DSP Lecture 2

introduction to Discrete (Time) Signal Processing

Tal Rosenwein

Lecture Overview

[Credit](#)

LINEAR SYSTEMS AND SIGNALS

THIRD EDITION

B. P. Lathi and R. A. Green

Lecture Overview

1. Signal & Systems
2. Time Domain Analysis of Discrete Time Systems
3. Discrete-Time System Analysis Using The Z-Transform
4. Frequency-Response and Filtering

LINEAR SYSTEMS AND SIGNALS

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Lecture Goals

1. Gain intuition about discrete signals and their properties
2. Understand the process of a signal when it passes through a (LTID) system.
3. Calculate the output the given the input and the system's transfer function
4. Nyquist–Shannon **Sampling Theorem**
5. **Non-uniqueness** of **discrete** time sinusoidal waveforms
6. Gain intuition about **aliasing** and **filtering**.

Lecture Overview

1. **Signal** & Systems
2. Time Domain Analysis of Discrete Time Systems
3. Discrete-Time System Analysis Using The Z-Transform
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TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

In this chapter we introduce the basic concepts of discrete-time signals and systems. Furthermore, we explore the time-domain analysis of linear, time-invariant, discrete-time (LTID) systems. We show how to compute the zero-input response, determine the unit impulse response, and use convolution to evaluate the zero-state response.

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

A *discrete-time signal* is basically a sequence of numbers. Such signals can be denoted by $x[n]$, $y[n]$, and so on, where the variable n takes integer values, and $x[n]$ denotes the nth number in the sequence labeled x. In this notation, the discrete-time variable n is enclosed in square brackets instead of parentheses, which we have reserved for enclosing continuous-time variables, such as t.

A discrete-time signal, when obtained by uniform sampling of a continuous-time signal $x(t)$, can also be expressed as $x(nT)$, where T is the sampling interval and n, the discrete variable taking on integer values. Thus, $x(nT)$ denotes the value of the signal $x(t)$ at $t = nT$. The signal $x(nT)$ is a sequence of numbers (sample values), and hence, by definition, is a discrete-time signal. Such a signal can also be denoted by the customary discrete-time notation $x[n]$, where $x[n] = x(nT)$. A typical discrete-time signal is depicted in Fig. 3.1, which shows both forms of notation. By way of an example, a continuous-time exponential $x(t) = e^{-t}$, when sampled every $T = 0.1$ seconds, results in a discrete-time signal $x(nT)$ given by

$$x(nT) = e^{-nT} = e^{-0.1n}$$

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7

# TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS
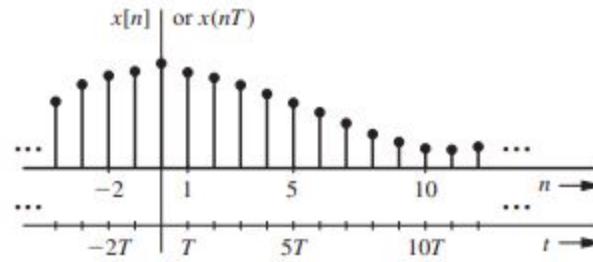

Figure 3.1 A discrete-time signal.
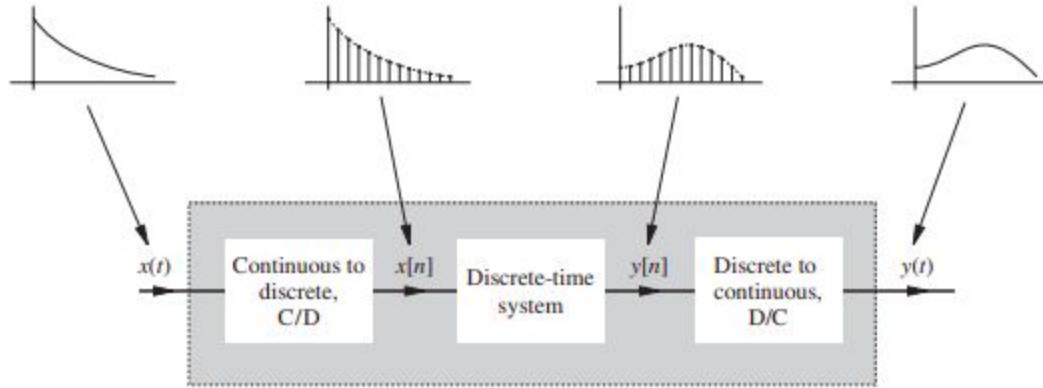
Figure 3.2 Processing a continuous-time signal by means of a discrete-time system.
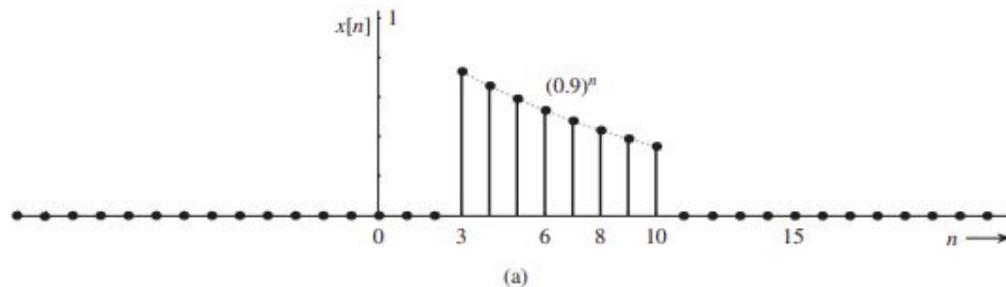
Not restricted to audio only, any signal being acquired: EEG, ECG, etc.

Signal [Acquisition](#)
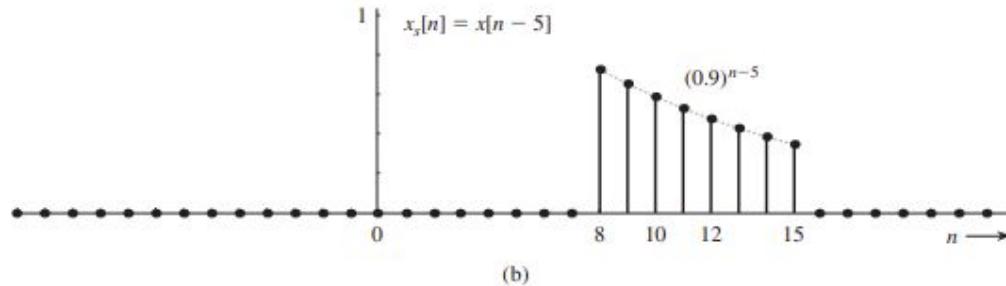
TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.2 USEFUL SIGNAL OPERATIONS

SHIFTING

(a)

(b)

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TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.2 USEFUL SIGNAL OPERATIONS

SAMPLING RATE ALTERATION: DOWNSAMPLING, UPSAMPLING, AND INTERPOLATION

Consider a signal $x[n]$ compressed by factor M. Compressing a signal $x[n]$ by factor M yields $x_d[n]$ given by

$$x_d[n] = x[Mn]$$
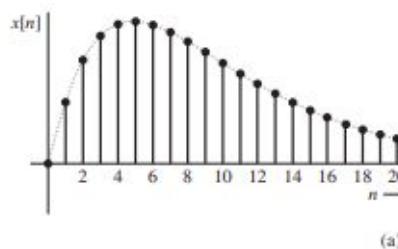
Because of the restriction that discrete-time signals are defined only for integer values of the argument, we must restrict M to integer values. The values of $x[Mn]$ at $n = 0, 1, 2, 3, \ldots$ are $x[0]$, $x[M], x[2M], x[3M], \ldots$. This means $x[Mn]$ selects every Mth sample of $x[n]$ and deletes all the samples in between. It reduces the number of samples by factor M. If $x[n]$ is obtained by sampling a continuous-time signal, this operation implies reducing the sampling rate by factor M. For this

An *interpolated* signal is generated in two steps; first, we expand $x[n]$ by an integer factor L to obtain the expanded signal $x_e[n]$, as
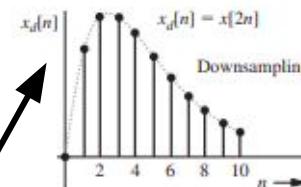
$$x_e[n] = \begin{cases} x[n/L] & n = 0, \pm L \pm 2L, \ldots, \\ 0 & \text{otherwise} \end{cases} \qquad (3.2)$$

Thus, the sampling rate of $x_e[n]$ is L times that of $x[n]$. Hence, this operation is commonly called *upsampling*. The upsampled signal $x_e[n]$ contains all the data of $x[n]$, although in an expanded form.
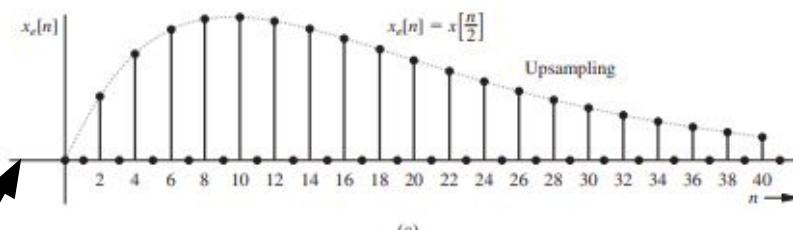
In the expanded signal in Fig. 3.5c, the missing (zero-valued) odd-numbered samples can be reconstructed from the non-zero-valued samples by using some suitable interpolation formula.
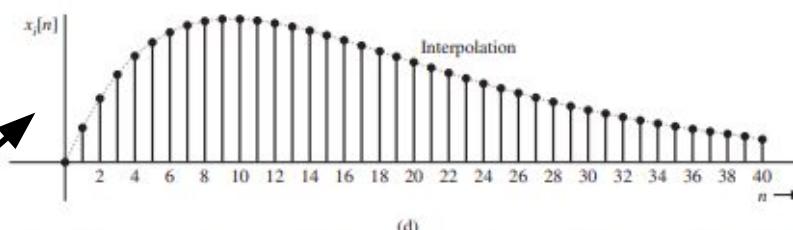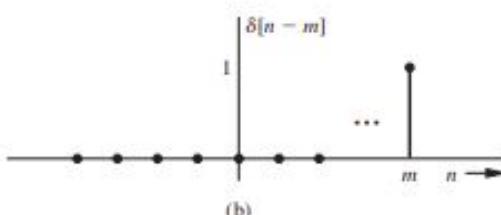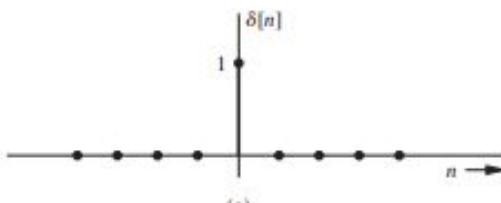

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TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

## 3.3 SOME USEFUL DISCRETE-TIME SIGNAL MODELS

3.3-1 Discrete-Time Impulse Function $\delta[n]$

The discrete-time counterpart of the continuous-time impulse function $\delta(t)$ is $\delta[n]$, a Kronecker delta function, defined by

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

3.3-2 Discrete-Time Unit Step Function $u[n]$

The discrete-time counterpart of the unit step function $u(t)$ is $u[n]$ (Fig. 3.7a), defined by

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

If we want a signal to start at $n = 0$ (so that it has a zero value for all $n < 0$), we need only multiply the signal by $u[n]$.
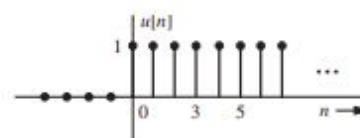
Figure 3.6 Discrete-time impulse function: (a) unit impulse sequence and (b) shifted impulse sequence.

