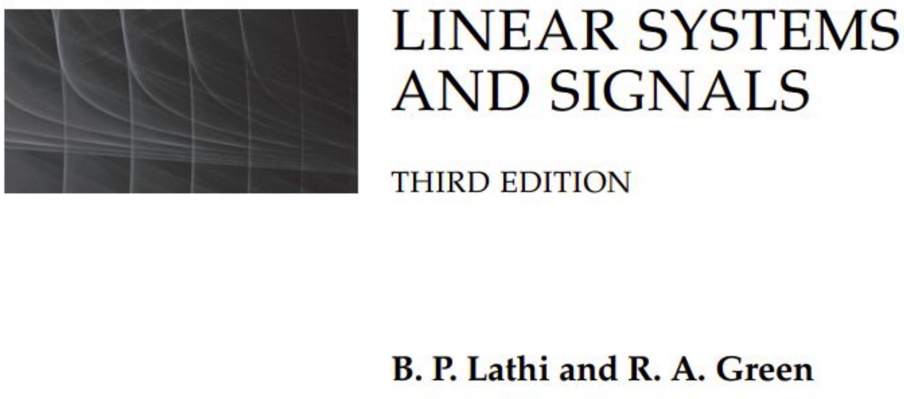
DSP Lecture 3

Continuous Time Signal Analysis- The Fourier Series & Fourier Transform

Tal Rosenwein

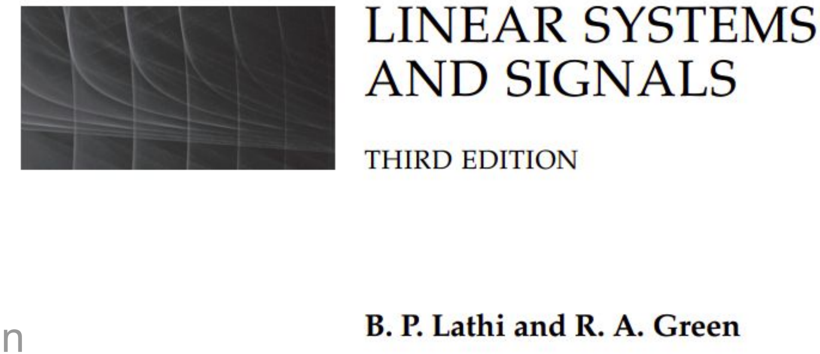
Lecture Overview

[Credit]



Lecture Overview

1. Preliminaries
2. Time Fourier Series
3. The Fourier Transform



Lecture Goals

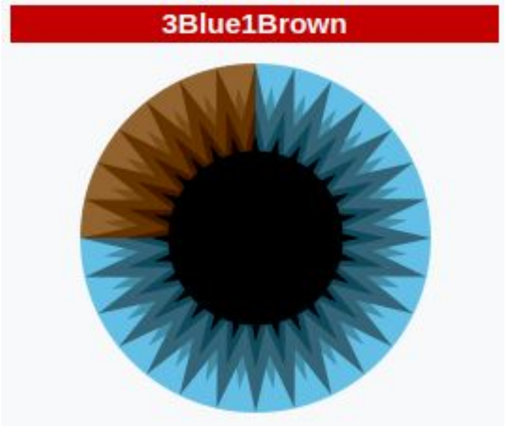
1. Gain intuition on ‘what frequency means?’
2. Similarity between Fourier and basis of vector space.
3. Get familiar with Fourier Series (for continuous and periodic signals) and some its properties
4. Get familiar with Fourier Transform (for continuous and non-periodic signals) and its properties
5. Gain Intuition on duality between time and frequency domains.

Lecture Overview

1. **Preliminaries**
2. The Fourier Series
3. The Fourier Transform

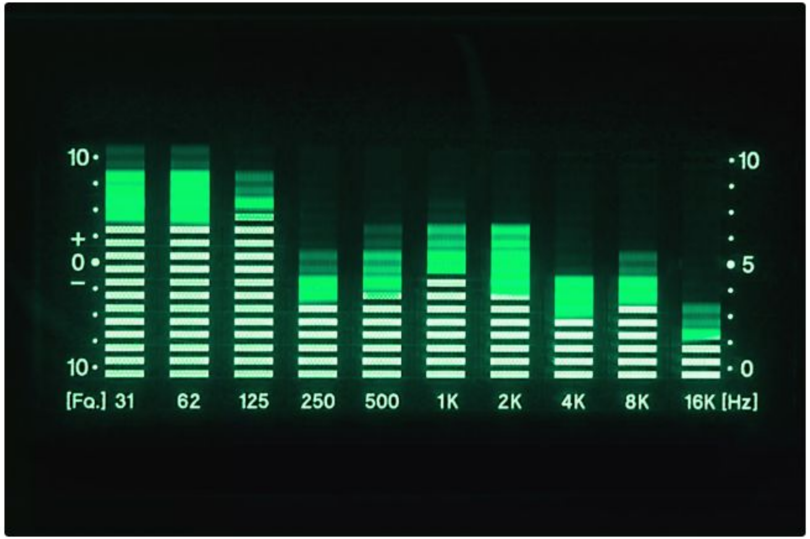
What is a frequency?

https://www.youtube.com/embed/spUNpyF58BY?start=50&end=150



What is a frequency?

https://www.szynalski.com/tone-generator/



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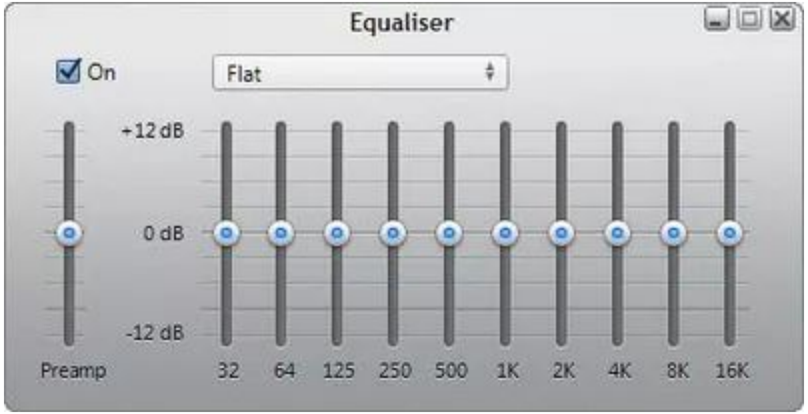
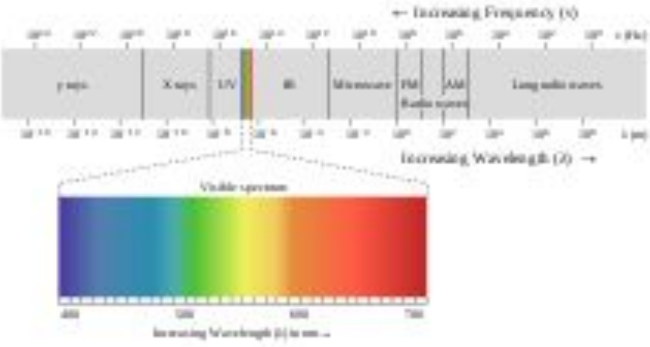
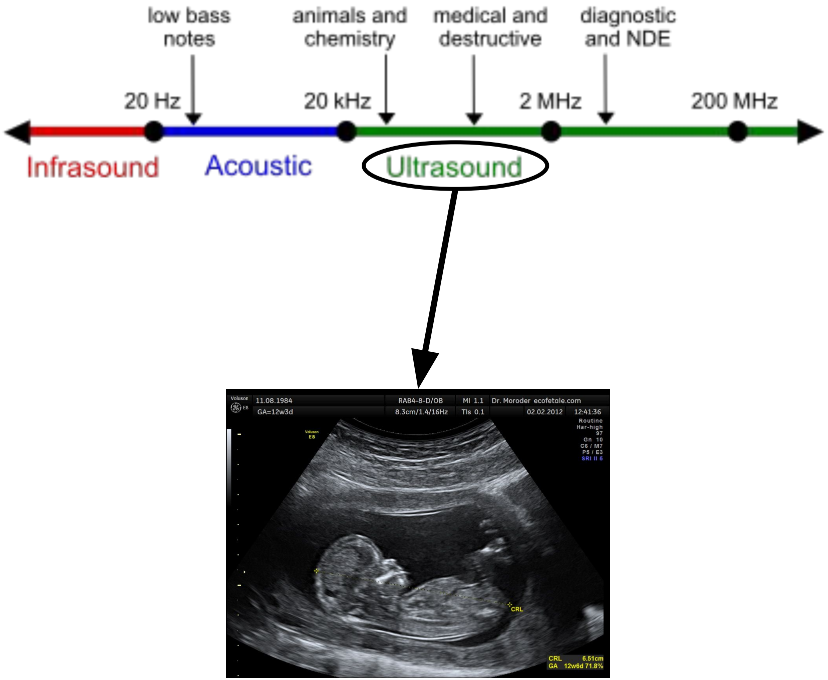


What is a frequency?

