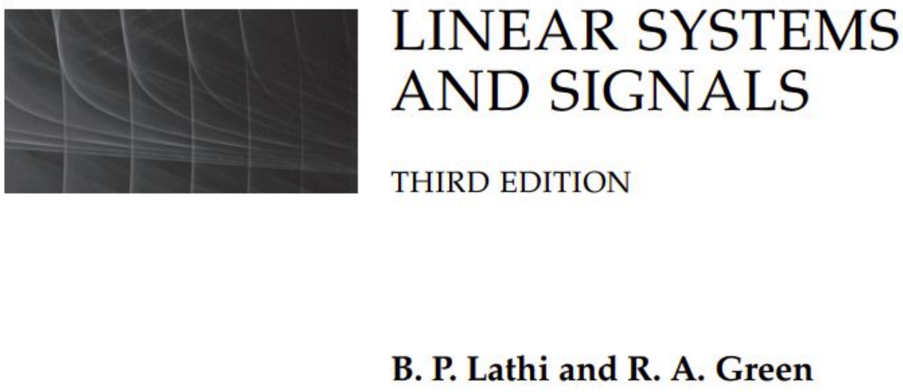
DSP Lecture 4

Discrete Time Signal Analysis- Sampling (& Reconstruction), Discrete Time Fourier Transform, Discrete Time Fourier Series and Fast Fourier Transform

Tal Rosenwein

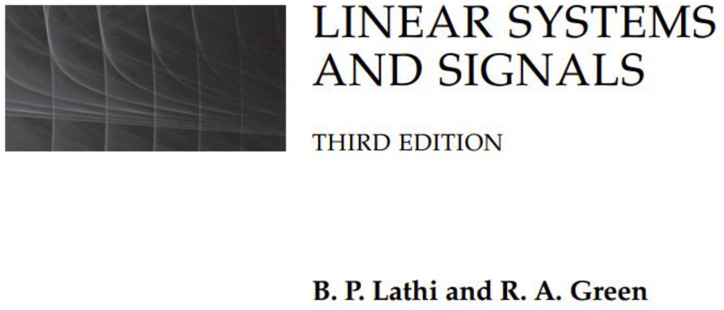
Lecture Overview

Credit



Lecture Overview

1. Analog → Digital → Analog:
    a. Preliminaries: Aliasing
    b. Sampling
    c. Reconstruction
2. Discrete Time Fourier Series (DTFS)
3. Discrete Time Fourier Transform (DTFT)
4. Discrete Fourier Transform (DFT)
5. Fast Fourier Transform (FFT)



Lecture Goals

1. Grasp some intuition on the process of Analog → Digital → Analog
2. Get familiar with Discrete Time Fourier Series (for discrete and periodic signals)
3. Get familiar with Discrete Time Fourier Transform (for discrete and non-periodic signals) and some intuition on it’s properties
4. Get familiar with Discrete Fourier Transform (for practical signals)

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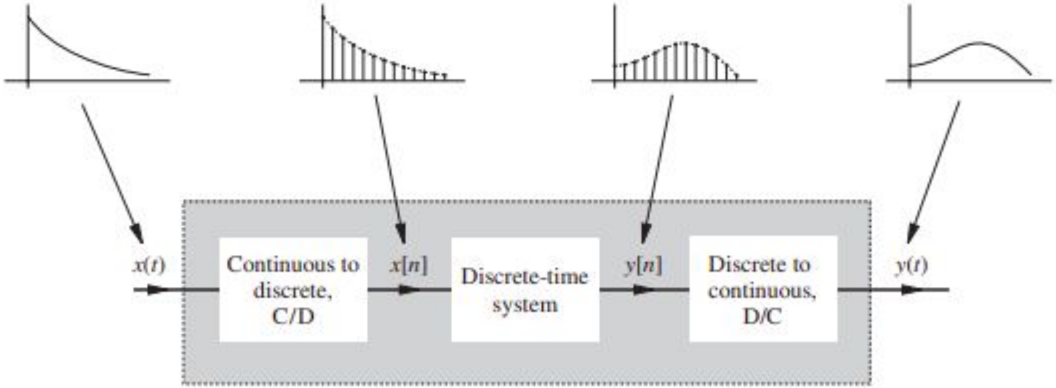

Figure 3.2 Processing a continuous-time signal by means of a discrete-time system.

