# FUNCTIONAL ANALYSIS - NOTES

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## Contents

# 1 Convexity

## 1.1 Locally convex spaces

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ be a field.

**Definition 1.1.** A *topological vector space* (TVS) is a $\mathbb{F}$-vector space that is also a topological space and the two structures are compatible. This means that the usual operations on vector spaces

$$V \times V \to v,\ (x, y) \mapsto x + y, \qquad \mathbb{F} \times V \to V,\ (\lambda, x) \mapsto \lambda x$$

are continuous maps.

**Example 1.2.** *Normed spaces are TVS.*

**Definition 1.3.** Let $V$ be a $\mathbb{F}$-space. Map $p : V \to \mathbb{R}$ is a *seminorm* if:
(1.) $p(x) \geq 0,\ \forall x \in V$ (positivity);
(2.) $p(\lambda x) = |\lambda| p(x),\ \forall x \in V,\ \forall \lambda \in \mathbb{F}$ (positive homogeneity);
(3.) $p(x + y) \leq p(x) + q(x),\ \forall x, y \in \mathbb{F}$ (triangle inequality).
A seminorm is therefore almost a norm, except that it's not necessarily positive definite.

Let $V$ be a $\mathbb{F}$-vector space and $\mathcal{P}$ a family of seminorms in $V$. Let $\mathcal{T}$ be the topology in $V$ with the following subbasis:

$$U(x_0, p, \varepsilon) = \{x \in V \mid p(x - x_0) < \varepsilon\};\ x_0 \in V,\ p \in \mathcal{P},\ \varepsilon > 0.$$

Basis of $\mathcal{T}$ are finite intersections of such sets. The set $U \subseteq V$ is open iff for every $x_0 \in U$ there exist seminorms $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$U \supset \bigcap_{j=1}^{n} U(x_0, p_j, \varepsilon_j).$$

The space $(V, \mathcal{T})$ is then a TVS. If $\mathcal{P}$ is a singleton and its element is a norm, then $(V, \mathcal{T})$ is a normed space.

**Definition 1.4.** A TVS $X$ is a *locally-convex space* (LCS) if its topology is generated by a family of seminorms $\mathcal{P}$ satisfying

$$\bigcap_{p \in \mathcal{P}} \{x \in X \mid p(x) = 0\} = \{0\}.$$

Equivalently, for every $x \in X \setminus \{0\}$ there exists a seminorm $p \in \mathcal{P}$ such that $p(x) \neq 0$.

**Corollary 1.5.** *Let $X$ be a space with a topology generated by a family of seminorms $\mathcal{P}$. Then $X$ is a LCS iff it is Hausdorff.*

*Proof.* Start with ($\Rightarrow$). Let $x, y \in X$ be two distinct points. There exists a seminorm $p \in \mathcal{P}$ such that $p(x-y) = b \neq 0$. Define the sets

$$V = U\left(x, p, \frac{b}{2}\right), \quad W = U\left(y, p, \frac{b}{2}\right).$$

By the triangle inequality property of a seminorm, $V$ and $W$ separate the points $x, y$. Now the converse ($\Leftarrow$). Choose a point $X \ni x \neq 0$. Then there exist open sets $0 \in V, x \in W$ that separate 0 from $x$. There exists an open basis set $\bigcap_{j=1}^{n} U(0, p_j, \varepsilon_j) \subseteq V$, so $x \notin U(0, p_j, \varepsilon_j)$ for some index $j$. Hence, $p_j(x-0) = p_j(x) \geq \varepsilon > 0$. $\square$

LCS generally aren't first-countable, so we need to go beyond the usual sequences to describe the topology.

**Definition 1.6.** Partially ordered set $(I, \leq)$ is *upwards-directed* if

$$\forall i', i'' \in I: \ \exists i \in I: \ i \geq i', i \geq i''.$$

**Example 1.7.** *(1.) Every linearly ordered set is upwards-directed.*

*(2.) Let $(X, \mathcal{T})$ be a topological space and $x_0 \in X$. Define a family of sets*

$$\mathcal{U} = \{U^{open} \subseteq X \mid x_0 \in U\}$$

*and a relation $U \geq V \Leftrightarrow U \subseteq V$. Then $(\mathcal{U}, \leq)$ is an upwards-directed set.*

*(3.) Let $S$ be a set and $\mathcal{F}$ a family of all finite subsets of $S$. Define $F_1 \geq F_2$ in $\mathcal{F}$ if $F_1 \supseteq F_2$. Then $(\mathcal{F}, \leq)$ is again an upwards-directed set.*

**Definition 1.8.** A *generalized sequence (net)* is $((I, \leq), x)$, where $(I, \leq)$ is upwards-directed and $x : I \to X$ is a function. We usually write $(x_i)_{i \in I}$ or $(x(i))_{i \in I}$.

**Example 1.9.** *(1.) Every sequence is a net.*

*(2.) Let $(X, \mathcal{T})$ be a topological space, $x_0 \in X$ and $\mathcal{U}$ a collection of all open sets which contain $x_0$ (see example 1.7). For each $U \in \mathcal{U}$ pick a $x_U \in U$. Then $(x_U)_{U \in \mathcal{U}}$ is a net.*

**Definition 1.10.** Let $X$ be a topological space. A net $(x_i)_{i \in I}$ *converges* to an $x \in X$ if

$$\forall U^{\text{open}} \subseteq X, \ x \in U: \ \exists i_0 \in I: \ \forall i \geq i_0: \ x_i \in U.$$

We write $\lim_{i \in I} x_i = x$, or alternatively, $x_i \xrightarrow[i \in I]{} x$. A point $x \in X$ is called a *cluster point* of a net $(x_i)_{i \in I}$ if

$$\forall U^{\text{open}} \subseteq X, \ x \in U: \ \forall i_0 \in I: \ \exists i \geq i_0: \ x_i \in U.$$

**Example 1.11.** *Take the net* $(x_U)_{U\in\mathcal{U}}$ *from example 1.9. It follows from the definition that* $x_U \xrightarrow[U\in\mathcal{U}]{} x_0$.

**Proposition 1.12.** *(1.) Let $X$ be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ iff there exists a net $(a_i)_{i\in I}$ in $A$ such that $a_i \to x$.*

*(2.) Let $X, Y$ be topological spaces and $f : X \to Y$. Then $f$ is continuous at $x_0 \in X$ iff $f(x_i) \to f(x_0)$ for every net $(x_i)_{i\in I}$ that converges to $x_0$.*

*Proof.* (1.) We begin with the implication to the left ($\Leftarrow$). Take any $U^{\text{open}} \subseteq X$ such that $x \in U$. Since $a_i \to x$, there exists an index $i_0 \in I$, such that for every $i \geq i_0$ we have $a_i \in U$. Hence $a_i \in A \cap U \neq \emptyset$ and $x \in \overline{A}$. The converse ($\Rightarrow$) is similar. Define $\mathcal{U} = \{U^{\text{open}} \subseteq X \mid x \in U\}$. Since $x \in \overline{A}$, for each $U \in \mathcal{U}$, we have $A \cap U \neq \emptyset$. Pick $a_U \in A \cap U$. Then the net $(a_U)_{U\in\mathcal{U}}$ in $A$ converges to $x$.

(2.) Start with the implication ($\Rightarrow$). Let $f$ be a continuous function and let $(x_i)_{i\in I}$ converge to $x_0$. Let $f(x_0) \in U^{\text{open}} \subseteq Y$. Then $x_0 \in f^{-1}(U)^{\text{open}} \subseteq X$, which means there exists an $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, $x_i \in f^{-1}(U)$. But that implies that for every $i \geq i_0$, $f(x_i) \in U$, which is what we wanted. Now we prove the converse ($\Leftarrow$). Let's say that for every net $(x_i)_{i\in I}$ that converges to $x_0$, we have $f(x_i) \xrightarrow[i\in I]{} f(x_0)$. So for every set $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$ (using the first item), which proves that $f$ is continuous. ☐

**Proposition 1.13.** *(a) A net $(x_i)_{i\in I}$ in a LCS converges to $x_0$ iff a net $(p(x_i - x_0))_{i\in I}$ converges to $0$ for all $p \in \mathcal{P}$.*

*(b) The topology in a LCS $X$ is the coarsest (smallest) topology in which all the maps $x \mapsto p(x - x_0)$ are continuous for every $x_0 \in X$ and $p \in \mathcal{P}$.*

*Proof.* (a) Start with the implication ($\Rightarrow$). Take any $p \in \mathcal{P}$. If we take $U = U(x_0, p, \varepsilon)$ in the definition of a limit of a net, we get

$$\forall \varepsilon > 0 : \ \exists i_0 \in I : \ \forall i \geq i_0 : \ p(x_i - x_0) \in (-\varepsilon, \varepsilon).$$

This proves our claim. Now for the opposite direction ($\Leftarrow$). For every $p \in \mathcal{P}$ and $\varepsilon_p > 0$ there exists an $i_p$ such that for every $i \geq i_p$, $x_i \in U(x_0, p, \varepsilon_p)$. Now let $U$ be an arbitrary basis set that includes the point $x_0$. That means $U$ is the finite intersection of the sets $U(x_0, p, \varepsilon_p)$. Now let $i_0$ be greater than all indices $i_p$. By our assumption, for every $i \geq i_0$ we have $x_i \in U$.

(b) Pick any point $x_0 \in X$ and a seminorm $p \in \mathcal{P}$. Denote

$$f_{x_0,p} : X \to \mathbb{R}, \quad f_{x_0,p}(x) = p(x - x_0).$$

We essentially have to prove that the sets

$$f_{x_0,p}^{-1}(V), \ V^{\text{open}} \subseteq \mathbb{R}, \ x_0 \in X, \ p \in \mathcal{P}$$

generate a subbasis for the seminorm topology of a LCS space. Since $f_{p,x_0}$ are continuous functions (by the first item and proposition 1.12), these are all open sets in the seminorm topology. But on the other hand, all subbasis sets $U(x_0, p, \varepsilon)$ of the seminorm topology are of this type, so the above subbasis generates the seminorm topology, thus concluding our proof. $\square$

**Example 1.14.** *Let $X$ be a topological space. For every $K^{compact} \subseteq X$ we define a seminorm*

$$p_K : C(X) \to \mathbb{R}, \quad f \mapsto \sup_{x \in K} |f(x)|.$$

*We endow $C(X)$ with the topology induced by the family of seminorms $\{p_K \mid K^{compact} \subseteq X\}$. It's trivial to see that $C(X)$ is then a LCS. Moreover, we notice that the induced seminorm topology coincides with the topology of compact convergence on $X$. In the future, we will require $X$ to be locally compact Hausdorff (this implies complete regularity) so that $C(X)$ has nice properties. There are examples of not completely regular spaces $X$ such that the only elements of $C(X)$ are constant maps.*

**Example 1.15.** *Let $D^{open} \subseteq \mathbb{C}$ and let $\mathcal{H}(D)$ be the set of all holomorphic functions on $D$. As in the example 1.14, we define $\mathcal{P} = \{p_K \mid K^{compact} \subseteq D\}$. This endows $\mathcal{H}(D)$ with a topology and makes $\mathcal{H}(D)$ into a LCS. Convergence in this topology concides with the uniform convergence on compacts in $D$.*

## 1.2 Weak topology

Let $X$ be a normed space and let $X^*$ be its dual. For every $f \in X^*$ we define a seminorm

$$p_f : X \to \mathbb{R}, \quad x \mapsto |f(x)|.$$

We claim that $\mathcal{P} = \{p_f \mid f \in X^*\}$ is a family of seminorms that induces a topology on $X$ which makes $X$ a LCS. Indeed, for any $x \in X \setminus \{0\}$ define a nonzero linear functional

$$f : \operatorname{span}(x) \to \mathbb{F}, \quad f(\lambda x) = \lambda$$

and extend it to $F : X \to \mathbb{F}$ using Hahn–Banach. Then $p_F(x) \neq 0$. The induced topology is the *weak topology* on $X$. We denote it as $\sigma(X, X^*)$.

**Proposition 1.16.** *A net $(x_i)_{i \in I}$ converges to $x_0 \in X$ with respect to the weak topology iff $f(x_i) \to f(x_0)$, $\forall f \in X^*$.*

*Remark.* We use the notation $x_i \xrightarrow{w} x_0$. Furthermore, a closure of a set $A \subseteq X$ in the weak topology will be denoted by $\overline{A}^w$.

*Remark.* The closure of a set $A \subseteq X$ in the weak topology will be denoted as $\overline{A}^w$.

**Example 1.17.** *Let $X = \mathbb{R}^n$. Then $X^* = \mathbb{R}^n$ and every linear functional $f$ is of the form $f(x) = \langle x, y \rangle$ for some $y \in X$ (Riesz' representation theorem). The subbasis sets are*

$$U(0, p_y, \varepsilon) = \{x \in \mathbb{R}^n \mid |\langle x, y \rangle| < \varepsilon\}.$$

*Weak topology in this case coincides with Euclidean topology.*

Let $X$ again be a normed space. To $x \in X$ we assign the seminorm

$$p_x : X^* \to \mathbb{R}, \quad f \mapsto |f(x)|.$$

The family $\{p_x \mid x \in X\}$ defines a topology in $X^*$ in which $X^*$ becomes a LCS. This topology is called the *weak-$*$ topology* and is denoted by $\sigma(X^*, X)$.

**Proposition 1.18.** *A net $(f_i)_{i\in I}$ converges to $f \in X^*$ with respect to the weak-$*$ topology iff $f_i(x) - f(x) \to 0, \forall x \in X$.*

*Remark.* We use the notation $f_i \xrightarrow{w^*} f$. Furthermore, a closure of a set $A \subseteq X^*$ in the weak-$*$ topology will be denoted by $\overline{A}^{w^*}$.

We can compare weak-$*$ topology on $X^*$ with its weak topology. As a consequence of Hahn–Banach, we have for every $x \in X$

$$\|x\| = \sup\{|f(x)|| \ f \in X^*, \ \|f\| \leq 1\},$$

which implies that the map

$$\iota : X \hookrightarrow X^{**}, \quad x \mapsto (f \mapsto f(x))$$

is an isometry and therefore injective. This means that every seminorm in the weak-$*$ topology is also a seminorm in a weak topology on $X^*$, so the weak topology is finer (stronger) than the weak-$*$ topology on $X^*$.

*Remark.* Weak and weak-$*$ topology can be defined even if $X$ is merely a LCS. In that case, $X^*$ is of course defined as the space of continuous linear functionals on $X$.

## 1.3 Banach–Alaoglu theorem

**Theorem 1.19** (Banach–Alaoglu).

*Let $X$ be a normed space. Then the closed unit ball in $X^*$ (denoted by $(X^*)_1$) is compact in the weak-$*$ topology in $X^*$.*

*Proof.* To $x \in X$ we assign $D_x = \{z \in \mathbb{F} \mid |z| \leq \|x\|\}$ and endow $D_x$ with the Euclidean topology. Then $D_x$ is clearly compact. The set $P = \prod_{x\in X} D_x$ is compact in the product topology (Tychonoff theorem). Now we construct a map

$$\Phi : (X^*)_1 \to P, \quad f \mapsto (f(x))_{x\in X} \in P.$$

Clearly, $\Phi$ is well-defined and injective. We start by proving that $\Phi$ is continuous. Let $(f_i)_{i\in I}$ be a net in $(X^*)_1$ that converges to $f \in X^*$ in the weak-$*$ topology. Then $f_i(x) \to f(x)$ for each $x \in X$. By the definition of the product topology in $P$, this means that $\Phi(f_i) \mapsto \Phi(f)$ in $P$. Hence $\Phi$ is continuous. Since $\Phi$ is injective, it induces an inverse map

$$\Phi^{-1} : \mathrm{im}(\Phi) \to (X^*)_1$$

that is also continuous (we read the previous argument backwards).
Finally, we prove that $\mathrm{im}(\Phi)$ is closed in $P$. Suppose that $(\Phi(f_i))_{i\in I}$ converges to $p = (p_x)_{x\in X} \in P$. By definition of the product topology, this means that $f_i(x) \to p_x$ for all

$x \in X$. Define

$$f : X \to \mathbb{F}, \quad x \mapsto p_x.$$

Then $f$ is linear and $f \in (X^*)_1$. Thus $p = \Phi(f) \in \operatorname{im}(\Phi)$. This in turn implies that $(\operatorname{im}\Phi)^{\text{closed}} \subseteq P^{\text{compact}}$. But we know that $(X^*)_1 \approx \operatorname{im}(\Phi)$, which implies that $(X^*)_1$ is also compact. $\square$

**Corollary 1.20.** *Every Banach space $X$ is isometrically isomorphic to a closed subspace of $C(K)$ for some compact $T_2$ space $K$.*

*Proof.* Denote $K = (X^*)_1$ endowed with the weak-$*$ topology. By the Banach–Alaoglu theorem, $K$ is compact and $T_2$. We now define the map

$$\Delta : X \to C(K), \quad x \mapsto (f \mapsto f(x)).$$

First, we prove that $\Delta$ is isometric. By Hahn–Banach, for every $x \in X \setminus \{0\}$ there exists an $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Then we have

$$\|\Delta(x)\|_\infty = \sup_{g \in K} |g(x)| = \|x\|.$$

Since $\Delta$ is an isometry, its image is complete and thus closed in $C(K)$. Obviously $\Delta$ is a linear map, so we are done. $\square$

## 1.4 Minkowski gauge

ADD MOTIVATION

**Definition 1.21.** Let $X$ be a $\mathbb{F}$-vector space. A set $A \subseteq X$ is

- balanced if:
$$\forall x \in A : \ \forall \alpha \in \mathbb{F}, \ |\alpha| \leq 1 : \ \alpha x \in A.$$
- absorbing if:
$$\forall x \in X : \ \exists \varepsilon > 0 : \ \forall t \in (0, \varepsilon) : \ tx \in A.$$
- absorbing in $a \in A$ if $A - a = \{x - a \mid x \in A\}$ is absorbing.

**Example 1.22.** *Let $X$ be a vector space and $p$ a seminorm in $X$. Then*

$$V = \{x \in X \mid p(x) < 1\}$$

*is convex, balanced, absorbing in each of its points.*

**Theorem 1.23.**

*Let $X$ be a vector space and $V \subseteq X$ convex, balanced and absorbing in each of its points. Then there exists a unique seminorm $p$ on $X$ such that*

$$V = \{x \in X \mid p(x) < 1\}.$$

*Proof.* To $V$ we associate the Minkowski gauge:

$$p(x) = \inf\{t \geq 0 \mid x \in t \cdot V\},$$

where $t \cdot V = \{t \cdot v \mid v \in V\}$. First we prove that $p$ is well defined. Since $V$ is absorbing, we have $X = \bigcup_{n \in \mathbb{N}} n \cdot V$, so for every $x \in X$ the set $\{t \geq 0 \mid x \in t \cdot V\}$ is nonempty. It's also clear to see that $p(0) = 0$. Next we check for homogeneity. Suppose $\alpha \neq 0$. Then

$$\begin{aligned}
p(\alpha x) &= \inf\{t \geq 0 \mid \alpha x \in t \cdot V\} \\
&= \inf\left\{t \geq 0 \mid x \in \frac{t}{\alpha} \cdot V\right\} \\
&= \inf\left\{t \geq 0 \mid x \in \frac{t}{|\alpha|} \cdot V\right\} \\
&= \inf|\alpha|\left\{\frac{t}{|\alpha|} \geq 0 \mid x \in \frac{t}{|\alpha|} \cdot V\right\} \\
&= |\alpha| p(x).
\end{aligned}$$

Now we do the same for triangle inequality: let $\alpha, \beta \geq 0$ so that $\alpha + \beta > 0$. Let $a, b \in V$. Then

$$\alpha a + \beta b = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b\right) \in (\alpha + \beta) \cdot V.$$

This means that $\alpha \cdot V + \beta \cdot V \subseteq (\alpha + \beta) \cdot V$. Now let $x, y \in X$ and $p(x) = \alpha, p(y) = \beta$. Take $\delta > 0$. Then $x \in (\alpha + \delta) \cdot V, y \in (\beta + \delta) \cdot V$. Hence

$$x + y \in (\alpha + \delta) \cdot V + (\beta + \delta) \cdot V \subseteq (\alpha + \beta + 2\delta) \cdot V,$$

and by definition, $p(x + y) \leq \alpha + \beta + 2\delta$. Since $\delta > 0$ was arbitrary, we have $p(x + y) \leq \alpha + \beta = p(x) + p(y)$. Now that we have proved that $p$ is a seminorm, we can show that

$$V = \{x \in X \mid p(x) < 1\}.$$

The inclusion ($\supseteq$) is easy: if $p(x) < 1$, then $x \in (p(x) + \varepsilon) \cdot V$ for all $\varepsilon > 0$. By choosing $\varepsilon = 1 - p(x) > 0$, we get $x \in V$. Now we prove the other inclusion ($\subseteq$). Let $x \in V$. Since $V$ is absorbing in $x$, there exists an $\varepsilon > 0$ such that $y = x + tx \in V$ for all $t \in (0, \varepsilon)$. This means that $x = \frac{1}{t+1} y$, where $y \in V$. This implies that

$$p(x) = \frac{1}{t+1} p(y) \leq \frac{1}{1+t} \leq 1,$$

which proves the equality. Lastly, we prove the $p$ is unique. Suppose there is some other seminorm $q$ such that

$$\{x \in X \mid p(x) < 1\} = \{x \in X \mid q(x) < 1\}.$$

Suppose $p \neq q$. W.l.o.g. there exists an $x \neq 0$ such that $p(x) > q(x)$. By homogeneity, we can assume that $p(x) = 1 > q(x)$, contradicting our assumption. $\square$

*Remark.* If $X$ is a TVS and $V$ is an open subset, then $V$ is absorbing at each of its points.

**Corollary 1.24.** *Let $X$ be a TVS and $\mathcal{U}$ a collection of all open convex balanced subsets of $X$. Then $X$ is locally convex iff $\mathcal{U}$ is a basis for the neighborhood system at $0$.*

## 1.5 Applications of Hahn–Banach

Recall: if $X$ is a $\mathbb{R}$-vector space then $p : X \to \mathbb{R}$ is a sublinear functional if

$$p(x + y) \leq p(x) + p(y), \ \forall x, y \in X$$

and

$$p(\alpha x) = \alpha x, \ \forall x \in X, \ \alpha > 0.$$

**Theorem 1.25** (Hahn–Banach theorem)**.**

$\mathbb{R}$: *Suppose $X$ is a $\mathbb{R}$-vector space and $p : X \to \mathbb{R}$ is a sublinear functional. Given a linear functional $f$ on $Y \leq X$ such that $f(y) \leq p(y)$ for every $y \in Y$, $f$ extends to a linear functional $F : X \to \mathbb{R}$ such that $F(x) \leq p(x)$ for every $x \in X$.*

$\mathbb{C}$: *Suppose $X$ is a $\mathbb{C}$-vector space and $p : X \to \mathbb{R}$ is a seminorm. Given a linear functional $f$ on $Y \leq X$ such that $|f(y)| \leq p(y)$ for every $y \in Y$, $f$ extends to a linear functional $F : X \to \mathbb{R}$ such that $|F(x)| \leq p(x)$ for every $x \in X$.*

**Corollary 1.26** (Hahn–Banach extension theorem)**.** *Let $X$ be a normed space, $f \in X^*$ and $Y \leq X$. Then there exists an $F \in X^*$ such that $F\big|_Y = f$ and $\|F\| = \|f\|$.*

**Corollary 1.27** (Hahn–Banach separation theorem)**.** *Suppose $X$ is a LCS and $A, B \subseteq X$ are disjoint closed convex sets. If $B$ is compact then there exists an $f \in X^*$ that seperates $A$ from $B$:*

$$\exists \alpha, \beta \in \mathbb{R} : \ \forall a, b \in B : \ \operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(b).$$

**Theorem 1.28.**

*Let $X$ be a LCS and $A \subseteq X$ convex. Then $\overline{A} = \overline{A}^w$.*

*Proof.* Since the weak topology is weaker than the original topology, we have $\overline{A} \subseteq \overline{A}^w$. Let $x \notin \overline{A}$. We now separate $\overline{A}$ and the compact set $\{x\}$: there exists $f \in X^*$ so that there exist $\alpha, \beta \in \mathbb{R}$ and we have

$$\operatorname{Re} f(a) \leq \alpha < \beta \leq \operatorname{Re} f(x)$$

for all $a \in \overline{A}$. This means that

$$\overline{A} \subseteq \{y \in X \mid \operatorname{Re} f(y) \leq \alpha\} = (\operatorname{Re} f)^{-1}(-\infty, \alpha] = C.$$

Since $C$ is closed in the weak topology, it follows from $A \subseteq C$ that $\overline{A}^w \subseteq \overline{C}^w = C$. Since $x \notin C$, we have $x \notin \overline{A}^w$. ☐

**Corollary 1.29.** *A convex set in a LCS is closed iff it is weakly closed.*

**Proposition 1.30.** *Let $X$ be a TVS and $f : X \to \mathbb{F}$ a linear functional. The following are equivalent:*

*(1.) $f$ is continuous;*

*(2.) $f$ is continuous in $0$;*

*(3.) $f$ is continuous in some point;*

*(4.) $\ker f$ is closed;*

*(5.) $x \mapsto |f(x)|$ is a seminorm.*

*If $X$ is a LCS, then these are also equivalent to*

*(6.) $\exists \alpha_1, \dots, \alpha_n \in \mathbb{R}_{>0}$ and $\exists p_1, \dots, p_n \in \mathcal{P}$ such that*

$$|f(x)| \leq \sum_{k=1}^{n} \alpha_k p_k(x), \ \forall x \in X.$$

*Proof.* Equivalence of the first five statements is routine. Assume that $X$ is a LCS. We prove the equivalence of (2) and (6). We start with (6) $\Rightarrow$ (2). Let $(x_i)_{i \in I}$ be a net in $X$ that converges to 0. Then we have

$$0 \leq |f(x_i)| \leq \sum_{k=1}^{n} \alpha_k p_k(x_i) \xrightarrow[i \in I]{} 0.$$

This implies that $f(x_i) \xrightarrow[i \in I]{} 0$, proving the implication. Now the opposite: (2) $\Rightarrow$ (6). We know that $f^{-1}(B_1^\circ(0)) = \{x \in X \mid |f(x)| < 1\}$ is an open neighborhood of 0 in $X$. Then there exist $p_1, \dots, p_r \in \mathcal{P}$ and an $\varepsilon > 0$ such that

$$0 \in \bigcap_{i=1}^{r} U(0, p_i, \varepsilon) \subseteq f^{-1}(B_1^\circ(0)).$$

If $p_i(x) < \varepsilon$ for all $i \leq r$, then $|f(x)| < 1$. Pick any $\delta > 0$. Then

$$p_i \left( x \cdot \frac{\varepsilon}{\sum p_i(x) + \delta} \right) = \frac{\varepsilon}{\delta + \sum p_i(x)} \cdot p_i(x) < \varepsilon,$$

which implies

$$\left| f \left( x \cdot \frac{\varepsilon}{\sum p_i(x) + \delta} \right) \right| < 1.$$

From this we get $|f(x)| < \frac{1}{\varepsilon} \left( \sum p_i(x) + \delta \right)$. Since $\delta > 0$ was arbitrary, we get

$$|f(x)| \leq \sum_{i=1}^{r} \frac{1}{\varepsilon} p_i(x).$$ □

Recall the following theorem from measure theory (theorem 2.14 in [4]).

**Theorem 1.31** (Riesz–Markoff)**.**

*Let $X$ be a compact $T_2$ space, $\Phi \in C(X)^*$. Then there exists a unique regular complex Borel measure $\mu$ such that*

$$\Phi(f) = \int_X f \, d\mu, \ \forall f \in C(X).$$

*Further, $\|\Phi\| = \|\mu\| = |\mu|(X)$.*

*Remark.* The above also works if $X$ is locally compact and $\Phi \in C_0(X)^*$.

As a corollary, we get the following proposition.

**Proposition 1.32.** *Let $X$ be completely regular. Endow $C(X)$ with a topology induced by its seminorms. If $L \in C(X)^*$ then there exists a compact $K \subseteq X$ and a regular Borel measure on $K$ such that*

$$L(f) = \int_K f \, d\mu, \ \forall f \in C(X).$$

*Conversely, every such pair $(K, \mu)$ defines $L \in C(K)^*$ with the above equation.*

*Proof.* Begin with the implication ($\Leftarrow$). Given $(K, \mu)$, we just need to prove that the induced functional $L$ is continuous on $X$. We have

$$|L(f)| = \left| \int_K f \, d\mu \right| \leq \|\mu\| \sup_K |f| = \|\mu\| p_K(f)$$

and $L$ is continuous. Now the converse ($\Rightarrow$). Let $L \in C(X)$. By the previous proposition, there exist compact sets $K_1, \dots, K_p \subseteq X$ and $\alpha_1, \dots, \alpha_p > 0$ such that

$$|L(f)| \leq \sum_{j=1}^{p} \alpha_j p_{K_j}(f).$$

Let $K = \bigcup_{j=1}^p K_j$ and $\alpha = \max\{\alpha_1, \dots, \alpha_p\}$. Then $\|f\| \leq \alpha p_K(f)$ for all $f \in C(X)$. Observe that if $f \in C(X)$ and $f\big|_K = 0$, then $L(f) = 0$. We now define a map $F : C(K) \to \mathbb{F}$. Since $X$ is completely regular, we have a Tietze-like extension theorem: for any compact $K \subseteq X$ and a continuous function $g \in C(K)$, there exists an extension $\widetilde{g} \in C(X)$. Define $F(g) := L(\widetilde{g})$. First we need to check that $F$ is well defined. Suppose we have two extensions $\widetilde{g}$ and $\widetilde{\widetilde{g}}$ of $g \in C(K)$. Since $\widetilde{g} - \widetilde{\widetilde{g}}$ is evidently zero on $K$, we have

$$L(\widetilde{g}) - L(\widetilde{\widetilde{g}}) = L(\widetilde{g} - \widetilde{\widetilde{g}}) = 0$$

and $F$ really is well defined. It is also clearly linear, so we just need to check continuity:

$$|F(g)| = |L(\widetilde{g})| \leq \alpha \cdot p_K(\widetilde{g}) = \alpha \cdot \|g\|_{\infty, K},$$

therefore $\|F\| \leq \alpha$ and $F$ is continuous. Lastly we apply Riesz–Markoff: there exists a regular Borel measure $\mu$ on $K$ so that $F(g) = \int_K g \, d\mu$. If $f \in C(X)$, then $g := f\big|_K \in C(K)$ and we have

$$L(f) = F(g) = \int_K g \, d\mu = \int_K f \, d\mu.$$

□

## 1.6 Krein–Milman theorem

**Definition 1.33.** Let $X$ be a vector space and $C \subseteq X$ a convex subset.

(a) A nonempty convex subset $F \subseteq C$ is a *face* if for any $x, y \in C$ we have

$$(\exists t \in (0,1):\ tx + (1-t)y \in F) \Rightarrow x, y \in F.$$

(b) A point $x \in C$ is a called an *extreme point* if $\{x\} \subseteq C$ is a face.

We use the notation $\operatorname{ext}(C)$ for the set of all extreme points of $C$.

**Example 1.34.** *If we consider spaces of real sequences, we have*
- $\operatorname{ext}((\ell^\infty)_1) = \{(\pm 1, \pm 1, \dots)\}$*;*
- $\operatorname{ext}((\ell^1)_1) = \{(0, 0, \dots, \pm 1, \dots)\}$.

**Example 1.35.** *We prove that for $c_0$ (the space of complex sequences that converge to $0$) we have $\operatorname{ext}(c_0)_1 = \emptyset$. Indeed, let $x = (x_n)_n \in (c_0)_1$. Since $\lim_n x_n = 0$, there exists $N \in \mathbb{N}$ such that $|x_n| < \frac{1}{2}$ for all $n > N$. Now define $y, z \in c_0$ by setting $y_n = z_n = x_n$ for $n \leq N$ and*

$$y_n = x_n + \frac{1}{2^n}, \quad z_n = x_n - \frac{1}{2^n}$$

*for $n > N$. Then $y, z \in (c_0)_1$ and $x = \frac{1}{2}(y+z)$, so $x \notin \operatorname{ext}(c_0)_1$.*

**Example 1.36.** *Let us show that $\operatorname{ext}(L^1[0,1])_1 = \emptyset$. Take any $f \in (L^1[0,1])_1$. Then $\int_0^1 |f(t)|\,dt = 1$, so there must exist an $x \in [0,1]$ such that $\int_0^x |f(t)|\,dt = 1/2$. Now define $g := 2 \cdot f \cdot \chi_{[0,x]}$ and $h := 2 \cdot f \cdot \chi_{[x,1]}$. Now we have $g, h \in (L^1[0,1])_1$ and $f = \frac{1}{2}g + \frac{1}{2}h$, so $f$ cannot be an extreme point.*

**Example 1.37.** *Finally, let us prove that $\operatorname{ext}(C[0,1])_1 = \{\pm 1\}$ for real valued functions. Take any $f \in (C[0,1])_1$. Then define functions $g(t) = \min\{2f(t)+1, 1\}$ and $h(t) = \max\{2f(t)-1, -1\}$. Clearly $g, h \in (C[0,1])_1$ and $f = \frac{1}{2}g + \frac{1}{2}h$. If $f$ is an extreme point, then $g = h$, which happens only if $f = \pm 1$.*

**Definition 1.38.** For a vector space $X$ and $A \subseteq X$, define a *convex hull* $\operatorname{co} A$ as the intersection of all convex sets in $X$ that contain $A$. If $X$ is a TVS, then define a *closed convex hull* $\overline{\operatorname{co}}A$ as the intersection of all closed convex sets that contain $A$.

Convex hull of a set $A$ can be given explicitly:

$$\operatorname{co} A = \left\{ \sum_{i=0}^{n} \alpha_i x_i \mid n \in \mathbb{N}, \alpha_i \geq 0, \sum_{i=0}^{n} \alpha_i = 1, x_i \in A \right\}.$$

If $X$ is a TVS, then $\overline{\operatorname{co}}A = \overline{\operatorname{co} A}$.

**Lemma 1.39.** *If $C \subseteq X$ is a convex subset of a vector space and $a \in C$, then the following are equivalent.*

*(a)* $a \in \operatorname{ext} C$.

*(b) If* $x_1, x_2 \in C$ *and* $a = \frac{1}{2}(x_1 + x_2)$*, then* $x_1 = x_2 = a$.

*(c) If* $x_1, x_2 \in C$, $t \in (0,1)$ *and* $a = tx_1 + (1-t)x_2$*, then* $x_1 = x_2 = a$.

*(d)* $C \setminus \{a\}$ *is a convex set.*

*(e) If* $x_1, \dots, x_n \in C$ *and* $a \in \operatorname{co}\{x_1, \dots, x_n\}$*, then* $a = x_k$ *for some index* $k$.

*Proof.* Items (a) and (c) are equivalent by definition.

(b) $\Rightarrow$ (c): Let $a = tx_1 + (1-t)x_2$. Then

$$a = \frac{1}{2}(2tx_1 + (1-2t)x_2) + \frac{1}{2}x_2,$$

so we get $2tx_1 + (1-2t)x_2 = x_2$, which gives us $x_1 = x_2$.

(c) $\Rightarrow$ (d): Take any $x_1, x_2 \in C \setminus \{a\}$. Since $C$ is convex, $tx_1 + (1-t)x_2 \in C$. Now if $a = tx_1 + (1-t)x_2 \in \operatorname{co}\{x_1, x_2\}$, then $a = x_1 = x_2$, which contradicts our assumption. So $tx_1 + (1-t)x_2 \in C \setminus \{a\}$ and $C \setminus \{a\}$ is convex.

(d) $\Rightarrow$ (e) If $x_1, \dots, x_n \in C \setminus \{a\}$, then $\operatorname{co}\{x_1, \dots, x_n\} \subseteq C \setminus \{a\}$ by convexity, contradiction.

(e) $\Rightarrow$ (b): Suppose $a = \frac{1}{2}(x_1 + x_2)$. Then either $x_1 = a$ or $x_2 = a$ by our assumption. W.l.o.g. assume $x_1 = a$. Then $a = \frac{1}{2}(a + x_2)$, which implies $a = x_2$. □

**Lemma 1.40.** *Let $X$ be a TVS and $C \subseteq X$ a nonempty compact convex set. Then for $\Phi \in X^*$ the set*

$$F = \{x \in C \mid \operatorname{Re}\Phi(x) = \min_C \operatorname{Re}\Phi\}$$

*is a closed face of $C$.*

*Proof.* Since $C$ is compact and $x \mapsto \operatorname{Re}\Phi(x)$ is continuous, it attains its minimum on $C$. Hence $F$ is nonempty. Since $F$ is a continuous preimage of a point, it is also closed. By the linearity of $\Phi$, $F$ is convex. Now suppose that $t \in (0,1)$ and $x, y \in C$ are such that $tx + (1-t)y \in F$. Then

$$\begin{aligned}\min_C \operatorname{Re}\Phi &= \operatorname{Re}\Phi(tx + (1-t)y) \\ &= t \cdot \operatorname{Re}\Phi(x) + (1-t)\operatorname{Re}\Phi(y) \\ &\geq t \cdot \min_C \operatorname{Re}\Phi + (1-t)\min_C \operatorname{Re} \\ &= \min_C \operatorname{Re}\Phi.\end{aligned}$$

Since we have the equality in the second-to-last line, we have $\operatorname{Re}\Phi(x) = \min_C \operatorname{Re}\Phi$ and $\operatorname{Re}\Phi(y) = \min_C \operatorname{Re}\Phi$, meaning that $x, y \in F$. □

*Remark.* Not all closed convex faces are of this form.

ADD A PICTURE

**Theorem 1.41** (Krein–Milman)**.**

*Let $X$ be a LCS and $C \subseteq X$ a nonempty compact convex subset. Then $C = \overline{\mathrm{co}}(\operatorname{ext} C)$. In particular,* $\operatorname{ext} C \neq \emptyset$.

*Proof.* Let $\mathcal{F} = \{\text{closed faces in } C\}$ be ordered with $\supset$. Since $C \in \mathcal{F}$, it is nonempty. The set $\mathcal{F}$ is then partially ordered. Since any increasing chain in $\mathcal{F}$ has the finite intersection property, $\mathcal{F}$ has a nonempty intersection due to $C$ being compact. As a result, any increasing chain in $\mathcal{F}$ has an upper bound. This tells us that we can apply Zorn's lemma to obtain a maximal element $F_0 \in \mathcal{F}$.

We prove that $F_0 = \{p\}$ for some $p \in X$. Assume for a contradiction that there are distinct $x, y \in F_0$. By Hahn–Banach, there exists a $\Phi \in X^*$ such that $\Phi(x) \neq \Phi(y)$. W.l.o.g. we assume that $\operatorname{Re} \Phi(x) < \operatorname{Re} \Phi(y)$. Define a set

$$F_1 = \{z \in F_0 \mid \operatorname{Re} \Phi(z) = \min_{F_0} \operatorname{Re} \Phi\}.$$

Then $F_1 \subsetneq F_0$, since $y \notin F_0$. By the previous lemma, $F_1$ is a closed face in $F_0$, so it is a closed face in $C$, contradicting maximality of $F_0$. As a result, $F_0 = \{p\}$, which implies that $p \in \operatorname{ext}(C)$ and the set of extreme points of $C$ is non-empty.

Since we have $C \supset \operatorname{ext} C$, we also have $C = \overline{\mathrm{co}}(C) \supseteq \overline{\mathrm{co}}(\operatorname{ext} C)$. Suppose $x \in C \setminus \overline{\mathrm{co}}(\operatorname{ext} C)$. By Hahn–Banach, there exists a $\Psi \in X^*$ such that $\operatorname{Re} \Psi(x) < \min_{\overline{\mathrm{co}}(\operatorname{ext} C)} \operatorname{Re} \Psi$. So the set

$$F = \{z \in C \mid \operatorname{Re} \Psi(z) = \min_{C} \operatorname{Re} \Psi\}$$

is a closed face in $C$. By the first part of this proof, there exists a $z \in \operatorname{ext} F \subseteq \operatorname{ext} C$. Hence

$$\min_C \operatorname{Re} \Psi = \operatorname{Re} \Psi(z) = \min_{\overline{\mathrm{co}}(\operatorname{ext} C)} \operatorname{Re} \Psi > \operatorname{Re} \Psi(x) \geq \min_C \operatorname{Re} \Psi,$$

which leads to a contradiction. Therefore $\overline{\mathrm{co}}(\operatorname{ext} C) = C$. $\square$

**Example 1.42.** *Let $\mathcal{H}$ be a Hilbert space. Then*

$$\operatorname{ext}(\mathcal{H})_1 = \{v \in \mathcal{H} \mid \|v\| = 1\}.$$

*First we prove the inclusion $(\supseteq)$. Suppose that $\|v\| = 1$ and $v = tx + (1-t)y$, where $t \in (0,1)$ and $x, y \in (\mathcal{H})_1$. We have*

$$\begin{aligned} 1 &= \|v\|^2 \\ &= \|tx + (1-t)y\|^2 \\ &= \langle tx + (1-t)y, tx + (1-t)y \rangle \\ &= t^2\|x\|^2 + (1-t)^2\|y\|^2 + 2t(1-t)\operatorname{Re}\langle x, y\rangle \\ &\leq t^2 + (1-t)^2 + 2t(1-t) = 1. \end{aligned}$$

*We get equality in the Cauchy-Schwartz inequality, so $x, y$ are linearly dependent and there-*

*fore equal. For the reverse inclusion, let $v \in \mathrm{ext}(\mathcal{H})_1$. If $\|v\| < 1$, then*

$$v = \frac{1}{2} \cdot \frac{v}{\|v\|} + \frac{1}{2} \cdot (2\|v\| - 1)\, \frac{v}{\|v\|},$$

*so $v$ cannot be an extreme point of $(\mathcal{H})_1$.*

**Example 1.43.** *It holds that*

$$\mathrm{ext}(\mathcal{B}(\mathcal{H}))_1 = \{V \in \mathcal{B}(\mathcal{H}) \mid V \text{ or } V^* \text{ is an isometry}\}.$$

*Here, we will just prove the inclusion $(\supseteq)$. Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry and suppose $V = tS + (1-t)T$ for $t \in (0,1)$ and $S, T \in (\mathcal{B}(\mathcal{H}))_1$. For $x \in \mathcal{H}$ we have:*

$$\begin{aligned} \|x\| &= \|Vx\| \\ &= \|tSx + (1-t)Tx\| \\ &\leq t\|Sx\| + (1-t)\|Tx\| \\ &\leq t\|S\|\|x\| + (1-t)\|T\|\|x\| \\ &\leq t\|x\| + (1-t)\|x\| = \|x\|. \end{aligned}$$

*Since we have equality, we get $\|S\| = \|T\| = 1$ and $\|Sx\| = \|Tx\| = \|x\|$. So $S, T$ are isometries. For every $x \in \partial(\mathcal{H})_1 = \mathrm{ext}(\mathcal{H})_1$, we have*

$$Vx = t \cdot Sx + (1-t)Tx$$

*and by the previous example that implies $Tx = Sx = Vx$, so we really have $S = T = V$. We use the same argument if $V^*$ is an isometry. For now, we lack some tools to prove the reverse inclusion. We will prove the equality in corollary 4.4.*

**Example 1.44.** *If $X$ be a Banach space, then $(X^*)_1$ is weak-$*$ compact (by Banach–Alaoglu), so Krein–Milman gives us $(X^*)_1 = \overline{\mathrm{co}}(\mathrm{ext}(X^*)_1)$. Hence $(X^*)_1$ has a lot of extreme points. As a corollary, $c_0$, $L^1[0,1]$ and $C[0,1]$ are not duals of Banach spaces.*

**Theorem 1.45** (Milman)**.**

*Let $X$ be a LCS, $K \subseteq X$ compact and assume $\overline{\mathrm{co}}(K)$ is compact. Then $\mathrm{ext}(\overline{\mathrm{co}}(K)) \subseteq K$.*

*Proof.* Assume there exists $x_0 \in \mathrm{ext}(\overline{\mathrm{co}}(K)) \setminus K$. Then there exists a basis neighborhood $V$ of 0 in $X$ such that $(x_0 + \overline{V}) \cap K = \emptyset$, or equivalently, $x_0 \notin K + \overline{V}$. If we write $K \subseteq \bigcup_{x \in K}(x + V)$, we get

$$K \subseteq \bigcup_{j=1}^{n} (x_j + V).$$

Form $K_j = \overline{\mathrm{co}}(K \cap (x_j + V))$. Then $K_j$ is convex and compact since $K_j \subseteq \overline{\mathrm{co}}(K)$. We also have $K_j \subseteq \overline{x_j + V} = x_j + \overline{V}$ since $V$ is convex. Also, $K \subseteq K_1 \cup \cdots \cup K_n$. Next we prove that $\mathrm{co}(K_1 \cup \cdots \cup K_n)$ is compact. Define

$$\Sigma = \{(t_1, \ldots, t_n) \in [0,1]^n \mid \sum_{j=1}^{n} t_j = 1\}$$

and the function

$$f : \Sigma \times K_1 \times \cdots \times K_n \to X, \quad (t, k_1, \ldots, k_n) \mapsto \sum_{j=1}^{k} t_j k_j.$$

Denote $C := \operatorname{im} f$. Obviously, $C \subseteq \mathrm{co}(K_1 \cup \cdots \cup K_n)$ and $C$ is a convex compact set. Furthermore, $C \supset K_j$ for each $j$, so $C = \mathrm{co}(K_1 \cup \cdots \cup K_n)$. From there, we get

$$\overline{\mathrm{co}}(K) \subseteq \overline{\mathrm{co}}(K_1 \cup \cdots \cup K_n) = \mathrm{co}(K_1 \cup \cdots \cup K_n).$$

But since $K_j \subseteq \overline{\mathrm{co}}(K)$ for all $j$, we deduce $\overline{\mathrm{co}}(K) = \overline{\mathrm{co}}(K_1 \cup \cdots \cup K_n)$. We know that $x_0 \in \overline{\mathrm{co}}(K)$, so

$$x_0 = t_1 y_1 + \cdots + t_n y_n$$

for some $t_i \in [0,1]$, $\sum t_i = 1$ and $y_j \in K_j$. But $x_0 \in \mathrm{ext}(\overline{\mathrm{co}})(K)$, so $y_j = x_0$ for some $j$. So we get $x_0 \in K_j \subseteq x_j + \overline{V} \subseteq K + \overline{V}$, a contradiction. $\square$

*Remark.* (1.) In finite dimensions, the convex hull of a compact set is compact. In infinite dimensions this fails.

(2.) The set $\mathrm{ext}(C)$ is not always closed, even if $C \subseteq \mathbb{R}^3$ is convex and compact.

ADD A PICTURE

# 2 $C^*$-algebras and continuous functional calculus

## 2.1 Spectrum

*Remark.* From here on, all algebras are over $\mathbb{C}$.

