FIBER BUNDLES - NOTES

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1 Fiber bundles

All manifolds are considered smooth (complex).

Definition 1.1. A fiber bundle is a quadruple (E, M, F, π), where (E, M, F) are smooth (complex) manifolds and $\pi : E \to M$ is a surjective submersion such that for all $x \in M$, there exists an open neighborhood $x \in U \subseteq M$ and a diffeomorphism (biholomorphism)

$$\varphi : \pi^{-1}(U) \to U \times F,$$

such that $\pi = \pi_1 \circ \varphi$.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

Definition 1.2. Let G be a Lie group. A fiber bundle (E, M, F, π) is a fiber bundle with structure group G if:

- G acts faithfully on F;
- there exists an open covering $\{U_\alpha\}$ of M such that for every index α, there exists a diffeomorphism (biholomorphism)
$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U \times F,$$
such that for $U_\alpha \cap U_\beta = \emptyset$, there exists a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying
$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, f) = (x, g_{\alpha\beta}(x) f), \quad \forall x \in U_\alpha \cap U_\beta, \ \forall f \in F.$$

The covering $\{U_\alpha\}$ is called the trivializing atlas, while $\{g_{\alpha\beta}\}$ are the local transition functions.

Transition functions satisfy the so-called cocycle conditions.

- $g_{\alpha\alpha}(x) = \mathrm{id}_F, \ \forall x \in U_\alpha$.
- $g_{\beta\alpha}(x) \cdot g_{\alpha\beta}(x) = \mathrm{id}_F, \ \forall x \in U_\alpha \cap U_\beta$.
- $g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = \mathrm{id}_F, \ \forall x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Definition 1.3. Let M, M' be manifolds and $\pi_E : E \to M$, $\pi_{E'} : E' \to M'$ fiber bundles. A morphism of bundles is a pair (f,φ) such that $f : M \to M'$, $\varphi : E \to E'$ and $\pi_{E'} \circ \varphi = f \circ \pi_E$.

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi_E"'] & E' \arrow[d, "\pi_{E'}"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

If f, φ are both diffeomorphisms (biholomorphisms), the bundles are equivalent.

Definition 1.4. Let $U \subseteq M$ open. The bundle E is trivial over U if $E\big|_U := \pi^{-1}(U)$ is equivalent to $U \times F$.

Proposition 1.5. *Let (E, M, F, π_E) be a bundle with trivializing atlas $\{(U_\alpha, \varphi^E_\alpha)\}$ and transition functions $\{g_{\alpha\beta}\}$ and $(E', M, F', \pi_{E'})$ be a bundle with trivializing atlas $\{(U_\alpha, \varphi^{E'}_\alpha)\}$ and transition functions $\{h_{\alpha\beta}\}$. Take any map $\psi : E \to E'$. For any α, denote*

$$\psi_\alpha := \varphi^{E'}_\alpha \circ \psi \circ (\varphi^E_\alpha)^{-1} = (\psi'_\alpha, \psi''_\alpha) : U_\alpha \times F \to U_\alpha \times F'.$$

If (id, ψ) is an equivalence, then:

1. $\psi'_\alpha = \mathrm{id}$;
2. $\psi_\alpha(x, \cdot) : \{x\} \times F \to \{x\} \times F'$ *is a diffeomorphism (biholomorphism);*
3. $\psi''_\beta(x,t) = h_{\beta\alpha}(x) \cdot \psi''_\alpha(x, g_{\alpha\beta}(x) \cdot t)$ *for any* $U_\alpha \cap U_\beta \neq \emptyset$.

Conversely, if there exists a family of smooth maps

$$\psi_\alpha = (\psi'_\alpha, \psi''_\alpha) : U_\alpha \times F \to U_\alpha \times F'$$

that satisfy the above three properties, then there exists a bundle equivalence (id, ψ) such that $\psi_\alpha = \varphi^{E'}_\alpha \circ \psi \circ (\varphi^E_\alpha)^{-1}$.

Proof. Let us first prove the right implication ($\Rightarrow$). From the commuting diagram

$$\begin{tikzcd} U_\alpha \times F \arrow[r, "(\varphi^E_\alpha)^{-1}"] \arrow[dr] & \pi_E^{-1}(U_\alpha) \arrow[r, "\psi"] \arrow[d, "\pi_E"'] & \pi_{E'}^{-1}(U_\alpha) \arrow[r, "\varphi^{E'}_\alpha"] \arrow[d, "\pi_{E'}"] & U_\alpha \times F' \arrow[dl] \\ & U_\alpha \arrow[r, "\mathrm{id}"'] & U_\alpha & \end{tikzcd}$$

we get the first item. Since (id, ψ) is an equivalence, ψ is a diffeomorphism and so is ψ_α for any α. The second item then follows. Finally, for $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$\begin{aligned} \psi_\beta &= \varphi^{E'}_\beta \circ \psi \circ (\varphi^E_\beta)^{-1} \\ &= \varphi^{E'}_\beta \circ (\varphi^{E'}_\alpha)^{-1} \circ \varphi^{E'}_\alpha \circ \psi \circ (\varphi^E_\alpha)^{-1} \circ \varphi^E_\alpha \circ (\varphi^E_\alpha)^{-1} \\ &= (\mathrm{id}, h_{\beta\alpha}) \circ \psi_\alpha \circ (\mathrm{id}, g_{\alpha\beta}). \end{aligned}$$

Now the converse direction ($\Leftarrow$). We define $\psi : E \to E'$ $\psi := (\varphi^{E'}_\alpha)^{-1} \circ \psi_\alpha \circ \varphi^E_\alpha$ on the open covering $\{\pi^{-1}(U_\alpha)\}$. This is well-defined because for $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$, we have the third item:

$$\varphi^{E'}_\beta \circ (\varphi^{E'}_\alpha)^{-1} \circ \psi_\alpha \circ \varphi^E_\alpha \circ (\varphi^E_\alpha)^{-1} = (\mathrm{id}, h_{\beta\alpha}) \circ \psi_\alpha \circ (\mathrm{id}, g_{\alpha\beta}) = (\mathrm{id}, \psi''_\beta) = \psi_\beta.$$

So we have proved that $\psi : E \to E'$ is a well-defined smooth map. Now since we have the first two items, ψ_α's are diffeomorphisms, so we can define the smooth inverse $\psi^{-1} : E' \to E$ in a similar manner. This means that ψ is a diffeomorphism. Finally, becuase of the diagram

$$\begin{tikzcd}
\pi_E^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha^E"] \arrow[dr, "\pi_E"'] & U_\alpha \times F \arrow[r, "\psi_\alpha"] \arrow[d] & U_\alpha \times F' \arrow[r, "(\varphi_\alpha^{E'})^{-1}"] \arrow[d] & \pi_{E'}^{-1}(U_\alpha) \arrow[dl, "\pi_{E'}"] \\
& U_\alpha \arrow[r, "\mathrm{id}"'] & U_\alpha &
\end{tikzcd}$$

we have the equivalence (id, ψ). $\square$

Up to an equivalence, a fiber bundle is defined by its transition functions.

Theorem 1.6.

Let M, F be manifolds and G a Lie group acting faithfully on F. Let $\{U_\alpha\}$ be an open cover of M with maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying the cocycle conditions. Then there exists a unique (up to equivalence) bundle E with base M, fiber F, structure group G and transition functions $\{g_{\alpha\beta}\}$.

Proof. Define $E := \bigsqcup_\alpha U_\alpha \times F /\!\sim$, where

$$(x, f) \sim (y, f') \Leftrightarrow x = y \text{ and } \exists \alpha, \beta : \ x \in U_\alpha, y \in U_\beta, \ f = g_{\alpha\beta(x)} f'.$$

This is an equivalence relation due to the cocycle conditions and E is a topological manifold. Next, we have to show that it is also a smooth one. Up to a refining, assume $\{(U_\alpha, \psi_\alpha)\}$ are local charts for M. Let $\{(W_j, \theta_j)\}$ be an atlas for F. Then $\{[U_\alpha \times W_j]\}_{\alpha,j}$ is an open cover of E. Define

$$\widetilde{\varphi}_{\alpha,j} : [U_\alpha \times W_j] \to \psi_\alpha(U_\alpha) \times \theta_j(W_j), \quad [x, f] \mapsto (\psi_\alpha(x), \theta_j(f)).$$

