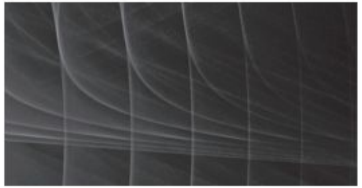
DSP Lecture 1

introduction to Continuous (Time) Signal Processing

Tal Rosenwein

Lecture Overview

[Credit](#)



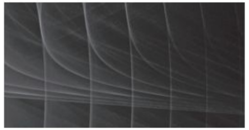
LINEAR SYSTEMS
AND SIGNALS

THIRD EDITION

B. P. Lathi and R. A. Green

Lecture Overview

1. Preliminaries
2. Signal and Systems
3. Time Domain Analysis of Continuous Time Systems
4. Continuous-Time System Analysis Using The Laplace Transform
5. Frequency-Response and Filtering



LINEAR SYSTEMS AND SIGNALS

THIRD EDITION

B. P. Lathi and R. A. Green

Lecture Goals

1. What ‘happens’ to a signal when it passes through a **(LTIC) system**.
2. Understand what system’s **stability** means
3. Find the output the given the input, and the system’s **transfer function**
4. Understand the **duality** between **time** and **frequency domains**
5. Get some basic intuition about **filtering**.

Preliminaries

B.1-2 Algebra of Complex Numbers

$$z = a + jb \tag{B.1}$$

This representation is the Cartesian (or rectangular) form of complex number z. The numbers a and b (the abscissa and the ordinate) of z are the *real part* and the *imaginary part*, respectively, of z. They are also expressed as

$$\operatorname{Re} z = a \quad \text{and} \quad \operatorname{Im} z = b$$

Note that in this plane all real numbers lie on the horizontal axis, and all imaginary numbers lie on the vertical axis.

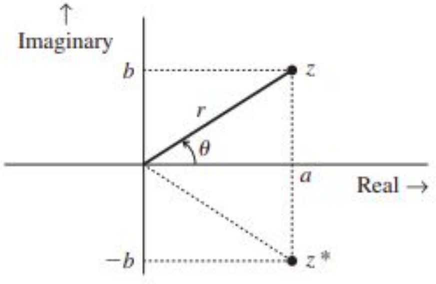


Figure B.2 Representation of a number in the complex plane.

Preliminaries

B.1-2 Algebra of Complex Numbers

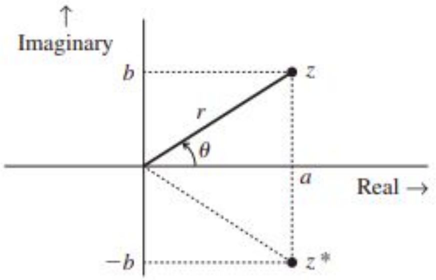


Figure B.2 Representation of a number in the complex plane.

Clearly, it follows that $e^{j\theta} = \cos\theta + j\sin\theta$. Using Eq. (B.3) in Eq. (B.2) yields

$$z = re^{j\theta} \tag{B.4}$$

This representation is the polar form of complex number z.

Summarizing, a complex number can be expressed in rectangular form $a + jb$ or polar form $re^{j\theta}$ with

$$\begin{aligned} a &= r\cos\theta \\ b &= r\sin\theta \end{aligned} \qquad \text{and} \qquad \begin{aligned} r &= \sqrt{a^2 + b^2} \\ \theta &= \tan^{-1}\left(\frac{b}{a}\right) \end{aligned} \tag{B.5}$$

Preliminaries

B.1-2 Algebra of Complex Numbers

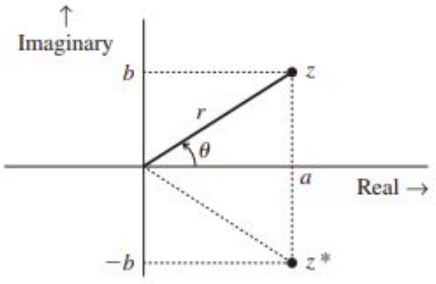


Figure B.2 Representation of a number in the complex plane.

Complex numbers may also be expressed in terms of polar coordinates. If (r, θ) are the polar coordinates of a point $z = a + jb$ (see Fig. B.2), then

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

Consequently,

$$z = a + jb = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta) \tag{B.2}$$

Euler's formula states that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{B.3}$$

Preliminaries

B.1-2 Algebra of Complex Numbers

$$z_1 = r_1 e^{j\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{j\theta_2}$$

then

$$z_1 z_2 = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

and

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$$

Moreover,

$$z^n = (re^{j\theta})^n = r^n e^{jn\theta}$$

and

$$z^{1/n} = (re^{j\theta})^{1/n} = r^{1/n} e^{j\theta/n}$$

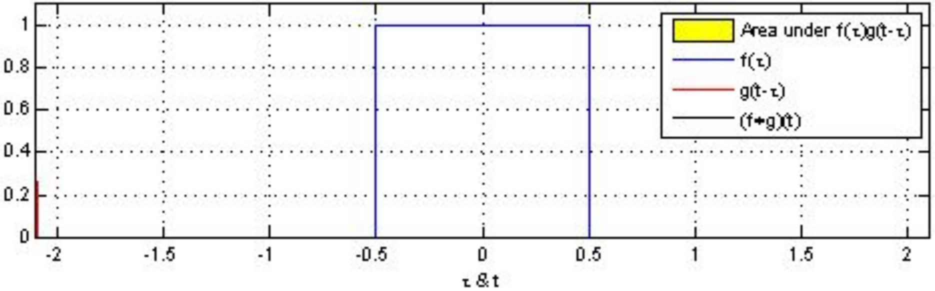
Preliminaries

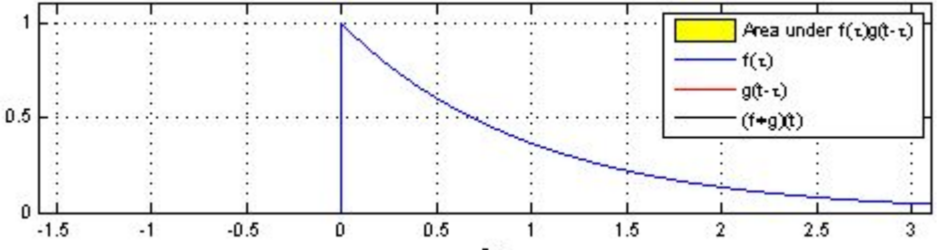
Convolution

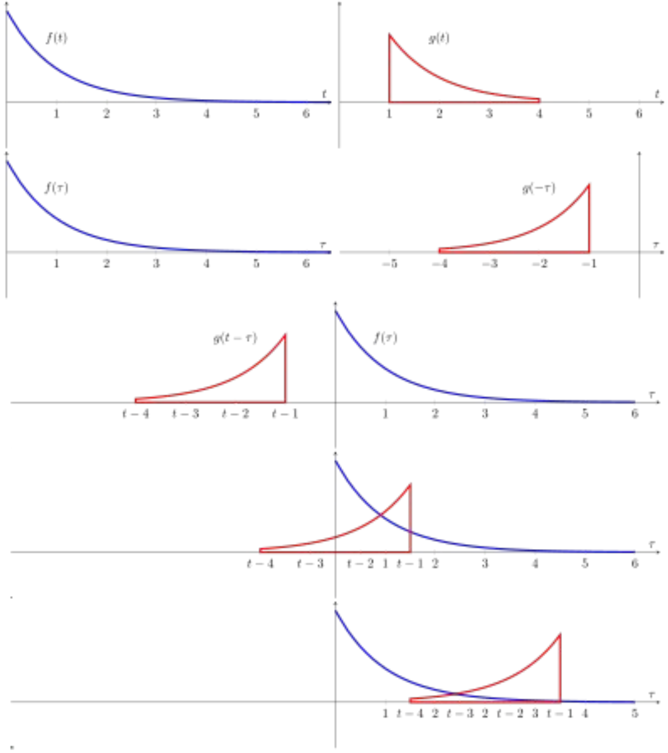
$$(f * g)(t) := \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau.$$