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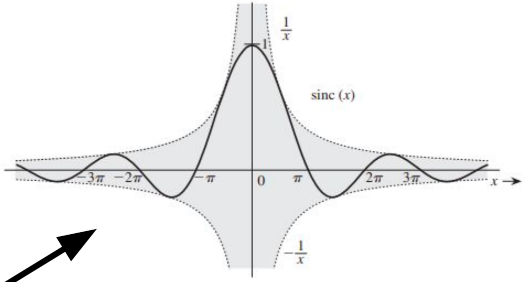
Summarizing Lecture 3 in 3 slides

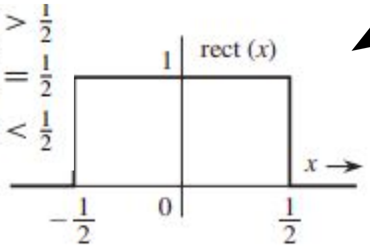
Continuous-Time Signal Analysis: The Fourier Transform

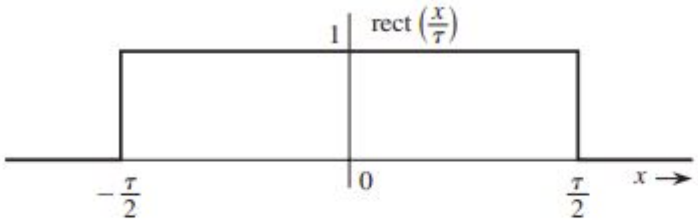
7.2 Transforms of Some Useful Functions



Unit Gate Function

$$\operatorname{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$





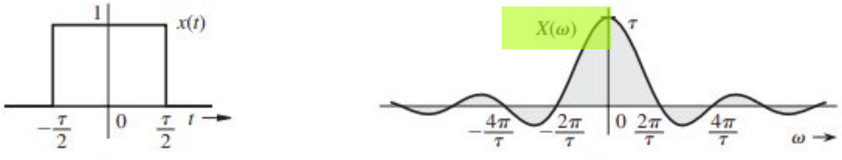
$$\operatorname{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$$

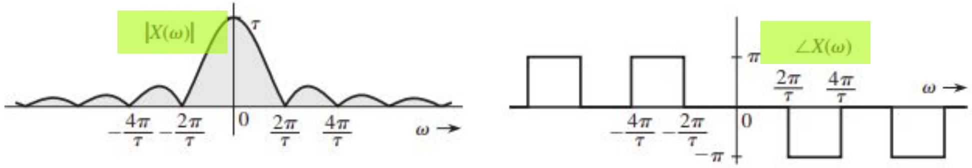
$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

$$= \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$



(a) (b)



Finite in time <-> infinite in spectra

CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER TRANSFORM

7.3 SOME PROPERTIES OF THE FOURIER TRANSFORM

DUALITY

The duality property states that if

$$x(t) \Longleftrightarrow X(\omega)$$

then

$$X(t) \Longleftrightarrow 2\pi x(-\omega) \tag{7.25}$$

Proof. From Eq. (7.10) we can write

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(u)e^{jut}\,du$$

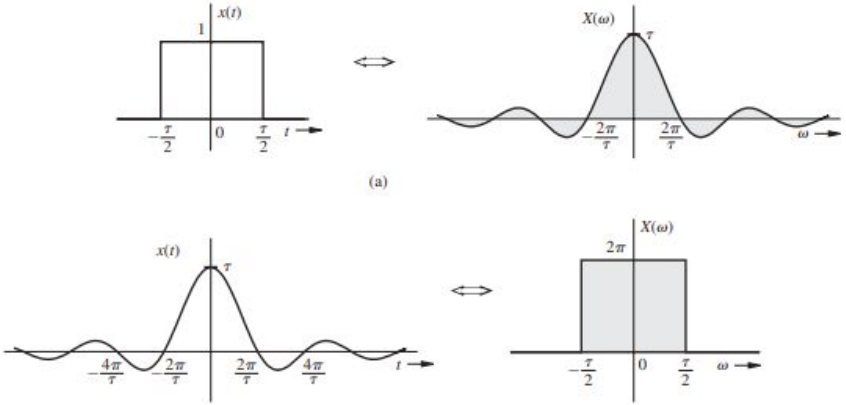
Hence,

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(u)e^{-jut}\,du$$

Changing t to ω yields Eq. (7.25).