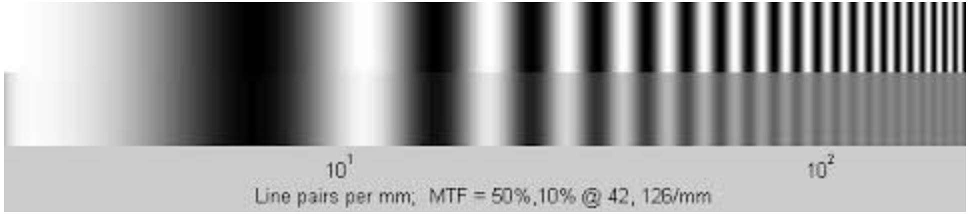


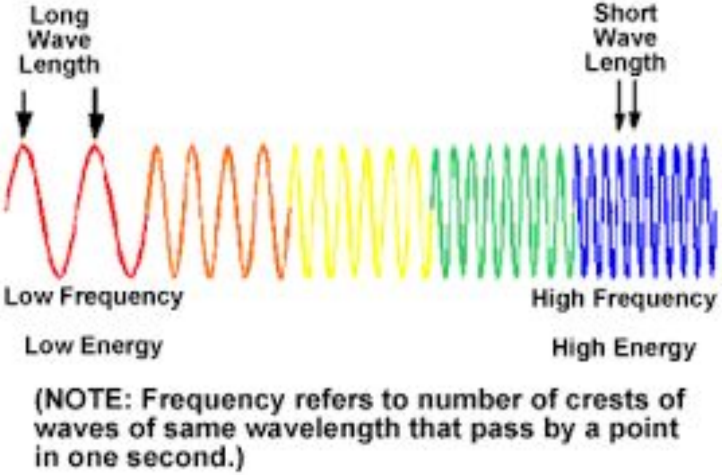


What is a frequency?

The frequency of a sinusoid $\cos 2\pi f_0 t$ or $\sin 2\pi f_0 t$ is f_0, and the period is $T_0 = 1/f_0$. These sinusoids can also be expressed as $\cos \omega_0 t$ or $\sin \omega_0 t$, where $\omega_0 = 2\pi f_0$ is the *radian frequency*,

What is a frequency?

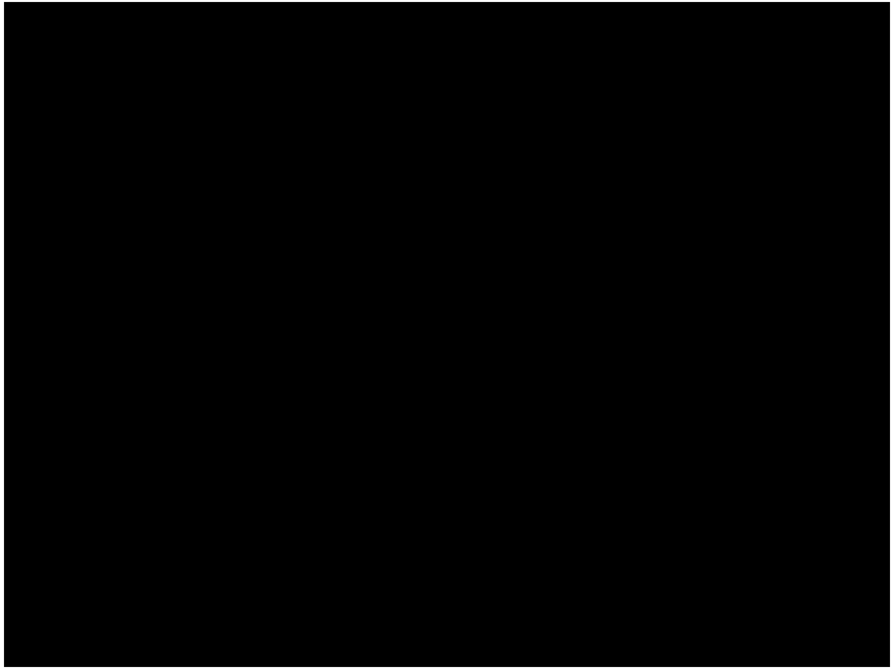




Source 1, 2

What is a frequency?

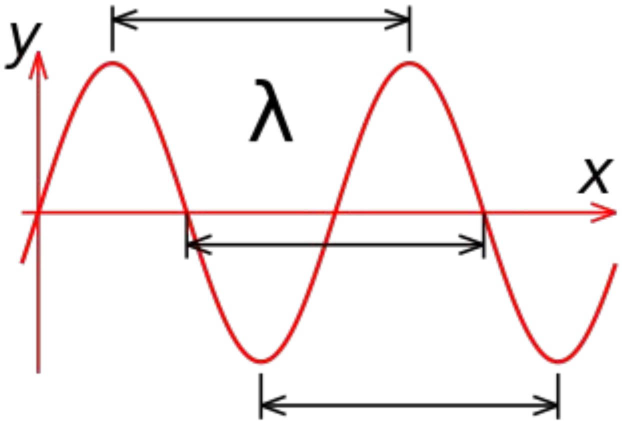
https://www.tiktok.com/@lewitt.audio/video/7213768652880022789



What is a frequency?

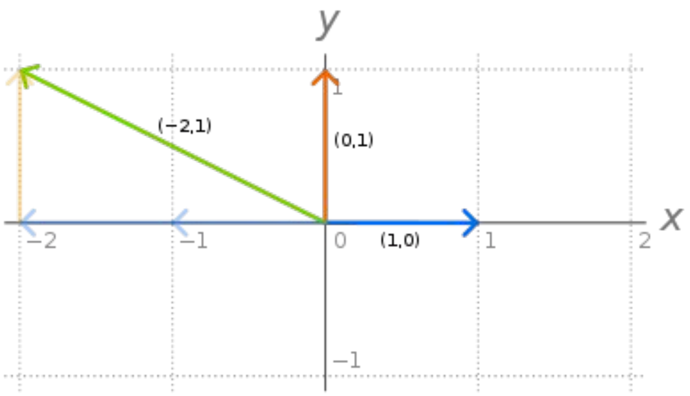
Cos/Sin with frequency of 10Hz has a **wavelength** of ~30 meters.

- Recall that the speed of sound is *c*=343 [m/s]
- $c[m/s]=\lambda[m]f[1/s]$
- If we set $f=10Hz;\ Hz=1/sec$
- $343[m/s]=\lambda[m]f[1/s]$ ->f = 10[1/s] $\rightarrow \lambda = 34.3[m]$



What is a frequency?

Basis of vector space



Each sample *Si* can be represented as a linear combination of the basis vectors.