An equivalent definition is (see commutativity):

$$(f * g)(t) := \int_{-\infty}^{\infty} f(t-\tau) g(\tau)\, d\tau.$$







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Lecture Overview

1. Preliminaries
2. **Signal** and Systems
3. Time Domain Analysis of Continuous Time Systems
4. Continuous-Time System Analysis Using The Laplace Transform
5. Frequency-Response and Filtering

1.4-1 The Unit Step Function $u(t)$

In much of our discussion, the signals begin at $t = 0$ (causal signals). Such signals can be conveniently described in terms of unit step function $u(t)$ shown in Fig. 1.14a. This function is defined by

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1.8}$$

$$x(t) = u(t-2) - u(t-4)$$

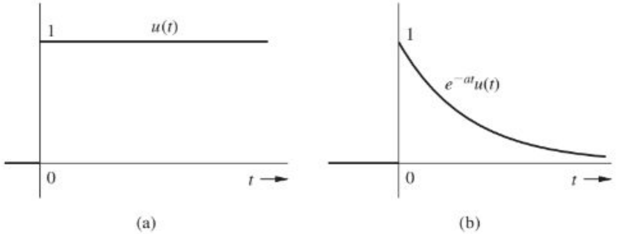

Figure 1.14 **(a)** Unit step function $u(t)$. **(b)** Exponential $e^{-at}u(t)$.

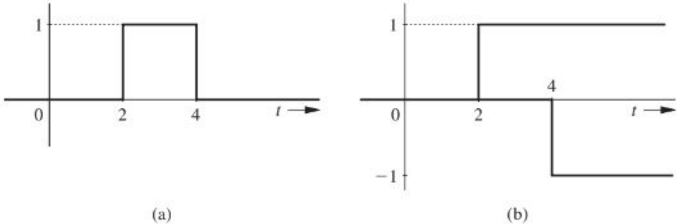

Figure 1.15 Representation of a rectangular pulse by step functions.

1.4-2 The Unit Impulse Function $\delta(t)$

The unit impulse function $\delta(t)$ is one of the most important functions in the study of signals and systems. This function was first defined in two parts by P. A. M. Dirac as

$$\delta(t) = 0 \quad t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{1.9}$$

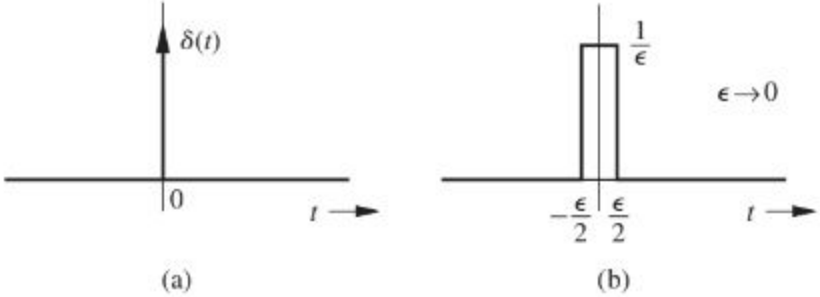

Figure 1.19 A unit impulse and its approximation.

$$\frac{du(t)}{dt} = \delta(t) \tag{1.12}$$

Consequently,

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$$

1.4-3 The Exponential Function e^{st}

Another important function in the area of signals and systems is the exponential signal e^{st}, where s is complex in general, given by

$$s = \sigma + j\omega$$

1. A constant $k = ke^{0t}$ $\quad (s = 0)$
2. A monotonic exponential $e^{\sigma t}$ $\quad (\omega = 0, s = \sigma)$
3. A sinusoid $\cos \omega t$ $\quad (\sigma = 0, s = \pm j\omega)$
4. An exponentially varying sinusoid $e^{\sigma t} \cos \omega t$ $\quad (s = \sigma \pm j\omega)$

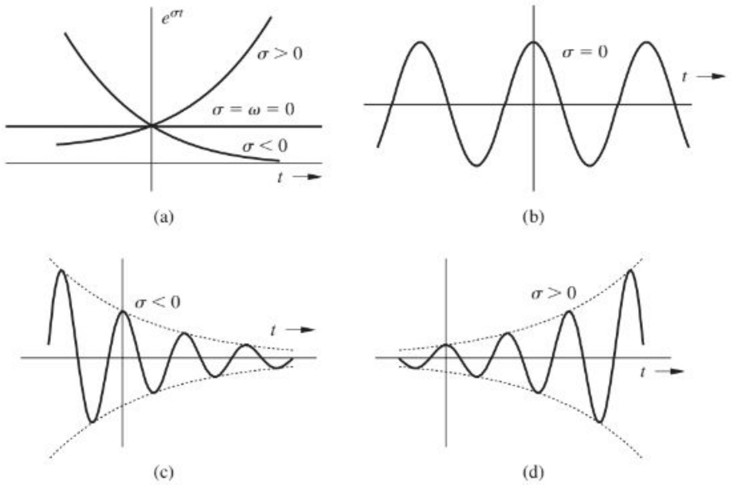

Figure 1.21 Sinusoids of complex frequency $\sigma + j\omega$.

Figure 1.22 Complex frequency plane.

Lecture Overview

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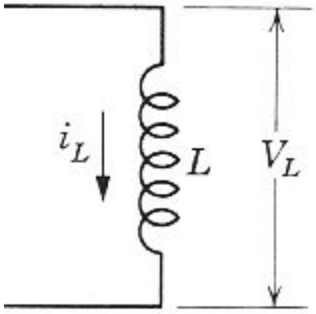
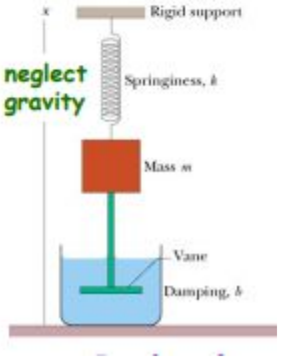
“Systems are all over the place”

$$V_L = L\frac{\partial I}{\partial t}$$



$$\frac{d^2x(t)}{dt^2} + \frac{b}{m}\frac{dx(t)}{dt} = -\frac{k}{m}x(t)$$



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1.6 Systems

As mentioned in Sec. 1.1, systems are used to process signals to allow modification or extraction of additional information from the signals. A system may consist of physical components (hardware realization) or of an algorithm that computes the output signal from the input signal (software realization).

Roughly speaking, a physical system consists of interconnected components, which are characterized by their terminal (input–output) relationships. In addition, a system is governed by laws of interconnection. For example, in electrical systems, the terminal relationships are the familiar voltage-current relationships for the resistors, capacitors, inductors, transformers, transistors, and so on, as well as the laws of interconnection (i.e., Kirchhoff's laws). We use these laws to derive mathematical equations relating the outputs to the inputs. These equations then represent a *mathematical model* of the system.

