |z|(2-|z|)$ coincide and are both bounded above by 1 on $\sigma(x)$, we have $\|V(2-|V|)\| = \||V|(2-|V|)\| \leq 1$ by continuous functional calculus. This implies that $V(2-|V|)$ is in the unit ball of $\mathcal{B}(\mathcal{H})$. The same can be said about $V|V|$ by the same argument. Now since V is an extreme point, we must have $V = V|V|$. Multiplying on the left with V^*, we get $|V|^2 = |V|^3$. This means that the functions $z \mapsto z^2$ and $z \mapsto z^3$ coincide on $\sigma(|V|)$, which implies that $\sigma(|V|) \subseteq \{0,1\}$. As a result, $|V|$ is a projection, so $P := |V| = |V|^2 = V^*V$. The same can be said about $Q := VV^*$, since we know that $\sigma(V^*V) \setminus \{0\} = \sigma(VV^*) \setminus \{0\}$. This means that V is a partial isometry. By the previous lemma, P is a projection onto the initial space of V, so $QV = VP = V$. Now suppose $W := (1-Q)Z(1-P) \neq 0$ for some Z in the unit ball. Then

$$\begin{aligned}
\|V+W\|^2 &= \|QVP + (1-Q)Z(1-P)\|^2 \\
&= \|(QVP + (1-Q)Z(1-P))^*(QVP + (1-Q)Z(1-P))\| \\
&= \|(PV^*Q + (1-P)Z^*(1-Q))(QVP + (1-Q)Z(1-P))\| \\
&= \|PV^*QVP + (1-P)Z^*(1-Q)Z(1-P)\| \\
&= \|PV^*VP + (1-P)W^*W(1-P)\| \\
&= \max\{\|V^*V\|, \|W^*W\|\} \\
&= \max\{\|V\|^2, \|W\|^2\} = 1
\end{aligned}$$

and similarly $\|V-W\|^2 = 1$. Therefore we have a decomposition

$$V = \frac{1}{2}(V+W) + \frac{1}{2}(V-W)$$

and V is not an extreme point, leading to a contradiction. Conversely, suppose that $(1 - VV^*)\,\mathcal{B}(\mathcal{H})(1-V^*V) = 0$. Then we have

$$0 = V^*(1-VV^*)V(1-V^*V) = V^*V(1-V^*V)^2.$$

This implies that the function $z \mapsto z(1-z)^2$ must be zero on $\sigma(V^*V)$, so $\sigma(V^*V) \subseteq \{0,1\}$ and $P := V^*V$ is a projection. By the same argument,

$$0 = (1-VV^*)V(1-V^*V)V^* = (1-VV^*)^2VV^*$$

and $Q := VV^*$ is a projection as well. Assume that $V = \frac{1}{2}U + \frac{1}{2}W$ for U, W in the unit ball. Again we have $V = VP = QV$, so

$$V = \frac{1}{2}UP + \frac{1}{2}WP$$

and

$$\begin{aligned}
4P = 4V^*V &= PU^*UP + PW^*WP + PU^*WP + PW^*UP \\
&= 2(PU^*UP + PW^*WP) - P(U-W)^*(U-W)P \\
&\leq 4P - P(U-W)^*(U-W)P.
\end{aligned}$$

This immediately implies that $(U-W)P = 0$. Similarly, we have $Q(U-W) = 0$. Now

$$\begin{aligned}
U - W &= Q(U-W)P + (1-Q)(U-W)P \\
&\quad + Q(U-W)(1-P) + (1-Q)(U-W)(1-P) = 0. \qquad \square
\end{aligned}$$

Remark. The above theorem holds for any C^*-algebra, not just $\mathcal{B}(\mathcal{H})$. We can identify any general C^*-algebra with an algebra of operators on some Hilbert space and then the above proof carries over verbatim.

Corollary 4.4.
$$\operatorname{ext}(\mathcal{B}(\mathcal{H}))_1 = \{V \in \mathcal{B}(\mathcal{H}) \mid V \textit{ or } V^* \textit{ is an isometry}\}.$$

Proof. We need to prove the inclusion ($\subseteq$). If V is an extreme point, then V^*V and VV^* are projections Therefore, V is a partial isometry with the initial space $(\ker V)^\perp$ and V^* is a partial isometry with the initial space $(\ker V^*)^\perp$. Assume neither V nor V^* are full isometries, so their initial spaces are proper subspaces of $\mathcal{H}$. This means that there exist vectors $0 \neq x \in \ker V$ and $0 \neq y \in \ker V^*$. Define P as a rank-one projection from x to y. Then
$$(1 - VV^*)P(1 - V^*V)x = y \neq 0,$$
so $(1 - VV^*)P(1 - V^*V) \neq 0$, contradiction. $\square$

4.2 Trace class operators

Definition 4.5. Let X, Y be Banach spaces. An operator $A \in \mathcal{B}(X, Y)$ has *finite rank* if $\operatorname{rank} A := \dim \overline{\operatorname{im} A} < \infty$. The set of finite rank operators is denoted by $\mathcal{F}(X, Y)$. We also denote $\mathcal{F}(X) := \mathcal{F}(X, X)$.

Remark. Let $A \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. We know that
$$\operatorname{im} A^* = \operatorname{im}(A^*|_{(\ker A^*)^\perp}) = \operatorname{im}(A^*|_{\overline{\operatorname{im} A}}).$$
From there, we can conclude that $\operatorname{rank} A < \infty$ iff $\operatorname{rank} A^* < \infty$.

If $\alpha, \beta \in \mathcal{H}$, then we can define the operator
$$\alpha \otimes \overline{\beta} : \mathcal{H} \to \mathcal{H}, \quad y \mapsto \langle y, \beta \rangle \cdot \alpha.$$
It is trivial to see that $\operatorname{rank}(\alpha \otimes \overline{\beta}) \leq 1$ and $(\alpha \otimes \overline{\beta})^* = \beta \otimes \overline{\alpha}$. By Riesz's representation theorem, we also know that every rank-one operator on $\mathcal{H}$ is of this form. If $\|\alpha\| = \|\beta\| = 1$, then $\alpha \otimes \overline{\beta}$ is a partial isometry with initial space $\mathbb{C}\beta$ and image $\mathbb{C}\alpha$. Then
$$\mathcal{F}(\mathcal{H}) = \operatorname{span}\, \{\alpha \otimes \overline{\beta} \mid \alpha, \beta \in \mathcal{H}\}.$$
For $x, y \in \mathcal{B}(\mathcal{H})$ we have
$$x(\alpha \otimes \overline{\beta})y = (x\alpha) \otimes \overline{(y^*\beta)}.$$

Lemma 4.6. *Let $x \in \mathcal{B}(\mathcal{H})$ have the polar decomposition $x = v \cdot |x|$. Then for all $y \in \mathcal{H}$, we have*
$$2\,|\langle xy, y \rangle| \leq \langle |x|y, y \rangle + \langle |x|v^*y, v^*y \rangle.$$

Proof. Let $\lambda \in \mathbb{T}$. Then

$$\begin{aligned} 0 &\leq \|(|x|^{\frac{1}{2}} - \lambda |x|^{\frac{1}{2}} v^*)y\|^2 \\ &= \||x|^{\frac{1}{2}} y\|^2 - 2\operatorname{Re} \overline{\lambda} \langle |x|^{\frac{1}{2}} y, |x|^{\frac{1}{2}} v^* y\rangle + \||x|^{\frac{1}{2}} v^* y\|^2. \end{aligned}$$

Now pick λ such that $\overline{\lambda}\langle |x|^{\frac{1}{2}} y, |x|^{\frac{1}{2}} v^* y\rangle \geq 0$ and we are done. □

Definition 4.7. Let $(e_i)_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. For $x \in \mathcal{B}(\mathcal{H})_+$, define the *trace*

$$\operatorname{Tr}(x) = \sum_{i\in I} \langle xe_i, e_i\rangle \in [0, \infty].$$

We call $x \in \mathcal{B}(\mathcal{H})$ *trace class* if

$$\|x\|_1 := \operatorname{Tr}(|x|) < \infty.$$

The set of trace class operators on $\mathcal{H}$ will be denoted by $L^1(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

Remark. Let $\{h_i \mid i \in I\} \subseteq \mathcal{H}$ be a set of vectors in a Hilbert space. We already know that the collection of finite sets $F \subseteq I$ forms a directed set. Then vectors $h_F := \sum_{i\in F} h_i$ form a net in $\mathcal{H}$. We define $\sum_{i\in I} h_i$ as the limit of the net (h_F), if it exists. Note that if I is countable, this definition of a convergent sum does not necessarily coincide with the usual one. In other words, for a set $\{h_n \mid n \in \mathbb{N}\}$ in a Hilbert space $\mathcal{H}$, the convergence of a sum $\sum_{n\in\mathbb{N}} h_n$ is not equivalent to the convergence of a sum $\sum_{n=1}^{\infty} h_n$ – in fact, the convergence of a former sum implies the convergence of the latter one (with the sums being equal). The converse holds if $\sum_{n=1}^{\infty} \|h_n\| < \infty$.

Remark. If $x \in \mathcal{B}(\mathcal{H})_+$ and $\operatorname{Tr}(x) = \sum_{i\in I}\langle xe_i, e_i\rangle < \infty$, then $\langle xe_i, e_i\rangle > 0$ holds for at most countably many e_i. Let $(e_n)_{n\in\mathbb{N}}$ be a set of such basis vectors. Then $\sum_{i\in I}\langle xe_i, e_i\rangle = \sum_{n=1}^{\infty}\langle xe_n, e_n\rangle$.

Lemma 4.8. *For all $x \in \mathcal{B}(\mathcal{H})$ we have* $\operatorname{Tr}(x^*x) = \operatorname{Tr}(xx^*)$.

Proof.

$$\begin{aligned} \operatorname{Tr}(x^*x) &= \sum_i \langle x^*xe_i, e_i\rangle = \sum_i \langle xe_i, xe_i\rangle \\ &= \sum_i \|xe_i\|^2 = \sum_i \sum_j \langle xe_i, e_j\rangle \overline{\langle xe_i, e_j\rangle} \\ &= \sum_j \sum_i \langle e_i, x^*e_j\rangle \overline{\langle e_i, x^*e_j\rangle} = \sum_j \sum_i \langle x^*e_j, e_i\rangle \overline{\langle x^*e_j, e_i\rangle} \\ &= \sum_j \|x^*e_j\|^2 = \sum_j \langle x^*e_j, x^*e_j\rangle \\ &= \sum_j \langle xx^*e_j, e_j\rangle = \operatorname{Tr}(xx^*) \end{aligned}$$

□

Corollary 4.9. *If $x \in \mathcal{B}(\mathcal{H})_+$ and $u \in \mathcal{U}(\mathcal{H})$, then*

$$\operatorname{Tr}(u^* x u) = \operatorname{Tr}(x).$$

In particular, the trace of a positive operator is independent of the choice of the orthonormal basis for $\mathcal{H}$.

Proof. Since $x \in \mathcal{B}(\mathcal{H})_+$, there exists a $y \in \mathcal{B}(\mathcal{H})$ such that $x = y^* y$. By Lemma 4.8, we have

$$\begin{aligned}\operatorname{Tr}(x) &= \operatorname{Tr}(y^* y) = \operatorname{Tr}(y y^*) \\ &= \operatorname{Tr}(u^* y^* y u) = \operatorname{Tr}(u^* x u).\end{aligned}$$

If (f_i) is another ONB for $\mathcal{H}$, then there exists $u \in \mathcal{U}(\mathcal{H})$ such that $u e_i = f_i$ for all indices i:

$$\begin{aligned}\sum_i \langle x f_i, f_i \rangle &= \sum_i \langle x u e_i, u e_i \rangle \\ &= \sum_i \langle u^* x u e_i, e_i \rangle \\ &= \operatorname{Tr}(u^* x u) = \operatorname{Tr}(x).\end{aligned}$$

□

Definition 4.10. If (e_i) is ONB for $\mathcal{H}$ and $x \in L^1(\mathcal{B}(\mathcal{H}))$, then its trace is

$$\operatorname{Tr}(x) := \sum_{i \in I} \langle x e_i, e_i \rangle.$$

By Lemma 4.6 and the proof below, we get

$$\begin{aligned}2|\operatorname{Tr}(x)| &\leq \sum_{i \in I} 2 |\langle x e_i, e_i \rangle| \\ &\leq \sum_{i \in I} \langle |x| e_i, e_i \rangle + \langle |x| v^* e_i, v^* e_i \rangle \\ &= \operatorname{Tr}(|x|) + \operatorname{Tr}(v|x|v^*) \\ &\leq \|x\|_1 + \|x\|_1 \\ &= 2\|x\|_1.\end{aligned}$$

Theorem 4.11.

(1.) $L^1(\mathcal{B}(\mathcal{H}))$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$ that is closed under involution.

(2.) $L^1(\mathcal{B}(\mathcal{H}))$ is a linear span of all positive operators of finite trace.

(3.) Trace is independent of the ONB and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.

Proof. Let $A, B \in L^1(\mathcal{B}(\mathcal{H}))$ and satisfy the polar decompositions:

$$A + B = U|A + B|, \quad A = V|A|, \quad B = W|B|.$$

Let (e_i) be an ONB. Then

$$\begin{aligned}\sum_{i=1}^{N} \langle |A+B|e_i, e_i\rangle &= \sum_{n=1}^{N} |\langle U^*(A+B)e_n, e_n\rangle| \\ &\leq \sum_{n=1}^{N} |\langle U^* A e_n, e_n\rangle| + \sum_{n=1}^{N} |\langle U^* B e_n, e_n\rangle| \\ &= \sum_{n=1}^{N} |\langle U^* V|A| e_n, e_n\rangle| + \sum_{n=1}^{N} |\langle U^* W|B| e_n, e_n\rangle|.\end{aligned}$$

We can bound the first term:

$$\begin{aligned}\sum_{n=1}^{N} |\langle U^* V|A|e_n, e_n\rangle| &= \sum_{n=1}^{N} |\langle |A|^{\frac{1}{2}} e_n, |A|^{\frac{1}{2}} V^* U e_n\rangle| \\ &\leq \sum_{n=1}^{N} \||A|^{\frac{1}{2}} e_n\| \||A|^{\frac{1}{2}} V^* U e_n\| \\ &\leq \left(\sum_{n=1}^{N} \||A|^{\frac{1}{2}} e_n\|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \||A|^{\frac{1}{2}} V^* U e_n\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\||A|^{\frac{1}{2}} e_n\|^2 = \langle |A|^{\frac{1}{2}} e_n, |A|^{\frac{1}{2}} e_n\rangle = \langle |A| e_n, e_n\rangle$, the expression in the first bracket goes to $\operatorname{Tr}|A|$. Next, we prove that the expression in the second bracket is less or equal to $\operatorname{Tr}|A|$:

$$\sum_{n=1}^{N} \langle |A|^{\frac{1}{2}} V^* U e_n, |A|^{\frac{1}{2}} V^* U e_n\rangle = \sum_{n=1}^{N} \langle U^* V|A|V^* U e_n, e_n\rangle \xrightarrow{N\to\infty} \operatorname{Tr}|A|.$$

Pick an ONB for $\mathcal{H}$ as follows: each f_j should be in $\ker U$ or $(\ker U)^{\perp}$. Then

$$\operatorname{Tr}(U^* V|A|V^* U) \leq \operatorname{Tr}(V|A|V^*).$$

By similar argument,

$$\operatorname{Tr}(V|A|V^*) \leq \operatorname{Tr}(|A|)$$

and we are done:

$$\sum_{n=1}^{N} |\langle U^* V|A|e_n, e_n\rangle| \leq \operatorname{Tr}|A|.$$

Similarly,

$$\sum_{n=1}^{N} |\langle U^* W|B|e_n, e_n\rangle| \leq \operatorname{Tr}|B|,$$

which implies $\operatorname{Tr}|A + B| \leq \operatorname{Tr}|A| + \operatorname{Tr}|B|$. We have proved that $L^1(\mathcal{B}(\mathcal{H}))$ is a vector space and $\|\cdot\|_1$ is a norm. Clearly, $L^1(\mathcal{B}(\mathcal{H}))$ contains all positive operators with finite trace, so also their linear span. Next we prove that it is a two-sided ideal of $\mathcal{B}(\mathcal{H})$. Let $A \in L^1(\mathcal{B}(\mathcal{H}))$ and $B \in \mathcal{B}(\mathcal{H})$. Since every operator is a linear combination of four unitaries, we can assume w.l.o.g. that $B = U$ is a unitary. Then

$$|UA| = (A^*U^*UA)^{\frac{1}{2}} = (A^*A)^{\frac{1}{2}} = |A|,$$

so $BA = UA \in L^1(\mathcal{B}(\mathcal{H}))$. Furthermore,

$$|AU| = (U^*A^*AU)^{\frac{1}{2}} = U^*|A|U,$$

which implies

$$\operatorname{Tr}|AU| = \operatorname{Tr}(U^*|A|U) = \operatorname{Tr}|A|$$

and $AB = AU \in L^1(\mathcal{B}(\mathcal{H}))$. Now we prove that $L^1(\mathcal{B}(\mathcal{H}))$ is closed under involution. Let $A = U|A|$ and $A^* = V|A^*|$ be polar decompositions. Then

$$|A^*| = V^*A^* = V^*(U|A|)^* = V^*|A|U^*.$$

If $A \in L^1(\mathcal{B}(\mathcal{H}))$, then $|A| \in L^1(\mathcal{B}(\mathcal{H}))$, so

$$|A^*| = V^*|A|U^* \in L^1(\mathcal{B}(\mathcal{H})).$$

This gives us $A^* \in L^1(\mathcal{B}(\mathcal{H}))$. Finally, we prove that $L^1(\mathcal{B}(\mathcal{H}))$ is the linear span of all positive operators of finite trace. Let $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $a \in \mathcal{B}(\mathcal{H})$. The following polarization identity holds:

$$4a|x| = \sum_{k=0}^{3} i^k \underbrace{(a+i^k)|x|(a+i^k)^*}_{\text{positive and finite trace}}.$$

If $a = v$ partial isometry from the polar decomposition theorem, then

$$x = v|x| = \sum_{k=0}^{3} \frac{i^k}{4}(v+i^k)|x|(v+i^k)^*.$$

is a linear combination of four positive operators with finite trace. □

Proposition 4.12. *Let $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $a, b \in \mathcal{B}(\mathcal{H})$. Then*

- $\|x\| \leq \|x\|_1$*;*
- $\|axb\|_1 \leq \|a\|\|b\|\|x\|_1$*;*
- $\operatorname{Tr}(ax) = \operatorname{Tr}(xa)$*.*

Proof. (1.)

$$\begin{aligned}\|x\| = \||x|\| &= \||x|^{\frac{1}{2}}\|^2 \\ &= \sup_{\|\alpha\|=1} \langle |x|^{\frac{1}{2}}\alpha, |x|^{\frac{1}{2}}\alpha\rangle = \sup_{\|\alpha\|=1} \langle |x|\alpha, \alpha\rangle \\ &\leq \operatorname{Tr}|x| = \|x\|_1.\end{aligned}$$

(2.) We begin with

$$|ax|^2 = x^*a^*ax \leq \|a^*a\|x^*x = \|a^*a\| \cdot |x|^2 = \|a\|^2 \cdot |x|^2$$

and since $|ax| \leq \|a\| \cdot |x|$ we get $\|ax\|_1 \leq \|a\| \cdot \|x\|_1$. But $\|x\|_1 = \|x^*\|_1$, so we also get $\|xb\|_1 \leq \|b\| \cdot \|x\|_1$.