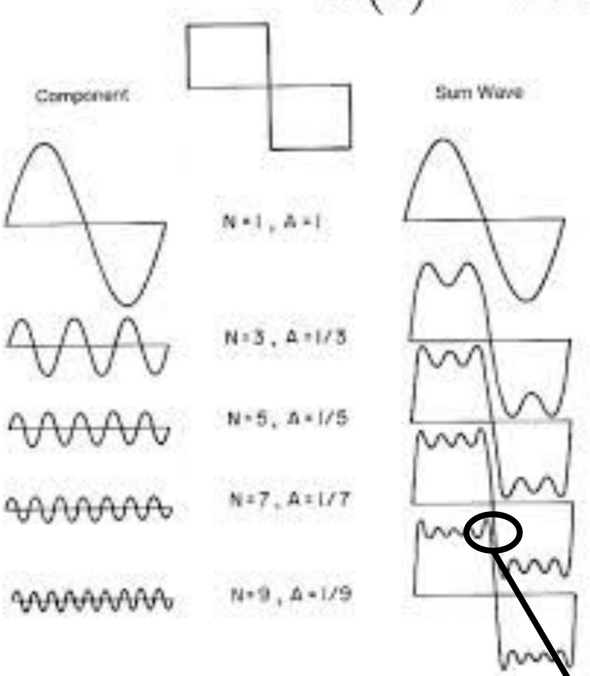
$$S_i = \Sigma \alpha_i V_i$$
$$V_i - basis\ vector$$
$$\alpha_i - weight\ of\ V_i$$

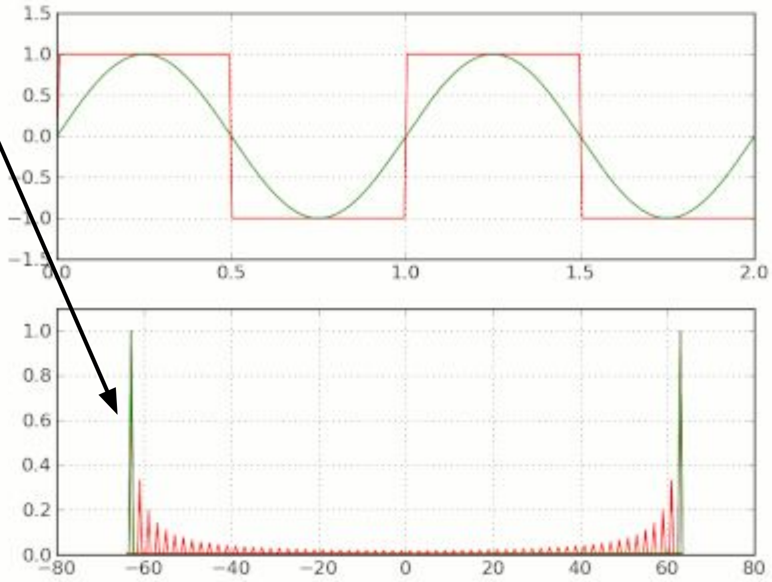
In order to find alpha_i we PROJECT the sample over Vi

$$\alpha_i =< V_i, S_j >= \Sigma_l V_{il} S_{jl}$$

What is a frequency?

$$x(t) = \Sigma \alpha_i * sin(w_i t)$$





Gibbs phenomenon- intuition

Lecture Overview

1. Preliminaries
2. **The Fourier Series**
3. The Fourier Transform

Continuous-Time Signal Analysis: The Fourier Series

In this chapter we show that a periodic signal can be represented as a sum of sinusoids (or exponentials) of various frequencies. These results are extended to aperiodic signals in Ch. 7 and to discrete-time signals in Ch. 9. The fascinating subject of sampling of continuous-time signals is discussed in Ch. 8, leading to A/D (analog-to-digital) and D/A conversion. Chapter 8 forms the bridge between the continuous-time and the discrete-time worlds.

Continuous-Time Signal Analysis: The Fourier Series

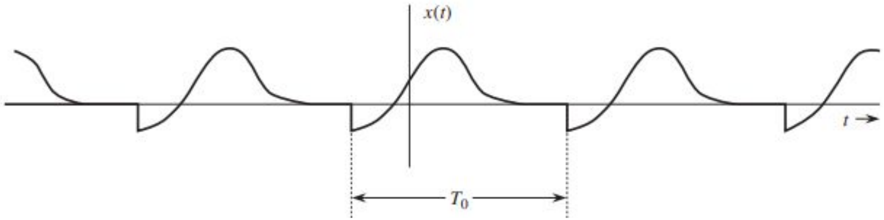
6.1 Periodic Signal Representation by Trigonometric Fourier Series

a periodic signal $x(t)$ with period T_0 (Fig. 6.1) has the property

$$x(t) = x(t + T_0) \qquad \text{for all } t$$

The *smallest* value of T_0 that satisfies this periodicity condition is the *fundamental period* of $x(t)$. As argued in Sec. 1.3-3, this equation implies that $x(t)$ starts at $-\infty$ and continues to ∞. Moreover, the area under a periodic signal $x(t)$ over any interval of duration T_0 is the same; that is, for any real numbers a and b

$$\int_a^{a+T_0} x(t)\,dt = \int_b^{b+T_0} x(t)\,dt$$



Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

Let us consider a signal $x(t)$ made up of a sines and cosines of frequency ω_0 and all of its harmonics (including the zeroth harmonic; i.e., dc) with arbitrary amplitudes†:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \tag{6.1}$$

The frequency ω_0 is called the *fundamental frequency.*

We now prove an extremely important property: $x(t)$ in Eq. (6.1) is a periodic signal with the same period as that of the fundamental, regardless of the values of the amplitudes a_n and b_n. Note that the period T_0 of the fundamental satisfies

$$T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0} \qquad \text{and} \qquad \omega_0 T_0 = 2\pi \tag{6.2}$$

CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER SERIES

6.1 PERIODIC SIGNAL REPRESENTATION BY TRIGONOMETRIC FOURIER SERIES

trigonometric Fourier series of a periodic signal $x(t)$. $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$

COMPUTING THE COEFFICIENTS OF A FOURIER SERIES

a periodic signal $x(t)$ with period T_0 can be expressed as a sum of a sinusoid of period T_0 and its harmonics:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \qquad (6.7)$$

where $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$ and

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt, \quad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt, \quad \text{and} \quad b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt \qquad (6.8)$$

CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER SERIES

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COMPUTING THE COEFFICIENTS OF A FOURIER SERIES

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt, \quad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt, \quad \text{and} \quad b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt \qquad (6.8)$$

DC - (n = 0) (offset)
Average
<u>Projection</u> over const

Even terms
<u>Projection</u> over cosine funcs

Odd terms
<u>Projection</u> over sine funcs

Fourier can be interpreted as reconstructing the original signal using cos/sin basis vectors

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

TABLE 6.1 Fourier Series Representation of a Periodic Signal of Period T_0 ($\omega_0 = 2\pi/T_0$)

Series Form	Coefficient Computation	Conversion Formulas
Trigonometric	$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt$	$a_0 = C_0 = D_0$
$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$	$a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos n\omega_0 t\,dt$	$a_n - jb_n = C_n e^{j\theta_n} = 2D_n$
	$b_n = \frac{2}{T_0}\int_{T_0} f(t)\sin n\omega_0 t\,dt$	$a_n + jb_n = C_n e^{-j\theta_n} = 2D_{-n}$
Compact trigonometric	$C_0 = a_0$	$C_0 = D_0$
$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$	$C_n = \sqrt{a_n{}^2 + b_n{}^2}$	$C_n = 2\|D_n\| \quad n \geq 1$
	$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$	$\theta_n = \angle D_n$
Exponential		
$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$	$D_n = \frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}\,dt$	

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

TABLE 6.1 Fourier Series Representa

Series Form
Trigonometric $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$
Compact trigonometric $f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$
Exponential $f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$

Cos / Sin / conj. Exp are basis vectors (orthogonal to each other)

$$\int_{T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \begin{cases} 0 & n \neq m \\ \dfrac{T_0}{2} & n = m \neq 0 \end{cases}$$

$$\int_{T_0} \cos n\omega_0 t \cos m\omega_0 t \, dt = \begin{cases} 0 & n \neq m \\ \dfrac{T_0}{2} & m = n \neq 0 \end{cases}$$

$$\int_{T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t} \, dt = \begin{cases} 0 & m \neq n \\ T_0 & m = n \end{cases}$$

Continuous-Time Signal Analysis: The Fourier Series

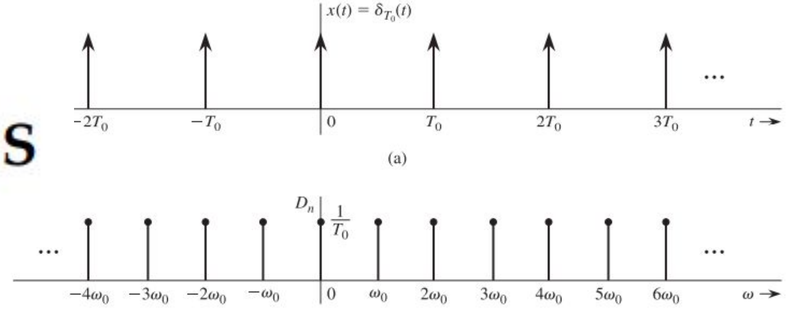
6.1 Periodic Signal Representation by Trigonometric Fourier Series

The frequency spectra of a signal constitute the *frequency-domain description* of $x(t)$, in contrast to the *time-domain description,* where $x(t)$ is specified as a function of time.