This is a homeomorphism by the same arguments as above. We need to see that $\widetilde{\varphi}_{\alpha,j} \circ (\widetilde{\varphi}_{\beta,k})^{-1}$ is smooth. Let $(p, t) \in \psi_\beta(U_\alpha \cap U_\beta) \times \theta_k(W_j \cap W_k)$. Then

$$\begin{aligned}
\widetilde{\varphi}_{\alpha,j} \circ (\widetilde{\varphi}_{\beta,k})^{-1}(p, t) &= \widetilde{\varphi}_{\alpha,j}([\psi_\beta^{-1}(p), \theta_k^{-1}(t)]) \\
&= \widetilde{\varphi}_{\alpha,j}([\psi_\beta^{-1}(p), g_{\alpha\beta}(\psi_\beta^{-1}(p)) \cdot \theta_k^{-1}(t)]) \\
&= (\psi_\alpha \circ \psi_\beta^{-1}(p), \theta_j(g_{\alpha\beta}(\psi_\beta^{-1}(p)) \cdot \theta_k^{-1}(t))).
\end{aligned}$$

The second component is smooth w.r.t. $t \in \theta_k(W_k)$ because the action of $g_{\alpha\beta}$ is smooth. It is also smooth w.r.t. $p \in \psi_\beta(U_\beta)$ because ψ_β^{-1} is smooth and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is smooth. Define

$$\pi : E \to M, \quad \pi([x, f]) = x$$

and this is a well-defined smooth submersion. Next, define

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F, \quad [\widetilde{x}, \widetilde{f}] \mapsto (x, f),$$

where $(x, f) \in U_\alpha \times F \subseteq \bigsqcup_\alpha U_\alpha \times F$ is a unique representative of $[\widetilde{x}, \widetilde{f}]$ in $U_\alpha \times F$. Then φ_α is bijective with inverse $\rho\big|_{U_\alpha \times F}$, where $\rho : \bigsqcup_\alpha U_\alpha \times F \to E$ is simply the quotient map. So $\rho\big|_{U_\alpha \times F}$ is continuous and φ_α is a homeomorphism. It's easy to show that it is even a diffeomorphism. Next, let $U_\alpha \cap U_\beta \neq \emptyset$. Then

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, f) = \varphi_\alpha([x, f]_\beta) = \varphi_\alpha([x, g_{\alpha\beta}(x) f]\alpha) = (x, g_{\alpha\beta}(x) f),$$

so $\{\varphi_\alpha\}$ is a trivializing atlas for E. Finally, suppose that $\pi' : E' \to M$ is another such bundle. Then the family of identity maps $U_\alpha \times F \to U_\alpha \times F$ satisfy the properties from the earlier proposition, so they induce a bundle equivalence between E and E'. $\square$

1.1 Vector and principal bundles

Definition 1.7. A bundle $(E, M, \mathbb{R}^k, \pi)$ with structure group $GL_k(\mathbb{R})$ is a vector bundle of rank k if there exists a trivializing atlas $\{(U_\alpha, \varphi_\alpha)\}$ for E such that for all $x \in U_\alpha$,

$$\varphi_\alpha\big|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k$$

is a vector space isomorphism.

The previous theorem produces local trivializations which are actually linear on the fibers, so it effectively gives us the fiber in the following lemma.

Lemma 1.8. *Let M be a manifold, E bundle with fibers $\mathbb{R}^k$ and structure group $GL_k(\mathbb{R})$. Then there exists a vector bundle over M which is equivalent to E as a bundle.*

Definition 1.9. Let M, M' be manifolds and E, E' vector bundles over M, M', respectively. A bundle morphism (f, φ) is a vector bundle morphism if for all $x \in M$, the maps

$$\varphi_x := \varphi\big|_{E_x} : E_x \to S'_{f(x)}$$

is a vector space morphism (i.e. it is linear).

Definition 1.10. A Lie group G acts on right on a manifold F if $R : G \to \mathrm{Diff}(F)$ is a group homomorphism such that

$$R(e) = \mathrm{id}, \quad R(g^{-1}) = (R(g))^{-1}, \quad R(gh) = R(h) \circ R(g).$$

Observe that if $L : G \to \mathrm{Diff}$ is a left action, then $R(g) := L(g^{-1})$ is a right action.

Example 1.11. *If $\theta : F \to G$ is a diffeomorphism, then we have a right action*

$$R_g(f) = \theta^{-1}(\theta(f) \cdot g) =: f \cdot g$$

for $f \in F$, $g \in G$.

Definition 1.12. G Lie group, a bundle (P, M, G, π) with structure group G (so fiber and structure group are the same) is a principal bundle if there exists a trivializing atlas $\{(U_\alpha, \varphi_\alpha)\}$ for P such that for all $x \in U_\alpha$, $\varphi_\alpha\big|_{P_x} : P_x \to \{x\} \times G$ is right G-equivariant: this means that if

$$\varphi_\alpha(v) = (x, \varphi''_\alpha(x, v))$$

for $v \in P_x$, then

$$\varphi''_\alpha(x, vg) = \varphi''_\alpha(x, v) \cdot g$$

for every $g \in G$.

Here, the right action of G on the bundles is the one described in fhe last example. Right G-equivariance tells us that no matter what the choice of the diffeomorphism $\varphi_\alpha\big|_{E_x} : E_x \to G$ is, we have essentially the same right action of G on E_x: for $x \in U_\alpha \cap U_\beta$ and $p \in P_x$, we get

$$p \cdot g := \varphi_\alpha^{-1}(\varphi_\alpha(p) \cdot g) = \varphi_\beta^{-1}(\varphi_\beta(p) \cdot g).$$

Also, observe that the right action commutes with the action, given by the fact that G is the structure group.

Definition 1.13. Let M, M' be manifolds, P, P' principal bundles over M, M' with structure group G, G'. Let $\rho : G \to G'$ be a Lie group morphism. A bundle group morphism (f, φ) is a principal bundle ρ-morphism if $\varphi_x(pg) = \varphi_x(p)\rho(g)$ for $p \in P_x$ and $g \in G$.

Lemma 1.14. *Let M be a manifold, P bundle over M with fiber G and structure group G. Then there exists a principal bundle P' on M which is equivalent to P as a bundle.*

Definition 1.15. Let G be a Lie group and $H \subseteq G$ a Lie subgroup, $\rho : M \hookrightarrow G$ an immersion. Let P be a principal bundle over M with fiber G and P' a principal bundle over M with fiber H. Then P' is a reduction of P if there exists a ρ-morphism of principal bundles (id, h) such that $h : P' \to P$ is injective.

Proposition 1.16. *Let P be a principal bundle with structure group G on M and $H \subseteq G$ a Lie subgroup. Then we can reduce G to H iff P is isomorphic to a G-principal bundle $\widetilde{P}$, which has transition functions in H.*

Proof. ($\Leftarrow$) Let $\{U_\alpha\}$ be a trivialization atlas for $\widetilde{P}$ (take a refinement if needed). and $h_{\alpha\beta} : U_\alpha \cap U_\beta \to H \subseteq G$ its transition maps. Now every G-principal bundle with this trivialization is isomorphic (as a principal bundle) to $\sqcup_\alpha U_\alpha \times G/\!\sim$, so WLOG we may take $\widetilde{P} = \sqcup_\alpha U_\alpha \times G/\!\sim$. Next, define a H-principal bundle $P' := \sqcup_\alpha U_\alpha \times H/\!\sim$ on M . Define the map

$$i_\alpha : U_\alpha \times H \to U_\alpha \times G, \quad (x, h) \mapsto (x, h).$$

This map passes to the quotients $P' \to P$ and the induced map is a morphism of principal bundles. Since $i : H \hookrightarrow G$ is injective, so is the resulting morphism of principal bundles. ($\Rightarrow$) Assume $h : P' \to P$ is a reduction and $\{U_\alpha\}$ is a trivializing atlas for P, P' (take a refinement, if needed). Let $\varphi'_\alpha : P'\big|_{U_\alpha} \to U_\alpha \times H$ be a local trivialization for P' with transition maps $h_{\alpha\beta}$ in H. For $p' \in P'_x$, let

$$\varphi'_\alpha(p') = (x, \widetilde{\varphi_\alpha}'(x, p')).$$

Similarly, let $\psi_\alpha : P\big|_{U_\alpha} \to U_\alpha \times H$ be a local trivialization for P with transition maps $g_{\alpha\beta}$ in G and

$$\psi_\alpha(p) = (x, \widetilde{\psi_\alpha}(x, p)).$$

For each α and $x \in U_\alpha$, let $e_{x,\alpha} = \widetilde{\psi_\alpha} \circ h \circ (\widetilde{\varphi_\alpha}')^{-1}(e)$ Now we define new maps $\chi_\alpha : P\big|_{U_\alpha} \to U_\alpha \times G$ such that for $p \in P_x$, we get

$$\chi_\alpha(p) = (x, \widetilde{\chi_\alpha}(x, p)), \quad \widetilde{\chi_\alpha}(x, p) = e_{x,\alpha}^{-1} \cdot \widetilde{\psi_\alpha}(x, p).$$