EXAMPLE 7.11 Applying the Duality Property of the Fourier Transform

Apply the duality property [Eq. (7.25)] of the Fourier transform to the pair in Fig. 7.19a.



$$\underbrace{\mathrm{rect}\left(\frac{t}{\tau}\right)}_{x(t)} \Longleftrightarrow \underbrace{\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)}_{X(\omega)}$$

Also, $X(t)$ is the same as $X(\omega)$ with ω replaced by t, and $x(-\omega)$ is the same as $x(t)$ with replaced by $-\omega$. Therefore, the duality property of Eq. (7.25) yields

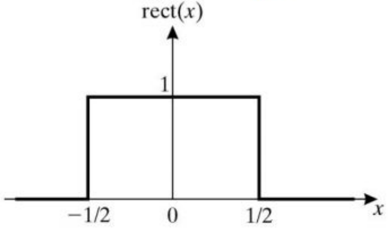
$$\underbrace{\tau\,\mathrm{sinc}\left(\frac{\tau t}{2}\right)}_{X(t)} \Longleftrightarrow \underbrace{2\pi\,\mathrm{rect}\left(\frac{-\omega}{\tau}\right)}_{2\pi x(-\omega)} = 2\pi\,\mathrm{rect}\left(\frac{\omega}{\tau}\right)$$

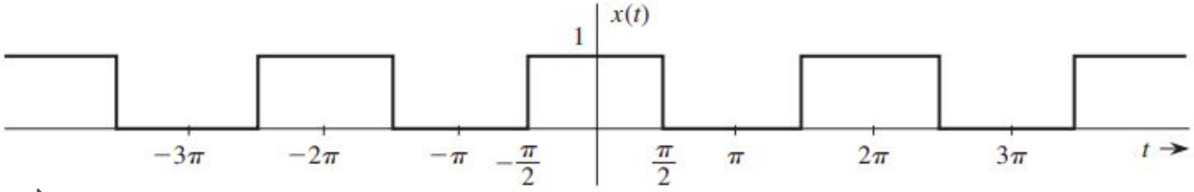
In this result, we used the fact that $\mathrm{rect}(-x) = \mathrm{rect}(x)$ because rect is an even function Figure 7.19b shows this pair graphically. Observe the interchange of the roles of t and ω (with the minor adjustment of the factor 2π). This result appears as pair 18 in Table 7.1 (with $\tau/2 = W$).

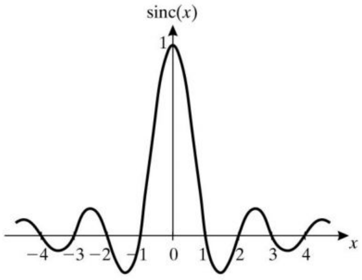
CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER TRANSFORM

7.2 TRANSFORMS OF SOME USEFUL FUNCTIONS

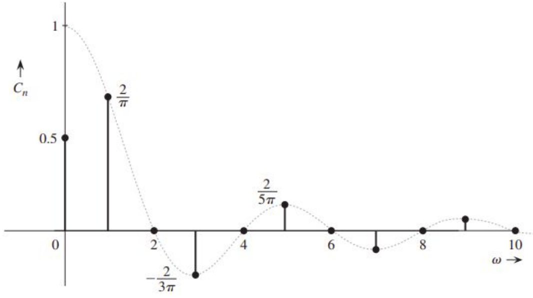
Periodic Signal < -- > Discrete Spectrum
Discrete Signal < -- > Periodic Spectrum







Conv in Time = Mul in Freq



Summarizing Lecture 2 in 3 slides

Discrete-Time System Analysis Using the z-Transform

$$F = \frac{\Omega}{2\pi}\left[\frac{rad}{sample}\,\frac{cycle}{rad}\right] = \frac{\Omega}{2\pi}\left[\frac{cycle}{sample}\right]$$

5.5 Frequency Response of Discrete-Time Systems

5.5-1 The Periodic Nature of Frequency Response

All Discrete-Time Signals Are Inherently Bandlimited

This discussion leads to the surprising conclusion that all discrete-time signals are inherently bandlimited, with frequencies lying in the range $-\pi$ to π radians per sample. In terms of frequency $\mathcal{F} = \Omega/2\pi$, where $\mathcal{F}$ is in cycles per sample, all frequencies $\mathcal{F}$ separated by an integer number are identical. For instance, all discrete-time sinusoids of frequencies 0.3, 1.3, 2.3, . . . cycles per sample are identical. The fundamental range of frequencies is -0.5 to 0.5 cycles per sample.

Any discrete-time sinusoid of frequency beyond the fundamental band, when plotted, appears and behaves, in every way, like a sinusoid having its frequency in the fundamental band. It is impossible to distinguish between the two signals. Thus, in a basic sense, discrete-time frequencies beyond $|\Omega| = \pi$ or $|\mathcal{F}| = 1/2$ do not exist. Yet, in a "mathematical" sense, we must admit the existence of sinusoids of frequencies beyond $\Omega = \pi$. What does this mean?