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.3 SOME USEFUL DISCRETE-TIME SIGNAL MODELS

3.3-3 Discrete-Time Exponential γ^n

A continuous-time exponential $e^{\lambda t}$ can be expressed in an alternate form as

$$e^{\lambda t} = \gamma^t \qquad (\gamma = e^\lambda \text{ or } \lambda = \ln\gamma)$$

For example, $e^{-0.3t} = (0.7408)^t$ because $e^{-0.3} = 0.7408$. Conversely, $4^t = e^{1.386t}$ because $e^{1.386} = 4$, that is, $\ln 4 = 1.386$. In the study of continuous-time signals and systems, we prefer the form $e^{\lambda t}$ rather than γ^t. In contrast, the exponential form γ^n is preferable in the study of discrete-time signals and systems, as will become apparent later. The discrete-time exponential γ^n can also be expressed by using a natural base, as

$$e^{\lambda n} = \gamma^n \qquad (\gamma = e^\lambda \text{ or } \lambda = \ln\gamma)$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS
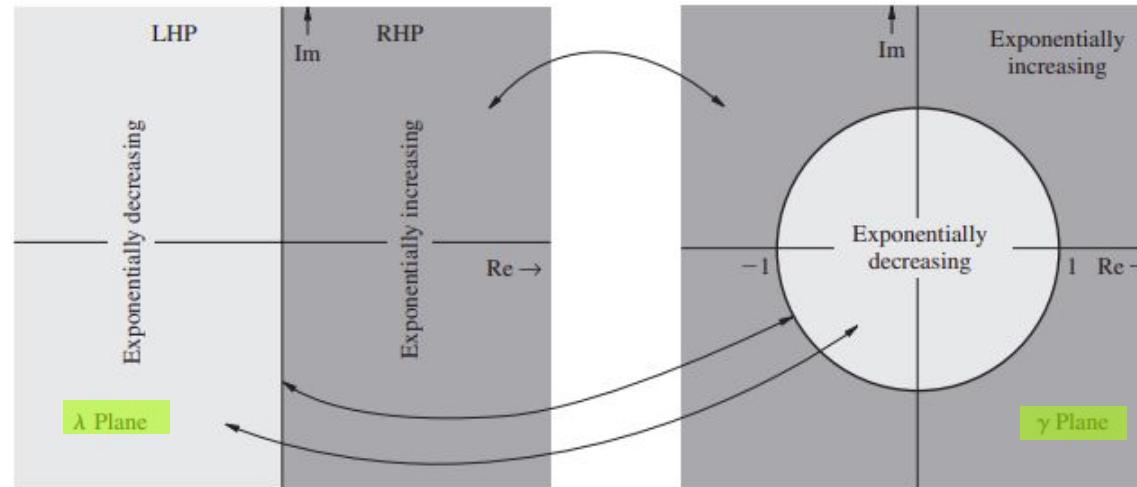
3.3 SOME USEFUL DISCRETE-TIME SIGNAL MODELS

3.3-3 Discrete-Time Exponential γ^n

$$e^{\lambda n} = \gamma^n$$

$$\gamma = e^{\lambda} = e^{a+jb} = e^a e^{jb}$$

$$|\gamma| = |e^a||e^{jb}| = e^a \qquad \text{because } |e^{jb}| = 1$$

Figure 3.8 The λ plane, the γ plane, and their mapping.

Nature of γ^n. The signal $e^{\lambda n}$ grows exponentially with n if $\text{Re}\,\lambda > 0$ (λ in the RHP), and decays exponentially if $\text{Re}\,\lambda < 0$ (λ in the LHP). It is constant or oscillates with constant amplitude if $\text{Re}\,\lambda = 0$ (λ on the imaginary axis). Clearly, the location of λ in the complex plane indicates whether the signal $e^{\lambda n}$ will grow exponentially, decay exponentially, or oscillate with constant amplitude (Fig. 3.8a). A constant signal ($\lambda = 0$) is also an oscillation with zero frequency. We now find a similar criterion for determining the nature of γ^n from the location of γ in the complex plane.

Time-Domain Analysis of Discrete-Time Systems

3.3 Some Useful Discrete-Time Signal Models
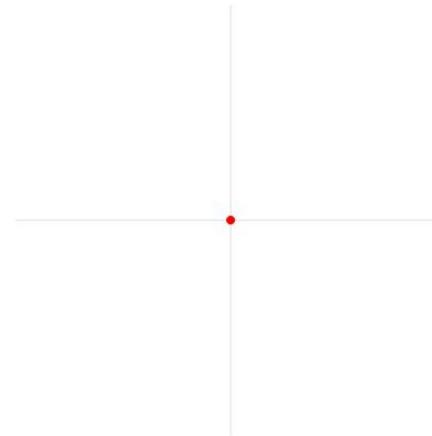
3.3-4 Discrete-Time Sinusoid $\cos(\Omega n + \theta)$

$\Omega n [rad]$

$\Omega [rad/sample]$

$There Are\ 2\pi [\dfrac{rad}{cycle}]$

$F = \dfrac{\Omega}{2\pi}[\dfrac{rad}{sample}\dfrac{cycle}{rad}] = \dfrac{\Omega}{2\pi}[\dfrac{cycle}{sample}]$

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TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

$$\Omega n [rad]$$
$$\Omega [rad/sample]$$
$$There Are\ 2\pi [\frac{rad}{cycle}]$$
$$F = \frac{\Omega}{2\pi}[\frac{rad}{sample}\frac{cycle}{rad}] = \frac{\Omega}{2\pi}[\frac{cycle}{sample}]$$

3.3 SOME USEFUL DISCRETE-TIME SIGNAL MODELS

3.3-4 Discrete-Time Sinusoid $\cos{(\Omega n + \theta)}$

A general discrete-time sinusoid can be expressed as $C\cos{(\Omega n + \theta)}$, where C is the *amplitude,* and θ is the *phase* in radians. Also, Ωn is an angle in radians. Hence, the dimensions of the frequency Ω are *radians per sample*. This sinusoid may also be expressed as

$$C\cos{(\Omega n + \theta)} = C\cos{(2\pi \mathcal{F} n + \theta)}$$

Figure 3.11 shows a discrete-time sinusoid $\cos{(\frac{\pi}{12}n + \frac{\pi}{4})}$. For this case, the frequency is $\Omega = \pi/12$ radians/sample. Alternately, the frequency is $\mathcal{F} = 1/24$ cycles/sample. In other words, there are 24 samples in one cycle of the sinusoid.
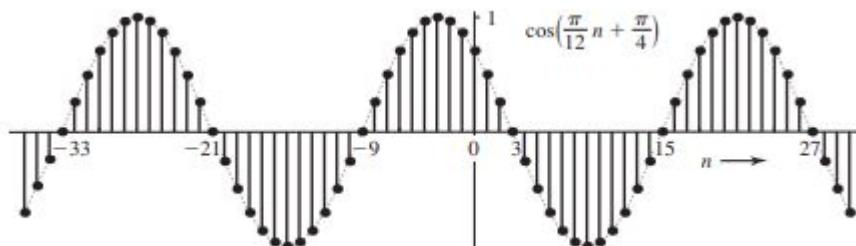

Figure 3.11 A discrete-time sinusoid $\cos{(\frac{\pi}{12}n + \frac{\pi}{4})}$.

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TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.3 SOME USEFUL DISCRETE-TIME SIGNAL MODELS

3.3-4 Discrete-Time Sinusoid $\cos(\Omega n + \theta)$

A continuous-time sinusoid $\cos \omega t$ is a periodic signal regardless of the value of ω. Such is not the case for the discrete-time sinusoid $\cos \Omega n$ (or exponential $e^{j\Omega n}$). A sinusoid $\cos \Omega n$ is periodic only if $\Omega/2\pi$ is a rational number. This can be proved by observing that if this sinusoid is N_0 periodic, then

$$\cos \Omega(n + N_0) = \cos \Omega n$$

This is possible only if

$$\Omega N_0 = 2\pi m \qquad m \text{ integer}$$

Here, both m and N_0 are integers. Hence, $\Omega/2\pi = m/N_0$ is a rational number. Thus, a sinusoid $\cos \Omega n$ (or exponential $e^{j\Omega n}$) is periodic only if

$$\frac{\Omega}{2\pi} = \frac{m}{N_0} \qquad \text{a rational number}$$

When this condition ($\Omega/2\pi$ a rational number) is satisfied, the period N_0 of the sinusoid $\cos \Omega n$ is given by

$$N_0 = m\left(\frac{2\pi}{\Omega}\right) \tag{9.1}$$

To compute N_0, we must choose the smallest value of m that will make $m(2\pi/\Omega)$ an integer. For example, if $\Omega = 4\pi/17$, then the smallest value of m that will make $m(2\pi/\Omega) = m(17/2)$ an

integer is 2. Therefore,

$$N_0 = m\left(\frac{2\pi}{\Omega}\right) = 2\left(\frac{17}{2}\right) = 17$$

However, a sinusoid $\cos(0.8n)$ is not a periodic signal because $0.8/2\pi$

is not a rational number.

