Relacija stroge inkluzije ($\subset$)

Def. $A \subset B \iff A \subseteq B \wedge A \neq B$

Beremo: A je prava podmnožica B

Trditev. $A \subset B \iff A \subseteq B \wedge \exists x : (x \in B \wedge x \notin A)$

Dokaz:
$$\begin{aligned}
A \subset B &\iff A \subseteq B \wedge A \neq B \\
&\iff A \subseteq B \wedge \neg(A \subseteq B \wedge B \subseteq A) \\
&\overset{IR}{\iff} A \subseteq B \wedge (A \nsubseteq B \vee B \nsubseteq A) \\
&\overset{IR}{\iff} (\underbrace{A \subseteq B \wedge A \nsubseteq B}_{0}) \vee (A \subseteq B \wedge B \nsubseteq A) \\
&\overset{IR}{\iff} A \subseteq B \wedge B \nsubseteq A \\
&\overset{\text{def.} \subseteq}{\iff} A \subseteq B \wedge \neg \forall x : (x \in B \Rightarrow x \in A) \\
&\overset{PR, IR}{\iff} A \subseteq B \wedge \exists x : (x \in B \wedge x \notin A) \quad \checkmark
\end{aligned}$$

Izrek. Za vse A, B, C velja:

(i) $A \not\subset A$ (irefleksivnost)

(ii) $A \subset B \Rightarrow B \not\subset A$ (asimetričnost)

(iii) $A \subset B \wedge B \subset C \Rightarrow A \subset C$ (tranzitivnost)

3.3. Operacije z množicami

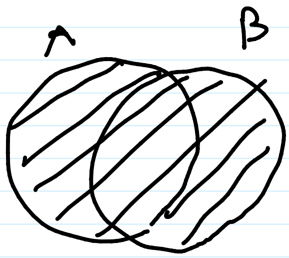
Def. $A \cup B = \{x ; x \in A \vee x \in B\}$ unija

$A \cap B = \{x ; x \in A \wedge x \in B\}$ presek

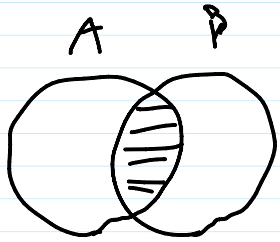
$A \setminus B = \{x ; x \in A \wedge x \notin B\}$ razlika, A brez B

$A \oplus B = \{x ; x \in A + x \in B\}$ Boolova vsota, simetrična razlika

A B A B A B

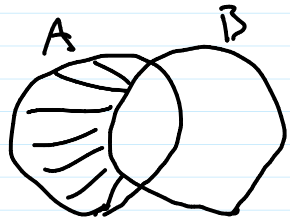


$A \cup B$

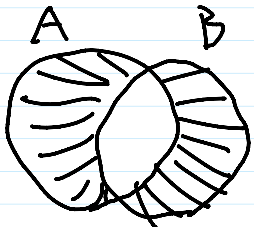


$A \cap B$

simetrična razlika



$A \setminus B$



$A \oplus B$

Trditev. Za poljubni A in B obstajata $A \cap B$ in $A \setminus B$.

Dokaz: Ker A obstaja, po ASP obstajata tudi

$$A \cap B = \{x;\ x \in A \wedge \varphi(x)\},$$
$$A \setminus B = \{x;\ x \in A \wedge \psi(x)\},$$

kjer je $\varphi(x) = x \in B$ in $\psi(x) = x \notin B$. ✓

Izrek. Za vse A, B, C velja:

$$A \cup \emptyset = A \qquad A \setminus \emptyset = A \qquad A \oplus \emptyset = A$$
$$A \cap \emptyset = \emptyset \qquad \emptyset \setminus A = \emptyset$$

$$\left.\begin{aligned} A \cup A &= A \\ A \cap A &= A \end{aligned}\right\} \text{idempotenca}$$

$$\left.\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \\ A \oplus B &= B \oplus A \end{aligned}\right\} \text{komutativnost}$$