Discrete-Time System Analysis Using the z-Transform

5.5 Frequency Response of Discrete-Time Systems

5.5-2 Aliasing and Sampling Rate

$$T_s\ [sec]$$
$$F_s = \frac{1}{T_s}\ [Hz]$$
$$\omega = 2\pi F$$

https://www.youtube.com/watch?v=ByTsISFXUoY&ab_channel=FABIOG.GUERRERO

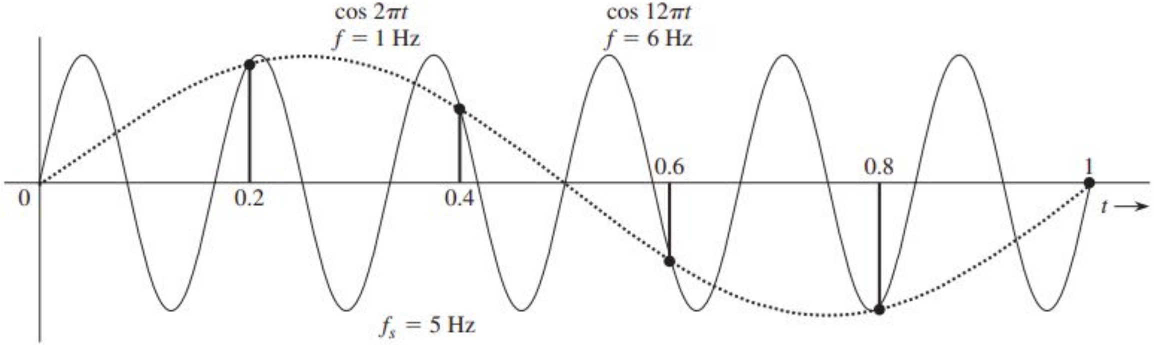


Figure 5.18 Demonstration of the aliasing effect.

$$\omega_{max} t \rightarrow \omega_{max} n T_s$$
$$\omega_{max} n T_s = \Omega_{max} n; \;\; \Omega_{max} = \omega_{max} T_s$$
$$\Omega_{max} < \pi$$
$$\omega_{max} T_s < \pi$$
$$\frac{\omega_{max}}{\pi} < \frac{1}{T_s}$$
$$\frac{\omega_{max}}{\pi} < F_s$$
$$2F_{max} < F_s$$
$$F_{max} < \frac{F_s}{2}$$

Discrete-Time System Analysis Using the z-Transform

5.5 Frequency Response of Discrete-Time Systems

5.5-2 Aliasing and Sampling Rate

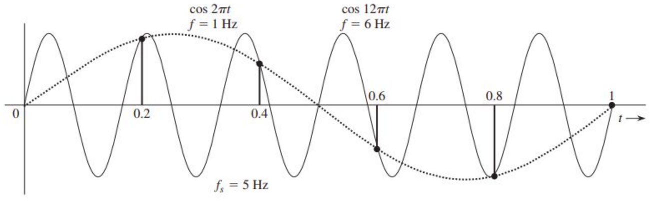
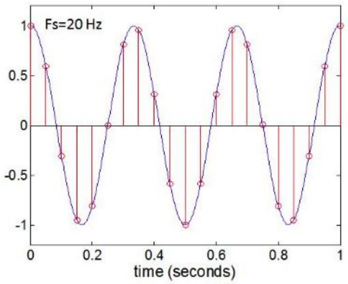


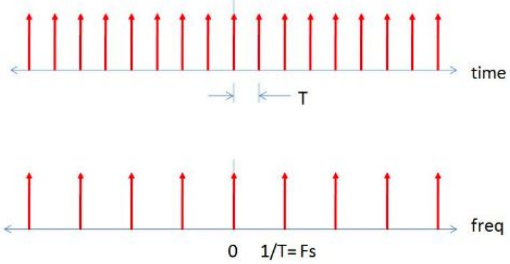
Figure 5.18 Demonstration of the aliasing effect.

Sampling



Multiplication (in time) with an impulse train

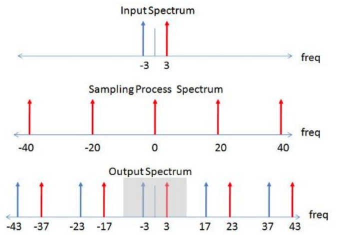
Review – FT of Impulse Train



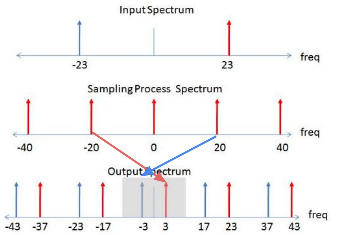
Impulses in time $\longrightarrow \mathfrak{F}\{\} \longrightarrow$ Impulses in frequency