We need to prove that this family of maps is a trivializing atlas on P. First, we prove that they are G-equivariant. For any $p \in P_x$, we get

$$\widetilde{\chi_\alpha}(p \cdot g) = e_{x,\alpha}^{-1} \cdot \widetilde{\psi_\alpha}(x, p \cdot g) = e_{x,\alpha}^{-1} \cdot \widetilde{\psi_\alpha}(x, p) \cdot g = \widetilde{\chi_\alpha}(p) \cdot g.$$

Now take any $t \in G$. Then for $x \in U_\alpha \cap U_\beta$, we have

$$\begin{aligned}
\widetilde{\chi_\alpha} \circ \widetilde{\chi_\beta}^{-1}(x, t) &= \widetilde{\chi_\alpha} \circ \widetilde{\chi_\beta}^{-1}(x, e_{x,\beta}^{-1} \cdot (e_{x,\beta} \cdot t)) \\
&= \widetilde{\chi_\alpha} \circ \widetilde{\chi_\beta}^{-1}(\widetilde{\chi_\beta}(x, \widetilde{\psi_\beta}^{-1}(e_{x,\beta} \cdot t))) \\
&= \widetilde{\chi_\alpha}(x, \widetilde{\psi_\beta}^{-1}(e_{x,\beta} \cdot t)) \\
&= \widetilde{\chi_\alpha}(x, \widetilde{\psi_\beta}^{-1}(e_{x,\beta}) \cdot t) \\
&= \widetilde{\chi_\alpha}(x, \widetilde{\psi_\beta}^{-1}(e_{x,\beta})) \cdot t \\
&= e_{x,\alpha}^{-1} \cdot \widetilde{\psi_\alpha}(x, \widetilde{\psi_\beta}^{-1}(e_{x,\beta})) \cdot t \\
&= (e_{x,\alpha}^{-1} \cdot g_{\alpha\beta}(x) \cdot e_{x,\beta}) \cdot t.
\end{aligned}$$

But now, notice that

$$\begin{aligned}
g_{\alpha\beta}(x) \cdot e_{x,\beta} &= \widetilde{\psi_\alpha}(x, \widetilde{\psi_\beta}^{-1}(e_{x,\beta})) \\
&= \widetilde{\psi_\alpha} \circ h \circ (\widetilde{\varphi_\beta}')^{-1}(e) \\
&= \widetilde{\psi_\alpha} \circ h \circ (\widetilde{\varphi_\alpha}')^{-1} \circ \widetilde{\varphi_\alpha}' \circ (\widetilde{\varphi_\beta}')^{-1}(e) \\
&= \widetilde{\psi_\alpha} \circ h \circ (\widetilde{\varphi_\alpha}')^{-1}(h_{\alpha\beta}(x) \cdot e) \\
&= \widetilde{\psi_\alpha} \circ h \circ (\widetilde{\varphi_\alpha}')^{-1}(e \cdot h_{\alpha\beta}(x)) \\
&= \widetilde{\psi_\alpha} \circ h((\widetilde{\varphi_\alpha}')^{-1}(e) \cdot h_{\alpha\beta}(x)) \\
&= \widetilde{\psi_\alpha}(h((\widetilde{\varphi_\alpha}')^{-1}(e)) \cdot h_{\alpha\beta}(x)) \\
&= \widetilde{\psi_\alpha}(h((\widetilde{\varphi_\alpha}')^{-1}(e))) \cdot h_{\alpha\beta}(x) = e_{x,\alpha} \cdot h_{\alpha\beta}(x).
\end{aligned}$$

Hence $\{\chi_\alpha\}$ is a trivialization atlas with transition functions $h_{\alpha\beta}$ in H. Let $\widetilde{P}$ be a G-principal bundle with this trivialization atlas. Then the maps

$$U_\alpha \times G \to U_\alpha \times G, \quad (x, g) \mapsto (x, e_{x,\alpha}^{-1} \cdot g)$$

induce a bundle isomorphism $P \to \widetilde{P}$ as per previous proposition. By definition (see the proof of proposition), the induced map is also a G-bundle isomorphism. □

Example 1.17. *Let E be a vector bundle over M with transition functions $f_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$. Then we can use these transition functions to build $GL_k(\mathbb{R})$-principal bundle $P(E)$. Conversely, given a $GL_k(\mathbb{R})$-bundle P, there exists a unique (up to isomorphism) vector bundle E such that $P(E)$ is equivalent to P.*

Definition 1.18. Let M be a manifold and E a vector bundle over M with fiber $\mathbb{C}^n$ and structure group $GL_k(\mathbb{C})$. Then E is a complex vector bundle of complex rank k. If M is a complex manifold and $\pi : E \to M$ is holomorphic, then E is a holomorphic bundle.

Proposition 1.19. *Let L, L' be holomorphic fiber bundles of rank 1 (also called line bundles) on a complex manifold M with transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$, respectively. Then there exists a holomorphic vector bundle isomorphism iff for all α, there exists a holomorphic map $f_\alpha : U_\alpha \to \mathbb{C}^*$ such that for $U_\alpha \cap U_\beta \neq \emptyset$, we get*

$$\frac{f_\alpha}{f_\beta}\Big|_{U_\alpha \cap U_\beta} = \frac{g_{\alpha\beta}}{g'_{\alpha\beta}}$$

Proof. Follows from the proposition on equivalence of bundles. □

1.2 Examples

Definition 1.20. The tangent bundle is defined as $TM := \bigsqcup_{p \in M} T_pM$.

Proposition 1.21. *$\pi : TM \to M$ is a vector bundle of rank $n = \dim M$.*

Proof. We define the smooth structure on TM with local charts

$$\pi^{-1}(U_\alpha) \to \mathbb{R}^n \times \mathbb{R}^n, \quad (p, v) \mapsto (\varphi_\alpha(p), (d\varphi_\alpha)_p(v)),$$

where $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ are local charts on M. Now if

$$\varphi_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

are the transition maps of the manifold, then we can take the trivialization atlas $\{U_\alpha\}$ with local transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$, which map p to the matrix of $(d\varphi_{\alpha\beta})_p$ in the standard basis of $\mathbb{R}^n$. $\square$

Definition 1.22. Let M be a manifold and E a vector bundle on M of rank k. For $p \in M$, let $F(E)_p$ be the set of ordered bases of E_p (equivalently: linear isomorphisms $\mathbb{R}^k \to E_p$) and define $F(E) := \bigsqcup_{p \in M} F(E)_p$ with

$$\pi_{F(F)} : F(E) \to M, \quad (p, \underbrace{(v_1, \ldots, v_k)}_{\in F(E)_p}) \mapsto p.$$

This is the frame bundle.

Proposition 1.23. *$F(E)$ is a principal bundle with fiber $GL_k(\mathbb{R})$, which is equivalent (as a principal bundle) to the associated principal bundle $P(E)$.*

Proof. As before, we have to introduce a principal bundle structure on $F(E)$. So far, we have done this using the fiber bundle construction theorem. Here, we do that in another way. Suppose that $\{U_\alpha\}$ is a trivializing atlas for the vector bundle $\pi : E \to M$ as well as the local atlas for M (take a refinement, if needed). Suppose that $\varphi'_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ are local trivializations and $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ are local charts on M. Also, let $\{V_\beta\}$ be a local atlas for a manifold $GL_k(\mathbb{R})$ and $\phi_\beta : V_\beta \to \mathbb{R}^{k^2}$ its local charts. Our goal is then to introduce a topology and a smooth structure on $F(E) := \bigsqcup_{p \in M} F(E)_p$ (so far, this is just a set) such that

$$\psi'_\alpha : \pi^{-1}_{F(E)}(U_\alpha) \to U_\alpha \times GL_k(\mathbb{R}), \quad (\underbrace{p}_{\in U_\alpha}, \underbrace{(v_1, \ldots, v_k)}_{\in F(E)_p}) \mapsto (p, \underbrace{(\varphi'_\alpha(v_1) \quad \cdots \quad \varphi'_\alpha(v_k))}_{\in GL_k(\mathbb{R})})$$

are local trivializations which make $F(E)$ into a $GL_k(\mathbb{R})$-principal bundle. For any indices α, β, define the map

$$\psi_{\alpha,\beta} : {\psi'_\alpha}^{-1}(U_\alpha \times V_\beta) \xrightarrow{\psi'_\alpha} U_\alpha \times V_\beta \xrightarrow{\varphi_\alpha \times \phi_\beta} \mathbb{R}^m \times \mathbb{R}^{k^2}.$$

These maps satisfy the smooth manifold chart lemma (Lemma 1.35 in Lee's *Introduction to smooth manifolds*), they induce a smooth manifold structure on $F(E)$ which makes ϕ'_α into trivialization maps. It is routine to prove that these trivialization maps make $F(E)$ into a $GL_k(\mathbb{R})$-principal bundle and that $F(E)$ has the same transition functions as the vector bundle E on trivialization atlas $\{\pi^{-1}_{F(E)}(U_\alpha)\}_\alpha$. By the uniqueness in the construction theorem for fiber bundles, it follows that $F(E)$ is equivalent as a $GL_k(\mathbb{R})$-principal bundle to $P(E)$. $\square$

Remark. The proof gives us a way to construct a fiber bundle structure on a set with chosen trivialization functions.