(3.) Since every element of $\mathcal{B}(\mathcal{H})$ is a linear combination of 4 unitaries, we can w.l.o.g. assume $a = u \in \mathcal{U}(\mathcal{H})$. Then

$$\begin{aligned}\operatorname{Tr}(xu) &= \sum_i \langle xue_i, e_i\rangle = \sum_i \langle xue_i, u^*ue_i\rangle \\ &= \sum_i \langle uxue_i, ue_i\rangle = \operatorname{Tr}(ux).\end{aligned}$$ □

Remark. We have the following identities:

(1.) $\operatorname{Tr}(\alpha \otimes \overline{\beta}) = \langle \alpha, \beta\rangle$;

(2.) $\mathcal{F}(\mathcal{H})$ is dense in $(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$.

Theorem 4.13.

$(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$ is a Banach space.

Proof. We only have to prove completeness. Let $(x_n)_n$ be a Cauchy sequence in $(L^1(\mathcal{B}(\mathcal{H})), \|\|_1)$. Since $\|\cdot\| \leq \|\cdot\|_1$, (x_n) is a Cauchy sequence in $(\mathcal{B}(\mathcal{H}), \|\|)$. But $(\mathcal{B}(\mathcal{H}), \|\|)$ is a Banach space, so there exists $x \in \mathcal{B}(\mathcal{H})$ such that $x_n \to x$ in norm-topology. Notice that

$$x^*x - x_n^*x_n = x^*(x - x_n) + (x - x_n)^*x_n.$$

By continuity of the continuous functional calculus, this implies $|x_n| \to |x|$, meaning that $\||x_n| - |x|\| \to 0$. Next we prove that $x \in L^1(\mathcal{B}(\mathcal{H}))$. For any ONB $(e_i)_i$, we have

$$\sum_{i=1}^k \langle |x|e_i, e_i\rangle = \lim_{n\to\infty} \sum_{i=1}^k \langle |x_n|e_i, e_i\rangle \leq \lim_{n\to\infty} \operatorname{Tr}|x_n| = \lim_{n\to\infty} \|x_n\|_1 < \infty.$$

Here, we used the fact that $\|x_n - x_k\|_1 \geq \|x_n\|_1 - \|x_k\|_1$, so the sequence $(\|x_n\|_1)_n$ is Cauchy and therefore has a limit. This proves that $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $\|x\|_1 \leq \lim_{n\to\infty} \|x_n\|_1$. Finally, we have to show that $\|x_n - x\|_1 \to 0$. Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that for every $n > N$, we get $\|x_n - x_N\|_1 < \frac{\varepsilon}{3}$. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a finite dimensional subspace such that

$$\|x_N P_{\mathcal{H}_0^\perp}\|_1,\ \|x P_{\mathcal{H}_0^\perp}\|_1 < \frac{\varepsilon}{3}.$$

Then for every $n > N$, we get that

$$\begin{aligned}
\|x - x_n\|_1 &\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|(x - x_n)P_{\mathcal{H}_0^\perp}\|_1 \\
&\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|xP_{\mathcal{H}_0^\perp} - x_N P_{\mathcal{H}_0^\perp}\|_1 + \|x_N P_{\mathcal{H}_0^\perp} - x_n P_{\mathcal{H}_0^\perp}\|_1 \\
&\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|x_N - x_n\|_1 \|P_{\mathcal{H}_0^\perp}\| \\
&< \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \varepsilon \\
&\leq \|(x - x_n)\| \|P_{\mathcal{H}_0}\|_1 + \varepsilon \xrightarrow[n\to\infty]{} \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows $x_n \xrightarrow[\|\cdot\|_1]{} x$. □

Theorem 4.14.

The map

$$\Psi : \mathcal{B}(\mathcal{H}) \to L^1(\mathcal{B}(\mathcal{H}))^*, \quad a \mapsto (\psi_a : x \mapsto \operatorname{Tr}(ax))$$

is an isometric isomorphism of Banach spaces.

Proof. We notice that Ψ is linear and a contraction because the norms $\|\cdot\|$ and $\|\cdot\|_1$ are comparable. We will first show that Ψ is surjective. Let $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$. Notice that

$$(\alpha, \beta) \mapsto \varphi(\alpha \otimes \overline{\beta})$$

is a bounded sesquilinear form in $\mathcal{H}$. By the introductory course, there exists an $a \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(\alpha \otimes \overline{\beta}) = \langle a\alpha, \beta\rangle = \operatorname{Tr}(a\alpha \otimes \overline{\beta}) = \operatorname{Tr}(a(\alpha \otimes \overline{\beta})) = \psi_a(\alpha \otimes \overline{\beta}).$$

So φ and ψ_a agree on $\mathcal{F}(\mathcal{H})$, so by bounded density $\varphi = \psi_a$. Finally,

$$\|a\| = \sup_{\alpha,\beta\in(\mathcal{H})_1} |\langle a\alpha, \beta\rangle| = \sup_{\alpha,\beta\in(\mathcal{H})_1} |\operatorname{Tr}(a(\alpha \otimes \overline{\beta}))| \leq \|\psi_a\|_1.$$

But since

$$\|\psi_a\|_1 = \sup_{x\in(L^1(\mathcal{B}(\mathcal{H})))_1} |\operatorname{Tr}(ax)| = \sup_{x\in(L^1(\mathcal{B}(\mathcal{H})))_1} \|ax\|_1 \leq \sup_{x\in(L^1(\mathcal{B}(\mathcal{H})))_1} \|a\|\|x\|_1 = \|a\|,$$

we have $\|a\| = \|\psi_a\|_1$ and ψ is isometric. □

Corollary 4.15. *The map*

$$\Phi : L^1(\mathcal{B}(\mathcal{H})) \to \mathcal{K}(\mathcal{H})^*, \quad x \mapsto (\varphi_x : a \mapsto \operatorname{Tr}(ax))$$

is an isometric isomorphism of Banach spaces.

Proof. Same as that of Theorem 4.14. □

Definition 4.16. Let X, Y be Banach spaces. An operator $T \in \mathcal{B}(X, Y)$ is said to be *compact* if $\overline{T((X)_1)}$ is compact. The space of compact operators is $\mathcal{K}(X, Y)$. We also write $\mathcal{K}(X) := \mathcal{K}(X, X)$.

From the introductory course, we know the following statements about compact operators.

Proposition 4.17. *Let $T \in \mathcal{B}(X, Y)$. The following are equivalent.*

(1.) T is compact;

(2.) T maps bounded maps in X into relatively compact maps in Y;

(3.) T maps bounded sequences in X into sequences in Y that have an accumulation point.

If X, Y are Hilbert spaces, then this is also equivalent to the following.

(4.) $T \in \overline{\mathcal{F}(X, Y)}$.

Remark. $\mathcal{K}(\mathcal{H})$ is a closed ideal in $\mathcal{B}(\mathcal{H})$.

Theorem 4.18 (Singular value decomposition)**.**

For $K \in \mathcal{K}(\mathcal{H})$, there exists orthonormal bases $(e_i)_i$ and $(f_j)_j$ for $\mathcal{H}$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ such that

$$Kx = \sum_{n=1}^{\infty} \sigma_n \langle x, e_n \rangle f_n. \tag{1}$$

As a result,

$$|K|x = \sum \sigma_n \langle x, e_n \rangle e_n.$$

Proposition 4.19. *Let X be a Banach space. Then the following statements are equivalent:*

(1.) $\mathrm{id} : X \to X$ is compact;

(2.) $(X)_1$ is compact;

(3.) $\dim X < \infty$.

The equivalence of the last two items is also known as the Riesz lemma.

Theorem 4.20.

(1.) $L^1(\mathcal{B}(\mathcal{H})) \subseteq K(\mathcal{H})$.

(2.) $K \in \mathcal{H}$ is a $L^1(\mathcal{B}(\mathcal{H}))$ iff $\sum_{k=1}^{\infty} \sigma_n < \infty$.

Proof. (1.) If $x \in L^1(\mathcal{B}(\mathcal{H}))$ then there exists $(x_n)_n$ in $\mathcal{F}(\mathcal{H})$ such that $\|x_n - x\|_1 \to 0$. Since $\|\cdot\| \leq \|\cdot\|_1$, we get $\|x_n - x\| \to 0$ and $x \in \overline{(\mathcal{F}, \|\cdot\|)} = \mathcal{K}(\mathcal{H})$.

(2.) This follows from $\operatorname{Tr}|K| = \sum \sigma_n$ for K as in (1). $\square$

4.3 Hilbert-Schmidt operators

Definition 4.21. An element $x \in \mathcal{B}(\mathcal{H})$ is a *Hilbert-Schmidt operator* if

$$|x|^2 = x^*x \in L^1(\mathcal{B}(\mathcal{H})).$$

The set of all such elements is denoted by $L^2(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

Proposition 4.22. *(1.) $L^2(\mathcal{B}(\mathcal{H})) \lhd \mathcal{B}(\mathcal{H})$ and is closed under $*$.*

(2.) If $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$ and $\operatorname{Tr}(xy) = \operatorname{Tr}(yx)$.

Remark. Beware: there exist $a, b \in \mathcal{B}(\mathcal{H})$ such that $ab \in L^1(\mathcal{B}(\mathcal{H}))$ and $ba \notin L^1(\mathcal{B}(\mathcal{H}))$. However, if $ab, ba \in L^1(\mathcal{B}(\mathcal{H}))$, then $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$.

Proof. For $\alpha \in \mathbb{C}$ and $x, y \in \mathcal{B}(\mathcal{H})$, we have $|\alpha x|^2 = |\alpha|^2 |x|^2$. Similarly, $|x+y|^2 \leq |x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$, so $L^2(\mathcal{B}(\mathcal{H}))$ is a complex vector space. Since $|ax|^2 \leq \|a\|^2 \cdot |x|^2$, we have $L^2(\mathcal{B}(\mathcal{H}))$ is a left ideal of $\mathcal{B}(\mathcal{H})$. From

$$\operatorname{Tr}|x|^2 = \operatorname{Tr}(x^*x) = \operatorname{Tr}(xx^*) = \operatorname{Tr}|x^*|^2,$$

we deduce that $L^2(\mathcal{B}(\mathcal{H}))$ is closed under involution. If $x \in L^2(\mathcal{B}(\mathcal{H}))$ and $b \in \mathcal{B}(\mathcal{H})$, then $x^* \in L^2(\mathcal{B}(\mathcal{H}))$, which implies $b^*x^* \in L^2(\mathcal{B}(\mathcal{H}))$ and finally $xb = (b^*x^*)^* \in L^2(\mathcal{B}(\mathcal{H}))$, so $L^2(\mathcal{B}(\mathcal{H})) \lhd \mathcal{B}(\mathcal{H})$. Next, we use the polarization identity

$$4y^*x = \sum_{k=0}^{3} i^k |x + i^k y|^2.$$

If $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then this shows $y^*x \in L^1(\mathcal{B}(\mathcal{H}))$ and

$$\begin{aligned} 4\operatorname{Tr}(y^*x) &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x+i^k y)^*(x+i^k y)) \\ &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x+i^k y)(x+i^k y)^*) \\ &= 4\operatorname{Tr}(xy^*). \end{aligned}$$

$\square$

On $L^2(\mathcal{B}(\mathcal{H}))$ we have the sesquilinear form $\langle x, y\rangle_2 := \operatorname{Tr}(y^*x)$. It is well-defined and positive definite, so it is a scalar product. The induced norm is denoted by $\|\cdot\|_2$. For every $y \in L^2(\mathcal{B}(\mathcal{H}))$, we have

$$\|y\| = \|y^*y\|^{\frac{1}{2}} \leq \|y^*y\|_1^{\frac{1}{2}} = \|y\|_2.$$

Similarly, we have

$$\|axb\|_2 = \|a\| \cdot \|x\|_2 \cdot \|b\|$$

for all $x \in L^2(\mathcal{B}(\mathcal{H}))$ and $a, b \in \mathcal{B}(\mathcal{H})$. As before, $\mathcal{F}(\mathcal{H})$ are dense in $L^2(\mathcal{B}(\mathcal{H}))$ with respect to $\|\cdot\|_2$ and $L^2(\mathcal{B}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$. Using singular values $(\sigma_n)_n$ of a compact $K \in \mathcal{K}(\mathcal{H})$, we have $K \in L^2(\mathcal{B}(\mathcal{H}))$ iff $\sum_{k=0}^{\infty} \sigma_j^2 < \infty$. For every $x \in L^1(\mathcal{B}(\mathcal{H}))$, we have

$$\|x\|_2 = \sup_{y \in L^2(\mathcal{B}(\mathcal{H})),\ \|y\|_2=1} |\operatorname{Tr}(y^* x)| \leq \sup_{y \in L^2(\mathcal{B}(\mathcal{H})),\ \|y\|_2=1} \|y\| \cdot \|x\|_1 \leq \|x\|_1.$$

As a result, $(L^2(\mathcal{B}(\mathcal{H})), \langle\cdot\rangle_2)$ is a Hilbert space.

Theorem 4.23 (Hölder's inequality)**.**

For all $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ we have

$$\|xy\|_1 \leq \|x\|_2 \|y\|_2.$$

Proof. Let $xy = v|xy|$ be the polar decomposition of xy. Then

$$\begin{aligned} \|xy\|_1 &= \operatorname{Tr}|xy| = \operatorname{Tr}(v^* xy) \\ &= |\langle y, x^* v\rangle_2| \leq \|x^* v\|_2 \|y\|_2 \\ &\leq \|x^*\|_2 \|v\| \|y\|_2 \leq \|x\|_2 \cdot \|y\|_2. \end{aligned}$$

□

4.4 Hilbert-Schmidt integral operators

Let (X, μ) be a σ-finite measure space. This means that X is a countable union of finite-measure sets:

$$X = \bigcup_{j=1}^{\infty} A_j, \quad \mu(A_j) < \infty.$$

For $K \in L^2(X \times X, \mu \times \mu)$, we can define a Hilbert-Schmidt integral operator with kernel K:

$$T_K : L^2(X, \mu) \to L^2(X, \mu), \quad f \mapsto \left(y \mapsto \int_X K(x, y) f(y)\, d\mu(x) \right).$$

Suppose $(\varphi_\alpha)_\alpha$ is an ONB for $L^2(K, \mu)$. By Fubini, $\left(\overline{\varphi_a(x)}\varphi_\beta(y)\right)_{\alpha,\beta}$ is an orthonormal basis for $L^2(X \times X, \mu \times \mu)$. Since $K \in L^2(X \times X, \mu \times \mu)$, there exist $c_{ij} \in \mathbb{C}$ such that

$$K(x, y) = \sum_{i,j} c_{ij} \overline{\varphi_i(x)} \varphi_j(y), \quad \|K\|_{L^2(X \times X)}^2 = \sum |c_{ij}|^2 < \infty.$$

We show that T_K is well-defined: for $f \in L^2(X, \mu)$, we have $T_K f \in L^2(X, \mu)$. Indeed,

$$T_k f(y) = \sum_{i,j} c_{ij} \langle f, \varphi_i \rangle \varphi_j(y),$$

which implies

$$\begin{aligned}
\|T_K f\|^2_{L^2(X)} &\le \sum_{i,j} |c_{ij}|^2 |\langle f, \varphi_j\rangle|^2 \|\varphi_j\|^2_{L^2(X)} \\
&\le \|f\|^2_{L^2} \sum_{i,j} |c_{ij}|^2 \|\varphi\|^2_{L^2} \|\varphi_j\|^2_{L^2} \\
&= \|f\|^2_{L^2} \sum |c_{ij}|^2 \\
&= \|f\|^2_{L^2} \|K\|^2_{L^2(X\times X)}
\end{aligned}$$

and finally $\|T_K\| \le \|K\|_{L^2}$. We claim that $T_K^* : L^2(X,\mu) \to L^2(X,\mu)$ is the integral operator with kernel

$$K^*(y,x) := \overline{K(x,y)}.$$

Indeed,

$$\begin{aligned}
\langle T_K f, g\rangle &= \int_Y \left(\int_X K(x,y) f(x)\, d\mu(x)\right) \cdot \overline{g(y)}\, d\mu(y) \\
&= \int_X f(x) \cdot \overline{\left(\int_Y \overline{K(x,y)} g(y)\, d\mu(y)\right)}\, d\mu(x) \\
&= \langle f, T_{K^*} g\rangle.
\end{aligned}$$

Remark (Fubini's theorem). If $(X,\mu), (Y,\lambda)$ are σ-finite measure spaces and $\int_{X\times Y} |f|\, d(\mu \times \lambda)(x,y) < \infty$, then

$$\int_{X\times Y} f\, d(\mu\times\lambda)(x,y) = \int_Y \left(\int_X f\, d\mu(x)\right) d\lambda(y) = \int_X \left(\int_Y f\, d\lambda(y)\right) d\mu(x).$$

Theorem 4.24.