$$\left.\begin{aligned} (A \cup B) \cup C &= A \cup (B \cup C) \\ (A \cap B) \cap C &= A \cap (B \cap C) \\ (A \oplus B) \oplus C &= A \oplus (B \oplus C) \end{aligned}\right\} \text{asociativnost}$$

$$\left.\begin{aligned} A \cup (A \cap B) &= A \\ A \cap (A \cup B) &= A \end{aligned}\right\} \text{absorpcija}$$

$$\left.\begin{aligned} A\cup(B\cap C) &= (A\cup B)\cap(A\cup C)\\ A\cap(B\cup C) &= (A\cap B)\cup(A\cap C)\\ A\cap(B\oplus C) &= (A\cap B)\oplus(A\cap C)\end{aligned}\right\}\text{ distributivnost}$$

$$A\oplus B = (A\setminus B)\cup(B\setminus A) = (A\cup B)\setminus(A\cap B)$$

$$A\cap B \subseteq A \subseteq A\cup B$$
$$A\cap B \subseteq B \subseteq A\cup B$$

$$\left.\begin{aligned} A\subseteq B &\Rightarrow A\cup C \subseteq B\cup C\\ A\subseteq B &\Rightarrow A\cap C \subseteq B\cap C\end{aligned}\right\}\text{ monotonost } \cup,\cap \text{ glede na } \subseteq$$

$$A\subseteq B \Leftrightarrow A\cup B = B$$
$$A\subseteq B \Leftrightarrow A\cap B = A$$
$$A\subseteq B \Leftrightarrow A\setminus B = \emptyset \quad (*)$$

Dokaz $(*)$:

$$\begin{aligned} A\subseteq B &\overset{\text{def.}\subseteq}{\Longleftrightarrow} \forall x: (x\in A \Rightarrow x\in B)\\ &\overset{\text{IR}}{\Longleftrightarrow} \forall x: \neg\neg(x\in A \Rightarrow x\in B)\\ &\overset{\text{IR}}{\Longleftrightarrow} \forall x: \neg(x\in A \land x\notin B)\\ &\overset{\text{def.}\setminus}{\Longleftrightarrow} \forall x: \neg(x\in A\setminus B)\\ &\Longleftrightarrow \forall x: x\notin A\setminus B\\ &\Longleftrightarrow A\setminus B = \emptyset \quad \checkmark\end{aligned}$$

Za krajše pisanje vpedemo omejene kvantifikatorje

Def. A množica, φ izj. formula

Potem:
formula $\forall x\in A: \varphi$ je okrajšava za $\forall x: (x\in A \Rightarrow \varphi)$
" $\exists x\in A: \varphi$ " $\exists x: (x\in A \land \varphi)$

Pripomba. Če $A\neq\emptyset$, za kvantifikatorja $\forall x\in A$ in $\exists x\in A$ veljajo analogne enakovrednosti kot za $\forall x$ in $\exists x$, npr.:

$$\underline{\neg\forall x\in A: \varphi} \sim \neg\forall x: (x\in A \Rightarrow \varphi)$$
$$\sim \exists x: \neg(x\in A \Rightarrow \varphi)$$

$$\underbrace{\neg \forall x \in A : \varphi} \sim \neg \forall x : (x \in A \Rightarrow \varphi)$$
$$\sim \exists x : \neg (x \in A \Rightarrow \varphi)$$
$$\sim \exists x : (x \in A \land \neg \varphi)$$
$$\sim \underbrace{\exists x \in A : \neg \varphi}$$

Zgled. $\forall x : \varphi \Rightarrow \exists x : \varphi$ je logično veljavna.

Če je $A = \emptyset$: $\forall x \in \emptyset : \varphi \sim \forall x : (\underbrace{\overbrace{x \in \emptyset}^{0} \Rightarrow \varphi}_{1})$

~ 1

toda $\exists x \in \emptyset : \varphi \sim \exists x : (\underbrace{\overbrace{x \in \emptyset}^{0} \land \varphi}_{0})$

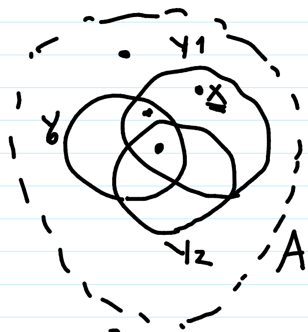
~ 0 //

Def. A poljubna.

$$\bigcup A = \{x ; \exists y \in A : x \in y\}$$

Def. $A \neq \emptyset$.

$$\bigcap A = \{x ; \forall y \in A : x \in y\}.$$



Zgled. (i) $A = \{\underbrace{\{\cancel{1}, \cancel{2}, 3\}}, \{2,3,4\}, \{3,4,5\}\}$.