Let $A$ be a complex algebra with a unit 1 and

$$\mathrm{GL}(A) = \{a \in A \mid a \text{ is invertible}\}.$$

If $x \in A$, we define the spectrum

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \cdot 1 \notin \mathrm{GL}(A)\}.$$

**Proposition 2.1.** *Let $A$ be a complex algebra with unity 1 and $x, y \in A$. Then*

$$\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}.$$

*Proof.* Suppose $1 - xy \in \mathrm{GL}(A)$. Formally, we can write

$$(1 - xy)^{-1} = 1 + xy + (xy)^2 + \cdots$$

and

$$(1 - yx)^{-1} = 1 + yx + (yx)^2 + \cdots = 1 + y(1 - xy)^{-1}x.$$

From this, we claim that indeed $1 - yx \in \mathrm{GL}(A)$ and

$$(1 - yx)^{-1} = 1 + y(1 - xy)^{-1}x.$$

The proof is straightforward: we have

$$\begin{aligned}
(1 + y(1 - xy)^{-1}x)(1 - yx) &= (1 - yx) + y(1 - xy)^{-1}(x - xyx) \\
&= (1 - yx) + y(1 - xy)^{-1}(1 - xy)x \\
&= (1 - yx) + yx = 1
\end{aligned}$$

and

$$\begin{aligned}
(1 - yx)(1 + y(1 - xy)^{-1}x) &= (1 - yx) + (y - yxy)(1 - xy)^{-1}x \\
&= (1 - yx) + y(1 - xy)(1 - xy)^{-1}x \\
&= (1 - yx) + yx = 1.
\end{aligned}$$

Now the proof of the statement is at hand: if $\lambda \in \sigma_A(xy) \setminus \{0\}$, then

$$\lambda - xy \notin \mathrm{GL}(A) \Rightarrow 1 - \frac{x}{\lambda}y \notin \mathrm{GL}(A) \Rightarrow 1 - y\frac{x}{\lambda} \notin \mathrm{GL}(A) \Rightarrow \lambda - yx \notin \mathrm{GL}(A).$$

Thus, $\lambda \in \sigma_A(yx)$. Similarly, if $\lambda \in \sigma_A(yx) \setminus \{0\}$, then $\lambda \in \sigma_A(xy)$. □

**Example 2.2.** *Let $S, S^* \in \mathcal{B}(\ell^2)$ be the right and left shift operators, respectively. Then $SS^* = I$, but*

$$SS^*(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

*This implies that $0 \in \sigma(SS^*)$, but $0 \notin \sigma(S^*S)$.*

## 2.2 Banach and $C^*$-algebras

**Definition 2.3.**

- A Banach algebra is a Banach space $A$ that is also an algebra, satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. If a Banach algebra has a unit 1, we also demand $\|1\| = 1$.
- An involution on a Banach algebra $A$ is a skew-linear map

$$* : A \to A, \quad a \mapsto a^*$$

satisfying

$$(xy)^* = y^* x^*, \quad (x^*)^* = x, \quad \|x^*\| = \|x\|.$$

A $C^*$-algebra is a Banach $*$-algebra $A$ that also satisfies $\|x^* x\| = \|x\|^2$ for all $x \in A$.

Unless otherwise mentioned, all algebras in this section will be unital.

**Proposition 2.4.** *We collect some basic properties of Banach algebras.*

*(1.) If $A$ is a Banach $*$-algebra, then $(x^*)^{-1} = (x^{-1})^*$ and $\sigma_A(x^*) = (\sigma_A(x))^*$.*

*(2.) Let $A$ be a Banach algebra. If $\|x\| < 1$, then $1 - x \in \mathrm{GL}(A)$ and*

$$(1 - x)^{-1} = 1 + x + x^2 + \cdots$$

*As a consequence, if $\|1 - x\| < 1$, then $x \in \mathrm{GL}(A)$.*

*(3.) Let $A$ be a Banach algebra. Then $\mathrm{GL}(A) \subseteq A$ is open, and the map $x \mapsto x^{-1}$ is continuous on $\mathrm{GL}(A)$.*

*(4.) If $A$ is a Banach algebra and $x \in A$, then $\sigma_A(x)$ is a nonempty compact set.*

*Proof.* (1.) Suppose that the inverse $(x^*)^{-1}$ exists. Then $(x^*)^{-1} \cdot (x^*) = 1$, so starring gives us $(x^*)^* \cdot ((x^*)^{-1})^* = 1$ and $x \cdot ((x^*)^{-1})^* = 1$. Similarly, we have $(x^*) \cdot (x^*)^{-1} = 1$, which implies $((x^*)^{-1})^* \cdot x = 1$. This means that $x$ is invertible and $((x^*)^{-1})^* = x^{-1}$. Starring this equation now gives us $(x^*)^{-1} = (x^{-1})^*$. For the opposite direction, suppose that $x$ is invertible. Then

$$(x^{-1})^* \cdot x^* = (x \cdot x^{-1})^* = 1^* = 1$$

and

$$x^* \cdot (x^{-1})^* = (x^{-1} \cdot x)^* = 1^* = 1,$$

which means that $x^*$ is invertible and $(x^*)^{-1} = (x^{-1})^*$. The rest is a matter of simple computation:

$$\begin{aligned}\lambda \in \sigma_A(x^*) &\Leftrightarrow x^* - \lambda \notin \mathrm{GL}(A) \Leftrightarrow (x-\overline{\lambda})^* \notin \mathrm{GL}(A)\\ &\Leftrightarrow (x-\overline{\lambda}) \notin \mathrm{GL}(A) \Leftrightarrow \overline{\lambda} \in \sigma_A(x)\\ &\Leftrightarrow \lambda \in (\sigma_A(x))^*.\end{aligned}$$

If $\|x\| \le 1$, then the series $\sum_{n=0}^{\infty} x^n$ converges in norm to some $x'$. Since multiplication between elements of a Banach algebra is norm-continuous, we get we get

$$(1-x)x' = (1-x)\cdot \lim_{k\to\infty}\sum_{n=1}^{k} x^n = \lim_{k\to\infty}(1-x)\cdot\sum_{n=1}^{k} x^n = \lim_{k\to\infty} 1 - x^{k+1} = 1$$

and similarly for $x'(1-x)$.
Let $y \in \mathrm{GL}(A)$. If $\|x-y\| \le \frac{1}{\|y^{-1}\|}$, then

$$\|1 - xy^{-1}\| = \|(y-x)y^{-1}\| \le \|y-x\|\|y^{-1}\| \le 1,$$

which implies that $xy^{-1} \in \mathrm{GL}(A)$, and thus $x = xy^{-1}\cdot y \in \mathrm{GL}(A)$. We have shown that $\mathrm{GL}(A)$ is open. Using the same notation and noting that $(xy^{-1})^{-1} = (1-(1-xy^{-1}))^{-1}$, we get

$$\|(xy^{-1})^{-1}\| \le \sum_{n=0}^{\infty}\|(1-xy^{-1})\|^n \le \sum_{n=0}^{\infty}\|y^{-1}\|^n\|x-y\|^n \le \frac{1}{1-\|y^{-1}\|\cdot\|x-y\|}.$$

Now,

$$\begin{aligned}\|x^{-1}-y^{-1}\| &= \|x^{-1}(y-x)y^{-1}\|\\ &\le \|y^{-1}(xy^{-1})^{-1}\|\|y-x\|\|y^{-1}\|\\ &\le \|(xy^{-1})^{-1}\|\|y-x\|\|y^{-1}\|^2\\ &\le \frac{\|y^{-1}\|^2}{1-\|y^{-1}\|\cdot\|x-y\|}\|y-x\|.\end{aligned}$$

Since the function $t \mapsto \frac{\|y^{-1}\|^2}{1-\|y^{-1}\|\cdot t}t$ is continuous at $t=0$, the map $x \mapsto x^{-1}$ is continuous.
First, we prove compactness by showing that $\sigma_A(x)$ is bounded and closed. Suppose there exists $\lambda \in \sigma_A(x)$, such that $|\lambda| > \|x\|$. Then $\left(1-\frac{x}{\lambda}\right)$ is invertible by (2.), so $(-\lambda)\cdot\left(1-\frac{x}{\lambda}\right) = x-\lambda$ is invertible as well. But this contradicts the fact that $\lambda \in \sigma_A(x)$, so we have shown that $\sigma_A(x) \subseteq \overline{B(0,\|x\|)}$. Next, we prove that the spectrum is closed. Define a continuous map

$$\mathbb{C} \to A, \quad \lambda \mapsto x - \lambda$$

and notice that the inverse image of $\mathrm{GL}(A)$ (which is open by (3.)) is exactly $\mathbb{C}\setminus\sigma_A(x)$. This means that $\mathbb{C}\setminus\sigma_A(x)$ is open and $\sigma_A(x)$ is closed. For non-emptyness, we have to employ some standard Banach algebra techniques. We say that a function from $f$ from a domain $\Omega \subseteq \mathbb{C}$ to a Banach space $X$ is analytical if there exists a limit

$$f'(z_0) := \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$$

for every $z_0 \in \Omega$ and the function $f'$ is continuous on $\Omega$. A lot of theory for complex analytic functions also applies to Banach space-valued analytic functions; in particular, we have Cauchy's integral formula, Liouville's theorem and the fact that every vector valued analytic function can be locally expressed as a power series with coefficients in $X$. Now we can define the resolvent function

$$F : \mathbb{C} \setminus \sigma_A(x) \to A, \quad F(z) = (z - x)^{-1}.$$

It's routine to show that $F$ is analytic and its derivative is $F'(z) = (z - x)^{-2}$. Now for $z \in \mathbb{C} \setminus \overline{B(0, \|x\|)}$, we have $F(z) = z^{-1} \cdot (1 - a/z)$, which goes to 0 as $z \to \infty$. Now if $\sigma_A(x) = \emptyset$, then $F$ would be an entire function that vanishes at $\infty$. By Liouville's theorem, $F$ is constant and so $F' = 0$. This is a contradiction. □

**Theorem 2.5** (Gelfand–Mazur)**.**

*If $A$ is Banach algebra that is also a division ring, then $A = \mathbb{C}$.*

*Proof.* Let $x \in A$ and $\lambda \in \sigma_A(x)$. Then $x - \lambda \cdot 1 \notin \mathrm{GL}(A)$, implying $x - \lambda = 0$, hence $x = \lambda \in \mathbb{C}$. □

**Definition 2.6.** If $f(x) = \sum_{j=0}^{n} a_j x^j$ is a polynomial and $a \in A$, we define $f(a) = \sum_{j=0}^{n} a_j a^j \in A$.

**Theorem 2.7** (Spectral mapping theorem for polynomials)**.**

*Let $A$ be a complex unitary algebra and $f \in \mathbb{C}[x]$. Then $f(\sigma_A(a)) = \sigma_A(f(a))$ for all $a \in A$.*

*Proof.* First, we prove the inclusion $(\subseteq)$. If $\lambda \in \sigma_A(a)$ and $f(x) = \sum_{j=0}^{n} a_j x^j$, then

$$f(x) - f(\lambda) = \sum_{j=1}^{n} a_j (x^j - \lambda^j) = (x - \lambda) \cdot \sum_{j=1}^{n} a_j \sum_{k=0}^{j-1} x^k \lambda^{j-1-k}.$$

Substituting $x = a$, we obtain

$$f(a) - f(\lambda) = (a - \lambda) \left( \sum_{j=1}^{n} a_j \sum_{k=0}^{j-1} a^k \lambda^{j-1-k} \right).$$

Since $a - \lambda$ commutes with the second factor, $f(a) - f(\lambda)$ is not invertible and $f(\lambda) \in \sigma_A(f(a))$. For the converse inclusion $(\supseteq)$, assume $\mu \notin f(\sigma_A(a))$. We factor

$$f(x) - \mu = a_n (x - \lambda_1) \cdots (x - \lambda_n).$$

Since $f(\lambda) - \mu \neq 0$ for any $\lambda \in \sigma_A(a)$, it follows that $\lambda_i \notin \sigma_A(a)$ for all $i$. Therefore, $f(a) - \mu \in \mathrm{GL}(A)$. □

**Definition 2.8.** Let $A$ be a Banach algebra and $x \in A$. The spectral radius of $x$ is

$$r(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|.$$

*Remark.* By proposition 2.1, we have $r(xy) = r(yx)$.

In the introductory course, we proved the following.

**Theorem 2.9** (Spectral radius formula)**.**

*Let $A$ be a Banach algebra and $x \in A$. Then $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and is equal to $r(x)$.*

**Definition 2.10.** Let $A$ be a Banach $*$-algebra and $x \in A$.
- $x$ is *normal* iff $xx^* = x^*x$.
- $x$ is *self-adjoint* iff $x^* = x$.
- $x$ is *skew self-adjoint* iff $x^* = -x$.

The set of all self-adjoint operators is denoted as $A_{\mathrm{sa}}$.

*Remark.* Every $a \in A$ can be uniquely expressed as a sum of a self-adjoint and skew self-adjoint element:

$$a = \frac{a + a^*}{2} + \frac{a - a^*}{2}.$$

Alternatively, we can uniquely write it in the form of

$$a = \left(\frac{a + a^*}{2}\right) + i \cdot \left(\frac{a - a^*}{2i}\right)$$

where both terms in parentheses are self-adjoint.

**Corollary 2.11.** *Let $A$ be a Banach $*$-algebra and $x \in A$ normal. Then $r(x^*x) \leq r(x)^2$. If $A$ is a $C^*$-algebra, then $r(x^*x) = r(x)^2$.*

*Proof.* We use the spectral radius formula:

$$\begin{aligned} r(x^*x) &= \lim_{n\to\infty} \|(x^*x)^n\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|(x^*)^n x^n\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|(x^n)^* x^n\|^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} \|x^n\|^{\frac{2}{n}} = r(x)^2. \end{aligned}$$

If $A$ is a $C^*$-algebra, we have an equality in the last line of the above calculation. □

**Proposition 2.12.** *Let $A$ be a $C^*$-algebra and $x \in A$ normal. Then $r(x) = \|x\|$.*

*Proof.* First, assume $x$ is self-adjoint. Then

$$\|x^2\| = \|xx^*\| = \|x\|^2.$$

By induction, we get $\|x^{2^n}\| = \|x\|^{2^n}$ for every $n \in \mathbb{N}$. Therefore,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|.$$

If $x$ is only normal, then

$$\|x\|^2 = \|x^*x\| = r(x^*x) = r(x)^2,$$

which implies $\|x\| = r(x)$. □

**Corollary 2.13.** *Let $A, B$ be $C^*$-algebras and $\Phi : A \to B$ a $*$-homomorphism ($\Phi(x^*) = \Phi(x)^*$). Then $\Phi$ is a contraction. Furthermore, if $\Phi$ is a $*$-isomorphism, then it is isometric.*

*Proof.* Clearly, $\Phi$ maps invertible elements to invertible elements, so $\Phi(\mathrm{GL}(A)) \subseteq \mathrm{GL}(B)$. This implies $\sigma_B(\Phi(x)) \subseteq \sigma_A(x)$, hence $r(\Phi(x)) \leq r(x)$. Then

$$\begin{aligned}\|\Phi(x)\|^2 &= \|\Phi(x)\Phi(x)^*\| = \|\Phi(x)\Phi(x^*)\| \\ &= \|\Phi(xx^*)\| = r(\Phi(xx^*)) \\ &\leq r(xx^*) = \|xx^*\| = \|x\|^2.\end{aligned}$$

If $\Phi$ is a $*$-isomorphism, we apply the same reasoning to its inverse, which implies that $\Phi$ must be an isometry. □

**Corollary 2.14.** *If $A$ is a $*$-algebra, then there exists at most one norm on $A$ that makes it into a $C^*$-algebra.*

*Proof.* Considering the identity map

$$(A, \|\|_1) \to (A, \|\|_2),$$

it is a $*$-isomorphism, so it preserves the norm by the previous corollary. □

**Lemma 2.15.** *Let $A$ be a $C^*$-algebra and $x \in A$ self-adjoint. Then $\sigma_A(x) \subseteq \mathbb{R}$.*

*Proof.* Suppose $\lambda = \alpha + i\beta \in \sigma_A(x)$ for some $\alpha, \beta \in \mathbb{R}$. Define $y = x - \alpha + it$ for $t \in \mathbb{R}$. Then $i(\beta + t) \in \sigma_A(y)$ and $y$ is normal. Thus,

$$\begin{aligned}|i(\beta + t)|^2 = (\beta + t)^2 &\leq r(y) \\ &= \|y\|^2 = \|yy^*\| \\ &= \|(x - \alpha)^2 + t^2\| \leq \|x - \alpha\|^2 + t^2.\end{aligned}$$

Simplifying, we get $\beta^2 + 2\beta t \leq \|x - \alpha\|^2$, and since $t \in \mathbb{R}$ was arbitrary, we have $\beta = 0$. □

**Lemma 2.16.** *Let $A$ be a Banach algebra and $x \notin \mathrm{GL}(A)$. If $(x_n)_n \subseteq \mathrm{GL}(A)$ satisfies $x_n \to x$, then $\|x_n^{-1}\| \to \infty$.*

*Proof.* If the sequence $\|x_n^{-1}\|$ is bounded, then

$$\|1 - xx_n^{-1}\| = \|(x_n - x)x_n^{-1}\| \leq \|x_n - x\| \cdot \|x_n^{-1}\| \to 0.$$

In particular, there exists some $n \in \mathbb{N}$ such that $\|1 - xx_n^{-1}\| < 1$, which implies $xx_n^{-1} \in \mathrm{GL}(A)$ and therefore $x = (xx_n^{-1})x_n \in \mathrm{GL}(A)$, a contradiction. □

**Proposition 2.17.** *Let $B$ be a $C^*$-algebra and $A \subseteq B$ a unital $C^*$-subalgebra. Then for all $x \in A$, we have $\sigma_A(x) = \sigma_B(x)$.*

*Proof.* Obviously, $\mathrm{GL}(A) \subseteq \mathrm{GL}(B)$. For a self adjoint $x \in A \setminus \mathrm{GL}(A)$, we have $it \notin \sigma_A(x)$ for $t \in \mathbb{R}$. So there exists $(x - it)^{-1} \in A$. Clearly,

$$x - it \in \mathrm{GL}(A) \xrightarrow[t \to 0]{} x \notin \mathrm{GL}(A),$$

thus $\|(x - it)^{-1}\| \to \infty$. Since the inverse function is continuous, this immediately yields $x \notin \mathrm{GL}(B)$. For general $x \in A$: if $x \in \mathrm{GL}(B)$, then $x^*x \in \mathrm{GL}(B)$ is self-adjoint. By the first part of the proof, $x^*x \in \mathrm{GL}(A)$. It follows that

$$x^{-1} = (x^*x)^{-1}x^* \in A,$$

so $x \in \mathrm{GL}(A)$. □

**Example 2.18.** *Let $X$ be a Hausdorff topological space and $C_b(X)$ be the set of continuous bounded complex functions on $X$, endowed with the sup metric. Then $C_b(X)$ is a unital abelian $C^*$-algebra (where $f^*(x) = \overline{f(x)}$).*

**Example 2.19.** *Let $X$ be a locally compact Hausdorff space and*

$$C_0(X) = \{f \in C(X) \mid \forall \varepsilon > 0: \ \exists K^{compact} \subseteq X: \ |f(x)| \leq \varepsilon, \ \forall x \in X \setminus K\}$$

*the set of all complex continuous functions on $X$ that vanish at infinity. Then $C_0(X)$ is an abelian $C^*$-algebra. In some sense, it is the natural abelian $C^*$-algebra. The algebra $C_0(X)$ is unital iff $X$ is compact – in that case, $C_0(X) = C_b(X) = C(X)$.*

**Example 2.20.** *Let $(X, \mu)$ be a measure space. Then $L^\infty(X, \mu)$, the set of essentially bounded functions on $X$ endowed with the essential supremum norm, is a unital abelian $C^*$-algebra.*

**Example 2.21.** *For a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ is a non-abelian $C^*$-algebra: for all $x \in \mathcal{B}(\mathcal{H})$ we have $\|x^*x\| = \|x\|^2$.*

**Example 2.22.** *If $\Gamma$ is a group, we define*

$$\ell^1(\Gamma) = \{(\alpha_s)_{s\in\Gamma} \mid \alpha_s \in \mathbb{C},\ \sum_{s\in\Gamma} |\alpha_s| < \infty\}.$$

*We can then introduce the convolution multiplication on $\ell^1(\Gamma)$:*

$$(\alpha * \beta)_s = \sum_{t\in\Gamma} \alpha_{st}\beta_{t^{-1}}.$$

*This is a Banach algebra; it is even a Banach $*$-algebra with involution $(\alpha^*)_s = \overline{\alpha_{s^{-1}}}$. However, it is not a $C^*$-algebra if the group $\Gamma$ has more than one element. In that case, there exists $z \in \Gamma$ such that $z \neq 1$. Define $\alpha = (\alpha_s) \in \ell^1(G)$ such that*

$$\alpha_s = \begin{cases} 1; & s = 1 \\ i; & s = z, z^{-1} \\ 0; & \textit{otherwise} \end{cases}.$$

*If $z \neq z^{-1}$, we have*

$$\begin{aligned}
\|\alpha\alpha^*\| &= \sum_{s\in\Gamma} \left| \sum_{t\in\Gamma} \alpha_{st}\overline{\alpha_t} \right| \\
&= \sum_{s\in\Gamma} (3 \cdot \mathbf{1}_{s=1} + \mathbf{1}_{s=z^2} + \mathbf{1}_{s=z^{-2}}) \\
&< \sum_{s\in\Gamma} (3 \cdot \mathbf{1}_{s=1} + 2 \cdot \mathbf{1}_{s=z} + 2 \cdot \mathbf{1}_{s=z^{-1}} + \mathbf{1}_{s=z^2} + \mathbf{1}_{s=z^{-2}}) \\
&= \sum_{s\in\Gamma} \sum_{t\in\Gamma} |\alpha_{st}\alpha_t| = \sum_{t\in\Gamma} |\alpha_t| \cdot \sum_{s\in\Gamma} |\alpha_{st}| \\
&= \sum_{t\in\Gamma} |\alpha_t| \cdot \sum_{s\in\Gamma} |\alpha_s| = \|\alpha\|^2.
\end{aligned}$$

*Otherwise, we get*

$$\begin{aligned}
\|\alpha\alpha^*\| &= \sum_{s\in\Gamma} \left| \sum_{t\in\Gamma} \alpha_{st}\overline{\alpha_t} \right| \\
&= \sum_{s\in\Gamma} (2 \cdot \mathbf{1}_{s=1}) \\
&< \sum_{s\in\Gamma} (2 \cdot \mathbf{1}_{s=1} + 2 \cdot \mathbf{1}_{s=z}) \\
&= \sum_{s\in\Gamma} \sum_{t\in\Gamma} |\alpha_{st}\alpha_t| = \sum_{t\in\Gamma} |\alpha_t| \cdot \sum_{s\in\Gamma} |\alpha_{st}| \\
&= \sum_{t\in\Gamma} |\alpha_t| \cdot \sum_{s\in\Gamma} |\alpha_s| = \|\alpha\|^2.
\end{aligned}$$

*Therefore, $\ell^1(\Gamma)$ is not a $C^*$-algebra if $\Gamma$ has order greater than one.*

## 2.3 Gelfand transform

**Definition 2.23.** Let $A$ be an abelian Banach algebra. The *spectrum* of $A$ is defined as

$$\sigma(A) := \{\varphi : A \to \mathbb{C} \mid \varphi \neq 0 \text{ continuous algebra homomorphism}\} \subseteq A^*$$

endowed with a weak-$*$ topology. Its elements are called *characters*.

If $\varphi \in \sigma(A)$, then $\ker \varphi \cap \mathrm{GL}(A) = \emptyset$. For $x \in A$, we have

$$\begin{aligned} \varphi(x - \varphi(x)) &= \varphi(x) - \varphi(\varphi(x) \cdot 1) \\ &= \varphi(x) - \varphi(x)\varphi(1) \\ &= \varphi(x) - \varphi(x) = 0, \end{aligned}$$

which implies that $\varphi(x) \in \sigma_A(x)$. Consequently, $|\varphi(x)| \leq r(x) \leq \|x\|$, giving us the bound $\|\varphi\| \leq 1$. But since $\varphi(1) = 1$, we get $\|\varphi\| = 1$. We know that $\sigma(A)$ is closed in $(A^*)_1$, making $\sigma(A)$ is a compact Hausdorff space by Banach–Alaoglu.

**Proposition 2.24.** *Let $A$ be a $C^*$-algebra and $h : A \to \mathbb{C}$ a non-zero homomorphism (not necessarily a $*$-homomorphism). Then the following statements hold:*

*(1.)* $h(a) \in \mathbb{R}$ *for self-adjoint* $a$*;*

*(2.)* $h(a^*) = \overline{h(a)}$ *for all* $a \in A$*;*

*(3.)* $h(aa^*) \geq 0$ *for all* $a \in A$*;*

*(4.) if* $uu^* = 1$ *or* $u^*u$*, then* $|h(u)| = 1$.

*Remark.* The first three item also hold for non-unital algebras.

*Proof.* (1.) Since $h(a) \in \sigma_A(a)$ and self-adjoint elements have real spectrum, this is trivial.
(2.) Let $a = a_1 + ia_2$, where $a_1, a_2$ are self-adjoint. Then $a^* = a_1 - ia_2$ and

$$h(a^*) = h(a_1 - ia_2) = h(a_1) - ih(a_2) = \overline{h(a_1) + ih(a_2)} = \overline{h(a)}.$$

(3.) Follows from (b).
(4.) If $u$ is unitary, then $|h(u)|^2 = h(u)h(u^*) = h(uu^*) = h(1) = 1$. $\square$

**Corollary 2.25.** *Every nonzero algebra homomorphism $h : A \to \mathbb{C}$ is a character.*

**Proposition 2.26.** *Let $A$ be an abelian Banach algebra. Then the map $\varphi \mapsto \ker \varphi$ is a bijection from $\sigma(A)$ to the set of all maximal ideals of $A$.*

*Proof.* If $\varphi \in \sigma(A)$, then $\ker \varphi \lhd A$. Suppose that $\ker \varphi \subsetneq I \lhd A$. Then there exists an element $x \in I \setminus \ker \varphi$. Thus, $\varphi(x) \neq 0$ and from $1 - \frac{x}{\varphi(x)} \in \ker \varphi$. From there, it follows that

$$1 = \left(1 - \frac{x}{\varphi(x)}\right) + \frac{1}{\varphi(x)} \cdot x \in I.$$

Hence, $\ker \varphi$ is a maximal ideal. Conversely, let $I \lhd A$ be a maximal ideal. Then $I \cap \mathrm{GL}(A) = \emptyset$

and since $\mathrm{GL}(A)$ is open, we also have $\overline{I} \cap \mathrm{GL}(A) = \emptyset$. Thus, $\overline{I} \lhd A$ and $1 \notin \overline{I}$, so $I \subseteq \overline{I} \subsetneq A$. By maximality, $\overline{I} = I$. Then $A/I$ is a Banach algebra and since $I$ is maximal, every nonzero element in $A/I$ is invertible. By Gelfand–Mazur, $A/I \cong \mathbb{C}$. The projection $\pi : A \to A/I \cong \mathbb{C}$ is in $\sigma(A)$ and $\ker \pi = I$. $\square$

**Corollary 2.27.** *Let $A$ be an abelian Banach algebra and $x \in A \setminus \mathrm{GL}(A)$. Then there exists $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$. In particular, $\sigma(A) \neq 0$.*

*Proof.* If $x \notin \mathrm{GL}(A)$, then it generates an ideal $\langle x \rangle \subsetneq A$. By Zorn's lemma, $\langle x \rangle$ has to be included in some maximal ideal $I \lhd A$. By the previous proposition, there exists a character $\varphi : A \to \mathbb{C}$ in $\sigma(A)$ such that $x \in I = \ker \varphi$. $\square$

**Theorem 2.28** (Stone–Čech compactification)**.**

*Let $X$ be a topological space. For $x \in X$, let $\beta_x : C_b(X) \to \mathbb{C}$ be the evaluation homomorphism $f \mapsto f(x)$. Then*

$$\beta : X \to \sigma(C_b(X)), \quad x \mapsto \beta_x$$

*is a continuous map whose image is dense in the codomain and has the following universal property: if $\pi : X \to K^{T_2,\ compact}$ is continuous, then there exists a unique continuous mapping*

$$\beta_\pi : \sigma(C_b(X)) \to K$$

*such that $\pi(x) = \beta_\pi(\beta_x)$ for all $x \in X$. In particular, if $X$ is compact $T_2$, then $\beta$ is a homeomorphism.*

$$\begin{tikzcd}
X \arrow[r, "\pi"] \arrow[d, "\beta"] & K^{T_2,\ compact} \\
\sigma(C_b(X)) \arrow[ur, dashed, "\exists! \beta_\pi"'] &
\end{tikzcd}$$

*Proof.* (1.) First, we prove that $\beta$ is continuous. Let $(x_i)_i$ ibes a net in $X$ and $x_i \to x$, then for all $f \in C_b(X)$ we have $\beta_{x_i} = f(x_i) \to f(x) = \beta_x(f)$. Hence $\beta_{x_i} \to \beta_x$ in the weak-$*$ topology.

(2.) Next, we prove that $\operatorname{im} \beta$ is dense. Assume otherwise and pick $\varphi \in \sigma(C_b(X)) \setminus \overline{\beta(X)}$. Define $I := \ker \varphi$. For all $\psi \in \overline{\beta(X)}$, there exists $f_\psi \in I$ such that $f_\psi \in \ker \psi$. Hence, there exists $c_\psi$ and a neighborhood $U_\psi$ of $\psi$ such that $|\widetilde{\psi}(f)| > c_\psi$ for all $\widetilde{\psi} \in U_\psi$. Thus, $\overline{\beta(X)} \subseteq U_{\psi \in \overline{\beta(X)}} U_\psi$. By compactness, there exists a finite subcovering of $\overline{\beta(X)}$, so $\overline{\beta(X)} \subseteq \bigcup_{i=1}^n U_{\psi_j}$. Then there exist $f_{\psi_1}, \ldots, f_{\psi_n} \in I$ and $c > 0$ such that

$$\sum_{i=1}^n \psi(|f_{\psi_i}|^2) > c, \quad \forall \psi \in \overline{\beta(X)}.$$

Hence,

$$\sum_{i=1}^n |f_{\psi_i}|^2(x) = \sum_{i=1}^n \beta(x)(|f_{\psi_i}|^2) > c, \quad \forall x \in X.$$

It follows that $\sum_{i=1}^n |f_{\psi_i}|^2 \in I$ and $(\sum |f_{\psi_i}|^2)^{-1} \in C_b(X)$. As a result, $I = C_b(X)$.

(3.) If $X$ is compact and Hausdorff, then $\beta$ is surjective since $\beta(X)$ is dense and compact. Also, $\beta$ is injective since $C_b(X)$ separates points. In that case, $\beta$ is a continuous bijection between compact Hausdorff spaces, and therefore a homeomorphism.

(4.) For the universal property: let $\pi : X \to K$, where $K$ is compact Hausdorff. Then there exists a continuous map

$$\pi^* : C(K) \to C_b(X), \quad f \mapsto f \circ \pi.$$

This induces a continuous map

$$\widetilde{\pi} : \sigma(C_b(X)) \to \sigma(C(K)), \quad \varphi \mapsto \varphi \circ \pi^*.$$

Since $K$ is compact Hausdorff, the map $\beta^K : K \to \sigma(C(K))$ is a homeomorphism. Define

$$\beta_\pi : \sigma(C_b(X)) \to K, \quad \beta_\pi = (\beta^K)^{-1} \circ \widetilde{\pi}.$$

Then we have

$$\widetilde{\pi}(\beta_x)(g) = \beta_x(\pi^*(g)) = \pi^*(g)(x) = g(\pi(x)) = \beta^K_{\pi(x)}(g).$$

By left multiplying by $(\beta^K)^{-1}$, we get $\beta_\pi(\beta_x) = \pi(x)$. $\square$

**Definition 2.29.** Let $A$ be an abelian Banach algebra. The Gelfand transform of $A$ is the map

$$\Gamma : A \to C(\sigma(A)), \quad x \mapsto (\varphi \mapsto \varphi(x)).$$

**Theorem 2.30.**

*Let $A$ be an abelian Banach algebra. Then $\Gamma$ is a homomorphism, contraction and for $x \in A$ we have*

$$\Gamma(x) \in \mathrm{GL}(C(\sigma(A))) \Leftrightarrow x \in \mathrm{GL}(A).$$

*Proof.* The homomorphism part is routine. We prove that $\Gamma$ is a contraction as follows:

$$\|\Gamma(x)\| = \sup_{\varphi \in \sigma(A)} \|\Gamma(x)\varphi\| = \sup_{\varphi} |\varphi(x)| \leq \|x\|.$$

Next, we prove the equivalence. The right implication ($\Rightarrow$) is trivial, since

$$\Gamma(x^{-1})\Gamma(x) = \Gamma(x^{-1}x) = \Gamma(1) = 1.$$

Now the converse ($\Leftarrow$): if $x \notin \mathrm{GL}(A)$, then by corollary 2.27 there exists $\varphi \in \sigma(A)$ such that $\varphi(x) = 0$. Then $\Gamma(x)(\varphi) = \varphi(x) = 0$, so the continuous map $\Gamma(x)$ is not invertible. $\square$

**Corollary 2.31.** *Let $A$ be an abelian Banach algebra. Then we have*

$$\sigma(\Gamma(x)) = \sigma(x)$$

*and*

$$\|\Gamma(x)\| = r(\Gamma(x)) = r(x).$$

**Theorem 2.32** (Gelfand)**.**

*Let $A$ be an abelian $C^*$-algebra. Then $\Gamma$ is an isometric $*$-isomorphism.*

*Proof.* For a self-adjoint $x \in A$ we have $\sigma(\Gamma(x)) = \sigma(x) \subseteq \mathbb{R}$. Then $\overline{\Gamma(x)} = \Gamma(x)$. An arbitrary $x \in A$ can be written as $x = a + ib$ for self-adjoint $a = \frac{x+x^*}{2}$ and $b = \frac{i(x^*-x)}{2}$. Then

$$\Gamma(x^*) = \Gamma(a - ib) = \Gamma(a) - i\Gamma(b) = \overline{\Gamma(a) + i\Gamma(b)} = \overline{\Gamma(x)}.$$

This implies that $\Gamma$ is a $*$-homomorphism. Since $A$ is abelian, each $x \in A$ is normal so

$$\|x\| = r(x) = r(\Gamma(x)) = \|\Gamma(x)\|$$

and $\Gamma$ is an isometry. In particular, $\Gamma$ is injective. We know that $\Gamma(A)$ is closed under $*$. Since $\Gamma$ is isometric, the subalgebra $\Gamma(A) \subseteq C(\sigma(A))$ is complete in the norm, so it is closed. It can be easily checked that $\Gamma(A)$ separates points. By Stone–Weierstrass, $\Gamma(A) = C(\sigma(A))$. □

*Remark.* Let $A$ be a $C^*$-algebra. If $x \in A$ is normal, then it generates an abelian $C^*$-subalgebra of $A$:

$$C^*(x) = \overline{\{p(x, x^*) \mid p \in \mathbb{C}[x, y]\}}.$$

**Corollary 2.33.** *Let $A$ be an abelian $C^*$-algebra, generated by $x \in A$. Then $\sigma(A) \cong \sigma(x)$.*

*Proof.* Let $\Gamma : A \to C(\sigma(A))$ be the Gelfand transform. Define

$$\tau : \sigma(A) \to \sigma(x), \quad \varphi \mapsto \varphi(x) = \Gamma(x)(\varphi).$$

Clearly, $\tau$ is well-defined since $\varphi(x) \in \sigma(x)$ for all $\varphi \in \sigma(A)$. Next we show that $\tau$ is onto. For $\lambda \in \sigma(x)$ we have $x - \lambda \notin \mathrm{GL}(A)$, so there exists $\psi \in \sigma(A)$ such that $\psi(x) - \psi(\lambda) = \psi(x - \lambda) = 0$. We show that $\tau$ is injective. Let $\tau(\varphi_1) = \tau(\varphi_2)$. Then $\varphi_1(x) = \varphi_2(x)$. Since

$$\varphi_j(x^*) = \Gamma(x^*)(\varphi_j) = \overline{\Gamma(x)(\varphi_j)} = \overline{\varphi_j(x)},$$

we have $\varphi_1(x^*) = \varphi_2(x^*)$. Hence $\varphi_1(p(x, x^*)) = \varphi_2(p(x, x^*))$ for every polynomial $p \in \mathbb{C}[x, y]$. Since $\{p(x, x^*) \mid p \text{ polynomial}\}$ is dense in $A$, we have $\varphi_1 = \varphi_2$. Finally, we prove the continuity of $\tau$. Let $(\varphi_\alpha)_\alpha$ be a net in $\sigma(A)$ such that $\varphi_\alpha \to \varphi$. Then $\varphi_\alpha(y) \to \varphi(y)$ for all $y \in A$, so in particular $\varphi_\alpha(x) \to \varphi(x)$, which proves that $\tau(\varphi_\alpha) \to \tau(\varphi)$. Since $\tau$ is a continuous bijection between compact Hausdorff spaces, it is a homeomorphism. □

*Remark.* Since $\varphi \in \sigma(A)$ is an algebra homomorphism, we have $\varphi(p(x, x^*)) = p(\varphi(x), \overline{\varphi(x)})$ for a complex polynomial $p(z, \overline{z})$ in $z$ and $\overline{z}$. Using the notation from above proof, we get $\Gamma(p(x, x^*)) = p \circ \tau$.

## 2.4 Continuous functional calculus

Now let $A$ be any $C^*$-algebra and $x \in A$ normal. Then $C^*(x)$ is an abelian $C^*$-subalgebra of $A$. Since $\sigma(x) = \sigma_{C^*(x)}$, we have the map

$$\tau^\# : C(\sigma(x)) \to C(C^*(x)), \quad f \mapsto f \circ \tau,$$

which is a $*$-isomorphism and an isometry. Define a map $\rho = \Gamma^{-1} \circ \tau^{\#} : C(\sigma(x)) \to C^*(x)$.

$$\begin{tikzcd} C^*(x) \arrow[rr, "\Gamma"] & & C(\sigma(A)) \\ & C(\sigma(x)) \arrow[ul, "\rho"] \arrow[ur, "\tau^{\#}"'] & \end{tikzcd}$$

We know that $C^*(x) = \overline{\{p(x, x^*) \mid p(z, \overline{z}) \text{ polynomial}\}}$ and $\Gamma(p(x, x^*)) = \tau^{\#}(p)$, which means that $\rho(p) = p(x, x^*)$ for any polynomial $p \in \mathbb{C}[x, y]$. This map $\rho : C(\sigma(x)) \to C^*(x) \subseteq A$ is called the *continuous functional calculus*. We use the notation $f(x) := \rho(f)$.

**Theorem 2.34** (Continuous functional calculus)**.**

*Let $A, B$ be $C^*$-algebras and let $x \in A$ be normal.*

*(1.) $f \mapsto f(x)$ is an isometric $*$-isomorphism $C(\sigma(x)) \to A$ and if*

$$f = \sum_{j,k=0}^{n} a_{jk} z^j \overline{z}^k$$

*is a polynomial, then*

$$f(x) = \sum_{j,k=0}^{n} a_{jk} x^j (x^*)^k.$$

*In particular, if $f(z) = z$ is the identity polynomial, then $f(x) = x$.*

*(2.) For $f \in C(\sigma(x))$, we have $\sigma(f(x)) = f(\sigma(x))$.*

*(3.) **(Spectral mapping theorem)** If $\Phi : A \to B$ is a $*$-homomorphism, then $\Phi(f(x)) = f(\Phi(x))$.*

*(4.) Let $(x_n)_n$ be a sequence of normal elements of $A$ that converge to $x$, $\Omega$ a compact neighborhood of $\sigma(x)$, and $f \in C(\Omega)$. Then for any sufficiently large $n$, we have $\sigma(x_n) \subseteq \Omega$ and $\|f(x_n) - f(x)\| \to 0$.*

*Proof.* The items (1) and (2) follow directly from Gelfand's theorem and properties of continuous functions on compact sets. The item (3) is obvious for polynomials $f$ and the general case follows from Stone–Weierstrass. We prove the item (4). Let $C = \sup_n \|x_n\| < \infty$. First we need to show that $\sigma(x_n) \subseteq \Omega$ for large enough $n$. If that wasn't the case, then for every $n \in \mathbb{N}$ there would exist $N_n > n$ such that there exists $\lambda_n \in \sigma(x_{N_n}) \setminus \Omega \subseteq \overline{B_C(0)}$. Thus there exists a convergent subsequence $(\lambda_{n_k})_k$ such that $\lambda_{n_k} \to \lambda \in U$, where $U$ is an open neighborhood of $\sigma(x)$ and $\lambda \notin \sigma(x)$. But then

$$\underbrace{x_{n_k} - \lambda_{n_k}}_{\notin \mathrm{GL}(A)} \to \underbrace{x - \lambda}_{\in \mathrm{GL}(A)},$$

which contradicts the openness of $\mathrm{GL}(A)$. For every $\varepsilon > 0$ there exists a polynomial $g : \Omega \to$

$\mathbb{C}$ such that $\|f - g\|_\infty < \varepsilon$. Now

$$\begin{aligned}\limsup_n \|f(x_n) - g(x_n)\| + \|g(x_n) - g(x)\| + \|g(x) - f(x)\| \\ &\leq 2 \cdot C \cdot \varepsilon + \limsup_n \|g(x_n) - g(x)\| \\ &= 2C\varepsilon.\end{aligned}$$

Since $\varepsilon$ was arbitrary, we conclude that $\lim_{n\to\infty} \|f(x_n) - f(x)\| = 0$. □

We illustrate the use of continuous functional calculus to obtain the strengthening of corollary 2.13.

**Corollary 2.35.** *If $A, B$ are $C^*$-algebras and $\Phi : A \to B$ is a $*$-monomorphism, then it is an isometry.*

*Proof.* Let $a \in A$ be self-adjoint. Then $\Phi(a) \in B$ is self-adjoint as well. As in the proof of 2.13, we observe that $\sigma_B(\Phi(a)) \subseteq \sigma_A(a)$. Suppose that $\sigma_B(\Phi(a)) \neq \sigma_A(a)$. Since $\sigma_B(\Phi(a))$ is compact, it is closed in $\sigma_A(a)$. This implies that $U := \sigma_A(a) \setminus \sigma_B(\Phi(a))$ is a nonempty open set. It follows that there exists a function $f$ which is zero on $\sigma_B(\Phi(a))$, but not identically zero on $\sigma_A(a)$ (take for example any bump function on $U$). Then $f(\Phi(a)) = 0$, but $f(a) \neq 0$. By Stone–Weierstrass, we can approximate $f$ uniformly on $\sigma_A(a)$ by polynomials $\{p_n\}_{n\in\mathbb{N}}$. Thus $p_n(a) \to f(a)$ and $p_n(\Phi(a)) \to f(\Phi(a)) = 0$. On the other hand, $p_n(\Phi(a)) = \Phi(p_n(a)) \to \Phi(f(a))$, which implies that $\Phi(f(a)) = f(\Phi(a)) = 0$. But $\Phi$ was assumed injective, so $f(a) = 0$, contradiction. Therefore, $\sigma_B(\Phi(a)) = \sigma_A(a)$ for self-adjoint $a$ and

$$\|a\| = r(a) = r(\Phi(a)) = \|\Phi(a)\|.$$

Now for a completely arbitrary $a \in A$, we have

$$\|a\|^2 = \|a^*a\| = \|\Phi(a^*a)\| = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a)\|^2,$$

concluding our proof. □

The argument in this proof is very common. We first approximate some function on the spectrum with polynomials using Stone–Weierstrass. Then we observe that the continuous functional calculus of a polynomial has desired properties and deduce the same for the continuous functional calculus of the original function.

## 2.5 Applications of the continuous functional calculus

**Definition 2.36.** Let $A$ be a $C^*$-algebra and $x \in A$.

- $x$ is *positive* if $x = y^*y$ for some $y \in A$ (i.e., $x$ is a hernitian square). The set of positive elements is denoted $A_+$.
- $x$ is a *projection* if $x^2 = x^* = x$.
- $x$ is *unitary* if $xx^* = x^*x = 1$. The set of positive elements is denoted $U(A)$.
- $x$ is an *isometry* if $x^*x = 1$.
- $x$ is a *partial isometry* if $x^*x$ is a projection.

*Remark.* The first three are automatically normal (the first two are even self-adjoint).

The set of all positive operators (denoted as $A_+$) induces a partial ordering on $A_{\text{sa}}$: for two elements $a, b \in A_{\text{sa}}$ we define

$$a \leq b \Leftrightarrow b - a \in A_+.$$

We notice that $x^* A_+ x \subseteq A_+$ for every $x \in A$. For any $a, b \in A_{\text{sa}}$ and $x \in A$, we have

$$a \leq b \Rightarrow x^* a x \leq x^* b x.$$

**Proposition 2.37.** *Let $A$ be a $C^*$-algebra and $x \in A$. Then $x$ is a linear combination of four unitaries.*

*Proof.* Since $x = \operatorname{Re} x + i \operatorname{Im} x$, where $\operatorname{Re} x, \operatorname{Im} x \in A_{\text{sa}}$, it's enough to show that every self-adjoint element is a linear combination of two unitaries. Without loss of generality, assume $\|x\| \leq 1$, so $\sigma(x) \subseteq [-1, 1]$. Consider the continuous function

$$f : [-1, 1] \to \mathbb{T}, \quad z \mapsto z + i(1 - z^2)^{\frac{1}{2}}.$$

Since $f \cdot \overline{f} \equiv 1$ on $[-1, 1]$, it follows from continuous functional calculus that

$$f(x) f(x)^* = f(x)^* f(x) = 1.$$

Consequently, $f(x) = u$ is unitary and $x = \frac{1}{2}(f(x) + f(x)^*)$ is a linear combination of two unitaries. $\square$

*Remark.* We use the notation $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

**Definition 2.38.** Let $x \in A_{\text{sa}}$. Then $\sigma(x) \subseteq \mathbb{R}$ and we can define

$$x_+ = \max\{0, z\}(x) \in A, \quad x_- = -\min\{0, z\}(x) \in A.$$

Then $\sigma(x_+), \sigma(x_-) \subseteq [0, \infty)$, $x = x_+ - x_-$ and $x_+ x_- = x_- x_+ = 0$.