A signal, therefore, has a dual identity: the time-domain identity $x(t)$ and the frequency-domain identity (Fourier spectra). The two identities complement each other; taken together, they provide a better understanding of a signal.

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series



The unit impulse train shown in Fig. 6.15a can be expressed as

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

Following Papoulis, we shall denote this function as $\delta_{T_0}(t)$ for the sake of notational brevity.

The exponential Fourier series is given by

$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} \tag{6.23}$$

where

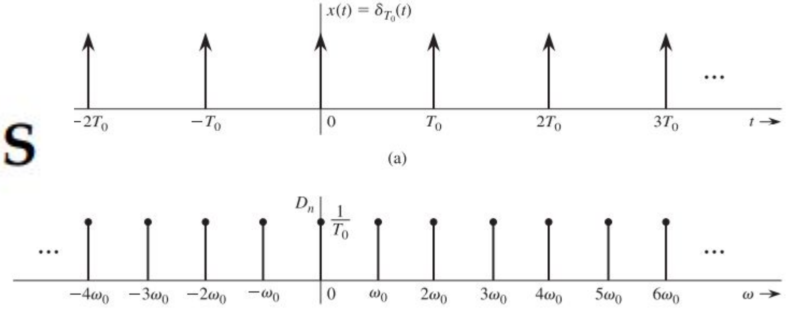
$$D_n = \frac{1}{T_0} \int_{T_0} \delta_{T_0}(t) e^{-jn\omega_0 t}\, dt$$

Choosing the interval of integration $(-T_0/2, T_0/2)$ and recognizing that over this interval $\delta_{T_0}(t) = \delta(t)$, we get

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t}\, dt$$

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series



In this integral, the impulse is located at $t = 0$. From the sampling property of Eq. (1.11), the integral on the right-hand side is the value of $e^{-jn\omega_0 t}$ at $t = 0$ (where the impulse is located). Therefore,

$$D_n = \frac{1}{T_0} \tag{6.24}$$

From this result, we see that the exponential spectrum is constant for all frequencies, as shown in Fig. 6.15b. The spectrum, being real, requires only the amplitude plot. All phases are zero.

Substituting $D_n = \frac{1}{T_0}$ into Eq. (6.23) yields the desired exponential Fourier series

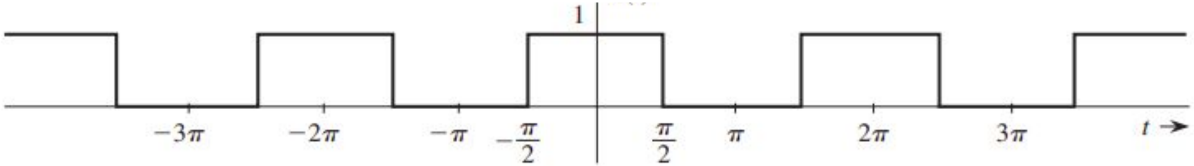
$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

EXAMPLE 6.4 Compact Trigonometric Fourier Series of a Periodic Square Wave

Direct transform



$a_0 =$?

DC

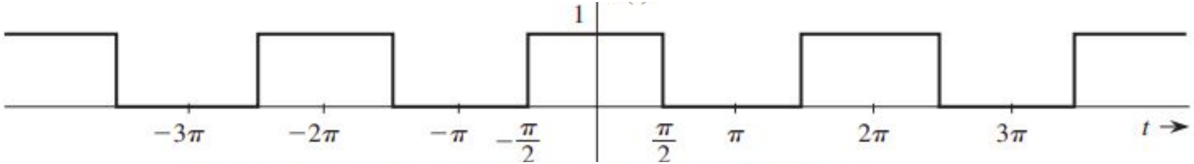
Average

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

EXAMPLE 6.4 Compact Trigonometric Fourier Series of a Periodic Square Wave

Direct transform



Here the period is $T_0 = 2\pi$ and $\omega_0 = 2\pi/T_0 = 1$. Therefore,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

where

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$$

From Fig. 6.6a, it is clear that a proper choice of region of integration is from $-\pi$ to π. But since $x(t) = 1$ only over $(-\pi/2, \pi/2)$, and $x(t) = 0$ over the remaining segment,

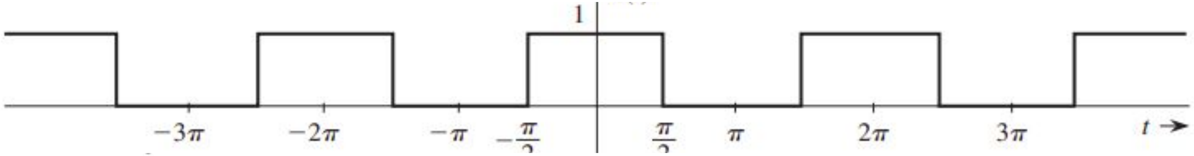
$$a_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dt = \frac{1}{2}$$

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

EXAMPLE 6.4 Compact Trigonometric Fourier Series of a Periodic Square Wave

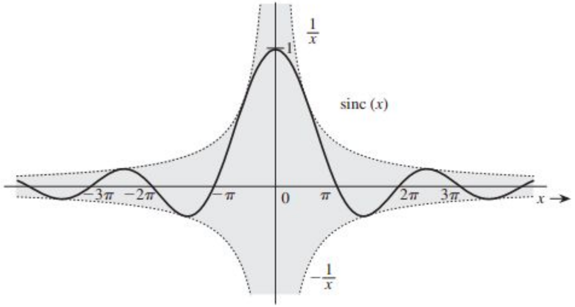
Direct transform



$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nt\, dt = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$(sampled)\ sinc(\frac{\pi n}{2})$$

$$\mathrm{sinc}\,(x) = \frac{\sin x}{x}$$

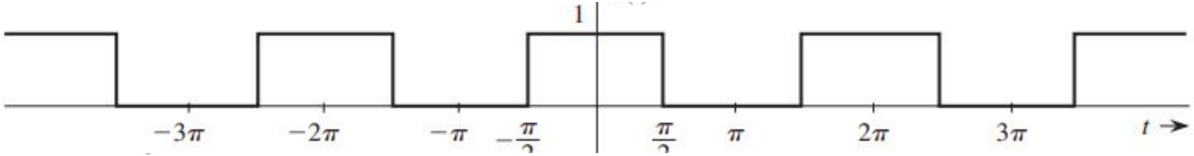


CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER SERIES

6.1 PERIODIC SIGNAL REPRESENTATION BY TRIGONOMETRIC FOURIER SERIES

EXAMPLE 6.4 Compact Trigonometric Fourier Series of a Periodic Square Wave

Direct transform



$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nt\, dt = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$(sampled)\ sinc(\frac{\pi n}{2})$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n = 1, 5, 9, 13, \ldots \\ -\frac{2}{\pi n} & n = 3, 7, 11, 15, \ldots \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nt\, dt = 0$$

Therefore

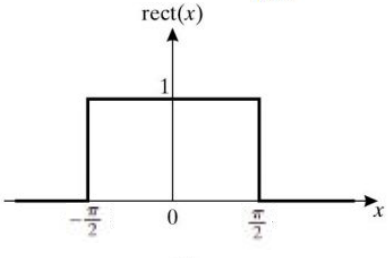
$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right) \tag{6.13}$$