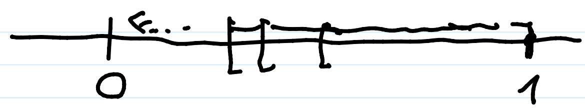
podmnožice A: $\emptyset$, $\{\{1,2,3\}\}$, ..., $\{\{1,2,3\},\{2,3,4\}\}$, ..., A

$$\bigcup A = \{1,2,3,4,5\}$$
$$\bigcap A = \{3\}$$

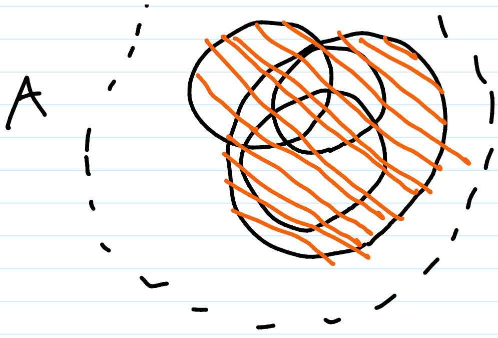
(ii) $A = \{[\frac{1}{n}, 1] ; n \in \mathbb{N} \land n > 0\}$

$$= \{[1,1], [\tfrac{1}{2}, 1], [\tfrac{1}{3}, 1], \dots\}$$

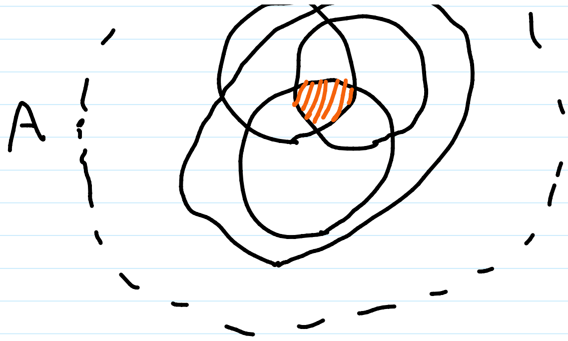
$$\bigcup A = (0, 1]$$



$$\bigcap A = \{1\}$$



$\bigcup A$



$\bigcap A$

Trditev. $\forall y \in A: y \subseteq \bigcup A$.

$A \neq \emptyset$: $\forall y \in A: \bigcap A \subseteq y$.

Aksiom o uniji (AU). Za vsako mn. A obstaja množica $\bigcup A$, oziroma:

$$\forall A\, \exists B\, \forall x: (x \in B \Leftrightarrow \exists y \in A: x \in y)$$

Posledica. Za poljubni množici A in B obstajata $A \cup B$ in $A \oplus B$.

Dokaz: (i) Po AP obstaja mn. $C = \{A, B\}$.

Torej po AU obstaja množica

$$\bigcup C = A \cup B. \checkmark$$

(ii) $A \oplus B = (A \setminus B) \cup (B \setminus A)$

Ker $A \setminus B$ in $B \setminus A$ obstajata, po (i) obstaja $(A \setminus B) \cup (B \setminus A) = A \oplus B$. ✓

Trditev. Naj bo $A \neq \emptyset$. Potem obstaja $\bigcap A$.