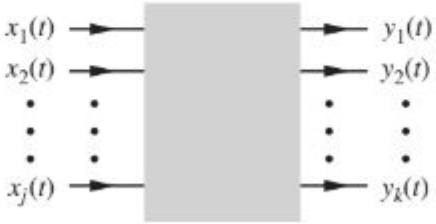

Figure 1.25 Representation of a system.

$$y(t) = H[x(t)]$$

$where:$

$x(t) - input,$

$y(t) - ouput,$

$h(t) - system$

1.7 Classification of Systems

Systems may be classified broadly in the following categories:

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

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$$x_1 \longrightarrow y_1 \qquad \text{and} \qquad x_2 \longrightarrow y_2$$

then for all inputs x_1 and x_2 and all constants k_1 and k_2,

$$k_1x_1 + k_2x_2 \longrightarrow k_1y_1 + k_2y_2$$

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1.7-2 Time-Invariant and Time-Varying Systems

Systems whose parameters do not change with time are *time-invariant* (also *constant-parameter*) systems. For such a system, if the input is delayed by T seconds, the output is the same as before but delayed by T (assuming initial conditions are also delayed by T). This property is expressed graphically in Fig. 1.28.

$$H[x(t-\tau)] = y(t-\tau)$$

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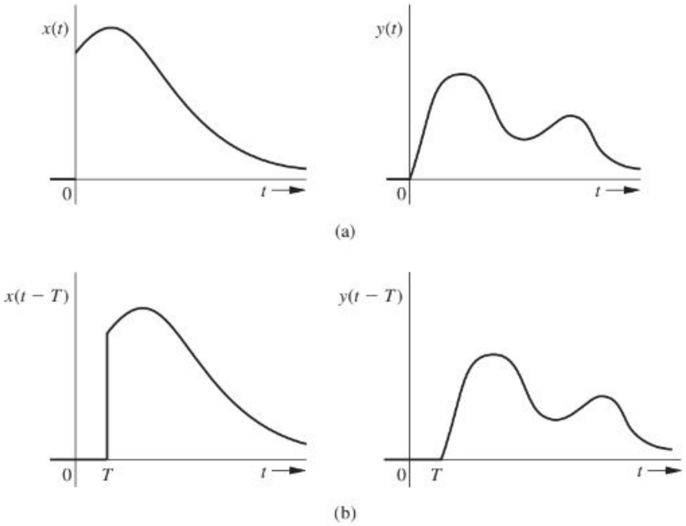

Figure 1.28 Time-invariance property.

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We will focus on *Linear Time Invariant* (LTI) systems

1.7 Classification of Systems

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1.7-4 Causal and Noncausal Systems

A *causal* (also known as a *physical* or *nonanticipative*) system is one for which the output at any instant t_0 depends only on the value of the input $x(t)$ for $t \leq t_0$. In other words, the value of the output at the present instant depends only on the past and present values of the input $x(t)$, not on its future values. To put it simply, in a causal system the output cannot start before the input is

$$causal : y(t) = x(t-2) \rightarrow y(0) = x(-2)$$

$$non\!-\!causal : y(t) = x(t+2) \rightarrow y(0) = x(2)$$

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$$H[x(t)] = y(t) \to H[x[n]] = y[n]$$
$$H[x(t)] = ax(t) \to H[x[n]] = ay[n]$$
$$H[x(t)] = \frac{\partial x(t)}{\partial t} \to H[x[n]] = x[n] - x[n-1]$$
$$H[x(t)] = \int x(t)dt \to H[x[n]] = \sum x[n]$$

1.7 Classification of Systems

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1.7-8 Stable and Unstable Systems

Systems can also be classified as *stable* or *unstable* systems. Stability can be *internal* or *external*. If every *bounded input* applied at the input terminal results in a *bounded output*, the system is said to be stable *externally*. External stability can be ascertained by measurements at the external terminals (input and output) of the system. This type of stability is also known as the stability in the BIBO (bounded-input/bounded-output) sense.

$$y(t) = \frac{\partial x(t)}{\partial t} \rightarrow \frac{\partial u(t)}{\partial t} = \delta(t) - unbounded \qquad y(t) = cos(t) \rightarrow bounded$$

Lecture Overview

1. Preliminaries
2. Signal and Systems
3. **Time Domain Analysis of Continuous Time Systems**
4. Continuous-Time System Analysis Using The Laplace Transform
5. Frequency-Response and Filtering

For the purpose of analysis, we shall consider *linear differential systems*. This is the class of LTIC systems introduced in Ch. 1, for which the input $x(t)$ and the output $y(t)$ are related by linear differential equations of the form

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t)$$
$$= b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t) \tag{2.1}$$

where all the coefficients a_i and b_i are constants. Using operator notation D to represent d/dt, we can express this equation as

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t)$$
$$= (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

or

$$Q(D) y(t) = P(D) x(t) \tag{2.2}$$

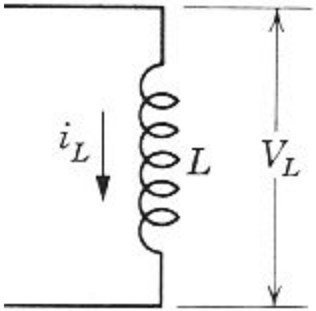
where the polynomials $Q(D)$ and $P(D)$ are

$$Q(D) = D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N$$
$$P(D) = b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N$$

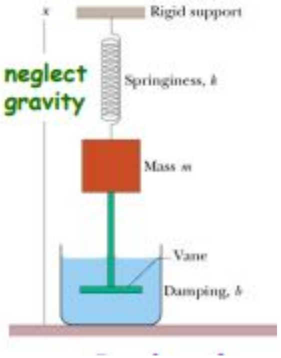
Theoretically the powers M and N in the foregoing equations can take on any value. However, practical considerations make $M > N$ undesirable for two reasons.

“Systems are all over the place”

$$V_L = L\frac{\partial I}{\partial t}$$



$$\frac{d^2x(t)}{dt^2} + \frac{b}{m}\frac{dx(t)}{dt} = -\frac{k}{m}x(t)$$



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In Ch. 1, we demonstrated that a system described by Eq. (2.2) is linear. Therefore, its response can be expressed as the sum of two components: the zero-input response and the zero-state response (decomposition property).[‡] Therefore,

$$\text{total response} = \text{zero-input response} + \text{zero-state response}$$

The zero-input response is the system output when the input $x(t) = 0$, and thus it is the result of internal system conditions (such as energy storages, initial conditions) alone. It is independent of the external input $x(t)$. In contrast, the zero-state response is the system output to the external input $x(t)$ when the system is in zero state, meaning the absence of all internal energy storages: that is, all initial conditions are zero.

$$Q(D)y(t) = P(D)x(t)$$

$$TotalResponse = Y_{ZIR} + Y_{ZSR}$$

$$Y_{ZIR} \to x(t) = 0, y_0(t) \neq 0 \to Q(D)y_0(t) = 0$$

$$Y_{ZSR} \to x(t) \neq 0, y_0(t) = 0 \to Q(D)h(t) = P(D)\delta(t)$$

In practical problems, we must derive such conditions from the physical situation. For instance, in an *RLC* circuit, we may be given the conditions (initial capacitor voltages, initial inductor currents, etc.).

Finding Yzir

can readily show that a general solution is given by the sum of these N solutions[†] so that

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t} \tag{2.6}$$

where $c_1, c_2, \ldots, c_N$ are arbitrary constants determined by N constraints (the auxiliary conditions) on the solution.