$$F_s = 20[Hz]$$

Sampling of 3 Hz Cosine Wave

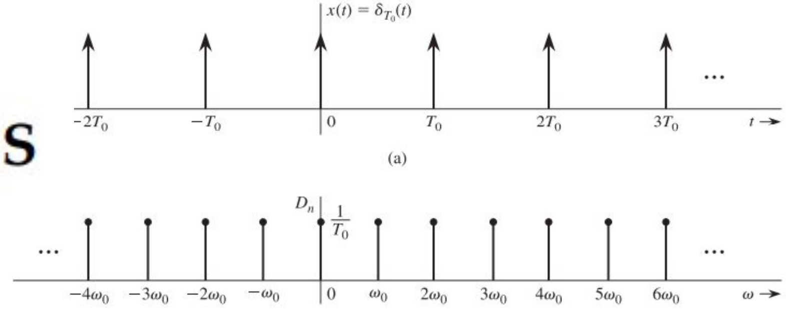


Sampling of 23 Hz Cosine Wave



Continuous-Time Signal Analysis: The Fourier Series

6.1 Periodic Signal Representation by Trigonometric Fourier Series



The unit impulse train shown in Fig. 6.15a can be expressed as

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

Following Papoulis, we shall denote this function as $\delta_{T_0}(t)$ for the sake of notational brevity.

The exponential Fourier series is given by

$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} \tag{6.23}$$

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

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Sampling: The Bridge from Continuous to Discrete

8.1 The Sampling Theorem

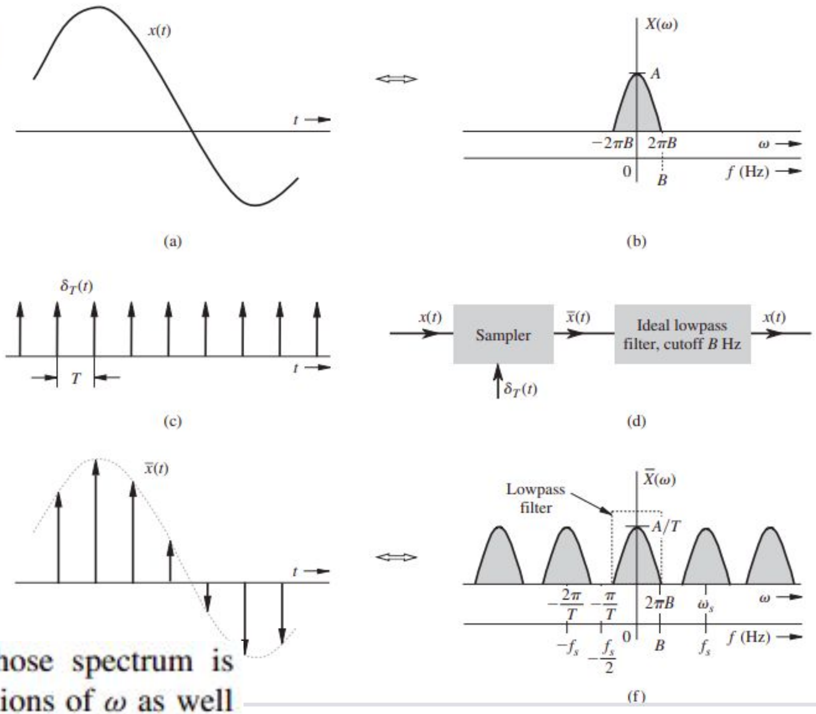
Sampling theory is the bridge between the continuous-time and discrete-time worlds.

We now show that a real signal whose spectrum is bandlimited to B Hz [$X(\omega) = 0$ for $|\omega| > 2\pi B$] can be reconstructed exactly (without any error) from its samples taken uniformly at a rate $f_s > 2B$ samples per second. In other words, the minimum sampling frequency is $f_s = 2B$ Hz.†

$$B = F_{max}$$

Sampling: The Bridge from Continuous to Discrete

8.1 The Sampling Theorem



To prove the sampling theorem, consider a signal $x(t)$ (Fig. 8.1a) whose spectrum is bandlimited to B Hz (Fig. 8.1b).‡ For convenience, spectra are shown as functions of ω as well as of f (hertz). Sampling $x(t)$ at a rate of f_s Hz (f_s samples per second) can be accomplished by multiplying $x(t)$ by an impulse train $\delta_T(t)$ (Fig. 8.1c), consisting of unit impulses repeating periodically every T seconds, where $T = 1/f_s$. The schematic of a sampler is shown in Fig. 8.1d. The resulting sampled signal $\bar{x}(t)$ is shown in Fig. 8.1e. The sampled signal consists of impulses spaced every T seconds (the sampling interval). The nth impulse, located at $t = nT$, has a strength $x(nT)$, the value of $x(t)$ at $t = nT$.