Dokaz: Vzemimo $y_0 \in A$. Po AsP obstaja

$$P = \{x;\ x \in y_0 \wedge \forall y \in A: x \in y\}$$

Velja: $\forall y \in A: x \in y \xRightarrow{y_0 \in A} x \in y_0$,

Zato velja: $x \in y_0 \wedge \forall y \in A: x \in y \Longleftrightarrow \forall y \in A: x \in y$, Zato velja: $x \in y_0 \land \forall y \in A: x \in y \iff \forall y \in A: x \in y$,

torej je $P = \{x;\ \forall y \in A: x \in y\} = \bigcap A$. ✓

<u>Pripomba.</u> Kaj je $\bigcap \emptyset$?

$$\bigcap \emptyset = \{x;\ \forall y \in \emptyset: x \in y\}$$
$$= \{x;\ \forall y: \underbrace{(\overset{0}{y \in \emptyset} \Rightarrow x \in y)}_{1}\} = V$$

<u>pravi</u> razred vseh množic

<u>Komplement množice</u>

Pogosto se omejimo na neko <u>univerzalno množico</u> ali <u>svet</u> S in gledamo le njene elemente in podmnožice.

<u>Def.</u> Naj bo $A \subseteq S$.

$$A^c = \{x \in S;\ x \notin A\} = \{x;\ x \in S \land x \notin A\}$$
$$= S \setminus A \ ✓$$

<u>Izrek.</u> Za vse $A, B \subseteq S$ velja:

1. $(A^c)^c = A$
2. $(A \cup B)^c = A^c \cap B^c$
   $(A \cap B)^c = A^c \cup B^c$
3. $A \setminus B = A \cap B^c$
4. $A \subseteq B \iff B^c \subseteq A^c$
5. $\underline{A \cap B = \emptyset} \iff A \subseteq B^c \iff B \subseteq A^c$

tuji množici

6. $A \cup A^c = S$, $A \cap A^c = \emptyset$

6. $A \cup A^c = S, \quad A \cap A^c = \emptyset$

7. $A \cup S = S, \quad A \cap S = A$

<u>Dokaz:</u> $A \cap B = \emptyset \overset{\text{def.}\,\emptyset}{\iff} \forall x: x \notin A \cap B$

$$\iff \forall x: \neg(x \in A \wedge x \in B)$$
$$\iff \forall x: (x \notin A \vee x \notin B)$$
$$\iff \forall x: (x \in A \Rightarrow x \notin B)$$
$$\iff \forall x: (x \in A \Rightarrow x \in B^c)$$
$$\iff A \subseteq B^c \checkmark$$

<u>Pripomba.</u> Za $A \subseteq S$ definiramo:

$$\bigcap A = \{x \in S;\ \forall y \in A: x \in y\}$$

Potem je $\bigcap \emptyset = \{x;\ x \in S \wedge \underbrace{\forall y \in \emptyset: x \in y}_{1}\}$

$$= \{x;\ x \in S\} = S$$

Potenčna množica

<u>Def.</u> $\mathcal{P}A = \{X;\ X \subseteq A\}$ je <u>potenčna množica</u> ali <u>mn. vseh podmnožic</u> mn. A.

Velja: $X \in \mathcal{P}A \iff X \subseteq A$.

<u>Zgled.</u>
$$\mathcal{P}\emptyset = \{\emptyset\}$$
$$\mathcal{P}\mathcal{P}\emptyset = \mathcal{P}\{\emptyset\} = \{\emptyset, \{\emptyset\}\}$$
$$\mathcal{P}\mathcal{P}\mathcal{P}\emptyset = \mathcal{P}\{\emptyset, \{\emptyset\}\} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

Kolikšno elementov ima končna mn. z n elementi? Odg.: 2^n.

Koliko elementov ima končna mn. z n elementi? Odg.: 2^n.

$\emptyset$	ima	0 eltov
$\mathcal{P}\emptyset$	〃	1 elt
$\mathcal{P}\mathcal{P}\emptyset$	〃	2 elta
$\mathcal{P}\mathcal{P}\mathcal{P}\emptyset$	〃	4 elte
$\mathcal{P}\mathcal{P}\mathcal{P}\mathcal{P}\emptyset$	〃	$2^4 = 16$ eltov
$\mathcal{P}\mathcal{P}\mathcal{P}\mathcal{P}\mathcal{P}\emptyset$	〃	$2^{16} = 65536$ eltov
$\mathcal{P}\mathcal{P}\mathcal{P}\mathcal{P}\mathcal{P}\mathcal{P}\emptyset$	〃	2^{65536} eltov