Not all discrete sin / cos signals are periodic

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.3 SOME USEFUL DISCRETE-TIME SIGNAL MODELS

SAMPLED CONTINUOUS-TIME SINUSOID YIELDS A DISCRETE-TIME SINUSOID

A continuous-time sinusoid $\cos \omega t$ sampled every T seconds yields a discrete-time sequence whose nth element (at $t = nT$) is $\cos \omega nT$. Thus, the sampled signal $x[n]$ is given by

$$x[n] = \cos \omega nT = \cos \Omega n \qquad \text{where } \Omega = \omega T$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.3 SOME USEFUL DISCRETE-TIME SIGNAL MODELS

KINSHIP OF DIFFERENCE EQUATIONS TO DIFFERENTIAL EQUATIONS

$$H[x(t)] = \frac{\partial x(t)}{\partial t} \rightarrow H[x[n]] = x[n] - x[n-1]$$

$$H[x(t)] = \int x(t)dt \rightarrow H[x[n]] = \sum x[n]$$

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TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.4-1 Classification of Discrete-Time Systems

LINEARITY AND TIME INVARIANCE

$$x_1 \longrightarrow y_1 \quad \text{and} \quad x_2 \longrightarrow y_2$$

then for all inputs x_1 and x_2 and all constants k_1 and k_2,

$$k_1 x_1 + k_2 x_2 \longrightarrow k_1 y_1 + k_2 y_2$$

$$H[x[n - M]] = y[n - M]$$

CAUSAL AND NONCAUSAL SYSTEMS

$$y[n] \ depends \ on \ x[m]; m \leq n$$

STABLE AND UNSTABLE SYSTEMS

Internal (system's modes) & External (BIBO) stability

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

In this section we discuss time-domain analysis of LTID (linear, time-invariant, discrete-time systems). With minor differences, the procedure is parallel to that for continuous-time systems.

DIFFERENCE EQUATIONS

can be written in two forms: the first form uses delay terms such as $y[n-1]$, $y[n-2]$, $x[n-1]$, $x[n-2]$, and so on; and the alternate form uses advance terms such as $y[n+1]$, $y[n+2]$, and so on. Although the delay form is more natural, we shall often prefer the advance form, not just for the general notational convenience, but also for resulting notational uniformity with the operator form for differential equations. This facilitates the commonality of the solutions and concepts for continuous-time and discrete-time systems.

We start here with a general difference equation, written in advance form as

$$y[n+N]+a_1y[n+N-1]+\cdots+a_{N-1}y[n+1]+a_Ny[n] =$$
$$b_{N-M}x[n+M]+b_{N-M+1}x[n+M-1]+\cdots+b_{N-1}x[n+1]+b_Nx[n] \qquad (3.14)$$

This is a linear difference equation whose order is $\max(N,M)$. We have assumed the coefficient of $y[n+N]$ to be unity ($a_0=1$) without loss of generality. If $a_0 \neq 1$, we can divide the equation throughout by a_0 to normalize the equation to have $a_0=1$.

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

DIFFERENCE EQUATIONS

Advanced form:

$$y[n+N]+a_1y[n+N-1]+\cdots+a_{N-1}y[n+1]+a_Ny[n] =$$
$$b_0x[n+N]+b_1x[n+N-1]+\cdots+b_{N-1}x[n+1]+b_Nx[n] \qquad (3.15)$$

Delay form:

replace n by $n-N$

$$y[n]+a_1y[n-1]+\cdots+a_{N-1}y[n-N+1]+a_Ny[n-N] =$$
$$b_0x[n]+b_1x[n-1]+\cdots+b_{N-1}x[n-N+1]+b_Nx[n-N] \qquad (3.16)$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

OPERATOR NOTATION

Advance

Delay- *D* operator

$$Ex[n] \equiv x[n+1]$$

$$E^2x[n] \equiv x[n+2]$$

$$\vdots$$

$$E^Nx[n] \equiv x[n+N]$$

Similarly, the second-order book sales estimate described by Eq. (3.6) as

$$y[n+2] + \tfrac{1}{4}y[n+1] + \tfrac{1}{16}y[n] = x[n+2]$$

can be expressed in operator notation as

$$\left(E^2 + \tfrac{1}{4}E + \tfrac{1}{16}\right)y[n] = E^2x[n]$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

OPERATOR NOTATION

The general Nth-order advance-form difference equation of Eq. (3.15) can be expressed as

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N)y[n] = (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1}E + b_N)x[n]$$

or

$$Q[E]y[n] = P[E]x[n] \qquad (3.20)$$

where $Q[E]$ and $P[E]$ are Nth-order polynomial operators

$$Q[E] = E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N$$
$$P[E] = b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1}E + b_N$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

OPERATOR NOTATION

The general Nth-order advance-form difference equation of Eq. (3.15) can be expressed as

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$$Q[E] y[n] = P[E] x[n] \qquad (3.20)$$

where $Q[E]$ and $P[E]$ are Nth-order polynomial operators

$$Q[E] = E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N$$
$$P[E] = b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t)$$
$$= (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

or

$$Q(D) y(t) = P(D) x(t)$$

where the polynomials $Q(D)$ and $P(D)$ are

$$Q(D) = D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N$$
$$P(D) = b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N$$

Discrete Time

Continuous Time

Lecture Overview

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Time-Domain Analysis of Discrete-Time Systems

3.5 Discrete-Time System Equations

Response of Linear Discrete-Time Systems

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.6 System Response to Internal Conditions: The Zero-Input Response

The zero-input response $y_0[n]$ is the solution of Eq. (3.20) with $x[n] = 0$; that is,

$$Q[E]y_0[n] = 0$$

or

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N)y_0[n] = 0 \qquad (3.21)$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.6 SYSTEM RESPONSE TO INTERNAL CONDITIONS: THE ZERO-INPUT RESPONSE

Such a situation is possible *if and only if* $y_0[n]$ and advanced $y_0[n]$ have the same form. Only an exponential function γ^n has this property, as the following equation indicates:

$$E^k\{\gamma^n\} = \gamma^{n+k} = \gamma^k \gamma^n$$

This expression shows that γ^n advanced by k units is a constant (γ^k) times γ^n. Therefore, the solution of Eq. (3.21) must be of the form[†]

$$y_0[n] = c\gamma^n \qquad (3.22)$$

To determine c and γ, we substitute this solution in Eq. (3.21). Since $E^k y_0[n] = y_0[n+k] = c\gamma^{n+k}$, this produces

$$c(\gamma^N + a_1\gamma^{N-1} + \cdots + a_{N-1}\gamma + a_N)\gamma^n = 0$$

For a nontrivial solution of this equation,

$$\gamma^N + a_1\gamma^{n-1} + \cdots + a_{N-1}\gamma + a_N = 0 \qquad (3.23)$$

or

$$Q[\gamma] = 0$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N)y_0[n] = 0$$

$$e^{\lambda n} = \gamma^n \qquad (\gamma = e^\lambda$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.6 SYSTEM RESPONSE TO INTERNAL CONDITIONS: THE ZERO-INPUT RESPONSE

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y_0[n] = 0$$

Our solution $c\gamma^n$ [Eq. (3.22)] is correct, provided γ satisfies Eq. (3.23). Now, $Q[\gamma]$ is an Nth-order polynomial and can be expressed in the factored form (assuming all distinct roots):

$$e^{\lambda n} = \gamma^n \qquad (\gamma = e^{\lambda}$$

$$(\gamma - \gamma_1)(\gamma - \gamma_2) \cdots (\gamma - \gamma_N) = 0$$

$$y_0[n] = c_1 \gamma_1^n + c_2 \gamma_2^n + \cdots + c_n \gamma_N^n$$

where γ_1, γ_2, ..., γ_n are the roots of Eq. (3.23) and c_1, c_2, ..., c_n are arbitrary constants determined from N auxiliary conditions, generally given in the form of initial conditions. The

Characteristic modes of the system

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS
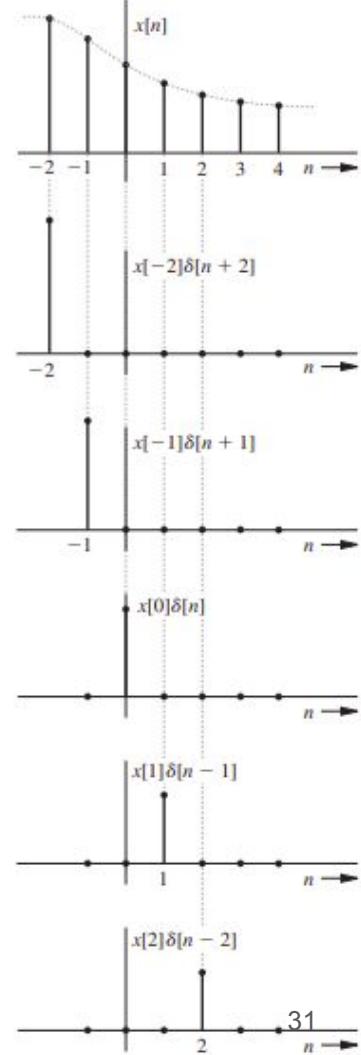
## 3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.8 SYSTEM RESPONSE TO EXTERNAL INPUT: THE ZERO-STATE RESPONSE

Here we follow the procedure parallel to that used in the continuous-time case by expressing an arbitrary input $x[n]$ as a sum of impulse components. A signal $x[n]$ in Fig. 3.20a can be expressed as a sum of impulse components, such as those depicted in Figs. 3.20b–3.20f. The component of $x[n]$ at $n = m$ is $x[m]\delta[n-m]$, and $x[n]$ is the sum of all these components summed from $m = -\infty$ to ∞.

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$
$$+ x[-1]\delta[n+1] + x[-2]\delta[n+2] + \cdots$$
$$= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

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TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.8 SYSTEM RESPONSE TO EXTERNAL INPUT: THE ZERO-STATE RESPONSE

$$x[n] \Longrightarrow y[n]$$

to indicate the input and the corresponding response of the system. Thus, if

$$\delta[n] \Longrightarrow h[n]$$

then because of time invariance

$$\delta[n-m] \Longrightarrow h[n-m]$$

and because of linearity

$$x[m]\delta[n-m] \Longrightarrow x[m]h[n-m]$$

and again because of linearity

$$\underbrace{\sum_{m=-\infty}^{\infty} x[m]\delta[n-m]}_{x[n]} \quad \Longrightarrow \quad \underbrace{\sum_{m=-\infty}^{\infty} x[m]h[n-m]}_{y[n]}$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.8 SYSTEM RESPONSE TO EXTERNAL INPUT: THE ZERO-STATE RESPONSE

$$\underbrace{\sum_{m=-\infty}^{\infty} x[m]\delta[n-m]}_{x[n]} \quad \Longrightarrow \quad \underbrace{\sum_{m=-\infty}^{\infty} x[m]h[n-m]}_{y[n]}$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \tag{3.31}$$

The summation on the right-hand side is known as the *convolution sum* of $x[n]$ and $h[n]$, and is represented symbolically by $x[n] * h[n]$