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**Lemma 2.39.** *Suppose $x, y \in A_{\text{sa}}$ satisfy $\sigma(x), \sigma(y) \subseteq [0, \infty)$. Then $\sigma(x + y) \subseteq [0, \infty)$.*

*Proof.* Let $a := \|x\|$ and $b := \|y\|$. Since $x = x^*$ and $\sigma(x) \subseteq [0, a]$, we deduce that $\sigma(a - x) \subseteq [0, a]$, where $\|a - x\| = r(a - x) \leq a$. Likewise, $\|b - y\| \leq b$. Then

$$\begin{aligned}
\sup_{\lambda \in \sigma(x+y)} \{a + b - \lambda\} &= r(a + b - (x + y)) \\
&= \|(a + b) - (x + y)\| \\
&\leq \|a - x\| + \|b - y\| \\
&\leq a + b.
\end{aligned}$$

$\square$

**Theorem 2.40.**

*Let $A$ be a $C^*$-algebra and $x \in A$ normal. Then:*

*(1.)* $x \in A_{\mathrm{sa}} \Leftrightarrow \sigma(x) \subseteq \mathbb{R}$;

*(2.)* $x \in A_+ \Leftrightarrow \sigma(x) \subseteq [0, \infty)$;

*(3.)* $x \in U(A) \Leftrightarrow \sigma(x) \subseteq \mathbb{T}$;

*(4.)* $x^2 = x^* = x \Leftrightarrow \sigma(x) \subseteq \{0, 1\}$.

*Proof.* Throughout this proof, let $f(z) = z$ denote the identity polynomial.

(1.)

$$\begin{aligned} x = x^* &\Leftrightarrow f(x) = \overline{f}(x) \\ &\Leftrightarrow f \equiv \overline{f} \text{ on } \sigma(x) \\ &\Leftrightarrow z = \overline{z} \text{ for all } z \in \sigma(x) \\ &\Leftrightarrow \sigma(x) \subseteq \mathbb{R}. \end{aligned}$$

(2.) ($\Rightarrow$) Let $x = y^*y$ for some $y \in A$. Write $x = x_+ - x_-$ and let $z := y \cdot x_-$. Then

$$z^*z = x_-y^*yx_- = x_-xx_- = -x_-^3.$$

From there we get

$$\sigma(zz^*) \subseteq \sigma(z^*z) \cup \{0\} \subseteq (-\infty, 0].$$

Let $z = a + ib$ for $a, b \in A_{\mathrm{sa}}$. Then $zz^* + z^*z = 2a^2 + 2b^2$, which implies that $\sigma(zz^* + z^*z) \subseteq [0, \infty)$. It follows that

$$\sigma(z^*z) = \sigma((2a^2 + 2b^2) - zz^*) \subseteq [0, \infty).$$

As a result,

$$\sigma(-x_-^3) = \sigma(z^*z) \subseteq \{0\},$$

so $x_-^3 = 0$ and $x_- = 0$. This proves that $x = x_+$ has nonnegative spectrum. For the converse implication ($\Leftarrow$), apply the function $\sqrt{\cdot} : [0, \infty) \to \mathbb{R}$. Then

$$x = (\sqrt{x})^2 = (\sqrt{x})^* \cdot \sqrt{x} \in A_+.$$

(3.)

$$\begin{aligned} xx^* = 1 &\Leftrightarrow f(x) \cdot \overline{f}(x) = 1 \\ &\Leftrightarrow f \cdot \overline{f} \equiv 1 \text{ on } \sigma(x) \\ &\Leftrightarrow |z|^2 = 1 \text{ for all } z \in \sigma(x) \\ &\Leftrightarrow \sigma(x) \subseteq \mathbb{T}. \end{aligned}$$

(4.)

$$\begin{aligned} x^2 = x^* = x &\Leftrightarrow f(x) \cdot \overline{f}(x) = \overline{f}(x) = f(x) \\ &\Leftrightarrow f \cdot \overline{f} \equiv \overline{f} \equiv f \text{ on } \sigma(x) \\ &\Leftrightarrow |z|^2 = \overline{z} = z \text{ for all } z \in \sigma(x) \\ &\Leftrightarrow \sigma(x) \subseteq \{0, 1\}. \end{aligned}$$

$\square$

**Corollary 2.41.** *Let $A$ be a $C^*$-algebra and $x \in A$. Then $x$ is a partial isometry iff $x^*$ is a partial isometry.*

*Proof.*

$$
\begin{aligned}
x \text{ partial isometry} &\Leftrightarrow x^*x \text{ projection} \\
&\Leftrightarrow \sigma(x^*x) \subseteq \{0,1\} \\
&\Leftrightarrow \sigma(xx^*) \subseteq \{0,1\} \\
&\Leftrightarrow xx^* \text{ projection} \\
&\Leftrightarrow x^* \text{ partial isometry.}
\end{aligned}
$$

□

**Corollary 2.42.** *Let $A$ be a $C^*$-algebra.*

*(1.) $A_+$ is a closed convex cone ($\lambda A_+ \subseteq A_+$ for $\lambda \in \mathbb{R}_{\geq 0}$).*

*(2.) If $a \in A_{\mathrm{sa}}$, then $a \leq \|a\|$.*

**Proposition 2.43.** *Let $A$ be a $C^*$-algebra and $x, y \in A_+$.*

*(1.) If $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.*

*(2.) If $x, y \in \mathrm{GL}(A)$ and $x \leq y$, then $y^{-1} \leq x^{-1}$.*

*Proof.* Let us prove the second point first. Suppose $x, y \in \mathrm{GL}(A)$. Then we have $y^{-\frac{1}{2}} x y^{-\frac{1}{2}} \leq 1$ and

$$
\begin{aligned}
x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}} &\leq \|x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}}\| \\
&= r(x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}}) \\
&= r(y^{-\frac{1}{2}} x y^{-\frac{1}{2}}) \\
&\leq 1.
\end{aligned}
$$

Multiplying on both sides by $x^{-\frac{1}{2}}$, we get $y^{-1} \leq x^{-1}$. Now we prove the first point. For invertible $x \leq y$, we have

$$
\begin{aligned}
\|y^{-\frac{1}{2}} x^{\frac{1}{2}}\|^2 &= \|(y^{-\frac{1}{2}} x^{\frac{1}{2}})(y^{-\frac{1}{2}} x^{\frac{1}{2}})^*\| \\
&= \|y^{-\frac{1}{2}} x y^{-\frac{1}{2}}\| \\
&\leq 1,
\end{aligned}
$$

which implies

$$
\begin{aligned}
y^{-\frac{1}{4}} x^{\frac{1}{2}} y^{-\frac{1}{4}} &\leq \|y^{-\frac{1}{4}} x^{\frac{1}{2}} y^{-\frac{1}{4}}\| \\
&= r(y^{-\frac{1}{4}} x^{\frac{1}{2}} y^{-\frac{1}{4}}) \\
&= r(y^{-\frac{1}{2}} x^{\frac{1}{2}}) \\
&= \|y^{-\frac{1}{2}} x^{\frac{1}{2}}\| \leq 1.
\end{aligned}
$$

Multiplying on both sides by $y^{\frac{1}{4}}$, we get $y^{\frac{1}{2}} \le x^{\frac{1}{2}}$. For general non-invertible $x \le y$, pick $\varepsilon > 0$ and notice that

$$0 \le x + \varepsilon \le y + \varepsilon.$$

However, since $x, y$ are positive, we also have $x + \varepsilon, y + \varepsilon \in \mathrm{GL}(A)$. We use the above calculation to obtain $(x+\varepsilon)^{\frac{1}{2}} \le (y+\varepsilon)^{\frac{1}{2}}$. If we send $\varepsilon \to 0$, we get $x^{\frac{1}{2}} \le y^{\frac{1}{2}}$. □

*Remark.* Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ be continuous. Then the function $f$ is operator monotone if for every $C^*$-algebra $A$ and $a, b \in A_{\mathrm{sa}}$ with $a \le b$ and $\sigma(a), \sigma(b) \subseteq I$, we have $f(a) \le f(b)$. By the above proposition, $z \mapsto \sqrt{z}$ and $z \mapsto \frac{1}{z}$ are operator monotone on $[0, \infty)$. Actually, this is also true for functions $z \mapsto z^r$ for $r \in [0, 1]$, but not for $r > 1$.

**Definition 2.44.** Absolute value of $x \in A$ is defined as

$$|x| = (x^* x)^{\frac{1}{2}} \in A_+.$$

**Corollary 2.45.** *For $x, y \in A$, we have $|xy| \le \|x\| |y|$.*

*Proof.* Notice that

$$|xy|^2 = y^* x^* x y \le y^* \|x^* x\| y = \|x\|^2 (y^* y)$$

and now apply the operator-monotone $\sqrt{\cdot}$ and the previous proposition. □

**Theorem 2.46.**

*Let $A$ be a $C^*$-algebra.*

*(1.)* $\mathrm{ext}(A_+)_1 = \{$*projections in* $A\}$.

*(2.)* $\mathrm{ext}(A)_1 \subseteq \{$*partial isometries in* $A\}$.

*(3.)* $\mathrm{ext}(A_{\mathrm{sa}})_1 = U(A) \cap A_{\mathrm{sa}}$.

*Proof.* (1.) Let $x \in (A_+)_1$. Then $x^2 \le 2x$, since $z^2 - 2z \le 0$ on $[0, 1] \supseteq \sigma(x)$. So $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$. If $x$ is an extreme point, then $x = x^2$ and $x \in A_+ \subseteq A_{\mathrm{sa}}$, so $x$ is a projection. For the converse, assume $A$ is abelian, meaning $A = C(K)$ for some compact Hausdorff space $K$ (by Gelfand). If $x \in A = C(K)$ is a projection, then $x = \chi_E$ for some clopen $E \subseteq K$. Since $\mathrm{ext}([0, 1]) = \{0, 1\}$, $\chi_E$ is an extreme point. Let $A$ now be a general $C^*$-algebra and $p \in A^*$ a projection. Suppose $p = \frac{1}{2}(a + b)$ for some $a, b \in (A_+)_1$. Then $\frac{1}{2}a = p - \frac{1}{2}b \le p$. Hence

$$0 \le (1-p)a(1-p) \le (1-p)2p(1-p) = 0,$$

so

$$(\sqrt{a}(1-p))^*(\sqrt{a}(1-p)) = (1-p)a(1-p) = 0.$$

This implies that $\sqrt{a}(1-p) = 0$ and $a(1-p) = 0$. It follows that

$$ap = a = a^* = (ap)^* = p^* a^* = pa.$$

Similarly, we can show that $a, b, p$ all commute, so the $C^*$-subalgebra $C^*(a, b, p)$ is abelian and we can just use the previous observation.

(2.) Suppose $x \in (A)_1$ is not a partial isometry (alternatively, $x^*x$ is not a projection). First, we notice that $\|x^*x\| = \|x\|^2 \leq 1$. Since $x$ is not a projection, $\sigma(x^*x) \cap (0,1) \neq \emptyset$. Then we apply the continuous functional calculus to obtain a function $f : \sigma(x^*x) \to [0,1]$ such that $|t(1 \pm f(t))^2| \leq 1$ for $t \in \sigma(x^*x)$ (for example, $f$ can be a small bump function on an interval $[a, b] \subseteq (0,1)$, where $[a,b] \cap \sigma(x^*x) \neq \emptyset$). Then $y := f(x^*x) \in A_+$ gives us $yx^*x = x^*xy \neq 0$ and $\|x^*x(1 \pm y)^2\| \leq 1$. Hence, $\|x(1 \pm y)\|^2 \leq 1$ and

$$x = \frac{1}{2}((x + xy) + (x - xy)) \notin \operatorname{ext}(A)_1.$$

(3.) If $u \in U(A) \cap A_{\text{sa}}$, then $x \mapsto ux$ is an isometry. As in the case of $\mathcal{B}(\mathcal{H})$, $u$ is an extreme point, so $A_{\text{sa}} \cap U(A) \subseteq \operatorname{ext}(A_{\text{sa}})_1$. For the converse, assume $x \in \operatorname{ext}(A_{\text{sa}})_1$ and $x_+ = \frac{1}{2}(a + b)$ for $a, b \in (A_+)_1$. Then

$$0 = x_- x_+ x_- = \frac{1}{2}(x_- a x_- + x_- b x_-) \geq 0.$$

From $x_- a x_- = 0$, we get $(\sqrt{a} x_-)^*(\sqrt{a} x_-) = 0$, which implies that $\sqrt{a} x_- = 0$ and $a x_- = 0$. Likewise, $x_- a = b x_- = x_- b = 0$. By Gelfand, the commutative $C^*$-algebra $C^*(a, b, x_-)$ is isometrically $*$-isomorphic to $C(K)$ for some compact $K$. This means that $a$ and $x_-$ are functions such that for every point in $K$, at least one of them is zero. Thus, $a - x_-$ is bounded above by 1, and we have $a - x_- \in (A_{\text{sa}})_1$. Similarly, $b - x_- \in (A_{\text{sa}})_1$, so

$$x = \frac{1}{2}((a - x_-) + (b - x_-)) \in (A_{\text{sa}})_1.$$

But since $x$ is an extreme point, we have $a - x_- = b - x_-$ and $a = b = x_+$. Thus, $x_+ \in \operatorname{ext}(A_+)_1$ is a projection by (1.), and by symmetry, so is $x_-$. Now we prove that $x$ is unitary:

$$x^*x = x^2 = (x_+ - x_-)^2 = x_+^2 + x_-^2 = x_+ + x_- = |x|.$$

This implies that $|x|$ is a projection. Now set $q := 1 - |x|$. Then $x + q$ and $x - q$ are both in $(A_{\text{sa}})_1$. But since

$$x = \frac{1}{2}((x + q) + (x - q)),$$

we obtain $q = 0$, which further implies $|x| = 1$ and $x^*x = xx^* = 1$. $\square$

# 3 Representations of $C^*$-algebras and states

## 3.1 States

Let $A$ be a $C^*$-algebra, then $A^*$ can be given an $A$-bimodule structure: if $\psi \in A^*$ and $a, b \in A$, then

$$(a \cdot \psi \cdot b)(x) = \psi(bxa), \quad \forall x \in A.$$

We have

$$\|a \cdot \psi \cdot b\| = \sup_{x \in (A)_1} \|\psi(bxa)\| \leq \sup_{x \in (A)_1} \|\psi\| \|bxa\| \leq \|\psi\| \|a\| \|b\|.$$

**Definition 3.1.** Let $A$ be a $C^*$-algebra and $\varphi \in A^*$.

- We say that $\varphi$ is *positive* if $\varphi(x) \geq 0$, $\forall x \in A_+$. If $\varphi$ is positive and $a \in A$, then $a\varphi a^*$ is also positive.
- A positive element $\varphi \in A^*$ is *faithful* if $\varphi(x) \neq 0$, $\forall x \in A_+ \setminus \{0\}$.
- An element $\varphi \in A^*$ is a state if it is *positive* and $\|\varphi\| = 1$. The set of states is denoted $S(A) \subseteq (A^*)_1$.

*Remark.* The set $S(A)$ is compact Hausdorff in the weak-$*$ topology.

We notice that if $\varphi \in A^*$ is positive and $x \in A_{\mathrm{sa}}$, then

$$\varphi(x) = \varphi(x_+ - x_-) = \varphi(x_+) - \varphi(x_-) \in \mathbb{R}.$$

If $y \in A$, then $y = y_1 + iy_2$, where $y_1, y_2$ are self-adjoint. Then

$$\begin{aligned} \varphi(y^*) &= \varphi((y_1 + iy_2)^*) = \varphi(y_1 - iy_2) \\ &= \varphi(y_1) - i\varphi(y_2) = \overline{\varphi(y_1) + i\varphi(y_2)} \\ &= \overline{\varphi(y_1 + iy_2)} = \overline{\varphi(y)} \end{aligned}$$

Such a functional $\varphi \in A^*$ is called *hermitian*. For any $\varphi \in A^*$, $\varphi^*(y) = \overline{\varphi(y^*)}$. Then $\varphi + \varphi^*$ and $i(\varphi - \varphi^*)$ are hermitian. One can, of course, define these notions also for unbounded linear functionals. However, positivity implies continuity: for every $a \in A_{\mathrm{sa}}$ we have $-\|a\| \cdot 1 \leq a \leq \|a\| \cdot 1$, which implies

$$-\|a\|\varphi(1) \leq \varphi(a) \leq \|a\|\varphi(1)$$

and $\varphi$ is bounded. For $a \in A$, we can of course write $a = b + ic$ for $b, c \in A_{\mathrm{sa}}$. Here,

$$\|b\| = \left\| \frac{a + a^*}{2} \right\| \leq \frac{\|a\|}{2} + \frac{\|a^*\|}{2} = \|a\|$$

and likewise $\|c\| \leq \|a\|$. Let $\varphi(1) = C$. Then

$$|\varphi(a)|^2 = |\varphi(b) + i\varphi(c)|^2 = \varphi(b)^2 + \varphi(c)^2 \leq C^2(\|b\|^2 + \|c\|^2) \leq 2C^2\|a\|^2.$$

**Lemma 3.2.** *Let $\varphi \in A^*$ be positive. Then $\forall x, y \in A$:*

$$|\varphi(y^*x)|^2 \leq \varphi(y^*y) \cdot \varphi(x^*x).$$

*Proof.* Consider the sesquilinear form $\langle x, y\rangle = \varphi(y^*x)$. Since $\varphi$ is positive, this is a positive sesquilinear form and we can apply Cauchy-Schwartz. $\square$

**Theorem 3.3.**

*An element $\varphi \in A^*$ is positive iff $\|\varphi\| = \varphi(1)$.*

*Remark.* This implies that the set of states $S(A)$ is convex.

*Proof.* First we prove the right implication ($\Rightarrow$). We know that $x^*x \leq \|x^*x\|$, so

$$\begin{aligned}|\varphi(x)|^2 &\leq \varphi(1)\varphi(x^*x)\\ &\leq \varphi(1)\varphi(\|x^*x\|)\\ &= \varphi(1)^2\|x^*x\|\\ &= \varphi(1)^2\|x\|^2,\end{aligned}$$

so $|\varphi(x)| \leq \varphi(1)\|x\|$. From there we get $\|\varphi\| \leq \varphi(1) \leq \|\varphi\|$, so $\varphi(1) = \|\varphi\|$. Now the converse ($\Leftarrow$). Suppose $x \in A_+$ and $\varphi(x) = \alpha + i\beta$. For each $t \in \mathbb{R}$, we have

$$\begin{aligned}\alpha^2 + (\beta + t\|\varphi\|)^2 &= |\alpha + i(\beta + t\varphi(1))|^2\\ &= |\varphi(x + it)|^2\\ &\leq \|x + it\|^2 \cdot \|\varphi\|^2\\ &= \left(\|x\|^2 + t^2\right)\|\varphi\|^2.\end{aligned}$$

From this it directly follows $2\beta t\|\varphi\| \leq \|x\|^2 \cdot \|\varphi\|^2$. Since $t \in \mathbb{R}$ was arbitrary, we have $\beta = 0$ and $\varphi(x) = \alpha \in \mathbb{R}$. Lastly, we derive

$$\begin{aligned}\|x\| \cdot \|\varphi\| - \varphi(x) &= \varphi(\|x\| - x)\\ &\leq \|\|x\| - x\| \cdot \|\varphi\|\\ &\leq \|x\| \cdot \|\varphi\|,\end{aligned}$$

so $\varphi(x) \geq 0$. $\square$

**Proposition 3.4.** *Let $A$ be a $C^*$-algebra and $x \in A$. Then $\forall \lambda \in \sigma(x)$ there exists a $\varphi \in S(A)$ such that $\varphi(x) = \lambda$.*

*Proof.* We know that $\mathbb{C}x + \mathbb{C} \cdot 1 \subseteq A$. Define

$$\varphi_0 : \mathbb{C}x + \mathbb{C}1 \to \mathbb{C}, \quad \alpha x + \beta \mapsto \alpha \cdot \lambda + \beta.$$

Since $\varphi_0(\alpha x + \beta) \in \sigma(\alpha x + \beta)$, we have

$$\|\varphi_0\| \leq 1 = \varphi_0(1),$$

therefore $\|\varphi_0\| = 1$. Now we apply Hahn–Banach to get an extension $\varphi \in A^*$ such that $\varphi\big|_{\mathbb{C}x+\mathbb{C}1} = \varphi_0$ and $\|\varphi\| = 1 = \varphi(1)$, so $\varphi \in S(A)$ by theorem 3.3. $\square$

**Proposition 3.5.** *Let $A$ be a $C^*$-algebra and $x \in A$.*

*(1.)* $x = 0$ *iff* $\varphi(x) = 0,\ \forall \varphi \in S(A)$.

*(2.)* $x \in A_{\mathrm{sa}}$ *iff* $\varphi(x) \in \mathbb{R},\ \forall \varphi \in S(A)$.

*(3.)* $x \in A_+$ *iff* $\varphi(x) \geq 0,\ \forall \varphi \in S(A)$.

*Proof.* (1.) If $\varphi(x) = 0$ for all $\varphi \in S(A)$, then writing $x = x_1 + ix_2$ for self-adjoint $x_1, x_2$ gives us

$$0 = \varphi(x) = \varphi(x_1) + i\varphi(x_2),$$

which implies $\varphi(x_1) = \varphi(x_2) = 0$. Now use the proposition 3.5 to get $\sigma(x_1) = \sigma(x_2) = \{0\}$, which can only imply $x_1 = x_2 = 0$, whence $x = 0$.

(2.) If $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(A)$, then

$$\varphi(x - x^*) = \varphi(x) - \varphi(x^*) = \varphi(x) - \overline{\varphi(x)} = 0$$

and we use the previous item to show that $x - x^*$. The converse implication follows from the fact that every positive functional is hermitian.

(3.) If $\varphi(x) \geq 0$ for all $\varphi \in S(A)$, then $x \in A_{\mathrm{sa}}$ by previous item and $\sigma(x) \subseteq [0, \infty)$, so $x \in A_+$. The converse once again follows from positivity of $\varphi$. ☐

## 3.2 Gelfand-Naimark-Segal construction

**Definition 3.6.**

- A *representation* of a $C^*$-algebra $A$ is a $*$-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.
- If $\mathcal{K}^{\text{closed}} \leq \mathcal{H}$ and $\pi(x)\mathcal{K} \subseteq \mathcal{K},\ \forall x \in A$ (we say that $\mathcal{K}$ is *invariant* under $\pi$), then the restriction of $\pi$ to $\mathcal{K}$ is a *subrepresentation.*
- If a representation has no other subrepresentations besides $\mathcal{K} = (0)$ and $\mathcal{K} = \mathcal{H}$ (equivalently, $\pi(A)$ only has $(0)$ and $\mathcal{H}$ as closed invariant subspaces), then $\pi$ is called irreducible.
- Representations $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\rho : A \to \mathcal{B}(\mathcal{K})$ are *equivalent* if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that

$$U\pi(x) = \rho(x)U, \quad \forall x \in A.$$

- Vector $\zeta \in \mathcal{H}$ is *cyclic* for a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ if

$$\pi(A)\zeta := \{\pi(a)\zeta \mid a \in A\}$$

is dense in $\mathcal{H}$ (this means that $\overline{\pi(A)\zeta} = \mathcal{H}$).

**Example 3.7.** *Each $w \in \mathcal{H}$ defines a subrepresentation on $K := \overline{\pi(A)w}$.*

**Example 3.8.** *Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation and $\mu \in \mathcal{H}$, $\|\mu\| = 1$. Then*

$$\varphi_\mu : A \to \mathbb{C}, \quad x \mapsto \langle \pi(x)\mu, \mu \rangle$$

*is a state. Indeed,*

$$\varphi_\mu(1) = \langle 1 \cdot \mu, \mu\rangle = \|\mu\|^2 = 1$$

*and*

$$\begin{aligned}\varphi_\mu(x^*x) &= \langle \pi(x^*x)\mu, \mu\rangle \\ &= \langle \pi(x^*)\pi(x)\mu, \mu\rangle \\ &= \langle \pi(x)^*\pi(x)\mu, \mu\rangle \\ &= \langle \pi(x)\mu, \pi(x)\mu\rangle \\ &= \|\pi(x)\mu\|^2 \geq 0.\end{aligned}$$

**Theorem 3.9** (Gelfand-Naimark-Segal construction)**.**

*Let $A$ be a $C^*$-algebra and $\rho \in S(A)$. Then there exists a Hilbert space $L^2(A, \varphi)$ and a unique (up to equivalence) representation $\pi : A \to \mathcal{B}(L^2(A, \varphi))$ and a unit cyclic vector $1_\varphi$ such that*

$$\varphi(x) = \langle \pi(x)1_\varphi, 1_\varphi\rangle, \quad \forall x \in A.$$

*Proof.* (1.) We start by defining

$$N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$$

whose elements we call nullvectors of $\varphi$. By the Cauchy-Schwartz lemma, we have

$$N_\varphi = \{x \in A \mid \varphi(yx) = 0, \ \forall y \in A\}.$$

Thus $N_\varphi$ is a closed subspace of $A$.

(2.) We prove that $N_\varphi$ is a left ideal: for $x \in N_\varphi$ and $a \in A$, we have $ax \in N_\varphi$. Indeed,

$$\varphi((ax)^*ax) = \varphi((x^*a^*a)x) = 0.$$

(3.) Now $\mathcal{H}_0 = A/N_\varphi$ is a vector space and we can endow it with the dot product $\langle [x], [y]\rangle := \varphi(y^*x)$ for $x, y \in A$. It can easily be checked that this is a well-defined dot product in $\mathcal{H}_0$. We denote the completion of $\mathcal{H}_0$ by $L^2(A, \varphi)$.

(4.) To an arbitrary $a \in A$, we associate the map

$$\pi_0(a) : \mathcal{H}_0 \to \mathcal{H}_0, \quad [x] \mapsto [ax].$$

Since $N_\varphi$ is a left ideal of $A$, $\pi_0(a)$ is a well-defined linear map. We have

$$\begin{aligned}\|\pi_0(a)[x]\|^2 &= \|[ax]\|^2 \\ &= \langle [ax], [ax]\rangle \\ &= \varphi((ax)^*ax) \\ &= \varphi(x^*a^*ax) \\ &\leq \|a\|^2 \cdot \varphi(x^*x) \leq \|a\|^2\|x\|^2.\end{aligned}$$

Since $\pi_0(a)$ is a bounded linear map, it exdends uniquely to $\pi(a) \in \mathcal{B}(L^2(A, \varphi))$ with $\|\pi(a)\| \leq \|a\|$. Then we get

$$\pi : A \to \mathcal{B}(L^2(A, \varphi)), \quad a \mapsto \pi(a),$$

which is a homomorphism and has the property

$$\begin{aligned}\langle [x], \pi(a^*)[y] \rangle &= \langle [x], [a^* y] \rangle \\ &= \varphi((a^* y)^* x) \\ &= \varphi(y^* a x) \\ &= \langle [ax], [y] \rangle \\ &= \langle \pi(a)[x], [y] \rangle.\end{aligned}$$

So $\pi(a)^* = \pi(a^*)$ and $\pi$ is a representation.

(5.) We define $1_\varphi := [1] \in \mathcal{H}_0 \subseteq L^2(A, \varphi)$ and notice that

$$\langle \pi(a) 1_\varphi, 1_\varphi \rangle = \langle \pi(a)[1], [1] \rangle = \langle [a], [1] \rangle = \varphi(a).$$

Since $\{\pi(a) 1_\varphi \mid a \in A\} = \mathcal{H}_0$ is dense in $L^2(A, \varphi)$, the vector $1_\varphi$ is cyclic for $\pi$.

(6.) Next we prove uniqueness: let $\rho : A \to \mathcal{B}(\mathcal{K})$ be a representation, $\mu \in \mathcal{K}$ a unit cyclic vector and assume $\varphi(a) = \langle \rho(a)\mu, \mu \rangle$, $\forall a \in A$. We will prove that $\rho$ is equivalent to $\pi$. Define

$$U_0 : \mathcal{H}_0 \to \mathcal{K}, \quad [x] \mapsto \rho(x)\mu.$$

Then we have

$$\begin{aligned}\langle U_0[x], U_0[y] \rangle_{\mathcal{K}} &= \langle \rho(x)\mu, \rho(y)\mu \rangle \\ &= \langle \rho(y)^* \rho(x)\mu, \mu \rangle \\ &= \langle \rho(y^* x)\mu, \mu \rangle = \varphi(y^* x) = \langle [x], [y] \rangle_{L^2(A,\varphi)},\end{aligned}$$

so $U_0$ really is a well-defined isometry. For all $a, x \in A$:

$$U_0(\pi(a)[x]) = U_0([ax]) = \rho(ax)\mu = \rho(a)\rho(x)\mu = \rho(a) U_0[x].$$

Therefore, $U_0$ induces an isometry $U : L^2(A, \varphi) \to \mathcal{K}$ such that $U\pi(a) = \rho(a)U$ for all $a \in A$. Since $\mu$ is cyclic and $\rho(a)\mu \subseteq \operatorname{im} U$, the range of $U$ is dense in $\mathcal{K}$. It is also closed since $U$ is isometric. We just proved that $U$ is isometric and onto, so it is unitary. $\square$

**Corollary 3.10.** *Every $C^*$-algebra has a faithful (i.e. injective) representation. In particular, every $C^*$-algebra is isometrically $*$-isomorphic to a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space $\mathcal{H}$.*

*Proof.* Let $\pi$ be a direct sum of all representations from GNS construction over all states. Then the proposition 3.5 tells us that $\pi$ is injective. An injective $*$-monomorphism is isometric and we are done. $\square$

The preceding corollary enables us to view abstract $C^*$-algebras as concrete algebras of operators on some Hilbert space.

**Definition 3.11.** If $S \subseteq A$, then

$$S' := \{x \in A \mid \forall s \in S : \ xs = sx\}$$

is its *commutant.*

**Proposition 3.12** (Radon-Nikodym for linear functionals)**.** *Let $\varphi, \psi$ be positive linear functionals on a $C^*$-algebra $A$ and $\varphi \in S(A)$. Then $\varphi \leq \psi$ iff there exists a unique $y \in \pi_\psi(A)'$ such that $0 \leq y \leq 1$ and*

$$\varphi(a) = \langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle, \quad \forall a \in A.$$

*Proof.* Start with ($\Leftarrow$). For $a \in A_+$ we have

$$\pi_\psi(a) y = \pi_\psi(a)^{\frac{1}{2}} y \pi_\psi(a)^{\frac{1}{2}} \leq \pi_\psi(a).$$

Then

$$\varphi(a) = \langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle \leq \langle \pi_\psi(a) 1_\psi, 1_\psi \rangle = \psi(a).$$

Now the converse ($\Rightarrow$). By Cauchy-Schwartz,

$$\begin{aligned} |\varphi(b^* a)|^2 &\leq \varphi(a^* a)\varphi(b^* b) \\ &\leq \psi(a^* a)\psi(b^* b) \\ &= \|\pi_\psi(a) 1_\psi\|^2 \cdot \|\pi_\psi(b) 1_\psi\|^2. \end{aligned}$$

This means that $\langle \pi_\psi(a) 1_\psi, \pi_\psi(b) 1_\psi \rangle_\varphi := \varphi(b^* a)$ is a nonnegative sesquilinear form on $\pi_\varphi(A) 1_\psi^{\text{dense}} \subseteq L^2(A, \psi)$, which is bounded by 1. This further implies that it is continuous and we can extend it to $L^2(A, \psi)$. By Riesz, there exists $y \in \mathcal{B}(L^2(A, \psi))$ such that

$$\varphi(b^* a) = \langle y \pi_\psi(a) 1_\psi, \pi_\psi(b) 1_\psi \rangle, \quad \forall a, b \in A$$

and $0 \leq y \leq 1$. For $a, b, c \in A$ we have

$$\begin{aligned} \langle y \pi_\psi(a) \pi_\psi(b) 1_\psi, \pi_\psi(c) 1_\psi \rangle &= \langle y \pi_\psi(ab) 1_\psi, \pi_\psi(c) 1_\psi \rangle \\ &= \varphi(c^* \cdot ab) = \varphi((a^* c)^* b) \\ &= \langle y \pi_\psi(b) 1_\psi, \pi_\psi(a^*) \pi_\psi(c) 1_\psi \rangle \\ &= \langle \pi_\psi(a) y \pi_\psi(b) 1_\psi, \pi_\psi(c) 1_\psi \rangle, \end{aligned}$$

so $y \pi_\psi(a) = \pi_\psi(a) y$ for all $a \in A$ and $y \in \pi_\psi(A)'$. Finally, the uniqueness. Say that there exists a $z \in \pi_\psi(A)'$ such that $0 \leq z \leq 1$ and

$$\langle \pi_\psi(a) y 1_\psi, 1_\psi \rangle = \langle \pi_\psi(a) z 1_\psi, 1_\psi \rangle, \quad \forall a \in A.$$

Then

$$\begin{aligned} \langle \pi_\psi(b^* a) z 1_\psi, 1_\psi \rangle &= \langle \pi_\psi(b^* a) y 1_\psi, 1_\psi \rangle \\ &= \langle y \pi_\psi(a) 1_\psi, \pi_\psi(b) 1_\psi \rangle \\ &= \langle z \pi_\psi(a) 1_\psi, \pi_\psi(b) 1_\psi \rangle, \end{aligned}$$

which implies $y = z$. □

**Proposition 3.13.** *Suppose that $A$ is a separable $C^*$-algebra. Then $A$ has a faithful cyclic representation on a separable Hilbert space.*

*Proof.* If $A$ is separable, then it has a dense subset $\{a_i\}_{i=1}^{\infty}$. We can embed $S(A)$ into the space $\prod_{i=1}^{\infty} \overline{B_1(0)}$, where $\overline{B_1(0)}$ is a closed unit ball in $\mathbb{C}$, by sending $\varphi \in S(A)$ to $(\varphi(a_i))_{i=1}^{\infty}$. The latter topological space is metrizable by the metric $\rho(x,y) = \sum_{i=1}^{\infty} \frac{\rho_i(x_i,y_i)}{2^i(\rho_i(x_i,y_i)+1)}$, and so is $S(A)$. Therefore, $S(A)$ with the weak-$*$ topology is a metrizable compact, therefore separable. Let $\{f_i\}_i^{\infty}$ countable weak-$*$ dense subset of $S(A)$. Then

$$f(a) := \sum_{i=1}^{\infty} 2^{-i} f_i(a)$$

defines a faithful ($f(a^*a) = 0$ iff $a = 0$) state on $A$. Then the GNS construction $\pi_f$ is faithful: if $\pi_f(a) = 0$, then

$$f(b^*a^*ab) = \langle \pi_f(a)[b], \pi_f(a)[b] \rangle = 0$$

for every $b \in A$. In particular for $b = 1$, we get $f(a^*a) = 0$ and so $a = 0$. Since $a \mapsto [a]$ is a continuous map of $A$ onto a dense subspace of some Hilbert space $\mathcal{H}_f$ (induced by $\pi_f : A \to \mathcal{B}(\mathcal{H}_f)$), the latter space is separable. □

**Proposition 3.14.** *Every representation of a $C^*$-algebra is equivalent to a direct sum of cyclic representations.*

*Proof.* Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be some representation of $A$. Let $\mathcal{E}$ be the collection of all subsets $E$ of nonzero vectors in $\mathcal{H}$ such that $\pi(A)e \perp \pi(A)f$ for any $e, f \in E$. If we order $\mathcal{E}$ by inclusion, then Zorn's lemma tells us that $\mathcal{E}$ has a maximal element $E_0$. Let $\mathcal{H}_0 = \bigoplus_{e \in E_0} \overline{\pi(A)e}$. Take $h \in \mathcal{H}_0^{\perp}$ in $\mathcal{H}$. Then for any $a, b \in A$ and $e \in E_0$ we have

$$\langle \pi(a)e, \pi(b)h \rangle = \langle \pi(b)^*\pi(a)e, h \rangle = \langle \pi(b^*a)e, h \rangle = 0,$$

so $\pi(A)e \perp \pi(A)h$ for each $e \in E_0$. By maximality, $h = 0$ and $\mathcal{H} = \mathcal{H}_0$. For $e \in E_0$, define $\mathcal{H}_e := \overline{\pi(A)e}$. Obviously, $\mathcal{H}_e$ is invariant for $\pi$, so $\pi_e := \pi\big|_{\mathcal{H}_e}$ is a cyclic representation of $A$. Clearly, $\pi = \bigoplus_{e \in E_0} \pi_e$. □

## 3.3 Pure states and irregular representations

**Definition 3.15.** A state $\varphi \in S(A)$ is called *pure* if it's an extreme point of $S(A)$.

**Proposition 3.16.** *A state $\varphi \in S(A)$ is pure iff the representation GNS $\pi_\varphi : A \to \mathcal{B}(L^2(A,\varphi))$ with cyclic vector $1_\varphi$ is irreducible.*

*Proof.* ($\Rightarrow$) Let $\mathcal{K} \leq L^2(A,\varphi)$ be a closed invariant subspace. Then $\mathcal{K}^{\perp}$ is also a closed invariant subspace: for $a \in A$, $x \in \mathcal{K}^{\perp}$ and $k \in \mathcal{K}$ we have

$$\langle \pi_\varphi(a)x, k \rangle = \langle x, \pi_\varphi(a^*)k \rangle = 0.$$

Since $L^2(A,\varphi) = \mathcal{K} \oplus \mathcal{K}^\perp$ we write $1_\varphi = \underbrace{\mu_1}_{\in\mathcal{K}} + \underbrace{\mu_2}_{\in\mathcal{K}^\perp}$ and form

$$\varphi_j := \frac{\langle \pi_\varphi(x)\mu_j, \mu_j\rangle}{\|\mu_j\|^2}, \quad j = 1, 2.$$

These are states and so is

$$\varphi(x) = \|\mu_1\|^2\varphi_1(x) + \|\mu_2\|^2\varphi_2(x)$$

because $1 = \|1_\varphi\|^2 = \|\mu_1\|^2 + \|\mu_2\|^2$. Since $\varphi \in \operatorname{ext} S(A)$, we either have $\mu_1 = 0$ or $\mu_2 = 0$, which implies that $\mathcal{K}$ is either (0) or $L^2(A,\varphi)$.

($\Leftarrow$) Suppose $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2)$ for $\varphi_1, \varphi_2 \in S(A)$. Define a linear map

$$U : L^2(A,\varphi) \to L^2(A,\varphi_1) \oplus L^2(A,\varphi_2), \quad \pi_\varphi(x)1_\varphi \mapsto \frac{1}{\sqrt{2}}\pi_{\varphi_1}(x)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(x)1_{\varphi_2}.$$

First we notice that $U$ preserves the scalar product:

$$\begin{aligned}
\langle \pi_\varphi(x)1_\varphi, \pi_\varphi(y)1_\varphi\rangle &= \varphi(x^*y)\\
&= \frac{1}{2}\varphi_1(x^*y) + \frac{1}{2}\varphi_2(x^*y)\\
&= \langle \frac{1}{\sqrt{2}}\pi_{\varphi_1}(x)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(x)1_{\varphi_2}, \frac{1}{\sqrt{2}}\pi_{\varphi_1}(y)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(y)1_{\varphi_2}\rangle\\
&= \langle U\pi_\varphi(x)1_\varphi, U\pi_\varphi(y)1_\varphi\rangle.
\end{aligned}$$

Additionally, $U$ intertwines: for all $x \in A$, we have

$$\begin{aligned}
U\pi_\varphi(x)(\pi_\varphi(y)1_\varphi) &= U\pi_\varphi(xy)1_\varphi\\
&= \frac{1}{\sqrt{2}}\pi_{\varphi_1}(xy)1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}\pi_{\varphi_2}(xy)1_{\varphi_2}\\
&= (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))\,(\pi_{\varphi_1}(y)1_{\varphi_1} \oplus \pi_{\varphi_2}(y)1_{\varphi_2})\\
&= (\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))\,U(\pi_\varphi(y)1_\varphi).
\end{aligned}$$

If we star the intertwining identity, we get

$$\pi_\varphi(x^*)U^* = U^*\left(\pi_{\varphi_1}(x^*) \oplus \pi_{\varphi_2}(x^*)\right), \quad \forall x^* \in A.$$

If we plug in $x$ instead of $x^*$, we get

$$\pi_\varphi(x)U^* = U^*\left(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)\right), \quad \forall x \in A.$$

Now let

$$p_1 \in \mathcal{B}(L^2(A,\varphi_1) \oplus L^2(A,\varphi_2))$$

be orthogonal projection onto the first direct summand. Clearly, we have

$$p_1\left(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)\right) = \left(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)\right)p_1$$

Putting it all together, we get

$$\begin{aligned}\pi_\varphi(x)U^*p_1U &= U^*(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))p_1U \\ &= U^*p_1(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x))U \\ &= U^*p_1U(\pi_{\varphi_1}(x) \oplus \pi_{\varphi_2}(x)),\end{aligned}$$

so $U^*p_1U$ commutes with $\pi_\varphi(x)$ for all $x \in A$. If $\sigma(U^*p_1U)$ has more than one element, then $\exists t \in (0,1]$ such that $\sigma(U^*p_1U - t)$ has both positive and negative elements. By definition 2.38, we can write $U^*p_1U = a - b$ for positive $0 \neq a, b$ such that $ab = ba = 0$. Then $a, b$ commute with $\pi_\varphi(A)$, so $\ker a \neq 0$ is an closed subspace of $L^2(A, \varphi)$ that is invariant under $\pi_\varphi(x)$, which is a contradiction. So $U^*p_1U$ has a single element spectrum $\{\alpha\}$ and since $U^*p_1U$ is normal (because it is positive), so is $U^*p_1U - \alpha I$. But now we can write

$$\|U^*p_1U - \alpha I\| = r(U^*p_1U) = 0,$$

which proves that $U^*p_1U = \alpha I$. Then

$$\begin{aligned}\alpha &= \alpha\varphi(1) = \varphi(\alpha) \\ &= \langle \alpha 1_\varphi, 1_\varphi \rangle \\ &= \langle U^*p_1U1_\varphi, 1_\varphi \rangle \\ &= \left\langle \frac{1}{\sqrt{2}}1_{\varphi_1} \oplus 0, \frac{1}{\sqrt{2}}1_{\varphi_1} \oplus \frac{1}{\sqrt{2}}1_{\varphi_2} \right\rangle \\ &= \left\langle \frac{1}{\sqrt{2}}1_{\varphi_1}, \frac{1}{\sqrt{2}}1_{\varphi_1} \right\rangle_{\varphi_1} = \frac{1}{2}.\end{aligned}$$

This means that we can write

$$\left(\sqrt{2}p_1U\right)^*\left(\sqrt{2}p_1U\right) = 1,$$

so

$$u_1 = \frac{1}{\sqrt{2}}p_1U : L^2(A, \varphi) \to L^2(A, \varphi_1)$$

is an isometry. We also have the identities

$$u_1 1_\varphi = 1_{\varphi_1}, \quad u_1\pi_\varphi(x) = \pi_{\varphi_1}(x)u_1.$$

It follows that

$$\begin{aligned}\varphi(x) &= \langle \pi_\varphi(x)1_\varphi, 1_\varphi \rangle \\ &= \langle u_1^*u_1\pi_\varphi(x)1_\varphi, 1_\varphi \rangle \\ &= \langle u_1^*\pi_{\varphi_1}(x)u_1 1_\varphi, 1_\varphi \rangle \\ &= \langle \pi_{\varphi_1}(x)u_1 1_\varphi, u_1 1_\varphi \rangle \\ &= \langle \pi_{\varphi_1}(x)1_{\varphi_1}, 1_{\varphi_1} \rangle = \varphi_1(x)\end{aligned}$$

and we are done. □

**Theorem 3.17.**

*A representation $\pi : A \to \mathcal{B}(\mathcal{H})$ if irreducible iff $\pi(A)' = \mathbb{C} \cdot \mathrm{id}$.*

*Proof.* Start with ($\Leftarrow$). Suppose there exists a closed invariant subspace $(0) \neq \mathcal{K} \subsetneqq \mathcal{H}$. Let $p \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection onto $\mathcal{K}$. Then $p \notin \mathbb{C}\cdot\mathrm{id}$. Now we prove that $p \in \pi(A)'$. Let $a \in A$. For $\mu \in \mathcal{K}$, we have

$$(p\pi(a))\mu = p(\pi(a)\mu) = \pi(a)\mu = \pi(a)(p\mu) = (\pi(a)p)\mu.$$

Now for $\mu \in \mathcal{K}^\perp$, we get

$$(p\pi(a))\mu = p(\pi(a)\mu) = 0 = \pi(a)(0) = \pi(a)(p\mu) = (\pi(a)p)\mu.$$

For the converse ($\Rightarrow$), suppose there exists a non-scalar self-adjoint $h \in \pi(A)'$. Then $\sigma(h)$ has at least two elements. We can define two bump functions $f, g \in C(\sigma(h))$ in the respective neighborhoods of these two elements of $\sigma(h)$ such that $fg = 0$. Then $f(h) \neq 0$ since $f \neq 0$. Then also $\mathcal{K} := \overline{\mathrm{im}\, f(h)} \leq \mathcal{H}$ is nonzero. Also, $g(h) \neq 0$ and $g(h)\big|_{\mathcal{K}} = 0$ since $g(h) \cdot f(h) = 0$. In particular, $\mathcal{K} \subsetneqq \mathcal{H}$. Take any $x \in \pi(A)$. By Stone-Weierstrass, we can approximate $f(h)$ in norm by $(p_i(h))_{i=1}^\infty$, where $(p_i)_{i=1}^\infty$ are complex polynomials. Notice that since $h \in \pi(A)'$, we also have $p_i(h) \in \pi(A)'$. Therefore, we have $p_i(h)x = xp_i(h)$. By sending $i \to \infty$, the left side converges in norm to $f(h)a$, while the right one converges in norm to $af(h)$. As a result, $f(h)x = xf(h)$ and $f(h) \in \pi(A)'$. We claim that $\mathcal{K}$ is invariant; it's enough to show that $\mathrm{im}\, f(h)$ is invariant. For $a \in A, \mu \in \mathcal{H}$ we have

$$\pi(a)(f(h)\mu) = \pi(a)f(h)\mu = f(h)\pi(a)\mu \in \mathrm{im}\, f(h).$$

In general, if $q \in \pi(A)'$, then $q^* \in \pi(A)'$ and we can reduce the problem to the self-adjoint case handled above. $\square$

**Corollary 3.18.** *Irreducible representations of abelian $C^*$-algebras are 1-dimensional.*

*Proof.* Let $A$ be an abelian $C^*$-algebra and $\pi : A \to \mathcal{B}(\mathcal{H})$ an irrep. Then by theorem 3.17, $\pi(A)' = \mathbb{C}$. Moreover,

$$\pi(A) = Z(\pi(A)) = \pi(A)' \cap \pi(A) = \mathbb{C} \cdot \mathrm{id}\,. \qquad \square$$

**Corollary 3.19.** *If $A$ is an abelian $C^*$-algebra, then $\mathrm{ext}\, S(A) = \sigma(A)$.*

*Proof.* Let $\varphi \in \sigma(A)$. Then $\varphi$ is 1-dimensional (therefore irreducible) representation and so $\varphi \in \mathrm{ext}\, S(A)$. For the converse, take $\varphi \in \mathrm{ext}\, S(A)$. Then the GNS construction $\pi_\varphi$ is irreducible, therefore 1-dimensional. So $L^2(A, \varphi) = \mathbb{C}$ with the standard scalar product and $\varphi(x) = \langle \pi_\varphi(x) 1_\varphi, 1_\varphi \rangle = \pi_\varphi(x)$. $\square$

**Proposition 3.20.** *Let $A$ be a $C^*$-algebra. Then $\mathrm{co}\,\mathrm{ext}\, S(A)$ is weak-$*$ dense in $S(A)$.*

*Proof.* We know that $S(A)$ is compact Hausdorff with respect to the weak-$*$ topology. The conclusion follows from Krein–Milman. $\square$

**Corollary 3.21.** *Let $A$ be a $C^*$-algebra and $x \in A\setminus(0)$. Then there exist an irrep $\pi : A \to \mathcal{B}(\mathcal{H})$ such that $\pi(x) \neq 0$.*

*Proof.* By proposition 3.5, there exists $\varphi \in S(A)$ such that $\varphi(x) \neq 0$. By the previous proposition (Krein–Milman), there exists a $\tau \in \operatorname{ext} S(A)$ such that $\tau(x) \neq 0$. Then apply GNS: $\pi_\tau$ is irreducible and $\pi_\tau(x) \neq 0$. □

**Theorem 3.22** (Jordan decomposition for linear functionals)**.**

*Let $A$ be a $C^*$-algebra and $\varphi \in A^*$ hermitian. Then there exist (unique - without proof) positive linear functionals $\varphi_+, \varphi_- \in A^*$ such that $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| = \|\varphi_-\|$.*

*Proof.* W.l.o.g. $\|\varphi\| = 1$. Let $\Sigma$ denote the set of positive linear functionals with norm $\leq 1$. By Banach–Alaoglu, $\Sigma$ is weak-$*$ compact and Hausdorff. Consider

$$\gamma : A \to C(\Sigma), \quad a \mapsto (\psi \mapsto \psi(a)).$$

This is an isometry and $\gamma(A_+) \subseteq C(\Sigma)_+$. By Hahn–Banach, there exists a $\widetilde{\varphi} : C(\Sigma) \to \mathbb{C}$ such that $\|\widetilde{\varphi}\| = \|\varphi\|$ and $\varphi = \widetilde{\varphi}\circ\gamma$. Assume $\widetilde{\varphi}$ is hermitian (otherwise, we can replace it by $\frac{\widetilde{\varphi}+\widetilde{\varphi}^*}{2}$). By Riesz–Markoff, there exists a regular Radon Measure $\mu$ on $\Sigma$ such that $\widetilde{\varphi}(f) = \int f\, d\mu$ for all $f \in C(\Sigma)$. Then we use Jordan decomposition for measures to obtain $\mu_+, \mu_-$ such that $\mu = \mu_+ - \mu_-$ and $\|\mu\| = \|\mu_+\| = \|\mu_-\|$. Now we just define $\varphi_\pm(a) := \int a\, d\mu_\pm$. □

**Corollary 3.23.** *For a $C^*$-algebra $A$, $A^*$ is the span of positive linear functionals on $A$.*

**Corollary 3.24.** *Let $A$ be a $C^*$-algebra and $\varphi \in A^*$. Then there exists a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\mu, \theta \in \mathcal{H}$ such that $\varphi(a) = \langle \pi(a)\theta, \mu\rangle$.*

*Proof.* Write $\varphi = \sum_{i=1}^n \alpha_i \psi_i$ for some $\psi_j \in S(A)$. Let $\pi_i$ be the GNS representation of $\psi_i$. Define $\pi := \bigoplus_i \pi_i$, $\theta := \bigoplus_i \alpha_i 1_{\psi_i}$ and $\mu = \bigoplus_i 1_{\psi_i}$. The result then follows immediately. □

## 3.4 Examples of $C^*$-algebras

**Example 3.25.** *The most canonical example of a $C^*$-algebra is $\mathcal{B}(\mathcal{H})$. Similarly, the algebra of compact operators $\mathcal{K}(\mathcal{H})$ is a $C^*$-algebra (if $\dim \mathcal{H} = \infty$, it is non-unital).*

**Definition 3.26.** A $C^*$-algebra $A$ is *simple* if it has no closed two-sided ideals.

The following results require some tools from nonunital $C^*$-algebras (namely, the approximate identity), so we state them without proofs. The reader can find additional information in the section VIII.4 of [1].