Observe that the polynomial $Q(\lambda)$, which is characteristic of the system, has nothing to do with the input. For this reason the polynomial $Q(\lambda)$ is called the *characteristic polynomial* of the system. The equation

$$Q(\lambda) = 0$$

is called the *characteristic equation* of the system. Equation (2.5) clearly indicates that $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the roots of the characteristic equation; consequently, they are called the *characteristic roots* of the system. The terms *characteristic values, eigenvalues,* and *natural frequencies* are also used for characteristic roots.[‡] The exponentials $e^{\lambda_i t}$ $(i = 1, 2, \ldots, n)$ in the zero-input response are the *characteristic modes* (also known as *natural modes* or simply as *modes*) of the system. There is a characteristic mode for each characteristic root of the system, and the *zero-input response is a linear combination of the characteristic modes of the system.*

An LTIC system's characteristic modes comprise its single most important attribute. Characteristic modes not only determine the zero-input response but also play an important role in determining the zero-state response. In other words, the entire behavior of a system is dictated primarily by its characteristic modes. In the rest of this chapter we shall see the pervasive presence

Finding Yzsr

We will use the properties of LTI systems:

$$\text{input} \Longrightarrow \text{output}$$

$$\delta(t) \Longrightarrow h(t)$$

Time invariance

$$\delta(t - n\Delta\tau) \Longrightarrow h(t - n\Delta\tau)$$

Linearity

$$[x(n\Delta\tau)\Delta\tau]\delta(t - n\Delta\tau) \Longrightarrow [x(n\Delta\tau)\Delta\tau]h(t - n\Delta\tau)$$

$$\underbrace{\lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)\delta(t - n\Delta\tau)\,\Delta\tau}_{x(t) \quad \text{[see Eq. (2.22)]}} \Longrightarrow \underbrace{\lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau}_{y(t)}$$

Then, using **superposition (convolution)** we can **invoke** the system at **every** timestep t (using shifted unit impulse times the input at time t), calculate it’s response and **sum** it over time to get the output.

$$y(t) = \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\,d\tau \qquad (2.23)$$

This is the result we seek. We have obtained the system response $y(t)$ to an arbitrary input $x(t)$ in terms of the unit impulse response $h(t)$. Knowing $h(t)$, we can determine the response $y(t)$ to any input.

Finding Yzsr- Intuition (Time Domain)

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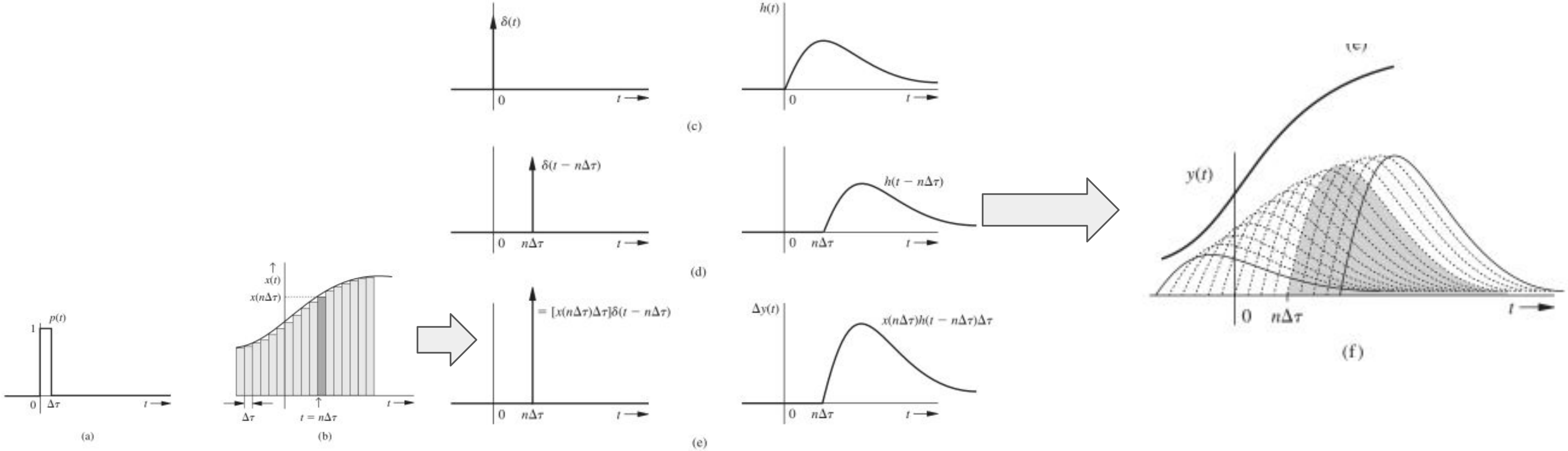
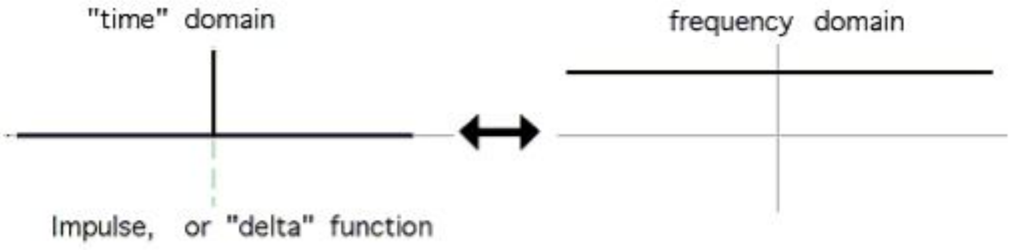


Finding Yzsr- Intuition (Spectral Domain)



4.2-6 Time Convolution and Frequency Convolution

Another pair of properties states that if

$$x_1(t) \Longleftrightarrow X_1(s) \qquad \text{and} \qquad x_2(t) \Longleftrightarrow X_2(s)$$

then (*time-convolution property*)

$$x_1(t) * x_2(t) \Longleftrightarrow X_1(s)X_2(s) \tag{4.17}$$

The system will “color” the flattened (unit impulse) input’s spectrum

Finding Total Response

$$\text{total response} = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{\text{ZIR}} + \underbrace{x(t) * h(t)}_{\text{ZSR}}$$

The Everlasting Exponential *exp(st) & Transfer Function* (1)

2.4-4 A Very Special Function for LTIC Systems: The Everlasting Exponential e^{st}

There is a very special connection of LTIC systems with the everlasting exponential function e^{st}, where s is a complex variable, in general. We now show that the LTIC system's (zero-state) response to everlasting exponential input e^{st} is also the same everlasting exponential (within a multiplicative constant). Moreover, no other function can make the same claim. Such an input for which the system response is also of the same form is called the *characteristic function* (also *eigenfunction*) of the system. Because a sinusoid is a form of exponential ($s = \pm j\omega$), everlasting sinusoid is also a characteristic function of an LTIC system. Note that we are talking here of an everlasting exponential (or sinusoid), which starts at $t = -\infty$.

If $h(t)$ is the system's unit impulse response, then system response $y(t)$ to an everlasting exponential e^{st} is given by

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$$

The integral on the right-most side is a function of a complex variable s and a constant with respect to t. Let us denote this term by $H(s)$, which is also complex, in general. Thus,

$$y(t) = H(s)e^{st} \tag{2.38}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau \tag{2.39}$$

The Everlasting Exponential *exp(st) & Transfer Function* (2)

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau \tag{2.39}$$

Equation (2.38) is valid only for the values of s for which $H(s)$ exists, that is, if $\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$ exists (or converges). The region in the s plane for which this integral converges is called the *region of convergence* for $H(s)$. Further elaboration of the region of convergence is presented in Ch. 4.

For a given s, note that $H(s)$ is a constant. Thus, the input and the output are the same (within a multiplicative constant) for the everlasting exponential signal.

$H(s)$, which is called the *transfer function* of the system, is a function of complex variable s. An alternate definition of the transfer function $H(s)$ of an LTIC system, as seen from Eq. (2.38), is

$$H(s) = \left.\frac{\text{output signal}}{\text{input signal}}\right|_{\text{input}=\text{everlasting exponential } e^{st}} = \frac{P(s)}{Q(s)} \tag{2.40}$$

The transfer function is defined for, and is meaningful to, LTIC systems only. It does not exist for nonlinear or time-varying systems, in general.