$$\bar{x}(t) = x(t)\delta_T(t) = \sum_n x(nT)\delta(t - nT)$$

Sampling: The Bridge from Continuous to Discrete

8.1 The Sampling Theorem

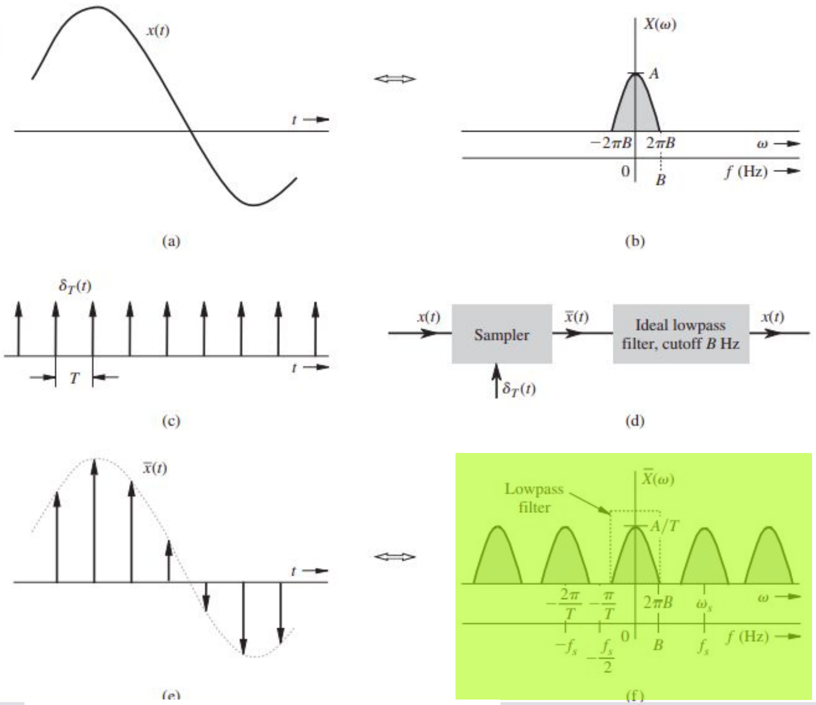
If we are to reconstruct $x(t)$ from $\overline{x}(t)$, we should be able to recover $X(\omega)$ from $\overline{X}(\omega)$. This recovery is possible if there is no overlap between successive cycles of $\overline{X}(\omega)$. Figure 8.1f indicates that this requires

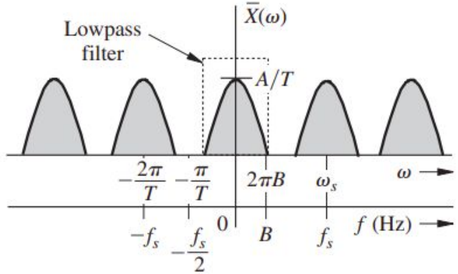
$$f_s > 2B \tag{8.3}$$

Also, the sampling interval $T = 1/f_s$. Therefore,

$$T < \frac{1}{2B}$$

Thus, as long as the sampling frequency f_s is greater than twice the signal bandwidth B (in hertz), $\overline{X}(\omega)$ consists of nonoverlapping repetitions of $X(\omega)$. Figure 8.1f shows that the gap between the two adjacent spectral repetitions is $f_s - 2B$ Hz, and $x(t)$ can be recovered from its samples $\overline{x}(t)$ by passing the sampled signal $\overline{x}(t)$ through an ideal lowpass filter having a bandwidth of any value between B and $f_s - B$ Hz. The minimum sampling rate $f_s = 2B$ required to recover $x(t)$ from its samples $\overline{x}(t)$ is called the *Nyquist rate* for $x(t)$, and the corresponding sampling interval $T = 1/2B$ is called the *Nyquist interval* for $x(t)$. Samples of a signal taken at its Nyquist rate are the *Nyquist samples* of that signal.