**Lemma 3.27.** *Any closed ideal $I$ of a $C^*$-algebra $A$ is closed for involution: if $a \in I$, then $a^* \in I$.*

**Proposition 3.28.** *If $I$ is a closed ideal of a $C^*$-algebra $A$, then $A/I$ is a $C^*$-algebra for involution*

$$(a+I)^* := a^* + I$$

*and the usual quotient norm.*

**Example 3.29.** *From the introductory course, we know that the set of compact operators $\mathcal{K}(\mathcal{H})$ on a Hilbert space forms a closed ideal in $\mathcal{B}(\mathcal{H})$. If $\dim \mathcal{H} = \infty$, then $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a* Calkin algebra*. If $\dim \mathcal{H} = |\mathbb{N}|$, then $\mathcal{K}(\mathcal{H})$ is the only proper nontrivial closed ideal in $\mathcal{B}(\mathcal{H})$ (a fact, later proved in corollary 6.15). In that case, Calkin algebra is simple and it does not admit a separable representation.*

**Example 3.30.** *The algebra of matrices $M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$ is a $C^*$-algebra.*

### 3.4.1 Structure theorem for finite-dimensional $C^*$-algebras

For finite-dimensional $C^*$-algebras, we have the Artin-Wedderburn type theorem.

**Proposition 3.31** (Structure theorem for finite-dimensional $C^*$-algebras)**.** *Every finite-dimensional $C^*$-algebra $A$ is*

$$A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$$

*for uniquely determined $n_1, \ldots, n_r$.*

To prove this, we need a few preliminary lemmas.

**Lemma 3.32.** *Every finite-dimensional $C^*$-algebra is unital.*

This fact, which we will not prove here, follows essentially from Krein–Milman theorem. The reader should consult theorem I.10.2 of [5].

**Corollary 3.33.** *If $A$ is a finite-dimensional $C^*$-algebra, then every ideal in $A$ is of the form $I = Ap$ for some central projection $p \in A$.*

*Proof.* If $I \lhd A$, then it is itself a (finite-dimensional) $C^*$-algebra, so it must have a unit $p$. As a result, we have $I = Ip \subseteq Ap$, but since $I$ is an ideal we also have $Ap \subseteq I$, so $Ap = I$. Since $p$ is a unit in $I$, we have $p^2 = p = p^*$, so $p$ is a projection. For any $x \in A$, we get $xp \in I$ and so $p \cdot (xp) = xp$. By starring this equation, we get $px^*p = px^*$. Combining the last two equations gives us

$$x^*p = p \cdot (x^*p) = px^*p = px^*,$$

so $p$ commutes with $x^*$. As a result, $p$ commutes with the entire $A$. □

**Lemma 3.34.** *If $A$ is a finite-dimensional abelian $C^*$-algebra, then its spectrum is finite.*

*Proof.* We know that by Gelfand, $A \cong C(\sigma(A))$, where $\sigma(A)$ is a compact Hausdorff (and therefore normal) space. Suppose that $\sigma(A)$ is infinite.

(1.) First, we will inductively construct an infinite subset $X = \{x_n\}_{n\in\mathbb{N}} \subseteq \sigma(A)$ that does not contain any of its accumulation points. Pick any point $x \in \sigma(A)$. If $x$ is not an accumulation point of $\sigma(A)$ (so it is an isolated point), then take $x_1 := x$ and choose any new point in $\sigma(A)\setminus\{x_1\}$ to repeat this process. Howevery, if $x$ is an accumulation point of $\sigma(A)$, then take any $x_1 \in \sigma(A) \setminus \{x\}$. Since $\sigma(A)$ is Hausdorff, there exist disjoint open neighborhoods $V_1 \ni x, U_1 \ni x_1$. Now since $x$ is an accumulation point of $\sigma(A)$, there must exist some $x_2 \in V_1$, such that $x_2 \neq x$. By the Hausdorff property, there must exist open disjoint neighborhoods $V_2 \ni x, U_2 \ni x_2$ inside $V_1$. Now repeat this process indefinitely to obtain a set $\{x_n\}_{n\in\mathbb{N}}$ which does not contain its accumulation points.

(2.) Notice that from the previous item, every point $x_n$ has an open neighborhood $U_n$, where $U_n \cap U_m = \emptyset$ for any $n \neq m$. By Urysson's lemma, there exists a continuous function $f_n : \sigma(A) \to [0,1]$ for every $n \in \mathbb{N}$ such that $f_n(x_n) = 1$ and $f_n = 0$ on $\sigma(A) \setminus U_n$.

(3.) Finally, we have an infinite linearly independent set $\{f_n\}_{n\in\mathbb{N}}$ in $C(\sigma(A))$, so the latter algebra must be infinite-dimensional. □

Let $A$ be any finite-dimensional $C^*$-algebra. Then its center, say $C$, is a finite-dimensional abelian $C^*$-algebra with spectrum $\{\omega_1, \dots, \omega_n\}$. Let $p_i \in C$ be an element that corresponds to the characteristic function $\chi_{\{\omega_i\}} \in C(\sigma(C))$. It follows from Gelfand that $C \cong \mathbb{C}p_1 \oplus \cdots \oplus \mathbb{C}p_n$, so $p_1 + \cdots + p_n = 1$. As a result,

$$A \cong Ap_1 \oplus \cdots \oplus Ap_n,$$

where each of $Ap_i$ has trivial center. By the previous lemma, $Ap_i$ is a simple finite-dimensional $C^*$-algebra. Therefore, it suffices to prove the structure theorem for simple finite-dimensional $C^*$-algebras.

*Proof of the structure theorem.* Assume $A$ is simple and finite-dimensional. We first note that $aAb \neq \{0\}$ for any nonzero $a, b \in A$. This follows from the observation that the set $AaA$ is an ideal of $A$ which must be the entire algebra $A$, since it is nonzero. Let $B$ be a maximal abelian $*$-subalgebra of $A$ and let $\sigma(B) = \{\omega_1, \omega_2, \dots, \omega_n\}$ be its spectrum. Let $e_i \in B$ denote the projection, corresponding to the characteristic function $\chi_{\{\omega_i\}}$. By our previous arguments, $e_i$ are orthogonal and $\sum_{i=1}^n e_i = 1$. Furthermore, $B \cong \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$. It follows that $e_iAe_i$ commutes with every $e_j$. Next, we prove that the $C^*$-algebra $e_iAe_i$ has dimension one. Take any normal element $x \in e_iAe_i$. By Gelfand, $\sigma(x)$ is homeomorphic to the spectrum of a $C^*$-algebra, generated by $x$. This $C^*$-algebra is finite-dimensional (since it lives inside the finite-dimensional $C^*$-algebra $e_iAe_i$), so its spectrum must be finite and as a result, $\sigma(x)$ is finite (and discrete). Suppose that there exists a normal element $x \in e_iAe_i$ such that $\sigma(x)$ has at least two distinct elements $\lambda_1, \lambda_2$. Then functional calculus gives us orthogonal projections $p = \chi_{\{\lambda_1\}}(x)$ and $q = \chi_{\{\lambda_2\}}(x)$ in $e_iAe_i$. Then $C^*(p, q, e_2, \dots, e_n)$ is an abelian $C^*$-algebra in $A$ of dimension $n+1$, which is in contradiction with assumption that $B$ is maximal abelian. This means that every normal element $x \in e_iAe_i$ has singleton spectrum $\sigma(x) = \lambda$, so it is of the form $x = \lambda \cdot e_i$ by functional calculus. Since $e_iAe_i$ is the span of self-adjoint (which are normal) elements, we have $e_iAe_i = \mathbb{C}e_i \subseteq B$. For fixed $i, j$, we know that $e_iAe_j \neq \{0\}$. Choose a nonzero $x \in e_iAe_j$ and notice that $x = e_ixe_j$. This implies that $x^*x = e_jx^*xe_j = \lambda e_j$ and $xx^* = e_ixx^*e_i = \mu e_j$ for some $\lambda, \mu \neq 0$. But from

$$\lambda = \|x^*x\| = \|x\|^2 = \|x^*\|^2 = \|xx^*\| = \mu$$

we get $\lambda = \mu > 0$. If we define $u = \lambda^{-\frac{1}{2}}x$, we get $u^*u = e_j$ and $uu^* = e_i$. For each $i$, let $u_i \in A$ be an element such that $u_i^*u_i = e_1 = u_iu_i^* = e_i$. Then, define $u_{i,j} = u_iu_j^*$. From our arguments, the following has to be true:

$$u_{i,j}^* = u_{j,i}, \quad \sum_{i=1}^{n} u_{i,i} = 1, \quad u_{i,j}u_{k,l} = \delta_{j,k}u_{i,l}$$

We claim that $e_iAe_j = \mathbb{C}u_{i,j}$. Indeed, if $x \in e_iAe_j$, then $xu_{i,j} \in e_iAe_i$, so $xu_{i,j} = \lambda e_i$ for some $\lambda \in \mathbb{C}$. Hence we get

$$x = xe_j = xu_{j,i}u_{i,j}\lambda e_iu_{i,j}\lambda u_{i,j}.$$

For each $x \in A$, let $\lambda_{i,j}(x)$ be a scalar such that $e_ixe_j = \lambda_{i,j}(x)u_{i,j}$. It follows that

$$x = \sum_{i,j=1}^{n} e_ixe_j = \sum_{i,j=1}^{n} \lambda_{i,j}(x)u_{i,j}.$$

Now the map

$$A \to M_n(\mathbb{C}), \quad x \mapsto (\lambda_{i,j}(x))_{i,j}$$

is a $*$-isomorphism of $A$ onto the algebra $M_n(\mathbb{C})$, where $\dim A = n^2$. $\square$

### 3.4.2 Group $C^*$-algebras

Let $G$ be a group. Then the (complex) group algebra $\mathbb{C}[G]$ is defined as the algebra with basis $\{u_g \mid g \in G\}$ and multiplication given by $u_g \cdot u_h = u_{gh}$. Multiplication is convolutive:

$$\begin{aligned}
\left(\sum_{g}^{\text{finite}} a_gu_g\right)\left(\sum_{h}^{\text{finite}} b_hu_h\right) &= \sum_{g,h} a_gb_hu_gu_h \\
&= \sum_{g,h} a_gb_hu_{gh} \\
&= \sum_{k}\left(\sum_{g} a_gb_{g^{-1}k}\right)u_k.
\end{aligned}$$

We can equip $\mathbb{C}[G]$ with an involution

$$\left(\sum_{g\in G}^{\text{finite}} a_gu_g\right)^* = \sum_{g} \overline{a_g}u_{g^{-1}}.$$

Given a representation (homomorphism of $*$-algebras) $\pi : \mathbb{C}[G] \to \mathcal{B}(\mathcal{H})$, we define the $C^*$-algebra

$$C_\pi^*(G) := \overline{\pi(\mathbb{C}[G])} \subseteq \mathcal{B}(\mathcal{H}).$$

For $g \in G$, we get

$$\begin{aligned}\pi(u_g)\pi(u_g)^* &= \pi(u_g) \cdot \pi(u_g^*) \\ &= \pi(u_g) \cdot \pi(u_{g^{-1}}) \\ &= \pi(u_g \cdot u_{g^{-1}}) = \pi(u_e) = 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\pi(u_g)^*\pi(u_g) &= \pi(u_g^*) \cdot \pi(u_g) \\ &= \pi(u_{g^{-1}}) \cdot \pi(u_g) \\ &= \pi(u_{g^{-1}} \cdot u_g) = \pi(u_e) = 1.\end{aligned}$$

We have thus proved that under any representation of $\mathbb{C}[G]$, each $u_g$ is mapped to a unitary.

**Example 3.35.** *Take $\mathcal{H} = \ell^2(G)$ (this is a Hilbert space with ONB $\{\delta_g \mid g \in G\}$). Then*

$$\lambda : \mathbb{C}[G] \to \mathcal{B}(\ell^2(G)), \quad u_g \mapsto (\delta_h \mapsto \delta_{gh})$$

*is a faithful representation. We call it the left regular representation of $G$. The closure of its image is called the reduced group $C^*$-algebra $C_r^*(G) := \overline{\lambda(\mathbb{C}[G])} \subseteq \mathcal{B}(\ell^2(G))$.*

**Definition 3.36.** *Universal* (or *full*) group $C^*$-algebra is the completion $\mathbb{C}[G]$, where the norm of an element $a \in \mathbb{C}[G]$ is $\|a\|_u = \sup\{\|\pi(a)\| \mid \pi \text{ representation of } \mathbb{C}[G]\}$.

**Lemma 3.37.** *If $\pi$ is a representation of $\mathbb{C}[G]$ and $a = \sum_{g \in G}^{\text{finite}} a_g u_g \in \mathbb{C}[G]$, then $\|\pi(a)\| \leq \sum |a_g|$.*

*Proof.* Then $\pi(a) = \sum a_g \cdot \pi(u_g)$. Then

$$\|\pi(a)\| = \left\|\sum a_g \pi(u_g)\right\| \leq \sum |a_g| \cdot \|\pi(u_g)\| = \sum |a_g|. \qquad \square$$

This implies that $\|\cdot\|_u$ is indeed a norm on $\mathbb{C}[G]$, making the universal $C^*$-algebra $C(G)$ of $G$ in definition 3.36 well-defined.

*Remark.* The group algebra $\mathbb{C}[G]$ is dense in both $C_\pi^*(G)$ and $C^*(G)$.

**Theorem 3.38** (Universal property).

*For each representation $\pi$ of $\mathbb{C}[G]$ there exists a surjective $*$-homomorphism $\widehat{\pi} : C^*(G) \to C_\pi^*(G)$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathbb{C}[G] \arrow[r, "\pi"] \arrow[d, hook] & C_\pi^*(G) \\ C^*(G) \arrow[ur, dashed, two heads, "\exists!\widehat{\pi}"'] & \end{tikzcd}$$

*Proof.* Define first $\widehat{\pi}$ on $\mathbb{C}[G] \subseteq C^*G$ by $\widehat{\pi}(a) := \pi(a) \in C^*_\pi(G)$. Firstly, $\widehat{\pi}$ on $\mathbb{C}[G]$ is contractive:

$$\|\widehat{\pi}(a)\| = \|\pi(a)\| \leq \|a\|_u.$$

By density, $\widehat{\pi}$ uniquely extends to a continuous $*$-homomorphism $\widehat{\pi} : C^*(G) \to C^*_\pi(G)$. This $\widehat{\pi}$ is also contractive and $\operatorname{im}\pi$ is dense, so $\widehat{\pi}$ is onto. $\square$

**Example 3.39.** *Let $G$ be abelian and $|G| = n$. Then $\mathbb{C}[G] \cong \mathbb{C}^{|G|}$ as a vector space. Hence $C^*G = \mathbb{C}[G] = C^*_r(G)$. Furthermore, $\mathbb{C}[G]$ is commutative, so by the structure theorem we have*

$$\mathbb{C}[G] \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ times}}$$

*as a $C^*$-algebra. For instance, $\mathbb{C}\left[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\right] \cong \mathbb{C}\left[\mathbb{Z}/4\mathbb{Z}\right]$.*

**Example 3.40.** *Let $G = S_3$. Then $|G| = 6$ and once again $C^*(G) = \mathbb{C}[G] = C^*_r(G)$. By structure theorem and the dimension consideration, $\mathbb{C}[G] \cong M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ (otherwise it would be commutative).*

**Example 3.41.** *Let $G = S_4$. Again, $C^*(G) = \mathbb{C}[G] = C^*_r(G)$. By Maschke's theorem, $\mathbb{C}[G]$ is semisimple, therefore it is isomorphic (as an $\mathbb{C}$-algebra) to a direct sum of matrix algebras over $\mathbb{C}$. This decomposition is unique. Since $S_4$ has five conjugacy classes, there are five factors (see, for example, theorem IX.7.9. in [2]). Adding up all the dimensions, the only combination that works is $9 + 9 + 4 + 1 + 1 = 24$, therefore*

$$\mathbb{C}[G] \cong M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$$

*as a $\mathbb{C}$-algebra. By the structure theorem, it is also isomorphic to this direct sum as a $C^*$-algebra.*

**Example 3.42.** *What is $C^*(\mathbb{Z})$? Representations $\pi(\mathbb{C}[Z]) \to \mathcal{B}(\mathcal{H})$ are determined by choice of unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\pi(u_1) = U$. By universal property, for every $\mathcal{H}$ and $U \in \mathcal{B}(\mathcal{H})$ there exists a unique $*$-homomorphism*

$$\widehat{\pi} : C^*(\mathbb{Z}) \to C^*(\{U\}),$$

*where the latter is a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$, generated by $U$. We call $C^*(\mathbb{Z})$ the universal $C^*$-algebra, generated by a unitary.*

## 3.5 Abelian group $C^*$-algebras

If $G$ is abelian, then $\mathbb{C}[G]$ is commutative and $C^*_r(G)$ is abelian. By Gelfand, there exists a compact Hausdorff space $\Sigma$ such that $C^*_r(G) \cong C(\Sigma)$ and $\Sigma = \sigma(C^*_r(G))$.

**Definition 3.43.** To each abelian group $G$ we associate its *Pontryagin dual*

$$\widehat{G} = \{w : G \to \mathbb{T} \text{ group homomorphism}\}.$$

Then $\widehat{G}$ is a group under pointwise multiplication. We endow $\widehat{G}$ with the compact-open topology induced from $\widehat{G} \subseteq \mathbb{T}^G$. Recall that the basis sets for this topology are

$$B_{\varepsilon,F}(w) = \{\eta \in \widehat{G} \mid |\eta(h) - w(h)| < \varepsilon, \ \forall h \in F\}$$

for $\varepsilon > 0$, $w \in \widehat{G}$ and $F \subseteq G$ finite.

*Remark.* A net $(w_i)_{i\in I} \subseteq \widehat{G}$ is Cauchy iff $(w_i(g))_{i\in I} \subseteq \mathbb{T}$ is Cauchy for all $g \in G$.

Let us calculate the Pontryagin dual of some very simple abelian groups.

**Example 3.44.** *Let us prove that $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ as a topological group. Take any $k \in \mathbb{Z}/n\mathbb{Z}$ and define a group homomorphism*

$$\varphi_k : \mathbb{Z}/n\mathbb{Z} \to \mathbb{T}, \quad l \mapsto e^{\frac{2\pi i k l}{n}}.$$

*Then*

$$\Phi : \mathbb{Z}/n\mathbb{Z} \to \widehat{\mathbb{Z}/n\mathbb{Z}}, \quad k \mapsto \varphi_k$$

*is easily seen to be a group isomorphism. Since both groups are obviously endowed with discrete topology, the above map is also a homeomorphism, hence $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ as topological groups.*

**Example 3.45.** *Next, we prove that $\widehat{\mathbb{Z}} \cong \mathbb{T}$. Take any $\xi \in \mathbb{T}$ and define*

$$\varphi_\xi : \mathbb{Z} \to \mathbb{T}, \quad k \mapsto \xi^k.$$

*Then*

$$\Phi : \mathbb{T} \to \widehat{\mathbb{Z}}, \quad \xi \mapsto \varphi_\xi$$

*is a group isomorphism. If $(\xi_i)_{i\in I}$ is a net in $\mathbb{T}$ such that $\xi_i \to \xi$, then $\Phi\xi_i = \varphi_{\xi_i} \to \varphi_\xi = \Phi\xi$. As a result, $\Phi$ is continuous. But since $\Phi$ maps from a compact to a Hausdorff space, it is also closed and thus a homeomorphism.*

**Theorem 3.46.**

*The map*

$$h : \widehat{G} \to \sigma(C_r^*(G)), \quad w \mapsto \left(\sum a_g u_g \mapsto \sum a_g w(g)\right)$$

*is a homeomorphism.*

*Proof.* First, we prove that $h(w) \in \sigma(C_r^*(G))$ for all $w \in \widehat{G}$. We begin by showing $h(w)$ :

$\mathbb{C}[G] \to \mathbb{C}$ is a homomorphism. Take $b = \sum b_k u_k \in \mathbb{C}[G]$. Then

$$h(w)(a \cdot b) = \sum_g \left( \sum_h a_h b_{h^{-1}g} \right) \cdot w(g)$$

and

$$h(w)(a) \cdot h(w)(b) = \left( \sum_g a_g w(g) \right) \cdot \left( \sum_h b_h w(h) \right) = \sum_k \left( \sum_h a_{kh^{-1}} b_h \right) w(k),$$

so $h(w)$ is multiplicative. To extend it to $C_r^*$, we must prove that $|h(w)a| \leq \|a\|_r$ for all $a \in \mathbb{C}[G]$. To $\chi \in \sigma(C_r^*(G))$ and $a \in \mathbb{C}[G]$ we associate

$$\widetilde{a} = \sum a_g w(g) \cdot \overline{\chi(u_g)} u_g,$$

so $h(w)a = \chi(\widetilde{a})$. By Gelfand,

$$\|\widetilde{a}\|_r = \sup\{|\mu(\widetilde{a})| \mid \mu \in \sigma(C_r^*(G))\} \geq |\chi(\widetilde{a})| = |h(w)a|.$$

Next, we show that $\|\widetilde{a}\|_r = \|a\|_r$: to $\theta \in \ell^2(G)$ assign $\widetilde{\theta}$ by $\widetilde{\theta_h} := \chi(u_{h^{-1}})\overline{w(h)}\theta_h$. Then $\|\theta\|_2 = \|\widetilde{\theta}\|_2$. Further, $\|\lambda(\widetilde{a})\widetilde{\theta}\|_2 = \|\lambda(a)\theta\|_2$ (short calculation), so

$$\|\widetilde{a}\|_r = \sup\{\|\lambda(\widetilde{a})\widetilde{\theta}\| \mid \|\widetilde{\theta}\|_2 = 1\} = \sup\{\|\lambda(\widetilde{a})\theta\| \mid \|\theta\|_2 = 1\} = \|a\|_r.$$

Next, we prove that $h$ is continuous. Suppose the net $(w_i)_{i\in I} \subseteq \widehat{G}$ is Cauchy. We prove that for every $a \in C_r^*(G)$ the net $(h(w_i)(a))_{i\in I}$ is Cauchy. Pick $\varepsilon > 0$. There exists $J$ such that for every $i, j \geq J$, we have

$$|w_i(g) - w_j(g)| < \frac{\varepsilon}{|\{g \mid a_g \neq 0\}|}, \quad \forall g \in G.$$

Then for all $i, j \geq J$ we get $|h(w_i)(g) - h(w_j)(g)| < \varepsilon$. For the general case $a \in C_r^*(G)$, we can take $a$ as a limit of a sequence $(a_n)_n \subseteq \mathbb{C}[G]$, approximate $a$ with $a_n$ and use the triangle inequality to establish that $(h(w_i)(a))_i$ is Cauchy. Now on to bijectivity of $h$. It's enough to check that it is surjective: take $\phi \in \sigma(C_r^*(G))$. Define

$$w_\phi : G \to \mathbb{C}, \quad g \mapsto \phi(u_g).$$

Since $\phi$ is a $*$-homomorphism, $\operatorname{im} w_\phi \subseteq \mathbb{T}$. We have to prove that $w_\phi \in \widehat{G}$. We just check the multiplicativity:

$$w_\phi(g) \cdot w_\phi(h) = \phi(u_g)\phi(u_h) = \phi(u_g u_h) = \phi(u_{gh}) = w_\phi(gh).$$

For every $w \in \widehat{G}$, we get $w_{h(w)} = w$. So $w_\phi = w_{h(w_\phi)}$, which gives us $h(w_\phi) = \phi$. Now since $h$ is a bijective continuous map between compact Hausdorff spaces, it is a homeomorphism. □

**Example 3.47.** *We can now explicitly calculate Gelfand transform of a reduced group*

*$C^*$-algebra $C_r^*(\mathbb{Z}/n\mathbb{Z})$:*

$$C_r^*(\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\Gamma} C(\sigma(C_r^*(\mathbb{Z}/n\mathbb{Z}))) \to C(\widehat{\mathbb{Z}/n\mathbb{Z}}) \to C(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{C}^n,$$

*which maps an element $\sum_{l=0}^{n-1} a_l u_l \in C_r^*(\mathbb{Z}/n\mathbb{Z})$ to*

$$\begin{bmatrix} \sum_{l=0}^{n-1} a_l \\ \sum_{l=0}^{n-1} a_l e^{\frac{2\pi i l}{n}} \\ \vdots \\ \sum_{l=0}^{n-1} a_l e^{\frac{2\pi i(n-1)l}{n}}. \end{bmatrix}$$

**Example 3.48.** *Similarly, we can explicitly calculate Gelfand transform of a reduced group algebra $C_r^*(\mathbb{Z})$:*

$$C_r^*(\mathbb{Z}) \xrightarrow{\Gamma} C(\sigma(C_r^*(\mathbb{Z}))) \to C(\widehat{\mathbb{Z}}) \to C(\mathbb{T}),$$

*which maps $\sum_{l\in\mathbb{Z}} a_l u_l$ into a function*

$$f \in C(\mathbb{T}), \quad f(\xi) = \sum_{l\in\mathbb{Z}} a_l \xi^l.$$

*Therefore, the inverse Gelfand transform maps a function $f \in C(\mathbb{T})$ into its Fourier series.*

In operator algebras, we usually consider topological groups $G$ that are locally compact and Hausdorff. For such groups, there exists a so-called *Haar measure* $\mu$ on $G$. This measure allows us to consider the $C^*$-algebras $L^2(G)$, $C_r^*(G)$ and $C^*(G)$ for general locally compact and Hausdorff topological groups. If a group $G$ is equipped with discrete topology, then these notions coincide with the ones from the previous subsection.

**Example 3.49.** *It turns out that the Pontryagin dual of a locally compact Hausdorff abelian group is itself a locally compact Hausdorff abelian group. As in the above examples, we can prove the following:*

- $\widehat{\mathbb{T}} = \mathbb{Z}$;
- $\widehat{\mathbb{R}} = \mathbb{R}$.

We notice that for each of the groups that we have seen so far in the examples, the dual of a dual is the original group. This is not a coincidence.

**Theorem 3.50** (Pontryagin)**.**

*If $G$ is a locally compact Hausdorff abelian group, then $G \cong \widehat{\widehat{G}}$.*

# 4 Bounded operators on Hilbert spaces

## 4.1 Polar decomposition

Let $\mathcal{H}$ be a complex Hilbert space. Then $\mathcal{B}(\mathcal{H})$ is a $C^*$-algebra with the operator norm

$$\|A\| = \sup_{\mu\in\mathcal{H},\mu\neq 0} \frac{\|A\mu\|}{\|\mu\|} = \sup_{\mu\in\mathcal{H},\|\mu\|=1} \|A\mu\| = \sup_{\mu\in\mathcal{H},\|\mu\|\leq 1} \|A\mu\|$$

*Remark.* Recall that $A \in \mathcal{B}(\mathcal{H})$ is:

(1.) normal $\Leftrightarrow A^*A = AA^* \Leftrightarrow \|A\mu\| = \|A^*\mu\|,\ \forall \mu \in \mathcal{H}$;

(2.) self-adjoint $\Leftrightarrow A^* = A \Leftrightarrow \langle A\mu, \mu\rangle \in \mathbb{R},\ \forall \mu \in \mathcal{H}$;

(3.) positive $\Leftrightarrow A = B^*B$ for some $B \in \mathcal{B}(\mathcal{H}) \Leftrightarrow \langle A\mu, \mu\rangle \geq 0,\ \forall \mu \in \mathcal{H}$;

(4.) isometry $\Leftrightarrow A^*A = I \Leftrightarrow \|A\mu\| = \|\mu\|,\ \forall \mu \in \mathcal{H}$;

(5.) projection $\Leftrightarrow A^2 = A = A^* \Leftrightarrow A$ is an orthogonal projection onto some closed subspace of $\mathcal{H}$.

**Lemma 4.1.** *An operator $A \in \mathcal{B}(\mathcal{H})$ is a partial isometry iff there exists a closed subspace $\mathcal{K} \leq \mathcal{H}$ such that $A\big|_{\mathcal{K}}$ is an isometry and $A\big|_{\mathcal{K}^\perp} = 0$.*

*Proof.* We first prove ($\Leftarrow$). Obviously, $\mathcal{K}^\perp \subseteq \ker A$. From $Ax = 0$, where $x = y + z$ and $y \in \mathcal{K}, z \in \mathcal{K}^\perp$, we have

$$0 = Ax = A(y + z) = Ay + Az = Ay.$$

But since $A\big|_{\mathcal{K}}$ is an isometry, $\|Ay\| = \|y\| = 0$, so $y = 0$ and $x \in \mathcal{K}^\perp$. Now we prove that $P = A^*A$ is the projection onto $\mathcal{K}$. For $x \in \mathcal{K}$, we have

$$\langle Px, x\rangle = \langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 = \|x\|^2,$$

so

$$\|P\| = \|A^*A\| \leq \|A\|\|A^*\| = \|A\|^2 = 1.$$

From Cauchy-Schwartz:

$$\langle Px, x\rangle \leq \|Px\|\|x\| \leq \|P\|\|x\|^2 \leq \|x\|^2.$$

Since we have equality in Cauchy-Schwartz, there exists a $\lambda \in \mathbb{C}$ such that $Px = \lambda x$. But from $\langle Px, x\rangle = \|x\|^2$, it follows that $\lambda = 1$. So $P\big|_{\mathcal{K}} = \mathrm{id}$ and for $x \in \mathcal{K}^\perp$, $Px = A^*Ax = 0$. Therefore, $P = A^*A$ is indeed a projection. Now onto the opposite direction ($\Rightarrow$). Suppose $P = A^*A$ is a projection and denote $\mathcal{K} = \operatorname{im} P$. Since $\mathcal{K} = \ker(I - P)$, it is a closed subspace of $\mathcal{H}$. For $x \in \mathcal{K}$, we have

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle Px, x\rangle = \langle x, x\rangle = \|x\|^2.$$

But for $x \in \mathcal{K}^\perp$, we use the identity

$$(\operatorname{im} P)^\perp = \ker P^* = \ker P$$

to get $Px = 0$, so $\|Ax\|^2 = \langle Px, x\rangle = 0$ and $\|Ax\| = 0$. $\square$

**Theorem 4.2** (Polar decomposition)**.**

*Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$. Then there exists a partial isometry $v$ such that $x = v \cdot |x|$ and $\ker v = \ker |x| = \ker x$. This decomposition is unique: if $x = wy$ for $y \geq 0$ and partial isometry $w$ such that $\ker y = \ker w$, then $w = v$ and $y = |x|$.*

*Proof.* First we prove the existence. Define

$$v_0 : \operatorname{im} |x| \to \operatorname{im} x, \quad |x|y \mapsto xy.$$

Since

$$\begin{aligned} \| |x|y \|^2 &= \langle |x|y, |x|y \rangle \\ &= \langle |x|^2 y, y \rangle \\ &= \langle x^* x y, y \rangle \\ &= \langle xy, xy \rangle \\ &= \| xy \|^2. \end{aligned}$$

The above $v_0$ is well defined. It is also linear and isometric. By continuity, extend $v_0$ to a map $\overline{\operatorname{im} |x|} \to \overline{\operatorname{im} x}$. Now $v_0$ can be extended to $v : \mathcal{H} \to \mathcal{H}$ by setting $v\big|_{(\operatorname{im} |x|)^\perp} = 0$. By previous lemma, $v$ is a partial isometry. By definition, $x = v \cdot |x|$ and $\ker v = (\operatorname{im} |x|)^\perp = \ker |x| = \ker x$. Next, we prove uniqueness. If $x = wy$ as in the statement, then $\ker w = \ker y = (\operatorname{im} y)^\perp$, so $w$ is a partial isometry on $\overline{\operatorname{im} y}$. From there, we get

$$|x|^2 = (wy)^*(wy) = y^* w^* w y = y^* y = y^2,$$

which implies

$$|x| = (|x|^2)^{\frac{1}{2}} = (y^2)^{\frac{1}{2}} = y.$$

Now

$$w|x|\mu = wy\mu = x\mu$$

together with

$$\ker w = (\operatorname{im} y)^\perp = (\operatorname{im} |x|)^\perp$$

implies $w = v$. □

Now we can also prove the statement in the example 1.43.

**Proposition 4.3.** *The extreme points of the unit ball of $\mathcal{B}(\mathcal{H})$ are exactly the elements $V \in \mathcal{B}(\mathcal{H})$ such that*

$$(1 - VV^*)\, \mathcal{B}(\mathcal{H})(1 - V^*V) = 0.$$

*In particular, $V^*V$ and $VV^*$ are projections.*

*Proof.* Let $V \in A$ be an extreme point of the unit ball of $\mathcal{B}(\mathcal{H})$, so $\sigma(V) \subseteq [-1, 1]$. Write

$$V = \frac{1}{2} V(2 - |V|) + \frac{1}{2} V|V|.$$

Since the functions $z \mapsto z(2-|z|)$ and $z \mapsto |z|(2-|z|)$ coincide and are both bounded above by 1 on $\sigma(x)$, we have $\|V(2-|V|)\| = \||V|(2-|V|)\| \leq 1$ by continuous functional calculus. This implies that $V(2-|V|)$ is in the unit ball of $\mathcal{B}(\mathcal{H})$. The same can be said about $V|V|$ by the same argument. Now since $V$ is an extreme point, we must have $V = V|V|$. Multiplying on the left with $V^*$, we get $|V|^2 = |V|^3$. This means that the functions $z \mapsto z^2$ and $z \mapsto z^3$ coincide on $\sigma(|V|)$, which implies that $\sigma(|V|) \subseteq \{0,1\}$. As a result, $|V|$ is a projection, so $P := |V| = |V|^2 = V^*V$. The same can be said about $Q := VV^*$, since we know that $\sigma(V^*V) \setminus \{0\} = \sigma(VV^*) \setminus \{0\}$. This means that $V$ is a partial isometry. By the previous lemma, $P$ is a projection onto the initial space of $V$, so $QV = VP = V$. Now suppose $W := (1-Q)Z(1-P) \neq 0$ for some $Z$ in the unit ball. Then

$$\begin{aligned}
\|V+W\|^2 &= \|QVP + (1-Q)Z(1-P)\|^2 \\
&= \|(QVP + (1-Q)Z(1-P))^*(QVP + (1-Q)Z(1-P))\| \\
&= \|(PV^*Q + (1-P)Z^*(1-Q))(QVP + (1-Q)Z(1-P))\| \\
&= \|PV^*QVP + (1-P)Z^*(1-Q)Z(1-P)\| \\
&= \|PV^*VP + (1-P)W^*W(1-P)\| \\
&= \max\{\|V^*V\|, \|W^*W\|\} \\
&= \max\{\|V\|^2, \|W\|^2\} = 1
\end{aligned}$$

and similarly $\|V-W\|^2 = 1$. Therefore we have a decomposition

$$V = \frac{1}{2}(V+W) + \frac{1}{2}(V-W)$$

and $V$ is not an extreme point, leading to a contradiction. Conversely, suppose that $(1 - VV^*)\,\mathcal{B}(\mathcal{H})(1-V^*V) = 0$. Then we have

$$0 = V^*(1-VV^*)V(1-V^*V) = V^*V(1-V^*V)^2.$$

This implies that the function $z \mapsto z(1-z)^2$ must be zero on $\sigma(V^*V)$, so $\sigma(V^*V) \subseteq \{0,1\}$ and $P := V^*V$ is a projection. By the same argument,

$$0 = (1-VV^*)V(1-V^*V)V^* = (1-VV^*)^2VV^*$$

and $Q := VV^*$ is a projection as well. Assume that $V = \frac{1}{2}U + \frac{1}{2}W$ for $U, W$ in the unit ball. Again we have $V = VP = QV$, so

$$V = \frac{1}{2}UP + \frac{1}{2}WP$$

and

$$\begin{aligned}
4P = 4V^*V &= PU^*UP + PW^*WP + PU^*WP + PW^*UP \\
&= 2(PU^*UP + PW^*WP) - P(U-W)^*(U-W)P \\
&\leq 4P - P(U-W)^*(U-W)P.
\end{aligned}$$

This immediately implies that $(U-W)P = 0$. Similarly, we have $Q(U-W) = 0$. Now

$$\begin{aligned}
U - W &= Q(U-W)P + (1-Q)(U-W)P \\
&\quad + Q(U-W)(1-P) + (1-Q)(U-W)(1-P) = 0.
\end{aligned}$$

□

*Remark.* The above theorem holds for any $C^*$-algebra, not just $\mathcal{B}(\mathcal{H})$. We can identify any general $C^*$-algebra with an algebra of operators on some Hilbert space and then the above proof carries over verbatim.

**Corollary 4.4.**

$$\operatorname{ext}(\mathcal{B}(\mathcal{H}))_1 = \{V \in \mathcal{B}(\mathcal{H}) \mid V \text{ or } V^* \text{ is an isometry}\}.$$

*Proof.* We need to prove the inclusion ($\subseteq$). If $V$ is an extreme point, then $V^*V$ and $VV^*$ are projections Therefore, $V$ is a partial isometry with the initial space $(\ker V)^\perp$ and $V^*$ is a partial isometry with the initial space $(\ker V^*)^\perp$. Assume neither $V$ nor $V^*$ are full isometries, so their initial spaces are proper subspaces of $\mathcal{H}$. This means that there exist vectors $0 \neq x \in \ker V$ and $0 \neq y \in \ker V^*$. Define $P$ as a rank-one projection from $x$ to $y$. Then

$$(1 - VV^*)P(1 - V^*V)x = y \neq 0,$$

so $(1 - VV^*)P(1 - V^*V) \neq 0$, contradiction. □

## 4.2 Trace class operators

**Definition 4.5.** Let $X, Y$ be Banach spaces. An operator $A \in \mathcal{B}(X, Y)$ has *finite rank* if $\operatorname{rank} A := \dim \overline{\operatorname{im} A} < \infty$. The set of finite rank operators is denoted by $\mathcal{F}(X, Y)$. We also denote $\mathcal{F}(X) := \mathcal{F}(X, X)$.

*Remark.* Let $A \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. We know that

$$\operatorname{im} A^* = \operatorname{im}(A^*|_{(\ker A^*)^\perp}) = \operatorname{im}(A^*|_{\overline{\operatorname{im} A}}).$$

From there, we can conclude that $\operatorname{rank} A < \infty$ iff $\operatorname{rank} A^* < \infty$.

If $\alpha, \beta \in \mathcal{H}$, then we can define the operator

$$\alpha \otimes \overline{\beta} : \mathcal{H} \to \mathcal{H}, \quad y \mapsto \langle y, \beta \rangle \cdot \alpha.$$

It is trivial to see that $\operatorname{rank}(\alpha \otimes \overline{\beta}) \leq 1$ and $(\alpha \otimes \overline{\beta})^* = \beta \otimes \overline{\alpha}$. By Riesz's representation theorem, we also know that every rank-one operator on $\mathcal{H}$ is of this form. If $\|\alpha\| = \|\beta\| = 1$, then $\alpha \otimes \overline{\beta}$ is a partial isometry with initial space $\mathbb{C}\beta$ and image $\mathbb{C}\alpha$. Then

$$\mathcal{F}(\mathcal{H}) = \operatorname{span}\, \{\alpha \otimes \overline{\beta} \mid \alpha, \beta \in \mathcal{H}\}.$$

For $x, y \in \mathcal{B}(\mathcal{H})$ we have

$$x(\alpha \otimes \overline{\beta})y = (x\alpha) \otimes \overline{(y^*\beta)}.$$

**Lemma 4.6.** *Let $x \in \mathcal{B}(\mathcal{H})$ have the polar decomposition $x = v \cdot |x|$. Then for all $y \in \mathcal{H}$, we have*

$$2\,|\langle xy, y\rangle| \leq \langle |x|y, y\rangle + \langle |x|v^*y, v^*y\rangle.$$

*Proof.* Let $\lambda \in \mathbb{T}$. Then

$$\begin{aligned}0 &\leq \|(|x|^{\frac{1}{2}} - \lambda |x|^{\frac{1}{2}} v^*) y\|^2 \\ &= \||x|^{\frac{1}{2}} y\|^2 - 2 \operatorname{Re} \overline{\lambda} \langle |x|^{\frac{1}{2}} y, |x|^{\frac{1}{2}} v^* y\rangle + \||x|^{\frac{1}{2}} v^* y\|^2.\end{aligned}$$

Now pick $\lambda$ such that $\overline{\lambda}\langle |x|^{\frac{1}{2}} y, |x|^{\frac{1}{2}} v^* y\rangle \geq 0$ and we are done. □

**Definition 4.7.** Let $(e_i)_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. For $x \in \mathcal{B}(\mathcal{H})_+$, define the *trace*

$$\operatorname{Tr}(x) = \sum_{i\in I} \langle x e_i, e_i\rangle \in [0, \infty].$$

We call $x \in \mathcal{B}(\mathcal{H})$ *trace class* if

$$\|x\|_1 := \operatorname{Tr}(|x|) < \infty.$$

The set of trace class operators on $\mathcal{H}$ will be denoted by $L^1(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

*Remark.* Let $\{h_i \mid i \in I\} \subseteq \mathcal{H}$ be a set of vectors in a Hilbert space. We already know that the collection of finite sets $F \subseteq I$ forms a directed set. Then vectors $h_F := \sum_{i\in F} h_i$ form a net in $\mathcal{H}$. We define $\sum_{i\in I} h_i$ as the limit of the net $(h_F)$, if it exists. Note that if $I$ is countable, this definition of a convergent sum does not necessarily coincide with the usual one. In other words, for a set $\{h_n \mid n \in \mathbb{N}\}$ in a Hilbert space $\mathcal{H}$, the convergence of a sum $\sum_{n\in\mathbb{N}} h_n$ is not equivalent to the convergence of a sum $\sum_{n=1}^{\infty} h_n$ – in fact, the convergence of a former sum implies the convergence of the latter one (with the sums being equal). The converse holds if $\sum_{n=1}^{\infty} \|h_n\| < \infty$.

*Remark.* If $x \in \mathcal{B}(\mathcal{H})_+$ and $\operatorname{Tr}(x) = \sum_{i\in I}\langle x e_i, e_i\rangle < \infty$, then $\langle x e_i, e_i\rangle > 0$ holds for at most countably many $e_i$. Let $(e_n)_{n\in\mathbb{N}}$ be a set of such basis vectors. Then $\sum_{i\in I}\langle x e_i, e_i\rangle = \sum_{n=1}^{\infty}\langle x e_n, e_n\rangle$.