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

EXAMPLE 6.4 Compact Trigonometric Fourier Series of a Periodic Square Wave

Intuition



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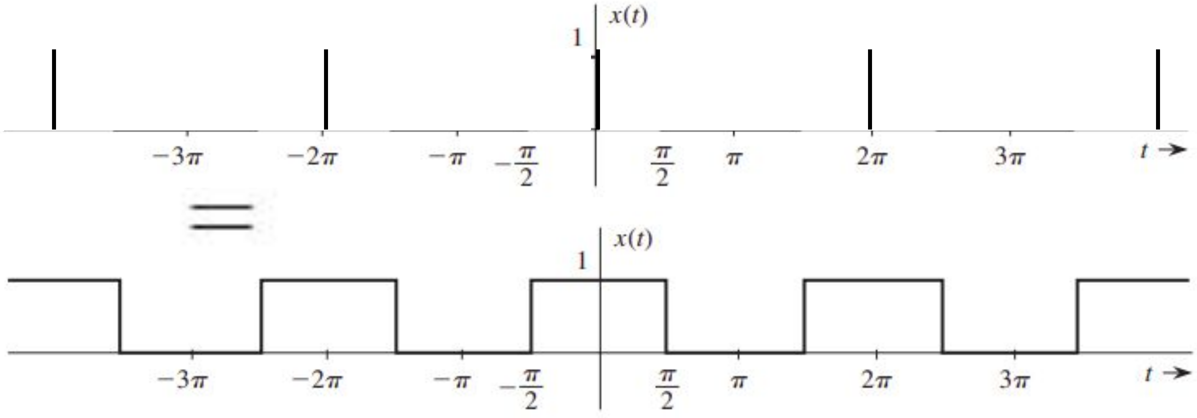


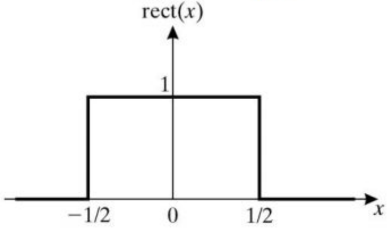
Time Domain

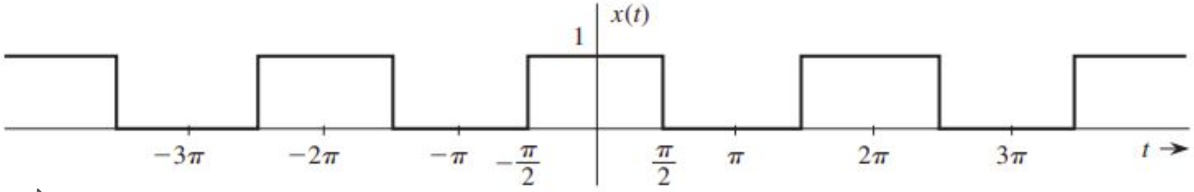
Continuous-Time Signal Analysis: The Fourier Series

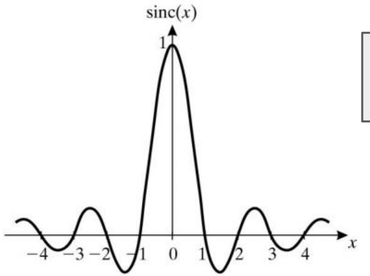
6.1 Periodic Signal Representation by Trigonometric Fourier Series

EXAMPLE 6.4 Compact Trigonometric Fourier Series of a Periodic Square Wave

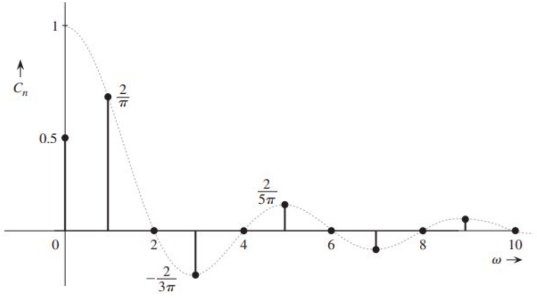
Intuition







Conv in Time =
Mul in Freq

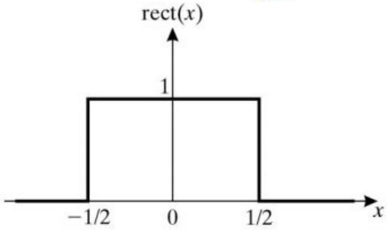


Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

$$\int_{-\infty}^{\infty} x(t)e^{-jwt}dt = \int_{-A/2}^{A/2} \frac{1}{A}e^{-jwt}dt = \frac{1}{A}\frac{e^{-jwt}}{-jw}\Big|_{-A/2}^{A/2} = \frac{e^{-jwA/2} - e^{jwA/2}}{-jwA} =$$

$$\frac{-2j sin(\frac{wA}{2})}{-jwA} = \frac{sin(\frac{wA}{2})}{\frac{wA}{2}} = sinc(\frac{wA}{2})$$



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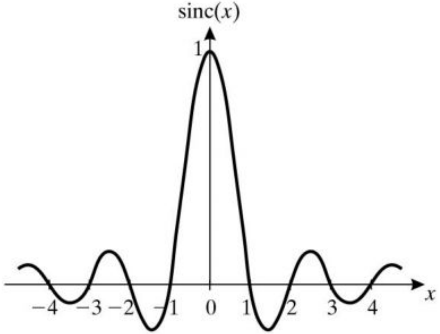


(b)

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

6.2-1 Convergence of a Series

$$\int_{T_0} |x(t)|\, dt < \infty \qquad (6.16)$$

Dirichlet Conditions

Dirichlet showed that if $x(t)$ satisfies certain conditions (*Dirichlet conditions*), its Fourier series is guaranteed to converge pointwise at all points where $x(t)$ is continuous. Moreover, at the points of discontinuities, $x(t)$ converges to the value midway between the two values of $x(t)$ on either side of the discontinuity. These conditions are:

1. The function $x(t)$ must be absolutely integrable; that is, it must satisfy Eq. (6.16).
2. The function $x(t)$ must have only a finite number of finite discontinuities in one period.
3. The function $x(t)$ must contain only a finite number of maxima and minima in one period.

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

Some intuition on the importance of the magnitude and phase

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

6.2-1 Convergence of a Series

Asymptotic Rate of Amplitude Spectrum Decay

The amplitude spectrum indicates the amounts (amplitudes) of various frequency components of $x(t)$.

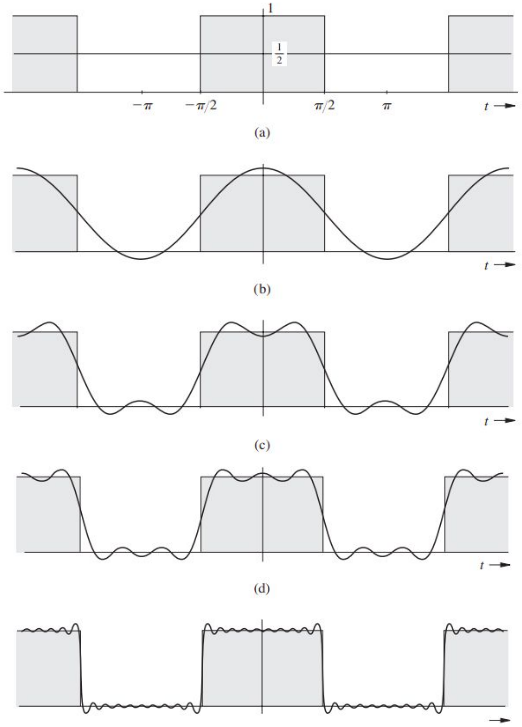


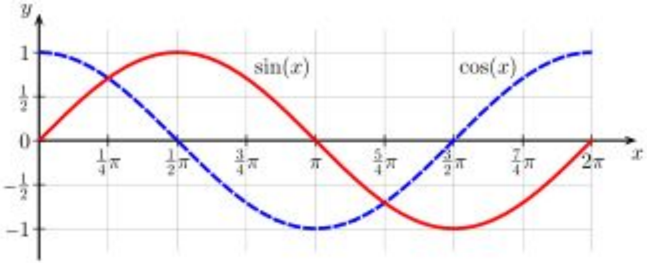
Figure 6.8 brings out one interesting aspect of the Fourier series. Lower frequencies in the Fourier series affect the large-scale behavior of $x(t)$, whereas the higher frequencies determine the fine structure such as rapid wiggling. Hence, sharp changes in $x(t)$, being a part of fine structure, necessitate higher frequencies in the Fourier series. The sharper the change [the higher the time derivative $\dot{x}(t)$], the higher are the frequencies needed in the series.