SAMPLING: THE BRIDGE FROM CONTINUOUS TO DISCRETE

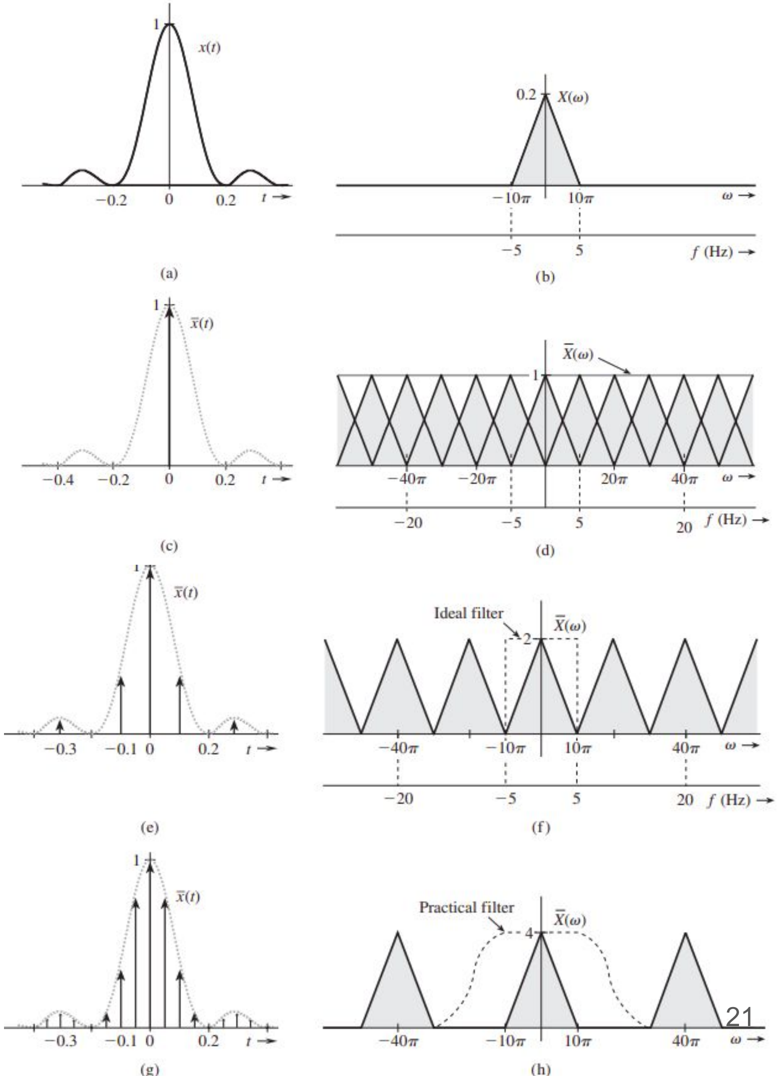
8.1 THE SAMPLING THEOREM

EXAMPLE 8.1 Sampling at, Below, and Above the Nyquist Rate

In this example, we examine the effects of sampling a signal at the Nyquist rate, below the Nyquist rate (undersampling), and above the Nyquist rate (oversampling). Consider a signal $x(t) = \text{sinc}^2(5\pi t)$ (Fig. 8.2a) whose spectrum is $X(\omega) = 0.2\,\Delta(\omega/20\pi)$ (Fig. 8.2b). The bandwidth of this signal is 5 Hz (10π rad/s). Consequently, the Nyquist rate is 10 Hz; that is, we must sample the signal at a rate no less than 10 samples/s. The Nyquist interval is $T = 1/2B = 0.1$ second.

Recall that the sampled signal spectrum consists of $(1/T)X(\omega) = (0.2/T)\,\Delta(\omega/20\pi)$ repeating periodically with a period equal to the sampling frequency f_s Hz. For the three sampling rates $f_s = 5$ Hz (undersampling), 10 Hz (Nyquist rate), and 20 Hz (oversampling), we see that

f_s (Hz)	$T = \frac{1}{f_s}$ (s)	$\frac{1}{T}X(\omega)$	Comments
5	0.2	$\Delta\left(\frac{\omega}{20\pi}\right)$	Undersampling
10	0.1	$2\Delta\left(\frac{\omega}{20\pi}\right)$	Nyquist rate
20	0.05	$4\Delta\left(\frac{\omega}{20\pi}\right)$	Oversampling