**Lemma 4.8.** *For all $x \in \mathcal{B}(\mathcal{H})$ we have* $\operatorname{Tr}(x^*x) = \operatorname{Tr}(xx^*)$.

*Proof.*

$$\begin{aligned}\operatorname{Tr}(x^*x) &= \sum_i \langle x^* x e_i, e_i\rangle = \sum_i \langle x e_i, x e_i\rangle \\ &= \sum_i \|x e_i\|^2 = \sum_i \sum_j \langle x e_i, e_j\rangle \overline{\langle x e_i, e_j\rangle} \\ &= \sum_j \sum_i \langle e_i, x^* e_j\rangle \overline{\langle e_i, x^* e_j\rangle} = \sum_j \sum_i \langle x^* e_j, e_i\rangle \overline{\langle x^* e_j, e_i\rangle} \\ &= \sum_j \|x^* e_j\|^2 = \sum_j \langle x^* e_j, x^* e_j\rangle \\ &= \sum_j \langle x x^* e_j, e_j\rangle = \operatorname{Tr}(xx^*)\end{aligned}$$

□

**Corollary 4.9.** *If $x \in \mathcal{B}(\mathcal{H})_+$ and $u \in \mathcal{U}(\mathcal{H})$, then*

$$\operatorname{Tr}(u^*xu) = \operatorname{Tr}(x).$$

*In particular, the trace of a positive operator is independent of the choice of the orthonormal basis for $\mathcal{H}$.*

*Proof.* Since $x \in \mathcal{B}(\mathcal{H})_+$, there exists a $y \in \mathcal{B}(\mathcal{H})$ such that $x = y^*y$. By lemma 4.8, we have

$$\begin{aligned}\operatorname{Tr}(x) &= \operatorname{Tr}(y^*y) = \operatorname{Tr}(yy^*) \\ &= \operatorname{Tr}(u^*y^*yu) = \operatorname{Tr}(u^*xu).\end{aligned}$$

If $(f_i)$ is another ONB for $\mathcal{H}$, then there exists $u \in \mathcal{U}(\mathcal{H})$ such that $ue_i = f_i$ for all indices $i$:

$$\begin{aligned}\sum_i \langle xf_i, f_i\rangle &= \sum_i \langle xue_i, ue_i\rangle \\ &= \sum_i \langle u^*xue_i, e_i\rangle \\ &= \operatorname{Tr}(u^*xu) = \operatorname{Tr}(x).\end{aligned}$$

□

**Definition 4.10.** If $(e_i)$ is ONB for $\mathcal{H}$ and $x \in L^1(\mathcal{B}(\mathcal{H}))$, then its trace is

$$\operatorname{Tr}(x) := \sum_{i\in I} \langle xe_i, e_i\rangle.$$

By lemma 4.6 and the proof below, we get

$$\begin{aligned}2|\operatorname{Tr}(x)| &\le \sum_{i\in I} 2|\langle xe_i, e_i\rangle| \\ &\le \sum_{i\in I} \langle |x|e_i, e_i\rangle + \langle |x|v^*e_i, v^*e_i\rangle \\ &= \operatorname{Tr}(|x|) + \operatorname{Tr}(v|x|v^*) \\ &\le \|x\|_1 + \|x\|_1 \\ &= 2\|x\|_1.\end{aligned}$$

**Theorem 4.11.**

*(1.) $L^1(\mathcal{B}(\mathcal{H}))$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$ that is closed under involution.*

*(2.) $L^1(\mathcal{B}(\mathcal{H}))$ is a linear span of all positive operators of finite trace.*

*(3.) Trace is independent of the ONB and $\|\cdot\|_1$ is a norm on $L^1(\mathcal{B}(\mathcal{H}))$.*

*Proof.* Let $A, B \in L^1(\mathcal{B}(\mathcal{H}))$ and satisfy the polar decompositions:

$$A + B = U|A + B|, \quad A = V|A|, \quad B = W|B|.$$

Let $(e_i)$ be an ONB. Then

$$\begin{aligned}
\sum_{i=1}^{N} \langle |A + B| e_i, e_i \rangle &= \sum_{n=1}^{N} |\langle U^*(A + B)e_n, e_n\rangle| \\
&\leq \sum_{n=1}^{N} |\langle U^* A e_n, e_n\rangle| + \sum_{n=1}^{N} |\langle U^* B e_n, e_n\rangle| \\
&= \sum_{n=1}^{N} |\langle U^* V |A| e_n, e_n\rangle| + \sum_{n=1}^{N} |\langle U^* W |B| e_n, e_n\rangle|.
\end{aligned}$$

We can bound the first term:

$$\begin{aligned}
\sum_{n=1}^{N} |\langle U^* V |A| e_n, e_n\rangle| &= \sum_{n=1}^{N} |\langle |A|^{\frac{1}{2}} e_n, |A|^{\frac{1}{2}} V^* U e_n\rangle| \\
&\leq \sum_{n=1}^{N} \||A|^{\frac{1}{2}} e_n\| \||A|^{\frac{1}{2}} V^* U e_n\| \\
&\leq \left( \sum_{n=1}^{N} \||A|^{\frac{1}{2}} e_n\|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{N} \||A|^{\frac{1}{2}} V^* U e_n\|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\||A|^{\frac{1}{2}} e_n\|^2 = \langle |A|^{\frac{1}{2}} e_n, |A|^{\frac{1}{2}} e_n\rangle = \langle |A| e_n, e_n\rangle$, the expression in the first bracket goes to $\operatorname{Tr}|A|$. Next, we prove that the expression in the second bracket is less or equal to $\operatorname{Tr}|A|$:

$$\sum_{n=1}^{N} \langle |A|^{\frac{1}{2}} V^* U e_n, |A|^{\frac{1}{2}} V^* U e_n\rangle = \sum_{n=1}^{N} \langle U^* V |A| V^* U e_n, e_n\rangle \xrightarrow{N \to \infty} \operatorname{Tr}|A|.$$

Pick an ONB for $\mathcal{H}$ as follows: each $f_j$ should be in $\ker U$ or $(\ker U)^\perp$. Then

$$\operatorname{Tr}(U^* V |A| V^* U) \leq \operatorname{Tr}(V |A| V^*).$$

By similar argument,

$$\operatorname{Tr}(V|A|V^*) \leq \operatorname{Tr}(|A|)$$

and we are done:

$$\sum_{n=1}^{N} |\langle U^* V |A| e_n, e_n\rangle| \leq \operatorname{Tr}|A|.$$

Similarly,

$$\sum_{n=1}^{N} |\langle U^* W |B| e_n, e_n\rangle| \leq \operatorname{Tr}|B|,$$

which implies $\operatorname{Tr}|A + B| \leq \operatorname{Tr}|A| + \operatorname{Tr}|B|$. We have proved that $L^1(\mathcal{B}(\mathcal{H}))$ is a vector space and $\|\cdot\|_1$ is a norm. Clearly, $L^1(\mathcal{B}(\mathcal{H}))$ contains all positive operators with finite trace, so also their linear span. Next we prove that it is a two-sided ideal of $\mathcal{B}(\mathcal{H})$. Let $A \in L^1(\mathcal{B}(\mathcal{H}))$ and $B \in \mathcal{B}(\mathcal{H})$. Since every operator is a linear combination of four unitaries, we can assume w.l.o.g. that $B = U$ is a unitary. Then

$$|UA| = (A^*U^*UA)^{\frac{1}{2}} = (A^*A)^{\frac{1}{2}} = |A|,$$

so $BA = UA \in L^1(\mathcal{B}(\mathcal{H}))$. Furthermore,

$$|AU| = (U^*A^*AU)^{\frac{1}{2}} = U^*|A|U,$$

which implies

$$\operatorname{Tr}|AU| = \operatorname{Tr}(U^*|A|U) = \operatorname{Tr}|A|$$

and $AB = AU \in L^1(\mathcal{B}(\mathcal{H}))$. Now we prove that $L^1(\mathcal{B}(\mathcal{H}))$ is closed under involution. Let $A = U|A|$ and $A^* = V|A^*|$ be polar decompositions. Then

$$|A^*| = V^*A^* = V^*(U|A|)^* = V^*|A|U^*.$$

If $A \in L^1(\mathcal{B}(\mathcal{H}))$, then $|A| \in L^1(\mathcal{B}(\mathcal{H}))$, so

$$|A^*| = V^*|A|U^* \in L^1(\mathcal{B}(\mathcal{H})).$$

This gives us $A^* \in L^1(\mathcal{B}(\mathcal{H}))$. Finally, we prove that $L^1(\mathcal{B}(\mathcal{H}))$ is the linear span of all positive operators of finite trace. Let $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $a \in \mathcal{B}(\mathcal{H})$. The following polarization identity holds:

$$4a|x| = \sum_{k=0}^{3} i^k \underbrace{(a + i^k)|x|(a + i^k)^*}_{\text{positive and finite trace}}.$$

If $a = v$ partial isometry from the polar decomposition theorem, then

$$x = v|x| = \sum_{k=0}^{3} \frac{i^k}{4}(v + i^k)|x|(v + i^k)^*.$$

is a linear combination of four positive operators with finite trace. □

**Proposition 4.12.** *Let $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $a, b \in \mathcal{B}(\mathcal{H})$. Then*

- $\|x\| \leq \|x\|_1$;
- $\|axb\|_1 \leq \|a\|\|b\|\|x\|_1$;
- $\operatorname{Tr}(ax) = \operatorname{Tr}(xa)$.

*Proof.* (1.)

$$\begin{aligned}\|x\| &= \||x|\| = \||x|^{\frac{1}{2}}\|^2 \\ &= \sup_{\|\alpha\|=1} \langle |x|^{\frac{1}{2}}\alpha, |x|^{\frac{1}{2}}\alpha\rangle = \sup_{\|\alpha\|=1} \langle |x|\alpha, \alpha\rangle \\ &\leq \operatorname{Tr}|x| = \|x\|_1.\end{aligned}$$

(2.) We begin with

$$|ax|^2 = x^*a^*ax \leq \|a^*a\|x^*x = \|a^*a\| \cdot |x|^2 = \|a\|^2 \cdot |x|^2$$

and since $|ax| \leq \|a\| \cdot |x|$ we get $\|ax\|_1 \leq \|a\| \cdot \|x\|_1$. But $\|x\|_1 = \|x^*\|_1$, so we also get $\|xb\|_1 \leq \|b\| \cdot \|x\|_1$.

(3.) Since every element of $\mathcal{B}(\mathcal{H})$ is a linear combination of 4 unitaries, we can w.l.o.g. assume $a = u \in \mathcal{U}(\mathcal{H})$. Then

$$\begin{aligned}\operatorname{Tr}(xu) &= \sum_i \langle xue_i, e_i\rangle = \sum_i \langle xue_i, u^*ue_i\rangle \\ &= \sum_i \langle uxue_i, ue_i\rangle = \operatorname{Tr}(ux).\end{aligned}$$ □

*Remark.* We have the following identities:

(1.) $\operatorname{Tr}(\alpha \otimes \overline{\beta}) = \langle \alpha, \beta\rangle$;

(2.) $\mathcal{F}(\mathcal{H})$ is dense in $(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$.

**Theorem 4.13.**

*$(L^1(\mathcal{B}(\mathcal{H})), \|\cdot\|_1)$ is a Banach space.*

*Proof.* We only have to prove completeness. Let $(x_n)_n$ be a Cauchy sequence in $(L^1(\mathcal{B}(\mathcal{H})), \|\|_1)$. Since $\|\cdot\| \leq \|\cdot\|_1$, $(x_n)$ is a Cauchy sequence in $(\mathcal{B}(\mathcal{H}), \|\|)$. But $(\mathcal{B}(\mathcal{H}), \|\|)$ is a Banach space, so there exists $x \in \mathcal{B}(\mathcal{H})$ such that $x_n \to x$ in norm-topology. Notice that

$$x^*x - x_n^*x_n = x^*(x - x_n) + (x - x_n)^*x_n.$$

By continuity of the continuous functional calculus, this implies $|x_n| \to |x|$, meaning that $\||x_n| - |x|\| \to 0$. Next we prove that $x \in L^1(\mathcal{B}(\mathcal{H}))$. For any ONB $(e_i)_i$, we have

$$\sum_{i=1}^k \langle |x|e_i, e_i\rangle = \lim_{n\to\infty} \sum_{i=1}^k \langle |x_n|e_i, e_i\rangle \leq \lim_{n\to\infty} \operatorname{Tr}|x_n| = \lim_{n\to\infty} \|x_n\|_1 < \infty.$$

Here, we used the fact that $\|x_n - x_k\|_1 \geq \|x_n\|_1 - \|x_k\|_1$, so the sequence $(\|x_n\|_1)_n$ is Cauchy and therefore has a limit. This proves that $x \in L^1(\mathcal{B}(\mathcal{H}))$ and $\|x\|_1 \leq \lim_{n\to\infty} \|x_n\|_1$. Finally, we have to show that $\|x_n - x\|_1 \to 0$. Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that for every $n > N$, we get $\|x_n - x_N\|_1 < \frac{\varepsilon}{3}$. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a finite dimensional subspace such that

$$\|x_N P_{\mathcal{H}_0^\perp}\|_1,\ \|xP_{\mathcal{H}_0^\perp}\|_1 < \frac{\varepsilon}{3}.$$

Then for every $n > N$, we get that

$$\begin{aligned}\|x - x_n\|_1 &\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|(x - x_n)P_{\mathcal{H}_0^\perp}\|_1 \\ &\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \|xP_{\mathcal{H}_0^\perp} - x_N P_{\mathcal{H}_0^\perp}\|_1 + \|x_N P_{\mathcal{H}_0^\perp} - x_n P_{\mathcal{H}_0^\perp}\|_1 \\ &\leq \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|x_N - x_n\|_1 \|P_{\mathcal{H}_0^\perp}\| \\ &< \|(x - x_n)P_{\mathcal{H}_0}\|_1 + \varepsilon \\ &\leq \|(x - x_n)\|\|P_{\mathcal{H}_0}\|_1 + \varepsilon \xrightarrow[n\to\infty]{} \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows $x_n \xrightarrow[\|\cdot\|_1]{} x$. ☐

**Theorem 4.14.**

*The map*

$$\Psi : \mathcal{B}(\mathcal{H}) \to L^1(\mathcal{B}(\mathcal{H}))^*, \quad a \mapsto (\psi_a : x \mapsto \operatorname{Tr}(ax))$$

*is an isometric isomorphism of Banach spaces.*

*Proof.* We notice that $\Psi$ is linear and a contraction because the norms $\|\cdot\|$ and $\|\cdot\|_1$ are comparable. We will first show that $\Psi$ is surjective. Let $\varphi \in L^1(\mathcal{B}(\mathcal{H}))^*$. Notice that

$$(\alpha, \beta) \mapsto \varphi(\alpha \otimes \overline{\beta})$$

is a bounded sesquilinear form in $\mathcal{H}$. By the introductory course, there exists an $a \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(\alpha \otimes \overline{\beta}) = \langle a\alpha, \beta\rangle = \operatorname{Tr}(a\alpha \otimes \overline{\beta}) = \operatorname{Tr}(a(\alpha \otimes \overline{\beta})) = \psi_a(\alpha \otimes \overline{\beta}).$$

So $\varphi$ and $\psi_a$ agree on $\mathcal{F}(\mathcal{H})$, so by bounded density $\varphi = \psi_a$. Finally,

$$\|a\| = \sup_{\alpha,\beta \in (\mathcal{H})_1} |\langle a\alpha, \beta\rangle| = \sup_{\alpha,\beta \in (\mathcal{H})_1} |\operatorname{Tr}(a(\alpha \otimes \overline{\beta}))| \leq \|\psi_a\|_1.$$

But since

$$\|\psi_a\|_1 = \sup_{x \in (L^1(\mathcal{B}(\mathcal{H})))_1} |\operatorname{Tr}(ax)| = \sup_{x \in (L^1(\mathcal{B}(\mathcal{H})))_1} \|ax\|_1 \leq \sup_{x \in (L^1(\mathcal{B}(\mathcal{H})))_1} \|a\|\|x\|_1 = \|a\|,$$

we have $\|a\| = \|\psi_a\|_1$ and $\psi$ is isometric. ☐

**Corollary 4.15.** *The map*

$$\Phi : L^1(\mathcal{B}(\mathcal{H})) \to \mathcal{K}(\mathcal{H})^*, \quad x \mapsto (\varphi_x : a \mapsto \operatorname{Tr}(ax))$$

*is an isometric isomorphism of Banach spaces.*

*Proof.* Same as that of theorem 4.14. ☐

**Definition 4.16.** Let $X, Y$ be Banach spaces. An operator $T \in \mathcal{B}(X, Y)$ is said to be *compact* if $\overline{T((X)_1)}$ is compact. The space of compact operators is $\mathcal{K}(X, Y)$. We also write $\mathcal{K}(X) := \mathcal{K}(X, X)$.

From the introductory course, we know the following statements about compact operators.

**Proposition 4.17.** *Let $T \in \mathcal{B}(X, Y)$. The following are equivalent.*

*(1.) $T$ is compact;*

*(2.) $T$ maps bounded maps in $X$ into relatively compact maps in $Y$;*

*(3.) $T$ maps bounded sequences in $X$ into sequences in $Y$ that have an accumulation point.*

*If $X, Y$ are Hilbert spaces, then this is also equivalent to the following.*

*(4.) $T \in \overline{\mathcal{F}(X, Y)}$.*

*Remark.* $\mathcal{K}(\mathcal{H})$ is a closed ideal in $\mathcal{B}(\mathcal{H})$.

**Theorem 4.18** (Singular value decomposition)**.**

*For $K \in \mathcal{K}(\mathcal{H})$, there exists orthonormal bases $(e_i)_i$ and $(f_j)_j$ for $\mathcal{H}$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ such that*

$$Kx = \sum_{n=1}^{\infty} \sigma_n \langle x, e_n \rangle f_n. \tag{4.1}$$

*As a result,*

$$|K|x = \sum \sigma_n \langle x, e_n \rangle e_n.$$

**Proposition 4.19.** *Let $X$ be a Banach space. Then the following statements are equivalent:*

*(1.) $\mathrm{id} : X \to X$ is compact;*

*(2.) $(X)_1$ is compact;*

*(3.) $\dim X < \infty$.*

*states The equivalence of the last two items is also known as the* Riesz lemma.

**Theorem 4.20.**

*(1.) $L^1(\mathcal{B}(\mathcal{H})) \subseteq K(\mathcal{H})$.*

*(2.) $K \in \mathcal{H}$ is a $L^1(\mathcal{B}(\mathcal{H}))$ iff $\sum_{k=1}^{\infty} \sigma_n < \infty$.*

*Proof.* (1.) If $x \in L^1(\mathcal{B}(\mathcal{H}))$, then there exists $(x_n)_n$ in $\mathcal{F}(\mathcal{H})$ such that $\|x_n - x\|_1 \to 0$. Since $\|\cdot\| \leq \|\cdot\|_1$, we get $\|x_n - x\| \to 0$ and $x \in \overline{(\mathcal{F}, \|\cdot\|)} = \mathcal{K}(\mathcal{H})$.

(2.) This follows from $\operatorname{Tr}|K| = \sum \sigma_n$ for $K$ as in (4.1). $\square$

## 4.3 Hilbert–Schmidt operators

**Definition 4.21.** An element $x \in \mathcal{B}(\mathcal{H})$ is a *Hilbert–Schmidt operator* if

$$|x|^2 = x^* x \in L^1(\mathcal{B}(\mathcal{H})).$$

The set of all such elements is denoted by $L^2(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$.

**Proposition 4.22.** *(1.) $L^2(\mathcal{B}(\mathcal{H})) \lhd \mathcal{B}(\mathcal{H})$ and is closed under $*$.*

*(2.) If $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then $xy, yx \in L^1(\mathcal{B}(\mathcal{H}))$ and $\operatorname{Tr}(xy) = \operatorname{Tr}(yx)$.*

*Remark.* Beware: there exist $a, b \in \mathcal{B}(\mathcal{H})$ such that $ab \in L^1(\mathcal{B}(\mathcal{H}))$ and $ba \notin L^1(\mathcal{B}(\mathcal{H}))$. However, if $ab, ba \in L^1(\mathcal{B}(\mathcal{H}))$, then $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$.

*Proof.* For $\alpha \in \mathbb{C}$ and $x, y \in \mathcal{B}(\mathcal{H})$, we have $|\alpha x|^2 = |\alpha|^2 |x|^2$. Similarly, $|x+y|^2 \le |x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$, so $L^2(\mathcal{B}(\mathcal{H}))$ is a complex vector space. Since $|ax|^2 \le \|a\|^2 \cdot |x|^2$, we have $L^2(\mathcal{B}(\mathcal{H}))$ is a left ideal of $\mathcal{B}(\mathcal{H})$. From

$$\operatorname{Tr}|x|^2 = \operatorname{Tr}(x^* x) = \operatorname{Tr}(x x^*) = \operatorname{Tr}|x^*|^2,$$

we deduce that $L^2(\mathcal{B}(\mathcal{H}))$ is closed under involution. If $x \in L^2(\mathcal{B}(\mathcal{H}))$ and $b \in \mathcal{B}(\mathcal{H})$, then $x^* \in L^2(\mathcal{B}(\mathcal{H}))$, which implies $b^* x^* \in L^2(\mathcal{B}(\mathcal{H}))$ and finally $xb = (b^* x^*)^* \in L^2(\mathcal{B}(\mathcal{H}))$, so $L^2(\mathcal{B}(\mathcal{H})) \lhd \mathcal{B}(\mathcal{H})$. Next, we use the polarization identity

$$4 y^* x = \sum_{k=0}^{3} i^k |x + i^k y|^2.$$

If $x, y \in L^2(\mathcal{B}(\mathcal{H}))$, then this shows $y^* x \in L^1(\mathcal{B}(\mathcal{H}))$ and

$$\begin{aligned} 4 \operatorname{Tr}(y^* x) &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x + i^k y)^* (x + i^k y)) \\ &= \sum_{k=0}^{3} i^k \operatorname{Tr}((x + i^k y)(x + i^k y)^*) \\ &= 4 \operatorname{Tr}(x y^*). \end{aligned}$$

$\square$

On $L^2(\mathcal{B}(\mathcal{H}))$ we have the sesquilinear form $\langle x, y \rangle_2 := \operatorname{Tr}(y^* x)$. It is well-defined and positive definite, so it is a scalar product. The induced norm is denoted by $\|\cdot\|_2$. For every $y \in L^2(\mathcal{B}(\mathcal{H}))$, we have

$$\|y\| = \|y^* y\|^{\frac{1}{2}} \le \|y^* y\|_1^{\frac{1}{2}} = \|y\|_2.$$

Similarly, we have

$$\|axb\|_2 = \|a\| \cdot \|x\|_2 \cdot \|b\|$$

for all $x \in L^2(\mathcal{B}(\mathcal{H}))$ and $a, b \in \mathcal{B}(\mathcal{H})$. As before, $\mathcal{F}(\mathcal{H})$ are dense in $L^2(\mathcal{B}(\mathcal{H}))$ with respect to $\|\cdot\|_2$ and $L^2(\mathcal{B}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$. Using singular values $(\sigma_n)_n$ of a compact $K \in \mathcal{K}(\mathcal{H})$, we have $K \in L^2(\mathcal{B}(\mathcal{H}))$ iff $\sum_{k=0}^{\infty} \sigma_j^2 < \infty$. For every $x \in L^1(\mathcal{B}(\mathcal{H}))$, we have

$$\|x\|_2 = \sup_{y \in L^2(\mathcal{B}(\mathcal{H})),\ \|y\|_2=1} |\operatorname{Tr}(y^* x)| \le \sup_{y \in L^2(\mathcal{B}(\mathcal{H})),\ \|y\|_2=1} \|y\| \cdot \|x\|_1 \le \|x\|_1.$$

As a result, $(L^2(\mathcal{B}(\mathcal{H})), \langle\cdot\rangle_2)$ is a Hilbert space.

**Theorem 4.23** (Hölder's inequality)**.**

*For all $x, y \in L^2(\mathcal{B}(\mathcal{H}))$ we have*

$$\|xy\|_1 \le \|x\|_2 \|y\|_2.$$

*Proof.* Let $xy = v|xy|$ be the polar decomposition of $xy$. Then

$$\begin{aligned} \|xy\|_1 &= \operatorname{Tr}|xy| = \operatorname{Tr}(v^* xy) \\ &= |\langle y, x^* v\rangle_2| \le \|x^* v\|_2 \|y\|_2 \\ &\le \|x^*\|_2 \|v\| \|y\|_2 \le \|x\|_2 \cdot \|y\|_2. \end{aligned}$$

□

## 4.4 Hilbert–Schmidt integral operators

In this section, we will make use of the following result from measure theory (see, for example, theorem 8.8 in [4]).

**Theorem 4.24** (Fubini's theorem)**.**

*If $(X, \mu), (Y, \lambda)$ are $\sigma$-finite measure spaces and $\int_{X \times Y} |f|\, d(\mu \times \lambda)(x, y) < \infty$, then*

$$\int_{X \times Y} f\, d(\mu \times \lambda)(x, y) = \int_Y \left( \int_X f\, d\mu(x) \right) d\lambda(y) = \int_X \left( \int_Y f\, d\lambda(y) \right) d\mu(x).$$

For $K \in L^2(X \times X, \mu \times \mu)$, we define a *Hilbert–Schmidt integral operator* with kernel $K$:

$$T_K : L^2(X, \mu) \to L^2(X, \mu), \quad f \mapsto \left( y \mapsto \int_X K(x, y) f(y)\, d\mu(x) \right).$$

Suppose $(\varphi_\alpha)_\alpha$ is an ONB for $L^2(K, \mu)$. By Fubini, $\left( \overline{\varphi_a(x)} \varphi_\beta(y) \right)_{\alpha, \beta}$ is an orthonormal basis for $L^2(X \times X, \mu \times \mu)$. Since $K \in L^2(X \times X, \mu \times \mu)$, there exist $c_{ij} \in \mathbb{C}$ such that

$$K(x, y) = \sum_{i,j} c_{ij} \overline{\varphi_i(x)} \varphi_j(y), \quad \|K\|^2_{L^2(X \times X)} = \sum |c_{ij}|^2 < \infty.$$

We show that $T_K$ is well-defined: for $f \in L^2(X, \mu)$, we have $T_K f \in L^2(X, \mu)$. Indeed,

$$T_k f(y) = \sum_{i,j} c_{ij} \langle f, \varphi_i \rangle \varphi_j(y),$$

which implies

$$\begin{aligned}
\|T_K f\|^2_{L^2(X)} &\le \sum_{i,j} |c_{ij}|^2 |\langle f, \varphi_j\rangle|^2 \|\varphi_j\|^2_{L^2(X)} \\
&\le \|f\|^2_{L^2} \sum_{i,j} |c_{ij}|^2 \|\varphi\|^2_{L^2} \|\varphi_j\|^2_{L^2} \\
&= \|f\|^2_{L^2} \sum |c_{ij}|^2 \\
&= \|f\|^2_{L^2} \|K\|^2_{L^2(X\times X)}
\end{aligned}$$

and finally $\|T_K\| \le \|K\|_{L^2}$. We claim that $T_K^* : L^2(X,\mu) \to L^2(X,\mu)$ is the integral operator with kernel

$$K^*(y,x) := \overline{K(x,y)}.$$

Indeed,

$$\begin{aligned}
\langle T_K f, g\rangle &= \int_Y \left(\int_X K(x,y) f(x)\, d\mu(x)\right) \cdot \overline{g(y)}\, d\mu(y) \\
&= \int_X f(x) \cdot \left(\overline{\int_Y \overline{K(x,y)} g(y)\, d\mu(y)}\right) d\mu(x) \\
&= \langle f, T_{K^*} g\rangle.
\end{aligned}$$

**Theorem 4.25.**

*(1.) For $K \in L^2(X\times X, \mu\times\mu)$ we have $T_K \in L^2(\mathcal{B}(L^2(X,\mu)))$.*

*(2.) The mapping $\Phi : K \mapsto T_K$ is a unitary $L^2(X\times X, \mu\times\mu) \to L^2(\mathcal{B}(L^2(X,\mu)))$.*

*Proof.* (1.) We will prove that $\|T_K\|_2 = \|K\|_{L^2}$. We want to approximate $T_K$ with finite rank operators, so we first approximate $K$:

$$K(x,y) = \sum_{i,j=1}^{\infty} c_{ij} \overline{\varphi_i(x)} \varphi_j(x)$$

for an orthonormal basis $(\varphi_\alpha)_\alpha$ for $L^2(X,\mu)$. For $N \in \mathbb{N}$ let $K_N(x,y) = \sum_{i,j}^N c_{ij}\overline{\varphi_i(x)}\varphi_j(x)$. Then

$$T_{K_N} f = \sum_{i,j=1}^N c_{ij} \langle f, \varphi_i\rangle \varphi_j \in \mathcal{F}(L^2(X,\mu)).$$

By the above inequality,

$$\|T_K - T_{K_N}\| \le \|K - K_N\|_{L^2} \to 0,$$

so $T_K \in \overline{(\mathcal{F}, \|\cdot\|)} = \mathcal{K}(\mathcal{H})$. Then

$$\|T_K\|_2^2 = \sum_i \|T_K \varphi_i\|^2_{L^2} = \sum_{i,j,k} \|c_{jk}\varphi_j(x)\delta_{ik}\|^2 = \sum |c_{ij}|^2 = \|K\|^2_{L^2}.$$

(2.) It remains to prove surjectivity. Since $\Phi$ is isometric, $\operatorname{im}\Phi$ is closed. So it suffices to show that $\operatorname{im}\Phi$ is dense. In particular, we will show that $\operatorname{im}\Phi \supseteq \mathcal{F}(L^2(X,\mu))$. Let $A \in \mathcal{F}(L^2(X,\mu))$, so $\operatorname{rank} A < \infty$. Let $(\psi_1,\dots,\psi_m)$ be an orthonormal basis for $\operatorname{im} A$. Then $A\varphi = c_1(\varphi)\psi_1 + \dots + c_m(\varphi)\psi_m$ for some bounded linear functionals $c_j$ on $L^2(X,\mu)$. By Riesz, there exist $\mu_j \in L^2(X,\mu)$ such that $c_j(\varphi) = \langle \varphi, \mu_j\rangle$. Hence

$$A\varphi(x) = \int_X \left(\sum_{j=1}^m \psi_j(x)\cdot\overline{\mu_j(y)}\cdot\varphi(y)\right) d\mu(y) = T_{\sum_{i=1}^m \psi_j(x)\overline{\mu_j(y)}} \in \operatorname{im}\Phi. \qquad \square$$

## 4.5 Tensor products of Hilbert spaces

Recall the usual construction of tensor product of Hilbert spaces: for Hilbert spaces $\mathcal{H},\mathcal{K}$, we first form the algebraic tensor product of vector spaces $\mathcal{H}\otimes\mathcal{K}$ and then equip it with a scalar product

$$\langle h_1\otimes k_1, h_2\otimes k_2\rangle_{\mathcal{H}\otimes\mathcal{K}} := \langle h_1,h_2\rangle_{\mathcal{H}}\cdot\langle k_1,k_2\rangle_{\mathcal{K}},$$

which we then extend linearly. Finally, we take the completion of $\mathcal{H}\otimes\mathcal{K}$ w.r.t. the above scalar product and denote it by $\mathcal{H}\overline{\otimes}\mathcal{K}$. Then $\mathcal{H}\otimes\mathcal{K}$ is the *tensor product of Hilbert spaces.* However, the machinery of Hilbert–Schmidt operators allows us to explicitly construct the tensor product of Hilbert spaces, without appealing to the metric space completion.

Let $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$. We associate to $A$ the map

$$\widetilde{A}\in\mathcal{B}(\mathcal{H}\oplus\mathcal{K},\mathcal{H}\oplus\mathcal{K}), \quad \alpha\oplus\beta\mapsto 0\oplus A\alpha,$$

or in matrix form,

$$\widetilde{A} = \begin{bmatrix} 0 & 0\\ A & 0\end{bmatrix}.$$

We denote the set of Hilbert–Schmidt operators $\mathcal{H}\to\mathcal{K}$ as

$$HS(\mathcal{H},\mathcal{K}) := \{A\in\mathcal{B}(\mathcal{H},\mathcal{K}) \mid \widetilde{A}\in L^2(\mathcal{B}(\mathcal{H}\oplus\mathcal{K}))\}.$$

It is trivial to show that this coincides with

$$\{A\in\mathcal{B}(\mathcal{H},\mathcal{K}) \mid A^*A\in L^1(\mathcal{B}(\mathcal{H}))\}.$$

With the usual scalar product $\langle A,B\rangle_2 = \operatorname{Tr}(B^*A)$, $HS(\mathcal{H},\mathcal{K})$ becomes a Hilbert space.

**Example 4.26.** *By Riesz's representation theorem, every functional in $\mathcal{H}^*$ is of the form $\overline{\alpha}: x\mapsto\langle x,\alpha\rangle$, where $\alpha\in\mathcal{H}$. This means that we can introduce a scalar product on $\mathcal{H}^*$ by $\langle\overline{\alpha},\overline{\beta}\rangle_{\mathcal{H}^*} := \langle\beta,\alpha\rangle_{\mathcal{H}}$. This scalar product induces the usual operator norm on $\mathcal{H}^*$, so it makes $\mathcal{H}^*$ into a Hilbert space.*

**Example 4.27.** *Now we show that the dual $\mathcal{H}^*$ is isomorphic as a Hilbert space to $HS(\mathcal{H},\mathbb{C})$. To prove this, it's enough to compare the scalar products. For any $\overline{\alpha},\overline{\beta}\in\mathcal{H}^*$,* *we have*

$$\begin{aligned}\langle\overline{\alpha},\overline{\beta}\rangle &= \mathrm{Tr}(\overline{\beta}^*\overline{\alpha})\\ &= \sum_{i\in I}\langle\overline{\alpha}e_i,\overline{\beta}e_i\rangle\\ &= \sum_{i\in I}\langle e_i,\alpha\rangle\overline{\langle e_i,\beta\rangle}\\ &= \sum_{i\in I}\langle e_i,\alpha\rangle\langle\beta,e_i\rangle\\ &= \langle\beta,\alpha\rangle = \langle\overline{\alpha},\overline{\beta}\rangle_{\mathcal{H}^*}.\end{aligned}$$

We can now explicitly define the tensor product of Hilbert spaces as $\mathcal{H}\overline{\otimes}\mathcal{K}$ as the Hilbert space $HS(\mathcal{H}^*,\mathcal{K})$. It's not hard to show that $HS(\mathcal{H}^*,\mathcal{K})$ is isomorphic as a Hilbert space to our previous definition of $\mathcal{H}\overline{\otimes}\mathcal{K}$. The elementary tensors which span the algebraic tensor product $\mathcal{H}\otimes\mathcal{K}$ correspond to operators

$$\alpha\otimes\beta : \mathcal{H}^*\to\mathcal{K}, \quad f\mapsto f(\alpha)\beta,$$

where $\alpha\in\mathcal{H}$ and $\beta\in\mathcal{K}$. The linear span of operators $\alpha\otimes\beta$ consists of all the finite-rank operators in $\mathcal{B}(\mathcal{H}^*,\mathcal{K})$.

## 4.6 Locally convex topologies on $\mathcal{B}(\mathcal{H})$

If $\mathcal{H}$ is a Hilbert space, then $(\mathcal{B}(\mathcal{H}),\|\cdot\|)$ is a Banach algebra with its norm topology.

**Definition 4.28.** (1.) The *weak operator topology* (WOT) is given by the seminorms

$$T\mapsto|\langle T\alpha,\beta\rangle|, \quad \forall\alpha,\beta\in\mathcal{H}.$$

(2.) The *strong operator topology* (SOT) is given by the seminorms

$$T\mapsto\|T\alpha\|, \quad \forall\alpha\in\mathcal{H}.$$

These topologies are comparable: WOT $\subseteq$ SOT $\subseteq$ norm topology.

- Norm topology has the subbasis

  $$\{S\in\mathcal{B}(\mathcal{H}) \mid \|S-T\|<\varepsilon\}$$

  for $T\in\mathcal{B}(\mathcal{H})$ and $\varepsilon>0$. The net $T_i$ converges to $T$ iff $\|T_i-T\|$ converges to 0.

- WOT topology has the subbasis

  $$\{S\in\mathcal{B}(\mathcal{H}) \mid \langle(S-T)\alpha,\beta\rangle<\varepsilon\}$$

  for $\alpha,\beta\in\mathcal{H}$, $T\in\mathcal{B}(\mathcal{H})$ and $\varepsilon>0$. The net $T_i$ converges to $T$ iff $\langle T_i\alpha,\beta\rangle$ converges to $\langle T\alpha,\beta\rangle$ for all $\alpha,\beta$.

- SOT topology has the subbasis

  $$\{S\in\mathcal{B}(\mathcal{H}) \mid \|(S-T)\alpha\|<\varepsilon\}$$

for $\alpha \in \mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. The net $T_i$ converges to $T$ iff $\|(T_i - T)\alpha\|$ converges to 0 for all $\alpha$.

**Example 4.29.** *Let $\mathcal{H} = \ell^2(\mathbb{N})$ and denote $T_n = \frac{1}{n} \cdot \mathrm{id}$. Then $T_n \to 0$ in the norm topology. Now if we introduce the operator*

$$S(x_1, x_2, \dots) = (0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots),$$

*then $S_n \to 0$ in SOT, but not in norm topology, since $\|S_n\| = 1$. Lastly, we define*

$$W_n(x_1, x_2, \dots) = (0, 0, \dots, x_1, x_2, \dots).$$

*We get that $W_n \to 0$ in WOT, but not in SOT or norm topology.*

**Example 4.30.** *Let $(y_n)_n$ be a countable dense subset of $\mathcal{H} = \ell^2$. Consider the following two metrics on $(\mathcal{B}(\mathcal{H}))_1$:*

$$d_S(A, B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|(A - B)y_n\|, \quad d_W(A, B) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle (A - B)y_n, y_n\rangle| .$$

*Then $d_S$ induces SOT and $d_W$ induces WOT on $(\mathcal{B}(\mathcal{H}))_1$.*

**Example 4.31.** *The multiplication*

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad (A, B) \mapsto A \cdot B$$

*is not jointly continuous with respect to SOT or WOT. Indeed, if $S : \ell^2 \to \ell^2$ is the right shift (and $S^*$ the left shift), then $S^n \to 0$ and $(S^*)^n \to 0$ in SOT and WOT, but $(S^*)^n S^n = I$. However, multiplication is WOT- and SOT-continuous in each factor separately. Suppose that $(x_\alpha)_\alpha \to x$ in WOT and $y \in \mathcal{B}(\mathcal{H})$. Then for each $v, w \in \mathcal{H}$, we have*

$$|\langle x_\alpha y v - x y v, w\rangle| \to 0,$$

*since $x_\alpha \to x$ in WOT. Similarly,*

$$|\langle y x_\alpha v - y x v, w\rangle| = |\langle x_\alpha v - x v, y^* w\rangle| \to 0,$$

*which implies $x_\alpha y \to xy$ and $y x_\alpha \to yx$ in WOT. Similarly, if $(x_\alpha)_\alpha \to x$ in SOT and $y \in \mathcal{B}(\mathcal{H})$, then for each $v \in \mathcal{H}$ we have*

$$\|(x_\alpha - x) y v\| \to 0, \quad \|y(x_\alpha - x)v\| \to 0,$$

*so $x_\alpha y \to xy$ and $y x_\alpha \to yx$ in SOT.*

**Example 4.32.** *The adjoint is isometric in the norm topology. It is also continuous in WOT:*

$$|\langle x^* v - y^* v, w\rangle| < \varepsilon \Leftrightarrow |\langle xw - yw, v\rangle| < \varepsilon.$$

*However, it is not continuous with respect to SOT. If $(e_n)_n$ is an ONB for $\mathcal{H}$, consider $e_1 \otimes \overline{e_n}$. Then for every $x \in \mathcal{H}$, we have*

$$\|(e_1 \otimes \overline{e_n})x\| = |\langle x, e_n\rangle| \xrightarrow[n\to\infty]{} 0,$$

*so $e_1 \otimes \overline{e_n} \to 0$ in SOT. However,*

$$\|(e_1 \otimes \overline{e_n})^* x\| = \|(e_n \otimes \overline{e_1})x\| = |\langle x, e_1\rangle|$$

*does not go to 0 for all $x \in \mathcal{H}$, which proves our statement.*

*Remark.* If $T : X \to Y$ is continuous, then $T$ remains continuous if $X$ is given a finer topology or $Y$ is given a coarser topology. But if both topologies are made coarser or both finer, nothing can be said in general. In particular, if $T : X \to X$ is continuous with respect to a given topology on $X$ in both domain and codomain, you cannot generally conclude anything about continuity of $T$ when $X$ is given a finer or coarser topology on both domain and codomain. The previous example illustrates this.

**Lemma 4.33.** *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be linear. The following are equivalent.*
*(1.) There exist $v_1, \dots, v_n \in \mathcal{H}$ and $w_1, \dots, w_n \in \mathcal{H}$ such that*

$$\varphi(T) = \sum_{i=1}^{n} \langle T v_i, w_i\rangle.$$

*(2.) $\varphi$ is WOT-continuous.*
*(3.) $\varphi$ is SOT-continuous.*

*Proof.* The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. Let us prove (3) $\Rightarrow$ (1). By proposition 1.30, there exists a $K > 0$ and $v_1, \dots, v_n \in \mathcal{H}$ such that

$$|\varphi(T)|^2 \leq K \cdot \sum_{i=1}^{n} \|T v_i\|^2.$$

Define

$$\mathcal{H}_0 := \overline{\left\{ \bigoplus_{i=1}^{n} T v_i \mid T \in \mathcal{B}(\mathcal{H}) \right\}} \leq \mathcal{H}^n.$$

The map

$$\mathcal{H}_0 \ni \bigoplus_{i=1}^{n} T v_i \mapsto \varphi(T) \in \mathbb{C}$$

is a well-defined and bounded linear functional, which by continuity extends to $\mathcal{H}_0 \to \mathbb{C}$. By Riesz, there exist $w_1, \dots, w_n \in \mathcal{H}$ such that

$$\varphi(T) = \sum_{i=1}^{n} \langle T v_i, w_i\rangle.$$

Recall that $v \otimes \overline{w} \in \mathcal{F}(\mathcal{H})$ and $\mathrm{Tr}(v \otimes \overline{w}) = \langle v, w \rangle$, so

$$\mathrm{Tr}(T(v \otimes \overline{w})) = \langle Tv, w \rangle.$$

The previous identity is really

$$\varphi(T) = \sum_{i=1}^{n} \mathrm{Tr}(T(v \otimes \overline{w})) = \mathrm{Tr}(T \cdot \sum_{i=1}^{n} v_i \otimes \overline{w}_i).$$

This means that $\varphi(T) = \mathrm{Tr}(T \cdot A)$ for $A \in \mathcal{F}(\mathcal{H})$. $\square$

**Corollary 4.34.** *If $K \subseteq \mathcal{B}(\mathcal{H})$ is convex, then*

$$\overline{K}^{WOT} = \overline{K}^{SOT}.$$

*Proof.* Consider $\mathcal{B}(\mathcal{H})$, equipped with WOT topology. This is a LCS, so $\overline{K}^{w,\mathrm{WOT}} = \overline{K}^{\mathrm{WOT}}$ by theorem 1.28. Similarly, we have that $\mathcal{B}(\mathcal{H})$ is a LCS when equipped with SOT topology, so $\overline{K}^{w,\mathrm{SOT}} = \overline{K}^{\mathrm{SOT}}$. Now

$$\begin{aligned} x \in \overline{K}^{w,\mathrm{WOT}} &\Leftrightarrow \exists \text{ a net } (x_\alpha)_\alpha \subseteq K, \text{ such that } x_\alpha \to x \text{ WOT-weakly} \\ &\Leftrightarrow f(x_\alpha) \to f(x) \text{ for all WOT-continuous functionals } f : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \\ &\Leftrightarrow f(x_\alpha) \to f(x) \text{ for all SOT-continuous functionals } f : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \\ &\Leftrightarrow \exists \text{ a net } (x_\alpha)_\alpha \subseteq K, \text{ such that } x_\alpha \to x \text{ SOT-weakly} \\ &\Leftrightarrow x \in \overline{K}^{w,\mathrm{SOT}}. \end{aligned}$$

Therefore, $\overline{K}^{w,\mathrm{WOT}} = \overline{K}^{w,\mathrm{SOT}}$ and we are done. $\square$

**Definition 4.35.** The *$\sigma$-weak operator topology* ($\sigma$-WOT or ultra-weak) is the topology in $\mathcal{B}(\mathcal{H})$ given by the seminorms

$$x \mapsto \left| \sum_{i=1}^{\infty} \langle x\alpha_i, \alpha_i \rangle \right|$$

for $\alpha_i \in \mathcal{H}$ with $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$. A subbasis of open sets is thus

$$\left\{ x \in \mathcal{B}(\mathcal{H}) \;\middle|\; \left| \sum_{i=1}^{\infty} \langle (x - x_0)\alpha_i, \alpha_i \rangle \right| < \varepsilon \right\}$$

for $\alpha_i \in \mathcal{H}$ with $\varepsilon > 0$, $x_0 \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$.

**Definition 4.36.** The *$\sigma$-strong operator topology* ($\sigma$-SOT or ultra-strong) is the topology in $\mathcal{B}(\mathcal{H})$ given by the seminorms

$$x \mapsto \left( \sum_{i=1}^{\infty} \|x\alpha_i\|^2 \right)^{\frac{1}{2}}$$

for $\alpha_i \in \mathcal{H}$ with $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$. A subbasis of open sets is thus

$$\left\{ x \in \mathcal{B}(\mathcal{H}) \;\middle|\; \left( \sum_{i=1}^{\infty} \|(x - x_0)\alpha_i\|^2 \right)^{\frac{1}{2}} < \varepsilon \right\}$$

for $\alpha_i \in \mathcal{H}$ with $\varepsilon > 0$, $x_0 \in \mathcal{B}(\mathcal{H})$ and $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$.

*Remark.* $\sigma$-WOT can also be given by seminorms

$$x \mapsto |\operatorname{Tr}(xa)|$$

for $a \in L^1(\mathcal{B}(\mathcal{H}))$ positive. Let $(f_i)_i$ be an ONB for $\mathcal{H}$ and define

$$b : \mathcal{H} \to \mathcal{H}, \quad f_i \mapsto \alpha_i.$$

Since $\sum_i^{\infty} \|\alpha_i\|^2 < \infty$, we can conclude $b \in L^2(\mathcal{B}(\mathcal{H}))$. Then:

$$\begin{aligned} \sum_i \langle x\alpha_i, \alpha_i \rangle &= \sum_i \langle xbf_i, bf_i \rangle \\ &= \sum_i \langle b^* xbf_i, f_i \rangle \\ &= \operatorname{Tr}(b^* xb) \\ &= \operatorname{Tr}(xbb^*), \end{aligned}$$

where $a := bb^* \in L^1(\mathcal{B}(\mathcal{H}))$. Since $\mathcal{B}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))^*$, the $\sigma$-WOT is just the weak-$*$ topology (with respect to this pairing).