Definition 1.24. When $E = TM$, we write $FM := F(TM)$.

Proposition 1.25. *FM admits a reduction to $GL_k(\mathbb{R})^+$ iff M is orientable.*

Proof. Follows directly from proposition (DOPOLNI). $\square$

Here's another important example. Define the map

$$\pi : SO(n) \to S^{n-1}, \quad (a_1, \ldots, a_n) \to a_n.$$

Now since $S^{n-1} = F^{-1}(0)$, where

$$F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, \quad x \mapsto \langle x, x \rangle - 1$$

and

$$T_p S^{n-1} = \{v \in \mathbb{R}^{n-1} : \ dF_p(v) = \langle p, v \rangle = 0\},$$

we get that if $A = (a_1, \ldots, a_n) \in SO(n)$, then $(a_1, \ldots, a_{n-1})$ is an orthonormal basis of $T_{a_n} S^{n-1}$. Denote this basis by $\mu(A)$. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas for S^{n-1} such that the basis $(e_1, \ldots, e_{n-1})$ in $T_{e_n} S^{n-1}$ has the same orientation as the basis, induced by the local chart of the atlas. Denote the basis from the local charts φ_α as

$$\left(\left(\frac{\partial}{\partial x_1} \right)_\alpha (p), \ldots, \left(\frac{\partial}{\partial x_{n-1}} \right)_\alpha (p) \right) \subseteq T_p S^{n-1},$$

then we can transform it using to an orthonormal basis

$$v^\alpha(p) = (v_1^\alpha(p), \ldots, v_{n-1}^\alpha(p)) \subseteq T_p S^{n-1}$$

(using Gram-Schmidt, for example). Therefore, given $A = (a_1, \ldots, a_n) \in SO(n)$, there exists a unique $A_\alpha \in SO(n-1)$ such that $\mu(A) = A_\alpha^\top v^\alpha(a_n)$. Now we can define $SO(n)$ as a principal bundle over S^{n-1} with structure group $SO(n-1)$ with local trivializations

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times SO(n-1), \quad A \mapsto (a_n, A_\alpha)$$

(obviously, transition map are in $SO(n-1)$). We can think of $SO(n)$ as a subbundle of FS^{n-1} made up of only oriented orthonormal bases: $SO(n)$ is a reduction of FS^{n-1} to S^{n-1}.

Example 1.26 (Hopf fibration)**.** *The map*

$$S^{2n+1} \subseteq \mathbb{C}^{n+1} \to \mathbb{C}P^n, \quad (z_0, \ldots, z_n) \mapsto [z_0 : \cdots : z_n]$$

is a principal bundle with fiber S^1.

Example 1.27 (Homogeneous spaces)**.** *Let G be a Lie group and $H \subseteq G$ its Lie subgroup, then $G \to G/H$ is a principal bundle with fiber H.*

1.3 Sections

Definition 1.28. Let M be a manifold, $\pi : E \to M$ a fiber bundle and $U \subseteq M$ open. A section of E over M is a smooth map $s : U \to E$ such that $\pi \circ s = \mathrm{id}$.

The set of sections of E over M is denoted by $C_\pi^\infty(U, E)$ or $\mathcal{O}_\pi(U, E)$ (if holomorphic). When $U = M$, we talk about a global section.

Remark. If E is a vector bundle, then $C_\pi^\infty(U, E)$ is a vector space.

Proposition 1.29. *Let E be a fiber bundle over M with fiber F and structure group G. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a trivializing atlas for E, $g_{\alpha\beta}$ transition maps and $U \subseteq M$ open. Then for $s \in C_\pi^\infty(U, E)$, we define*

$$s_\alpha = \pi_F \circ \varphi_\alpha \circ s : U \cap U_\alpha \to F$$

and we have $s_\alpha = g_{\alpha\beta} s_\beta$ whenever $U \cap U_\alpha \cap U_\beta \neq \emptyset$. Conversely, given a family of smooth $s_\alpha = \pi_F : U \cap U_\alpha \to F$ such that $s_\alpha = g_{\alpha\beta} \cdot s_\beta$, then there exists a unique $s \in C_\pi^\infty(U, E)$ such that $s_\alpha = \pi_F \circ \varphi_\alpha \circ s$.

Proof. ($\Rightarrow$) For $x \in U \cap U_\alpha \cap U_\beta$, we have

$$s(x) = \varphi_\alpha^{-1}(x, s_\alpha(x)).$$

Now for $x \in U \cap U_\alpha \cap U_\beta$, we get

$$\begin{aligned}(x, s_\alpha(x)) &= \varphi_\alpha(s(x)) = \varphi_\alpha(s(x)) \\ &= \varphi_\alpha(\varphi_\beta^{-1}(x, s_\beta(x))) = (x, g_{\alpha\beta}(x) \cdot s_\beta(x)).\end{aligned}$$

Conversely ($\Leftarrow$), given s_α, we can define $s(x) := \varphi_\alpha^{-1}(x, s_\alpha(x))$. By the same calculation as above, this gives us a well defined smooth map, which is a section. □

Definition 1.30. The maps $s_\alpha : U \cap U_\alpha \to F$ are called the local data of s.

Proposition 1.31. *Let E be a vector bundle of rank k over M and $U \subseteq M$ open. Then E is trivial over U iff there exist sections $s_1, \dots, s_k$ such that $\{s_1(x), \dots, s_k(x)\} \subseteq E_x$ is a basis for every $x \in U$.*

Proof. ($\Leftarrow$) For each $v \in E_x$, there exist coefficients $a_j \in \mathbb{R}$ such that $v = \sum_{j=1}^k a_j s_j(x)$. Then

$$\Phi : E\big|_U \to U \times \mathbb{R}^k, \quad v \mapsto (x, (a_1, \dots, a_k))$$

is a vector bundle isomorphism. Conversely ($\Rightarrow$), if $\Phi : E\big|_U \to U \times \mathbb{R}^k$ is a trivialization, then define $s_j(x) = \Phi^{-1}(x, e_j)$. □

Remark. $x \mapsto (x, 0)$ is a global section of a vector bundle.

Proposition 1.32. *A principal bundle P over M admits a global section iff P is equivalent (as a principal bundle) to $M \times G$.*

Proof. ($\Leftarrow$) If $\Phi : M \times G \to P$ is a principal bundle equivalence, then $x \mapsto \Phi(x, e)$ is a global section on P. Conversely ($\Rightarrow$), if $s : M \to P$ is a global section. For each $g \in P_x$, there exists a unique $h \in G$ such that $s(x)h = g$. Now define the principal bundle equivalence

$$\Phi : P \to M \times G, \quad g \mapsto (x, h).$$ □

Corollary 1.33. *A principal bundle P over M is equivalent to $M \times G$ (as a principal bundle) iff G is reducible to $\{e\}$.*

Proof. ($\Rightarrow$) By previous proposition, we have a global section $s : M \to P$, which also gives us an injective principal bundle morphism.

$$\begin{tikzcd} M \times \{e\} \arrow[rr, "s"] \arrow[dr] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

Conversely ($\Leftarrow$), a reduction $\Phi : M \times \{e\} \to P$ induces a global section $s : x \mapsto \Phi(x, e)$. □

1.4 Operations on vector bundles

Let E, F be vector bundles over a manifold M of rank k_1, k_2, respectively. WLOG assume that both have the same trivialization atlas $\{U_\alpha\}$ (take a refinement, if needed). Furthermore, they have trivialization maps $\varphi_\alpha^E, \varphi_\alpha^F$ and transition maps $g_{\alpha\beta}^E, g_{\alpha\beta}^F$, respectively.

Direct sum $E \oplus F$

Explicitly, we can construct the vector bundle $E \oplus F$ as a set

$$\bigsqcup_{p \in M} (E_p \oplus F_p) = \{(p, (u, v)) \mid p \in M,\ u \in E_p, v \in F_p\},$$

equipped with the smooth structure induced by trivializations

$$\varphi_\alpha^{E\oplus F} : \bigsqcup_{p \in U_\alpha} (E_p \oplus F_p) \to U_\alpha \times \mathbb{R}^{k_1+k_2}, \quad (p, (u, v)) \mapsto (p, (\varphi_p^E(u), \varphi_p^F(v))).$$

For $U_\alpha \cap U_\beta \neq \emptyset$, the transition maps are

$$g_{\alpha\beta}^{E\oplus F}(x) = \begin{pmatrix} g_{\alpha\beta}^E(x) & \\ & g_{\alpha\beta}^F(x) \end{pmatrix} : U_\alpha \cap U_\beta \to GL(k_1 + k_2).$$

Alternatively, we can construct $E \oplus F$ using the theorem for construction of vector bundles, by defining it as the vector bundle over M with fibers $\mathbb{R}^{k_1+k_2}$ and transition maps $g_{\alpha\beta}^{E\oplus F}$.