(1.) For $K \in L^2(X \times X, \mu\times\mu)$ we have $T_K \in L^2(\mathcal{B}(L^2(X,\mu)))$.

(2.) The mapping $\Phi : K \mapsto T_K$ is a unitary $L^2(X\times X, \mu\times\mu) \to L^2(\mathcal{B}(L^2(X,\mu)))$.

Proof. (1.) We will prove that $\|T_K\|_2 = \|K\|_{L^2}$. We want to approximate T_K with finite rank operators, so we first approximate K:

$$K(x,y) = \sum_{i,j=1}^{\infty} c_{ij} \overline{\varphi_i(x)} \varphi_j(x)$$

for an orthonormal basis $(\varphi_\alpha)_\alpha$ for $L^2(X,\mu)$. For $N \in \mathbb{N}$ let $K_N(x,y) = \sum_{i,j}^N c_{ij}\overline{\varphi_i(x)}\varphi_j(x)$. Then

$$T_{K_N} f = \sum_{i,j=1}^{N} c_{ij} \langle f, \varphi_i\rangle \varphi_j \in \mathcal{F}(L^2(X,\mu)).$$

By the above inequality,

$$\|T_K - T_{K_N}\| \le \|K - K_N\|_{L^2} \to 0,$$

so $T_K \in \overline{(\mathcal{F}, \|\cdot\|)} = \mathcal{K}(\mathcal{H})$. Then

$$\|T_K\|_2^2 = \sum_i \|T_K\varphi_i\|_{L^2}^2 = \sum_{i,j,k} \|c_{jk}\varphi_j(x)\delta_{ik}\|^2 = \sum |c_{ij}|^2 = \|K\|_{L^2}^2.$$

(2.) It remains to prove surjectivity. Since Φ is isometric, $\operatorname{im}\Phi$ is closed. So it suffices to show that $\operatorname{im}\Phi$ is dense. In particular, we will show that $\operatorname{im}\Phi \supseteq \mathcal{F}(L^2(X,\mu))$. Let $A \in \mathcal{F}(L^2(X,\mu))$, so $\operatorname{rank} A < \infty$. Let $(\psi_1, \dots, \psi_m)$ be an orthonormal basis for $\operatorname{im} A$. Then $A\varphi = c_1(\varphi)\psi_1 + \dots + c_m(\varphi)\psi_m$ for some bounded linear functionals c_j on $L^2(X,\mu)$. By Riesz, there exist $\mu_j \in L^2(X,\mu)$ such that $c_j(\varphi) = \langle \varphi, \mu_j\rangle$. Hence

$$A\varphi(x) = \int_X \left(\sum_{j=1}^m \psi_j(x)\cdot\overline{\mu_j(y)}\cdot\varphi(y)\right) d\mu(y) = T_{\sum_{i=1}^m \psi_j(x)\overline{\mu_j(y)}} \in \operatorname{im}\Phi. \qquad \square$$

4.5 Hilbert-Schmidt operators $\mathcal{H} \to \mathcal{K}$ and tensor products

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. We associate to A the map

$$\widetilde{A} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}, \mathcal{H} \oplus \mathcal{K}), \quad \alpha \oplus \beta \mapsto 0 \oplus A\alpha,$$

or in matrix form,

$$\widetilde{A} = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}.$$

We denote the set of Hilbert-Schmidt operators $\mathcal{H} \to \mathcal{K}$ as

$$HS(\mathcal{H}, \mathcal{K}) = \{A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \mid \widetilde{A} \in L^2(\mathcal{B}(\mathcal{H} \oplus \mathcal{K}))\} = \{A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \mid A^*A \in L^1(\mathcal{B}(\mathcal{H}))\}.$$

With the usual scalar product $\langle A, B\rangle_2 = \operatorname{Tr}(B^*A)$, this becomes a Hilbert space.

Remark. By Riesz's representation theorem, every functional in $\mathcal{H}^*$ is of the form $\overline{\alpha} : x \mapsto \langle x, \alpha\rangle$, where $\alpha \in \mathcal{H}$. This means that we can introduce a scalar product on $\mathcal{H}^*$ by $\langle \overline{\alpha}, \overline{\beta}\rangle_{\mathcal{H}^*} := \langle \beta, \alpha\rangle_{\mathcal{H}}$. This scalar product induces the usual operator norm on $\mathcal{H}^*$, so it makes $\mathcal{H}^*$ into a Hilbert space.

Example 4.25. *The dual $\mathcal{H}^*$ is isomorphic as a Hilbert space to $HS(\mathcal{H}, \mathbb{C})$. To prove* *this, it's enough to compare the scalar products. For any $\overline{\alpha}, \overline{\beta} \in \mathcal{H}^*$, we have*

$$\begin{aligned}\langle \overline{\alpha}, \overline{\beta} \rangle &= \operatorname{Tr}(\overline{\beta}^* \overline{\alpha}) \\ &= \sum_{i \in I} \langle \overline{\alpha} e_i, \overline{\beta} e_i \rangle \\ &= \sum_{i \in I} \langle e_i, \alpha \rangle \overline{\langle e_i, \beta \rangle} \\ &= \sum_{i \in I} \langle e_i, \alpha \rangle \langle \beta, e_i \rangle \\ &= \langle \beta, \alpha \rangle = \langle \overline{\alpha}, \overline{\beta} \rangle_{\mathcal{H}^*}.\end{aligned}$$

Definition 4.26. We define the *tensor product of Hilbert spaces* as $\mathcal{H} \overline{\otimes} \mathcal{K} := HS(\mathcal{K}^*, \mathcal{H})$.

Remark. For any $\alpha \in \mathcal{H}$ and $\beta \in \mathcal{K}$, we define the elementary tensors as

$$\alpha \otimes \beta : \mathcal{H}^* \to \mathcal{K}, \quad f \mapsto f(\alpha)\beta.$$

The span of these operators is the usual algebraic tensor product of vector spaces $\mathcal{H} \otimes \mathcal{K}$ and consists of all the finite-rank operators in $\mathcal{H} \overline{\otimes} \mathcal{K}$.

4.6 Locally convex topologies on $\mathcal{B}(\mathcal{H})$

If $\mathcal{H}$ is a Hilbert space, then $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ is a Banach algebra with its norm topology.

Definition 4.27. (1.) The *weak operator topology* (WOT) is given by the seminorms

$$T \mapsto |\langle T\alpha, \beta \rangle|, \quad \forall \alpha, \beta \in \mathcal{H}.$$

(2.) The *strong operator topology* (SOT) is given by the seminorms

$$T \mapsto \|T\alpha\|, \quad \forall \alpha \in \mathcal{H}.$$

These topologies are comparable: WOT $\subseteq$ SOT $\subseteq$ norm topology.

- Norm topology has the subbasis

  $$\{S \in \mathcal{B}(\mathcal{H}) \mid \|S - T\| < \varepsilon\}$$

  for $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. The net T_i converges to T iff $\|T_i - T\|$ converges to 0.

- WOT topology has the subbasis

  $$\{S \in \mathcal{B}(\mathcal{H}) \mid \langle (S - T)\alpha, \beta \rangle < \varepsilon\}$$

  for $\alpha, \beta \in \mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. The net T_i converges to T iff $\langle T_i\alpha, \beta \rangle$ converges to $\langle T\alpha, \beta \rangle$ for all α, β.

- SOT topology has the subbasis

$$\{S \in \mathcal{B}(\mathcal{H}) \mid \|(S-T)\alpha\| < \varepsilon\}$$

for $\alpha \in \mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. The net T_i converges to T iff $\|(T_i - T)\alpha\|$ converges to 0 for all α.

Example 4.28. *Let $\mathcal{H} = \ell^2(\mathbb{N})$ and denote $T_n = \frac{1}{n} \cdot \mathrm{id}$. Then $T_n \to 0$ in the norm topology. Now if we introduce the operator*

$$S(x_1, x_2, \dots) = (0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots),$$

then $S_n \to 0$ in SOT, but not in norm topology, since $\|S_n\| = 1$. Lastly, we define

$$W_n(x_1, x_2, \dots) = (0, 0, \dots, x_1, x_2, \dots).$$

We get that $W_n \to 0$ in WOT, but not in SOT or norm topology.

Example 4.29. *Let $(y_n)_n$ be a countable dense subset of $\mathcal{H} = \ell^2$. Consider the following two metrics on $(\mathcal{B}(\mathcal{H}))_1$:*

$$d_S(A, B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|(A-B)y_n\|, \quad d_W(A, B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left|\langle (A-B)y_n, y_n\rangle\right|.$$

Then d_S induces SOT and d_W induces WOT on $(\mathcal{B}(\mathcal{H}))_1$.

Example 4.30. *The multiplication*

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad (A, B) \mapsto A \cdot B$$

is not jointly continuous with regards to SOT or WOT. Indeed, if $S : \ell^2 \to \ell^2$ is the right shift (and S^* the left shift), then $S^n \to 0$ and $(S^*)^n \to 0$ in SOT and WOT, but $(S^*)^n S^n = I$. However, multiplication is WOT- and SOT-continuous in each factor separately. Suppose that $(x_\alpha)_\alpha \to x$ in WOT and $y \in \mathcal{B}(\mathcal{H})$. Then for each $v, w \in \mathcal{H}$, we have

$$|\langle x_\alpha y v - x y v, w\rangle| \to 0,$$

since $x_\alpha \to x$ in WOT. Similarly,

$$|\langle y x_\alpha v - y x v, w\rangle| = |\langle x_\alpha v - x v, y^* w\rangle| \to 0,$$

which implies $x_\alpha y \to xy$ and $y x_\alpha \to yx$ in WOT. Similarly, if $(x_\alpha)_\alpha \to x$ in SOT and $y \in \mathcal{B}(\mathcal{H})$, then for each $v \in \mathcal{H}$ we have

$$\|(x_\alpha - x)yv\| \to 0, \quad \|y(x_\alpha - x)v\| \to 0,$$

so $x_\alpha y \to xy$ and $y x_\alpha \to yx$ in SOT.

Example 4.31. *The adjoint is isometric in the norm topology. It is also continuous in WOT:*

$$|\langle x^*v - y^*v, w\rangle| < \varepsilon \Leftrightarrow |\langle xw - yw, v\rangle| < \varepsilon.$$

However, it is not continuous with respect to SOT. If $(e_n)_n$ *is an ONB for* $\mathcal{H}$*, consider* $e_1 \otimes \overline{e_n}$*. Then for every* $x \in \mathcal{H}$*, we have*

$$\|(e_1 \otimes \overline{e_n})x\| = |\langle x, e_n\rangle| \xrightarrow[n\to\infty]{} 0,$$

so $e_1 \otimes \overline{e_n} \to 0$ *in SOT. However,*

$$\|(e_1 \otimes \overline{e_n})^*x\| = \|(e_n \otimes \overline{e_1})x\| = |\langle x, e_1\rangle|$$

does not go to 0 *for all* $x \in \mathcal{H}$*, which proves our statement.*

Remark. If $T : X \to Y$ is continuous, then T remains continuous if X is given a finer topology or Y is given a coarser topology. But if both topologies are made coarser or both finer, nothing can be said in general. In particular, if $T : X \to X$ is continuous with respect to a given topology on X in both domain and codomain, you cannot generally conclude anything about continuity of T when X is given a finer or coarser topology on both domain and codomain. The previous example illustrates this.

Lemma 4.32. *Let* $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ *be linear. The following are equivalent.*

(1.) There exist $v_1, \dots, v_n \in \mathcal{H}$ *and* $w_1, \dots, w_n \in \mathcal{H}$ *such that*

$$\varphi(T) = \sum_{i=1}^{n} \langle Tv_i, w_i\rangle.$$

(2.) φ *is WOT-continuous.*

(3.) φ *is SOT-continuous.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. Let us prove (3) $\Rightarrow$ (1). By Proposition 1.30, there exists a $K > 0$ and $v_1, \dots, v_n \in \mathcal{H}$ such that

$$|\varphi(T)|^2 \le K \cdot \sum_{i=1}^{n} \|Tv_i\|^2.$$

Define

$$\mathcal{H}_0 := \overline{\left\{ \bigoplus_{i=1}^{n} Tv_i \mid T \in \mathcal{B}(\mathcal{H}) \right\}} \le \mathcal{H}^{\bigoplus n}.$$

The map

$$\mathcal{H}_0 \ni \bigoplus_{i=1}^{n} Tv_i \mapsto \varphi(T) \in \mathbb{C}$$

is a well-defined and bounded linear functional, which by continuity extends to $\mathcal{H}_0 \to \mathbb{C}$. By

Riesz, there exist $w_1, \ldots, w_n \in \mathcal{H}$ such that

$$\varphi(T) = \sum_{i=1}^{n} \langle Tv_i, w_i \rangle.$$

Recall that $v \otimes \overline{w} \in \mathcal{F}(\mathcal{H})$ and $\mathrm{Tr}(v \otimes \overline{w}) = \langle v, w \rangle$, so

$$\mathrm{Tr}(T(v \otimes \overline{w})) = \langle Tv, w \rangle.$$

The previous identity is really

$$\varphi(T) = \sum_{i=1}^{n} \mathrm{Tr}(T(v \otimes \overline{w})) = \mathrm{Tr}(T \cdot \sum_{i=1}^{n} v_i \otimes \overline{w}_i).$$

This means that $\varphi(T) = \mathrm{Tr}(T \cdot A)$ for $A \in \mathcal{F}(\mathcal{H})$. ◻

Corollary 4.33. *If $K \subseteq \mathcal{B}(\mathcal{H})$ is convex, then*

$$\overline{(K, WOT)} = \overline{(K, SOT)}.$$

Proof. Consider $(\mathcal{B}(\mathcal{H}), \mathrm{WOT})$. This is a LCS, so $\overline{(K, \mathrm{WOT})}^w = \overline{(K, \mathrm{WOT})}$ by Theorem 1.28. Similarly, we have that $(\mathcal{B}(\mathcal{H}), \mathrm{SOT})$ is a LCS and $\overline{(K, \mathrm{SOT})}^w = \overline{(K, \mathrm{SOT})}$. Now

$$\begin{aligned} x \in \overline{(K, \mathrm{WOT})}^w &\Leftrightarrow \exists \text{ a net } (x_\alpha)_\alpha \subseteq (K, \mathrm{WOT}), \text{ such that } x_\alpha \to x \text{ weakly} \\ &\Leftrightarrow f(x_\alpha) \to f(x) \text{ for all WOT-continuous functionals } f : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \\ &\Leftrightarrow f(x_\alpha) \to f(x) \text{ for all SOT-continuous functionals } f : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \\ &\Leftrightarrow \exists \text{ a net } (x_\alpha)_\alpha \subseteq (K, \mathrm{SOT}), \text{ such that } x_\alpha \to x \text{ weakly} \\ &\Leftrightarrow x \in \overline{(K, \mathrm{SOT})}^w. \end{aligned}$$

Therefore, $\overline{(K, \mathrm{WOT})}^w = \overline{(K, \mathrm{SOT})}^w$ and we are done. ◻

Definition 4.34. The *σ-weak operator topology* (σ-WOT or ultra-weak) is the topology in $\mathcal{B}(\mathcal{H})$ given by the seminorms

$$x \mapsto \left| \sum_{i=1}^{\infty} \langle x\alpha_i, \alpha_i \rangle \right|$$

for $\alpha_i \in \mathcal{H}$ with $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$. A subbasis of open sets is thus

$$\left\{ x \in \mathcal{B}(\mathcal{H}) \;\middle|\; \left| \sum_{i=1}^{\infty} \langle (x - x_0)\alpha_i, \alpha_i \rangle \right| < \varepsilon \right\}$$

for $\alpha_i \in \mathcal{H}$ with $\varepsilon > 0$, $x_0 \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$.