$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$
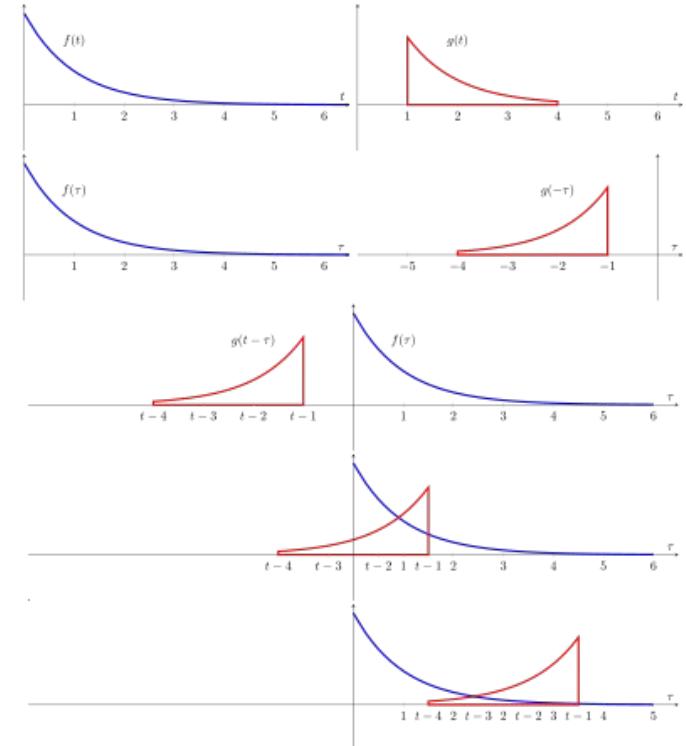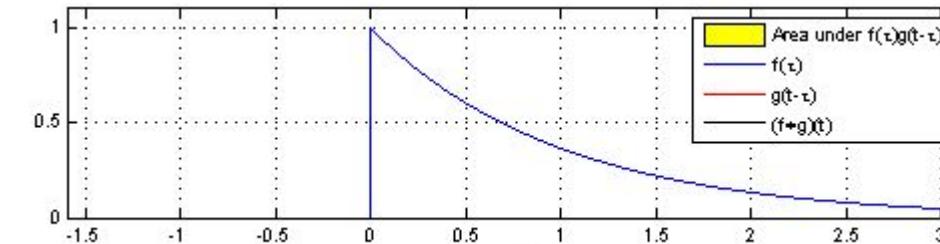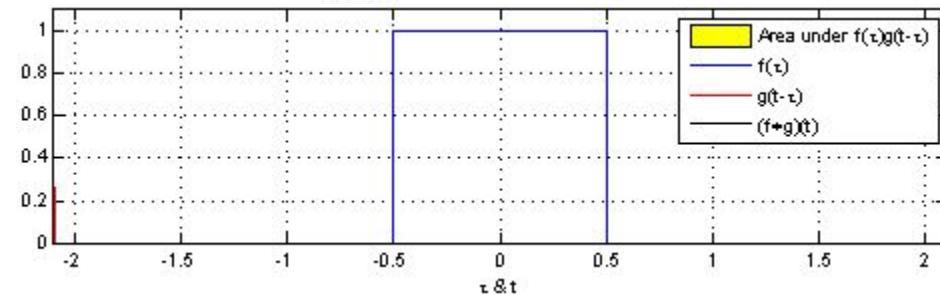
Preliminaries

Convolution

$$(f * g)(t) := \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\, d\tau.$$
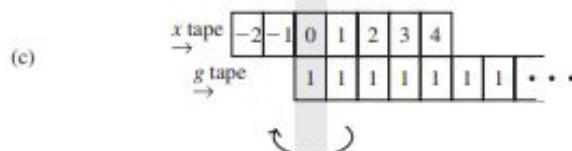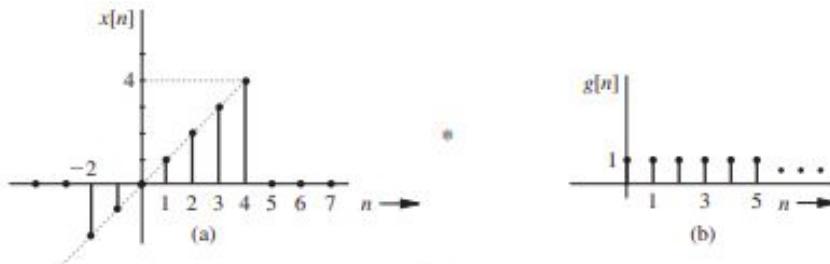
An equivalent definition is (see commutativity):

$$(f * g)(t) := \int_{-\infty}^{\infty} f(t - \tau)g(\tau)\, d\tau.$$

34

Preliminaries
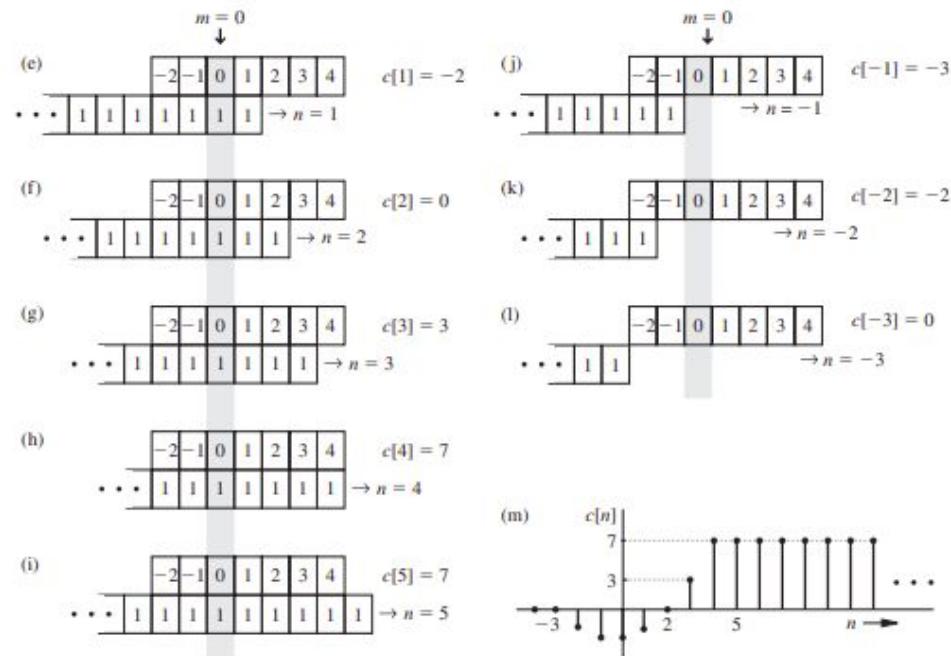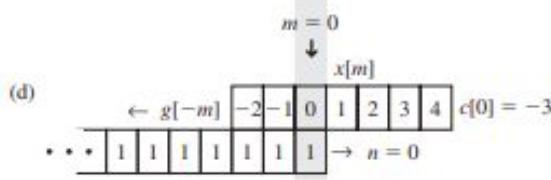
Convolution

Figure 3.23 Sliding-tape algorithm for discrete-time convolution.

TIME-DOMAIN ANALYSIS
OF DISCRETE-TIME SYSTEMS
3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.8 SYSTEM RESPONSE TO EXTERNAL INPUT:
THE ZERO-STATE RESPONSE

A VERY SPECIAL FUNCTION FOR LTID SYSTEMS: THE EVERLASTING EXPONENTIAL z^n

$$e^{\lambda n} = \gamma^n \qquad (\gamma = e^{\lambda}$$

In Sec. 2.4-4, we showed that there exists one signal for which the response of an LTIC system is the same as the input within a multiplicative constant. The response of an LTIC system to an everlasting exponential input e^{st} is $H(s)e^{st}$, where $H(s)$ is the system transfer function. We now show that for an LTID system, the same role is played by an everlasting exponential z^n. The system response $y[n]$ in this case is given by

$$y[n] = h[n] * z^n$$

$$= \sum_{m=-\infty}^{\infty} h[m]z^{n-m}$$

$$= z^n \sum_{m=-\infty}^{\infty} h[m]z^{-m}$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.8 SYSTEM RESPONSE TO EXTERNAL INPUT: THE ZERO-STATE RESPONSE

A VERY SPECIAL FUNCTION FOR LTID SYSTEMS: THE EVERLASTING EXPONENTIAL z^n

$$e^{\lambda n} = \gamma^n \qquad (\gamma = e^{\lambda}$$

$$y[n] = H[z]z^n \qquad (3.38)$$

where

$$H[z] = \sum_{m=-\infty}^{\infty} h[m]z^{-m} \qquad (3.39)$$

Equation (3.38) is valid only for values of z for which the sum on the right-hand side of Eq. (3.39) exists (converges). Note that $H[z]$ is a constant for a given z. Thus, the input and the output are the same (within a multiplicative constant) for the everlasting exponential input z^n.

Time-Domain Analysis of Discrete-Time Systems

3.5 Discrete-Time System Equations

Response of Linear Discrete-Time Systems

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.8 System Response to External Input: The Zero-State Response

A Very Special Function for LTID Systems: The Everlasting Exponential z^n

$$e^{\lambda n} = \gamma^n \qquad (\gamma = e^{\lambda}$$

$$y[n] = h[n] * z^n$$

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\, d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\, d\tau$$

$$y[n] = H[z]z^n$$

$$y(t) = H(s)e^{st}$$

$$H[z] = \sum_{m=-\infty}^{\infty} h[m]z^{-m}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\, d\tau$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.8 SYSTEM RESPONSE TO EXTERNAL INPUT: THE ZERO-STATE RESPONSE

A VERY SPECIAL FUNCTION FOR LTID SYSTEMS: THE EVERLASTING EXPONENTIAL z^n

$$e^{\lambda n} = \gamma^n \qquad (\gamma = e^{\lambda}$$

$H[z]$, which is called the *transfer function* of the system, is a function of the complex variable z. An alternate definition of the transfer function $H[z]$ of an LTID system from Eq. (3.38) is

$$H[z] = \left.\frac{\text{output signal}}{\text{input signal}}\right|_{\text{input=everlasting exponential } z^n} \qquad (3.40)$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.8-3 Total Response

The total response of an LTID system can be expressed as a sum of the zero-input and zero-state responses:

$$\text{total response} = \underbrace{\sum_{j=1}^{N} c_j \gamma_j^n}_{ZIR} + \underbrace{x[n] * h[n]}_{ZSR}$$

Discrete

$$\text{total response} = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{ZIR} + \underbrace{x(t) * h(t)}_{ZSR}$$

Continuous

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.9 SYSTEM STABILITY

3.9-1 External (BIBO) Stability

$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} h[m]x[n-m] \right| \leq \sum_{m=-\infty}^{\infty} |h[m]|\,|x[n-m]|$$

If $x[n]$ is bounded, then $|x[n-m]| < K_1 < \infty$, and

$$|y[n]| \leq K_1 \sum_{m=-\infty}^{\infty} |h[m]|$$

Clearly the output is bounded if the summation on the right-hand side is bounded; that is, if

$$\sum_{n=-\infty}^{\infty} |h[n]| < K_2 < \infty \qquad (3.43)$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.9 SYSTEM STABILITY