External (BIBO) Stability (1)

Clearly, when a system is in stable equilibrium, application of a small disturbance (input) produces a small response. In contrast, when the system is in unstable equilibrium, even a minuscule disturbance (input) produces an unbounded response. The BIBO-stability definition can be understood in the light of this concept. If every bounded input produces bounded output, the system is (BIBO) stable.† In contrast, if even one bounded input results in unbounded response, the system is (BIBO) unstable.

For an LTIC system,

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

Therefore,

$$|y(t)| \le \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\,d\tau$$

Moreover, if $x(t)$ is bounded, then $|x(t-\tau)| < K_1 < \infty$, and

$$|y(t)| \le K_1 \int_{-\infty}^{\infty} |h(\tau)|\,d\tau$$

Hence for BIBO stability,

$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty \tag{2.45}$$

This is a sufficient condition for BIBO stability. We can show that this is also a necessary condition (see Prob. 2.5-7). Therefore, for an LTIC system, if its impulse response $h(t)$ is absolutely integrable, the system is (BIBO) stable. Otherwise it is (BIBO) unstable.

Internal (Asymptotic) Stability (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t)$$
$$= b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t) \qquad (2.1)$$

where the polynomials $Q(D)$ and $P(D)$ are

$$Q(D) = D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N$$
$$P(D) = b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N$$

$$H(s) = \left.\frac{\text{output signal}}{\text{input signal}}\right|_{\text{input=everlasting exponential } e^{st}} = \frac{P(s)}{Q(s)} \qquad (2.40)$$

For a system characterized by Eq. (2.1), we can restate the internal stability criterion in terms of the location of the N characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_N$ of the system in a complex plane. The characteristic modes are of the form $e^{\lambda_k t}$ or $t^r e^{\lambda_k t}$. The locations of various roots in the complex plane and the corresponding modes are shown in Fig. 2.17. These modes $\to 0$ as $t \to \infty$ if Re $\lambda_k < 0$. In contrast, the modes $\to \infty$ as $t \to \infty$ if Re $\lambda_k > 0$.†

Internal (Asymptotic) Stability (2)

1. An LTIC system is asymptotically stable if, and only if, all the characteristic roots are in the LHP. The roots may be simple (unrepeated) or repeated.
2. An LTIC system is unstable if, and only if, one or both of the following conditions exist: (i) at least one root is in the RHP; (ii) there are repeated roots on the imaginary axis.
3. An LTIC system is marginally stable if, and only if, there are no roots in the RHP, and there are some unrepeated roots on the imaginary axis.

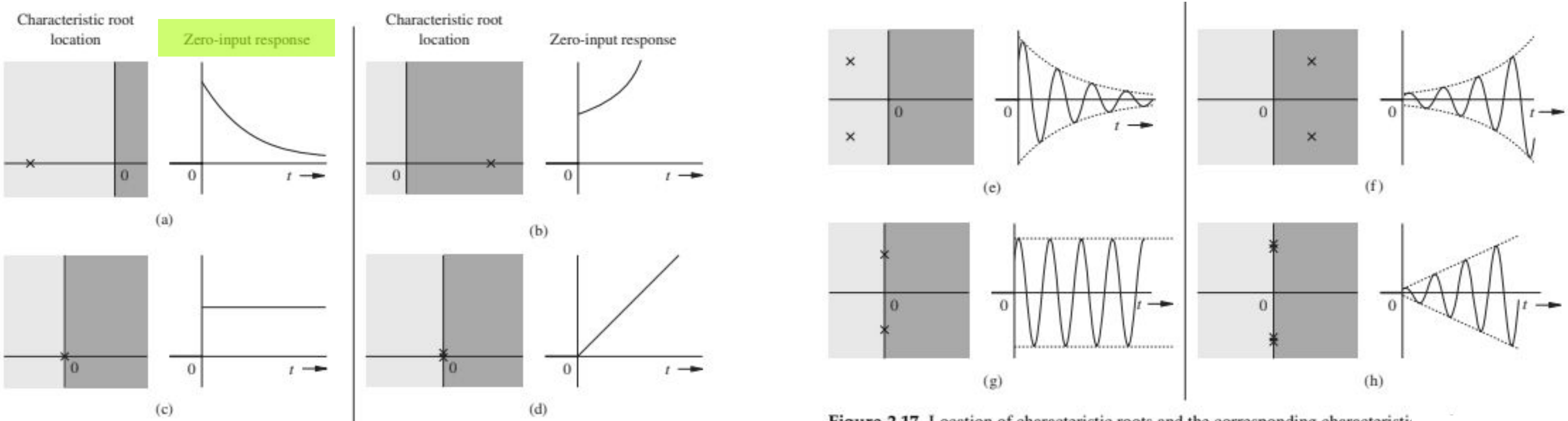


Figure 2.17 Location of characteristic roots and the corresponding characteristi

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Stability - Resonance (1)

https://www.youtube.com/watch?v=3mclp9QmCGs&ab_channel=SimonLesp%C3%A9rance

Lecture Overview

1. Preliminaries
2. Signal and Systems
3. Time Domain Analysis of Continuous Time Systems
4. **Continuous-Time System Analysis Using The Laplace Transform**
5. Frequency-Response and Filtering

Continuous-Time System Analysis Using the Laplace Transform

Pierre-Simon de Laplace

Because of the linearity (superposition) property of linear time-invariant systems, we can find the response of these systems by breaking the input $x(t)$ into several components and then summing the system response to all the components of $x(t)$. We have already used this procedure in time-domain analysis, in which the input $x(t)$ is broken into impulsive components. In the *frequency-domain analysis* developed in this chapter, we break up the input $x(t)$ into exponentials of the form e^{st}, where the parameter s is the complex frequency of the signal e^{st}, as explained in Sec. 1.4-3. This method offers an insight into the system behavior complementary to that seen in the time-domain analysis. In fact, the time-domain and the frequency-domain methods are duals of each other.

The tool that makes it possible to represent arbitrary input $x(t)$ in terms of exponential components is the *Laplace transform*, which is discussed in the following section.

4.1 The Laplace Transform

For a signal $x(t)$, its Laplace transform $X(s)$ is defined by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \tag{4.1}$$

The signal $x(t)$ is said to be the *inverse Laplace transform* of $X(s)$. It can be shown that

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds \tag{4.2}$$

where c is a constant chosen to ensure the convergence of the integral in Eq. (4.1), as explained later. See also [1].

This pair of equations is known as the *bilateral Laplace transform pair*, where $X(s)$ is the direct Laplace transform of $x(t)$ and $x(t)$ is the inverse Laplace transform of $X(s)$. Symbolically,

$$X(s) = \mathcal{L}[x(t)] \qquad \text{and} \qquad x(t) = \mathcal{L}^{-1}[X(s)]$$

Note that

$$\mathcal{L}^{-1}\{\mathcal{L}[x(t)]\} = x(t) \qquad \text{and} \qquad \mathcal{L}\{\mathcal{L}^{-1}[X(s)]\} = X(s)$$

The Laplace transform, defined in this way, can handle signals existing over the entire time interval from $-\infty$ to ∞ (causal and noncausal signals). For this reason it is called the *bilateral* (or *two-sided*) Laplace transform. Later we shall consider a special case—the *unilateral* or *one-sided* Laplace transform—which can handle only causal signals.