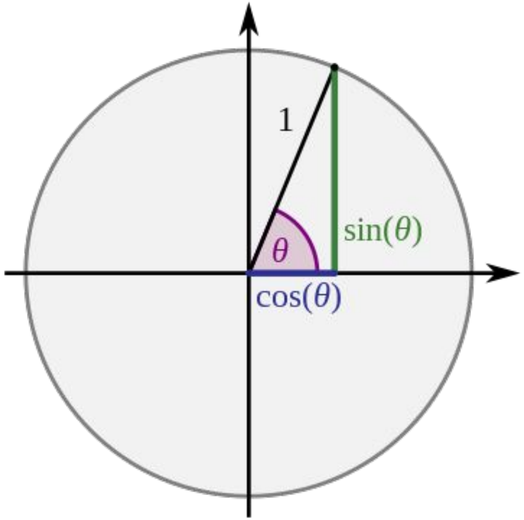
Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

6.2-1 Convergence of a Series

Phase Spectrum





CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER SERIES

6.1 PERIODIC SIGNAL REPRESENTATION BY TRIGONOMETRIC FOURIER SERIES

6.2-1 Convergence of a Series

PHASE SPECTRUM

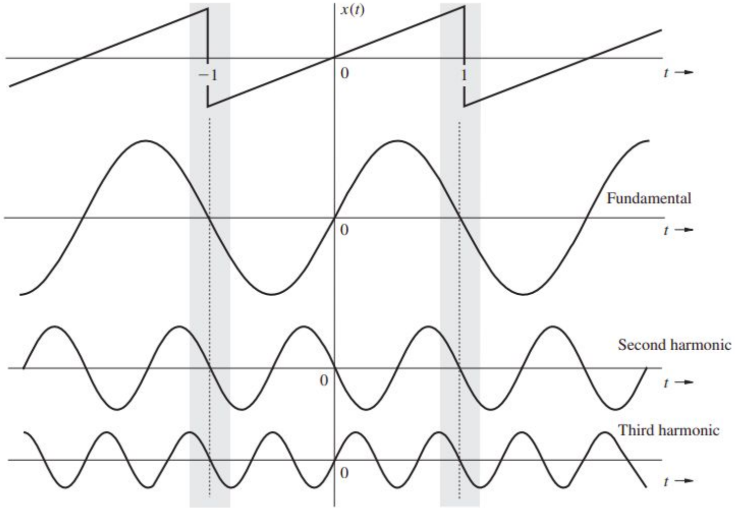


Figure 6.10 Role of the phase spectrum in shaping a periodic signal.

The role of the amplitude spectrum in shaping the waveform $x(t)$ is quite clear. However, the role of the phase spectrum in shaping this waveform is less obvious. Yet, the phase spectrum, plays an equally important role in waveshaping. We can explain this role by considering a signal $x(t)$ that has rapid changes, such as jump discontinuities. To synthesize an instantaneous change at a jump discontinuity, the phases of the various sinusoidal components in its spectrum must be such that all (or most) of the harmonic components will have one sign before the discontinuity and the opposite sign after the discontinuity. This will result in a sharp change in $x(t)$ at the point of discontinuity.

Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series

6.2-1 Convergence of a Series

Phase Spectrum

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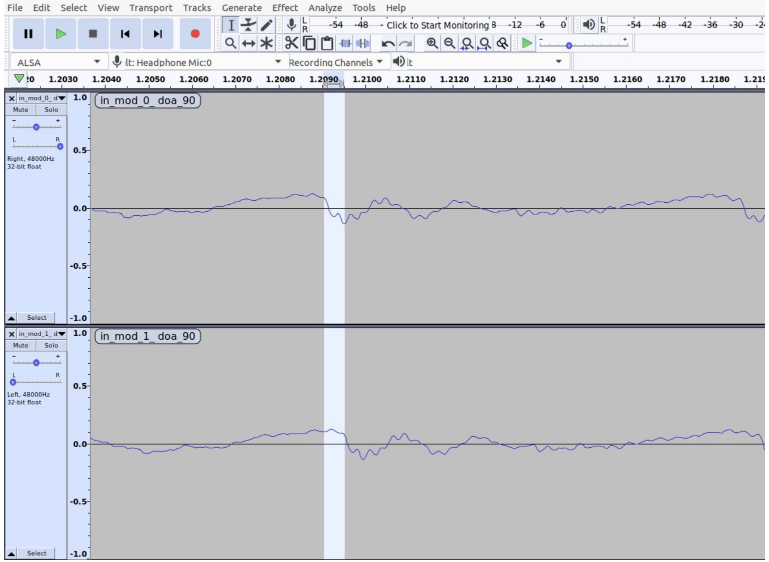


Lecture Overview

1. Preliminaries
2. The Fourier Series
3. **The Fourier Transform**

Continuous-Time Signal Analysis: The Fourier Transform

transform. In a sense, the Fourier transform may be considered to be a special case of the Laplace transform with $s = j\omega$. Although this view is true most of the time, it does not always hold because of the nature of convergence of the Laplace and Fourier integrals.

In Ch. 6, we succeeded in representing periodic signals as a sum of (everlasting) sinusoids or exponentials of the form $e^{j\omega t}$. The Fourier integral developed in this chapter extends this spectral representation to aperiodic signals.

Continuous-Time Signal Analysis: The Fourier Transform

7.1 Aperiodic Signal Representation by the Fourier Integral

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad \Longleftrightarrow \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Continuous-Time Signal Analysis: The Fourier Transform

7.1 Aperiodic Signal Representation by the Fourier Integral

Existence of the Fourier Transform

The Dirichlet conditions are as follows:

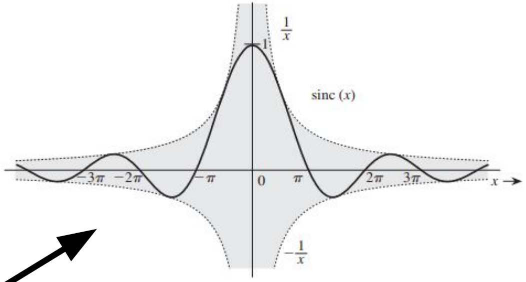
1. $x(t)$ should be absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty \qquad (7.14)$$

   If this condition is satisfied, we see that the integral on the right-hand side of Eq. (7.9) is guaranteed to have a finite value.
2. $x(t)$ must have only a finite number of finite discontinuities within any finite interval.
3. $x(t)$ must contain only a finite number of maxima and minima within any finite interval.

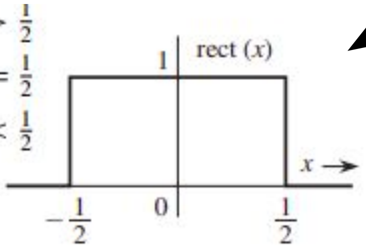
Continuous-Time Signal Analysis: The Fourier Transform

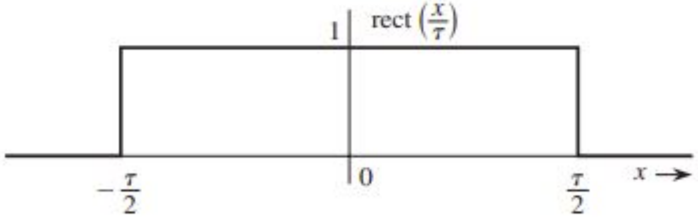
7.2 Transforms of Some Useful Functions



Unit Gate Function

$$\operatorname{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$





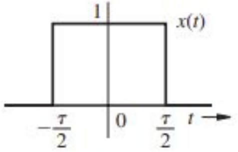
$$\operatorname{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$

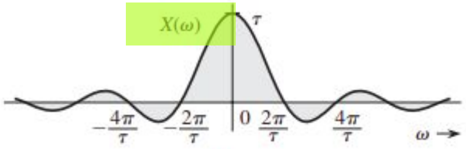
$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega}(e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

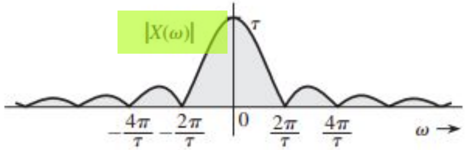
$$= \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$

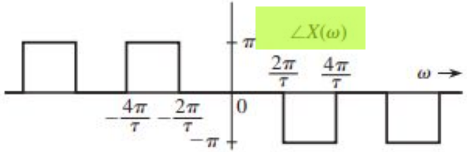


(a)



(b)





Finite in time <-> infinite in spectra

Continuous-Time Signal Analysis: The Fourier Transform

7.2 Transforms of Some Useful Functions

EXAMPLE 7.3 Fourier Transform of the Dirac Delta Function

Find the Fourier transform of the unit impulse $\delta(t)$.