3.9-2 Internal (Asymptotic) Stability

$$\underbrace{\sum_{j=1}^{N} c_j \gamma_j^n}_{ZIR} -$$

$$e^{\lambda n} = \gamma^n \qquad (\gamma = e^{\lambda} \longrightarrow \gamma = |\gamma| e^{j\beta} \qquad \text{and} \qquad \gamma^n = |\gamma|^n e^{j\beta n}$$

Since the magnitude of $e^{j\beta n}$ is always unity regardless of the value of n, the magnitude of γ^n is $|\gamma|^n$. Therefore,

$$\text{if } |\gamma| < 1, \qquad \text{then } \gamma^n \to 0 \text{ as } n \to \infty$$
$$\text{if } |\gamma| > 1, \qquad \text{then } \gamma^n \to \infty \text{ as } n \to \infty$$
$$\text{and if } |\gamma| = 1, \qquad \text{then } |\gamma|^n = 1 \text{ for all } n$$

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS

$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

3.9 SYSTEM STABILITY

### 3.9-2 Internal (Asymptotic) Stability
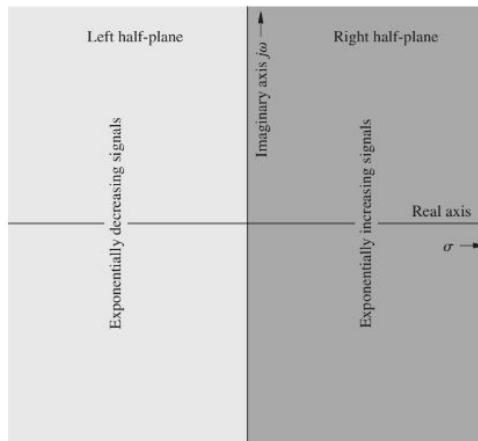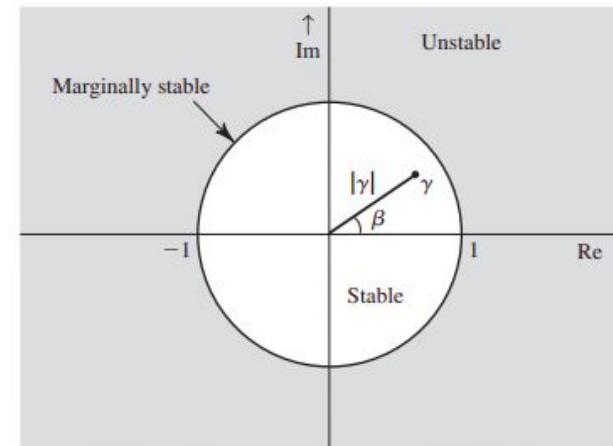
$$e^{\lambda n} = \gamma^n \qquad (\gamma = e^{\lambda}$$

Figure 1.22 Complex frequency plane.

TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

3.5 DISCRETE-TIME SYSTEM EQUATIONS

RESPONSE OF LINEAR DISCRETE-TIME SYSTEMS
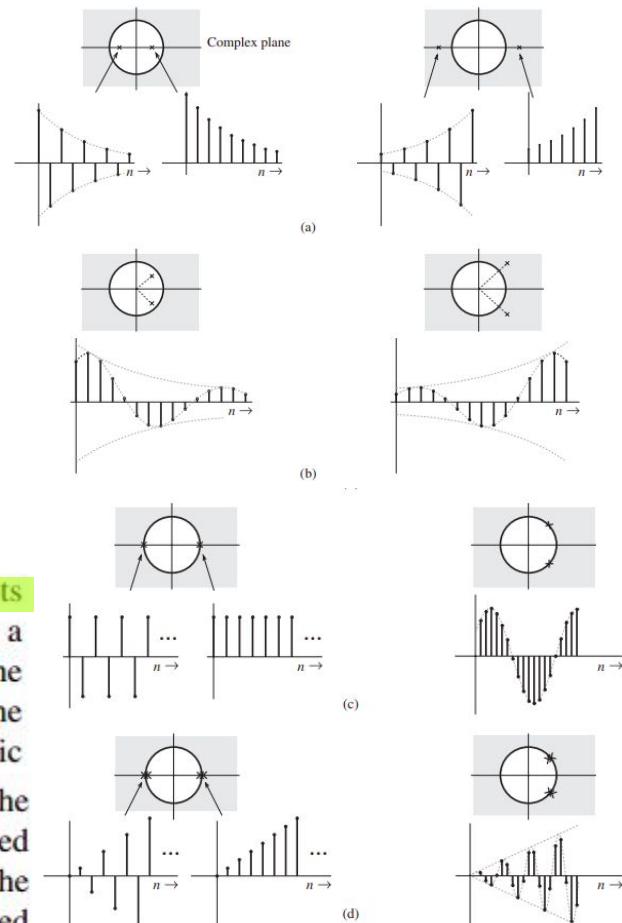
$$Total\,Response = Y_{ZIR} + Y_{ZSR}$$

## 3.9 SYSTEM STABILITY

3.9-2 Internal (Asymptotic) Stability

$$\underbrace{\sum_{j=1}^{N} c_j \gamma_j^n}_{ZIR} -$$

These results can be grasped more effectively in terms of the location of characteristic roots in the complex plane. Figure 3.28 shows a circle of unit radius, centered at the origin in a complex plane. Our discussion shows that if all characteristic roots of the system lie inside the unit circle, $|\gamma_i| < 1$ for all i and the system is asymptotically stable. On the other hand, even if one characteristic root lies outside the unit circle, the system is unstable. If none of the characteristic roots lie outside the unit circle, but some simple (unrepeated) roots lie on the circle itself, the system is marginally stable. If two or more characteristic roots coincide on the unit circle (repeated roots), the system is unstable. The reason is that for repeated roots, the zero-input response is of the form $n^{r-1}\gamma^n$, and if $|\gamma| = 1$, then $|n^{r-1}\gamma^n| = n^{r-1} \to \infty$ as $n \to \infty$.[†] Note, however, that repeated roots inside the unit circle do not cause instability.

Lecture Overview

1. Signals & Systems
2. Time Domain Analysis of Discrete Time Systems
3. **Discrete-Time System Analysis Using The Z-Transform**
4. Frequency-Response and Filtering

Discrete-Time System Analysis Using the z-Transform

The counterpart of the Laplace transform for discrete-time systems is the z-transform. The Laplace transform converts integro-differential equations into algebraic equations. In the same way, the z-transforms changes difference equations into algebraic equations, thereby simplifying the analysis of discrete-time systems. The z-transform method of analysis of discrete-time systems parallels the Laplace transform method of analysis of continuous-time systems, with some minor differences. In fact, we shall see that *the z-transform is the Laplace transform in disguise*.

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.1 THE z-TRANSFORM

We define $X[z]$, the direct z-transform of $x[n]$, as

Continuous (and periodic) $\longrightarrow$
$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{5.1}$$

where z is a complex variable. The signal $x[n]$, which is the inverse z-transform of $X[z]$, can be obtained from $X[z]$ by using the following inverse z-transformation:

Discrete $\longrightarrow$
$$x[n] = \frac{1}{2\pi j} \oint X[z]z^{n-1} \, dz \tag{5.2}$$

The symbol $\oint$ indicates an integration in counterclockwise direction around a closed path in the complex plane (see Fig. 5.1). We derive this z-transform pair later, in Ch. 9, as an extension of the discrete-time Fourier transform pair.

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.1 THE z-TRANSFORM

RELATIONSHIP BETWEEN $h[n]$ AND $H[z]$

For an LTID system, if $h[n]$ is its unit impulse response, then from Eq. (3.39), where we defined $H[z]$, the system transfer function, we write

$$H[z] = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \qquad (5.11)$$

For causal systems, the limits on the sum are from $n = 0$ to ∞. This equation shows that the transfer function $H[z]$ is the z-transform of the impulse response $h[n]$ of an LTID system; that is,

$$h[n] \Longleftrightarrow H[z]$$

This important result relates the time-domain specification $h[n]$ of a system to $H[z]$, the frequency-domain specification of a system. The result is parallel to that for LTIC systems.

Discrete-Time System Analysis Using the z-Transform

5.2 Some Properties of the z-Transform

5.2-1 Time-Shifting Properties

Right Shift (Delay)

If

$$x[n]u[n] \Longleftrightarrow X[z]$$

then

$$x[n-1]u[n-1] \Longleftrightarrow \frac{1}{z}X[z]$$

In general,

$$x[n-m]u[n-m] \Longleftrightarrow \frac{1}{z^m}X[z]$$

$$\mathcal{Z}\{x[n-m]u[n-m]\} = \sum_{n=0}^{\infty} x[n-m]u[n-m]z^{-n}$$

$$\mathcal{Z}\{x[n-m]u[n-m]\} = \sum_{n=m}^{\infty} x[n-m]z^{-n}$$

$$= \sum_{r=0}^{\infty} x[r]z^{-(r+m)}$$

$$= \frac{1}{z^m}\sum_{r=0}^{\infty} x[r]z^{-r} = \frac{1}{z^m}X[z]$$

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.2 SOME PROPERTIES OF THE z-TRANSFORM

5.2-5 Convolution Property

The time-convolution property states that if[‡]

$$x_1[n] \Longleftrightarrow X_1[z] \qquad \text{and} \qquad x_2[n] \Longleftrightarrow X_2[z],$$

then (*time convolution*)

$$x_1[n] * x_2[n] \Longleftrightarrow X_1[z]X_2[z] \qquad\qquad (5.19)$$

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.3-1 Zero-State Response of LTID Systems: The Transfer Function

Consider an Nth-order LTID system specified by the difference equation

$$Q[E]y[n] = P[E]x[n]$$

or

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N)y[n]$$
$$= (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1}E + b_N)x[n]$$

or

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1}y[n+1] + a_N y[n]$$
$$= b_0 x[n+N] + \cdots + b_{N-1}x[n+1] + b_N x[n]$$

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.3-1 Zero-State Response of LTID Systems: The Transfer Function

We now derive the general expression for the zero-state response: that is, the system response to input $x[n]$ when all the initial conditions $y[-1] = y[-2] = \cdots = y[-N] = 0$ (zero state). The input $x[n]$ is assumed to be causal so that $x[-1] = x[-2] = \cdots = x[-N] = 0$.

