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Aksiom o potenčni množici (APM).

Za vsako množico A obstaja $\mathcal{P}A$, oziroma:

$$\forall A\ \exists B\ \forall x: (x \in B \Leftrightarrow x \subseteq A).$$

Izrek. Za vse A, B velja:

1. $\emptyset \in \mathcal{P}A$, $A \in \mathcal{P}A$
2. $\bigcup \mathcal{P}A = A$, $\bigcap \mathcal{P}A = \emptyset$
3. $A \subseteq B \Rightarrow \mathcal{P}A \subseteq \mathcal{P}B$
4. $\mathcal{P}(A \cap B) = \mathcal{P}A \cap \mathcal{P}B$
5. $\mathcal{P}(A \cup B) \supseteq \mathcal{P}A \cup \mathcal{P}B$

Dokaz 4.

$$\begin{aligned} C \in \mathcal{P}(A \cap B) &\Rightarrow C \subseteq A \cap B \\ &\Rightarrow C \subseteq A \wedge C \subseteq B \\ &\Rightarrow C \in \mathcal{P}A \wedge C \in \mathcal{P}B \\ &\Rightarrow C \in \mathcal{P}A \cap \mathcal{P}B \end{aligned}$$

Torej: $\mathcal{P}(A \cap B) \subseteq \mathcal{P}A \cap \mathcal{P}B$.

$$\begin{aligned} C \in \mathcal{P}A \cap \mathcal{P}B &\Rightarrow C \in \mathcal{P}A \wedge C \in \mathcal{P}B \\ &\Rightarrow C \subseteq A \wedge C \subseteq B \\ &\Rightarrow A \cap C = C \wedge B \cap C = C \\ &\Rightarrow A \cap \underbrace{B \cap C}_{} = A \cap C = C \\ &\Rightarrow C \subseteq A \cap B \Rightarrow C \in \mathcal{P}(A \cap B) \end{aligned}$$

Torej: $\mathcal{P}A \cap \mathcal{P}B \subseteq \mathcal{P}(A \cap B)$. ✓

Urejeni pari in kartezični produkt

urejeni par: (a, b)

1. komponenta (a), 2. komponenta (b)

Osnovna lastnost ur. parov (OL):

$$(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d.)$$

$$(a,b)=(c,d) \Leftrightarrow a=c \wedge b=d.$$

Ali lahko vzamemo $(a,b) := \cancel{\{a,b\}}$?

NE: $a \neq b$: potem $(a,b) \neq (b,a)$,
toda $\{a,b\} = \{b,a\}$

Def. $(a,b) = \{\{a,b\},\{a\}\}$

Izrek. Za vse a,b,c,d velja
$$(a,b)=(c,d) \Leftrightarrow a=c \wedge b=d.$$

Dokaz: $(\Rightarrow)$ $(a,b)=(c,d)$

$$\Rightarrow \{\{a,b\},\{a\}\} = \{\{c,d\},\{c\}\} \quad (*)$$

Ločimo 2 primera:

1. $\{a\} = \{c,d\} \Rightarrow a=c=d$
2. $\{a\} = \{c\} \Rightarrow a=c$

$(*)$: $\{\{a,b\},\{a\}\} = \{\{a,d\},\{a\}\}$

Ločimo 2 primera:

1. $a=b$: $\{\{a,b\},\{a\}\} = \{\{a,a\},\{a\}\}$
$= \{\{a\},\{a\}\} = \{\{a\}\}$
$\Rightarrow \{a,d\} = \{a\} \Rightarrow a=d$
$\Rightarrow b=d$

2. $a \neq b$: $\{a,b\} \neq \{a\}$
$(*) \Rightarrow \{a,b\} = \{a,d\}$
$\Rightarrow b \in \{a,d\} \Rightarrow b=d$ ✓

Trditev. Za vse a,b obstaja (a,b).

Dokaz: Po AP obstaja $\{a,b\}$.
$\{a,a\} = \{a\}$.

$\#$ $\qquad \{\{a,b\},\{a\}\}$
$= \{\{a\},\{a,b\}\}.$ ✓

<u>Def.</u> Mn. vseh ur. parov, katerih 1. komponenta pripada A, 2. komponenta pa B, je <u>kartezični produkt</u> $A \times B$ množic A in B, oziroma:

$$A \times B = \{(u,v);\ u \in A \wedge v \in B\}$$
$$= \{x;\ \underline{\exists u \exists v : (u \in A \wedge v \in B \wedge x = (u,v))}\}$$

<u>Zgled.</u> $A = \{a,b\}$, $B = \{1,2,3\}$

$$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$$

$$B \times A = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$$

<u>Trditev.</u> Za poljubni A in B obstaja $A \times B$.