Definition 4.35. The *σ-strong operator topology* (σ-SOT or ultra-strong) is the topology in $\mathcal{B}(\mathcal{H})$ given by the seminorms

$$x \mapsto \left(\sum_{i=1}^{\infty} \|x\alpha_i\|^2\right)^{\frac{1}{2}}$$

for $\alpha_i \in \mathcal{H}$ with $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$. A subbasis of open sets is thus

$$\left\{ x \in \mathcal{B}(\mathcal{H}) \;\middle|\; \left(\sum_{i=1}^{\infty} \|(x - x_0)\alpha_i\|^2\right)^{\frac{1}{2}} < \varepsilon \right\}$$

for $\alpha_i \in \mathcal{H}$ with $\varepsilon > 0$, $x_0 \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$.

Remark. σ-WOT can also be given by seminorms

$$x \mapsto |\operatorname{Tr}(xa)|$$

for $a \in L^1(\mathcal{B}(\mathcal{H}))$ positive. Let $(f_i)_i$ be an ONB for $\mathcal{H}$ and define

$$b : \mathcal{H} \to \mathcal{H}, \quad f_i \mapsto \alpha_i.$$

Since $\sum_i^{\infty} \|\alpha_i\|^2 < \infty$, we can conclude $b \in L^2(\mathcal{B}(\mathcal{H}))$. Then:

$$\begin{aligned} \sum_i \langle x\alpha_i, \alpha_i\rangle &= \sum_i \langle xbf_i, bf_i\rangle \\ &= \sum_i \langle b^* xbf_i, f_i\rangle \\ &= \operatorname{Tr}(b^* xb) \\ &= \operatorname{Tr}(xbb^*), \end{aligned}$$

where $a := bb^* \in L^1(\mathcal{B}(\mathcal{H}))$. Since $\mathcal{B}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))^*$, the σ-WOT is just the weak-$*$ topology (with respect to this pairing).

Remark. The map

$$\operatorname{id} \otimes 1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2), \quad x \mapsto x \otimes 1$$

is an isometric *-isomorphism of C^*-algebras. It is neither SOT- nor WOT-continuous. Despite that, σ-WOT on $\mathcal{B}(\mathcal{H})$ is induced by WOT on $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2)$ and the σ-SOT on $\mathcal{B}(\mathcal{H})$ is induced by SOT on $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2)$. Indeed, if $(e_i)_{i \in \mathbb{N}}$ is an ONB for ℓ^2, define $\alpha := \sum_{i=1}^{\infty} \alpha_i \otimes e_i \in \mathcal{H} \overline{\otimes} \ell^2$. Then

$$\sum_{i \in \mathbb{N}} \langle x\alpha_i, \alpha_i\rangle_{\mathcal{H}} = \langle (\operatorname{id} \otimes 1)(x)\alpha, \alpha\rangle_{\mathcal{H} \overline{\otimes} \ell^2}$$

and similarly

$$\left(\sum_{i \in \mathbb{N}} \|x\alpha_i\|_{\mathcal{H}}^2\right)^{\frac{1}{2}} = \|(\operatorname{id} \otimes 1)(x)\alpha\|_{\mathcal{H} \overline{\otimes} \ell^2}$$

Lemma 4.36. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional operator. Then the following are equivalent.*
(1.) $\exists a \in L^1(\mathcal{B}(\mathcal{H}))$ such that $\varphi(x) = \mathrm{Tr}(ax),\ \forall x \in \mathcal{B}(\mathcal{H})$;
(2.) φ is σ-WOT continuous;
(3.) φ is σ-SOT continuous.

Proof. As previously, the implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious. Let us prove (3) $\Rightarrow$ (1). Assume φ is σ-SOT continuous. By identifying $\mathcal{B}(\mathcal{H})$ via $\mathrm{id} \otimes 1$ with a subspace in $\mathcal{B}(\mathcal{H} \otimes \ell^2)$, φ is SOT-continuous on this subspace. By Hahn-Banach, φ extends to a SOT-continuous linear functional on $\mathcal{B}(\mathcal{H} \otimes \ell^2)$. By the previous lemma, $\exists \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathcal{H} \overline{\otimes} \ell^2$.

$$\varphi(x) = \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x)\alpha_i, \beta_i \rangle.$$

With

$$\alpha_i \sum_{j=1}^{\infty} \alpha_{ij} \otimes e_j, \quad \sum_j \|\alpha_{ij}\|^2 < \infty$$

and

$$\beta_i \sum_{j=1}^{\infty} \beta_{ij} \otimes e_j, \quad \sum_j \|\beta_{ij}\|^2 < \infty.$$

Then

$$\begin{aligned}
\varphi(x) &= \sum_{i=1}^{n} \langle (x \otimes 1) \sum_{j=1}^{\infty} \alpha_{ij} \otimes e_j, \sum_{k=1}^{\infty} \beta_{ik} \otimes e_k \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle x\alpha_{ij}, \beta_{ik} \rangle \langle e_j, e_k \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{\infty} \langle x\alpha_{ij}, \beta_{ij} \rangle.
\end{aligned}$$

Define

$$A_i : \mathcal{H} \to \mathcal{H}, \quad A_i f_k = \alpha_{ik}$$

and

$$B_i : \mathcal{H} \to \mathcal{H}, \quad B_i f_k = \beta_{ik}$$

for an orthonormal basis $(f_k)_{k \in \mathbb{N}}$. By assumption, $A_i, B_i \in L^2(\mathcal{B}(\mathcal{H}))$. As before, this gives $\varphi(x) = \sum_i \mathrm{Tr}(B_i^* x A_i) = \mathrm{Tr}(x A_i B_i^*)$. $\square$

Corollary 4.37. *The unit disk $(\mathcal{B}(\mathcal{H}))_1$ is compact with respect to the σ-WOT topology.*

Proof. σ-WOT on $\mathcal{B}(\mathcal{H})$ is the weak-$*$ topology from $L^1(\mathcal{B}(\mathcal{H}))^* = \mathcal{B}(\mathcal{H})$. The statement now follows from Banach-Alaoglu. $\square$

Corollary 4.38. *WOT and σ-WOT topologies agree on bounded subsets $B \subseteq \mathcal{B}(\mathcal{H})$.*

Proof. W.l.o.g. $B = M \cdot (\mathcal{B}(\mathcal{H}))_1$ for some $M > 0$. Then the identity $(B, \sigma\text{-WOT}) \to (B, \text{WOT})$ is a continuous map from a Hausdorff compact space (previous corollary) to a Hausdorff space. Therefore the identity map is a closed continuous bijection, so it's a homeomorphism. $\square$

Definition 4.39. Let A be a vector space and $B \subseteq \mathcal{L}(A, \mathbb{C})$ a set of some of its linear functionals. Then we define $\sigma(A, B)$ as the weakest topology in A such that functionals in B are continuous.

Remark. σ-WOT topology is $\sigma(\mathcal{B}(\mathcal{H}), L^1(\mathcal{B}(\mathcal{H})))$.

Remark. Let us define the following topologies on $\mathcal{B}(\mathcal{H})$.

(1.) Weak Banach topology is $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^*)$.

(2.) Ultrastrong-$*$ topology is the weakest topology stronger than σ-SOT such that $*$ is continuous.

(3.) Strong-$*$ topology is generated by seminorms $x \mapsto \|x\alpha\|$ and $x \mapsto \|x^*\alpha\|$ for $\alpha \in \mathcal{H}$.

In the end, we get the following diagram that demonstrates which topologies are comparable.

$$
\begin{tikzcd}
& \text{norm} \arrow[dl] \arrow[dr] & \\
\text{weak Banach} \arrow[dd] & & \text{ultrastrong-}* \arrow[dl] \arrow[d] \\
& \sigma\text{-SOT} \arrow[dl] \arrow[dr] & \text{strong-}* \arrow[d] \\
\sigma\text{-WOT} \arrow[dr] & & \text{SOT} \arrow[dl] \\
& \text{WOT} &
\end{tikzcd}
$$

5 von Neumann algebras

5.1 Bicommutant theorem

Definition 5.1. A *von Neumann algebra* (on Hilbert space $\mathcal{H}$) is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is WOT-closed. Equivalently, it is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is SOT-closed.

Remark. To shorten the notation, we will abbreviate "von Neumann algebra" to vNa.

If $A \subseteq \mathcal{B}(\mathcal{H})$, then $W^*(A)$ denotes the vNa generated by A, or the smallest vNa in $\mathcal{B}(\mathcal{H})$ that contains A. This is well defined, since

$$W^*(A) = \bigcap\{W \mid A \subseteq W,\ W \subseteq \mathcal{B}(\mathcal{H}) \text{ is vNa}\}.$$

Lemma 5.2. *If $A \subseteq \mathcal{B}(\mathcal{H})$ is a vNa, then $(A)_1$ is WOT-compact.*

Proof. By Corollary 4.37, $(\mathcal{B}(\mathcal{H}))_1$ is compact in σ-WOT topology. By Corollary 4.38, the WOT and σ-WOT topologies on $(\mathcal{B}(\mathcal{H}))_1$ are equivalent, so $(\mathcal{B}(\mathcal{H}))_1$ is also compact in WOT topology. Next, we prove that $(A)_1$ is WOT-closed in $(\mathcal{B}(\mathcal{H}))_1$. Suppose that the net $(x_i)_i$ in (A_1) converges to some x. Since A is WOT-closed, we must have $x \in A$. Assume that $x \notin (A)_1$, so $\|x\| > 1$. Since $\|x\| = \sup_{\alpha,\beta\in(\mathcal{H})_1} |\langle x\alpha, \beta\rangle|$, there must exist some $\alpha, \beta \in (\mathcal{H})_1$ such that $|\langle x\alpha, \beta\rangle| > 1$. However, for every x_i we have $|\langle x_i\alpha, \beta\rangle| \leq \|x_i\| \cdot |\alpha| \cdot |\beta| \leq 1$, contradicting the fact that $\langle x_i\alpha, \beta\rangle \to \langle x\alpha, \beta\rangle$. Therefore, $x \in (A)_1$ and (A_1) is WOT-closed in $(\mathcal{B}(\mathcal{H}))_1$, so it is compact. $\square$

Corollary 5.3. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ vNa. Then $(A)_1$ and A_{sa} are SOT-closed and WOT-closed.*

Proof. We already know that the adjoint is continuous in WOT, so A_{sa} is closed in WOT. Since A_{sa} is convex, it is also SOT-closed. The same exact argument applies for $(A)_1$. $\square$

Definition 5.4. The *commutant* of a set $B \subseteq \mathcal{B}(\mathcal{H})$ is

$$B' := \{T \in \mathcal{B}(\mathcal{H}) \mid \forall S \in B:\ ST = TS\}$$

and its *bicommutant* is $B'' := (B')'$.

Remark. By definition, $B'' \supseteq B$.

Theorem 5.5.

Suppose $A \subseteq \mathcal{B}(\mathcal{H})$ is closed under $*$. Then A' is vNa.

Proof. Obviously, A' is also a subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed under $*$. We prove that it is WOT-closed. Let $(x_\alpha)_\alpha$ be a net in A' that WOT-converges to $x \in \mathcal{B}(\mathcal{H})$. Pick any $a \in A$

and $\varphi, \mu \in \mathcal{H}$. Then

$$\begin{aligned}
\langle [x, a]\varphi, \mu \rangle &= \langle (xa - ax)\varphi, \mu \rangle \\
&= \langle xa\varphi, \mu \rangle - \langle ax\varphi, \mu \rangle \\
&= \langle xa\varphi, \mu \rangle - \langle x\varphi, a^*\mu \rangle \\
&= \lim_\alpha \langle x_\alpha a\varphi, \mu \rangle - \langle x_\alpha \varphi, a^*\mu \rangle \\
&= \lim_\alpha \langle (x_\alpha a - a x_\alpha)\varphi, \mu \rangle \\
&= \lim_\alpha \langle [x_\alpha, a]\varphi, \mu \rangle = 0,
\end{aligned}$$

so $x \in A'$ and we are done. □

Corollary 5.6. *Every vNa is unital.*

Example 5.7. *For an infinitely-dimensional Hilbert space $\mathcal{H}$, the set of all compact operators $\mathcal{K}(\mathcal{H})$ is not a vNa, since it doesn't include the identity (by the Riesz lemma). In particular, $\mathcal{K}(\mathcal{H})$ is neither SOT- nor WOT-closed.*

Remark. As we will see later, the finite-rank projections on a Hilbert space converge strongly to identity.

Corollary 5.8. *Suppose that $A \subseteq \mathcal{B}(\mathcal{H})$ is a maximal commutative subalgebra. If A is closed under $*$, then it is a vNa.*

Proof. Since A is commutative, $A' \supseteq A$. Take $b \in A' \subseteq A$ and consider the subalgebra, generated by A and b. This is an abelian algebra, so by maximality we have we have $b \in A$ and $A = A'$. Then by Theorem 5.5, A is a vNa. □

Lemma 5.9. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra. For any $\mu \in \mathcal{H}$ and $x \in A''$ there exists a net $(x_\alpha)_\alpha$ in A such that $\lim_\alpha \|(x_\alpha - x)\mu\| = 0$.*

Proof. Define $\mathcal{K} := \overline{A\mu} \leq \mathcal{H}$. Let $p : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection onto $\mathcal{K}$. By definition, $a\mathcal{K} \subseteq \mathcal{K}$ for any $a \in A$. Equivalently, $pap = ap$ for any $a \in A$. Then

$$pa = (a^*p)^* = (pa^*p)^* = pap = ap,$$

so $p \in A'$. But $x \in A''$, so

$$xp = xp^2 = pxp$$

and $x\mathcal{K} \subseteq \mathcal{K}$. In particular, since $\mu \in \mathcal{K}$, we have $x\mu \in \mathcal{K} = \overline{A\mu}$. So there must exist some net in $A\mu$ that converges to $x\mu$. □

Theorem 5.10 (von Neumann's bicommutant theorem)**.**

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra. Then $\overline{A}^{\mathrm{WOT}} = A''$.

Proof. By the previous theorem, A'' is a vNa. In particular, it is WOT-closed. Since $A \subseteq A''$, it suffices to show that A is WOT-dense in A''. Because A is convex, it is enough to show that A is SOT-dense in A''. Let $x \in A''$ and $\mu_1, \dots, \mu_n \in \mathcal{H}$. Consider the matrix $*$-algebra $M_n(\mathcal{B}(\mathcal{H}))$ with the usual matrix involution. There exists a canonical $*$-isomorphism $M_n(\mathcal{B}(\mathcal{H})) \to \mathcal{B}(\mathcal{H}^n)$, which allows us to introduce a (unique) norm on $M_n(\mathcal{B}(\mathcal{H}))$, making it a C^*-algebra. Define

$$\widetilde{A} = \left\{ \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \in M_n(\mathcal{B}(\mathcal{H})) \mid a \in A \right\}.$$

Then $\widetilde{A}' = M_n(A')$. Hence we get

$$\widetilde{A''} \subseteq M_n(A')' = \widetilde{A}''.$$

This implies that

$$\begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} \in \widetilde{A''} \subseteq \widetilde{A}''.$$

Now we apply Lemma 5.9 to $\widetilde{A}$ to get a net $(a_i)_i$ in A such that

$$\lim_i \|(x - a_i)\mu_j\| = 0, \quad \forall j = 1, \dots, n.$$

Finally, we have to show that this implies that x is in the SOT-closure of A. Let U be an open neighborhood around x. Then U must contain some finite intersection of subbasis sets that generate the SOT topology. This means that there exists $\varepsilon > 0$ and $\mu_1, \dots, \mu_n \in \mathcal{H}$ such that

$$\bigcap_{j=1}^{n} \{y \in \mathcal{B}(\mathcal{H}) \mid \|(x-y)\mu_j\| < \varepsilon\} \subseteq U.$$

Now we can conclude that $U \cap A \neq \emptyset$ and x is in the SOT-closure of A. $\square$

Corollary 5.11. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra. Then A is a vNa iff $A = A''$.*

Remark. WOT-closed implies norm-closed. In particular, every vNa is a C^*-algebra. However, the converse is not always true: $\mathcal{C}([0,1])$ is a C^*-algebra that is not vNa. As we will see, this is because it does not contain nontrivial projections.

Corollary 5.12 (Polar decomposition in vNa)**.** *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $x \in A$. Suppose that $x = v|x|$ is the polar decomposition of x in $\mathcal{B}(\mathcal{H})$. Then $v \in A$.*

Proof. We know that

$$\ker v = (\operatorname{im}|x|)^{\perp} = \ker|x| = \ker x.$$

For $a \in A'$ and $\mu \in \ker x$ we have $a\mu \in \ker x$:

$$x(a\mu) = ax\mu = 0,$$

which implies $a \ker|x| \subseteq \ker|x|$. We know that $\mathcal{H} = \ker|x| \oplus \overline{\operatorname{im}|x|}$. Suppose that $|x|\mu \in \operatorname{im}|x|$. Then

$$\begin{aligned}
[a, v]|x|\mu &= (av - va)|x|\mu = av|x|\mu - va|x|\mu \\
&= ax\mu - v|x|a\mu = ax\mu - xa\mu \\
&= [a, x]\mu = 0.
\end{aligned}$$

But for $\beta \in \ker|x| = \ker v$, we have

$$[a, v]\beta = (av - va)\beta = av\beta - va\beta = 0.$$

Since av and va agree on $\ker|x| \oplus \overline{\operatorname{im}|x|} = \mathcal{H}$, we have $v \in A'' = A$. $\square$

Example 5.13 (Commutative vNa - IMPORTANT)**.** *Let (X, μ) be a σ-finite measure space and*

$$M : L^{\infty}(X, \mu) \to \mathcal{B}(L^2(X, \mu)), \quad g \mapsto M_g,$$

where we define

$$(M_g f)(x) = g(x)f(x).$$

Then M is an isometric $*$-isomorphism onto its image and $M(L^{\infty}(X, \mu))$ is a maximal commutative vNa in $\mathcal{B}(L^2(X, \mu))$.