*Remark.* The map

$$\operatorname{id} \otimes 1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2), \quad x \mapsto x \otimes 1$$

is an isometric *-isomorphism of $C^*$-algebras. It is neither SOT- nor WOT-continuous. Despite that, $\sigma$-WOT on $\mathcal{B}(\mathcal{H})$ is induced by WOT on $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2)$ and the $\sigma$-SOT on $\mathcal{B}(\mathcal{H})$ is induced by SOT on $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2)$. Indeed, if $(e_i)_{i \in \mathbb{N}}$ is an ONB for $\ell^2$, define $\alpha := \sum_{i=1}^{\infty} \alpha_i \otimes e_i \in \mathcal{H} \overline{\otimes} \ell^2$. Then

$$\sum_{i \in \mathbb{N}} \langle x\alpha_i, \alpha_i \rangle_{\mathcal{H}} = \langle (\operatorname{id} \otimes 1)(x)\alpha, \alpha \rangle_{\mathcal{H} \overline{\otimes} \ell^2}$$

and similarly

$$\left( \sum_{i \in \mathbb{N}} \|x\alpha_i\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} = \|(\operatorname{id} \otimes 1)(x)\alpha\|_{\mathcal{H} \overline{\otimes} \ell^2}$$

**Lemma 4.37.** *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional operator. Then the following are equivalent.*
*(1.)* $\exists a \in L^1(\mathcal{B}(\mathcal{H}))$ *such that* $\varphi(x) = \mathrm{Tr}(ax), \ \forall x \in \mathcal{B}(\mathcal{H})$*;*
*(2.)* $\varphi$ *is* $\sigma$*-WOT continuous;*
*(3.)* $\varphi$ *is* $\sigma$*-SOT continuous.*

*Proof.* As previously, the implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious. Let us prove (3) $\Rightarrow$ (1). Assume $\varphi$ is $\sigma$-SOT continuous. By identifying $\mathcal{B}(\mathcal{H})$ via $\mathrm{id} \otimes 1$ with a subspace in $\mathcal{B}(\mathcal{H} \otimes \ell^2)$, $\varphi$ is SOT-continuous on this subspace. By Hahn–Banach, $\varphi$ extends to a SOT-continuous linear functional on $\mathcal{B}(\mathcal{H} \otimes \ell^2)$. By the previous lemma, $\exists \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathcal{H} \overline{\otimes} \ell^2$.

$$\varphi(x) = \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x) \alpha_i, \beta_i \rangle.$$

With

$$\alpha_i \sum_{j=1}^{\infty} \alpha_{ij} \otimes e_j, \quad \sum_j \|\alpha_{ij}\|^2 < \infty$$

and

$$\beta_i \sum_{j=1}^{\infty} \beta_{ij} \otimes e_j, \quad \sum_j \|\beta_{ij}\|^2 < \infty.$$

Then

$$\begin{aligned} \varphi(x) &= \sum_{i=1}^{n} \langle (x \otimes 1) \sum_{j=1}^{\infty} \alpha_{ij} \otimes e_j, \sum_{k=1}^{\infty} \beta_{ik} \otimes e_k \rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle x \alpha_{ij}, \beta_{ik} \rangle \langle e_j, e_k \rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{\infty} \langle x \alpha_{ij}, \beta_{ij} \rangle. \end{aligned}$$

Define

$$A_i : \mathcal{H} \to \mathcal{H}, \quad A_i f_k = \alpha_{ik}$$

and

$$B_i : \mathcal{H} \to \mathcal{H}, \quad B_i f_k = \beta_{ik}$$

for an orthonormal basis $(f_k)_{k \in \mathbb{N}}$. By assumption, $A_i, B_i \in L^2(\mathcal{B}(\mathcal{H}))$. As before, this gives $\varphi(x) = \sum_i \mathrm{Tr}(B_i^* x A_i) = \mathrm{Tr}(x A_i B_i^*)$. □

**Corollary 4.38.** *The unit disk* $(\mathcal{B}(\mathcal{H}))_1$ *is compact with respect to the* $\sigma$*-WOT topology.*

*Proof.* $\sigma$-WOT on $\mathcal{B}(\mathcal{H})$ is the weak-$*$ topology from $L^1(\mathcal{B}(\mathcal{H}))^* = \mathcal{B}(\mathcal{H})$. The statement now follows from Banach–Alaoglu. □

**Corollary 4.39.** *WOT and* $\sigma$*-WOT topologies agree on bounded subsets* $B \subseteq \mathcal{B}(\mathcal{H})$*.*

*Proof.* W.l.o.g. $B = M \cdot (\mathcal{B}(\mathcal{H}))_1$ for some $M > 0$. Then the identity $(B, \sigma\text{-WOT}) \to (B, \text{WOT})$ is a continuous map from a Hausdorff compact space (previous corollary) to a Hausdorff space. Therefore the identity map is a closed continuous bijection, so it's a homeomorphism. $\square$

**Definition 4.40.** Let $A$ be a vector space and $B \subseteq \mathcal{L}(A, \mathbb{C})$ a set of some of its linear functionals. Then we define $\sigma(A, B)$ as the weakest topology in $A$ such that functionals in $B$ are continuous.

*Remark.* $\sigma$-WOT topology is $\sigma(\mathcal{B}(\mathcal{H}), L^1(\mathcal{B}(\mathcal{H})))$.

*Remark.* Let us define the following topologies on $\mathcal{B}(\mathcal{H})$.

(1.) Weak Banach topology is $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^*)$.

(2.) Ultrastrong-$*$ topology is the weakest topology that is stronger than $\sigma$-SOT such that $*$ is continuous.

(3.) Strong-$*$ topology is generated by seminorms $x \mapsto \|x\alpha\|$ and $x \mapsto \|x^*\alpha\|$ for $\alpha \in \mathcal{H}$.

In the end, we get the following diagram that demonstrates which topologies are comparable.

$$\begin{tikzcd}
 & \text{norm} \arrow[dl] \arrow[dr] & \\
\text{weak Banach} \arrow[dd] & & \text{ultrastrong-}* \arrow[dl] \arrow[d] \\
 & \sigma\text{-SOT} \arrow[dl] \arrow[dr] & \text{strong-}* \arrow[d] \\
\sigma\text{-WOT} \arrow[dr] & & \text{SOT} \arrow[dl] \\
 & \text{WOT} &
\end{tikzcd}$$

# 5 von Neumann algebras

## 5.1 Bicommutant theorem

**Definition 5.1.** A *von Neumann algebra* (on Hilbert space $\mathcal{H}$) is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is WOT-closed. Equivalently, it is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is SOT-closed.

*Remark.* To shorten the notation, we will abbreviate "von Neumann algebra" to vNa.

If $A \subseteq \mathcal{B}(\mathcal{H})$, then $W^*(A)$ denotes the vNa generated by $A$, or the smallest vNa in $\mathcal{B}(\mathcal{H})$ that contains $A$. This is well defined, since

$$W^*(A) = \bigcap \{W \mid A \subseteq W,\ W \subseteq \mathcal{B}(\mathcal{H}) \text{ is vNa}\}.$$

**Lemma 5.2.** *If $A \subseteq \mathcal{B}(\mathcal{H})$ is a vNa, then $(A)_1$ is WOT-compact.*

*Proof.* By corollary 4.38, $(\mathcal{B}(\mathcal{H}))_1$ is compact in $\sigma$-WOT topology. By corollary 4.39, the WOT and $\sigma$-WOT topologies on $(\mathcal{B}(\mathcal{H}))_1$ are equivalent, so $(\mathcal{B}(\mathcal{H}))_1$ is also compact in WOT topology. Next, we prove that $(A)_1$ is WOT-closed in $(\mathcal{B}(\mathcal{H}))_1$. Suppose that the net $(x_i)_i$ in $(A_1)$ converges to some $x$. Since $A$ is WOT-closed, we must have $x \in A$. Assume that $x \notin (A)_1$, so $\|x\| > 1$. Since $\|x\| = \sup_{\alpha,\beta \in (\mathcal{H})_1} |\langle x\alpha, \beta\rangle|$, there must exist some $\alpha, \beta \in (\mathcal{H})_1$ such that $|\langle x\alpha, \beta\rangle| > 1$. However, for every $x_i$ we have $|\langle x_i\alpha, \beta\rangle| \leq \|x_i\| \cdot |\alpha| \cdot |\beta| \leq 1$, contradicting the fact that $\langle x_i\alpha, \beta\rangle \to \langle x\alpha, \beta\rangle$. Therefore, $x \in (A)_1$ and $(A_1)$ is WOT-closed in $(\mathcal{B}(\mathcal{H}))_1$, so it is compact. $\square$

**Corollary 5.3.** *Let $A \subseteq \mathcal{B}(\mathcal{H})$ vNa. Then $(A)_1$ and $A_{\text{sa}}$ are SOT-closed and WOT-closed.*

*Proof.* We already know that the adjoint is continuous in WOT, so $A_{\text{sa}}$ is closed in WOT. Since $A_{\text{sa}}$ is convex, it is also SOT-closed. The same exact argument applies for $(A)_1$. $\square$

**Definition 5.4.** The *commutant* of a set $B \subseteq \mathcal{B}(\mathcal{H})$ is

$$B' := \{T \in \mathcal{B}(\mathcal{H}) \mid \forall S \in B:\ ST = TS\}$$

and its *bicommutant* is $B'' := (B')'$.

*Remark.* By definition, $B'' \supseteq B$.

**Theorem 5.5.**

*Suppose $A \subseteq \mathcal{B}(\mathcal{H})$ is closed under $*$. Then $A'$ is vNa.*

*Proof.* Obviously, $A'$ is also a subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed under $*$. We prove that it is WOT-closed. Let $(x_\alpha)_\alpha$ be a net in $A'$ that WOT-converges to $x \in \mathcal{B}(\mathcal{H})$. Pick any $a \in A$

and $\varphi, \mu \in \mathcal{H}$. Then

$$\begin{aligned}
\langle [x,a]\varphi, \mu\rangle &= \langle (xa - ax)\varphi, \mu\rangle \\
&= \langle xa\varphi, \mu\rangle - \langle ax\varphi, \mu\rangle \\
&= \langle xa\varphi, \mu\rangle - \langle x\varphi, a^*\mu\rangle \\
&= \lim_\alpha \langle x_\alpha a\varphi, \mu\rangle - \langle x_\alpha\varphi, a^*\mu\rangle \\
&= \lim_\alpha \langle (x_\alpha a - ax_\alpha)\varphi, \mu\rangle \\
&= \lim_\alpha \langle [x_\alpha, a]\varphi, \mu\rangle = 0,
\end{aligned}$$

so $x \in A'$ and we are done. □

**Corollary 5.6.** *Every vNa is unital.*

**Example 5.7.** *For an infinitely-dimensional Hilbert space $\mathcal{H}$, the set of all compact operators $\mathcal{K}(\mathcal{H})$ is not a vNa, since it doesn't include the identity (by the Riesz lemma). In particular, $\mathcal{K}(\mathcal{H})$ is neither SOT- nor WOT-closed.*

*Remark.* As we will see later, the finite-rank projections on a Hilbert space converge strongly to identity.

**Corollary 5.8.** *Suppose that $A \subseteq \mathcal{B}(\mathcal{H})$ is a maximal commutative subalgebra. If $A$ is closed under $*$, then it is a vNa.*

*Proof.* Since $A$ is commutative, $A' \supseteq A$. Take $b \in A' \subseteq A$ and consider the subalgebra, generated by $A$ and $b$. This is an abelian algebra, so by maximality we have we have $b \in A$ and $A = A'$. Then by theorem 5.5, $A$ is a vNa. □

**Lemma 5.9.** *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra. For any $\mu \in \mathcal{H}$ and $x \in A''$ there exists a net $(x_\alpha)_\alpha$ in $A$ such that $\lim_\alpha \|(x_\alpha - x)\mu\| = 0$.*

*Proof.* Define $\mathcal{K} := \overline{A\mu} \leq \mathcal{H}$. Let $p : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection onto $\mathcal{K}$. By definition, $a\mathcal{K} \subseteq \mathcal{K}$ for any $a \in A$. Equivalently, $pap = ap$ for any $a \in A$. Then

$$pa = (a^*p)^* = (pa^*p)^* = pap = ap,$$

so $p \in A'$. But $x \in A''$, so

$$xp = xp^2 = pxp$$

and $x\mathcal{K} \subseteq \mathcal{K}$. In particular, since $\mu \in \mathcal{K}$, we have $x\mu \in \mathcal{K} = \overline{A\mu}$. So there must exist some net in $A\mu$ that converges to $x\mu$. □

**Theorem 5.10** (von Neumann's bicommutant theorem)**.**

*Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra. Then $\overline{A}^{\mathrm{WOT}} = A''$.*

*Proof.* By the previous theorem, $A''$ is a vNa. In particular, it is WOT-closed. Since $A \subseteq A''$, it suffices to show that $A$ is WOT-dense in $A''$. Because $A$ is convex, it is enough to show that $A$ is SOT-dense in $A''$. Let $x \in A''$ and $\mu_1, \ldots, \mu_n \in \mathcal{H}$. Consider the matrix $*$-algebra $M_n(\mathcal{B}(\mathcal{H}))$ with the usual matrix involution. There exists a canonical $*$-isomorphism $M_n(\mathcal{B}(\mathcal{H})) \to \mathcal{B}(\mathcal{H}^n)$, which allows us to introduce a (unique) norm on $M_n(\mathcal{B}(\mathcal{H}))$, making it a $C^*$-algebra. Define

$$\widetilde{A} = \left\{ \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \in M_n(\mathcal{B}(\mathcal{H})) \mid a \in A \right\}.$$

Then $\widetilde{A}' = M_n(A')$. Hence we get

$$\widetilde{A''} \subseteq M_n(A')' = \widetilde{A}''.$$

This implies that

$$\begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} \in \widetilde{A''} \subseteq \widetilde{A}''.$$

Now we apply lemma 5.9 to $\widetilde{A}$ to get a net $(a_i)_i$ in $A$ such that

$$\lim_i \|(x - a_i)\mu_j\| = 0, \quad \forall j = 1, \ldots, n.$$

Finally, we have to show that this implies that $x$ is in the SOT-closure of $A$. Let $U$ be an open neighborhood around $x$. Then $U$ must contain some finite intersection of subbasis sets that generate the SOT topology. This means that there exists $\varepsilon > 0$ and $\mu_1, \ldots, \mu_n \in \mathcal{H}$ such that

$$\bigcap_{j=1}^{n} \{y \in \mathcal{B}(\mathcal{H}) \mid \|(x - y)\mu_j\| < \varepsilon\} \subseteq U.$$

Now we can conclude that $U \cap A \neq \emptyset$ and $x$ is in the SOT-closure of $A$. $\square$

**Corollary 5.11.** *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra. Then $A$ is a vNa iff $A = A''$.*

*Remark.* WOT-closed implies norm-closed. In particular, every vNa is a $C^*$-algebra. However, the converse is not always true: $\mathcal{C}([0,1])$ is a $C^*$-algebra that is not vNa. As we will see, this is because it does not contain nontrivial projections.

**Corollary 5.12** (Polar decomposition in vNa)**.** *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $x \in A$. Suppose that $x = v|x|$ is the polar decomposition of $x$ in $\mathcal{B}(\mathcal{H})$. Then $v \in A$.*

*Proof.* We know that

$$\ker v = (\operatorname{im}|x|)^{\perp} = \ker|x| = \ker x.$$

For $a \in A'$ and $\mu \in \ker x$ we have $a\mu \in \ker x$:

$$x(a\mu) = ax\mu = 0,$$

which implies $a \ker|x| \subseteq \ker|x|$. We know that $\mathcal{H} = \ker|x| \oplus \overline{\operatorname{im}|x|}$. Suppose that $|x|\mu \in \operatorname{im}|x|$. Then

$$\begin{aligned}[a,v]|x|\mu &= (av - va)|x|\mu = av|x|\mu - va|x|\mu \\ &= ax\mu - v|x|a\mu = ax\mu - xa\mu \\ &= [a,x]\mu = 0.\end{aligned}$$

But for $\beta \in \ker|x| = \ker v$, we have

$$[a,v]\beta = (av - va)\beta = av\beta - va\beta = 0.$$

Since $av$ and $va$ agree on $\ker|x| \oplus \overline{\operatorname{im}|x|} = \mathcal{H}$, we have $v \in A'' = A$. $\square$

The next example is of fundamental importance.

**Example 5.13** (Commutative vNa)**.** *Let $(X,\mu)$ be a $\sigma$-finite measure space and*

$$M : L^\infty(X,\mu) \to \mathcal{B}(L^2(X,\mu)), \quad g \mapsto M_g,$$

*where we define*

$$(M_g f)(x) = g(x)f(x).$$

*Then $M$ is an isometric $*$-isomorphism onto its image and $M(L^\infty(X,\mu))$ is a maximal commutative vNa in $\mathcal{B}(L^2(X,\mu))$.*

*Proof of the example.* Clearly, $M$ is injective, additive and multiplicative. First, we prove that $M$ is a $*$-homomorphism. This follows from the next calculation:

$$\begin{aligned}\langle M_{\overline{g}}\mu, \varphi\rangle &= \int_X M_{\overline{g}}\mu \cdot \overline{\varphi}\, d\mu \\ &= \int_X \overline{g}\mu\overline{\varphi} \\ &= \int_X \mu\overline{g\varphi}\, d\mu \\ &= \langle \mu, M_g\varphi\rangle = \langle M_g^*\mu, \varphi\rangle,\end{aligned}$$

so $M_{\overline{g}} = M_g^*$. Next, we prove that $M$ is isometric. For $g \in L^\infty(X,\mu)$, there exists a sequence $E_n \subseteq X$ such that $0 < \mu(E_n) < \infty$ and $|g|\big|_{E_n} \geq \|g\|_\infty - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\|M_g\| \geq \frac{\|M_g 1_{E_n}\|_2}{\|1_{E_n}\|_2} \geq \|g\|_\infty - \frac{1}{n}, \quad \forall n \in \mathbb{N},$$

which implies $\|M_g\| \geq \|g\|_\infty$. For the reverse, notice that

$$\begin{aligned}
\|M_g 1_{E_n}\|^2 &= \int_X |g \cdot 1_{E_n}|^2 \, d\mu \\
&= \int_{E_n} |g|^2 \, d\mu \\
&\geq \int_{E_n} (\|g\|_\infty - \frac{1}{n})^2 \, d\mu \\
&= (\|g\|_\infty - \frac{1}{n})^2 \cdot \mu(E_n)
\end{aligned}$$

and

$$\begin{aligned}
\|M_g\|^2 = \sup_{\|\mu\|_2 = 1} \|M_g \mu\|_2^2 &= \sup_{\|\mu\|_2=1} \int_X |g\mu|^2 \, d\mu \\
&\leq \|g\|_\infty^2 \cdot \sup_{\|\mu\|_2=1} \int_X |\mu|^2 \, d\mu = \|g\|_\infty^2.
\end{aligned}$$

We've just shown that $\|Mg\| = \|g\|_\infty$. Lastly, we prove that $M(L^\infty(X,\mu))$ is a maximal commutative subalgebra of $\mathcal{B}(L^2(X,\mu))$. Take $T \in \mathcal{B}(L^2(X,\mu))$ and assume it commutes with all $M_g$'s. Now pick a measurable sequence $E_n \subseteq X$ such that $0 < \mu(E_n) < \infty$, $E_n \subseteq E_{n+1}$ and $X = \bigcup_{n\in\mathbb{N}} E_n$. Define $f_n := T(1_{E_n}) \in (X,\mu)$. First we prove that $f_n \in L^\infty(X,\mu)$. If $A$ is measurable and $0 < \mu(A) < \infty$, then

$$\begin{aligned}
\frac{1}{\mu(A)} \int_X |f_n \cdot 1_A|^2 \, d\mu &= \frac{1}{\mu(A)} \cdot \|M_{1_A} T(1_{E_n})\|^2 \\
&= \frac{1}{\mu(A)} \cdot \|T(1_{A\cap E_n})\|^2 \\
&\leq \frac{1}{\mu(A)} \cdot \|T\|^2 \cdot \|1_A\|^2 = \|T\|^2.
\end{aligned}$$

If $f \notin L^\infty(X,\mu)$, then for all $M \in \mathbb{R}$ we have

$$0 < \mu(\underbrace{\{x \in X \mid |f_n(x)| > M\}}_{A_{n,M}}) < \infty,$$

since $f_n \in L^2(X,\mu)$. By above calculation,

$$M^2 \leq \frac{1}{\mu(A_{n,M})} \cdot \int_X |f \cdot 1_{A_{n,M}}|^2 \, d\mu \leq \|T\|^2,$$

which is of course a contradiction. This proves that $f_n \in L^\infty(X,\mu)$ and $\|f_n\|_\infty \leq \|T\|$. For $n \leq m$ we have

$$\begin{aligned}
1_{E_n} \cdot f_m &= 1_{E_n} \cdot T(1_{E_m}) \\
&= M_{1_{E_n}}(T(1_{E_m})) \\
&= T(M_{1_{E_n}} 1_{E_m}) \\
&= T(1_{E_n} 1_{E_m}) = f_n.
\end{aligned}$$

Therefore, $f_m\big|_{E_n} = f_n$. The sequence $(f_n)_n$ converges to a measurable $f : X \to \mathbb{C}$. From $\|f_n\|_\infty \leq \|T\|$ for all $n \in \mathbb{N}$ we also deduce $\|f\|_\infty \leq T$, so $f \in L^\infty(X,\mu)$. Lastly, we prove $T = M_f$. Note that simple functions $\sum_{j=i}^r \alpha_j 1_{A_j}$ are $L^2(X,\mu)$-dense. Let $A \subseteq X$ be measurable with $\mu(A) < \infty$. Then $\|1_{A\cap E_n} - 1_A\|_2 \xrightarrow{n\to\infty} 0$. Hence

$$\|(T - M_f)1_A\|_2 = \lim_{n\to\infty} \|(T - M_f)1_{A\cap E_n}\|_2 = 0,$$

as we shall prove.

$$\begin{aligned} T(1_{A\cap E_n}) &= T(1_A \cdot 1_{E_n}) = T(M_{1_A} 1_{E_n}) \\ &= M_{1_A}(T(1_{E_n})) = M_{1_A}(f_n) \\ &= 1_A \cdot f_n. \end{aligned}$$

On the other hand,

$$M_f(1_{A\cap E_n}) = f \cdot 1_{A\cap E_n} = f \cdot 1_{E_n} \cdot 1_A = 1_A \cdot f_n$$

and we are done. □

Another possible characterization of vNa's is given by the following.

**Theorem 5.14** (Sakai)**.**

*Let $A$ be a $C^*$-algebra such that for a Banach space $E$ there exists an isometric isomorphism $A \to E^*$. Then there exists a vNa $B \subseteq \mathcal{B}(\mathcal{H})$ such that $A \cong B$ as a $C^*$-algebra.*

For the proof, see the expository article [3].

## 5.2 Kaplansky's density theorem

**Lemma 5.15.** *The multiplication $(A, B) \mapsto A \cdot B$ is SOT-continuous on bounded sets.*

*Proof.* Let $(A_i)_i$ and $(B_i)_i$ be nets with $\sup\|A_i\|, \sup\|B_i\| < M$ for some $M \in \mathbb{R}$. Suppose $A_i \to A$ and $B_i \to B$ in SOT. For any $x$, we get

$$\begin{aligned} \|ABx - A_iB_ix\| &= \|ABx - A_iBx + A_iBx - A_iB_ix\| \\ &\leq \|ABx - A_iBx\| + \|A_iBx - A_iB_ix\| \\ &\leq \|A(Bx) - A_i(Bx)\| + \|A_i\| \cdot \|Bx - B_ix\| \\ &\leq \|A(Bx) - A_i(Bx)\| + M \cdot \|Bx - B_ix\| \to 0, \end{aligned}$$

so $A_iB_i \xrightarrow{\text{SOT}} AB$. □

**Proposition 5.16.** *Let $f \in C(\mathbb{C})$. Then $x \mapsto f(x)$ is SOT-continuous on each bounded set of normal operators in $\mathcal{B}(\mathcal{H})$.*

*Proof.* By Stone–Weierstrass, we can uniformly approximate $f$ by polynomials on a bounded subset $B_R(0) \subseteq \mathbb{C}$. By the previous lemma, multiplication is SOT-continuous on this bounded set of normal operators. But for a normal operator $A$, we have $\|Ax\| = \|A^*x\|$ for every $x \in \mathcal{H}$, so $*$ is also SOT-continuous on normal operators and we're done. $\square$

**Theorem 5.17** (Cayley transform)**.**

*Mapping* $x \mapsto (x-i)(x+i)^{-1}$ *is SOT-continuous* $\mathcal{B}(\mathcal{H})_{\text{sa}} \to \mathcal{U}(\mathcal{H})$.

*Proof.* If $x \in \mathcal{B}(\mathcal{H})_{\text{sa}}$, then $\sigma(x) \subseteq \mathbb{R}$ and $(x+i) \in \mathcal{B}(\mathcal{H})$ is invertible. We notice that $z \mapsto \frac{z-i}{z+i} : \mathbb{R} \to \mathbb{C}$ has its range in $\mathbb{T}$, so the Cayley transform does in fact map into the unitaries. Now onto the SOT-continuity: let $(x_k)_k$ be a net in $\mathcal{B}(\mathcal{H})_{\text{sa}}$ with $x_k \to x$ in SOT. By the spectral mapping theorem, $\|(x_k+i)^{-1}\| \leq 1$. For each $\alpha \in \mathcal{H}$, we have

$$
\begin{aligned}
\|(x-i)(x+i)^{-1}\alpha - (x_k-i)(x_k+i)^{-1}\alpha\| &= \|(x_k+i)^{-1}\left((x_k+i)(x-i)(x+i^{-1}) - (x_k-i)\right)\alpha\| \\
&= \|(x_k+i)^{-1}\left((x_k+i)(x-i) - (x_k-i)(x+i)\right)(x+i)^{-1}\alpha\| \\
&= \|(x_k+i)^{-1}2i(x-x_k)(x+i)^{-1}\alpha\| \\
&\leq 2\|(x_k+i)^{-1}\|\|(x-x_k)\underbrace{(x+i)^{-1}\alpha}_{\beta}\| \\
&\leq 2\|(x-x_k)\beta\| \to 0.
\end{aligned}
$$
$\square$

**Corollary 5.18.** *If* $f \in C_0(\mathbb{R})$, *then* $x \mapsto f(x)$ *is SOT-continuous on* $\mathcal{B}(\mathcal{H})_{\text{sa}}$.

*Proof.* Consider the continuous function

$$
g(t) = \begin{cases} f\left(i\frac{1+t}{1-t}\right); & t \neq 1 \\ 0; & t = 1 \end{cases}
$$

which maps $\mathbb{T} \to \mathbb{C}$. By the previous proposition, $x \mapsto g(x)$ is SOT-continuous on unitaries. Letting $U(z) = \frac{z-i}{z+i}$, denote the Cayley transform, we have that $f = g \circ U$ is a composite of two SOT-continuous maps, which is a SOT-continuous map of itself. $\square$

**Theorem 5.19** (Kaplansky's density theorem)**.**

*Let* $A \subseteq \mathcal{B}(\mathcal{H})$ *be a* $*$*-subalgebra and* $B = \overline{A}^{\text{SOT}}$, *then*

*(1.)* $\overline{A_{\text{sa}}}^{\text{SOT}} = B_{\text{sa}}$;

*(2.)* $\overline{(A)_1}^{\text{SOT}} = (B)_1$.

*Proof.* W.l.o.g. $A$ is a $C^*$-algebra, so norm-closed.
(1.) First we prove that $\overline{A_{\text{sa}}}^{\text{SOT}} \subseteq B_{\text{sa}}$. Since $\overline{A_{\text{sa}}}^{\text{SOT}} = \overline{A_{\text{sa}}}^{\text{WOT}}$, take $x \in \overline{A_{\text{sa}}}^{\text{SOT}}$ and a net

$(x_k)_k \subseteq A_{\text{sa}}$ that converges to $x$. Since $*$ is WOT continuous, $(x_k^*)_k = (x_k)_k$ converge to $x^*$, so $x = x^*$. Now the converse inclusion: suppose the net $(x_k)_k$ SOT-converges to $x \in B_{\text{sa}}$. Then $\frac{x_k + x_k^*}{2} \to x$ in the WOT-topology, which implies

$$B_{\text{sa}} \subseteq \overline{A_{\text{sa}}}^{\text{WOT}} = \overline{A_{\text{sa}}}^{\text{SOT}}.$$

(2.) Suppose the net $(y_i)_i$ in $A_{\text{sa}}$ SOT-converges to $x \in B_{\text{sa}}$. Take $f \in C_0(\mathbb{R})$ such that we have $f(t) = t$, $\forall |t| \leq \|x\|$ and $|f(t)| \leq \|x\|$, $\forall t \in \mathbb{R}$. By functional calculus, $\|f(y_k)\| \leq \|x\|$. By the previous corollary, $(f(y_i))_i \xrightarrow{\text{SOT}} f(x) = x$. This proves that $(A)_1 \cap A_{\text{sa}}$ is SOT-dense in $(B)_1 \cap B_{\text{sa}}$. Pass over to $M_2(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then $M_2(A)$ is SOT-dense in $M_2(B)$ by the first part of the proof. For $x \in (B)_1$, we have

$$\widetilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (M_2(B))_1 \cap (M_2(B))_{\text{sa}}.$$

That means there exists a net

$$\widetilde{x_i} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in (M_2(A))_1$$

such that $\widetilde{x_i} \to \widetilde{x}$ and therefore $b_i \in (A)_1$ SOT-converge to $x$. $\square$

**Corollary 5.20.** *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-algebra. Then $A$ is a vNa iff $(A)_1$ is SOT-closed.*

## 5.3 Examples of vNa's

**Definition 5.21.** A vNa $M$ is called a *factor* if $Z(M) = M \cap M' = \mathbb{C} \cdot 1$.

**Example 5.22.** *Clearly, $\mathcal{B}(\mathcal{H})$ is a factor. In particular, $M_n(\mathbb{C})$ is a factor.*

Let $\Gamma$ be a group and $\mathcal{H} = \ell^2(\Gamma)$. Consider the left regular representation

$$\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma)), \quad g \mapsto (\delta_h \mapsto \delta_{gh})$$

and extend it linearly to $\lambda : \mathbb{C}[\Gamma] \to \mathcal{B}(\ell^2(\Gamma))$. The group vNa of $\Gamma$ is $VN(\Gamma) := \lambda(\mathbb{C}[\Gamma])''$ in $\mathcal{B}(\ell^2(\Gamma))$. It has a *trace*, which is defined as the linear functional

$$\tau : VN(\Gamma) \to \mathbb{C}, \quad x \mapsto \langle x\delta_e, \delta_e \rangle.$$

For $g \in \Gamma$, $\tau(\lambda(g)) = 1$ if $g = e$, otherwise zero. For $g_1, \ldots, g_r \in \Gamma$, we have

$$g_1 \ldots g_r = e \Leftrightarrow \tau(\lambda(g_1) \ldots \lambda(g_r)) = 1.$$

Since $\tau$ is a positive linear functional and $\tau(1) = 1$, $\tau$ is a state. For any two elements $g, h \in \Gamma$ we have $gh = e \Leftrightarrow hg = e$, which together with the above line implies

$$\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g)).$$

By linearity, $\tau$ has the same cyclic property on $\lambda(\mathbb{C}[\Gamma])$. But since $\tau$ is, by definition, WOT-continuous and $VN(\Gamma) = (\lambda(\mathbb{C}[\Gamma]))'' = \overline{\lambda(\mathbb{C}[\Gamma])}^{\text{WOT}}$, $\tau$ is cyclic on the entire $VN(\Gamma)$. Now if

$|\Gamma| = \infty$, then $VN(\Gamma) \neq \mathcal{B}(\mathcal{H})$, since the latter does not have a trace if $\dim \mathcal{H} = \infty$. If $\Gamma$ is an abelian group, then $VN(\Gamma)$ is commutative.

**Definition 5.23.** Group $\Gamma$ has *icc* (infinite conjugacy classes), if for all $g \in \Gamma \setminus \{e\}$, the set $\{f^{-1}gf \mid f \in \Gamma\}$ is infinite.

**Example 5.24.** *The group*

$$S_\infty = \{\textit{bijections } \mathbb{N} \to \mathbb{N} \textit{ that only permute finitely many elements}\}$$

*has icc.*

**Example 5.25.** *Free groups $\mathbb{F}_n$ for $n > 1$ have icc.*

**Theorem 5.26.**

*If $\Gamma$ has icc, then $VN(\Gamma)$ is a factor.*

**Definition 5.27.** $VN(S_\infty) =: R$ is the *hyperfinite $II_1$-factor.*

Open problem: does $VN(\mathbb{F}_2) \cong VN(\mathbb{F}_3)$ hold?

## 5.4 Operations with vNa's

### 5.4.1 Direct sums

Let $M_i \subseteq \mathcal{B}(\mathcal{H}_i)$ be vNa's. Define the isometric embedding

$$\iota_j : \mathcal{B}(\mathcal{H}_j) \to \mathcal{B}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n), \quad x \mapsto ((\alpha_1, \ldots, \alpha_n) \mapsto (0, \ldots, 0, x\alpha_j, 0, \ldots, 0)).$$

This map is the $n \times n$ bounded matrix where the $(j, j)$-th element is $x$ and the rest are zero. Then

$$M_1 \oplus \cdots \oplus M_n := \operatorname{span}\{\iota_j(x) \mid j = 1, \ldots, n, \; x \in M_j\}$$

is the direct sum of vNa's. If $n \geq 2$, then from

$$Z(M_1 \oplus \cdots \oplus M_n) = Z(M_1) \oplus \cdots \oplus Z(M_n),$$

we deduce that $M_1 \oplus \cdots \oplus M_n$ is not a factor.

### 5.4.2 Tensor products

The algebraic tensor product $\mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n)$ acts on $\mathcal{H}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{H}_n$ by

$$(x_1 \otimes \cdots \otimes x_n)(\alpha_1 \otimes \cdots \otimes \alpha_n) = (x_1\alpha_1) \otimes \cdots \otimes (x_n\alpha_n)$$

for $x_j \in \mathcal{B}(\mathcal{H}_j)$ and $\alpha_j \in \mathcal{H}_j$, which implies

$$\mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n) \subseteq \mathcal{B}(\mathcal{H}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{H}_n).$$

Finally, we define the tensor product of vNa's as

$$M_1 \overline{\otimes} \cdots \overline{\otimes} M_n = (M_1 \otimes \cdots \otimes M_n)'' \cap \mathcal{B}(\mathcal{H}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{H}_n).$$

### 5.4.3 Compressions

**Definition 5.28.** Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $p \in \mathcal{B}(\mathcal{H})$ a projection. A compression of $M$ is $pMp = \{pxp \mid x \in M\}$. When $p \in M$, it is also called a corner.

If $\mathcal{H} = \operatorname{im} p \oplus (\operatorname{im} p)^\perp = \operatorname{im} p \oplus \operatorname{im}(1-p)$. In this basis, elements of $pMp$ have the matrix form

$$\begin{bmatrix} pxp & 0 \\ 0 & 0 \end{bmatrix}.$$

If $M \ni p \neq 1$, then $pMp$ is a $*$-algebra and $pMp \subseteq M$ but it is not a subalgebra since $1_M = 1_{\mathcal{B}(\mathcal{H})} \notin pMp$. However, $pMp$ is a subalgebra of $\mathcal{B}(p\mathcal{H})$ with identity element $p$.

**Definition 5.29.** Let $\mathcal{K} \subseteq \mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$.
(1.) $\mathcal{K}$ is invariant under $x$ if $x\mathcal{K} \subseteq \mathcal{K}$;
(2.) $\mathcal{K}$ is reducing under $x$ if $\mathcal{K}$ is invariant under both $x$ and $x^*$.
Now if $S \subseteq \mathcal{B}(\mathcal{H})$, then
(1.) $\mathcal{K}$ is invariant under $S$ if $x\mathcal{K} \subseteq \mathcal{K}$ under all $x \in S$;
(2.) $\mathcal{K}$ is reducing under $S$ if $\mathcal{K}$ is reducing under all $x \in S$.

If $S \subseteq \mathcal{B}(\mathcal{H})$ is closed under $*$, then $\mathcal{K}$ is invariant under $S$ iff it is reducing under $S$. The following lemma was proved in the introductory course.

**Lemma 5.30.** *Let $\mathcal{K}^{\text{closed}} \leq \mathcal{H}$ and $M \subseteq \mathcal{B}(\mathcal{H})$ an $*$-algebra. Let $p : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. Then $\mathcal{K}$ is reducing under $M$ iff $p \in M'$.*

**Theorem 5.31.**

*Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $p \in M$ a projection. Then $pMp$ and $M'p$ are vNa's in $\mathcal{B}(p\mathcal{H})$.*

*Proof.* We will show that

$$(M'p)' \cap \mathcal{B}(p\mathcal{H}) = pMp, \quad (pMp)' \cap \mathcal{B}(p\mathcal{H}) = M'p.$$

Then the bicommutant theorem will take care of the rest. It is obvious that $(M'p)' \cap \mathcal{B}(p\mathcal{H}) \supseteq pMp$. For the converse, pick $x \in (M'p)' \cap \mathcal{B}(p\mathcal{H})$. Define $\widetilde{x} = xp = px \in \mathcal{B}(\mathcal{H})$. For $y \in M'$, we have

$$y\widetilde{x} = ypx = xyp = xpy = \widetilde{x}y,$$

which implies $\widetilde{x} \in M'' = M$. Then $x = pxp = p\widetilde{x}p \in pMp$. As before, $(pMp)' \cap \mathcal{B}(p\mathcal{H}) \supseteq M'p$ is trivial and we just prove the converse. Take $y \in (pMp)' \cap \mathcal{B}(p\mathcal{H})$. Using continuous functional calculus, we can write $y$ as a linear combinations of 4 unitaries. Since $pMp$ is closed under $*$, $(pMp)'$ is a vNa (and therefore a $C^*$-algebra). So we can assume w.l.o.g. that $y = u$ a unitary. Set $\mathcal{K} := \overline{Mp\mathcal{H}}$ and let $q : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. Since $\mathcal{K}$ is reducing under $M$ and $M'$, which implies

$$q \in M' \cap M'' = M' \cap M = Z(M).$$

Next, we extend $u$ to $\mathcal{K}$:

$$\widetilde{u}(\sum_i \underbrace{x_i}_{\in M} p \underbrace{\alpha_i}_{\in \mathcal{H}}) = \sum_i x_i u p \alpha_i.$$

We shall show that this is a well-defined isometry in $Mp\mathcal{H}$:

$$\begin{aligned}
\|\widetilde{u}\sum_i x_i p\alpha_i\|^2 &= \sum_{i,j}\langle x_i u p\alpha_i, x_j u p\alpha_j\rangle \\
&= \sum_{i,j}\langle (p x_j^* x_i p) u\alpha_i, u\alpha_j\rangle \\
&= \sum_{i,j}\langle u p x_j^* x_i p\alpha_i, u\alpha_j\rangle \\
&= \sum_{i,j}\langle p x_j^* x_i p\alpha_i, \alpha_j\rangle = \|\sum_i x_i p\alpha_i\|^2.
\end{aligned}$$

So $\widetilde{u}$ extends to an isometry on $\mathcal{K} = \overline{Mp\mathcal{H}}$. By definition, $\widetilde{u}$ commutes with $M$ on $Mp\mathcal{H}$, so also on $\mathcal{K}$. Thus for every $x \in M$ and $\alpha \in \mathcal{H}$, we have

$$x(\widetilde{u}q)\alpha = \widetilde{u}xq\alpha = (\widetilde{u}q)x\alpha,$$

which implies $\widetilde{u}q \in M' \cap \mathcal{B}(\mathcal{H})$. Then

$$\widetilde{u}qp\alpha = \widetilde{u}1p\alpha = 1up\alpha,$$

which implies $u = \widetilde{u}qp \in \mathcal{B}(\mathcal{H})$ and $u \in M'p$. □

**Corollary 5.32.** *Suppose the vNa $M \subseteq \mathcal{B}(\mathcal{H})$ is a factor and let $p \in M$ be a projection. Then $pMp$ and $M'p$ are factors (in $\mathcal{B}(p\mathcal{H})$).*

*Proof.* Let $\mathcal{K} = \overline{Mp\mathcal{H}}$ and $q : \mathcal{H} \to \mathcal{K}$ the projection. From the previous proof, $q \in Z(M) = \mathbb{C}$. Then $q \in \{0, 1\}$. w.l.o.g. $p \neq 0$, so $q = 1$. Thus $\mathcal{K} = \mathcal{H}$, so $Mp\mathcal{H}$ is dense in $\mathcal{H}$. Consider

$$\psi : M' \to M'p, \quad y \mapsto yp.$$

We will prove that $\psi$ is an isomorphism of algebras. Obviously, it is additive. Since

$$\psi(xy) = xyp = xyp^2 = xpyp = \psi(x)\psi(y),$$

it is also multiplicative. Same calculation shows $\psi(y^*) = \psi(y)^*$. Obviously, $\psi$ is surjective. Finally, we prove injectivity. Suppose $y \in M'$ satisfies $yp = 0$. Then for every $x \in M$ and $\alpha \in \mathcal{H}$, we get $yxp\alpha = x(yp)\alpha = 0$. Hence $y\big|_{Mp\mathcal{H}} = 0$, so by continuity, $y\big|_{\overline{Mp\mathcal{H}}} = y\big|_{\mathcal{K}} = 0$. But because $\mathcal{K} = \mathcal{H}$, this yields $y\big|_{\mathcal{H}} = 0$. As a result, we get

$$Z(M'p) = Z(M')p = \mathbb{C} \cdot p,$$

so $M'p$ is a factor. Similarly,

$$Z(pMp) = (pMp) \cap (pMp)' = (M'p)' \cap M'p = Z(M'p) = \mathbb{C}p,$$

so $pMp$ is a factor. □

# 6 Spectral theorem and Borel functional calculus

## 6.1 Spectral theorem

Recall the spectral theorem for compact operators.

**Theorem 6.1** (Spectral theorem for compact operators)**.**

*If $T \in \mathcal{K}(\mathcal{H})$ is self-adjoint, then $T$ has only a countable number of distinct eigenvalues, where each nonzero eigenvalue has finite multiplicity. If $\{\lambda_1, \lambda_2, \dots\}$ are the distinct eigenvalues of $T$, and $P_n$ is the projection of $\mathcal{H}$ onto $\ker(T - \lambda_n)$, then $P_n P_m = 0$ for $n \neq m$ and*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n.$$

Our first goal is to generalize this result to non-compact self-adjoint operators.

**Theorem 6.2** (Vigier)**.**

*Let $(u_\lambda)$ be a net of increasing (decreasing) and bounded above (below) self-adjoint operators on $\mathcal{H}$. Then $(u_\lambda)$ converges.*

*Proof.* We prove the statement for an increasing net that is bounded above. We can assume $(u_\lambda)$ has a lower bound $m$ by considering a truncated net. Without loss of generality, we assume $u_\lambda$ is positive (otherwise, consider $u_\lambda - m$). There exists $M \geq 0$ such that $\|u_\lambda\| \leq M$ for all indices $\lambda$. So the net $\langle u_\lambda x, x\rangle$ is real, increasing, and bounded above by $M\|x\|^2$. Using the polarization identity,

$$\langle u_\lambda x, x\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle u_\lambda(x + i^k y), x + i^k y\rangle,$$

we see that $\langle u_\lambda x, y\rangle$ is a convergent net for all $x, y \in \mathcal{H}$. Letting $\sigma(x, y)$ denote its limit, we can easily check that $\sigma$ is a bounded sesquilinear form ($|\sigma(x, y)| \leq M\|x\|\|y\|$). By Riesz's representation theorem, there exists an operator $u \in \mathcal{B}(\mathcal{H})$ such that $\langle ux, y\rangle = \sigma(x, y)$. Then $u$ is self-adjoint, $\|u\| \leq M$, and $u_\lambda \leq u$. Note that

$$\begin{aligned}
\|(u - u_\lambda)x\|^2 &\leq \|(u - u_\lambda)^{\frac{1}{2}}(u - u_\lambda)^{\frac{1}{2}}x\|^2 \\
&\leq \|(u - u_\lambda)\|\|(u - u_\lambda)^{\frac{1}{2}}x\|^2 \\
&\leq 2M\langle (u - u_\lambda)x, x\rangle \to 0,
\end{aligned}$$

so $u_\lambda$ converges strongly to $u$. ∎

*Remark.* If $(p_\lambda)$ is a net of projections converging strongly to some operator $u$, then $u$ is also a projection. Clearly, $u$ is self-adjoint, and

$$\begin{aligned}
\langle ux, y\rangle &= \lim_\lambda \langle p_\lambda x, y\rangle = \lim_\lambda \langle p_\lambda x, p_\lambda y\rangle \\
&= \langle ux, uy\rangle = \langle u^2 x, y\rangle,
\end{aligned}$$

therefore, $u^2 = u$.

**Corollary 6.3.** *If $(p_n)_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal projections in $\mathcal{B}(\mathcal{H})$, then $\left(\sum_{n=1}^{N} p_n\right)$ SOT-converges as $N \to \infty$ (we denote the limit by $\sum_{n=1}^{\infty} p_n$).*

**Definition 6.4.** Let $X$ be a set, $\Omega$ a $\sigma$-algebra in $X$, and $\mathcal{H}$ a Hilbert space. A *projection-valued measure* (PVM) for $(X, \Omega, \mathcal{H})$ is a map $E : \Omega \to \mathcal{B}(\mathcal{H})$ such that

(1.) $E(S)$ is a projection for all $S \in \Omega$;
(2.) $E(\emptyset) = 0$ and $E(X) = 1$;
(3.) $E(S \cap T) = E(S)E(T)$ for all $S, T \in \Omega$;
(4.) If $(S_n)_{n\in\mathbb{N}} \subseteq \Omega$ is a sequence of pairwise disjoint sets, then

$$E\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} E(S_n).$$

*Remark.* The projections $E(S)$ commute with each other, which follows directly from the third point of the definition.

**Example 6.5.** *Let $(X, \Omega, \mu)$ be a $\sigma$-finite measure space. Let $\mathcal{H} = L^2(X, \mu)$, and for $S \in \Omega$, define $E(S) := \chi_S \in \mathcal{B}(L^2(X, \mu))$. Then $E : \Omega \to \mathcal{B}(L^2(X, \mu))$ is a PVM.*

**Lemma 6.6.** *Let $E$ be a PVM for $(X, \Omega, \mathcal{H})$. Then, for all $\alpha, \beta \in \mathcal{H}$, the mapping*

$$E_{\alpha,\beta} : \Omega \to \mathbb{C}, \quad S \mapsto \langle E(S)\alpha, \beta\rangle$$

*is a complex measure on $\Omega$ with total variation $\leq \|\alpha\|\|\beta\|$.*

*Proof.* Let $\alpha, \beta \in \mathcal{H}$. Then $E_{\alpha,\beta}$ is $\sigma$-additive (i.e., countably additive for disjoint sets) since $E$ is $\sigma$-additive by (4). The total variation of a complex measure is given by

$$\|E_{\alpha,\beta}\| := \sup \left\{ \sum_{S\in\pi} |E_{\alpha,\beta}(S)| \right\},$$

where the sum is taken over all partitions of $X$ into finitely many measurable sets. Let $\pi = \{S_1, \ldots, S_n\}$ be a partition of $X$ with $S_j \in \Omega$. For each $j$, pick $\alpha_j \in \mathbb{C}$ such that $|\alpha_j| = 1$ and

$$\alpha_j \cdot E_{\alpha,\beta}(S_j) = \alpha_j \langle E(S_j)\alpha, \beta\rangle = |\langle E(S_j)\alpha, \beta\rangle| = |E_{\alpha,\beta}(S_j)|.$$

Then,

$$\sum_{j=1}^{n} |E_{\alpha,\beta}(S_j)| = |\langle \sum \alpha_j E(S_j)\alpha, \beta\rangle| \leq \| \sum \alpha_j E(S_j)\alpha\| \cdot \|\beta\|.$$

For $i \neq j$, we have

$$E(S_i)E(S_j) = E(S_i \cap S_j) = E(\emptyset) = 0,$$

so $E(S_i)$ and $E(S_j)$ are pairwise orthogonal. Finally, applying the Pythagorean theorem, we get

$$\begin{aligned}\|\sum_{j=1}^{n}\alpha_j E(S_j)\alpha\|^2 &= \sum_{j=1}^{n}\|E(S_j)\alpha\|^2 \\ &= \|\sum_{j=1}^{n}E(S_j)\alpha\|^2 \\ &= \|E\left(\bigcup_{j=1}^{n}S_j\right)\alpha\|^2 \\ &= \|E(X)\alpha\|^2 = \|\alpha\|^2. \qquad \square\end{aligned}$$

*Remark.* Let $E$ be a PVM for $(X, \Omega, \mathcal{H})$, and let $\alpha \in \mathcal{H}$ and $S \in \Omega$. Then,

$$\begin{aligned}E_{\alpha,\alpha}(S) &= \langle E(S)\alpha, \alpha\rangle \\ &= \langle E(S)^2\alpha, \alpha\rangle \\ &= \langle E(S)\alpha, E(S)\alpha\rangle \geq 0,\end{aligned}$$

so $E_{\alpha,\alpha}$ is a positive measure on $X$. Furthermore, if $\|\alpha\| = 1$, then $E_{\alpha,\alpha}$ is a probability measure.