Using the sampling property of the impulse [Eq. (1.11)], we obtain

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = 1 \qquad \text{and} \qquad \delta(t) \Longleftrightarrow 1$$

Figure 7.11 shows $\delta(t)$ and its spectrum.

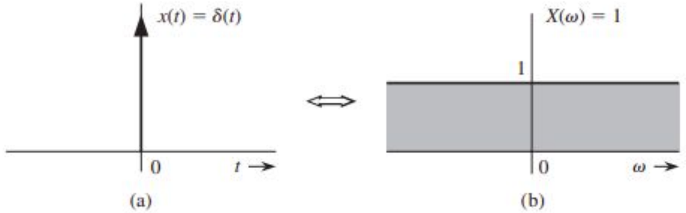


Figure 7.11 **(a)** Unit impulse and **(b)** its Fourier spectrum.

What can it be useful for?

Continuous-Time Signal Analysis: The Fourier Transform

7.2 Transforms of Some Useful Functions

EXAMPLE 7.4 Inverse Fourier Transform of the Dirac Delta Function

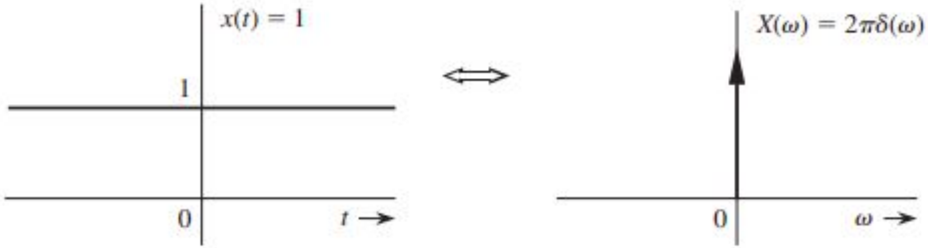
Find the inverse Fourier transform of $\delta(\omega)$.

On the basis of Eq. (7.10) and the sampling property of the impulse function,

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}$$

Therefore,

$$\frac{1}{2\pi} \Longleftrightarrow \delta(\omega) \qquad \text{and} \qquad 1 \Longleftrightarrow 2\pi\,\delta(\omega) \tag{7.20}$$



Intuition:
in time domain there is an offset (DC) -> only a single component in the spectrum should be "on" - DC component

Continuous-Time Signal Analysis: The Fourier Transform

7.2 Transforms of Some Useful Functions

Transform of sin / cos Recall Euler's formula

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

EXAMPLE 7.5 Inverse Fourier Transform of a Shifted Dirac Delta Function

Find the inverse Fourier transform of $\delta(\omega - \omega_0)$.

Using the sampling property of the impulse function, we obtain

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}\,d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

Therefore,

$$\frac{1}{2\pi}e^{j\omega_0 t} \Longleftrightarrow \delta(\omega - \omega_0) \qquad \text{and} \qquad e^{j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega - \omega_0) \tag{7.21}$$

This result shows that the spectrum of an everlasting exponential $e^{j\omega_0 t}$ is a single impulse at $\omega = \omega_0$. We reach the same conclusion by qualitative reasoning. To represent the everlasting

Continuous-Time Signal Analysis: The Fourier Transform

7.2 Transforms of Some Useful Functions

Transform of sin / cos

EXAMPLE 7.6 Fourier Transform of a Sinusoid

Find the Fourier transform of the everlasting sinusoid $\cos \omega_0 t$ (Fig. 7.13a).

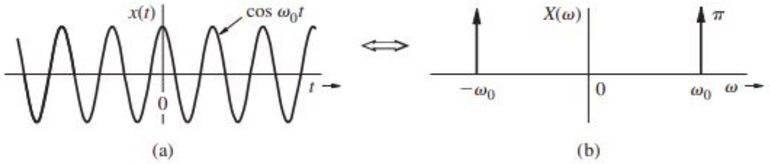


Figure 7.13 **(a)** A cosine signal and **(b)** its Fourier spectrum.

Recall Euler's formula

$$\cos \omega_0 t = \tfrac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

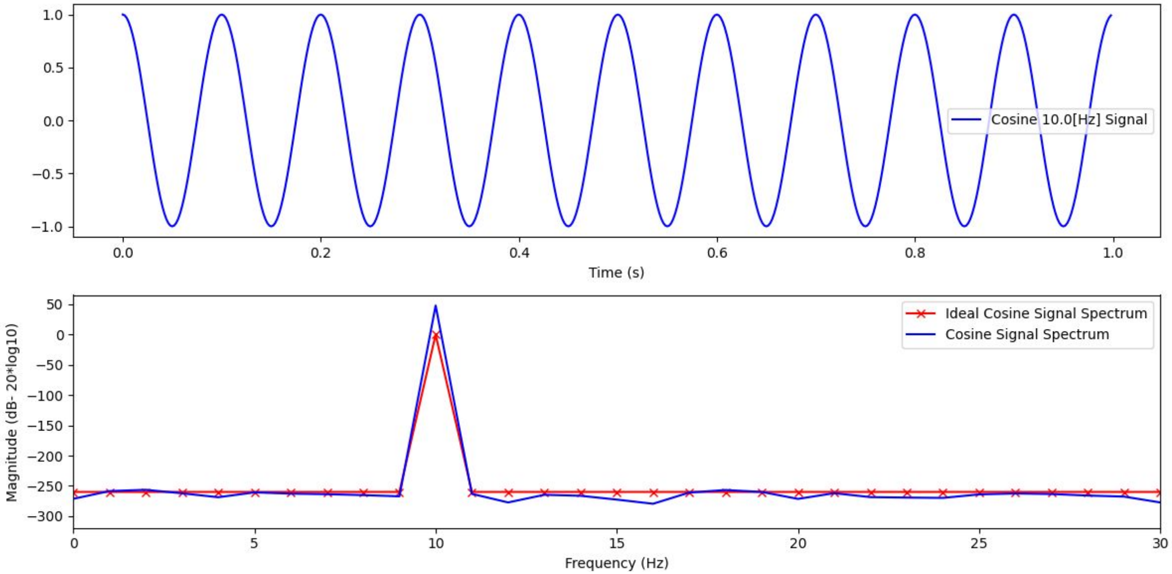
Applying Eq. (7.21), we obtain

$$\cos \omega_0 t \Longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

Continuous-Time Signal Analysis: The Fourier Transform

7.2 Transforms of Some Useful Functions

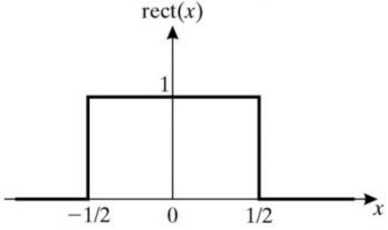
Transform of sin / cos

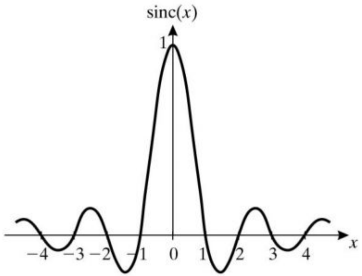


Continuous-Time Signal Analysis: The Fourier Transform

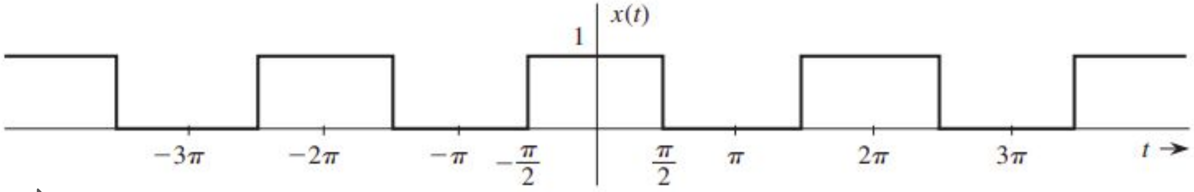
7.2 Transforms of Some Useful Functions

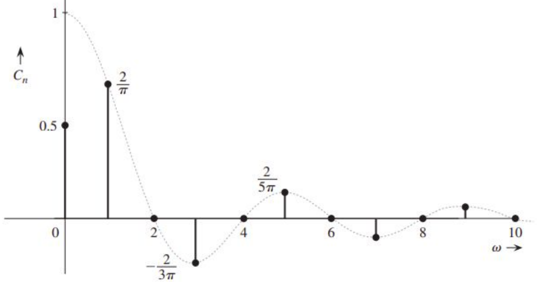
Periodic Signal < -- > Discrete Spectrum
Discrete Signal < -- > ?