Proof of the example. Clearly, M is injective, additive and multiplicative. First, we prove that M is a $*$-homomorphism. This follows from the next calculation:

$$\begin{aligned}
\langle M_{\overline{g}}\mu, \varphi\rangle &= \int_X M_{\overline{g}}\mu \cdot \overline{\varphi}\, d\mu \\
&= \int_X \overline{g}\mu\overline{\varphi} \\
&= \int_X \mu\overline{g\varphi}\, d\mu \\
&= \langle \mu, M_g\varphi\rangle = \langle M_g^*\mu, \varphi\rangle,
\end{aligned}$$

so $M_{\overline{g}} = M_g^*$. Next, we prove that M is isometric. For $g \in L^{\infty}(X, \mu)$, there exists a sequence $E_n \subseteq X$ such that $0 < \mu(E_n) < \infty$ and $|g|\big|_{E_n} \geq \|g\|_{\infty} - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\|M_g\| \geq \frac{\|M_g 1_{E_n}\|_2}{\|1_{E_n}\|_2} \geq \|g\|_{\infty} - \frac{1}{n}, \quad \forall n \in \mathbb{N},$$

which implies $\|M_g\| \geq \|g\|_\infty$. For the reverse, notice that

$$\begin{aligned}
\|M_g 1_{E_n}\|^2 &= \int_X |g \cdot 1_{E_n}|^2 \, d\mu \\
&= \int_{E_n} |g|^2 \, d\mu \\
&\geq \int_{E_n} (\|g\|_\infty - \frac{1}{n})^2 \, d\mu \\
&= (\|g\|_\infty - \frac{1}{n})^2 \cdot \mu(E_n)
\end{aligned}$$

and

$$\begin{aligned}
\|M_g\|^2 = \sup_{\|\mu\|_2=1} \|M_g \mu\|_2^2 &= \sup_{\|\mu\|_2=1} \int_X |g\mu|^2 \, d\mu \\
&\leq \|g\|_\infty^2 \cdot \sup_{\|\mu\|_2=1} \int_X |\mu|^2 \, d\mu = \|g\|_\infty^2.
\end{aligned}$$

We've just shown that $\|Mg\| = \|g\|_\infty$. Lastly, we prove that $M(L^\infty(X,\mu))$ is a maximal commutative subalgebra of $\mathcal{B}(L^2(X,\mu))$. Take $T \in \mathcal{B}(L^2(X,\mu))$ and assume it commutes with all M_g's. Now pick a measurable sequence $E_n \subseteq X$ such that $0 < \mu(E_n) < \infty$, $E_n \subseteq E_{n+1}$ and $X = \bigcup_{n \in \mathbb{N}} E_n$. Define $f_n := T(1_{E_n}) \in (X,\mu)$. First we prove that $f_n \in L^\infty(X,\mu)$. If A is measurable and $0 < \mu(A) < \infty$, then

$$\begin{aligned}
\frac{1}{\mu(A)} \int_X |f_n \cdot 1_A|^2 \, d\mu &= \frac{1}{\mu(A)} \cdot \|M_{1_A} T(1_{E_n})\|^2 \\
&= \frac{1}{\mu(A)} \cdot \|T(1_{A \cap E_n})\|^2 \\
&\leq \frac{1}{\mu(A)} \cdot \|T\|^2 \cdot \|1_A\|^2 = \|T\|^2.
\end{aligned}$$

If $f \notin L^\infty(X,\mu)$, then for all $M \in \mathbb{R}$ we have

$$0 < \mu(\underbrace{\{x \in X \mid |f_n(x)| > M\}}_{A_{n,M}}) < \infty,$$

since $f_n \in L^2(X,\mu)$. By above calculation,

$$M^2 \leq \frac{1}{\mu(A_{n,M})} \cdot \int_X |f \cdot 1_{A_{n,M}}|^2 \, d\mu \leq \|T\|^2,$$

which is of course a contradiction. This proves that $f_n \in L^\infty(X,\mu)$ and $\|f_n\|_\infty \leq \|T\|$. For $n \leq m$ we have

$$\begin{aligned}
1_{E_n} \cdot f_m &= 1_{E_n} \cdot T(1_{E_m}) \\
&= M_{1_{E_n}}(T(1_{E_m})) \\
&= T(M_{1_{E_n}} 1_{E_m}) \\
&= T(1_{E_n} 1_{E_m}) = f_n.
\end{aligned}$$

Therefore, $f_m\big|_{E_n} = f_n$. The sequence $(f_n)_n$ converges to a measurable $f : X \to \mathbb{C}$. From $\|f_n\|_\infty \leq \|T\|$ for all $n \in \mathbb{N}$ we also deduce $\|f\|_\infty \leq T$, so $f \in L^\infty(X, \mu)$. Lastly, we prove $T = M_f$. Note that simple functions $\sum_{j=i}^r \alpha_j 1_{A_j}$ are $L^2(X,\mu)$-dense. Let $A \subseteq X$ be measurable with $\mu(A) < \infty$. Then $\|1_{A\cap E_n} - 1_A\|_2 \xrightarrow{n\to\infty} 0$. Hence

$$\|(T - M_f)1_A\|_2 = \lim_{n\to\infty} \|(T - M_f)1_{A\cap E_n}\|_2 = 0,$$

as we shall prove.

$$\begin{aligned} T(1_{A\cap E_n}) &= T(1_A \cdot 1_{E_n}) = T(M_{1_A} 1_{E_n}) \\ &= M_{1_A}(T(1_{E_n})) = M_{1_A}(f_n) \\ &= 1_A \cdot f_n. \end{aligned}$$

On the other hand,

$$M_f(1_{A\cap E_n}) = f \cdot 1_{A\cap E_n} = f \cdot 1_{E_n} \cdot 1_A = 1_A \cdot f_n$$

and we are done. □

Another possible characterization of vNa's is given by the following.

Theorem 5.14 (Sakai)**.**

Let A be a C^*-algebra such that for a Banach space E there exists an isometric isomorphism $A \to E^*$. Then there exists a vNa $B \subseteq \mathcal{B}(\mathcal{H})$ such that $A \cong B$ as a C^*-algebra.

For the proof, see R.V.Kadison's *The von Neumann algebra characterization theorems* (1985).

5.2 Kaplansky's density theorem

Lemma 5.15. *The multiplication $(A, B) \mapsto A \cdot B$ is SOT-continuous on bounded sets.*

Proof. Let $(A_i)_i$ and $(B_i)_i$ be nets with $\sup \|A_i\|, \sup \|B_i\| < M$ for some $M \in \mathbb{R}$. Suppose $A_i \to A$ and $B_i \to B$ in SOT. For any x, we get

$$\begin{aligned} \|ABx - A_iB_ix\| &= \|ABx - A_iBx + A_iBx - A_iB_ix\| \\ &\leq \|ABx - A_iBx\| + \|A_iBx - A_iB_ix\| \\ &\leq \|A(Bx) - A_i(Bx)\| + \|A_i\| \cdot \|Bx - B_ix\| \\ &\leq \|A(Bx) - A_i(Bx)\| + M \cdot \|Bx - B_ix\| \to 0, \end{aligned}$$

so $A_iB_i \xrightarrow{\text{SOT}} AB$. □

Proposition 5.16. *Let $f \in C(\mathbb{C})$. Then $x \mapsto f(x)$ is SOT-continuous on each bounded set of normal operators in $\mathcal{B}(\mathcal{H})$.*

Proof. By Stone-Weierstrass, we can uniformly approximate f by polynomials on a bounded subset $B_R(0) \subseteq \mathbb{C}$. By the previous lemma, multiplication is SOT-continuous on this bounded set of normal operators. But for a normal operator A, we have $\|Ax\| = \|A^*x\|$ for every $x \in \mathcal{H}$, so $*$ is also SOT-continuous on normal operators and we're done. ☐

Theorem 5.17 (Cayley transform)**.**

Mapping $x \mapsto (x-i)(x+i)^{-1}$ *is SOT-continuous* $\mathcal{B}(\mathcal{H})_{\text{sa}} \to \mathcal{U}(\mathcal{H})$.

Proof. If $x \in \mathcal{B}(\mathcal{H})_{\text{sa}}$, then $\sigma(x) \subseteq \mathbb{R}$ and $(x+i) \in \mathcal{B}(\mathcal{H})$ is invertible. We notice that $z \mapsto \frac{z-i}{z+i} : \mathbb{R} \to \mathbb{C}$ has its range in $\mathbb{T}$, so the Cayley transform does in fact map into the unitaries. Now onto the SOT-continuity: let $(x_k)_k$ be a net in $\mathcal{B}(\mathcal{H})_{\text{sa}}$ with $x_k \to x$ in SOT. By the spectral mapping theorem, $\|(x_k+i)^{-1}\| \leq 1$. For each $\alpha \in \mathcal{H}$, we have

$$\begin{aligned}
\|(x-i)(x+i)^{-1}\alpha - (x_k-i)(x_k+i)^{-1}\alpha\| &= \|(x_k+i)^{-1}\left((x_k+i)(x-i)(x+i^{-1}) - (x_k-i)\right)\alpha\| \\
&= \|(x_k+i)^{-1}\left((x_k+i)(x-i) - (x_k-i)(x+i)\right)(x+i)^{-1}\alpha\| \\
&= \|(x_k+i)^{-1}2i(x-x_k)(x+i)^{-1}\alpha\| \\
&\leq 2\|(x_k+i)^{-1}\|\|(x-x_k)\underbrace{(x+i)^{-1}\alpha}_{\beta}\| \\
&\leq 2\|(x-x_k)\beta\| \to 0.
\end{aligned}$$

☐

Corollary 5.18. *If* $f \in C_0(\mathbb{R})$, *then* $x \mapsto f(x)$ *is SOT-continuous on* $\mathcal{B}(\mathcal{H})_{\text{sa}}$.

Proof. Consider the continuous function

$$g(t) = \begin{cases} f\left(i\frac{1+t}{1-t}\right); & t \neq 1 \\ 0; & t = 1 \end{cases}$$

which maps $\mathbb{T} \to \mathbb{C}$. By the previous proposition, $x \mapsto g(x)$ is SOT-continuous on unitaries. Letting $U(z) = \frac{z-i}{z+i}$, denote the Cayley transform, we have that $f = g \circ U$ is a composite of two SOT-continuous maps, which is a SOT-continuous map of itself. ☐

Theorem 5.19 (Kaplansky's density theorem)**.**

Let $A \subseteq \mathcal{B}(\mathcal{H})$ *be a* $*$*-subalgebra and* $B = \overline{A}^{\text{SOT}}$, *then*

(1.) $\overline{A_{\text{sa}}}^{\text{SOT}} = B_{\text{sa}}$;

(2.) $\overline{(A)_1}^{\text{SOT}} = (B)_1$.

Proof. WLOG A is a C^*-algebra, so norm-closed.
(1.) First we prove that $\overline{A_{\text{sa}}}^{\text{SOT}} \subseteq B_{\text{sa}}$. Since $\overline{A_{\text{sa}}}^{\text{SOT}} = \overline{A_{\text{sa}}}^{\text{WOT}}$, take $x \in \overline{A_{\text{sa}}}^{\text{SOT}}$ and a net

$(x_k)_k \subseteq A_{\text{sa}}$ that converges to x. Since $*$ is WOT continuous, $(x_k^*)_k = (x_k)_k$ converge to x^*, so $x = x^*$. Now the converse inclusion: suppose the net $(x_k)_k$ SOT-converges to $x \in B_{\text{sa}}$. Then $\frac{x_k + x_k^*}{2} \to x$ in the WOT-topology, which implies

$$B_{\text{sa}} \subseteq \overline{A_{\text{sa}}}^{\text{WOT}} = \overline{A_{\text{sa}}}^{\text{SOT}}.$$

(2.) Suppose the net $(y_i)_i$ in A_{sa} SOT-converges to $x \in B_{\text{sa}}$. Take $f \in C_0(\mathbb{R})$ such that we have $f(t) = t$, $\forall |t| \leq \|x\|$ and $|f(t)| \leq \|x\|$, $\forall t \in \mathbb{R}$. By functional calculus, $\|f(y_k)\| \leq \|x\|$. By the previous corollary, $(f(y_i))_i \xrightarrow{\text{SOT}} f(x) = x$. This proves that $(A)_1 \cap A_{\text{sa}}$ is SOT-dense in $(B)_1 \cap B_{\text{sa}}$. Pass over to $M_2(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then $M_2(A)$ is SOT-dense in $M_2(B)$ by the first part of the proof. For $x \in (B)_1$, we have

$$\widetilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (M_2(B))_1 \cap (M_2(B))_{\text{sa}}.$$

That means there exists a net

$$\widetilde{x_i} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in (M_2(A))_1$$

such that $\widetilde{x_i} \to \widetilde{x}$ and therefore $b_i \in (A)_1$ SOT-converge to x. □

Corollary 5.20. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-algebra. Then A is a vNa iff $(A)_1$ is SOT-closed.*

5.3 Examples of vNa's

Definition 5.21. A vNa M is called a *factor* if $Z(M) = M \cap M' = \mathbb{C} \cdot 1$.

Example 5.22. *Clearly, $\mathcal{B}(\mathcal{H})$ is a factor. In particular, $M_n(\mathbb{C})$ is a factor.*

Let Γ be a group and $\mathcal{H} = \ell^2(\Gamma)$. Consider the left regular representation

$$\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma)), \quad g \mapsto (\delta_h \mapsto \delta_{gh})$$

and extend it linearly to $\lambda : \mathbb{C}[\Gamma] \to \mathcal{B}(\ell^2(\Gamma))$. The group vNa of Γ is $VN(\Gamma) := \lambda(\mathbb{C}[\Gamma])''$ in $\mathcal{B}(\ell^2(\Gamma))$. It has a *trace*, which is defined as the linear functional

$$\tau : VN(\Gamma) \to \mathbb{C}, \quad x \mapsto \langle x\delta_e, \delta_e \rangle.$$

For $g \in \Gamma$, $\tau(\lambda(g)) = 1$ if $g = e$, otherwise zero. For $g_1, \dots, g_r \in \Gamma$, we have

$$g_1 \dots g_r = e \Leftrightarrow \tau(\lambda(g_1) \dots \lambda(g_r)) = 1.$$

Since τ is a positive linear functional and $\tau(1) = 1$, τ is a state. For any two elements $g, h \in \Gamma$ we have $gh = e \Leftrightarrow hg = e$, which together with the above line implies

$$\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g)).$$

By linearity, τ has the same cyclic property on $\lambda(\mathbb{C}[\Gamma])$. But since τ is, by definition, WOT-continuous and $VN(\Gamma) = (\lambda(\mathbb{C}[\Gamma]))'' = \overline{(\lambda(\mathbb{C}[\Gamma]), \text{WOT})}$, τ is cyclic on the entire $VN(\Gamma)$. Now if $|\Gamma| = \infty$, then $VN(\Gamma) \neq \mathcal{B}(\mathcal{H})$, since the latter does not have a trace if $\dim \mathcal{H} = \infty$. If Γ is an abelian group, then $VN(\Gamma)$ is commutative.

Definition 5.23. Group Γ has *icc* (infinite conjugacy classes) if for all $g \in \Gamma \setminus \{e\}$ the set $\{f^{-1}gf \mid f \in \Gamma\}$ is infinite.

Example 5.24. *The group*

$$S_\infty = \{\textit{bijections } \mathbb{N} \to \mathbb{N} \textit{ that only permute finitely many elements}\}$$

has icc.

Example 5.25. *Free groups $\mathbb{F}_n$ for $n > 1$ have icc.*

Theorem 5.26.

If Γ has icc, then $VN(\Gamma)$ is a factor.

Definition 5.27. $VN(S_\infty) =: R$ is the *hyperfinite II_1-factor.*

Open problem: does $VN(\mathbb{F}_2) \cong VN(\mathbb{F}_3)$ hold?

5.4 Operations with vNa's

5.4.1 Direct sums

Let $M_i \subseteq \mathcal{B}(\mathcal{H}_i)$ be vNa's. Define the isometric embedding

$$\iota_j : \mathcal{B}(\mathcal{H}_j) \to \mathcal{B}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n), \quad x \mapsto ((\alpha_1, \ldots, \alpha_n) \mapsto (0, \ldots, 0, x\alpha_j, 0, \ldots, 0)).$$

This map is the $n \times n$ bounded matrix where the (j, j)-th element is x and the rest are zero. Then

$$M_1 \oplus \cdots \oplus M_n := \operatorname{span}\{\iota_j(x) \mid j = 1, \ldots, n, \ x \in M_j\}$$

is the direct sum of vNa's. If $n \geq 2$, then from

$$Z(M_1 \oplus \cdots \oplus M_n) = Z(M_1) \oplus \cdots \oplus Z(M_n),$$

we deduce that $M_1 \oplus \cdots \oplus M_n$ is not a factor.