Equation (5.24) can be expressed in delay form as

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N] \qquad (5.25)$$

Because $y[-r] = x[-r] = 0$ for $r = 1, 2, \ldots, N$,

$$y[n-m]u[n] \Longleftrightarrow \frac{1}{z^m} Y[z]$$

$$x[n-m]u[n] \Longleftrightarrow \frac{1}{z^m} X[z] \qquad m = 1, 2, \ldots, N$$

Now the z-transform of Eq. (5.25) is given by

$$\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_N}{z^N}\right) Y[z] = \left(b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots + \frac{b_N}{z^N}\right) X[z]$$

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Discrete-Time System Analysis Using the z-Transform

5.3-1 Zero-State Response of LTID Systems: The Transfer Function

Now the z-transform of Eq. (5.25) is given by

$$\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_N}{z^N}\right) Y[z] = \left(b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots + \frac{b_N}{z^N}\right) X[z]$$

Multiplication of both sides by z^N yields

$$(z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N) Y[z]$$
$$= (b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N) X[z]$$

Therefore,

$$Y[z] = \left(\frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N}{z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}\right) X[z]$$
$$= \frac{P[z]}{Q[z]} X[z]$$

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DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.3-1 Zero-State Response of LTID Systems: The Transfer Function

We have shown in Eq. (5.20) that $Y[z] = X[z]H[z]$. Hence, it follows that

$$H[z] = \frac{P[z]}{Q[z]} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1}z + b_N}{z^N + a_1 z^{N-1} + \cdots + a_{N-1}z + a_N} \qquad (5.26)$$

As in the case of LTIC systems, this result leads to an alternative definition of the LTID system transfer function as the ratio of $Y[z]$ to $X[z]$ (assuming all initial conditions zero).

$$H[z] \equiv \frac{Y[z]}{X[z]} = \frac{\mathcal{Z}[\text{zero-state response}]}{\mathcal{Z}[\text{input}]}$$

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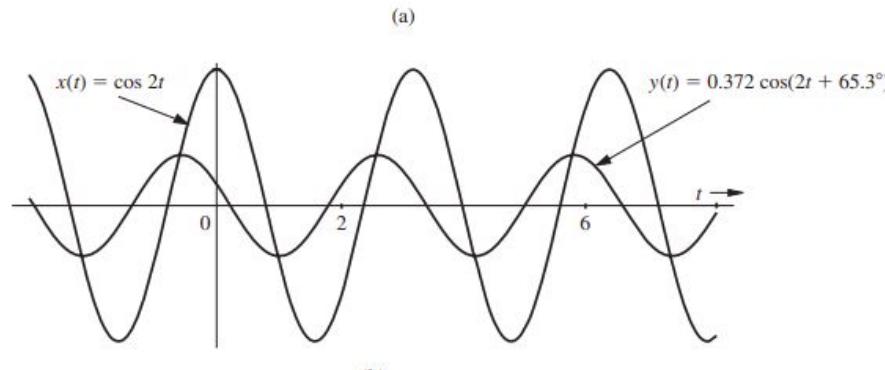
DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

For (asymptotically or BIBO-stable) continuous-time systems, we showed that the system response to an input $e^{j\omega t}$ is $H(j\omega)e^{j\omega t}$ and that the response to an input $\cos \omega t$ is $|H(j\omega)| \cos [\omega t + \angle H(j\omega)]$. Similar results hold for discrete-time systems. We now show that for an (asymptotically or BIBO-stable) LTID system, the system response to an input $e^{j\Omega n}$ is $H[e^{j\Omega}]e^{j\Omega n}$ and the response to an input $\cos \Omega n$ is $|H[e^{j\Omega}]| \cos (\Omega n + \angle H[e^{j\Omega}])$.

(a)

$x(t) = \cos 2t$

$y(t) = 0.372 \cos(2t + 65.3°)$

(b)

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

$$z^n \implies H[z]z^n \tag{5.30}$$

Setting $z = e^{j\Omega}$ in this relationship yields

$$e^{j\Omega n} \implies H[e^{j\Omega}]e^{j\Omega n} \tag{5.31}$$

Noting that $\cos \Omega n$ is the real part of $e^{j\Omega n}$, use of Eq. (3.34) yields

$$\cos \Omega n \implies \text{Re}\{H[e^{j\Omega}]e^{j\Omega n}\} \tag{5.32}$$

Expressing $H[e^{j\Omega}]$ in the polar form

$$H[e^{j\Omega}] = |H[e^{j\Omega}]|e^{j\angle H[e^{j\Omega}]}$$

Eq. (5.32) can be expressed as

$$\cos \Omega n \implies |H[e^{j\Omega}]| \cos(\Omega n + \angle H[e^{j\Omega}])$$

In other words, the system response $y[n]$ to a sinusoidal input $\cos \Omega n$ is given by

$$y[n] = |H[e^{j\Omega}]| \cos(\Omega n + \angle H[e^{j\Omega}])$$

Following the same argument, the system response to a sinusoid $\cos(\Omega n + \theta)$ is

$$y[n] = |H[e^{j\Omega}]| \cos(\Omega n + \theta + \angle H[e^{j\Omega}]) \tag{5.33}$$

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DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

$$y[n] = |H[e^{j\Omega}]| \cos(\Omega n + \theta + \angle H[e^{j\Omega}]) \tag{5.33}$$

This result is valid only for BIBO-stable or asymptotically stable systems. The frequency response is meaningless for BIBO-unstable systems (which include marginally stable and asymptotically unstable systems). This follows from the fact that the frequency response in Eq. (5.31) is obtained by setting $z = e^{j\Omega}$ in Eq. (5.30). But, as shown in Sec. 3.8-2 [Eqs. (3.38) and (3.39)], the relationship of Eq. (5.30) applies only for values of z for which $H[z]$ exists. For BIBO-unstable systems, the ROC for $H[z]$ does not include the unit circle where $z = e^{j\Omega}$. This means, for BIBO-unstable systems, that $H[z]$ is meaningless when $z = e^{j\Omega}$.†

This important result shows that the response of an asymptotically or BIBO-stable LTID system to a discrete-time sinusoidal input of frequency Ω is also a discrete-time sinusoid of the same frequency. *The amplitude of the output sinusoid is $|H[e^{j\Omega}]|$ times the input amplitude, and the phase of the output sinusoid is shifted by $\angle H[e^{j\Omega}]$ with respect to the input phase.* Clearly, $|H[e^{j\Omega}]|$ is the amplitude gain, and a plot of $|H[e^{j\Omega}]|$ versus Ω is the amplitude response of the discrete-time system. Similarly, $\angle H[e^{j\Omega}]$ is the phase response of the system, and a plot of $\angle H[e^{j\Omega}]$ versus Ω shows how the system modifies or shifts the phase of the input sinusoid. Note that $H[e^{j\Omega}]$ incorporates the information of both amplitude and phase responses and therefore is called the *frequency responses* of the system.

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

EXAMPLE 5.10 Sinusoidal Response of a Difference Equation System

For a system specified by the equation

$$y[n+1] - 0.8y[n] = x[n+1]$$

The system equation can be expressed as

$$(E - 0.8)y[n] = Ex[n]$$

Therefore, the transfer function of the system is

$$H[z] = \frac{z}{z - 0.8} = \frac{1}{1 - 0.8z^{-1}}$$

The frequency response is

$$H[e^{j\Omega}] = \frac{1}{1 - 0.8e^{-j\Omega}} = \frac{1}{(1 - 0.8\cos\Omega) + j0.8\sin\Omega}$$

Therefore,

$$|H[e^{j\Omega}]| = \frac{1}{\sqrt{(1 - 0.8\cos\Omega)^2 + (0.8\sin\Omega)^2}} = \frac{1}{\sqrt{1.64 - 1.6\cos\Omega}} \qquad (5.35)$$

and

$$\angle H[e^{j\Omega}] = -\tan^{-1}\left[\frac{0.8\sin\Omega}{1 - 0.8\cos\Omega}\right] \qquad (5.36)$$

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM
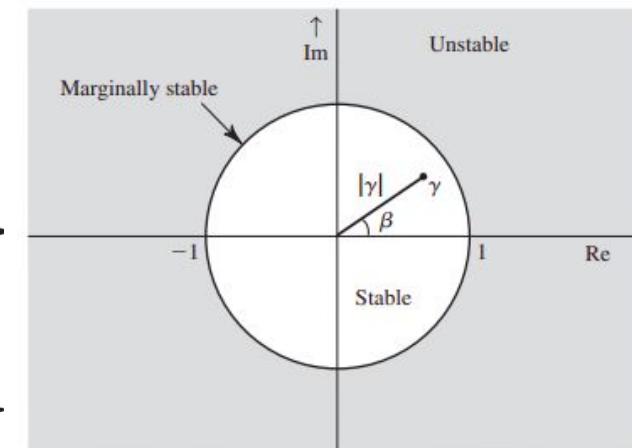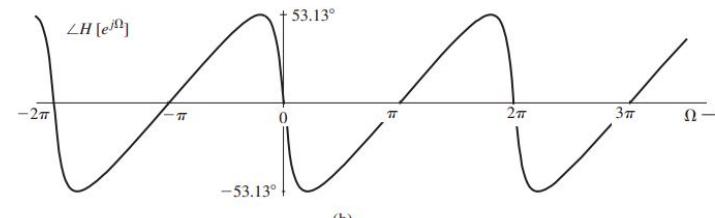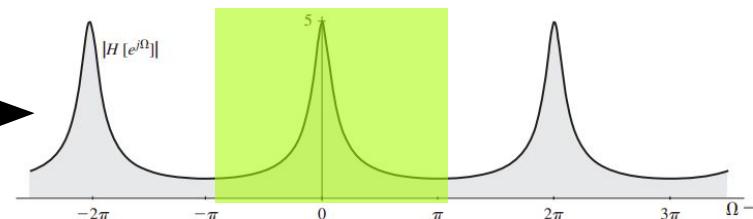
5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

EXAMPLE 5.10   Sinusoidal Response of a Difference Equation System

For a system specified by the equation

$$y[n+1] - 0.8y[n] = x[n+1]$$

Continuous (and periodic)

Figure 5.14 Frequency response of the LTID system.

Discrete-Time System Analysis Using the z-Transform

5.5 Frequency Response of Discrete-Time Systems

5.5-1 The Periodic Nature of Frequency Response

Because $e^{\pm j2\pi m} = 1$ for all integer values of m [see Eq. (B.10)],

$$H[e^{j\Omega}] = H\left[e^{j(\Omega+2\pi m)}\right] \qquad m \text{ integer}$$

Therefore, the frequency response $H[e^{j\Omega}]$ is a periodic function of Ω with a period 2π. This is the mathematical explanation of the periodic behavior.