Linearity of the Laplace Transform

We now prove that the Laplace transform is a linear operator by showing that the principle of superposition holds, implying that if

$$x_1(t) \Longleftrightarrow X_1(s) \qquad \text{and} \qquad x_2(t) \Longleftrightarrow X_2(s)$$

then

$$a_1x_1(t) + a_2x_2(t) \Longleftrightarrow a_1X_1(s) + a_2X_2(s)$$

4.2 Some Properties of the Laplace Transform

4.2-1 Time Shifting

The time-shifting property states that if

$$x(t)u(t) \Longleftrightarrow X(s)$$

then

$$x(t-t_0)u(t-t_0) \Longleftrightarrow X(s)e^{-st_0} \qquad t_0 \geq 0$$

4.2 Some Properties of the Laplace Transform

4.2-1 Time Shifting

Proof.

$$\mathcal{L}[x(t-t_0)u(t-t_0)] = \int_0^{\infty} x(t-t_0)u(t-t_0)e^{-st}\,dt$$

Setting $t - t_0 = \tau$, we obtain

$$\mathcal{L}[x(t-t_0)u(t-t_0)] = \int_{-t_0}^{\infty} x(\tau)u(\tau)e^{-s(\tau+t_0)}\,d\tau$$

Because $u(\tau) = 0$ for $\tau < 0$ and $u(\tau) = 1$ for $\tau \geq 0$, the limits of integration can be taken from 0 to ∞. Thus,

$$\begin{aligned}\mathcal{L}[x(t-t_0)u(t-t_0)] &= \int_0^{\infty} x(\tau)e^{-s(\tau+t_0)}\,d\tau \\ &= e^{-st_0}\int_0^{\infty} x(\tau)e^{-s\tau}\,d\tau \\ &= X(s)e^{-st_0}\end{aligned}$$

Phase

Note that $x(t-t_0)u(t-t_0)$ is the signal $x(t)u(t)$ delayed by t_0 seconds. The time-shifting property states that *delaying a signal by t_0 seconds amounts to multiplying its transform e^{-st_0}*.

4.2 Some Properties of the Laplace Transform

4.2-1 Time Shifting

The time-shifting property states that if

$$x(t)u(t) \Longleftrightarrow X(s)$$

then

$$x(t-t_0)u(t-t_0) \Longleftrightarrow X(s)e^{-st_0} \qquad t_0 \geq 0$$

4.2-3 The Time-Differentiation Property

The time-differentiation property states that if†

$$x(t) \Longleftrightarrow X(s)$$

then

$$\frac{dx(t)}{dt} \Longleftrightarrow sX(s) - x(0^-)$$

4.2-4 The Time-Integration Property

The time-integration property states that if†

$$x(t) \Longleftrightarrow X(s)$$

then

$$\int_{0^-}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{X(s)}{s} \qquad \text{and} \qquad \int_{-\infty}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s}$$

4.2 Some Properties of the Laplace Transform

4.2-5 The Scaling Property

The scaling property states that if

$$x(t) \Longleftrightarrow X(s)$$

Down-sampling 48KHz → 16kHz resulted in x3 factor in energy

then for $a > 0$

$$x(at) \Longleftrightarrow \frac{1}{a}X\left(\frac{s}{a}\right)$$

4.2-6 Time Convolution and Frequency Convolution

Another pair of properties states that if

$$x_1(t) \Longleftrightarrow X_1(s) \qquad \text{and} \qquad x_2(t) \Longleftrightarrow X_2(s)$$

then (*time-convolution property*)

$$x_1(t) * x_2(t) \Longleftrightarrow X_1(s)X_2(s) \tag{4.17}$$

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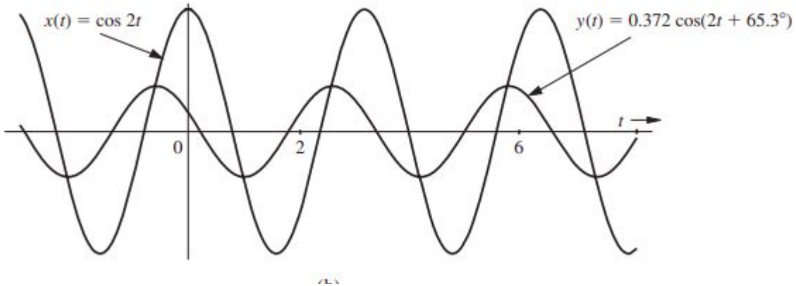
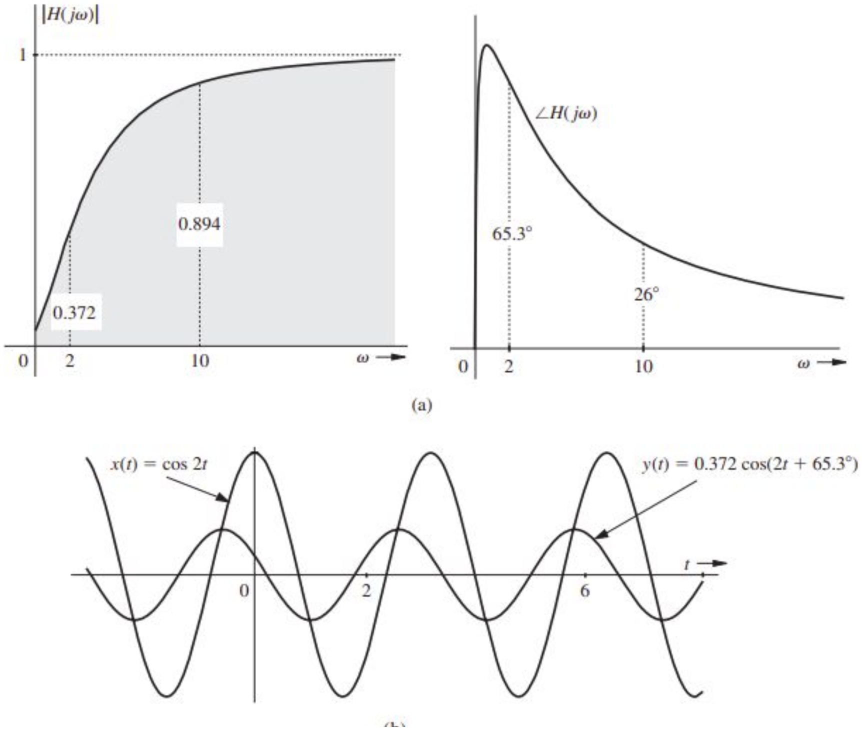
5. **Frequency-Response and Filtering**

4.8 Frequency Response of an LTIC System



4.8 Frequency Response of an LTIC System



4.8 Frequency Response of an LTIC System

Filtering is an important area of signal processing. Filtering characteristics of a system are indicated by its response to sinusoids of various frequencies varying from 0 to ∞. Such characteristics are called the frequency response of the system. In this section, we shall find the frequency response of LTIC systems.

In Sec. 2.4-4 we showed that an LTIC system response to an everlasting exponential input $x(t) = e^{st}$ is also an everlasting exponential $H(s)e^{st}$. As before, we use an arrow directed from the input to the output to represent an input–output pair:

$$e^{st} \Longrightarrow H(s)e^{st} \tag{4.40}$$

Setting $s = j\omega$ in this relationship yields

$$e^{j\omega t} \Longrightarrow H(j\omega)e^{j\omega t} \tag{4.41}$$

Noting that $\cos\omega t$ is the real part of $e^{j\omega t}$, use of Eq. (2.31) yields