5.4.2 Tensor products

The algebraic tensor product $\mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n)$ acts on $\mathcal{H}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{H}_n$ by

$$(x_1 \otimes \cdots \otimes x_n)(\alpha_1 \otimes \cdots \otimes \alpha_n) = (x_1\alpha_1) \otimes \cdots \otimes (x_n\alpha_n)$$

for $x_j \in \mathcal{B}(\mathcal{H}_j)$ and $\alpha_j \in \mathcal{H}_j$, which implies

$$\mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n) \subseteq \mathcal{B}(\mathcal{H}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{H}_n).$$

Finally, we define the tensor product of vNa's as

$$M_1 \overline{\otimes} \cdots \overline{\otimes} M_n = (M_1 \otimes \cdots \otimes M_n)'' \cap \mathcal{B}(\mathcal{H}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{H}_n).$$

5.4.3 Compressions

Definition 5.28. Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $p \in \mathcal{B}(\mathcal{H})$ a projection. A compression of M is $pMp = \{pxp \mid x \in M\}$. When $p \in M$, it is also called a corner.

If $\mathcal{H} = \operatorname{im} p \oplus (\operatorname{im} p)^{\perp} = \operatorname{im} p \oplus \operatorname{im}(1-p)$. In this basis, elements of pMp have the matrix form
$$\begin{bmatrix} pxp & 0 \\ 0 & 0 \end{bmatrix}.$$
If $M \ni p \neq 1$, then pMp is a $*$-algebra and $pMp \subseteq M$ but it is not a subalgebra since $1_M = 1_{\mathcal{B}(\mathcal{H})} \notin pMp$. However, pMp is a subalgebra of $\mathcal{B}(p\mathcal{H})$ with identity element p.

Definition 5.29. Let $\mathcal{K} \subseteq \mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$.
(1.) $\mathcal{K}$ is invariant under x if $x\mathcal{K} \subseteq \mathcal{K}$;
(2.) $\mathcal{K}$ is reducing under x if $\mathcal{K}$ is invariant under both x and x^*.
Now if $S \subseteq \mathcal{B}(\mathcal{H})$, then
(1.) $\mathcal{K}$ is invariant under S if $x\mathcal{K} \subseteq \mathcal{K}$ under all $x \in S$;
(2.) $\mathcal{K}$ is reducing under S if $\mathcal{K}$ is reducing under all $x \in S$.

If $S \subseteq \mathcal{B}(\mathcal{H})$ is closed under $*$, then $\mathcal{K}$ is invariant under S iff it is reducing under S.

Lemma 5.30. *Let $\mathcal{K}^{\text{closed}} \leq \mathcal{H}$ and $M \subseteq \mathcal{B}(\mathcal{H})$ an $*$-algebra. Let $p : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. Then $\mathcal{K}$ is reducing under M iff $p \in M'$.*

Theorem 5.31.

Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $p \in M$ a projection. Then pMp and $M'p$ are vNa's in $\mathcal{B}(p\mathcal{H})$.

Proof. We will show that
$$(M'p)' \cap \mathcal{B}(p\mathcal{H}) = pMp, \quad (pMp)' \cap \mathcal{B}(p\mathcal{H}) = M'p.$$
Then the bicommutant theorem will take care of the rest. It is obvious that $(M'p)' \cap \mathcal{B}(p\mathcal{H}) \supseteq pMp$. For the converse, pick $x \in (M'p)' \cap \mathcal{B}(p\mathcal{H})$. Define $\widetilde{x} = xp = px \in \mathcal{B}(\mathcal{H})$. For $y \in M'$, we have
$$y\widetilde{x} = ypx = xyp = xpy = \widetilde{x}y,$$
which implies $\widetilde{x} \in M'' = M$. Then $x = pxp = p\widetilde{x}p \in pMp$. As before, $(pMp)' \cap \mathcal{B}(p\mathcal{H}) \supseteq M'p$ is trivial and we just prove the converse. Take $y \in (pMp)' \cap \mathcal{B}(p\mathcal{H})$. Using continuous functional calculus, we can write y as a linear combinations of 4 unitaries. Since pMp is closed under $*$, $(pMp)'$ is a vNa (and therefore a C^*-algebra). So we can assume WLOG that $y = u$ a unitary. Set $\mathcal{K} := \overline{Mp\mathcal{H}}$ and let $q : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. Since $\mathcal{K}$ is reducing under M and M', which implies
$$q \in M' \cap M'' = M' \cap M = Z(M).$$
Next, we extend u to $\mathcal{K}$:
$$\widetilde{u}\Big(\sum_i \underbrace{x_i}_{\in M} p \underbrace{\alpha_i}_{\in \mathcal{H}}\Big) = \sum_i x_i u p \alpha_i.$$

We shall show that this is a well-defined isometry in $Mp\mathcal{H}$:

$$\begin{aligned}\|\widetilde{u}\sum_i x_i p\alpha_i\|^2 &= \sum_{i,j}\langle x_i up\alpha_i, x_j up\alpha_j\rangle\\ &= \sum_{i,j}\langle (px_j^* x_i p)u\alpha_i, u\alpha_j\rangle\\ &= \sum_{i,j}\langle upx_j^* x_i p\alpha_i, u\alpha_j\rangle\\ &= \sum_{i,j}\langle px_j^* x_i p\alpha_i, \alpha_j\rangle = \|\sum_i x_i p\alpha_i\|^2.\end{aligned}$$

So $\widetilde{u}$ extends to an isometry on $\mathcal{K} = \overline{Mp\mathcal{H}}$. By definition, $\widetilde{u}$ commutes with M on $Mp\mathcal{H}$, so also on $\mathcal{K}$. Thus for every $x \in M$ and $\alpha \in \mathcal{H}$, we have

$$x(\widetilde{u}q)\alpha = \widetilde{u}xq\alpha = (\widetilde{u}q)x\alpha,$$

which implies $\widetilde{u}q \in M' \cap \mathcal{B}(\mathcal{H})$. Then

$$\widetilde{u}qp\alpha = \widetilde{u}1p\alpha = 1up\alpha,$$

which implies $u = \widetilde{u}qp \in \mathcal{B}(\mathcal{H})$ and $u \in M'p$. □

Corollary 5.32. *Suppose the vNa $M \subseteq \mathcal{B}(\mathcal{H})$ is a factor and let $p \in M$ be a projection. Then pMp and $M'p$ are factors (in $\mathcal{B}(p\mathcal{H})$).*

Proof. Let $\mathcal{K} = \overline{Mp\mathcal{H}}$ and $q : \mathcal{H} \to \mathcal{K}$ the projection. From the previous proof, $q \in Z(M) = \mathbb{C}$. Then $q \in \{0, 1\}$. WLOG $p \neq 0$, so $q = 1$. Thus $\mathcal{K} = \mathcal{H}$, so $Mp\mathcal{H}$ is dense in $\mathcal{H}$. Consider

$$\psi : M' \to M'p, \quad y \mapsto yp.$$

We will prove that ψ is an isomorphism of algebras. Obviously, it is additive. Since

$$\psi(xy) = xyp = xyp^2 = xpyp = \psi(x)\psi(y),$$

it is also multiplicative. Same calculation shows $\psi(y^*) = \psi(y)^*$. Obviously, ψ is surjective. Finally, we prove injectivity. Suppose $y \in M'$ satisfies $yp = 0$. Then for every $x \in M$ and $\alpha \in \mathcal{H}$, we get $yxp\alpha = x(yp)\alpha = 0$. Hence $y\big|_{Mp\mathcal{H}} = 0$, so by continuity, $y\big|_{\overline{Mp\mathcal{H}}} = y\big|_{\mathcal{K}} = 0$. But because $\mathcal{K} = \mathcal{H}$, this yields $y\big|_{\mathcal{H}} = 0$. As a result, we get

$$Z(M'p) = Z(M')p = \mathbb{C} \cdot p,$$

so $M'p$ is a factor. Similarly,

$$Z(pMp) = (pMp) \cap (pMp)' = (M'p)' \cap M'p = Z(M'p) = \mathbb{C}p,$$

so pMp is a factor. □

6 Spectral theorem and Borel functional calculus

6.1 Spectral theorem

Recall the spectral theorem for $\mathcal{K}(\mathcal{H})$. Let $T \in \mathcal{K}(\mathcal{H})$ be self-adjoint and for $\lambda \in \sigma_p(T)$, define $E(\lambda)$ as an orthogonal projection onto the eigenspace $\ker(T-\lambda I)$. For $\mu \neq \lambda$, we get $E(\lambda)E(\mu) = 0$ and

$$T = \sum_{\lambda \in \sigma_p(T)\setminus\{0\}} \lambda E(\lambda).$$

Our first goal will be to generalize this to non-compact self-adjoint operator.

Theorem 6.1 (Vigier)**.**

Let (u_λ) be a net of increasing (decreasing) and above (below) bounded self-adjoint operators on $\mathcal{H}$. Then (u_λ) converges.

Proof. We prove the statement for above-bounded increasing net. We can assume (u_λ) has a lower bound m by considering a truncated net. WLOG we can assume u_λ are positive (otherwise we can consider $u_\lambda - m$). There exists $M \geq 0$ such that $\|u_\lambda\| \leq M$ for indices λ. So the net $\langle u_\lambda x, x\rangle$ is real and increasing and bounded above by $M\|x\|^2$. Using the polarization identity

$$\langle u_\lambda x, x\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle u_\lambda(x + i^k y), x + i^k y\rangle,$$

we see that $\langle u_\lambda x, y\rangle$ is an convergent net for all $x, y \in \mathcal{H}$. Letting $\sigma(x, y)$ denote its limit, we can eaasily check that σ is a bounded sesquilinear form ($|\sigma(x, y)| \leq M\|x\|\|y\|$). By Riesz, there exists an operator $u \in \mathcal{B}(\mathcal{H})$ such that $\langle ux, y\rangle = \sigma(x, y)$. Then u is self-adjoint, $\|u\| \leq M$ and $u_\lambda \leq u$. Note that

$$\begin{aligned}\|(u - u_\lambda)x\|^2 &\leq \|(u - u_\lambda)^{\frac{1}{2}}(u - u_\lambda)^{\frac{1}{2}}x\|^2\\ &\leq \|(u - u_\lambda)\|\|(u - u_\lambda)^{\frac{1}{2}}x\|^2\\ &\leq 2M\langle (u - u_\lambda)x, x\rangle \to 0,\end{aligned}$$

so u_λ converge strongly to u. □

Remark. If (p_λ) is a net of projections converging strongly to some operator u, then u is also a projection. Clearly, u is self-adjoint and

$$\begin{aligned}\langle ux, y\rangle &= \lim_\lambda \langle p_\lambda x, y\rangle = \lim_\lambda \langle p_\lambda x, p_\lambda y\rangle\\ &= \langle ux, uy\rangle = \langle u^2 x, y\rangle,\end{aligned}$$

therefore $u^2 = u$.

Corollary 6.2. *If $(p_n)_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal orthogonal projections in $\mathcal{B}(\mathcal{H})$, then $\left(\sum_{n=1}^{N} p_n\right)$ SOT-converges for $N \to \infty$ (we denote the limit by $\sum_{n=1}^{\infty} p_n$).*

Definition 6.3. Let X be a set, Ω a σ-algebra in X and $\mathcal{H}$ a Hilbert space. Then we define a projection-valued measure (PVM) for $(X, \Omega, \mathcal{H})$ is a map $E : \Omega \to \mathcal{B}(\mathcal{H})$ such that

(1.) $E(S)$ is a projection for all $S \in \Omega$;
(2.) $E(\emptyset) = 0$ and $E(X) = 1$;
(3.) $E(S \cap T) = E(S)E(T)$ for all $S, T \in \Omega$;
(4.) If $(S_n)_{n\in\mathbb{N}} \subseteq \Omega$ is a sequence of pairwise disjoint sets, then

$$E(\bigcup_{n=1}^{\infty} S_n) = \sum_{n=1}^{\infty} E(S_n).$$

Remark. Projections $E(S)$ commute with each other (follows directly from the third point of the definition).

Example 6.4. *Let (X, Ω, μ) be a σ-finite measure space. Let $\mathcal{H} = L^2(X, \mu)$ and $S \in \Omega$, then $\chi_S \in L^\infty(X, \mu) \subseteq \mathcal{B}(L^2(X, \mu))$ is a projection (with pointwise multiplication in $\mathcal{B}(\mathcal{H})$). Letting $E(S) := \chi_S \in \mathcal{B}(L^2(X, \mu))$, we get a PVM $E : \Omega \to \mathcal{B}(L^2(X, \mu))$.*

Lemma 6.5. *Let E be a PVM for $(X, \Omega, \mathcal{H})$. Then for all $\alpha, \beta \in \mathcal{H}$ the mapping*

$$E_{\alpha,\beta} : \Omega \to \mathbb{C}, \quad S \mapsto \langle E(S)\alpha, \beta\rangle$$

is a complex measure in Ω with total variation $\leq \|\alpha\|\|\beta\|$.

Proof. Let $\alpha, \beta \in \mathcal{H}$. Then $E_{\alpha,\beta}$ is σ-additive (countably-additive for disjoint sets) since E is σ-additive by (4). Total variation of a complex measure is

$$\|E_{\alpha,\beta}\| := \sup\{\sum_{S\in\pi} |E_{\alpha,\beta}(S)|\},$$

where the sum is over all partitions of X into finitely many pieces of measurable sets. Let $\pi = \{S_1, \dots, S_n\}$ be a partition of X with $S_j \in \Omega$. For each j pick $\alpha_j \in \mathbb{C}$ such that $|\alpha_j| = 1$ and

$$\alpha_j \cdot E_{\alpha,\beta}(S_j) = \alpha_j\langle E(S_j)\alpha, \beta\rangle = |\langle E(S_j)\alpha, \beta\rangle| = |E_{\alpha,\beta}(S_j)|.$$

Then

$$\sum_{j=1}^{n} |E_{\alpha,\beta}(S_j)| = \langle \sum \alpha_j E(S_j)\alpha, \beta\rangle| \leq \| \sum \alpha_j E(S_j)\alpha\| \cdot \|\beta\|.$$

For $i \neq j$ we have

$$E(S_i)E(S_j) = E(S_i \cap S_j) = E(\emptyset) = 0,$$

so $E(S_i)$ are pairwise orthogonal. Finally, we can use Pythagoras to get

$$\begin{aligned}
\|\sum_{j=1}^{n}\alpha_j E(S_j)\alpha\|^2 &= \sum_{j=1}^{n}\|E(S_j)\alpha\|^2 \\
&= \|\sum_{j=1}^{n}E(S_j)\alpha\|^2 \\
&= \|E\left(\bigcup_{j=1}^{n}S_j\right)\alpha\|^2 \\
&= \|E(X)\alpha\|^2 = \|\alpha\|^2.
\end{aligned}$$

□

Remark. Let E be a PVM for $(X,\Omega,\mathcal{H})$, $\alpha\in\mathcal{H}$ and $S\in\Omega$. Then

$$\begin{aligned}
E_{\alpha,\alpha}(S) &= \langle E(S)\alpha,\alpha\rangle \\
&= \langle E(S)^2\alpha,\alpha\rangle \\
&= \langle E(S)\alpha, E(S)\alpha\rangle \geq 0,
\end{aligned}$$

so $E_{\alpha,\alpha}$ is a positive measure on X. Furthermore, if $\|\alpha\|=1$, then $E_{\alpha,\alpha}$ is a probability measure.

Let

$$(\alpha,\beta)\mapsto\int_X 1 dE_{\alpha,\beta}.$$

Since $E_{\alpha+\lambda\alpha',\beta}=E_{\alpha,\beta}+\lambda E_{\alpha',\beta}$ and $E_{\alpha,\beta+\lambda\beta'}=E_{\alpha,\beta}+\overline{\lambda}E_{\alpha,\beta'}$, the above is a sesquilinear form on $\mathcal{H}$. In particular, it is bounded

$$\|\int_X dE_{\alpha,\beta}\|\leq\|E_{\alpha,\beta}\|\leq\|\alpha\|\|\beta\|.$$

Suppose $f:X\to\mathbb{C}$ is a bounded Ω-measurable function. Then

$$(\alpha,\beta)\mapsto\int_X f dE_{\alpha,\beta}$$

is a bounded sesquilinear form:

$$\|\int_X f\,dE_{\alpha,\beta}\|\leq\|f\|_\infty\|E_{\alpha,\beta}\|\leq\|f\|_\infty\|\alpha\|\|\beta\|.$$

So there exists an $x\in\mathcal{B}(\mathcal{H})$ such that $\|x\|\leq\|f\|_\infty$ and $\langle x\alpha,\beta\rangle=\int_X f\,dE_{\alpha,\beta}$. If $f=\chi_S$ for $S\in\Omega$, then $x=E(S)$:

$$\int_X \chi_S\,dE_{\alpha,\beta}=E_{\alpha,\beta}(S)=\langle E(S)\alpha,\beta\rangle.$$

Definition 6.6. Let E be a PVM for $(X,\Omega,\mathcal{H})$ and $f:X\to\mathbb{C}$ be a bounded Ω-measurable function and $x \in \mathcal{B}(\mathcal{H})$. We call x the integral of f with regards to E if

$$\langle x\alpha, \beta \rangle = \int_X f \, dE_{\alpha,\beta}, \quad \forall \alpha, \beta \in \mathcal{H}.$$

We denote it by $x := \int_X f \, dE$.