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

5.5-1 The Periodic Nature of Frequency Response

NON-UNIQUENESS OF DISCRETE-TIME SINUSOID WAVEFORMS

A continuous-time sinusoid $\cos \omega t$ has a unique waveform for every real value of ω in the range 0 to ∞. Increasing ω results in a sinusoid of ever-increasing frequency. Such is not the case for the discrete-time sinusoid $\cos \Omega n$ because

$$\cos[(\Omega \pm 2\pi m)n] = \cos \Omega n \qquad m \text{ integer}$$

and

$$e^{j(\Omega \pm 2\pi m)n} = e^{j\Omega n} \qquad m \text{ integer}$$

$$H[e^{j\Omega}] = H[e^{j(\Omega + 2\pi m)}] \qquad m \text{ integer}$$

This shows that the discrete-time sinusoids $\cos \Omega n$ (and exponentials $e^{j\Omega n}$) separated by values of Ω in integral multiples of 2π are identical. The reason for the periodic nature of the frequency response of an LTID system is now clear. Since the sinusoids (or exponentials) with frequencies separated by interval 2π are identical, the system response to such sinusoids is also identical and, hence, is periodic with period 2π.

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

5.5-1 The Periodic Nature of Frequency Response

NON-UNIQUENESS OF DISCRETE-TIME SINUSOID WAVEFORMS

This discussion shows that the discrete-time sinusoid $\cos \Omega n$ has a unique waveform only for the values of Ω in the range $-\pi$ to π. This band is called the *fundamental band*. Every frequency Ω, no matter how large, is identical to some frequency, Ω_a, in the fundamental band ($-\pi \leq \Omega_a < \pi$), where

$$\Omega_a = \Omega - 2\pi m \qquad -\pi \leq \Omega_a < \pi \quad \text{and} \quad m \text{ integer} \qquad (5.38)$$

The integer m can be positive or negative. We use Eq. (5.38) to plot the fundamental band frequency Ω_a versus the frequency Ω of a sinusoid (Fig. 5.17a). The frequency Ω_a is modulo 2π value of Ω.
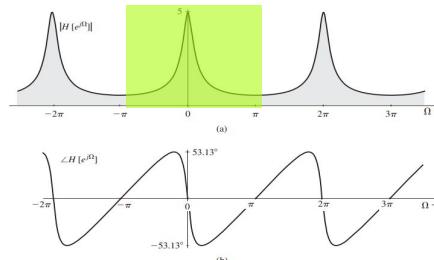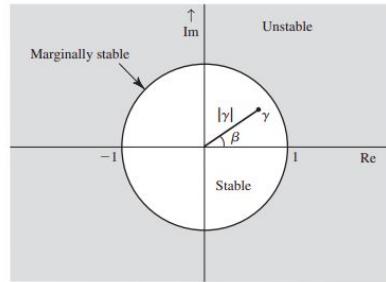

Figure 5.14 Frequency response of the LTID system.

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

$$F = \frac{\Omega}{2\pi}\left[\frac{rad}{sample}\frac{cycle}{rad}\right] = \frac{\Omega}{2\pi}\left[\frac{cycle}{sample}\right]$$

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

5.5-1 The Periodic Nature of Frequency Response

ALL DISCRETE-TIME SIGNALS ARE INHERENTLY BANDLIMITED

This discussion leads to the surprising conclusion that all discrete-time signals are inherently bandlimited, with frequencies lying in the range $-\pi$ to π radians per sample. In terms of frequency $\mathcal{F} = \Omega/2\pi$, where $\mathcal{F}$ is in cycles per sample, all frequencies $\mathcal{F}$ separated by an integer number are identical. For instance, all discrete-time sinusoids of frequencies 0.3, 1.3, 2.3, . . . cycles per sample are identical. The fundamental range of frequencies is -0.5 to 0.5 cycles per sample.

Any discrete-time sinusoid of frequency beyond the fundamental band, when plotted, appears and behaves, in every way, like a sinusoid having its frequency in the fundamental band. It is impossible to distinguish between the two signals. Thus, in a basic sense, discrete-time frequencies beyond $|\Omega| = \pi$ or $|\mathcal{F}| = 1/2$ do not exist. Yet, in a "mathematical" sense, we must admit the existence of sinusoids of frequencies beyond $\Omega = \pi$. What does this mean?

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DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

5.5-2 Aliasing and Sampling Rate

*https://www.youtube.com/watch?v=ByTsISFXUoY&ab
_channel=FABIOG.GUERRERO*
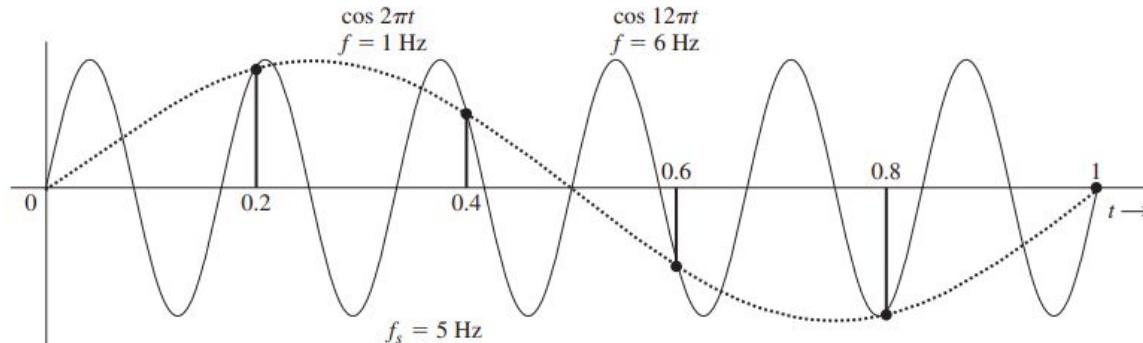
Figure 5.18 Demonstration of the aliasing effect.

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

$$T_s \ [sec]$$
$$F_s = \frac{1}{T_s} \ [Hz]$$
$$\omega = 2\pi F$$

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

5.5-2 Aliasing and Sampling Rate

a continuous-time signal that contains two distinct components of frequencies ω_1 and ω_2. The samples of these components appear as discrete-time sinusoids of frequencies $\Omega_1 = \omega_1 T$ and $\Omega_2 = \omega_2 T$. If Ω_1 and Ω_2 happen to differ by an integer multiple of 2π (if $\omega_2 - \omega_1 = 2k\pi/T$), the two frequencies will be read as the same (lower of the two) frequency by the digital processor.[‡] As a result, the higher-frequency component ω_2 not only is lost for good (by losing its identity to ω_1), but also it reincarnates as a component of frequency ω_1, thus distorting the true amplitude of the original component of frequency ω_1. Hence, the resulting processed signal will be distorted.

$$f_s = \frac{1}{T} > 2f_h \qquad \text{or} \qquad f_h < \frac{f_s}{2}$$

$$F_{max} < \frac{F_s}{2}$$

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Discrete-Time System Analysis Using the z-Transform

5.5 Frequency Response of Discrete-Time Systems

### 5.5-2 Aliasing and Sampling Rate

Anti-aliasing Filter
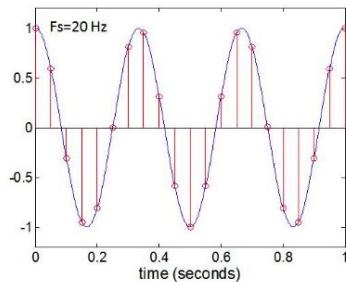
If the sampling rate fails to satisfy Eq. (5.40), aliasing occurs, causing the frequencies beyond $f_s/2$ Hz to masquerade as lower frequencies to corrupt the spectrum at frequencies below $f_s/2$. To avoid such a corruption, a signal to be sampled is passed through an *anti-aliasing* filter of bandwidth $f_s/2$ prior to sampling. This operation ensures the condition of Eq. (5.40). The drawback of such a filter is that we lose the spectral components of the signal beyond frequency $f_s/2$, which is preferable to the aliasing corruption of the signal at frequencies below $f_s/2$. Chapter 8 presents a detailed analysis of the aliasing problem.

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

Figure 5.18 Demonstration of the aliasing effect.

## 5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS
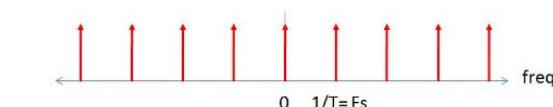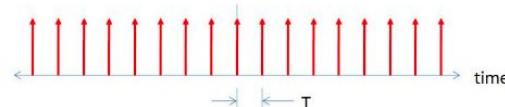
5.5-2 Aliasing and Sampling Rate

Sampling

Multiplication (in time) with an impulse train

Review – FT of Impulse Train

Impulses in time $\longrightarrow \mathscr{F}\{\} \longrightarrow$ Impulses in frequency
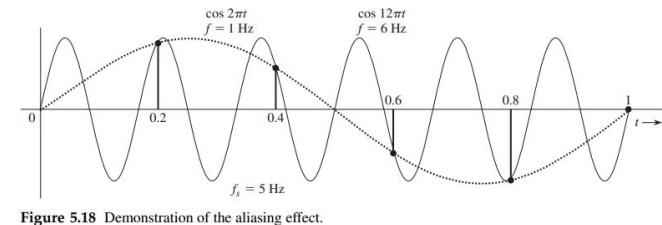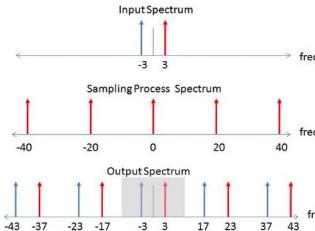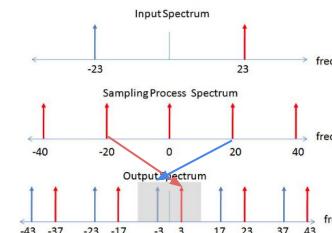
$F_s = 20[Hz]$

Sampling of 3 Hz Cosine Wave
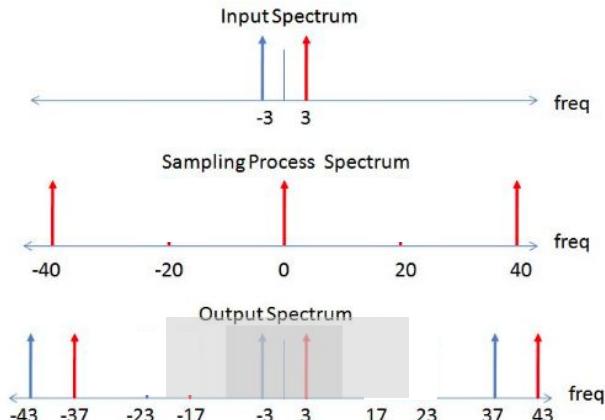
Sampling of 23 Hz Cosine Wave

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.5 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