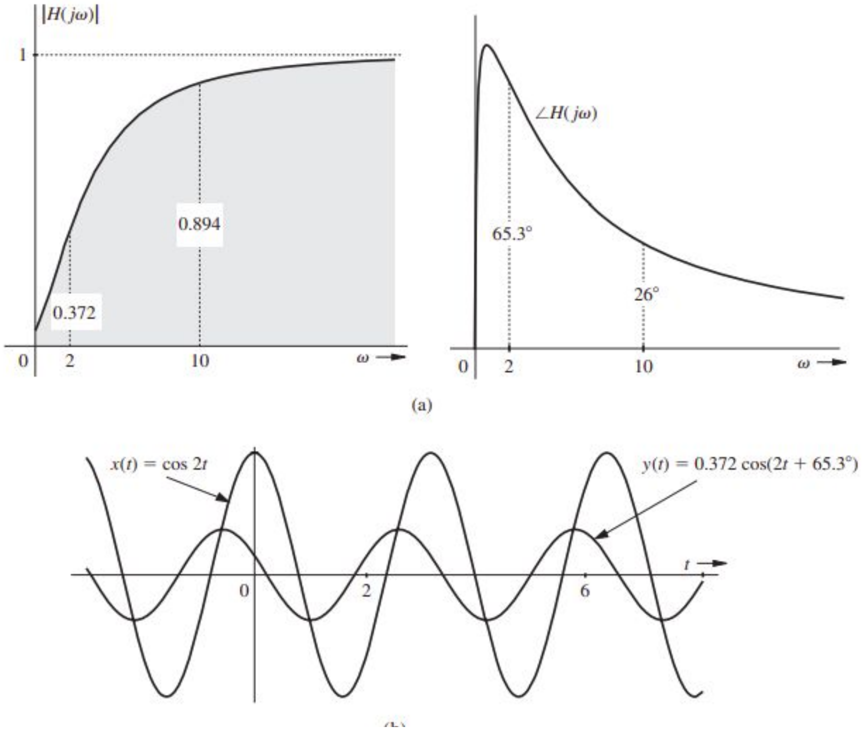
$$\cos\omega t \Longrightarrow \mathrm{Re}[H(j\omega)e^{j\omega t}] \tag{4.42}$$

We can express $H(j\omega)$ in the polar form as

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

With this result, Eq. (4.42) becomes

$$\cos\omega t \Longrightarrow |H(j\omega)|\cos[\omega t + \angle H(j\omega)]$$


|H(jω)|
1
0.894
0.372
0
2
10
ω
∠H(jω)
65.3°
26°
0
2
10
ω
(a)
x(t) = cos 2t
y(t) = 0.372 cos(2t + 65.3°)
0
2
6
t

4.8 Frequency Response of an LTIC System

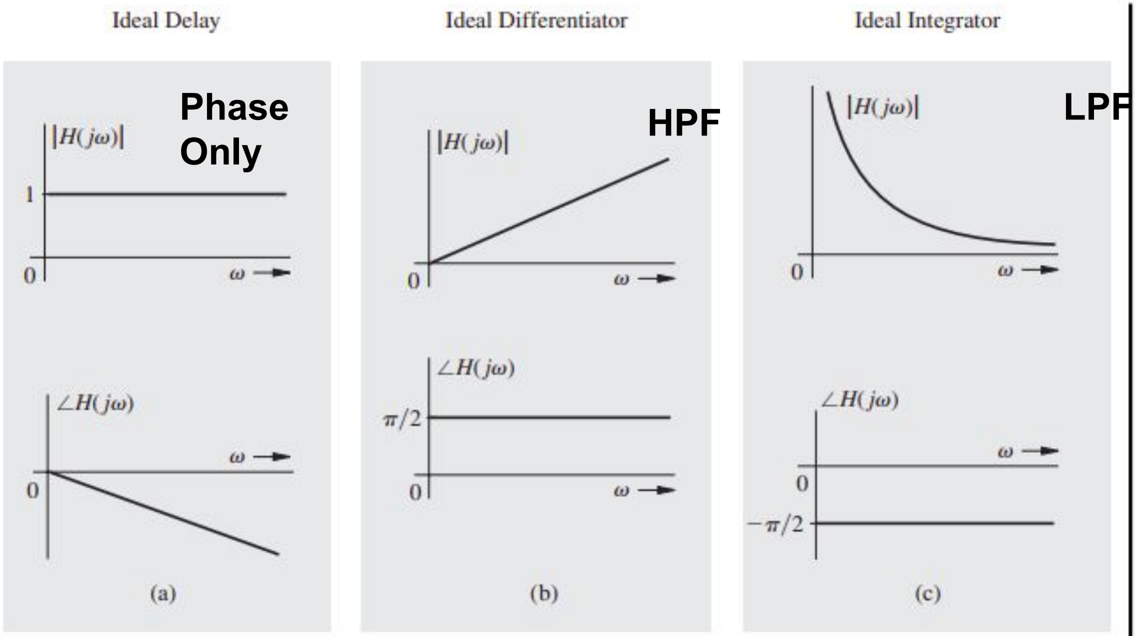


Figure 4.39 Frequency response of an ideal **(a)** delay, **(b)** differentiator, and **(c)** integrator.

$H(s) = e^{-sT}$

$H(j\omega) = e^{-j\omega T}$

$|H(j\omega)| = 1 \quad \text{and} \quad \angle H(j\omega) = -\omega T$

$H(s) = s$

$H(j\omega) = j\omega = \omega e^{j\pi/2}$

$|H(j\omega)| = \omega \quad \text{and} \quad \angle H(j\omega) = \frac{\pi}{2}$

$H(s) = \frac{1}{s}$

$H(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega} = \frac{1}{\omega} e^{-j\pi/2}$

$|H(j\omega)| = \frac{1}{\omega} \quad \text{and} \quad \angle H(j\omega) = -\frac{\pi}{2}$

Values of s for which $X(s) = 0$ are called the *zeros* of $X(s)$; the values of s for which $X(s) \to \infty$ are called the *poles* of $X(s)$. If $X(s)$ is a rational function of the form $P(s)/Q(s)$, the roots of $P(s)$ are the zeros and the roots of $Q(s)$ are the poles of $X(s)$.

4.10 Filter Design by Placement of Poles and Zeros of $H(s)$

In this section we explore the strong dependence of frequency response on the location of poles and zeros of $H(s)$. This dependence points to a simple intuitive procedure to filter design.

4.10-1 Dependence of Frequency Response on Poles and Zeros of $H(s)$

Frequency response of a system is basically the information about the filtering capability of the system. A system transfer function can be expressed as

$$H(s) = \frac{P(s)}{Q(s)} = b_0 \frac{(s - z_1)(s - z_2) \cdots (s - z_N)}{(s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_N)}$$

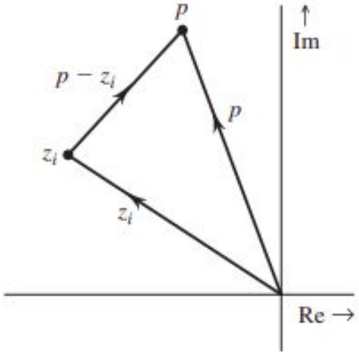
where $z_1, z_2, \ldots, z_N$ are $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the poles of $H(s)$. Now the value of the transfer function $H(s)$ at some frequency $s = p$ is

$$H(s)|_{s=p} = b_0 \frac{(p - z_1)(p - z_2) \cdots (p - z_N)}{(p - \lambda_1)(p - \lambda_2) \cdots (p - \lambda_N)} \tag{4.53}$$

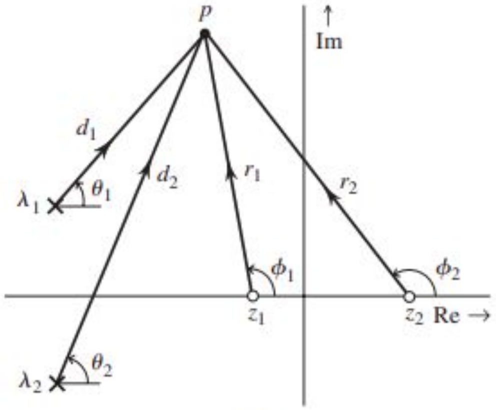
4.10 Filter Design by Placement of Poles and Zeros of $H(s)$

$$H(s)|_{s=p} = b_0 \frac{(p-z_1)(p-z_2)\cdots(p-z_N)}{(p-\lambda_1)(p-\lambda_2)\cdots(p-\lambda_N)} \tag{4.53}$$