Remark. Define $B(X, \Omega)$ as the set of all bounded Ω-measurable complex functions on X and endow it with the sup norm. If X is a topological space and $\Omega = \mathcal{B}_X$ is the Borel σ-algebra on X, then $B(X) = B(X, \mathcal{B}_X)$.

Proposition 6.7. *Let E be a PVM for $(X, \Omega, \mathcal{H})$. Then*

$$\rho : B(X, \Omega) \to \mathcal{B}(\mathcal{H}), \quad f \mapsto \int_X f \, dE.$$

*is a *-homomorphism and contractive. Furthermore:*

(1.) If $(f_n)_n \subseteq B(X, \Omega)$ is an increasing sequence of nonnegative functions and $f = \sup_n f_n \in B(X, \Omega)$, then $\int_X f_n dE \to \int_X f \, dE$ in SOT.

(2.) If X is compact and T_2, then $\rho(B(X)) \subseteq \rho(C(X))''$.

Proof. We already saw that $\|\rho(f)\| \leq \|f\|_\infty$, hence ρ is contractive. It is also clear that ρ is linear and $\rho(f)^* = \rho(\overline{f})$. Next, we prove multiplicativity: $\rho(\chi_S) = E(S)$ for $S \in \Omega$. Then

$$\rho(\chi_S) \cdot \rho(\chi_T) = E(S) \cdot E(T) = E(S \cap T) = \rho(\chi_{S \cap T}) = \rho(\chi_S \cdot \chi_T).$$

Since ρ is linear, it is also multiplicative on simple functions (these are finite linear combinations of characteristic functions). Since each $f \in B(X, \Omega)$ is a uniform limit of a uniformly bounded sequence of simple functions, we deduce that $\rho(fg) = \rho(f)\rho(g)$ for all $f, g \in B(X, \Omega)$.

(1.) Let f, f_n be as in the statement. Since ρ is a *-homomorphism, $(\rho(f_n))_n$ is an increasing sequence of positive operators and $\sup_n \|\rho(f_n)\| \leq \sup_n \|f\|_\infty = \|f\|$. By Vigier, there exists $x \in \mathcal{B}(\mathcal{H})$ such that $\rho(f_n) \xrightarrow{\text{SOT}} x$. This x is a natural candidate for $\rho(f)$. Indeed, for $\alpha, \beta \in \mathcal{H}$, we have

$$\begin{aligned} \langle \rho(f)\alpha, \beta \rangle &= \int_X f \, dE_{\alpha,\beta} \\ &= \lim_{n\to\infty} \int_X f_n \, dE_{\alpha,\beta} \\ &= \lim_{n\to\infty} \langle \rho(f_n)\alpha, \beta \rangle, \end{aligned}$$

so $\rho(f_n) \xrightarrow{\text{WOT}} \rho(f)$ and therefore $\rho(f) = x$.

(2.) Let X be compact Hausdorff and $a \in \rho(C(X))'$. Take $\alpha, \beta \in \mathcal{H}$. Then for all $f \in C(X)$, we have

$$
\begin{aligned}
0 &= \langle (a\rho(f) - \rho(f)a)\alpha, \beta\rangle \\
&= \langle \rho(f)\alpha, a^*\beta\rangle - \langle \rho(f)(a\alpha), \beta\rangle \\
&= \int_X f\, dE_{\alpha,a^*\beta} - \int_X f\, dE_{a\alpha,\beta},
\end{aligned}
$$

so by uniqueness from Riesz-Markoff we get $E_{\alpha,a^*\beta} = E_{a\alpha,\beta}$. This same calculation backwards tells us that a commutes with all $\rho(g) = \int_X g\, dE$ for $g \in B(X)$, so $\rho(B(X)) \subseteq \rho(C(X))''$ $\square$

Remark. The map ρ is not necessarily isometric. However, we can define E-null sets

$$\{S \in \Omega \mid E(S) = 0\},$$

which gives us an equivalence relation on $B(X,\Omega)$ as follows: $f \sim_E g$ if $f(x) = g(s)$ except possibly on some E-null set. Then we have

$$\ker \rho = \{f \in B(X,\Omega) \mid f \sim_E 0\}$$

and an essential supremum

$$\|\rho(f)\| = \|f\|_\infty := \inf\{t > 0 \mid E(\{x \in X \mid |f(x)| \geq t\}) = 0\}.$$

Define $L^\infty(X,E) = \frac{B(X,\Omega)}{\sim_E}$ with norm induced by an essential supremum above. Then the map ρ from the above proposition induces an isometric *-isomorphism $\widetilde{\rho} : L^\infty(X,E) \to \mathcal{B}(\mathcal{H})$.

Recall that for a commutative C^*-algebra A the Gelfand transform

$$\Gamma : A \to C(\sigma(A))$$

is an isometric *-isomorphism.

Theorem 6.8 (Spectral theorem)**.**

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a commutative C^*-algebra and $\mathcal{B}_{\sigma(A)}$ be the Borel σ-algebra on $\sigma(A)$. Then there exists a PVM E for $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$ such that

$$x = \int_{\sigma(A)} \Gamma(x)\, dE$$

for all $x \in A$.

Proof. For all $\alpha, \beta \in \mathcal{H}$ and

$$\varphi : C(\sigma(A)) \to \mathbb{C}, \quad f \mapsto \langle \Gamma^{-1}(f)\alpha, \beta\rangle$$

is a bounded linear functional. Indeed, since Γ is an isometry we get

$$\langle \Gamma^{-1}(f)\alpha, \beta\rangle \leq \|f\|_\infty \|\alpha\| \|\beta\|.$$

By Riesz-Markoff, there exists a unique regular Borel measure $\mu_{\alpha,\beta}$ such that

$$\langle \Gamma^{-1}(f)\alpha, \beta\rangle = \int_{\sigma(A)} f \, d\mu_{\alpha,\beta}.$$

We will show that $\mu_{\alpha,\beta} = E_{\alpha,\beta}$ for a PVM E. For $f, g \in C(\sigma(A))$ we have

$$\int_{\sigma(A)} fg \, d\mu_{\alpha,\beta} = \langle \Gamma^{-1}(fg)\alpha, \beta\rangle = \langle \Gamma^{-1}(f)\Gamma(g)\alpha, \beta\rangle = \int_{\sigma(A)} f \, d\mu_{\Gamma^{-1}(g)\alpha,\beta}.$$

Now we notice that this is equal to

$$\langle \Gamma^{-1}(f)\alpha, \Gamma^{-1}(\overline{g})\beta\rangle = \int_{\sigma(A)} f \, d\mu_{\alpha,\Gamma^{-1}(\overline{g})\beta}.$$

From the uniqueness of Riesz-Markoff, we get

$$g \, d\mu_{\alpha,\beta} = d\mu_{\Gamma^{-1}(g)\alpha,\beta} = d\mu_{\alpha,\Gamma^{-1}(\overline{g})\beta}.$$

Finally, we have

$$\begin{aligned}
\int_{\sigma(A)} f \, d\overline{\mu_{\alpha,\beta}} &= \overline{\int \overline{f} \, d\mu_{\alpha,\beta}} \\
&= \overline{\langle \Gamma^{-1}(\overline{f})\alpha, \beta\rangle} \\
&= \overline{\langle \alpha, \Gamma^{-1}(f)\beta\rangle} \\
&= \langle \Gamma^{-1}(f)\beta, \alpha\rangle \\
&= \int_{\sigma(A)} f \, d\mu_{\beta,\alpha}
\end{aligned}$$

for all $f \in C(\sigma(A))$, which implies $\overline{\mu_{\alpha,\beta}} = \mu_{\beta,\alpha}$. To each $S \in \mathcal{B}_{\sigma(A)}$ we assign the sesquilinear form

$$\mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_{\sigma(A)} \chi_S \, d\mu_{\alpha,\beta}.$$

This form is bounded by $\|\alpha\|\|\beta\| = \|\mu_{\alpha,\beta}\|$. Thus there exists $E(S) \in \mathcal{B}(\mathcal{H})$ such that

$$\int_{\sigma(A)} \chi_S \, d\mu_{\alpha,\beta} = \langle E(S)\alpha, \beta\rangle.$$

Now notice that

$$\begin{aligned}
\langle E(S)^*\alpha, \beta\rangle &= \langle \alpha, E(S)\beta\rangle \\
&= \overline{\langle E(S)\beta, \alpha\rangle} \\
&= \overline{\int_{\sigma(A)} \chi_S \, d\mu_{\beta,\alpha}} \\
&= \int_{\sigma(A)} \chi_S \, d\overline{\mu_{\beta,\alpha}} \\
&= \int_{\sigma(A)} \chi_S d\mu_{\alpha,\beta} \\
&= \langle E(S)\alpha, \beta\rangle,
\end{aligned}$$

so $E(S) = E(S)^*$. For any $f \in C(\sigma(A))$, we get

$$\begin{aligned}\langle \Gamma^{-1}(f)E(S)\alpha, \beta\rangle &= \langle E(S)\alpha, \Gamma^{-1}(\overline{f})\beta\rangle \\ &= \int \chi_S \, d\mu_{\alpha,\Gamma^{-1}(\overline{f})} \\ &= \int \chi_S f \, d\mu_{\alpha,\beta}.\end{aligned}$$

By the lemma below, $C(\sigma(A))$ is weak-* dense in $C(\sigma(A))^{**}$. Furthermore, the latter set contains $B(\sigma(A))$: indeed, given any $\psi \in C(\sigma(A))^*$, we have that $\psi(\cdot) = \int \cdot \, d\mu$ for some measure μ. For any $r \in B(\sigma(A))$, we have $r(\psi) = \int_{\sigma(A)} r \, d\mu$. Hence, $i : C \to C^{**}$ extends to $\widehat{i} : B \to C^{**}$. In particular, for $T \in \mathcal{B}_{\sigma(A)}$, there exists a net $(f_i)_i \subseteq C(\sigma(A))$ such that $f_i \xrightarrow{\text{weak-*}} \chi_T$. As a result, $\int_{\sigma(A)} f_i \, d\mu_{\alpha,\beta} \to \int_{\sigma(A)} \chi_T \, d\mu_{\alpha,\beta}$ for all $\alpha, \beta \in \mathcal{H}$, so $\Gamma^{-1}(f_i) \xrightarrow{\text{WOT}} E(T)$. Now

$$\begin{aligned}\langle E(T) \cdot E(S)\alpha, \beta\rangle &= \lim_i \langle \Gamma^{-1}(f_i)E(S)\alpha, \beta\rangle \\ &= \lim_i \langle E(S)\alpha, \Gamma^{-1}(\overline{f_i})\beta\rangle \\ &= \lim_i \int_{\sigma(A)} \chi_S f_i \, d\mu_{\alpha,\beta} \\ &= \int_{\sigma(A)} \chi_S \cdot \chi_T \, d\mu_{\alpha,\beta} \\ &= \int_{\sigma(A)} \chi_{S\cap T} \, d\mu_{\alpha,\beta} = \langle E(S \cap T)\alpha, \beta\rangle.\end{aligned}$$

Since α, β were arbitrary, we get $E(S) \cdot E(T) = E(S \cap T)$ for all $S, T \in \mathcal{B}_{\sigma(A)}$. As a consequence, $E(S)^2 = E(S)$, so $E(S)$ is a projection. Obviously, $E(\emptyset) = 0$. Further,

$$\langle E(\sigma(A))\alpha, \beta\rangle = \int_{\sigma(A)} 1 \, d\mu_{\alpha,\beta} = \langle \Gamma^{-1}(1)\alpha, \beta\rangle = \langle 1\alpha, \beta\rangle,$$

so $E(\sigma(A)) = 1$. Since all $\mu_{\alpha,\beta}$ are σ-additive, we have

$$\begin{aligned}\langle E\left(\bigcup_{i=1}^{\infty} S_i\right)\alpha, \beta\rangle &= \mu_{\alpha,\beta}\left(\bigcup_{i=1}^{\infty} S_i\right) \\ &= \sum_{i=1}^{\infty} \mu_{\alpha,\beta}(S_i) \\ &= \langle \sum_{i=1}^{\infty} E(S_i)\alpha, \beta\rangle\end{aligned}$$

for each sequence $(S_i)_i \subseteq \mathcal{B}_{\sigma(A)}$ of pairwise disjoint sets, so $E\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} E(S_i)$. We have proved that E is a PVM for $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$. For any $\alpha, \beta \in \mathcal{H}$, we get

$$E_{\alpha,\beta}(S) = \langle E(S)\alpha, \beta\rangle = \int \chi_S \, d\mu_{\alpha,\beta} = \mu_{\alpha,\beta}(S),$$

so $E_{\alpha,\beta} = \mu_{\alpha,\beta}$ and therefore $\int f \, dE_{\alpha,\beta} = \langle \Gamma^{-1}\alpha, \beta \rangle$. But this proves that $x = \int_{\sigma(A)} \Gamma(x) \, dE$ for all $x \in A$. Uniqueness: suppose that E' is another such PVM. For $\alpha, \beta \in \mathcal{H}$ and $f \in C(\sigma(A))$, we have

$$\int_{\sigma(A)} f \, dE_{\alpha,\beta} = \langle \Gamma^{-1}(f)\alpha, \beta \rangle = \int_{\sigma(A)} f \, dE'_{\alpha,\beta},$$

which proves that $E_{\alpha,\beta} = E'_{\alpha,\beta}$ as elements in $C(\sigma(A))^*$. Thus

$$\langle E(S)\alpha, \beta \rangle = E_{\alpha,\beta}(S) = E'_{\alpha,\beta}(S) = \langle E'(S)\alpha, \beta \rangle.$$

Since α, β were arbitrary, $E(S) = E'(S)$. But since S was also arbitrary, $E = E'$. □

Lemma 6.9 (Goldstine's theorem). *Let X be a Banach space. Then the image of*

$$\iota : X \to X^{**}, \quad x \mapsto (f \mapsto f(x))$$

is dense in weak- topology.*

Proof. Let $\beta \in X^{**}$ and $f_1, \ldots, f_r \in X^*$ (WLOG linearly independent). Then

$$U = \{\alpha \in X^{**} \mid |(\alpha - \beta)(f_j)| < \varepsilon, \ j = 1, \ldots, r\}.$$

is a basic open set in weak-* topology in X^{**}. WLOG assume X is infinitely-dimensional. Consider the linear map

$$\Phi : X \to \mathbb{C}^r, \quad x \mapsto (f_1(x), \ldots, f_r(x)).$$

This map is surjective. In particular, there exists $x_0 \in X$ such that

$$\Phi(x_j) = (f_1(x_0), \ldots, f_r(x_0)) = (\rho_0(f_1), \ldots, \rho_0(f_r)),$$

hence $\iota(x_0) \in U \cap \iota(X)$. □

6.2 Borel functional calculus

Let $x \in \mathcal{B}(\mathcal{H})$ be normal ($x^*x = xx^*$) and $A = C^*(x) \subseteq \mathcal{B}(\mathcal{H})$. Then $\sigma(A) \cong \sigma(x)$. By the spectral theorem, there exists a PVM E for $(\sigma(x), \mathcal{B}_{\sigma(x)}, \mathcal{H})$ and

$$B(\sigma(x)) \to \mathcal{B}(\mathcal{H}), \quad f \mapsto \int_{\sigma(x)} f \, dE$$

is a *-homomorphism and a contraction (proposition above). Furthermore, $f \in C(\sigma(x))$ maps into $\int_{\sigma(x)} f \, dE = \Gamma^{-1}(f)$, so the above map, when restricted to $C(\sigma(x))$, coincides with Γ^{-1}. Furthermore, If $f = \mathrm{id} \in B(\sigma(x))$, meaning $f(z) = z$ for $z \in \sigma(x)$, then

$$\int_{\sigma(x)} \mathrm{id} \; dE = \Gamma^{-1}(\mathrm{id}) = x.$$

For $f \in B(\sigma(x))$, define
$$f(x) := \int_{\sigma(x)} f\, dE \in A'' = W^*(x),$$
which is a vNa generated by x.

Definition 6.10. Let $x \in \mathcal{B}(\mathcal{H})$ be normal. The mapping
$$B(\sigma(x)) \to W^*(x), \quad f \mapsto f(x)$$
is the Borel functional calculus.

Theorem 6.11 (Spectral mapping theorem)**.**

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and let $x \in A$ be normal. Then the Borel functional calculus has the following properties:

(1.) The map
$$B(\sigma(x)) \to A, \quad f \mapsto f(x)$$
*is a bounded *-homomorphism.*

(2.) If $f \in C(\sigma(x))$, then this $f(x)$ is the same $f(x)$ as in continuous functional calculus.

(3.) For $f \in B(\sigma(x))$, we have $\sigma(f(x)) \subseteq f(\sigma(x))$.

Proof. (1.) This is the above proposition.
(2.) Obvious.
(3.) For $\lambda \notin f(\sigma(x))$, then $f - \lambda \in \mathcal{B}(\sigma(x))$ is invertible in $\mathcal{B}(\sigma(x))$, so there exists $g \in \mathcal{B}(\sigma^{-1})$ such that $(f - \lambda)g = \mathrm{id}$. By Borel functional calculus, $(f(x) - \lambda I) \cdot g(x) = I$, so $\lambda \notin \sigma(f(x))$. ☐

Corollary 6.12. *Every vNa is the norm-closure of the linear span of the projections.*

Proof. Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $x \in M$. By using $\operatorname{Re} x, \operatorname{Im} x \in M_{\mathrm{sa}}$, we may WLOG assume $x \in A_{\mathrm{sa}}$. Hence x is normal and for all $f \in B(\sigma(x))$ we have $f(x) \in M$. For $S \in \mathcal{B}_{\sigma(x)}$, $\chi_S(x) \in M$ is a projection. Now we can uniformly approximate id on $\sigma(x)$ by using simple functions. By BFC, x is uniformly approximated by linear combinations of projections. ☐

Remark. There exist C^*-algebras without nontrivial projections. For example, for a compact Hausdorff connected X, the algebra $C(X)$ only has trivial projections 0 and 1. There exist noncommutative examples, too.