Define

$$(\alpha, \beta) \mapsto \int_X 1\, dE_{\alpha,\beta}.$$

Since

$$E_{\alpha+\lambda\alpha',\beta} = E_{\alpha,\beta} + \lambda E_{\alpha',\beta}$$

and

$$E_{\alpha,\beta+\lambda\beta'} = E_{\alpha,\beta} + \overline{\lambda} E_{\alpha,\beta'},$$

the above defines a sesquilinear form on $\mathcal{H}$. In particular, it is bounded:

$$\left\|\int_X dE_{\alpha,\beta}\right\| \leq \|E_{\alpha,\beta}\| \leq \|\alpha\|\|\beta\|.$$

Suppose $f : X \to \mathbb{C}$ is a bounded $\Omega$-measurable function. Then

$$(\alpha, \beta) \mapsto \int_X f\, dE_{\alpha,\beta}$$

defines a bounded sesquilinear form:

$$\left\|\int_X f\, dE_{\alpha,\beta}\right\| \leq \|f\|_\infty \|E_{\alpha,\beta}\| \leq \|f\|_\infty \|\alpha\|\|\beta\|.$$

So there exists an $x \in \mathcal{B}(\mathcal{H})$ such that $\|x\| \leq \|f\|_\infty$ and

$$\langle x\alpha, \beta\rangle = \int_X f\, dE_{\alpha,\beta}.$$

If $f = \chi_S$ for $S \in \Omega$, then $x = E(S)$, i.e.,

$$\int_X \chi_S\, dE_{\alpha,\beta} = E_{\alpha,\beta}(S) = \langle E(S)\alpha, \beta\rangle.$$

**Definition 6.7.** Let $E$ be a PVM for $(X, \Omega, \mathcal{H})$, and let $f : X \to \mathbb{C}$ be a bounded $\Omega$-measurable function. We call $x \in \mathcal{B}(\mathcal{H})$ the *integral of $f$ with respect to $E$* if

$$\langle x\alpha, \beta \rangle = \int_X f \, dE_{\alpha,\beta}, \quad \forall \alpha, \beta \in \mathcal{H}.$$

We denote it by

$$x := \int_X f \, dE.$$

*Remark.* Define $B(X, \Omega)$ as the set of all bounded $\Omega$-measurable complex functions on $X$, endowed with the supremum norm. If $X$ is a topological space and $\Omega = \mathcal{B}_X$ is the Borel $\sigma$-algebra on $X$, then $B(X) = B(X, \mathcal{B}_X)$.

**Proposition 6.8.** *Let $E$ be a PVM for $(X, \Omega, \mathcal{H})$. Then, the mapping*

$$\Phi : B(X, \Omega) \to \mathcal{B}(\mathcal{H}), \quad f \mapsto \int_X f \, dE$$

*is a $*$-homomorphism and contractive. Furthermore:*

*(1.) If $(f_n)_n \subseteq B(X, \Omega)$ is an increasing sequence of nonnegative functions and $f = \sup_n f_n \in B(X, \Omega)$, then $\int_X f_n dE \to \int_X f \, dE$ in SOT.*

*(2.) If $X$ is compact and $T_2$, then $\Phi(B(X)) \subseteq \Phi(C(X))''$.*

*Proof.* We already saw that $\|\Phi(f)\| \leq \|f\|_\infty$; hence, $\Phi$ is contractive. It is also clear that $\Phi$ is linear and that $\Phi(f)^* = \Phi(\overline{f})$. Next, we prove multiplicativity: $\Phi(\chi_S) = E(S)$ for $S \in \Omega$. Then,

$$\Phi(\chi_S) \cdot \Phi(\chi_T) = E(S) \cdot E(T) = E(S \cap T) = \Phi(\chi_{S \cap T}) = \Phi(\chi_S \cdot \chi_T).$$

Since $\Phi$ is linear, it is also multiplicative on simple functions (which are finite linear combinations of characteristic functions). Since each $f \in B(X, \Omega)$ is a uniform limit of a uniformly bounded sequence of simple functions, we deduce that $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in B(X, \Omega)$.

(1.) Let $f, f_n$ be as in the statement. Since $\Phi$ is a $*$-homomorphism, $(\Phi(f_n))_n$ is an increasing sequence of positive operators, and $\sup_n \|\Phi(f_n)\| \leq \sup_n \|f_n\|_\infty = \|f\|$. By Vigier, there exists $x \in \mathcal{B}(\mathcal{H})$ such that $\Phi(f_n) \xrightarrow{\text{SOT}} x$. This $x$ is a natural candidate for $\Phi(f)$. Indeed, for $\alpha, \beta \in \mathcal{H}$, we have

$$\begin{aligned} \langle \Phi(f)\alpha, \beta \rangle &= \int_X f \, dE_{\alpha,\beta} \\ &= \lim_{n \to \infty} \int_X f_n \, dE_{\alpha,\beta} \\ &= \lim_{n \to \infty} \langle \Phi(f_n)\alpha, \beta \rangle, \end{aligned}$$

so $\Phi(f_n) \xrightarrow{\text{WOT}} \Phi(f)$, and therefore $\Phi(f) = x$.

(2.) Let $X$ be compact Hausdorff and $a \in \Phi(C(X))'$. Take $\alpha, \beta \in \mathcal{H}$. Then, for all

$f \in C(X)$, we have

$$\begin{aligned} 0 &= \langle (a\Phi(f) - \Phi(f)a)\alpha, \beta \rangle \\ &= \langle \Phi(f)\alpha, a^*\beta \rangle - \langle \Phi(f)(a\alpha), \beta \rangle \\ &= \int_X f \, dE_{\alpha,a^*\beta} - \int_X f \, dE_{a\alpha,\beta}, \end{aligned}$$

so by uniqueness from Riesz–Markoff, we get $E_{\alpha,a^*\beta} = E_{a\alpha,\beta}$. Reversing this calculation shows that $a$ commutes with all $\Phi(g) = \int_X g \, dE$ for $g \in B(X)$, so $\Phi(B(X)) \subseteq \Phi(C(X))''$. □

*Remark.* The map $\Phi$ is not necessarily isometric. In fact, it is not injective in general.

Recall that for an abelian $C^*$-algebra $A$, the Gelfand transform

$$\Gamma : A \to C(\sigma(A))$$

is an isometric $*$-isomorphism.

**Theorem 6.9** (Spectral theorem).

*Let $A \subseteq \mathcal{B}(\mathcal{H})$ be an abelian $C^*$-algebra, and let $\mathcal{B}_{\sigma(A)}$ be the Borel $\sigma$-algebra on $\sigma(A)$. Then, there exists a PVM $E$ for $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$ such that*

$$x = \int_{\sigma(A)} \Gamma(x) \, dE$$

*for all $x \in A$.*

In the proof, we will need the following lemma.

**Lemma 6.10.** *Let $X$ be a compact Hausdorff space and $\mu$ a regular finite Borel measure on $X$. Then the space of continuous functions $C(X)$ is weak-$*$ dense in $L^\infty(X, \mu)$.*

Recall the classic result from measure theory (we refer to, for example, theorem 2.24 in [4]).

**Theorem 6.11** (Luzin).

*Let $\mu$ be a regular finite Borel measure on $X$ and $f : X \to \mathbb{C}$ measurable. Then, for any $\varepsilon > 0$, there exists a $g \in C(X)$ such that*

$$\mu(\{x \in X \mid f(x) \neq g(x)\}) < \varepsilon$$

*and*

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|.$$

*Remark.* Luzin's theorem may be stated in greater generality for $X$ locally compact Hausdorff and $\mu$ a Radon measure.

*Proof of lemma.* Take any $f \in L^\infty(X,\mu)$. Then for any $n \in \mathbb{N}$, there exists $f_n \in C(X)$ such that $\|f_n\|_\infty \le \|f\|_\infty$ and $\mu(\{x \in X \mid f(x) \neq f_n(x)\}) < \frac{1}{n}$. We prove that $f_n \to f$ in weak-$*$ topology. Take any $g \in L^1(X,\mu)$.

(1.) In the first step, we will show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any Borel set $B \subseteq X$ with $\mu(B) < \frac{1}{N}$, we have $\int_B |g|\,d\mu < \varepsilon$. First of all, there exists a step function $\phi$ such that $0 \le \phi \le |g|$ and $\int |g|\,d\mu - \int \phi\,d\mu < \frac{\varepsilon}{2}$. Take $M := \sup \phi$ and $\delta := \frac{\varepsilon}{2M}$. Then for any Borel $B \subseteq X$ with $\mu(B) < \delta$, we have

$$\int_B |g|\,d\mu \le \int_B \phi\,d\mu + \int (|g| - \phi)\,d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(2.) For any $n \ge N$, we have

$$\int |(f - f_n)g|\,d\mu \le 2 \cdot \|f\|_\infty \cdot \varepsilon$$

and we are done. $\square$

*Proof.* For all $\alpha, \beta \in \mathcal{H}$, define

$$\varphi : C(\sigma(A)) \to \mathbb{C}, \quad f \mapsto \langle \Gamma^{-1}(f)\alpha, \beta\rangle.$$

This is a bounded linear functional. Indeed, since $\Gamma$ is an isometry, we get

$$\langle \Gamma^{-1}(f)\alpha, \beta\rangle \le \|f\|_\infty \|\alpha\|\|\beta\|.$$

By the Riesz–Markoff theorem, there exists a unique regular Borel measure $\mu_{\alpha,\beta}$ such that

$$\langle \Gamma^{-1}(f)\alpha, \beta\rangle = \int_{\sigma(A)} f\,d\mu_{\alpha,\beta}.$$

We will show that $\mu_{\alpha,\beta} = E_{\alpha,\beta}$ for a PVM $E$. For $f, g \in C(\sigma(A))$, we have

$$\int_{\sigma(A)} fg\,d\mu_{\alpha,\beta} = \langle \Gamma^{-1}(fg)\alpha, \beta\rangle = \langle \Gamma^{-1}(f)\Gamma(g)\alpha, \beta\rangle = \int_{\sigma(A)} f\,d\mu_{\Gamma^{-1}(g)\alpha,\beta}.$$

This is also equal to

$$\langle \Gamma^{-1}(f)\alpha, \Gamma^{-1}(\overline{g})\beta\rangle = \int_{\sigma(A)} f\,d\mu_{\alpha,\Gamma^{-1}(\overline{g})\beta}.$$

By the uniqueness in Riesz–Markoff, we obtain

$$g\,d\mu_{\alpha,\beta} = d\mu_{\Gamma^{-1}(g)\alpha,\beta} = d\mu_{\alpha,\Gamma^{-1}(\overline{g})\beta}.$$

Finally, we have

$$\begin{aligned}
\int_{\sigma(A)} f\,d\overline{\mu_{\alpha,\beta}} &= \overline{\int \overline{f}\,d\mu_{\alpha,\beta}} \\
&= \overline{\langle \Gamma^{-1}(\overline{f})\alpha, \beta\rangle} \\
&= \overline{\langle \alpha, \Gamma^{-1}(f)\beta\rangle} \\
&= \langle \Gamma^{-1}(f)\beta, \alpha\rangle \\
&= \int_{\sigma(A)} f\,d\mu_{\beta,\alpha}
\end{aligned}$$

for all $f \in C(\sigma(A))$, which implies $\overline{\mu_{\alpha,\beta}} = \mu_{\beta,\alpha}$. To each $S \in \mathcal{B}_{\sigma(A)}$, we assign the sesquilinear form

$$\mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_{\sigma(A)} \chi_S \, d\mu_{\alpha,\beta}.$$

This form is bounded by $\|\alpha\|\|\beta\| = \|\mu_{\alpha,\beta}\|$. Thus, there exists $E(S) \in \mathcal{B}(\mathcal{H})$ such that

$$\int_{\sigma(A)} \chi_S \, d\mu_{\alpha,\beta} = \langle E(S)\alpha, \beta\rangle.$$

Now notice that

$$\begin{aligned}
\langle E(S)^*\alpha, \beta\rangle &= \langle \alpha, E(S)\beta\rangle \\
&= \overline{\langle E(S)\beta, \alpha\rangle} \\
&= \overline{\int_{\sigma(A)} \chi_S \, d\mu_{\beta,\alpha}} \\
&= \int_{\sigma(A)} \chi_S \, d\overline{\mu_{\beta,\alpha}} \\
&= \int_{\sigma(A)} \chi_S d\mu_{\alpha,\beta} \\
&= \langle E(S)\alpha, \beta\rangle,
\end{aligned}$$

so $E(S) = E(S)^*$. We now show that $E$ is a projection-valued measure. Using the weak-$*$ density of $C(\sigma(A))$ in $L^\infty(\sigma(A), \mu)$, we have that for any $T \in \mathcal{B}_{\sigma(A)}$, there exists a net $(f_i)_i \subseteq C(\sigma(A))$ such that $f_i \xrightarrow{w^*} \chi_T$, which in turn implies that $\Gamma^{-1}(f_i)$ converge to $E(T)$ in SOT. Now for any $\alpha, \beta \in \mathcal{H}$, we have

$$\begin{aligned}
\langle E(T)E(S)\alpha, \beta\rangle &= \lim_i \langle \Gamma^{-1}(f_i)E(S)\alpha, \beta\rangle \\
&= \lim_i \int_{\sigma(A)} \chi_S f_i \, d\mu_{\alpha,\beta} \\
&= \int_{\sigma(A)} \chi_S \cdot \chi_T \, d\mu_{\alpha,\beta} \\
&= \int_{\sigma(A)} \chi_{S\cap T} \, d\mu_{\alpha,\beta} \\
&= \langle E(S \cap T)\alpha, \beta\rangle.
\end{aligned}$$

As a result, we have

$$E(S)E(T) = E(S \cap T)$$

for any $S, T \in \mathcal{B}_{\sigma(A)}$. This shows that $E(S)$ is a projection. Further,

$$E(\sigma(A)) = 1,$$

and since $\mu_{\alpha,\beta}$ is $\sigma$-additive, we conclude that

$$E\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} E(S_i)$$

for pairwise disjoint sets $(S_i)_i$. This proves that $E$ is a PVM. Finally, for all $\alpha, \beta \in \mathcal{H}$, we get

$$E_{\alpha,\beta}(S) = \langle E(S)\alpha, \beta\rangle = \mu_{\alpha,\beta}(S),$$

so $\mu_{\alpha,\beta} = E_{\alpha,\beta}$, and

$$\int f\, dE_{\alpha,\beta} = \langle \Gamma^{-1}(f)\alpha, \beta\rangle.$$

This proves that

$$x = \int_{\sigma(A)} \Gamma(x)\, dE \quad \forall x \in A.$$

Uniqueness: Suppose $E'$ is another such PVM. Then, for all $f \in C(\sigma(A))$,

$$\int_{\sigma(A)} f\, dE_{\alpha,\beta} = \langle \Gamma^{-1}(f)\alpha, \beta\rangle = \int_{\sigma(A)} f\, dE'_{\alpha,\beta}.$$

Thus, $E_{\alpha,\beta} = E'_{\alpha,\beta}$ in $C(\sigma(A))^*$, which implies $E = E'$. $\square$

## 6.2 Borel functional calculus

Let $x \in \mathcal{B}(\mathcal{H})$ be normal and $A := C^*(x) \subseteq \mathcal{B}(\mathcal{H})$ an abelian $C^*$-algebra, generated by $x$. By the spectral theorem, there exists a PVM $E$ for $(\sigma(A), \mathcal{B}_{\sigma(A)}, \mathcal{H})$ and

$$\Phi : B(\sigma(A)) \to \mathcal{B}(\mathcal{H}), \quad f \mapsto \int_{\sigma(A)} f\, dE$$

is a $*$-homomorphism and a contraction by proposition 6.8. Since $\sigma(x) = \sigma_{C^*(x)}$, we have the homeomorphism $\tau : \sigma(C^*(x)) \to \sigma(x)$, which induces the isometric $*$-isomorphism

$$\tau^{\#} : B(\sigma(x)) \to B(C^*(x)), \quad f \mapsto f \circ \tau.$$

We can now define a map

$$\rho = \Phi \circ \tau^{\#} : B(\sigma(x)) \to \mathcal{B}(\mathcal{H}),$$

called the *Borel functional calculus.*

$$\begin{tikzcd}
\mathcal{B}(\mathcal{H}) & & C(\sigma(A)) \arrow[ll, "\Phi"'] \\
& C(\sigma(x)) \arrow[ul, "\rho"] \arrow[ur, "\tau^{\#}"'] &
\end{tikzcd}$$

As with the continuous functional calculus, we will employ the notation $f(x) := \rho(f)$. This notation makes sense because the Borel functional calculus actually extends the continuous functional calculus. Indeed, for $f \in C(\sigma(x))$, we have

$$\rho(f) = \int_{\sigma(x)} f\, dE = \int_{\sigma(A)} \tau^{\#}(f)\, dE = \Gamma^{-1}(\tau^{\#}(f)) = (\Gamma^{-1} \circ \tau^{\#})(f),$$

which implies that $\rho$, when restricted to $C(\sigma(x))$, coincides with the continuous functional calculus. As a corollary, if $f = \mathrm{id} \in B(\sigma(x))$, then

$$\rho(\mathrm{id}) = \int_{\sigma(x)} z\, dE = x.$$

Since $\rho$ extends continuous functional calculus, we have by the second item of proposition 6.8

$$\rho(B(\sigma(x))) = \Phi(B(\sigma(A))) \subseteq \Phi(C(\sigma(A)))'' = \rho(C(\sigma(x)))'' = A''.$$

By the bicommutant theorem, we have $A'' = W^*(x)$, where $W^*(x)$ is the vNa, generated by $x$.

**Theorem 6.12** (Spectral mapping theorem)**.**

*Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and let $x \in A$ be normal. Then the Borel functional calculus has the following properties:*

*(1.) The map*

$$B(\sigma(x)) \to W^*(x), \quad f \mapsto f(x)$$

*is a contractive $*$-homomorphism.*

*(2.) If $f \in C(\sigma(x))$, then this $f(x)$ coincides with the $f(x)$ from the continuous functional calculus.*

*(3.) For $f \in B(\sigma(x))$, we have $\sigma(f(x)) \subseteq f(\sigma(x))$.*

*Proof.* The first and second item follow directly from the above discussion. Let us prove the third item. Suppose $\lambda \notin f(\sigma(x))$. Then $f - \lambda \in \mathcal{B}(\sigma(x))$ is invertible in $\mathcal{B}(\sigma(x))$, so there exists $g \in \mathcal{B}(\sigma(x))$ such that $(f - \lambda)g = \mathrm{id}$. By the Borel functional calculus, $(f(x) - \lambda I) \cdot g(x) = I$, so $\lambda \notin \sigma(f(x))$. □

The spectral theorem is a powerful result that enables us to answer many questions about normal operators. Let us mention some of its consequences.

**Corollary 6.13.** *Every vNa is the norm-closure of the linear span of its projections.*

*Proof.* Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a vNa and $x \in M$. Using that $\operatorname{Re} x, \operatorname{Im} x \in M_{\mathrm{sa}}$, we may assume without loss of generality that $x \in M_{\mathrm{sa}}$. Hence $x$ is normal, and for all $f \in B(\sigma(x))$, we have $f(x) \in M$. For $S \in \mathcal{B}_{\sigma(x)}$, the characteristic function $\chi_S(x) \in M$ is a projection. Now we can uniformly approximate id on $\sigma(x)$ using simple functions. By the Borel functional calculus, $x$ is uniformly approximated by linear combinations of projections. □

*Remark.* There exist $C^*$-algebras without nontrivial projections. For example, for a compact Hausdorff connected space $X$, the algebra $C(X)$ only has trivial projections 0 and 1. There also exist nonabelian examples.

We can now prove the statement in example 3.29. We begin with a lemma.

**Lemma 6.14.** *If $x \in \mathcal{B}(\mathcal{H})$ is a normal operator and $x = \int_{\sigma(x)} z \, dE$, then $N$ is compact iff for every $\varepsilon > 0$, $E(\{z \mid |z| > \varepsilon\})$ has finite rank.*

*Proof.* If $\varepsilon > 0$, then define $\Delta_\varepsilon := \{z \mid |z| > \varepsilon\}$ and $e_\varepsilon := E(\Delta_\varepsilon)$. Then

$$\begin{aligned} x - xe_\varepsilon &= \int_{\sigma(x)} z\,dE - \int_{\sigma(x)} z\chi_{\Delta_\varepsilon}(z)\,dE \\ &= \int_{\sigma(x)} z\chi_{\mathbb{C}\setminus\Delta_\varepsilon}(z)\,dE = \phi(x), \end{aligned}$$

where $\phi(z) = z\chi_{\mathbb{C}\setminus\Delta_\varepsilon}$. Since the Borel functional calculus is a contraction, this implies that

$$\|x - xe_\varepsilon\| = \|\phi(x)\| \leq \|\phi\|_\infty \leq \varepsilon.$$

Therefore, we can approximate $x$ with operators $xe_\varepsilon$. If every $e_\varepsilon$ has finite rank, then every $xe_\varepsilon$ has finite rank and $x$ can be approximated using finite rank operators, so it must be compact. Conversely, let $x$ be compact and take an arbitrary $\varepsilon > 0$. Define the function $\psi(z) = z^{-1}\chi_{\Delta_\varepsilon}$. Then

$$x\psi(x) = \int_{\sigma(x)} zz^{-1}\chi_{\Delta_\varepsilon}\,dE = \int_{\sigma(x)} \chi_{\Delta_\varepsilon} = E_\varepsilon$$

is compact. Define a map $\iota_\varepsilon : \operatorname{im} e_\varepsilon \hookrightarrow \mathcal{H}$. Of course, $\operatorname{im} e_\varepsilon$ is closed (and therefore Hilbert) in $\mathcal{H}$ since $e_\varepsilon$ is a projection. Now

$$E_\varepsilon \circ \iota_\varepsilon = \mathrm{id} : \operatorname{im} e_\varepsilon \to \operatorname{im} e_\varepsilon$$

must be compact, which can happen only if $\operatorname{im} e_\varepsilon$ has finite dimension. As a result, $e_\varepsilon$ is a finite rank operator. □

**Corollary 6.15.** *If $\mathcal{H}$ is separable and $I$ is an ideal of $\mathcal{B}(\mathcal{H})$ that contains a noncompact operator, then $I = \mathcal{B}(\mathcal{H})$.*

*Proof.* Suppose that $x \in I$ is noncompact and let

$$x^*x = \int_{\sigma(x^*x)} z\,dE$$

by the spectral theorem. By the previous lemma, there exists such an $\varepsilon > 0$ such that $e_\varepsilon = E(\varepsilon, \infty)$ has infinite rank. Now notice that

$$e_\varepsilon = \int_{\sigma(x)} \chi_{(\varepsilon,\infty)}\,dE = \int_{\sigma(x)} z^{-1}\chi_{(\varepsilon,\infty)} z\,dE = \left(\int_{\sigma(x)} z^{-1}\chi_{(\varepsilon,\infty)}\,dE\right)(x^*x) \in I.$$

Let $U : \mathcal{H} \to e_\varepsilon\mathcal{H}$ be a unitary (such an $U$ exists because $\dim \mathcal{H} = \dim e_\varepsilon\mathcal{H} = |\mathbb{N}|$). Then $U^* e_\varepsilon U = \mathrm{id}_{\mathcal{H}} \in I$, hence $I = \mathcal{B}(\mathcal{H})$. □

*Remark.* The statement is not true for $\mathcal{H}$ with dimension of higher cardinality. Notice that $\mathcal{K}(\mathcal{H})$ is a closed ideal, generated by the operators of finite rank. If $\mathcal{H}$ is not separable, then operators with separable image (or any cardinality less than the dimension of $\mathcal{H}$) also generate a closed proper ideal in $\mathcal{B}(\mathcal{H})$.

## 6.3 Abelian vNa's

**Definition 6.16.** Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a subalgebra. A vector $\alpha \in \mathcal{H}$ is:
(1.) *Cyclic* for $A$ if $A\alpha$ is dense in $\mathcal{H}$.
(2.) *Separating* for $A$ if $x\alpha = 0$ for $x \in A$ implies $x = 0$.

**Proposition 6.17.** *Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a subalgebra.*

*(1.) If $\alpha \in \mathcal{H}$ is cyclic for $A$, then it is separating for $A'$.*

*(2.) Assume $A$ is a $*$-subalgebra. Then, if $\alpha$ is separating for $A'$, it is cyclic for $A$.*

*(3.) Suppose $W \subseteq \mathcal{B}(\mathcal{H})$ is a vNa. Then $\alpha$ is cyclic for $W$ iff it is separating for $W'$, and separating for $W$ iff it is cyclic for $W'$.*

*Proof.* (1.) Let $\alpha$ be cyclic for $A$. Let $y \in A'$ satisfy $y\alpha = 0$. Pick any $\beta \in \mathcal{H}$. There exists a sequence $(x_n)_n \subseteq A$ such that $\|x_n\alpha - \beta\| \to 0$. Hence,

$$y\beta = \lim_{n\to\infty} yx_n\alpha = \lim_{n\to\infty} x_n(y\alpha) = 0,$$

and $\alpha$ is separating for $A'$.
(2.) Define $\mathcal{K} := (A\alpha)^\perp \leq \mathcal{H}$. Let $p : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. For $x_1, x_2 \in A$ and $\beta \in \mathcal{K}$, we have

$$\langle x_1\beta, x_2\alpha\rangle = \langle \beta, x_1^* x_2\alpha\rangle = 0,$$

so $x_1\beta \in \mathcal{K}$, and $\mathcal{K}$ is an invariant subspace for $A$. But since $A$ is $*$-closed, $\mathcal{K}$ is reducing, and by lemma 5.30, $p \in A'$. Of course, $\alpha \in A\alpha$ and $p\alpha = 0$. Now we use the fact that $\alpha$ is separating for $A'$, and therefore $p = 0$. This implies $\mathcal{K} = (0)$.
(3.) This follows immediately from $W = W''$ and the previous two points. $\square$

**Example 6.18.** *Recall that $VN(\Gamma) := \lambda(\mathbb{C}[\Gamma])'' \subseteq \mathcal{B}(\ell^2(\Gamma))$. Similarly, we can use the right regular map*

$$\rho : \Gamma \to \mathcal{B}(\ell^2(\Gamma)), \quad g \mapsto (\rho_g : \delta_k \mapsto \delta_{kg^{-1}})$$

*to define $VN_{\text{right}}(\Gamma) = \rho(\Gamma)'' \subseteq \mathcal{B}(\ell^2(\Gamma))$. Notice that $\delta_e \in \ell^2(\Gamma)$ is cyclic for both $\lambda(\mathbb{C}[\Gamma])$ and $\rho(\mathbb{C}[\Gamma])$. This means that it is cyclic for both $VN(\Gamma)$ and $VN_{\text{right}}(\Gamma)$. It's easy to see that $VN(\Gamma)' = VN_{\text{right}}(\Gamma)$, so $\delta_e$ is separating for both $VN(\Gamma)$ and $VN_{\text{right}}(\Gamma)$.*

**Corollary 6.19.** *If $A \subseteq \mathcal{B}(\mathcal{H})$ is abelian, then each cyclic vector for $A$ is also separating for $A$.*

*Proof.* If $\alpha \in \mathcal{H}$ is cyclic for $A$, then it is separating for $A'$, but since $A \subseteq A'$, it is also separating for $A$. $\square$

Recall from example 5.13 that for a $\sigma$-finite measure on $X$, the map

$$M : L^\infty(X, \mu) \to \mathcal{B}(L^2(X, \mu))$$

is an isometric $*$-isomorphism onto its image. In the remainder of this chapter, we will identify $L^\infty(X, \mu)$ as the image in $\mathcal{B}(L^2(X, \mu))$. Also note that the WOT on $L^\infty(X, \mu)$ is generated by seminorms

$$f \mapsto |\langle M_f \alpha, \beta \rangle| = \left| \int_X f \alpha \beta \, d\mu \right|$$

for any $\alpha, \beta \in L^2(X, \mu)$. Now recall the well-known theorem from measure theory (theorem 6.16 in [4]).

**Theorem 6.20.**

*Suppose $1 \leq p < \infty$ and $\mu$ is a $\sigma$-finite positive measure on $X$, and $\Phi$ is a bounded linear functional on $L^p(X, \mu)$. Then there is a unique $g \in L^q(X, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that*

$$\Phi(f) = \int_X f g \, d\mu.$$

*Moreover, $\|\Phi\| = \|g\|_q$.*

Theorem 6.20 tells us that the weak-$*$ topology on $L^\infty(X, \mu)$ is generated by the seminorms

$$f \mapsto \left| \int_X f g \, d\mu \right|$$

for all $g \in L^1(X, \mu)$. By Hölder's inequality, WOT and weak-$*$ topologies coincide on $L^\infty(X, \mu)$.

**Theorem 6.21** (Classification of abelian vNa's)**.**

*Let $A \subseteq \mathcal{B}(\mathcal{H})$ be an abelian vNa with a cyclic vector $\alpha_0 \in \mathcal{H}$. Suppose $A_0 \subseteq A$ is a $C^*$-algebra that is SOT-dense. Then, there exists a finite regular positive Borel measure $\mu$ on $\sigma(A_0)$ and an isomorphism*

$$\widetilde{\Gamma} : A \to L^\infty(\sigma(A_0), \mu) \subseteq \mathcal{B}(L^2(\sigma(A_0), \mu))$$

*that extends the Gelfand transform $\Gamma : A_0 \to C(\sigma(A_0))$. Furthermore, $\widetilde{\Gamma}$ is spatial, that is, it is induced by conjugation with a unitary $U : \mathcal{H} \to L^2(\sigma(A_0), \mu)$.*

*Remark.* Applying this theorem to $A_0 = A$, we get

$$L^\infty(\sigma(A), \mu) = \widetilde{\Gamma}(A) = C(\sigma(A)).$$

*Proof.* Since $A_0$ is an abelian $C^*$-algebra, the Gelfand transform $\Gamma : A_0 \to C(\sigma(A_0))$ is an isometric $*$-isomorphism. Define $\varphi_0 : A \to \mathbb{C}$ by $x \mapsto \langle x\alpha_0, \alpha_0 \rangle$. Then $\varphi_0 \Gamma^{-1} : C(\sigma(A_0)) \to \mathbb{C}$ is a bounded linear functional, so by the Riesz–Markoff theorem, there exists a regular Borel measure $\mu$ on $\sigma(A_0)$ such that

$$\varphi_0 \Gamma^{-1}(f) = \int_{\sigma(A_0)} f \, d\mu.$$

For every positive function $f \in C(\sigma(A_0))$, we have

$$\begin{aligned}\int_{\sigma(A_0)} f\,d\mu = \int \sqrt{f}^2\,d\mu = \varphi_0\Gamma^{-1}(\sqrt{f}^2) &= \langle \Gamma^{-1}(\sqrt{f}^2)\alpha_0, \alpha_0\rangle \\ &= \langle \Gamma^{-1}(\sqrt{f})^2\alpha_0, \alpha_0\rangle = \langle \Gamma^{-1}(\sqrt{f})\alpha_0, \Gamma^{-1}(\sqrt{f})\alpha_0\rangle \\ &= \|\Gamma^{-1}(\sqrt{f})\alpha_0\|^2 \geq 0\end{aligned}$$

and $\mu$ is a positive measure. Furthermore, $\mu$ is finite, since

$$\mu(\sigma(A_0)) = \varphi_0(1) = \|\alpha_0\|^2 < \infty.$$

Now we prove that $\operatorname{supp}\mu = \sigma(A_0)$. If $\operatorname{supp}\mu \subsetneq \sigma(A_0)$, then there exists a non-empty open set $S \subseteq \sigma(A_0)$ with $\mu(S) = 0$. Consider a nonnegative function $f \in C(\sigma(A_0)) \setminus \{0\}$ with $f\big|_{\sigma(A_0)\setminus S} = 0$. Then

$$\|\Gamma^{-1}(\sqrt{f})\alpha_0\|^2 = \int_{\sigma(A_0)} f\,d\mu = \int_S f\,d\mu = 0.$$

We get $\Gamma^{-1}(\sqrt{f})\alpha_0 = 0$, which, by the cyclicity of $\alpha_0$, implies $\Gamma^{-1}(\sqrt{f}) = 0$, $\sqrt{f} = 0$, and $f = 0$, a contradiction. Define

$$U_0 : A_0\alpha_0 \to C(\sigma(A_0)) \subseteq L^2(\sigma(A_0), \mu), \quad x\alpha_0 \mapsto \Gamma(x).$$

Since $\alpha_0$ is separating for $A_0$, this $U_0$ is a well-defined linear map. For $x, y \in A_0$, we have

$$\begin{aligned}\langle U_0(x\alpha_0), U_0(y\alpha_0)\rangle &= \langle \Gamma(x), \Gamma(y)\rangle_2 \\ &= \int_{\sigma(A_0)} \overline{\Gamma(y)}\Gamma(x)\,d\mu \\ &= \int_{\sigma(A_0)} \Gamma(y^*x)\,d\mu \\ &= \varphi(y^*x) = \langle y^*x\alpha_0, \alpha_0\rangle = \langle x\alpha_0, y\alpha_0\rangle\end{aligned}$$

and so $U_0$ is an isometry! Since $\alpha_0$ is cyclic for $A$ and $A_0$ is SOT-dense in $A$, $\alpha_0$ is cyclic for $A_0$. Thus, $A_0\alpha_0$ is dense in $\mathcal{H}$ and the image of $U_0$ is the entire $C(\sigma(A_0))$. By continuity, $U_0$ extends to a surjective isometry

$$U : \mathcal{H} \to L^2(\sigma(A_0), \mu) = \overline{C(\sigma(A_0), \mu)}^{\langle\cdot,\cdot\rangle_2},$$

where $U$ is unitary. Next, define

$$\widetilde{\Gamma} : A \to \mathcal{B}(L^2(\sigma(A_0), \mu)), \quad x \mapsto UxU^*.$$

We claim that $\widetilde{\Gamma}$ is an isometric $*$-homomorphism. Since $U$ is unitary, the isometric part is obvious, and the homomorphism property soon follows. Now we claim that $\widetilde{\Gamma}(A) = L^\infty(\sigma(A_0), \mu)$. For $x \in A_0$ and $g \in C(\sigma(A_0))$, we have

$$\begin{aligned}\widetilde{\Gamma}(x)g = UxU^*g &= UxU^{-1}(\Gamma(\Gamma^{-1}(g))) \\ &= Ux(\Gamma^{-1}(g)\alpha_0) = \Gamma(x\Gamma^{-1}(g)) \\ &= \Gamma(x)g\end{aligned}$$

and since $C(\sigma(A_0))$ is dense in $L^2(\sigma(A_0), \mu)$, we get $\widetilde{\Gamma}(x) = M_{\Gamma(x)}$. It follows that

$$\widetilde{\Gamma}(A_0) = C(\sigma(A_0)) \subseteq L^\infty(\sigma(A_0), \mu).$$

Then we use the fact that $\widetilde{\Gamma}$ is SOT-continuous (by definition) and that $L^\infty(X, \mu)$ is a vNa to get

$$\widetilde{\Gamma}(A) = \widetilde{\Gamma}(\overline{A_0}^{\text{SOT}}) \subseteq \overline{\widetilde{\Gamma}(A_0)}^{\text{SOT}} \subseteq \overline{L^\infty(\sigma(A_0), \mu)}^{\text{SOT}} = L^\infty(\sigma(A_0), \mu).$$

The reverse inclusion is proved by nets. Suppose $(\widetilde{\Gamma}(x_i))_i \subseteq \widetilde{\Gamma}(A_0)$ WOT-converges to $T \in B(L^2(\sigma(A_0), \mu))$. Then, for all $\beta, \mu \in \mathcal{H}$, we have

$$\begin{aligned}\langle TU\beta, U\mu\rangle &= \lim_i \langle \widetilde{\Gamma}(x_i)U\beta, U\mu\rangle \\ &= \lim_i \langle Ux_iU^*U\beta, U\mu\rangle \\ &= \lim_i \langle x_i\beta, \mu\rangle\end{aligned}$$

and $(x_i)_i \xrightarrow{\text{WOT}} U^*TU \in \mathcal{B}(\mathcal{H})$. Since $\overline{A_0}^{\text{WOT}} = A$, we get $x = U^*TU \in A$ and $\widetilde{\Gamma}(x) = T$, so $\overline{\widetilde{\Gamma}(A_0)}^{\text{WOT}} \subseteq \widetilde{\Gamma}(A)$. Finally, we ask: what is $\overline{C(\sigma(A_0))}^{\text{WOT}}$? By lemma 6.10, $C(\sigma(A_0))$ is weak-$*$ dense in $L^\infty(\sigma(A_0), \mu)$, so we have

$$L^\infty(\sigma(A_0), \mu) = \overline{C(\sigma(A_0))}^{w^*} = \overline{C(\sigma(A_0))}^{\text{WOT}} = \overline{\widetilde{\Gamma}(A_0)}^{\text{WOT}} \subseteq \widetilde{\Gamma}(A)$$

and finally $\widetilde{\Gamma}(A) = L^\infty(\sigma(A_0), \mu)$. □

*Remark.* In the above proof, we also showed that

$$\widetilde{\Gamma} : (A, \text{WOT}) \to (L^\infty(\sigma(A_0), \mu), w^*)$$

is a homeomorphism.

How crucial was the cyclicity assumption in the previous theorem? Let $A \subseteq \mathcal{B}(\mathcal{H})$ be an abelian vNa. Let $\{\alpha_i \mid i \in I\}$ be a maximal set of nonzero vectors in $\mathcal{H}$ such that $\overline{A\alpha_i} \perp \overline{A\alpha_j}$ for $i \neq j$ (such a set must exist by Zorn's lemma). For every $i \in I$, define the orthogonal projection $p_i : \mathcal{H} \to \overline{A\alpha_i} =: \mathcal{K}_i$. By maximality, we must have $\bigoplus_{i\in I} \mathcal{K}_i = \mathcal{H}$. Due to the reducibility of $\mathcal{K}_i$, we get $p_i \in A'$. Therefore, $p_iAp_i = Ap_i \subseteq \mathcal{B}(\mathcal{K}_i)$ is an abelian vNa with a cyclic vector $\alpha_i \in \mathcal{K}_i$. Also, $A \cong \bigoplus_{i\in I} Ap_i$. For every $i \in I$, we have by theorem 6.21 $Ap_i \cong L^\infty(X_i, \mu_i)$ for some Borel measure space $(X_i, \mu_i)$. Therefore, we have

$$A \cong \bigoplus_{i\in I} L^\infty(X_i, \mu_i).$$

However, if $\mathcal{H}$ is assumed to be separable, we have even stronger results.

**Proposition 6.22.** *Let $\mathcal{H}$ be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ be an abelian vNa. Then there exists a separating vector for $A$.*

*Proof.* By Zorn's lemma, there exists a maximal set of unit vectors $(\alpha_k)_k$ such that $A\alpha_k \perp A\alpha_l$ for $k \neq l$. By maximality, $\sum_k A\alpha_k$ is dense in $\mathcal{H}$. Define $\alpha = \sum_{n=1}^\infty \frac{1}{2^n}\alpha_n$. We claim that $\alpha$ is separating for $A$. Indeed, let $x \in A$ such that $x\alpha = 0$. Then $\sum_{n=1}^{\infty} \frac{1}{2^n} x\alpha_n = 0$. By orthogonality, $x\alpha_n = 0$ for all indices $n$. For all $y \in A$, we get $xy\alpha_n = yx\alpha_n = 0$, so $x\big|_{A\alpha_n} = 0$ for all $n$. But since $\sum_n A\alpha_n$ is dense in $A$, we get $x = 0$. ☐

**Corollary 6.23.** *Let $\mathcal{H}$ be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ be a maximal abelian vNa. Then there exists a cyclic vector for $A$.*

*Proof.* By proposition 6.22, there exists a separating vector $\alpha$ for $A$, which is then cyclic for $A'$. But since $A$ is maximal, we get $A = A'$. ☐

**Theorem 6.24.**

*Let $\mathcal{H}$ be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ an abelian vNa. Then there exists a compact Hausdorff space $X$ and a finite regular Borel measure $\mu$ on $X$ such that $A \cong L^\infty(X, \mu)$.*

*Proof.* By proposition 6.22, there exists a separating vector $\alpha \in \mathcal{H}$ for $A$. Form $\mathcal{K} := \overline{A\alpha}$. Then the algebra $\{x\big|_{\mathcal{K}} \mid x \in A\} \subseteq \mathcal{B}(\mathcal{K})$ is $*$-isomorphic to $A$, has cyclic vector $\alpha$, and the above theorem applies. ☐

As a corollary, if $\mathcal{H}$ is a separable Hilbert space and $x \in \mathcal{B}(\mathcal{H})$ is a normal element, then $W^*(x)$ has a cyclic vector $\alpha_0$ and there exists a finite regular positive Borel measure $\mu$ on $\sigma(x)$ such that

$$W^*(x) \cong L^\infty(\sigma(x), \mu).$$

Following the proof of theorem 6.21, we notice that the measure $\mu$ is exactly $E_{\alpha_0,\alpha_0}$, where $E$ is the PVM from theorem 6.9.

**Proposition 6.25.** *(1.) If $S \in \mathcal{B}_{\sigma(x)}$, then $\mu(S) = 0$ iff $E(S) = 0$.*

*(2.) If $f \in B(\sigma(x))$, then $f = 0$ a.e. w.r.t. $\mu$ iff $\int_{\sigma(x)} f\,dE = 0$.*

*(3.) If $f \in L^\infty(\sigma(x), \mu)$, then $\widetilde{\Gamma}^{-1}(f) = \int_{\sigma(x)} f\,dE$.*

*Proof.* (1.) If $\mu(S) = 0$, then $\langle E(S)\alpha_0, \alpha_0\rangle = \|E(S)\alpha_0\|^2 = 0$ and hence $E(S)\alpha_0 = 0$. Since $\alpha_0$ is a separating vector for $W^*(x)$, we have $E(S) = 0$. The converse implication is trivial.

(2.) If $f = 0$ a.e., then $f = f\chi_S$ for $\mu(S) = 0$. By the spectral theorem, we get

$$\int_{\sigma(x)} f\,dE = \int_{\sigma(x)} f\chi_S\,dE = E(S)\int_{\sigma(x)} f\,dE = 0.$$

Conversely, let $\int_{\sigma(x)} f\,dE = 0$ and define the Borel set $S = \{f \neq 0\}$. Then

$$E(S) = \int_{\sigma(x)} \chi_S\,dE = \int_{\sigma(x)} f \cdot \frac{1}{f}\chi_S\,dE = \int_{\sigma(x)} f\,dE \cdot \int_{\sigma(x)} \frac{1}{f}\chi_S\,dE = 0,$$

so $\mu(S) = 0$.

(3.) By the second item, every function $f$ from the same equivalence class in $L^\infty(X,\mu)$ defines the same operator $\int_{\sigma(x)} f\,dE$, which means that the Borel functional calculus

$$L^\infty(X,\mu) \to W^*(x), \quad f \mapsto \rho(f)$$

is well-defined. To show that $\widetilde{\Gamma}^{-1}(f) = \rho(f)$, it suffices to prove that

$$\langle \widetilde{\Gamma}^{-1}(f)\alpha, \beta\rangle = \langle \rho(f)\alpha, \beta\rangle \tag{6.1}$$

for all $\alpha, \beta \in \mathcal{H}$. Since $\alpha_0$ is cyclic for $W^*(x)$ and $C^*(x)$ is WOT-dense in $W^*(x)$, it is also cyclic for $C^*(x)$. Therefore, $C^*(x)\alpha_0$ is dense in $\mathcal{H}$. So it is enough to prove the equation (6.1) for $\alpha = g(x)\alpha_0$ and $\beta = h(x)\alpha_0$, where $g, h \in C(\sigma(x))$. Now use the proof of theorem 6.9 to get

$$\begin{aligned}
\langle \widetilde{\Gamma}^{-1}(f)\alpha, \beta\rangle &= \langle \widetilde{\Gamma}^{-1}(f)\widetilde{\Gamma}^{-1}(g)\alpha_0, \widetilde{\Gamma}^{-1}(h)\alpha_0\rangle \\
&= \langle \widetilde{\Gamma}^{-1}(\overline{h}fg)\alpha_0, \alpha_0\rangle \\
&= \int_{\sigma(x)} \overline{h}fg\,d\mu \\
&= \int_{\sigma(x)} \overline{h}fg\,dE_{\alpha_0,\alpha_0} \\
&= \int_{\sigma(x)} f\,dE_{\Gamma^{-1}(g)\alpha_0, \Gamma^{-1}(h)\alpha_0} \\
&= \left\langle \rho(f)\Gamma^{-1}(g)\alpha_0, \Gamma^{-1}(h)\alpha_0 \right\rangle \\
&= \langle \rho(f)\alpha, \beta\rangle\,.
\end{aligned}$$

□

In the preceding proposition, we proved that the Borel functional calculus

$$L^\infty(\sigma(x),\mu) \to W^*(x), \quad f \mapsto \rho(f)$$

is an isometric $*$-isomorphism. Even further, it is a homeomorphism

$$(L^\infty(\sigma(x),\mu), w^*) \to (W^*(x), \mathrm{WOT})$$

and

$$W^*(x) = \{\rho(f) \mid f \in L^\infty(X,\mu)\} = \{\rho(f) \mid f \in B(\sigma(x))\}.$$

We have another easy corollary of theorem 6.21, which is also known as the spectral theorem.