Figure 5.18 Demonstration of the aliasing effect.
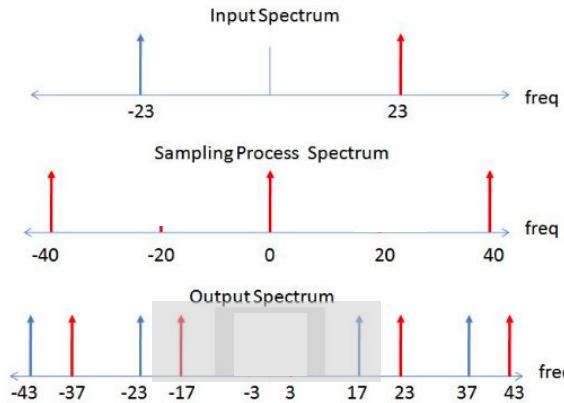
5.5-2 Aliasing and Sampling Rate

$$F_s = 40[Hz]$$

Sampling of 3 Hz Cosine Wave

Sampling of 23 Hz Cosine Wave


$looking\ through\ F$
$Vs.$
$looking\ thorugh\ \pi$
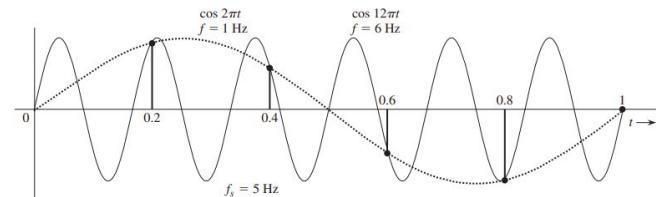
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Discrete-Time System Analysis Using the z-Transform

5.6 Frequency Response from Pole-Zero Locations

The frequency responses (amplitude and phase responses) of a system are determined by pole-zero locations of the transfer function $H[z]$. Just as in continuous-time systems, it is possible to determine quickly the amplitude and the phase response and to obtain physical insight into the filter characteristics of a discrete-time system by using a graphical technique. The general Nth-order transfer function $H[z]$ in Eq. (5.26) can be expressed in factored form as

$$H[z] = b_0 \frac{(z - z_1)(z - z_2) \cdots (z - z_N)}{(z - \gamma_1)(z - \gamma_2) \cdots (z - \gamma_N)}$$

Discrete-Time System Analysis Using the z-Transform

5.6 Frequency Response from Pole-Zero Locations

Continuous

$$H(s)|_{s=p} = b_0 \frac{(p - z_1)(p - z_2) \cdots (p - z_N)}{(p - \lambda_1)(p - \lambda_2) \cdots (p - \lambda_N)} \qquad (4.53)$$

$$|H(s)|_{s=p} = b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N} = b_0 \frac{\text{product of distances of zeros to } p}{\text{product of distances of poles to } p} \qquad (4.54)$$

$$\angle H(s)|_{s=p} = (\phi_1 + \phi_2 + \cdots + \phi_N) - (\theta_1 + \theta_2 + \cdots + \theta_N)$$
$$= \text{sum of angles of zeros to } p - \text{sum of angles of poles to } p \qquad (4.55)$$

Discrete

$$H[z] = b_0 \frac{(z - z_1)(z - z_2) \cdots (z - z_N)}{(z - \gamma_1)(z - \gamma_2) \cdots (z - \gamma_N)}$$

$$|H[e^{j\Omega}]| = b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N} = b_0 \frac{\text{product of the distances of zeros to } e^{j\Omega}}{\text{product of distances of poles to } e^{j\Omega}}$$

$$\angle H[e^{j\Omega}] = (\phi_1 + \phi_2 + \cdots + \phi_N) - (\theta_1 + \theta_2 + \cdots + \theta_N)$$
$$= \text{sum of zero angles to } e^{j\Omega} - \text{sum of pole angles to } e^{j\Omega}$$

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

5.6 FREQUENCY RESPONSE FROM POLE-ZERO LOCATIONS

$$H[z] = b_0 \frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-\gamma_1)(z-\gamma_2)\cdots(z-\gamma_N)}$$

To compute the frequency response $H[e^{j\Omega}]$ we evaluate $H[z]$ at $z = e^{j\Omega}$. But for $z = e^{j\Omega}$, $|z| = 1$ and $\angle z = \Omega$ so that $z = e^{j\Omega}$ represents a point on the unit circle at an angle Ω with the horizontal. We now connect all zeros $(z_1, z_2, \ldots, z_N)$ and all poles $(\gamma_1, \gamma_2, \ldots, \gamma_N)$ to the point $e^{j\Omega}$, as indicated in

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5.6 FREQUENCY RESPONSE FROM POLE-ZERO LOCATIONS

$$H[z] = b_0 \frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-\gamma_1)(z-\gamma_2)\cdots(z-\gamma_N)}$$

Fig. 5.19b. Let $r_1, r_2, \ldots, r_N$ be the lengths and $\phi_1, \phi_2, \ldots, \phi_N$ be the angles, respectively, of the straight lines connecting $z_1, z_2, \ldots, z_N$ to the point $e^{j\Omega}$. Similarly, let $d_1, d_2, \ldots, d_N$ be the lengths and $\theta_1, \theta_2, \ldots, \theta_N$ be the angles, respectively, of the lines connecting $\gamma_1, \gamma_2, \ldots, \gamma_N$ to $e^{j\Omega}$. Then
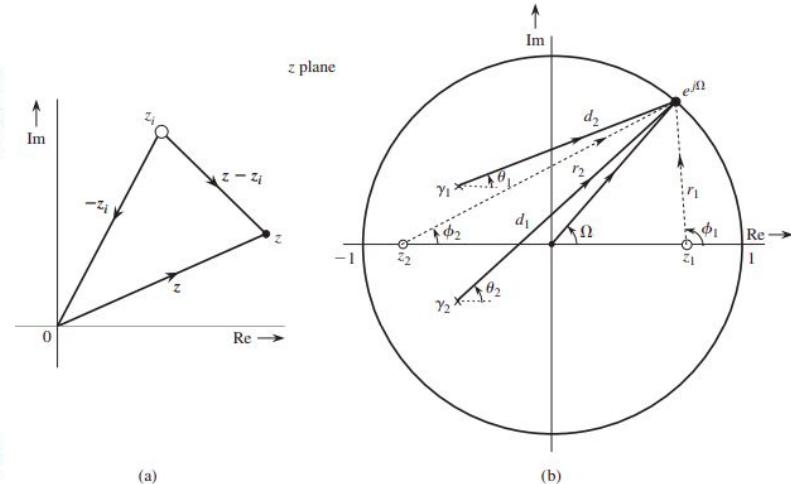
$$H[e^{j\Omega}] = H[z]|_{z=e^{j\Omega}} = b_0 \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2})\cdots(r_N e^{j\phi_N})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2})\cdots(d_N e^{j\theta_N})}$$

$$= b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N} e^{j[(\phi_1+\phi_2+\cdots+\phi_N)-(\theta_1+\theta_2+\cdots+\theta_N)]}$$

Therefore (assuming $b_0 > 0$),

$$|H[e^{j\Omega}]| = b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N} = b_0 \frac{\text{product of the distances of zeros to } e^{j\Omega}}{\text{product of distances of poles to } e^{j\Omega}} \qquad (5.41)$$

and

$$\angle H[e^{j\Omega}] = (\phi_1 + \phi_2 + \cdots + \phi_N) - (\theta_1 + \theta_2 + \cdots + \theta_N)$$
$$= \text{sum of zero angles to } e^{j\Omega} - \text{sum of pole angles to } e^{j\Omega}$$

(a)

(b)

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM
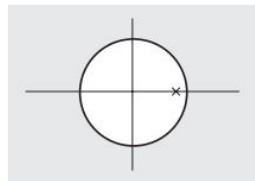
5.6 FREQUENCY RESPONSE FROM POLE-ZERO LOCATIONS

CONTROLLING GAIN BY PLACEMENT OF POLES AND ZEROS
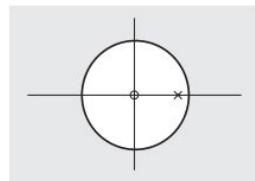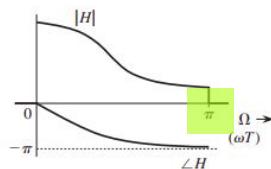
The nature of the influence of pole and zero locations on the frequency response is similar to that observed in continuous-time systems, with minor differences. In place of the imaginary axis of the continuous-time systems, we have the unit circle in the discrete-time case. The nearer the pole (or zero) is to a point $e^{j\Omega}$ (on the unit circle) representing some frequency Ω, the more influence that pole (or zero) wields on the amplitude response at that frequency because the length of the vector joining that pole (or zero) to the point $e^{j\Omega}$ is small. The proximity of a pole (or a zero) has a similar effect on the phase response. From Eq. (5.41), it is clear that to enhance the amplitude response at

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM
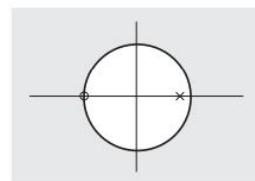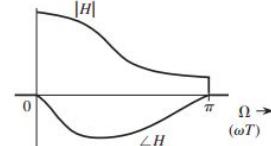
5.6 FREQUENCY RESPONSE FROM POLE-ZERO LOCATIONS
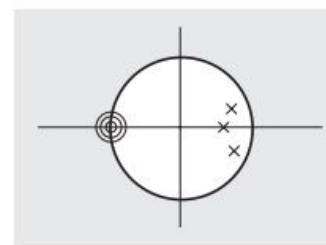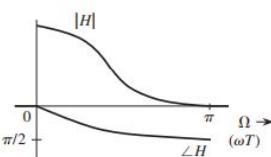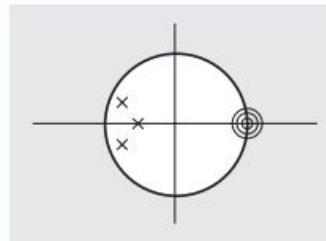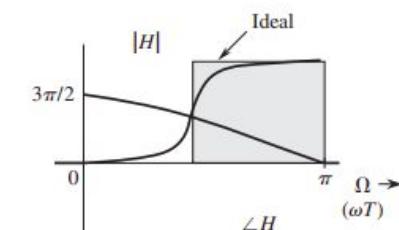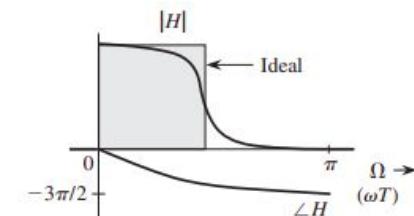
Figure 5.20 Various pole-zero configurations and the corresponding frequency responses.

Lecture Goals

1. Gain intuition about discrete signals and their properties
2. Nyquist–Shannon **Sampling Theorem**
3. **Non-uniqueness** of **discrete** time sinusoidal waveforms
4. Gain intuition about **aliasing** and **filtering**.