This equation consists of factors of the form $p-z_i$ and $p-\lambda_i$. The factor $p-z_i$ is a complex number represented by a vector drawn from point z to the point p in the complex plane, as illustrated in Fig. 4.48a. The length of this line segment is $|p-z_i|$, the magnitude of $p-z_i$. The angle of this directed line segment (with the horizontal axis) is $\angle(p-z_i)$. To compute $H(s)$ at $s=p$, we draw line segments from all poles and zeros of $H(s)$ to the point p, as shown in Fig. 4.48b. The vector connecting a zero z_i to the point p is $p-z_i$. Let the length of this vector be r_i, and let its angle with the horizontal axis be ϕ_i. Then $p-z_i = r_i e^{j\phi_i}$. Similarly, the vector connecting a pole λ_i to the point p is $p-\lambda_i = d_i e^{j\theta_i}$, where d_i and θ_i are the length and the angle (with the horizontal axis),



(a)



(b)

4.10 Filter Design by Placement of Poles and Zeros of $H(s)$

$$H(s)|_{s=p} = b_0 \frac{(p - z_1)(p - z_2)\cdots(p - z_N)}{(p - \lambda_1)(p - \lambda_2)\cdots(p - \lambda_N)} \tag{4.53}$$

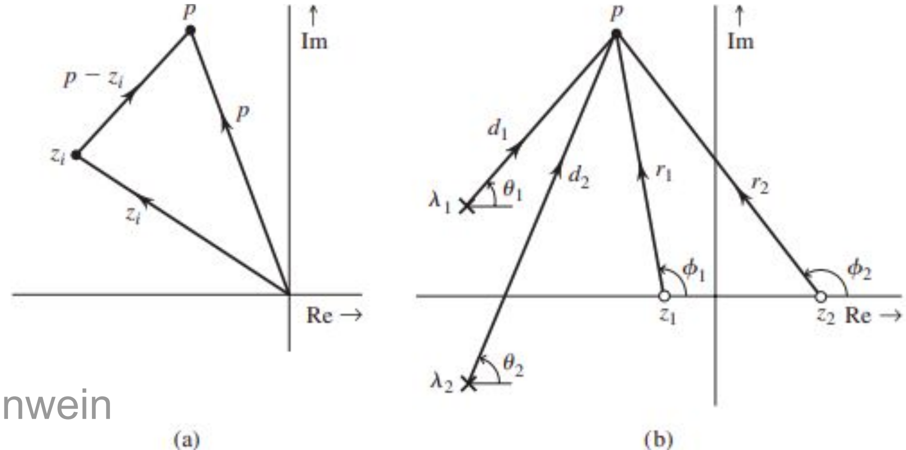
$$\begin{aligned} H(s)|_{s=p} &= b_0 \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2})\cdots(r_N e^{j\phi_N})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2})\cdots(d_N e^{j\theta_N})} \\ &= b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N} e^{j[(\phi_1+\phi_2+\cdots+\phi_N)-(\theta_1+\theta_2+\cdots+\theta_N)]} \end{aligned}$$

Therefore

$$|H(s)|_{s=p} = b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N} = b_0 \frac{\text{product of distances of zeros to } p}{\text{product of distances of poles to } p} \tag{4.54}$$

and

$$\begin{aligned} \angle H(s)|_{s=p} &= (\phi_1 + \phi_2 + \cdots + \phi_N) - (\theta_1 + \theta_2 + \cdots + \theta_N) \\ &= \text{sum of angles of zeros to } p - \text{sum of angles of poles to } p \end{aligned} \tag{4.55}$$



4.10 Filter Design by Placement of Poles and Zeros of $H(s)$

$$|H(s)|_{s=p} = b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N} = b_0 \frac{\text{product of distances of zeros to } p}{\text{product of distances of poles to } p} \tag{4.54}$$

and

$$\begin{aligned} \angle H(s)|_{s=p} &= (\phi_1 + \phi_2 + \cdots + \phi_N) - (\theta_1 + \theta_2 + \cdots + \theta_N) \\ &= \text{sum of angles of zeros to } p - \text{sum of angles of poles to } p \end{aligned} \tag{4.55}$$

To compute the frequency response $H(j\omega)$, we use $s = j\omega$ (a point on the imaginary axis), connect all poles and zeros to the point $j\omega$, and determine $|H(j\omega)|$ and $\angle H(j\omega)$ from Eqs. (4.54) and (4.55). We repeat this procedure for all values of ω from 0 to ∞ to obtain the frequency response.

4.10 Filter Design by Placement of Poles and Zeros of $H(s)$

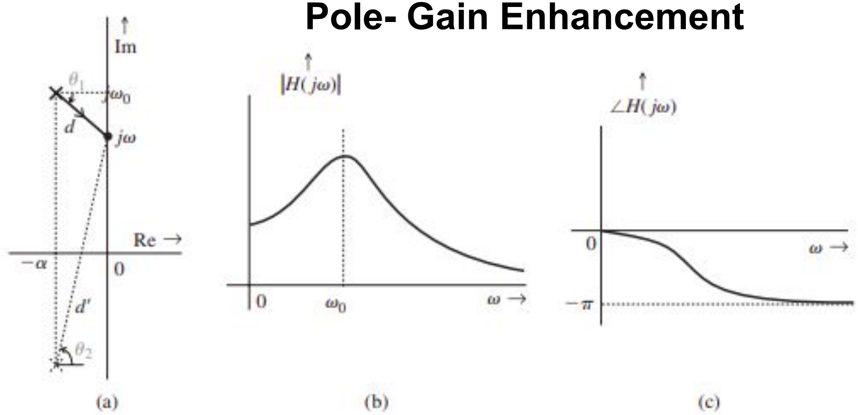
$$|H(s)|_{s=p} = b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N} = b_0 \frac{\text{product of distances of zeros to } p}{\text{product of distances of poles to } p} \qquad (4.54)$$

To understand the effect of poles and zeros on the frequency response, consider a hypothetical case of a single pole $-\alpha + j\omega_0$, as depicted in Fig. 4.49a. To find the amplitude response $|H(j\omega)|$ for a certain value of ω, we connect the pole to the point $j\omega$ (Fig. 4.49a). If the length of this line is d, then $|H(j\omega)|$ is proportional to $1/d$,

$$|H(j\omega)| = \frac{K}{d} \qquad (4.56)$$

where the exact value of constant K is not important at this point.

Pole- Gain Enhancement



Zero - Gain Suppression

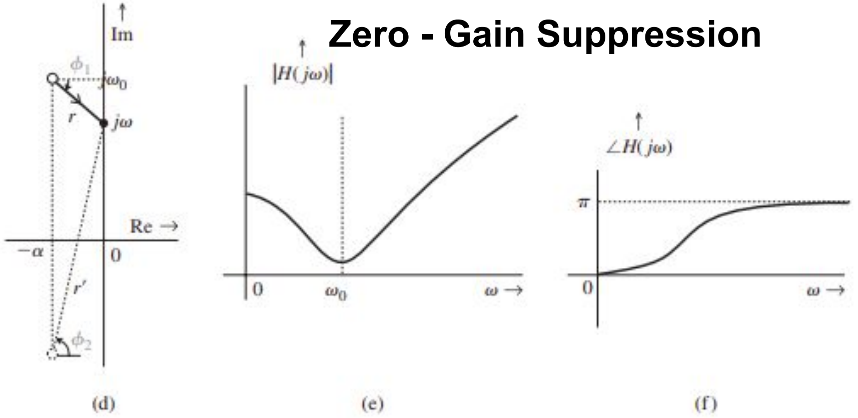


Figure 4.49 The role of poles and zeros in determining the frequency response of an LTIC system.

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