**Theorem 6.26.**

*Let $\mathcal{H}$ be a separable Hilbert space. If $x \in \mathcal{B}(\mathcal{H})$ is a normal operator, then there exists a measure space $(Y,\nu)$, a function $\varphi \in L^\infty(Y,\nu)$ and a unitary $U : \mathcal{H} \to L^2(Y,\nu)$ such that*

$$x = U^* M_\varphi U,$$

*where $M_\varphi \in L^\infty(Y,\nu) \subseteq \mathcal{B}(L^2(Y,\nu))$ is the multiplication operator.*

We can use theorem 6.21 to find $VN(G)$ of an abelian group $G$. Recall that by definition, $C_r^*(G)$ is a SOT-dense $C^*$-subalgebra in $VN(G)$. Moreover, $VN(G)$ has a cyclic vector $\delta_0 \in \ell^2(G)$.

Therefore, there exists a positive Borel measure $\mu$ on $\sigma_r^*(G)$ such that

$$VN(G) \cong L^\infty(\sigma(C_r^*(G)), \mu) \cong L^\infty(\widehat{G}, \mu).$$

**Example 6.27.** *Let us find $VN(\mathbb{Z}/n\mathbb{Z})$. Following the proof of theorem 6.21, all we need to do is find a positive Borel measure $\mu$ on $\mathbb{Z}/n\mathbb{Z}$, such that for any $x \in C_r^*(G)$, we have*

$$\langle x\delta_0, \delta_0\rangle = \int_{\mathbb{Z}/n\mathbb{Z}} \Gamma(x)\, d\mu.$$

*Now if $x = \sum_{l=0}^{n-1} a_l u_l$, the LHS of this equation equals $a_0$. On the other hand, example 3.47 tells us that the RHS is equal to*

$$\sum_{k=0}^{n-1} \mu(\{k\}) \cdot \left(\sum_{l=0}^{n-1} a_l e^{\frac{2\pi i k l}{n}}\right) = \sum_{l=0}^{n-1} a_l \cdot \left(\sum_{k=0}^{n-1} \mu(\{k\}) e^{\frac{2\pi i k l}{n}}\right).$$

*The above expression equals $a_0$ if we set $\mu(\{k\}) = \frac{1}{n}$ for any $k \in \mathbb{Z}/n\mathbb{Z}$. Therefore, we can take $\mu$ to be the normalized counting measure on $\mathbb{Z}/n\mathbb{Z}$ and*

$$VN(\mathbb{Z}/n\mathbb{Z}) \cong L^\infty(\mathbb{Z}/n\mathbb{Z}, \mu)$$

*as a $C^*$-algebra. In particular, $VN(\mathbb{Z}/n\mathbb{Z}) \cong C(\mathbb{Z}/n\mathbb{Z})$, so we have $VN(\mathbb{Z}/n\mathbb{Z}) = C_r^*(\mathbb{Z}/n\mathbb{Z})$ and $VN(\mathbb{Z}/n\mathbb{Z})$ is generated by*

$$u_1 = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \ell^2(\mathbb{Z}/n\mathbb{Z}).$$

**Example 6.28.** *Similar to the previous example, we can find $VN(\mathbb{Z})$. Again, it suffices to find a positive Borel measure $\mu$ on $\mathbb{T}$ such that*

$$\langle x\delta_0, \delta_0\rangle = \int_{\mathbb{T}} \Gamma(x)\, d\mu$$

*for any $x = \sum_{l\in\mathbb{Z}} a_l u_l \in C_r^*(\mathbb{Z})$. Now the LHS of the above equation equals $a_0$, while the RHS is equal, by example 3.48, to*

$$\int_{\mathbb{T}} \sum_{l\in\mathbb{Z}} a_l \xi^l\, d\mu = \sum_{l\in\mathbb{Z}} a_l \int_{\mathbb{T}} \xi^l\, d\mu$$

*(here, we used Lebesgue's dominated convergence theorem). If we set $\mu$ to be the normalized Lebesgue measure $m$ on $\mathbb{T}$, then the above sum equals $a_0$. Therefore, we have*

$$VN(\mathbb{Z}) \cong L^\infty(\mathbb{T}, m)$$

*as a $C^*$-algebra by theorem 6.21.*

# 7 Completely positive maps

## 7.1 Dilations

**Definition 7.1.** The *dilation* of $T \in \mathcal{B}(\mathcal{H})$ is an operator $S \in \mathcal{B}(\mathcal{K})$, where $\mathcal{H}$ is a subspace of Hilbert space $\mathcal{K}$, $P_{\mathcal{H}} : \mathcal{H} \hookrightarrow \mathcal{K}$ and $T = P_{\mathcal{H}} S\big|_{\mathcal{H}}$. We say that $T$ is a *compression* of $S$.

If we write $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$, the operator $S$ has matrix form

$$S = \begin{bmatrix} T & * \\ * & * \end{bmatrix}.$$

**Example 7.2.** *We can show that every isometry has a unitary dilation. Indeed, let $V \in \mathcal{B}(\mathcal{H})$ be an isometry and $P := I - VV^*$ a projection onto $(\operatorname{im} V)^{\perp}$. Now let $\mathcal{K} := \mathcal{H} \oplus \mathcal{H}$ and define $U \in \mathcal{B}(\mathcal{K})$ as*

$$U := \begin{bmatrix} V & P \\ 0 & V^* \end{bmatrix}.$$

*Obviously, $U$ is a dilation of $V$. To show that it is unitary, simply calculate*

$$\begin{aligned} U^*U &= \begin{bmatrix} V^* & 0 \\ P & V \end{bmatrix} \begin{bmatrix} V & P \\ 0 & V^* \end{bmatrix} \\ &= \begin{bmatrix} V^*V & V^*P \\ PV & P^2 + VV^* \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

*Similarly, we get $UU^* = I$ and $U$ is a unitary dilation of $V$. In fact, even more is true: $U$ is a* power dilation *of $V$. This means that for any $n \in \mathbb{N}$, the operator*

$$U^n = \begin{bmatrix} V^n & * \\ 0 & (V^*)^n \end{bmatrix}$$

*is a dilation of $V^n$.*

**Example 7.3.** *We can go even further and show that every contraction has an isometric dilation. Indeed, take $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leq 1$. Then*

$$D_T := (I - T^*T)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$$

*and for every $h \in \mathcal{H}$, we have*

$$\begin{aligned} \|Th\|^2 + \|D_T h\|^2 &= \langle Th, Th\rangle + \langle D_T h, D_T h\rangle \\ &= \langle T^*Th, h\rangle + \langle D_T^2 h, h\rangle \\ &= \langle T^*Th, h\rangle + \langle (I - T^*T)h, h\rangle \\ &= \langle h, h\rangle = \|h\|^2. \end{aligned}$$

*Define the Hilbert space* $\mathcal{K} := \bigoplus_{n\in\mathbb{N}} \mathcal{H}$, *which is the sequence space*

$$\{(h_1, h_2, \dots) \mid h_n \in \mathcal{H},\ \sum_{n=1}^{\infty} \|h_n\|^2 < \infty\}$$

*with the scalar product*

$$\langle (h_1, h_2, \dots), (k_1, k_2, \dots) \rangle := \sum_{n=1}^{\infty} \langle h_i, k_i \rangle.$$

*Now define the operator*

$$V : \mathcal{K} \to \mathcal{K}, \quad (h_1, h_2, h_3, \dots) \mapsto (Th_1, D_T h_2, h_3, \dots),$$

*which is an isometry, since*

$$\|V(h_1, h_2, h_3, \dots)\|^2 = \|Th_1\|^2 + \|D_T h_2\|^2 + \sum_{n=3}^{\infty} \|h_n\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2 = \|(h_1, h_2, h_3, \dots)\|^2.$$

*If we identify* $\mathcal{H}$ *with* $\mathcal{H} \oplus 0 \oplus 0 \oplus \cdots \subseteq \mathcal{K}$, *then*

$$T^n = P_{\mathcal{H}} V^n\big|_{\mathcal{H}}$$

*for all* $n \in \mathbb{N}$.

By combining the examples 7.2 and 7.3, we obtain the following theorem.

**Theorem 7.4** (Sz.-Nagy)**.**

*Let* $T \in \mathcal{B}(\mathcal{H})$ *be a contraction. Then there exists a Hilbert space* $\mathcal{K} \supseteq \mathcal{H}$ *and* $U \in \mathcal{B}(\mathcal{K})$ *unitary such that*

$$T^n = P_{\mathcal{H}} U^n\big|_{\mathcal{H}}$$

*for all* $n \in \mathbb{N}$.

Sz.-Nagy theorem allows us to effectively reduce a statement about contractions to a statement about unitary operators. As an example of this approach, we prove the following corollary.

**Corollary 7.5** (von Neumann inequality)**.** *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a contraction and* $p \in \mathbb{C}[z]$ *a complex polynomial. Then we have*

$$\|p(T)\| \leq \sup\{|p(z)| \mid z \in \mathbb{T}\}.$$

*Proof.* Let $U$ be a power dilation of $T$, so $T^n = P_{\mathcal{H}} U^n\big|_{\mathcal{H}}$ for all $n \in \mathbb{N} \cup \{0\}$. Then $p(T) = P_{\mathcal{H}} p(U)\big|_{\mathcal{H}}$ and so $\|p(T)\| \leq \|p(U)\|$. Note that $U$ is normal, so by the spectral theorem, we have

$$\|p(U)\| = \sup\{|p(\lambda)| \mid \lambda \in \sigma(U)\}.$$

But since $U$ is unitary, we have $\sigma(U) \subseteq \mathbb{T}$ and so

$$\sup\{|p(\lambda)| \mid \lambda \in \sigma(U)\} \leq \sup\{|p(\lambda)| \mid \lambda \in \mathbb{T}\}.$$

By combining all of this, we get

$$\|p(T)\| \leq \|p(U)\| = \sup\{|p(\lambda)| \mid \lambda \in \sigma(U)\} \leq \sup\{|p(\lambda)| \mid \lambda \in \mathbb{T}\}. \qquad \square$$

## 7.2 Stinespring and Choi theorems

In general, dilations allow us to prove a statement about not-so-nice operators by focusing on the nicer ones. The machinery of completely bounded maps provides necessary and sufficient conditions for the existence of dilations.

If $A$ is a $*$-star algebra, then the set $M_n(A)$ of $n \times n$ square matrices with elements in $A$ is a $*$-algebra under regular matrix addition and multiplication, together with an involution $[a_{ij}]_{i,j}^* = [a_{ji}^*]_{i,j}$.

**Definition 7.6.** Let $A, B$ be $*$-algebras and $\varphi : A \to B$ a linear map. For any $n \in \mathbb{N}$, we define the $n$-th *ampliation* of $\varphi$ as

$$\varphi^{(n)} : M_n(A) \to M_n(B), \quad [a_{ij}]_{i,j} \mapsto [\varphi(a_{ij})]_{i,j}.$$

Let $A$ be a $C^*$-algebra. By GNS, there exists a Hilbert space $\mathcal{H}$ and a faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$, which induces an injective $*$-homomorphism

$$\pi^{(n)} : M_n(A) \to M_n(\mathcal{B}(\mathcal{H}))$$

for every $n \in \mathbb{N}$. Recall that $M_n(\mathcal{B}(\mathcal{H}))$ is isomorphic as a $*$-algebra to $\mathcal{B}(\mathcal{H}^n)$, which induces a (unique) norm on $M_n(\mathcal{B}(\mathcal{H}))$ that makes it a $C^*$-algebra (see also: proof of the bicommutant theorem). Therefore, we can identify $M_n(\mathcal{A})$ as a $*$-subalgebra of a $C^*$-algebra $M_n(\mathcal{B}(\mathcal{H}))$. Furthermore, it is trivial to show that the image of $\pi^{(n)}$ is closed in $M_n(\mathcal{B}(\mathcal{H}))$, so $M_n(A)$ is a closed $*$-subalgebra of $M_n(\mathcal{B}(\mathcal{H}))$. As a result, $M_n(A)$ is itself a $C^*$-algebra. Since every $*$-algebra admits at most one norm that makes it into a $C^*$-algebra, our norm on $M_n(A)$ is completely independent on the chosen GNS representation $\pi$. We see that every $C^*$-algebra $A$ carries along this extra "baggage" of canonically defined norms on $M_n(A)$. Keeping track of how this extra structure behaves yields additional information about the original $C^*$-algebra $A$.

**Definition 7.7.** Let $A, B$ be $C^*$-algebras and $\varphi : A \to B$ a linear map.
(1.) $\varphi$ is *positive* if $\varphi(A_+) \subseteq B_+$.
(2.) $\varphi$ is *$n$-positive* if $\varphi^{(n)}$ is positive.
(3.) $\varphi$ is *completely positive* (cp) if it is $n$-positive for all $n \in \mathbb{N}$.

**Lemma 7.8.** *Every positive linear functional $\varphi$ on a $C^*$-algebra $A$ is cp.*

*Proof.* Take any $n \in \mathbb{N}$. We attempt to show that the map

$$\varphi^{(n)} : M_n(A) \to M_n(\mathbb{C})$$

is positive. Take any $[a_{ij}]_{i,j} \in M_n(A)_+$ and $\alpha \in \mathbb{C}^n$. Since $[a_{ij}]_{i,j} \geq 0$, we have

$$\begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n & 0 & \cdots & 0 \end{bmatrix}^* \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \sum_{i,j=1}^n \overline{\alpha_i}\alpha_j a_{ij} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \geq 0$$

and so $\sum_{i,j=1}^n \overline{\alpha_i}\alpha_j a_{ij} \geq 0$ in $A$. By positivity of $\varphi$, we get

$$\begin{aligned} \langle \varphi^{(n)}([a_{ij}]_{i,j})\alpha, \alpha\rangle &= \langle [\varphi(a_{ij})]_{i,j}\alpha, \alpha\rangle \\ &= \left\langle \begin{bmatrix} \sum_{j=1}^n \varphi(a_{1j})\alpha_j \\ \vdots \\ \sum_{j=1}^n \varphi(a_{nj})\alpha_j \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \right\rangle \\ &= \sum_{i,j} \overline{\alpha_i}\alpha_j \varphi(a_{ij}) \\ &= \varphi\left( \sum_{i,j} \overline{\alpha_i}\alpha_j a_{ij} \right) \geq 0, \end{aligned}$$

which implies that $\varphi^{(n)}([a_{ij}]_{i,j}) \geq 0$. $\square$

**Lemma 7.9.** *If $\varphi : A \to B$ is positive, then it is* $*$-linear*, i.e.*

$$\varphi(a^*) = \varphi(a)^*, \quad \forall a \in A.$$

*Proof.* Obviously, this lemma holds for $a \in A_+$. Let us first prove the statement for $a \in A_{\text{sa}}$. We know from continuous functional calculus that $a = a_+ - a_-$ for some $a_+, a_- \in A_+$, so we have

$$\begin{aligned} \varphi(a)^* &= \varphi(a_+ - a_-)^* \\ &= \varphi(a_+)^* - \varphi(a_-)^* \\ &= \varphi(a_+) - \varphi(a_-) \\ &= \varphi(a_+ - a_-) \\ &= \varphi(a) = \varphi(a^*). \end{aligned}$$

Now, for a general $a \in A$, we have

$$\begin{aligned} \varphi(a)^* &= \varphi(\operatorname{Re} a + i \cdot \operatorname{Im} a)^* \\ &= \varphi(\operatorname{Re} a)^* - i\varphi(\operatorname{Im} a)^* \\ &= \varphi(\operatorname{Re} a) - i\varphi(\operatorname{Im} a) \\ &= \varphi(\operatorname{Re} a - i \cdot \operatorname{Im} a) = \varphi(a^*). \end{aligned}$$ $\square$

**Example 7.10.** *Every $*$-homomorphism $\varphi : A \to B$ is positive. Furthermore, for each $n \in \mathbb{N}$ the map $\varphi^{(n)} : M_n(A) \to M_n(B)$ is a $*$-homomorphism (and therefore positive as well). As a result, every $*$-homomorphism between $C^*$-algebras is cp.*

**Example 7.11.** *Let us construct a positive map that is not cp. Define*

$$\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C}), \quad A \mapsto A^\top.$$

*This map is positive, however it is not cp. Indeed, the 2-nd ampliation*

$$\varphi^{(2)} : M_4(\mathbb{C}) \to M_4(\mathbb{C}), \quad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \mapsto \begin{bmatrix} A_{11}^\top & A_{12}^\top \\ A_{21}^\top & A_{22}^\top \end{bmatrix}$$

*maps a positive matrix*

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^* \geq 0$$

*into a matrix*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

*which has a negative eigenvalue $-1$ (and hence cannot be positive).*

**Example 7.12.** *Let $\psi : A \to B$ be cp and $b \in B$. Define*

$$\varphi : A \to B, \quad a \mapsto b^*\psi(a)b.$$

*Then $\varphi$ is cp. Indeed, for any $n \in \mathbb{N}$ and $[a_{ij}]_{i,j} \in M_n(A)_+$, we have*

$$\begin{aligned} \varphi^{(n)}([a_{ij}]_{i,j}) = [\varphi(a_{ij})]_{i,j} &= \begin{bmatrix} b^*\psi(a_{11})b & \cdots & b^*\psi(a_{1n})b \\ \vdots & \ddots & \vdots \\ b^*\psi(a_{n1})b & \cdots & b^*\psi(a_{nn})b \end{bmatrix} \\ &= \begin{bmatrix} b & & \\ & \ddots & \\ & & b \end{bmatrix}^* (\psi^{(n)}[(a_{ij})]_{i,j}) \begin{bmatrix} b & & \\ & \ddots & \\ & & b \end{bmatrix} \geq 0. \end{aligned}$$

The next theorem proves that every completely positive map $\varphi : A \to \mathcal{B}(\mathcal{H})$ is of this form.

**Theorem 7.13** (Stinespring)**.**

*Let $A$ be a $C^*$-algebra and $\varphi : A \to \mathcal{B}(\mathcal{H})$ cp. Then there exists a Hilbert space $\mathcal{K}$, a bounded* *operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ such that*

$$\varphi(a) = V^* \pi(a) V, \quad \forall a \in A.$$

*Proof.* Consider the algebraic tensor product of vector spaces $A \otimes \mathcal{H}$. Define a form on this vector space by

$$\langle x \otimes \alpha, y \otimes \beta \rangle = \langle \varphi(y^* x)\alpha, \beta \rangle$$

and extend it linearly. For any $u = \sum_{j=1}^n x_j \otimes \alpha_j \in A \otimes \mathcal{H}$, we have

$$\begin{aligned}
\langle u, u \rangle &= \sum_{i,j}^n \langle \varphi(x_i^* x_j)\alpha_j, \alpha_i \rangle \\
&= \left\langle \varphi^{(n)} \left( \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^* \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \right\rangle \geq 0,
\end{aligned}$$

which means that $\langle \cdot, \cdot \rangle$ is positive definite on $A \otimes \mathcal{H}$. To each $x \in A$, we assign the map

$$\pi_0(x) : A \otimes \mathcal{H} \to A \otimes \mathcal{H}, \quad \sum_{j=1}^n x_j \otimes \alpha_j \mapsto \sum_{j=1}^n x x_j \otimes \alpha_j.$$

This map has the following property: for $u = \sum_{j=1}^n x_j \otimes \alpha_j$ and $u = \sum_{i=1}^m y_i \otimes \beta_i$, we have

$$\begin{aligned}
\langle u, \pi_0(x) v \rangle &= \langle \sum_{j=1}^n x_j \otimes \alpha_j, \sum_{i=1}^n x_j \otimes \alpha_j, \sum_{i=1}^m x y_i \otimes \beta_i \rangle \\
&= \sum_{i,j} \langle \varphi((x y_i)^* x_j)\alpha_j, \beta_i \rangle \\
&= \sum_{i,j} \langle \varphi(y_i^* x^* x_j^*)\alpha_j, \beta_i \rangle \\
&= \langle \pi_0(x^*) u, v \rangle. \qquad (7.1)
\end{aligned}$$

Define

$$\mathcal{N} := \{u \in A \otimes \mathcal{H} \mid \langle u, u \rangle = 0\}.$$

By Cauchy-Schwartz, $\mathcal{N}$ is a subspace in $A \otimes \mathcal{H}$ and $\langle \cdot, \cdots \rangle$ induces a scalar product on $A \otimes \mathcal{H} / \mathcal{N}$. Upon completion, we obtain a Hilbert space $\mathcal{K}$. For any $u \in A \otimes \mathcal{A}$, $f(x) := \langle \pi_0(x) u, u \rangle$ is a positive linear functional on $A$ by (7.1):

$$\begin{aligned}
f(x^* x) &= \langle \pi_0(x^* x) u, u \rangle \\
&= \langle \pi_0(x^*) \pi_0(x) u, u \rangle \\
&= \langle \pi_0(x) u, \pi_0(x) u \rangle \geq 0.
\end{aligned}$$

But from our discussion on states, we know that

$$\langle \pi_0(x) u, \pi_0(x) u \rangle = f(x^* x) \leq \|x^* x\| \cdot f(1) = \|x\|^2 \cdot \langle u, u \rangle.$$

As a result, we get $\pi_0(x)(\mathcal{N}) \subseteq \mathcal{N}$ and so $\pi_0(x)$ induces a bounded operator on $A \otimes \mathcal{H}/\mathcal{N}$, which can be extended to a bounded operator on $\mathcal{K}$. Therefore, $\pi_0$ induces a map (which is also a $*$-homomorphism) $\pi : A \to \mathcal{B}(\mathcal{K})$ such that

$$\pi(x)(u + \mathcal{N}) = \pi_0(x)u + \mathcal{N}.$$

Lastly, define

$$V : \mathcal{H} \to \mathcal{K}, \quad \alpha \mapsto 1 \otimes \alpha + \mathcal{N}.$$

Since

$$\|V\alpha\|^2 = \langle 1 \otimes \alpha, 1 \otimes \alpha \rangle_{\mathcal{K}} = \langle \varphi(1)\alpha, \alpha \rangle_{\mathcal{H}} \leq \|\varphi(1)\| \cdot \|\alpha\|^2,$$

$V$ is a bounded operator. For any $\alpha, \beta \in \mathcal{H}$ and $x \in A$, we have

$$\begin{aligned} \langle V\alpha, x \otimes \beta + \mathcal{N} \rangle &= \langle 1 \otimes \alpha + \mathcal{N}, x \otimes \beta + \mathcal{N} \rangle \\ &= \langle \varphi(x^*)\alpha, \beta \rangle \\ &= \langle \alpha, \varphi(x^*)^*\beta \rangle \\ &= \langle \alpha, \varphi(x)\beta \rangle, \end{aligned}$$

which directly implies that $V^*(x \otimes \beta + \mathcal{N}) = \varphi(x)\beta$. But now

$$V^*\pi(x)V\beta = V^*\pi(x)(1 \otimes \beta + \mathcal{N}) = V^*(x \otimes \beta + \mathcal{N}) = \varphi(x)\beta,$$

which concludes our proof. □

Note that if $\psi$ is unital ($\psi(I) = I$), then $V$ is an isometry and we may identify $\mathcal{H}$ with the subspace $V\mathcal{H} \leq \mathcal{K}$. Under this identification, $V^*$ becomes the projection $P_{\mathcal{H}}$ of $\mathcal{K}$ to $\mathcal{H}$. Thus, we see that

$$\varphi(a) = P_{\mathcal{H}}\pi(a)$$

*Remark.* Stinespring's theorem is a natural generalization of GNS representation of states. Indeed, if we take $\mathcal{H} = \mathbb{C}$, then $\mathcal{B}(\mathcal{C}) \cong \mathbb{C}$ and $A \otimes \mathcal{C} = A$, so the isometry $V : \mathbb{C} \to \mathcal{K}$ is determined by $V(1) = x$. Therefore, we have

$$\varphi(a) = \varphi(a)(1) \cdot 1 = V^*\pi(a)V(1) \cdot 1 = \langle \pi(a)V(1), V(1) \rangle_{\mathcal{K}} = \langle \pi(a)x, x \rangle.$$

In fact, if we take $\mathcal{H} = \mathbb{C}$, then the above proof is a proof of GNS.

**Theorem 7.14** (Choi–Kraus)**.**

*Let $\varphi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ cp. Then there exists $r \leq m \cdot n$ and $n \times m$ complex matrices $V_1, \ldots, V_r$, such that*

$$\varphi(A) = \sum_{k=1}^{r} V_k^* A V_k, \quad \forall A \in M_n(\mathbb{C}).$$

**Lemma 7.15.** *Let $A$ be a $C^*$-algebra. Then every positive element of $M_n(A)$ is a sum of $n$ positive elements of the form $[a_i^* a_j]_{i,j}$ for some $\{a_1, \ldots, a_n\} \subseteq A$.*

*Proof.* Let $R$ be the element of $M_n(A)$ whose $k$-th row is

$$\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

and whose other entries are zero, then $[a_i^* a_j]_{i,j}$, so such an element is positive. Now if $P \in M_n(A)$ is positive, then it is of the form $P = B^*B$ for $B \in M_n(A)$. Then write $B = R_1 + \cdots + R_n$, where $R_k$ is the $k$-th row of $B$ and 0 elsewhere. Now notice that $R_i^* R_j = 0$ for $i \neq j$, yielding

$$P = B^*B = (R_1^* + \cdots + R_n^*)(R_1 + \cdots + R_n) = R_1^* R_1 + \cdots + R_n^* R_n$$

and we are done. □

*Proof of Choi–Kraus.* Let $E_{ij} \in M_n(\mathbb{C})$ be the standard matrix units. First, we prove that

$$[E_{ij}]_{i,j} \in M_n(M_n(\mathbb{C})) = M_{n^2}(\mathbb{C})$$

is positive. Notice that $E_{ij} = e_i e_j^*$. Take any $x_1, \ldots, x_n \in \mathbb{C}^n$ and let $x_i = \sum_{j=1}^n \lambda_{ij} e_j$. Then

$$\begin{aligned}
\sum_{i,j} \langle E_{ij} x_j, x_i \rangle &= \sum_{i,j} \langle e_j^* x_j, e_i^* x_i \rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n \lambda_{jj} \lambda_{ii} \\
&= \left( \sum_{i=1}^n \lambda_{ii} \right) \left( \sum_{j=1}^n \lambda_{jj} \right) \geq 0.
\end{aligned}$$

Since $\varphi$ is cp, the *Choi matrix* $[\varphi(E_{ij})]_{i,j} = \varphi^{(n)}([E_{ij}]_{i,j}) \in M_{mn}(\mathbb{C})$ is positive. By the lemma 7.15, there exists $r \leq n \cdot m$ and rows $v_1, \ldots, v_r \in \mathbb{C}^{1 \times mn}$ such that

$$[\varphi(E_{ij})]_{i,j} = \sum_{k=1}^r v_k^* v_k \in M_n(M_m(\mathbb{C})) = M_{mn}(\mathbb{C}).$$

To each row

$$v_k = \begin{bmatrix} x_1^{(k)} & \cdots & x_n^{(k)} \end{bmatrix} \in \mathbb{C}^{1 \times mn}, \quad x_j^{(k)} \in \mathbb{C}^{1 \times m},$$

assign the $n \times m$ matrix

$$V_k = \begin{bmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}$$

and notice that

$$[V_k^* E_{ij} V_k]_{1 \leq i,j \leq n} = [x_i^{(k)*} x_j^{(k)}]_{i,j} = v_k^* v_k.$$

Therefore,

$$[\varphi(E_{ij})]_{i,j} = \sum_{k=1}^r [V_k^* E_{ij} V_k].$$

Now for any $A = \sum_{i,j=1}^{n} a_{ij} E_{ij} \in M_{n^2}(\mathbb{C})$, we get

$$
\begin{aligned}
\varphi(A) &= \varphi\left(\sum_{i,j=1}^{n} a_{ij} E_{ij}\right) \\
&= \sum_{i,j=1}^{n} a_{ij} \varphi\left(E_{ij}\right) \\
&= \sum_{i,j=1}^{n} a_{ij} \sum_{k=1}^{r} V_k^* E_{ij} V_k \\
&= \sum_{k=1}^{r} \sum_{i,j=1}^{n} a_{ij} V_k^* E_{ij} V_k \\
&= \sum_{k=1}^{r} V_k^* \left(\sum_{i,j=1}^{n} a_{ij} E_{ij}\right) V_k \\
&= \sum_{k=1}^{r} V_k^* A V_k.
\end{aligned}
$$

□

The following proposition gives a neat characterization of positive linear maps from the matrix space to an arbitrary $C^*$-algebra.

**Proposition 7.16.** *Let $B$ be a $C^*$-algebra and $\varphi : M_n(\mathbb{C}) \to B$ a linear map. The following statements are equivalent:*

*(1.) $\varphi$ is cp;*

*(2.) $\varphi$ is $n$-positive;*

*(3.) the Choi matrix $[\varphi(E_{ij})]_{i,j} \in M_n(B)$ is positive.*

*Proof.* We only need to prove the implication (3.) $\Rightarrow$ (1.). By GNS, we can reduce this statement to the $C^*$-algebra $B = \mathcal{B}(\mathcal{H})$. The majority of work was already done in the preceding proof, so we just follow the argument with minor adjustments. Take any positive matrix $A \in M_k(M_n(\mathbb{C})) = M_{kn}(\mathbb{C})$, which can be expressed (by lemma 7.15) as a sum of $k$ matrices of the form $[B_i^* B_j]_{1 \leq i,j \leq k}$ for $B_1, \ldots, B_k \in M_n(\mathbb{C})$. It suffices to prove that $\varphi^{(k)}([B_i^* B_j]_{i,j})$ is positive. Now let $B_l = \sum_{r,s=1}^{n} b_{r,s,l} E_{r,s}$ and

$$
B_i^* B_j = \sum_{r,s,t=1}^{n} \bar{b}_{r,s,l} b_{r,t,j} E_{r,s}.
$$

Define $y_{t,r} = \sum_{j=1}^k b_{r,t,j} x_j$ and then

$$\sum_{i,j}^{n} \langle \varphi(B_i^* B_j) x_j, x_i \rangle = \sum_{r=1}^{n} \sum_{s,t=1}^{n} \left\langle \varphi(E_{st}) \left( \sum_{i,j} \bar{b}_{r,s,i} b_{r,t,j} x_j \right), x_i \right\rangle$$
$$= \sum_{r=1}^{n} \sum_{s,t=1}^{n} \langle \varphi(E_{st}) y_{t,r}, y_{s,r} \rangle$$

is a sum of $r$ positive numbers, so it has to be positive. □

## 7.3 Arveson extension theorem

Let $M$ be a vector subspace of a $C^*$-algebra $A$. Let $\varphi : M \to M_n(\mathbb{C})$ be a linear map. Define a linear functional

$$s_\varphi : M_n(M) \to \mathbb{C}, \quad s_\varphi([a_{ij}]_{1 \le i,j \le n}) = \frac{1}{n} \sum_{i,j}^{n} \varphi(a_{i,j}).$$

Equivalently, if $e := (e_1, \ldots, e_n) \in \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{n} = \mathbb{C}^{n^2}$, then

$$s_\varphi([a_{ij}]_{i,j}) = \frac{1}{n} \langle \varphi^{(n)}([a_{ij}]_{i,j}) e, e \rangle.$$

Thus, we get a linear map

$$\mathcal{L}(M, M_n(\mathbb{C})) \to \mathcal{L}(M_n(M), \mathbb{C}), \quad \varphi \mapsto s_\varphi.$$

*Remark.* If $1 \in M$ and $\varphi(1) = 1$, then $s_\varphi(1) = 1$.

Conversely, if $s : M_n(M) \to \mathbb{C}$ is linear, then we define

$$\varphi_s : M \to M_n(\mathbb{C}), \quad \varphi_s(a)_{ij} = n \cdot s(a \otimes E_{ij}).$$

This induces a linear map

$$\mathcal{L}(M_n(M), \mathbb{C}) \to \mathcal{L}(M, M_n(\mathbb{C})), \quad s \mapsto \varphi_s,$$

which is inverse to $\varphi \mapsto s_\varphi$ as defined above.

**Definition 7.17.** Let $A$ be a $C^*$-algebra. A vector subspace $S \subseteq A$ with $S^* \subseteq S$ and $1 \in S$ is called an *operator system.*

**Theorem 7.18** (Krein–Riesz)**.**

*Let $S$ be an operator system in a $C^*$-algebra $A$ and let $\varphi_0 : S \to \mathbb{C}$ be a linear functional such that $\varphi(S \cap A_+) \subseteq [0, \infty)$. Then there exists an extension of $\varphi_0$ to a positive linear functional $\varphi : A \to \mathbb{C}$.*

*Proof.* The functional $\varphi_0$ is positive, hence $*$-linear, i.e. $\varphi_0(s^*) = \overline{\varphi_0(s)}$. This implies that is determined by $\varphi_0\big|_{S\cap A_{sa}} : S \cap A_{\text{sa}} \to \mathbb{R}$. We with to extend $\varphi_0\big|_S \cap A_{\text{sa}}$ to $\varphi : A_{\text{sa}} \to \mathbb{R}$. Similar to the proof of Hahn–Banach, it suffices to extend $\varphi_0\big|_{S\cap A_{\text{sa}}}$ to $S \cap A_{\text{sa}} + \mathbb{R} \cdot x_0$ for $x_0 \in A_{\text{sa}} \setminus S$. Define

$$C := \{y \in S \cap A_{\text{sa}} \mid y \leq x_0\}, \quad D := \{y \in S \cap A_{\text{sa}} \mid y \geq x_0\}.$$

Since $1 \in S \cap A_{\text{sa}}$, none of the above sets are empty. For each $y' \in C$ and $Y'' \in D$, we have

$$y'' - y' = \underbrace{(y'' - x_0)}_{\geq 0} + \underbrace{(x_0 - y')}_{\geq 0} \geq 0,$$

so $\varphi_0(y'') - \varphi_0(y') = \varphi_0(y'' - y') \geq 0$. Therefore, there must exist a constant $\alpha \in \mathbb{R}$ such that

$$\sup\{\varphi_0(y') \mid y' \in C\} \leq \alpha \leq \inf\{\varphi_0(y'') \mid y'' \in D\}.$$

Define $\varphi'$ on $(S \cap A_{\text{sa}}) + \mathbb{R} \cdot x_0$ as

$$y + t \cdot x_0 \mapsto \varphi_0(y) + t \cdot \alpha.$$

We have to prove that this map is positive. Let $y + tx_0 \geq 0$. If $t > 0$, then $x_0 \geq -\frac{1}{t}y$ and $-\frac{1}{t}y \in C$. But then $\varphi_0\left(-\frac{1}{t}y\right) \leq \alpha$ and $\varphi'(y + tx_0) \geq 0$. However, if $t < 0$, then $x_0 \leq -\frac{1}{t}y$ and $-\frac{1}{t}y \in D$. This implies $\varphi_0\left(-\frac{1}{t}y\right) \geq \alpha$ and again $\varphi'(y + tx_0) \geq 0$. $\square$

**Proposition 7.19.** *Let $A$ be a $C^*$-algebra and $S \subseteq A$ an operator system. Suppose that $\varphi : S \to M_n(\mathbb{C})$ is linear map. Then the following statements are equivalent:*

*(1.) $\varphi$ is cp;*

*(2.) $\varphi$ is $n$-positive;*

*(3.) $s_\varphi$ is a positive linear functional.*

*Proof.* The only nontrivial implication is (3.) $\Rightarrow$ (1.). Take $s_\varphi : M_n(S) \to \mathbb{C}$ and notice that $M_n(S)$ is an operator system in $M_n(A)$. By Krein–Riesz, we can extend $s_\varphi$ to the positive functional $s : M_n(A) \to \mathbb{C}$. Then we have $\varphi_s : A \to M_n(\mathbb{C})$ such that $s_{\varphi_s} = s$. Also, $\varphi_s$ extends $\varphi$. It suffices to show that $\varphi_s$ is cp. Take any $m \in \mathbb{N}$, let $a_1, \ldots, a_m \in A$ and $x_1, \ldots, x_m \in \mathbb{C}^m$, where $x_j = \sum_{k=1}^n \lambda_{jk} e_k$. Also, define $A_i \in M_n(\mathbb{C})$ which has the first row $\lambda_{i1}, \ldots, \lambda_{in}$ and zero everywhere else. Then

$$A_i^* A_j = \sum_{k,l=1}^n \lambda_{jk} \overline{\lambda_{il}} E_{lk}.$$

Now we have

$$
\begin{aligned}
\sum_{i,j}^{n} \langle \varphi_s(a_i^* a_j) x_j, x_i \rangle &= \sum_{i,j,k,l}^{n} \lambda_{jk} \overline{\lambda_{il}} \langle \varphi_s(a_i^* a_j) e_k, e_l \rangle \\
&= \sum_{i,j,k,l}^{n} \lambda_{jk} \overline{\lambda_{il}} s(a_i^* a_j \otimes E_{lk}) \\
&= \sum_{i,j}^{n} s(a_i^* a_j \otimes A_i^* A_j) \\
&= s\left( (\sum_i a_i \otimes A_i)^* (\sum_j a_j \otimes A_j) \right) \geq 0. \qquad \square
\end{aligned}
$$

**Corollary 7.20.** *Let $A$ be a $C^*$-algebra, $S \leq A$ an operator system and $\varphi : S \to M_n(\mathbb{C})$ cp. Then there exists a cp map $\psi : A \to M_n(\mathbb{C})$ that extends $\varphi$.*

Let $X, Y$ be Banach spaces. We wish to introduce a weak-$*$ topology on $\mathcal{B}(X, Y^*)$, so we need to find a Banach space $Z$ such that $Z^*$ is isomorphic to $\mathcal{B}(X, Y^*)$. For any $x \in X$ and $y \in Y$, define a linear functional on $\mathcal{B}(X, Y^*)$ by

$$x \otimes y(L) := L(x)(y).$$

Notice that

$$|x \otimes y(L)| \leq \|L\| \cdot \|x\| \cdot \|y\|,$$

which implies that $\|x \otimes y\| \leq \|x\| \cdot \|y\|$ and $x \otimes y \in \mathcal{B}(X, Y^*)^*$. Furthermore, we have $\|x \otimes y\| = \|x\| \cdot \|y\|$. To see this, use Hahn–Banach to obtain a linear functional $\varphi_x \in X^*$ such that $\varphi_x(x) = \|x\|$ and $\|\varphi_x\| = 1$. Similarly, define $\varphi_y \in Y^*$ such that $\varphi_y(y) = \|y\|$ and $\|\varphi_y\| = 1$. Now define $L_{x,y} \in \mathcal{B}(X, Y^*)$ by $L_{x,y}(\cdot) = \varphi_x(\cdot)\varphi_y$ and notice that $\|L_{x,y}\| = 1$. Finally,

$$x \otimes y(L_{x,y}) = \|x\| \cdot \|y\| = \|x\| \cdot \|y\| \cdot \|L_{x,y}\|$$

and the maximum is obtained, hence $\|x \otimes y\| = \|x\| \cdot \|y\|$. Define the space

$$Z := \overline{\operatorname{span}\{x \otimes y \mid x \in X, y \in Y\}} \leq \mathcal{B}(X, Y^*)^*.$$

**Lemma 7.21.** *$\mathcal{B}(X, Y^*)$ is isometrically isomorphic to $Z^*$ via the map*

$$\Phi : \mathcal{B}(X, Y^*) \to Z^*, \quad L \mapsto (x \otimes y \mapsto x \otimes y(L)).$$

*Proof.* Firstly, we prove that $\Phi$ is an isometry. Take any $L \in \mathcal{B}(X, Y^*)$. For any $x \otimes y \in Z$, we get

$$\begin{aligned}
|\Phi(L)(x\otimes y)| &= |x\otimes y(L)| \\
&= |L(x)(y)| \\
&\leq \|L(x)\|\cdot\|y\| \\
&\leq \|L\|\cdot\|x\|\cdot\|y\| \\
&= \|L\|\cdot\|x\otimes y\|,
\end{aligned}$$

which shows that $\|\Phi(L)\| \leq \|L\|$. But on the other hand, we have

$$\begin{aligned}
\|L\| &= \sup_{\|x\|=1} \|L(x)\| \\
&= \sup_{\|x\|=1}\sup_{\|y\|=1} |L(x)(y)| \\
&= \sup_{\|x\|=1}\sup_{\|y\|=1} |x\otimes y(L)| \\
&= \sup_{\|x\|=1}\sup_{\|y\|=1} |\Phi(L)(x\otimes y)| \\
&\leq \sup_{\|x\otimes y\|=1} |\Phi(L)(x\otimes y)\| = \|\Phi(L)\|.
\end{aligned}$$

Secondly, we prove surjectivity. Take any $f \in Z^*$. For all $x \in X$, we define

$$f_x : Y \to \mathbb{C}, \quad y \mapsto f(x\otimes y).$$

Since $|f_x(y)| \leq \|f\|\cdot\|x\|\cdot\|y\|$, this is a bounded functional and so $f_x \in Y^*$. Now define

$$L : X \to Y^*, \quad L(x) = f_x.$$

This is a bounded map and $\Phi(L) = f$. $\square$

By identifying $\mathcal{B}(X, Y^*)$ with $Z^*$, we can endow the former space with the weak-$*$ topology. This topology is called the *bounded weak (BW) topology*.

**Lemma 7.22.** *Let $(L_\lambda)_\lambda$ be a bounded net in $\mathcal{B}(X, Y^*)$. Then $L_\lambda \xrightarrow{BW} L$ iff $L_\lambda(x) \xrightarrow{w^*} L(x)$ for all $x \in X$.*

*Proof.* First, we prove ($\Rightarrow$). If $L_\lambda \to L$ in BW topology, then

$$L_\lambda(x)(y) = \Phi(L_\lambda)(x\otimes y) \to \Phi(L)(x\otimes y) = L(x)(y)$$

for all $x \in X$ and $y \in Y$, so $L_\lambda(x) \to L(x)$ in the weak-$*$ topology on $Y^*$. Conversely ($\Leftarrow$), if $L_\lambda(x) \xrightarrow{w^*} L(x)$ for all $x \in X$, then

$$\Phi(L_\lambda)(x\otimes y) = L_\lambda(x)(y) \to L(x)(y) = \Phi(L)(x\otimes y)$$

for all $x\otimes y$. Hence $\Phi(L_\lambda)(z) \to \Phi(L)(z)$ for $z$ in the linear span of $x\otimes y$, which is (by definition) a dense subset of $Z$. But since $(L_\lambda)_\lambda$ is norm bounded, so is $(\Phi(L_\lambda))_\lambda$ and therefore $\Phi(L_\lambda)(z) \to \Phi(L)(z)$ for all $z \in Z$. $\square$

Since $\mathcal{B}(\mathcal{H})$ is a dual of $L^1(\mathcal{B}(\mathcal{H}))$, we can equip $\mathcal{B}(X, \mathcal{B}(\mathcal{H}))$ with a BW topology for any Banach space $X$. Now if $(L_\lambda)_\lambda \subseteq \mathcal{B}(X, Y^*)$ is a bounded net, then $L_\lambda \to L$ in a BW topology iff for all $x \in X$ and $h, k \in \mathcal{H}$, we have $\langle L_\lambda(x)h, k\rangle \to \langle L(x)h, k\rangle$.

**Lemma 7.23.** *Let $A$ be a $C^*$-algebra and $S \subseteq A$ a closed operator system. Then*

$$CP_r(S, \mathcal{B}(\mathcal{H})) := \{L \in \mathcal{B}(S, \mathcal{B}(\mathcal{H})) \mid L \text{ cp and } \|L\| \leq r\}$$

*is a compact set in BW topology.*

*Proof.* We know that

$$B_r(S, \mathcal{B}(\mathcal{H})) := \{L \in \mathcal{B}(S, \mathcal{B}(\mathcal{H})) \mid \|L\| \leq r\}$$

is a compact set in BW by Banach–Alaoglu. We have to show that $CP_r$ is closed in $B_r$. Let $(L_\lambda)_\lambda$ be a net in $CP_r$ that converges to some $L \in L \in B_r$. We have to show that $L$ is also cp. Fix $n \in \mathbb{N}$ and take any positive $[a_{ij}]_{i,j} \in M_n(S)$ and $x_1, \ldots, x_n \in \mathcal{H}$. Then

$$\sum_{i,j}^{n} \langle L_\lambda^{(n)}([a_{ij}]_{i,j})x_j, x_i\rangle \to \sum_{i,j}^{n} \langle L^{(n)}([a_{ij}]_{i,j})x_j, x_i\rangle$$

and since the sum on the left is positive for every $\lambda$, so too must be the sum on the right. □

**Theorem 7.24** (Arveson extension theorem)**.**

*Let $A$ be a $C^*$-algebra, $S \subseteq A$ an operator system and $\varphi : S \to \mathcal{B}(\mathcal{H})$ a cp map. Then there exists a cp map $\psi : A \to \mathcal{B}(\mathcal{H})$ that is an extension of $\varphi$.*

*Proof.* W.l.o.g. assume that $S$ is closed. Let $\mathcal{F} \leq \mathcal{H}$ be a finite-dimensional subspace. Define

$$\varphi_{\mathcal{F}} : S \to \mathcal{B}(\mathcal{F}), \quad a \mapsto P_{\mathcal{F}}\varphi(a)\big|_{\mathcal{F}},$$

where $P_{\mathcal{F}} : \mathcal{H} \to \mathcal{F}$ is a projection. Since $\dim(\mathcal{F}) = n < \infty$, then $\mathcal{B}(\mathcal{F}) = M_n(\mathbb{C})$. By corollary 7.20, there exists a cp map $\psi_{\mathcal{F}}$ that is an extension of $\varphi_{\mathcal{F}}$. Define a map $\psi'_{\mathcal{F}}$ such that $\psi'_{\mathcal{F}} = \psi_{\mathcal{F}}$ on $\mathcal{F}$ and zero on $\mathcal{F}^\perp$. It's trivial to see that this map is also cp. Since $\{\mathcal{F} \leq \mathcal{H} \mid \dim \mathcal{F} < \infty\}$ is a directed set, $(\psi'_{\mathcal{F}})_{\mathcal{F}}$ is a net in $PP_{\|\varphi\|}(A, \mathcal{B}(\mathcal{H}))$. By the lemma 7.23, there exists a subnet that converges to $\psi \in PP_{\|\varphi\|}(A, \mathcal{B}(\mathcal{H}))$. Now we just have to prove that $\psi$ is the desired extension. Take any $a \in S$ and $x, y \in \mathcal{H}$. Define $\mathcal{F} := \operatorname{span}\{x, y\}$. Now for all finite-dimensional $\mathcal{F}_1 \supseteq \mathcal{F}$, we have

$$\langle \varphi(a)x, y\rangle = \langle \psi'_{\mathcal{F}_1}(a)x, y\rangle.$$

But there exists a subnet such that $\langle \psi'_{\mathcal{F}_1}(a)x, y\rangle \to \langle \psi(a)x, y\rangle$. Therefore, we have $\langle \varphi(a)x, y\rangle = \langle \psi(a)x, y\rangle$ for all $x, y \in \mathcal{H}$. Hence, $\varphi(a) = \psi(a)$ for all $a \in S$. □

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