Tensor product $E \oplus F$

Explicitly, we define the vector bundle $E \otimes F$ as the set

$$\bigsqcup_{p \in M} (E_p \otimes F_p),$$

equipped by the smooth structure induced by trivializations

$$\varphi_\alpha^{E\otimes F} : \bigsqcup_{p \in U_\alpha} (E_p \otimes F_p) \to U_\alpha \times \mathbb{R}^{k_1 \cdot k_2}, \quad (p, (\underbrace{u}_{\in E_p} \otimes \underbrace{v}_{\in F_p})) \mapsto (p, \varphi_p^F(v) \cdot (\varphi_\alpha^E(u))^\top)$$

(when extended linearly). For $A \in \mathbb{R}^{k_1 \times k_1}$ and $B \in \mathbb{R}^{k_2 \times k_2}$, we define the Cauchy product of matrices as

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1k_1}B \\ \vdots & & \vdots \\ a_{k_11}B & \cdots & a_{k_1k_1}B \end{pmatrix}.$$

Now again for $U_\alpha \cap U_\beta \neq \emptyset$, the transition maps for $E \otimes F$ are

$$g_{\alpha\beta}^{E\otimes F}(x) = \left(g_{\alpha\beta}^E(x) \otimes g_{\alpha\beta}^F(x)\right) : U_\alpha \cap U_\beta \to GL(k_1 \cdot k_2).$$

Using the construction theorem, we can also define $E \otimes F$ as the vector bundle over M with fibers $\mathbb{R}^{k_1 \cdot k_2}$ and transition maps $g_{\alpha\beta}^{E\otimes F}$.

External product $\Lambda^r(E)$

For a finite dimensional real vector space V, we define $V^{\otimes r} = \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}}$ and $A \in \operatorname{End}(V^{\otimes r})$ as

$$A(v_1 \otimes \cdots \otimes v_r) := \frac{1}{r!} \sum_{\sigma \in S(r)} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

Since $A^2 = A$, we have $V^{\otimes r} = \ker A \oplus \operatorname{im} A$. Define $\Lambda^r(V) = A(V^{\otimes r}) = \operatorname{im} A$ and write

$$v_1 \wedge \cdots \wedge v_r := A(v_1 \otimes \cdots \otimes v_r).$$

If $\{v_1, \ldots, v_k\}$ is the basis of V, then $\{v_{i_1} \wedge \cdots \wedge v_{i_r}\}$ is the basis for $\lambda^r(V)$ (if $r > k$, then $\lambda^r(V)$ is a trivial vector space). In particular, we have the canonical isomorphism $\Lambda^r(\mathbb{R}^k) \cong \mathbb{R}^{\binom{k}{r}}$. Now we define $\Lambda^r(E)$ as the set

$$\bigsqcup_{p \in M} \Lambda^r(E_p),$$

equipped with the smooth structure induced by trivializations

$$\varphi_\alpha^{\Lambda^r(E)}: \bigsqcup_{p\in U_\alpha} (\Lambda^r(E_p)) \to U_\alpha \times \mathbb{R}^{\binom{k_1}{r}}, \quad (p, \underbrace{v_{i_1}}_{\in E_p} \wedge \cdots \wedge \underbrace{v_{i_n}}_{\in E_p}) \mapsto (p, \varphi_p^E(v_{i_1}) \wedge \cdots \wedge \varphi_p^E(v_{i_n}))$$

The transition maps are given by $g_{\alpha\beta}^{\Lambda^r(E)}(x)$, which is the matrix in $\mathbb{R}^{n\times n}$, comprised of r-minors of $g_{\alpha\beta}^E(x)$. If $r = \operatorname{rank} E$, then $\Lambda^r(E) = \det(E)$ is the determinant bundle of E. If $r > \operatorname{rank}(E)$, then $\Lambda^r(E) = M$ is the trivial vector bundle.

The Hom bundle $\operatorname{Hom}(E, F)$

Define the vector bundle $\operatorname{Hom}(E, F)$ as the set

$$\bigsqcup_{p\in M} \operatorname{Hom}(E_p, F_p),$$

equipped by the smooth structure induced by trivializations

$$\varphi_\alpha^{\operatorname{Hom}(E,F)}: \bigsqcup_{p\in U_\alpha} (\operatorname{Hom}(E_p, F_p)) \to U_\alpha \times \mathbb{R}^{k_1\cdot k_2}, \quad (p, f) \mapsto (p, \varphi_p^F \circ f \circ \varphi_p^{E^{-1}}),$$

where we view $\varphi_p^F \circ f \circ \varphi_p^{E^{-1}} : \mathbb{R}^{k_1} \to \mathbb{R}^{k_2}$ as a matrix in $\mathbb{R}^{k_2\times k_1}$. Then $\operatorname{Hom}(E, F)$ is the vector bundle over M with transition maps $h_{\alpha\beta}$, where $h_{\alpha\beta}(x)$ (for any $x \in U_\alpha \cap U_\beta$) is given by the matrix of the linear map

$$\mathbb{R}^{k_2\times k_1} \to \mathbb{R}^{k_2\times k_1}, \quad A \mapsto g_{\alpha\beta}^F(x) \cdot A \cdot g_{\beta\alpha}^E(x).$$

The dual bundle E^*

The dual bundle is defined by $E^* = \operatorname{Hom}(E, M\times\mathbb{R})$. This is equivalent to constructing a vector bundle over M with transition maps $g_{\alpha\beta}^{E^*} = (g_{\alpha\beta}^E)^{-\top}$. As a corollary, $T^*M \cong (TM)^*$.

Proposition 1.34. $\operatorname{Hom}(E, F) \cong E^* \otimes F$

Proof. For any $p \in M$, we have $\operatorname{Hom}(E_p, F_p) \cong E_p^* \otimes F_p$ as vector spaces. Then calculate the trivialization maps and determine that the resulting vector bundles are equivalent. □

1.5 Pullbacks, subbundles and quotient bundles

Let $f \in C^\infty(M)$, then $df_x : T_xM \to T_{f(x)}M$ is a vector bundle morphism. Identifying $T\mathbb{R} \cong \mathbb{R}\times\mathbb{R}$, write $df_x(v) = (f(x), A_x(v))$. The operator $A_x : T_xM \to \mathbb{R}$ is linear, hence

$$M \to T^*M, \quad x \mapsto (x, A_x)$$

is a section of T^*M. This section is also called df_x. So every differential gives us a section on T^*M. Given a local chart $(U, (x_1, \dots, x_n))$ on M, the set $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ is a local basis of $TM\big|_U$. Let $(dx_1, \dots, dx_n)$ be its dual basis in $T^*M\big|_U$, then

$$df\big|_U = \sum_{j=1}^n \frac{\partial}{\partial x_j} \cdot dx_j$$

and for each $x \in M$, we have $df_x(v) = (x, v(f))$. Notice that these dx_j are precisely the sections in $C^\infty(U, E)$ that we obtain as differentials of the $x_j : U \to \mathbb{R}$ as above. Since differentials give us sections on T^*M, it is natural to ask whether every section on T^*M is a differential of a function on M. This is one of the motivations for geometric PDE's: for a $u \in C^\infty(M, T^*M)$, we are looking for an $f \in C^\infty(M)$ such that $df = u$.

Definition 1.35. Let M, N be manifolds and $f : M \to N$ smooth. Let $\pi : E \to N$ be a bundle with fiber F and structire group G. Define the pullback f^*E with fiber F and structure group G over M as

$$f^*E := \{(m, e) \in M \times E : \; f(m) = \pi(e)\}$$

and map

$$\pi' : f^*E \to M, \quad (m, e) \mapsto m.$$

Proposition 1.36. $\pi' : f^*E \to M$ *is a fiber bundle with fiber* F*, structure group* G *and transition maps* $g_{\alpha\beta} \circ f$.

Proof. (DOPOLNI) □

Example 1.37. *Let* $x \in M$ *and* $i : \{x\} \hookrightarrow M$ *an immersion. Then* $i^*E = Ex$*. More generally, if* $N \subseteq M$ *is a submanifold and* $i : N \to M$ *is the inclusion, then* $E\big|_N := i^*E$ *is the restriction of* E *to* N.

The map

$$\widetilde{f} : f^*E \to E, \quad (m, e) \mapsto e$$

is a fiber bundle morphism, which makes the following diagram commutative.

$$\begin{array}{ccc} f^*E & \xrightarrow{\widetilde{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[f]{} & N \end{array}$$

Up to equivalence, f^*E is the only bundle that makes the above diagram commutative when f is a diffeomorphism.