6.3 Commutative von Neumann algebras

Definition 6.13. Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a subalgebra. Vector $\alpha \in \mathcal{H}$ is:
(1.) cyclic if A if $A\alpha$ is dense in $\mathcal{H}$.
(2.) separating for A if $x\alpha = 0$ for $x \in A$ implies $x = 0$.

Proposition 6.14. *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a subalgebra.*

(1.) If $\alpha \in \mathcal{H}$ is cyclic for A, then it is separating for A'.

*(2.) Assume A is a *-subalgebra. Then if α is separating for A', it is cyclic for A.*

(3.) Suppose $W \subseteq \mathcal{B}(\mathcal{H})$ is a vNa. Then α is cyclic for M iff it is separating for M' and separating for M iff it is cyclic for M'.

Proof. (1.) Let α be cyclic for A. Let $y \in A'$ satisfy $y\alpha = 0$. Pick any $\beta \in \mathcal{H}$. there exists a sequence $(x_n)_n \subseteq A$ such that $\|x_n\alpha - \beta\| \to 0$. Hence

$$y\beta = \lim_{n\to\infty} yx_n\alpha = \lim_{n\to\infty} x_n(y\alpha) = 0$$

and α is separating for A'.
(2.) Define $\mathcal{K} := (A\alpha)^\perp \leq \mathcal{H}$. Let $p : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. For $x_1, x_2 \in A$ and $\beta \in \mathcal{K}$ we have

$$\langle x_1\beta, x_2\alpha\rangle = \langle \beta, x_1^* x_2\alpha\rangle = 0,$$

so $x_1\beta \in \mathcal{K}$ and $\mathcal{K}$ is an invariant subspace for A. But since A is *-closed, $\mathcal{K}$ is reducing and by lemma 5.30 $p \in A'$. Of course, $\alpha \in A\alpha$ and $p\alpha = 0$. Now we use the fact that α is separating for A' and therefore $p = 0$. This implies $\mathcal{K} = (0)$.
(3.) This follows immediately from $M = M''$ and the previous two points. □

Example 6.15. *Recall $VN(\Gamma) := \lambda(\mathbb{C}[\Gamma])'' \subseteq \mathcal{B}(\ell^2(\Gamma))$. Similarly, we can use the right regular map*

$$\rho : \Gamma \to \mathcal{B}(\ell^2(\Gamma)), \quad g \mapsto (\rho_g : \delta_k \mapsto \delta_{kg^{-1}})$$

to define $VN_{\text{right}}(\Gamma) = \rho(\Gamma)'' \subseteq \mathcal{B}(\ell^2(\Gamma))$. Notice that $\delta_e \in \ell^2(\Gamma)$ is cyclic for $\lambda(\mathbb{C}[\Gamma])$ as well as $\rho(\mathbb{C}[\Gamma])$. This means that it is cyclic for both $VN(\Gamma)$ and $VN_{\text{right}}(\Gamma)$. It's easy to see that $VN(\Gamma)' = VN_{\text{right}}(\Gamma)$, so δ_e is separating for $VN(\Gamma)$ and $VN_{\text{right}}(\Gamma)$.

Corollary 6.16. *If $A \subseteq \mathcal{B}(\mathcal{H})$ is commutative, then each cyclic vector for A is also separating for A.*

Proof. If $\alpha \in \mathcal{H}$ is cyclic for A, then it is separating for A', but since $A \subseteq A'$ it is also separating for A. □

Theorem 6.17 (Classification of commutative vNa's)**.**

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a commutative vNa with a cyclic vector $\alpha \in \mathcal{H}$. Suppose $A_0 \subseteq A$ is a C^*-algebra that is SOT-dense. Then there exists a finite regular positive Borel measure μ on $\sigma(A_0)$ and an isomorphism

$$\widetilde{\Gamma} : A \to L^\infty(\sigma(A_0), \mu) \subseteq \mathcal{B}(L^2(\sigma(A_0), \mu))$$

that extends the Gelfand transform $\Gamma : A_0 \to C(\sigma(A_0))$. Furthermore, $\widetilde{\Gamma}$ is spacial, that is induced by conjugation with a unitary $U : \mathcal{H} \to L^2(\sigma(A_0), \mu)$.

Proof. Since A_0 is a commutative C^*-algebra, the Gelfand transform $\Gamma : A_0 \to C(\sigma(A_0))$ is an isometric *-isomorphism. Define $\varphi_0 : A \to \mathbb{C}$ by $x \mapsto \langle x\alpha_0, \alpha_0 \rangle$. Then $\varphi_0 \Gamma^{-1} : C(\sigma(A_0)) \to \mathbb{C}$ is a bounded linear functional, so by Riesz-Markoff there exists a regular Borel measure μ on $\sigma(A_0)$ such that

$$\varphi_0 \Gamma^{-1}(f) = \int_{\sigma(A_0)} f \, d\mu.$$

For every positive function $f \in C(\sigma(A_0))$ we have

$$\begin{aligned}\int_{\sigma(A_0)} f \, d\mu &= \int \sqrt{f}^2 \, d\mu = \varphi_0 \Gamma^{-1}(\sqrt{f}^2) = \langle \Gamma^{-1}(\sqrt{f}^2)\alpha_0, \alpha_0 \rangle \\ &= \langle \Gamma^{-1}(\sqrt{f})^2 \alpha_0, \alpha_0 \rangle = \langle \Gamma^{-1}(\sqrt{f})\alpha_0, \Gamma^{-1}(\sqrt{f})\alpha_0 \rangle \\ &= \|\Gamma^{-1}(\sqrt{f})\alpha_0\|^2 \geq 0\end{aligned}$$

and μ is a positive measure. Furthermore, μ is finite, since

$$\mu(\sigma(A_0)) = \varphi(1) = \|\alpha_0\|^2 < \infty.$$

Now we prove that $\operatorname{supp} \mu = \sigma(A_0)$. If $\operatorname{supp} \mu \subsetneq \sigma(A_0)$, then there exists $\emptyset \neq S^{\text{open}} \subseteq \sigma(A_0)$ with $\mu(S) = 0$. Consider a nonnegative $f \in C(\sigma(A_0)) \setminus (0)$ with $f\big|_{S^c} = 0$. Then

$$\|\Gamma^{-1}(\sqrt{f})\alpha_0\|^2 = \int_{\sigma(A_0)} f \, d\mu = \int_S f \, d\mu = 0.$$

We get $\Gamma^{-1}(\sqrt{f})\alpha_0 = 0$, which by cyclicity of α_0 implies $\Gamma^{-1}(\sqrt{f}) = 0$, $\sqrt{f} = 0$ and $f = 0$, a contradiction. Define

$$U_0 : A_0\alpha_0 \to C(\sigma(A_0)) \subseteq L^2(\sigma(A_0), \mu), \quad x\alpha_0 \mapsto \Gamma(x).$$

Since α_0 is separating for A_0, this U_0 is a well-defined linear map. For $x, y \in A_0$, we have

$$\begin{aligned}\langle U_0(x\alpha_0), U_0(y\alpha_0) \rangle &= \langle \Gamma(x), \Gamma(y) \rangle_2 \\ &= \int_{\sigma(A_0)} \overline{\Gamma(y)}\Gamma(x) \, d\mu \\ &= \int_{\sigma(A_0)} \Gamma(y^* x) \, d\mu \\ &= \varphi(y^* x) = \langle y^* x\alpha_0, \alpha_0 \rangle = \langle x\alpha_0, y\alpha_0 \rangle\end{aligned}$$

and so U_0 is an isometry! Since α_0 is cyclic for A and A_0 is SOT-dense in A, α_0 is cyclic for A_0. So $A_0\alpha_0$ is dense in $\mathcal{H}$ and the image of U_0 is the entire $C(\sigma(A_0))$. By continuity, U_0 extends to a surjective isometry

$$U : \mathcal{H} \to L^2(\sigma(A_0), \mu) = \overline{C(\sigma(A_0), \mu)}^{\langle \cdot, \cdot \rangle_2},$$

where U is unitary. Next, define

$$\widetilde{\Gamma} : A \to \mathcal{B}(L^2(\sigma(A_0), \mu)), \quad x \mapsto UxU^*.$$

We claim that $\widetilde{\Gamma}$ is an isometric *-homomorphism. Since U is unitary, the isometric part is obvious and the homomorphism soon follows. Now we claim that $\widetilde{\Gamma}(A) = M(L^\infty(\sigma(A_0), \mu))$. For $x \in A_0$ and $g \in C(\sigma(A_0))$, we have

$$\begin{aligned}\widetilde{\Gamma}(x)g &= UxU^*g = UxU^{-1}(\Gamma(\Gamma^{-1}(g))) \\ &= Ux(\Gamma^{-1}(g)\alpha_0) = \Gamma(x\Gamma^{-1}(g)) \\ &= \Gamma(x)g = M_{\Gamma(x)}g\end{aligned}$$

and since $C(\sigma(A_0))$ is dense in $L^2(\sigma(A_0), \mu)$, we get $\widetilde{\Gamma}(x) = M_{\Gamma(x)}$. It follows that

$$\widetilde{\Gamma}(A_0) = M(C(\sigma(A_0))) \subseteq M(L^\infty(\sigma(A_0), \mu)).$$

Then we use the fact that $\widetilde{\Gamma}$ is SOT-continuous (by definition) and $M(L^\infty)$ is a vNa (example 5.13) to get

$$\widetilde{\Gamma}(A) = \widetilde{\Gamma}(\overline{A_0}^{\text{SOT}}) \subseteq \overline{\widetilde{\Gamma}(A_0)}^{\text{SOT}} \subseteq \overline{M(L^\infty(\sigma(A_0), \mu))}^{\text{SOT}} = M(L^\infty(\sigma(A_0), \mu)).$$

The reverse inclusion is done by nets. Suppose $(\widetilde{\Gamma}(x_i))_i \subseteq \widetilde{\Gamma}(A_0)$ WOT-converges to $T \in B(L^2(\sigma(A_0), \mu))$. Then for all $\beta\mu \in \mathcal{H}$ we have

$$\begin{aligned}\langle TU\beta, U\mu\rangle &= \lim_i \langle \widetilde{\Gamma}(x_i)U\beta, U\mu\rangle \\ &= \lim_i \langle Ux_iU^*U\beta, U\mu\rangle \\ &= \lim_i \langle x_i\beta, \mu\rangle\end{aligned}$$

and $(x_i)_i \xrightarrow{\text{WOT}} U^*TU \in \mathcal{B}(\mathcal{H})$. Since $\overline{A_0}^{\text{WOT}} = A$, we get $x = U^*TU \in A$ and $\widetilde{\Gamma}(x) = T$, so $\overline{\widetilde{\Gamma}(A_0)}^{\text{WOT}} \subseteq \widetilde{\Gamma}(A)$. Finally, we ask what is $\overline{M(C(\sigma(A_0)))}^{\text{WOT}}$? WOT on that set is generated by seminorms

$$|\langle M_g\alpha, \beta\rangle| = \left| \int_{\sigma(A_0)} g\alpha\beta \, d\mu \right|,$$

where $g \in C(\sigma(A_0))$ and $\alpha, \beta \in L^2(\sigma(A_0), \mu)$. Accordingly, weak-* topology on $L^\infty(\sigma(A_0), \mu)$ is generated by seminorms

$$\left| \int_{\sigma(A_0)} g\gamma \, d\mu \right|,$$

where $\mu \in L^1(\sigma(A_0), \mu)$ (this is because for a σ-finite X, $(L^1(X))^* = L^\infty(X)$). By Hölder, L^1 function is a product of two L^2 functions, so these two topologies coincide. By Goldstine's theorem, $C(\sigma(A_0))$ are weak-* dense in $L^\infty(\sigma(A_0), \mu)$. Then we have

$$M(L^\infty(\sigma(A_0), \mu)) = \overline{M(C(\sigma(A_0)))}^{\mathrm{WOT}} = \overline{\widetilde{\Gamma}(A_0)}^{\mathrm{WOT}} \subseteq \widetilde{\Gamma}(A)$$

and finally $\widetilde{\Gamma}(A) = M(L^\infty(\sigma(A_0), \mu))$. ☐

Remark. Applying this theorem to $A_0 = A$ we get

$$M(L^\infty(\sigma(A), \mu)) = \widetilde{\Gamma}(A) = M(C(\sigma(A))).$$

The following statement from the above proof is important in its own right.

Lemma 6.18. *$C(\sigma(A_0))$ are weak-* dense in $L^\infty(\sigma(A_0), \mu)$.*

Proof. For any bounded measurable function $f \in B(\sigma(A_0))$, there exists a net $(f_i)_i \subseteq C(\sigma(A_0))$ such that $f_i \xrightarrow{\text{weak}-*} f$ by Goldstine (see the proof for theorem 6.8). ☐

In the proof, we used the following theorem.

Theorem 6.19.

Suppose $1 \leq p < \infty$ and μ is a σ-finite positive measure on X, and Φ is a bounded linear functional on $L^p(X, \mu)$. Then there is a unique $g \in L^q(X, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\Phi(f) = \int_X fg \, d\mu.$$

Moreover, $\|\Phi\| = \|g\|_q$.

The theorem tells us that under these conditions, $L^q(X, \mu)$ is isometrically isomorphic to the dual space of $L^q(X, \mu)$. In particular, we used the fact that $(L^1(X, \mu))^* = L^\infty(X, \mu)$. This is theorem 6.16 in W.Rudin's *Real and complex analysis.*

How crucial is the cyclicity assumption? Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a commutative vNa. Pick $0 \neq \alpha \in \mathcal{H}$. Define $\mathcal{K} := \overline{A\alpha}$ and let $p : \mathcal{H} \to \mathcal{K}$ be an orthogonal projection, so by reducibility of $\mathcal{K}$ we get $p \in A'$. Therefore $pAp = Ap \subseteq \mathcal{B}(\mathcal{H})$ is a commutative vNa with a cyclic vector $\alpha \in \mathcal{K}$. Then, again by theorem, $Ap \cong L^\infty(X, \mu)$ for some (X, μ). By Zorn's lemma, $A \cong L^\infty(Y, \nu)$ for some disjoint union of measure spaces (Y, ν).

Proposition 6.20. *Let $\mathcal{H}$ be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ a commutative vNa. Then there exists a separating vector for A.*

Proof. By Zorn, there exists a maximal set of unit vectors $(\alpha_k)_k$ such that $A\alpha_k \perp A\alpha_l$ for $k \neq l$. By maximality, $\sum_k A\alpha_k$ is dense in $\mathcal{H}$. Define $\alpha = \sum_{n=1}^\infty \frac{1}{2^n}\alpha_n$. We claim that

α is separating for A. Indeed, let $x \in A$ such that $x\alpha = 0$. Then $\sum_{n=1}^{\infty} \frac{1}{2^n} x\alpha_n = 0$. By orthogonality, $x\alpha_n = 0$ for all indices n. For all $y \in A$, we get $xy\alpha_n = yx\alpha_n = 0$, so $x\big|_{A\alpha_n} = 0$ for all n. But since $\sum_n A\alpha_n$ is dense in A, we get $x = 0$. $\square$

Corollary 6.21. *Let $\mathcal{H}$ be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ is a maximal commutative vNa. Then there exists a cyclic vector for A.*

Proof. By the proposition, there exists a separating vector α for A, which is then cyclic for A'. But since A is maximal, we get $A = A'$. $\square$

Theorem 6.22.

Let $\mathcal{H}$ be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ a commutative vNa. Then there exists a compact Hausdorff space X and a finite regular Borel measure μ on X such that $A \cong L^\infty(X, \mu)$.

Proof. By proposition, there exists a separating vector $\alpha \in \mathcal{H}$ for A. Form $\mathcal{K} := \overline{A\alpha}$. Then the algebra $\{x\big|_{\mathcal{K}} \mid x \in A\} \subseteq \mathcal{B}(\mathcal{K})$ is *-isomorphic to A, has cyclic vector α and the above theorem applies. $\square$

Example 6.23. *Let $\Gamma = \mathbb{Z}/n\mathbb{Z}$. Then the spectrum is $\sigma(VN(\Gamma)) = \{e^{\frac{2k\pi i}{n}} \mid 0 \leq k < n\}$ and $\mu(e^{\frac{2k\pi i}{n}}) = \frac{1}{n}$. Then*

$$VN(\Gamma) = L^\infty(\sigma(VN(\Gamma)), \mu) \cong \mathbb{C}^n$$

as an algebra. The generator for $VN(\Gamma)$ is the matrix

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}.$$

Example 6.24. *Let $\Gamma = \mathbb{Z}$. Then $\mathbb{T}$ is the Pontryagin dual of Γ, so $C^*(\Gamma) = C(\mathbb{T})$ and $VN(\Gamma) = L^\infty(\mathbb{T}, m)$, where m is the normalized Lebesgue measure.*