Conv in Time = Mul in Freq





CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER TRANSFORM

7.2-1 Connection Between the Fourier and Laplace Transforms

The general (bilateral) Laplace transform of a signal $x(t)$, according to Eq. (4.1), is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \tag{7.24}$$

Setting $s = j\omega$ in this equation yields

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

where $X(j\omega) = X(s)|_{s=j\omega}$. But, the right-hand-side integral defines $X(\omega)$, the Fourier transform of $x(t)$. Does this mean that the Fourier transform can be obtained from the corresponding Laplace transform by setting $s = j\omega$? In other words, is it true that $X(j\omega) = X(\omega)$?

Continuous-Time Signal Analysis: The Fourier Transform

7.3 Some Properties of the Fourier Transform

Linearity

The linearity property, already introduced as Eq. (7.15), states that if $x_1(t) \Longleftrightarrow X_1(\omega)$ and $x_2(t) \Longleftrightarrow X_2(\omega)$, then $a_1x_1(t) + a_2x_2(t) \Longleftrightarrow a_1X_1(\omega) + a_2X_2(\omega)$.

CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER TRANSFORM

7.3 SOME PROPERTIES OF THE FOURIER TRANSFORM

THE SCALING PROPERTY

If

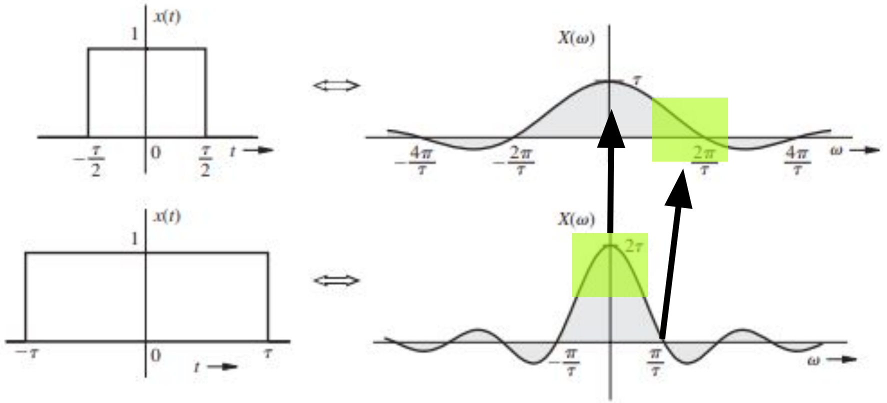
$$x(t) \Longleftrightarrow X(\omega)$$

then, for any real constant a,

$$x(at) \Longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof. For a positive real constant a,

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t}\, dt = \frac{1}{a} \int_{-\infty}^{\infty} x(u) e^{(-j\omega/a)u}\, du = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$



The scaling property states that time compression of a signal results in its spectral expansion, and time expansion of the signal results in its spectral compression. Intuitively, compression in time by factor a means that the signal is varying faster by factor a.† To synthesize such a signal, the frequencies of its sinusoidal components must be increased by the factor a, implying that its frequency spectrum is expanded by the factor a. Similarly, a signal expanded in time varies more slowly; hence the frequencies of its components are lowered, implying that its frequency spectrum is compressed. For instance, the signal $\cos 2\omega_0 t$ is the same as the signal $\cos \omega_0 t$ time-compressed by a factor of 2. Clearly, the spectrum of the former (impulse at $\pm 2\omega_0$) is an expanded version of the spectrum of the latter (impulse at $\pm\omega_0$). The effect of this scaling is demonstrated in Fig. 7.20.

Continuous-Time Signal Analysis: The Fourier Transform

7.3 Some Properties of the Fourier Transform

The Time-Shifting Property

If

$$x(t) \Longleftrightarrow X(\omega)$$

then

$$x(t - t_0) \Longleftrightarrow X(\omega)e^{-j\omega t_0} \tag{7.29}$$

Continuous-Time Signal Analysis: The Fourier Transform

7.3 Some Properties of the Fourier Transform

Convolution

The time-convolution property and its dual, the frequency-convolution property, state that if

$$x_1(t) \Longleftrightarrow X_1(\omega) \qquad \text{and} \qquad x_2(t) \Longleftrightarrow X_2(\omega)$$

then

$$x_1(t) * x_2(t) \Longleftrightarrow X_1(\omega)X_2(\omega) \quad \text{(time convolution)} \tag{7.33}$$

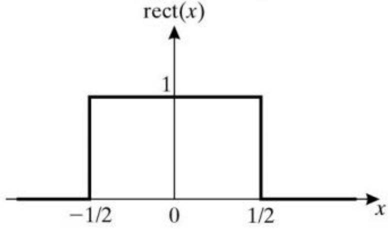
and

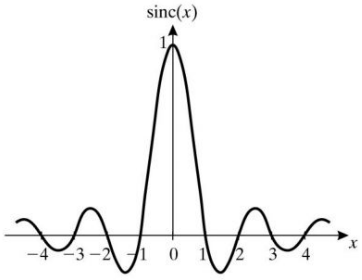
$$x_1(t)x_2(t) \Longleftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega) \quad \text{(frequency convolution)} \tag{7.34}$$

Continuous-Time Signal Analysis: The Fourier Transform

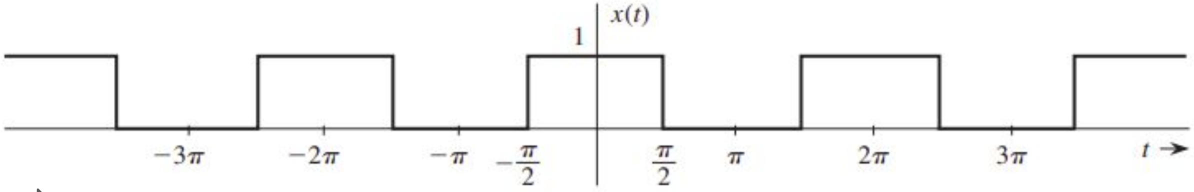
7.2 Transforms of Some Useful Functions

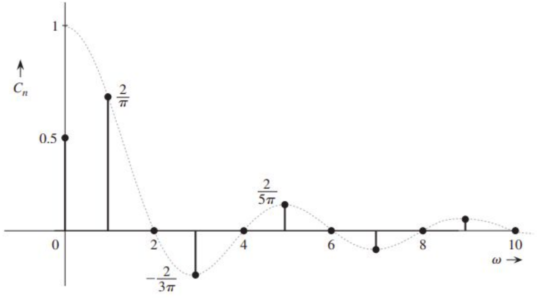
Periodic Signal < -- > Discrete Spectrum
Discrete Signal < -- > Periodic Spectrum





Conv in Time = Mul in Freq





Lecture Goals

1. Gain intuition on ‘what frequency means?’
2. Similarity between Fourier and basis of vector space.
3. Get familiar with Fourier Series (for continuous and periodic signals) and some its properties
4. Get familiar with Fourier Transform (for continuous and non-periodic signals) and its properties
5. Gain Intuition on duality between time and frequency domains.