Proposition 1.38. *Let* $f : N \to M$ *smooth and* $\pi : E \to N$ *a bundle with fiber* F *and structure group* G*. Let* $\rho : E' \to M$ *be another bundle with fiber* F *and structure group* G*. If* $(f, g) : E' \to E$ *is a fiber bundle equivalence*

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[f]{} & N \end{array},$$

then $E' \cong f^*E$.

Proof. Define the map

$$\Phi : E' \to f^*E, \quad e' \mapsto (\rho(e'), g(e')).$$

Since $f \circ \rho = \pi \circ g$, this is well-defined and the image is exactly f^*E. □

Definition 1.39. Let E, F' be vector bundles over M. If there exists a vector bundle morphism $i : F' \subseteq E$ that is injective on the fibers, we call $F := i(F')$ a subbundle of E.

Proposition 1.40. *Let* M *be a manifold and* $\pi : E \to M$ *a vector bundle of rank* k *on* M*. Suppose* $i : F' \hookrightarrow E$ *and* $F := i(F')$ *is a subbundle of rank* $l \subseteq k$*. Then* i *is an embedding and* $F \subseteq E$ *is a submanifold of* E*. Moreover,* $\pi\big|_F : F \to M$ *is a vector bundle and there exists a trivializing atlas* $(U_\alpha, \varphi_\alpha)$ *of* E *such that if*

$$P_l(x, v_1, \ldots, v_k) = (x, v_1, \ldots, v_l),$$

$\{(U_\alpha, P_l \circ \varphi_\alpha)\}$ *is a trivializing atlas for* F.

Example 1.41. *Let $S \subseteq M$ be a submanifold and $i : S \hookrightarrow M$ a inclusion, then*

$$di : TS \to TM$$

is fiber-wise injective, so $TS \subseteq TM\big|_S$ is a subbundle over S.

Remark. In the adapted trivialization

$$g^E_{\alpha\beta} \begin{pmatrix} g^F_{\alpha\beta} & \kappa_{\alpha\beta} \\ 0 & h_{\alpha\beta} \end{pmatrix},$$

$g^E_{\alpha\beta}$ satisfy the cocycle conditions so the same must be true for $g^F_{\alpha\beta}$ and $h_{\alpha\beta}$.

Definition 1.42. Let M be a manifold and $E \to M$ a vector bundle of rank k. Let $F \subseteq E$ a subbundle of rank l. Define an equivalence relation

$$e \sim e' \Leftrightarrow e, e' \in E_x \text{ and } e - e' \in F_x.$$

We denote by E/F the set of its equivalence classes and the map

$$\pi' : E/F \to M, \quad [e] \mapsto \pi(e).$$

Proposition 1.43. *E/F is a vector bundle such that*

$$\rho : E \to E/F, \quad e \mapsto [e]$$

is a morphism and $(E/F)_x = E_x/F_x$ for each $x \in M$. If $Q \to M$ is a vector bundle with fibers E_x/F_x such that there exists a morphism

$$g' : E \to Q, \quad v \mapsto [v],$$

then $Q \cong E/F$.

Definition 1.44. We define $\pi' : E/F \to M$ as the quotient bundle.

Proof. (DOPOLNI) □

Example 1.45. *Let $S \subseteq M$ be a submanifold. Since $TS \subseteq TM\big|_S$ is a subbundle, we can define the normal bundle to S in M as*

$$NS := TM\big|_S / TS.$$

1.6 Cartier divisors

Definition 1.46. Let M be a complex manifold. An effective Cartier divisor D on M is given by an open cover $\{U_\alpha\}$ and holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}$ which are not identically zero and such that if $U_\alpha \cap U_\beta \neq \emptyset$, then

$$f_{\alpha\beta} := \frac{f_\alpha}{f_\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$$

is holomorphic and without zeros.

The functions $\{f_{\alpha\beta}\}$ satisfy cocycle conditions, so they define a line bundle, denoted by $\mathcal{O}(D)$. Furthermore, the functions f_α are the local data for a non-zero holomorphic section of $\mathcal{O}(D)$. Conversely, given a line bundle and a non-zero global holomorphic section on $\mathcal{O}(D)$, then its local data define a divisor. Another way to obtain divisors is from complex submanifolds of codimension 1. Let $S \subseteq M$ be a submanifold and $(U_\alpha, \varphi_\alpha)$ an atlas, adapted to S:

$$\varphi_\alpha(S \cap U_\alpha) = \{z^\alpha = (z_1^\alpha, \ldots, z_n^\alpha) \in \mathbb{C}^n \mid z_n^\alpha = 0\}.$$

If $U_\alpha \cap U_\beta \neq \emptyset$, then on $U_\alpha \cap U_\beta$ we have $z_n^\alpha = 0$ iff $z_n^\beta = 0$. On this intersection, we may regard $z^\alpha = \varphi_\alpha \circ \varphi_\beta^{-1}(z_\beta) = z^\alpha(z^\beta)$ as a function of z^β, so

$$z_n^\alpha(z_1^\beta, \ldots, z_n^\beta) = (z_n^\beta)^k \cdot u(z_1^\beta, \ldots, z_n^\beta),$$

where $k \geq 1$ and $u \in \mathcal{O}(U_\alpha \cap U_\beta)$. Now since

$$\det(d(\varphi_\alpha \circ \varphi_\beta^{-1})) = \det \begin{pmatrix} \frac{\partial z_1^\alpha}{\partial z_1^\beta} & \cdots & \frac{\partial z_1^\alpha}{\partial z_n^\beta} \\ \vdots & & \vdots \\ \frac{\partial z_n^\alpha}{\partial z_1^\beta} & \cdots & \frac{\partial z_n^\alpha}{\partial z_n^\beta} \end{pmatrix} \neq 0$$

and $\frac{\partial z_n^\alpha}{\partial z_k^\beta}(p) = 0$ for $p \in S$ and $k = 1, \ldots, n-1$, we get $\frac{\partial z_n^\alpha}{\partial z_n^\beta}(p) \neq 0$ for all $p \in S$. Compute

$$\frac{\partial z_n^\alpha}{\partial z_n^\beta} = k \cdot (z_n^\beta)^{k-1} \cdot u + (z_n^\beta)^k \cdot \frac{\partial u}{\partial z_n^\beta}.$$

Since the above must be nonzero on S, we must have $k = 1$ and $u \neq 0$ on S. Therefore, $u = \frac{z_n^\alpha}{z_n^\beta}$ is non-zero and holomorphic. The functions $f_\alpha := z_n^\alpha : U_\alpha \to \mathbb{C}$ now define a Cartier divisor called $[S]$. On S, we have

$$\frac{z_n^\alpha}{z_n^\beta}\Big|_S = u\Big|_S = \frac{\partial z_n^\alpha}{\partial z_n^\beta}\Big|_S,$$

which implies that $\mathcal{O}([S])\big|_S \cong NS$. Also observe that f_α is zero to order 1 on S and nowhere else.

1.7 Kernel, image and exact sequences

Given vector bundles E, F over M, a morphism of vector bundles $\varphi : E \to F$ over the identity is a sextion of $\operatorname{Hom}(E, F)$.

Proposition 1.47. $\ker\varphi := \bigcup_{x \in M} \ker \varphi_x \subseteq E$ *and* $\operatorname{im}\varphi := \bigcup_{x \in M} \operatorname{im} \varphi_x \subseteq F$ *are vector subbundles iff* $\operatorname{rank}\varphi_x$ *is constant w.r.t.* $x \in M$*.*

Proof. (DOPOLNI) ☐

Let $E; E', E''$ be vector bundles over M. Let $\alpha : E' \to E$ and $\beta : E \to E''$ be morphisms.

Definition 1.48. If α is injective, β is surjective and $\operatorname{im}\alpha = \ker\beta$, then we say that the sequence

$$0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0$$

is a short exact sequence.

In that case, $E'' \cong E/E'$ and $\operatorname{rank} E = \operatorname{rank} E' + \operatorname{rank} E''$.

Proposition 1.49. *If*

$$0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0$$

is a short exact sequence and F is a vector bundlle over M, then the following sequences are exact:

- $0 \to E' \otimes F \xrightarrow{\alpha\otimes\mathrm{id}} E \otimes F \xrightarrow{\beta\otimes\mathrm{id}} E'' \otimes F \to 0$

- $0 \to \mathrm{Hom}(F, E') \xrightarrow{\alpha\circ} \mathrm{Hom}(F, E) \xrightarrow{\beta\circ} \mathrm{Hom}(F, E'') \to 0$
- $0 \to \mathrm{Hom}(E'', F) \xrightarrow{\circ\beta} \mathrm{Hom}(E, F) \xrightarrow{\circ\alpha} \mathrm{Hom}(E', F) \to 0$

In particular, for $M \times \mathbb{R}$ we get the short exact sequence

$$0 \to (E'')^* \to E^* \to (E')^* \to 0.$$

If $f : N \to M$ is smooth, we also have

$$0 \to f^*(E') \to f^*(E) \to f^*(E'') \to 0.$$

We omit the proof (highly technical).