<u>Dokaz:</u> Sestavimo mn. M, ki obstaja in velja: $A \times B \subseteq M$.

$$\underline{(a,b) \in A \times B} \Rightarrow a \in A \wedge b \in B$$
$$\overset{AP,\ AU}{\Longrightarrow} \{a,b\} \subseteq A \cup B \wedge \{a\} \subseteq A \cup B$$
$$\overset{APM}{\Longrightarrow} \{a,b\} \in \mathcal{P}(A \cup B) \wedge \{a\} \in \mathcal{P}(A \cup B)$$
$$\overset{(*)}{\Longrightarrow} \{\{a,b\},\{a\}\} \subseteq \mathcal{P}(A \cup B)$$
$$\overset{APM}{\Longrightarrow} \{\{a,b\},\{a\}\} \in \mathcal{P}\mathcal{P}(A \cup B)$$
$$\Longrightarrow \underline{(a,b) \in \mathcal{P}\mathcal{P}(A \cup B)}$$

Sledi: $A \times B \subseteq \mathcal{P}\mathcal{P}(A \cup B)$

Sledi: $A \times B \subseteq \underbrace{\mathcal{P}\mathcal{P}(A \cup B)}_{M}$

Po ASP obstaja

$$C = \{x;\ x \in \mathcal{P}\mathcal{P}(A \cup B) \wedge \underbrace{\exists u \exists v : (u \in A \wedge v \in B \wedge x = (u,v)}_{\varphi(x)}\}$$

Ker zaradi $(*)$ iz $\varphi(x) \Rightarrow x \in \mathcal{P}\mathcal{P}(A \cup B)$,

je $C = \{x;\ \exists u \exists v \,|\, u \in A \wedge v \in B \wedge x = (u,v)\}$

$= A \times B$ ✓

<u>Izrek</u>. Za vse A, B, C, D velja:

1. $A \times \emptyset = \emptyset \times A = \emptyset$
2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$ distrib.
   $(A \cup B) \times C = (A \times C) \cup (B \times C)$ glede na $\cup$
3. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
   superdistributivnost glede na $\cap$
4. $A \times B = \emptyset \iff A = \emptyset \vee B = \emptyset$
5. $A \subseteq C \wedge B \subseteq D \Rightarrow A \times B \subseteq C \times D$
   monotonost
6. $A \times B \subseteq C \times D \wedge A \times B \neq \emptyset$
   $\Rightarrow A \subseteq C \wedge B \subseteq D$

<u>Dokaz</u>: 3. $(x,y) \in \underline{(A \cap B) \times (C \cap D)} \iff$

$\iff x \in A \cap B \wedge y \in C \cap D$

$\iff x \in A \wedge x \in B \wedge y \in C \wedge y \in D$

$\iff (x \in A \wedge y \in C) \wedge (x \in B \wedge y \in D)$

$\iff (x,y) \in A \times C \wedge (x,y) \in B \times D$

$\iff (x,y) \in \underline{(A \times C) \cap (B \times D)}$

$\Leftrightarrow (x,y) \in (A\times C)\cap(B\times D)$

Posledica: $A\times(B\cap C) = (A\cap A)\times(B\cap C)$
$= (A\times B)\cap(A\times C)$

$(A\cap B)\times C = (A\cap B)\times(C\cap C)$
$= (A\times C)\cap(B\times C)$ ✓
distrib. glede na $\cap$

Def. $n\in\mathbb{N}$, $n\geq 3$. Induktivno definiramo urejeno n-terico $a_1, a_2, \ldots, a_n$ takole:

$$(a_1, a_2, \ldots, a_n) = ((a_1, a_2, \ldots, a_{n-1}), a_n)$$

Zgled: $(a_1, a_2, a_3, a_4) = ((a_1, a_2, a_3), a_4)$
$= (((a_1, a_2), a_3), a_4)$

Izrek. Za vse $n\in\mathbb{N}$, $n\geq 2$, velja:
$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \Leftrightarrow$
$\forall i\in\{1,2,\ldots,n\}: a_i = b_i$.

Dokaz: Z indukcijo po n.