Theorem 1.50.

If

$$0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0$$

is a short exact sequence, then $\det E \cong \det E' \otimes \det E''$.

Proof. If we take a trivialization of E, adapted to $\alpha(E')$, we have

$$g^E_{\alpha\beta} = \begin{pmatrix} g^{E'}_{\alpha\beta} & \kappa_{\alpha\beta} \\ 0 & g^{E''}_{\alpha\beta} \end{pmatrix}.$$

As a result, $\det g^E_{\alpha\beta} = \det g^{E'}_{\alpha\beta} \cdot \det g^{E''}_{\alpha\beta}$. □

Definition 1.51. If M is a manifold (usually complex), then $K_M := \det(T^*M)$ is the canonical bundle of M.

If L is a line bundle, then $\mathrm{Hom}(L, L)$ is a line bundle that admits a global nowhere zero section $x \mapsto \mathrm{id}_{L_x}$ and so $\mathrm{Hom}(L, L) \cong M \times \mathbb{C}$ is trivial. Then also $L \otimes L^* \cong \mathrm{Hom}(L, L) \cong M \times \mathbb{C}$ is trivial. As a result, we get that for any holomorphic vector bundles E, F, we have

$$E \otimes L \cong F \Leftrightarrow E \cong F \otimes L^*.$$

Theorem 1.52.

Let $S \subseteq M$ be a complex submanifold of codimension 1. Then

$$K_S \cong (K_M \otimes \mathcal{O}([S]))\big|_S.$$

Proof. We have a short exact sequence over S:

$$0 \to TS \to TM\big|_S \to NS \to 0.$$

By dualizing, we get

$$0 \to NS^* \to T^*M\big|_S \to T^*S \to 0$$

and so

$$K_M\big|_S \cong K_S \otimes NS^* \cong K_S \otimes \mathcal{O}([S])\big|^*_S.$$

Now our previous observation gives us $K_S \cong K_M\big|_S \otimes \mathcal{O}(S)\big|_S \cong (K_M \otimes \mathcal{O}(S))\big|_S$. □

1.8 Line bundles and the Picard group

For this section, let M be a complex manifold of dimension n.

Definition 1.53. The set of all line bundles modulo equivalence with tensor product operation is the Picard group of M. We denote it as $\mathrm{Pic}(M)$.

Remark. Often in algebraic geometry, they produce $\mathcal{O}([S])^*$ instead of $\mathcal{O}([S])$.

Proposition 1.54. $\mathrm{Pic}(M)$ *is an abelian group with* $L^{-1} = L^*$ *and a neutral element* $M \times \mathbb{C}$.

Denote by $\mathcal{T}(M)$ the set of all open covers $\{U_\alpha\}$ of M together with functions $\{g_{\alpha\beta}\} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ satisfying cocycle conditions. Every element of $\mathcal{T}(M)$ defines a line bundle. After passing to a common refinement $\{U_\alpha\}$, two line bundles with trivializing functions $\{g_{\alpha\beta}\}$ in $\{h_{\alpha\beta}\}$ are equivalent iff there exist functions $f_\alpha : U_\alpha \to \mathbb{C}^*$ such that $\frac{f_\beta}{f_\alpha} = \frac{g_{\alpha\beta}}{h_{\alpha\beta}}$ on $U_\alpha \cap U_\beta \neq \emptyset$. This defines an equivalence relation $\sim$ on $\mathcal{T}(M)$ and $\mathrm{Pic}(M) \cong \frac{\mathcal{T}(M)}{\sim}$. This can be generalized to obtain a cohomology theory.

Definition 1.55. The tautological bundle over $\mathbb{C}P^n$ is

$$\mathcal{O}(-1) := \{([p], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid v \in \mathbb{C} \cdot p\}.$$

The name comes from the fact that the fiber over $[p]$ is the line passing through 0 and p in $\mathbb{C}^{n+1}$.

Let $U_\alpha = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_\alpha \neq 0\}$ and define the map

$$\varphi_\alpha : \mathcal{O}(-1)\big|_{U_\alpha} \to U_\alpha \times \mathbb{C}, \quad ([z_0 : \cdots : z_n], (v_0, \ldots, v_n)) \mapsto ([z_0 : \cdots : z_n], v_\alpha).$$

To prove that it's a trivialization, we give the inverse. If $p = (z_0, \ldots, z_n)$, then $v \in \mathbb{C}p$ iff there exists a $\lambda \in \mathbb{C}$, such that $v_j = \lambda z_j$ for each j. On U_α, $\lambda = \frac{v_\alpha}{z_\alpha}$ so we can define

$$U_\alpha \times \mathbb{C} \to \mathcal{O}(-1)\big|_{U_\alpha}, \quad ([z], \lambda) \to ([z], \frac{\lambda}{z_\alpha}(z_0, \ldots, z_n)).$$

For $[z] \in U_\alpha \cap U_\beta$, we have

$$\begin{aligned}\varphi_\alpha \circ \varphi_\beta^{-1}([z], \lambda) &= \varphi_\alpha([z], \frac{\lambda}{z_p}(z_0, \ldots, z_n)) \\ &= ([z], \lambda \cdot \frac{z_\alpha}{z_\beta}) \\ &= ([z], g_{\alpha\beta}([z]) \cdot \lambda),\end{aligned}$$

so $g_{\alpha\beta}([z]) = \frac{z_\alpha}{z_\beta}$.

Definition 1.56. Let $\mathcal{O}(1) := \mathcal{O}(-1)^*$ and

$$\mathcal{O}(k) := \begin{cases} \mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1); & k > 0 \\ \mathcal{O}(-1) \otimes \cdots \otimes \mathcal{O}(-1); & k < 0 \\ \mathbb{C}P^n \times \mathbb{C}; & k = 0. \end{cases}$$

Some immediate properties:

- $\mathcal{O}(k) \otimes \mathcal{O}(m) \cong \mathcal{O}(k+m)$ for any $k, m \in \mathbb{Z}$.
- $g_{\alpha\beta}^{\mathcal{O}(k)}([z]) = \left(\frac{z_\alpha}{z_\beta}\right)^{-k}$.

Let $H := \{[z] \in \mathbb{C}P^n \mid a_0 z_0 + \cdots + a_n z_n = 0,\ a_0, \dots a_n \in \mathbb{C} \setminus \{0\}\}$. Then H is a codimension 1 submanifold, so it defines an effective Cartier divisor $[H]$. Now $\{(U_\alpha \cap H, \psi_\alpha)\}$ is an adapted atlas to H, where

$$\psi_\alpha : U_\alpha \cap H \to \mathbb{C}^{n-1} \times \{0\}, \quad [z_0 : \cdots : z_n] \mapsto \left(\frac{z_0}{z_\alpha}, \dots, \widehat{z_\alpha}, \dots, \frac{z_{\alpha'}}{z_\alpha}, \sum_{k=0}^{n} a_k \frac{z_k}{z_\alpha} \right)$$

and $\alpha' = \min\{2k - \alpha - 1, k\}$. By our derivation of Cartier divisors, we see that $[H]$ is defined by the last component functions

$$f_\alpha : U_\alpha \to \mathbb{C}, \quad [z] \mapsto \sum_{k=0}^{n} a_k \frac{z_k}{z_\alpha}.$$

The transition functions of $\mathcal{O}([H])$ are therefore

$$g_{\alpha\beta}^{\mathcal{O}([H])}[z] = \frac{f_\alpha([z])}{f_\beta([z])} = \frac{z_\beta}{z_\alpha}.$$

This implies that $\mathcal{O}([H]) \cong \mathcal{O}(1)$.

Proposition 1.57. *The space of global holomorphic sections on $\mathcal{O}(k)$ is*

$$\mathcal{O}(\mathbb{C}P^n, \mathcal{O}(k)) \cong \begin{cases} \{0\}; & k < 0 \\ \mathrm{Pol}_k(\mathbb{C}^{n+1}); & k \geq 0 \end{cases},$$

where $\mathrm{Pol}_k(\mathbb{C}^{n+1})$ *is the space of k-homogeneous complex polynomials in $n+1$ variables.*

Proof. Let $s \in \mathcal{O}(\mathbb{C}P^n, \mathcal{O}(k))$ be a nonzero global section and $s_\alpha : U_\alpha \cong \mathbb{C}^n \to \mathbb{C}$ its local data. On $U_\alpha \cap U_\beta$, we have

$$s_\alpha \left(\frac{z_0}{z_\alpha}, \dots, \widehat{z_\alpha}, \dots, \frac{z_n}{z_\alpha} \right) = \left(\frac{z_\alpha}{z_\beta} \right)^{-k} \cdot s_\beta \left(\frac{z_0}{z_\beta}, \dots, \widehat{z_\beta}, \dots, \frac{z_n}{z_\beta} \right),$$

which implies $z_\alpha^k \cdot s_\alpha = z_\beta^k \cdot s_\beta$ on $\mathbb{C}^{n+1} \setminus \{z_\alpha \cdot z_\beta = 0\}$. The left function is holomorphic on $\mathbb{C}^{n+1} \setminus \{z_\alpha = 0\}$ and the right one is holomorphic on $\mathbb{C}^{n+1} \setminus \{z_\beta = 0\}$ and they agree on the (open) intersection, so they can be extended to $\mathbb{C}^{k+1} \setminus \{0\}$. Now by Hartog's extension theorem, this can be extended to an entire map $\mathbb{C}^{n+1} \to \mathbb{C}$. Now notice that for any $z \in \mathbb{C}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, we have $P(\lambda z) = \lambda^k P(z)$. This implies that $P(0) = 0$ and so $P(\lambda z) = \lambda^k P(z)$ for any $z \in \mathbb{C}^{n+1}, \lambda \in \mathbb{C}$. But since P is entire and not zero everywhere, we have $k \geq 0$. Since P is entire, it can be written as a power sum around $0 \in \mathbb{C}^{n+1}$, and since it is homogeneous of degree k, its power series has to be finite, so it is a polynomial. Conversely, let $p \in \mathrm{Pol}_k(\mathbb{C}^{n+1})$. Define

$$s_\alpha \left(\frac{z_0}{z_\alpha}, \dots, \widehat{z_\alpha}, \dots, \frac{z_n}{z_\alpha} \right) = P \left(\frac{z_0}{z_\alpha}, \dots, 1, \dots, \frac{z_n}{z_\alpha} \right).$$

Since $s_\alpha = \left(\frac{z_\alpha}{z_\beta} \right)^{-k} s_\beta$, they obey the transition functions for $\mathcal{O}(k)$ and hence they're the local data of a section of $\mathcal{O}(k)$. □

Corollary 1.58. $\mathcal{O}(k) \cong \mathcal{O}(m) \Leftrightarrow k = m$.

Proof. Assume $k > m$, then

$$\mathcal{O}(m-k) \cong \mathcal{O}(-k) \otimes \mathcal{O}(m) \cong \mathcal{O}(-k) \otimes \mathcal{O}(k) \cong \mathbb{C}P^n \times \mathbb{C}.$$

The left bundle only has a trivial section by the above proposition, while the bundle on the right has $\mathbb{C}$-many sections. □

Theorem 1.59 (Euler exact sequence)**.**

$$0 \to \mathcal{O}(-1) \to \mathbb{C}P^n \times \mathbb{C}^{n+1} \xrightarrow{R} T\mathbb{C}P^n \otimes \mathcal{O}(-1) \to 0$$

is a short exact sequence, where the first map is the inclusion and the second map is

$$R([v], a) := d\pi_v(a) \otimes ([v], v).$$

Here,

$$d\pi_v(\mathbb{C}^{n+1} \setminus \{0\}) \cong \mathbb{C}^{n+1} \to T_{[v]}\mathbb{C}P^n$$

is the differential of

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n.$$

Proof. Obviously, $d\pi_v$ is surjective and $\ker d\pi_v = \mathbb{C} \cdot v$. Now if (U_0, φ_0) is a local chart on $\mathbb{C}P^n$, then

$$(\varphi_0 \circ \pi)(v) = \left(\frac{v_1}{v_0}, \dots, \frac{v_n}{v_0}\right)$$

and

$$d(\varphi_0)_{\pi(v)} \circ d(\pi)_v = d(\varphi_0 \circ \pi)_v = \begin{pmatrix} -\frac{v_1}{v_0^2} & \frac{1}{v_0} & 0 & \cdots & 0 \\ -\frac{v_1}{v_0^2} & 0 & \frac{1}{v_0} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ -\frac{v_1}{v_0^2} & 0 & \cdots & 0 & \frac{1}{v_0} \end{pmatrix}.$$

Notice that for $\lambda \in \mathbb{C} \setminus \{0\}$, $d(\varphi_0 \circ \pi)_{\lambda v} = \frac{1}{\lambda} d(\varphi_0 \circ \pi)_v$, which implies that

$$\begin{aligned} d(\varphi_0)_{\pi(v)} \circ d(\pi)_{\lambda v} &= d(\varphi_0)_{\pi(\lambda v)} \circ d(\pi)_{\lambda v} \\ &= d(\varphi_0 \circ \pi)_{\lambda v} = \frac{1}{\lambda} d(\varphi_0 \circ \pi)_v \\ &= \frac{1}{\lambda} d(\varphi_0)_{\pi(v)} \circ d(\pi)_v \end{aligned}$$

and so $d(\pi)_{\lambda v} = \frac{1}{\lambda} d(\pi)_v$. This means that $R([v], u)$ is well-defined. It is routine to check that it is smooth and since the diagram

$$\begin{tikzcd} ([v], a) \arrow[r, mapsto, "R"] \arrow[d, mapsto] & d\pi_v(a) \otimes ([v], v) \arrow[d, mapsto] \\ {[v]} \arrow[r, mapsto, "\mathrm{id}"'] & {[v]} \end{tikzcd}$$

commutes, R is a bundle morphism. Since $d\pi_v$ is linear, R is also a vector bundle morphism. Furthermore, R is surjective since for any $d\pi_v(a) \otimes ([v], \lambda v)$ (where $\lambda \in \mathbb{C} \setminus \{0\}$), we have

$$R([v], \lambda a) = d\pi_v(\lambda a) \otimes ([v], v) = \lambda \cdot (d\pi_v(a) \otimes ([v], v)) = d\pi_v(a) \otimes ([v], \lambda v). \qquad \square$$

Also, for any $v \in \mathbb{C}^{n+1} \setminus \{0\}$ we have

$$\ker R([v], \cdot) = \ker(a \mapsto d\pi_v(a)) = \mathbb{C} \cdot v,$$

so

$$([v], a) \in \ker R \Leftrightarrow a \in \ker R([v], \cdot) \Leftrightarrow a \in \mathbb{C} \cdot v \Leftrightarrow ([v], a) \in \mathcal{O}(-1).$$

If we multiply this exact sequence with $\mathcal{O}(1)$, we get

$$0 \to \mathbb{C}P^n \times \mathbb{C} \to (\mathbb{C}P^n \times \mathbb{C}^{n+1}) \otimes \mathcal{O}(1) \to \mathbb{C}P^n \times \mathbb{C} \to 0.$$

By dualizing, we get

$$0 \to T^*\mathbb{C}P^n \to (\mathbb{C}P^n \times \mathbb{C}^{n+1}) \otimes \mathcal{O}(-1) \to \mathbb{C}P^n \times \mathbb{C} \to 0.$$

By taking determinants, we get

$$\begin{aligned}\det((\mathbb{C}P^n \times \mathbb{C}^{n+1}) \otimes \mathcal{O}(-1)) &= \det(T^*\mathbb{C}P^n) \otimes \det(\mathbb{C}P^n \times \mathbb{C}) \\ &= \det(T^*\mathbb{C}P^n) \\ &= K_{\mathbb{C}P^n}.\end{aligned}$$

Now notice that the transition functions of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ are just the identity I_{n+1} and the transition functions of $\mathcal{O}(-1)$ are $\left(\frac{z_\beta}{z_\alpha}\right)^{-1}$, so their Cauchy product is

$$\begin{pmatrix} \left(\frac{z_\beta}{z_\alpha}\right)^{-1} & & \\ & \ddots & \\ & & \left(\frac{z_\beta}{z_\alpha}\right)^{-1} \end{pmatrix} = g_{\alpha\beta},$$

so $\det g_{\alpha\beta} = \left(\frac{z_\beta}{z_\alpha}\right)^{-(n+1)}$. As a result,

$$\det((\mathbb{C}P^n \times \mathbb{C}^{n+1}) \otimes \mathcal{O}(-1)) \cong \mathcal{O}(-(n+1))$$

and so $K_{\mathbb{C}P^n} \cong \mathcal{O}(-(n+1))$.

Corollary 1.60. *Let $S \subseteq \mathbb{C}P^n$ be a complex submanifold of codimension 1, given by*

$$S = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid p(z_0, \ldots, z_n) = 0\},$$

where $p \in \mathrm{Pol}_k(\mathbb{C}^{n+1})$. Then

$$K_S \cong (K_{\mathbb{C}P^n} \otimes \mathcal{O}([S]))\big|_S \cong \mathcal{O}(-(n+1)) \otimes \mathcal{O}(k) \cong \mathcal{O}(k-n-1).$$