Def. Za $n\in\mathbb{N}$, $n\geq 2$, je
$A_1\times A_2\times\cdots\times A_n = \{(a_1, a_2, \ldots, a_n);\ \forall i\in\{1,2,\ldots,n\}: a_i\in A_i\}$
kart. produkt n množic

4. Relacije in funkcije

Def. a) Množica R je dvomestna relacija, če $\forall x\in R\ \exists a\ \exists b: x = (a,b)$.

[illegible]

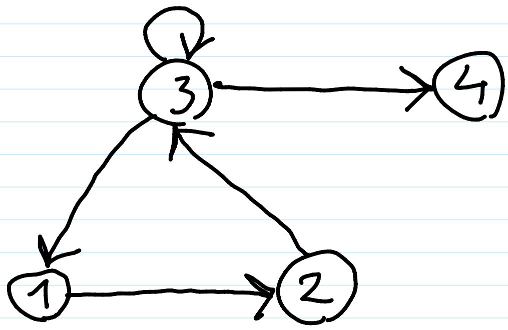
če $\forall x \in R\ \exists a\ \exists b:\ x = (a,b)$.

b) Množica R je <u>dvomestna relacija v A</u>, če $R \subseteq A \times A$.

<u>Zgled.</u> $A = \{1,2,3,4\}$

$$R = \{(1,2),\ (2,3),\ (3,1),\ (3,3),\ (3,4)\} \subseteq A \times A$$

usmerjeni graf relacije R:



<u>Zgled.</u> $A = \mathbb{N}$, R naj bo relacija „$\leq$"

$$R = \{(x,y) \in \mathbb{N} \times \mathbb{N};\ x \leq y\}$$
$$= \{(x,y) \in \mathbb{N} \times \mathbb{N};\ \exists z \in \mathbb{N}:\ x + z = y\}$$
$$= \{(0,0),\ (0,1),\ (1,1),\ (0,2),\ (1,2),\ (2,2), \ldots\}$$

<u>Zgled.</u> A polj. množica

1. $\emptyset \subseteq A \times A$ ... <u>prazna relacija</u>
2. $A \times A \subseteq A \times A$ ... <u>univerzalna relacija</u>
3. $id_A = \{(x,x);\ x \in A\}$

   relacija <u>enakosti</u> ali <u>identitete</u> v A

Relacijska pisava:

relacija [illegible] [illegible]

v A

Relacijska pisava:

namesto $(x,y)\in R$ lahko pišemo xRy

Def. Naj bo $R \subseteq A\times A$.

$$\mathcal{D}_R = \{x\in A;\ \exists y\in A:\ xRy\}$$

domena ali def. območje R

$$\mathcal{Z}_R = \{y\in A;\ \exists x\in A:\ xRy\}$$

zaloga vrednosti R

Zgled. 1. $A=\{1,2,3,4\}$, $R=\{(1,3),(2,2),(4,3)\}$.

Potem: $\mathcal{D}_R=\{1,2,4\}$, $\mathcal{Z}_R=\{2,3\}$.

2. $A=\mathbb{N}$, $R=\{(x,y)\in\mathbb{N}\times\mathbb{N};\ \exists z\in\mathbb{N}: x+z+1=y\}$,

torej relacija $<$.

$\mathcal{D}_R=\mathbb{N}$, $\mathcal{Z}_R=\mathbb{N}\setminus\{0\}$

Def. Naj bo $R\subseteq A\times A$. Rel. R je:

1. refleksivna, če $\forall x\in A: xRx$.
2. irefleksivna, če $\forall x\in A: \neg xRx$.
3. simetrična, če $\forall x,y\in A: (xRy \Rightarrow yRx)$
4. antisimetrična, če $\quad$ " $\quad (xRy \wedge yRx \Rightarrow x=y)$
5. asimetrična, če $\quad$ " $\quad (xRy \Rightarrow \neg yRx)$
6. tranzitivna, če $\forall x,y,z\in A:$
   $(xRy\wedge yRz \Rightarrow xRz)$
7. intranzitivna, če $\forall x,y,z\in A:$
   $(xRy\wedge yRz \Rightarrow \neg xRz)$
8. strogo sovisna, če $\forall x,y\in A: (xRy \vee yRx)$

$(xRy \land yRz \Rightarrow xRz)$

8. strogo sovisna, če $\forall x, y \in A: (xRy \lor yRx)$

9. sovisna, če $\forall x, y \in A: (x \neq y \Rightarrow xRy \lor yRx)$

10. enolična, če $\forall x, y, z \in A: (xRy \land xRz \Rightarrow y = z)$.