SAMPLING: THE BRIDGE FROM CONTINUOUS TO DISCRETE

8.1 THE SAMPLING THEOREM

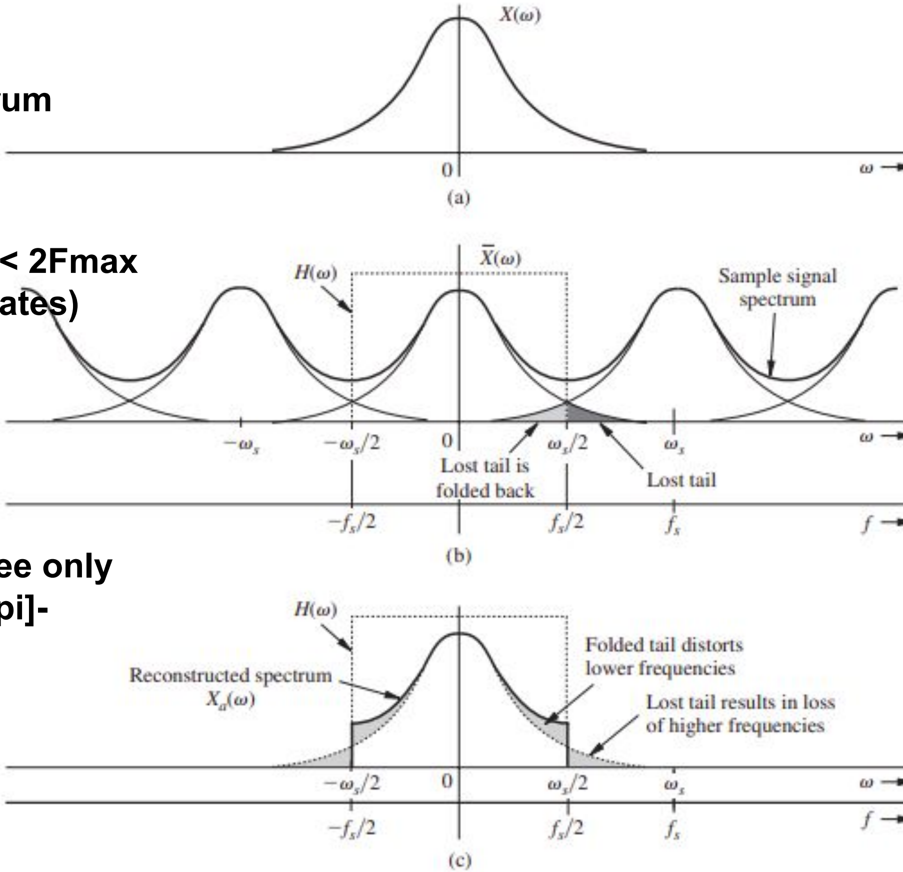
8.2 SIGNAL RECONSTRUCTION

8.2-1 Practical Difficulties in Signal Reconstruction

Orig spectrum

Sample Fs < 2Fmax (with replicates)

De facto we see only between [-ip, pi]- aliasing



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Sampling: The Bridge from Continuous to Discrete

8.1 The Sampling Theorem

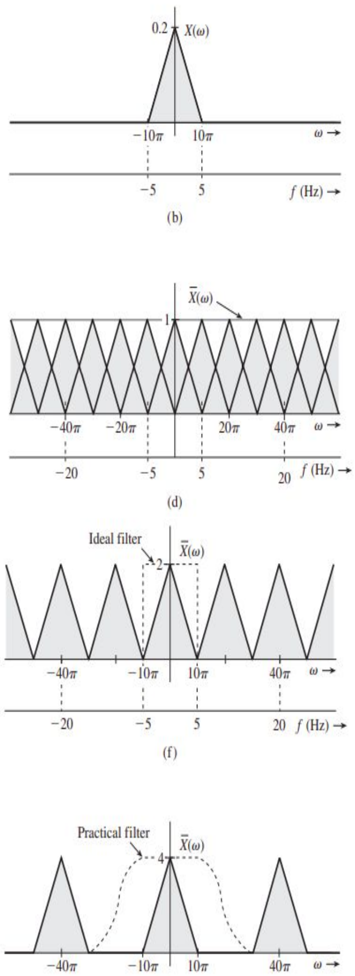
8.2 Signal Reconstruction

$$\bar{X}(w) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(w - nw_s)$$

The process of reconstructing a continuous-time signal $x(t)$ from its samples is also known as *interpolation*. In Sec. 8.1, we saw that a signal $x(t)$ bandlimited to B Hz can be reconstructed (interpolated) exactly from its samples if the sampling frequency f_s exceeds $2B$ Hz or the sampling interval T is less than $1/2B$. This reconstruction is accomplished by passing the sampled signal through an ideal lowpass filter of gain T and having a bandwidth of any value between B and $f_s - B$ Hz. From a practical viewpoint, a good choice is the middle value $f_s/2 = 1/2T$ Hz or π/T rad/s. This value allows for small deviations in the ideal filter characteristics on either side of the cutoff frequency. With this choice of cutoff frequency and gain T, the ideal lowpass filter required for signal reconstruction (or interpolation) is

$$H(\omega) = T\,\mathrm{rect}\left(\frac{\omega}{2\pi f_s}\right) = T\,\mathrm{rect}\left(\frac{\omega T}{2\pi}\right) \tag{8.4}$$

The interpolation process here is expressed in the frequency domain as a filtering operation. Now we shall examine this process from the time-domain viewpoint.



Sampling: The Bridge from Continuous to Discrete

8.1 The Sampling Theorem

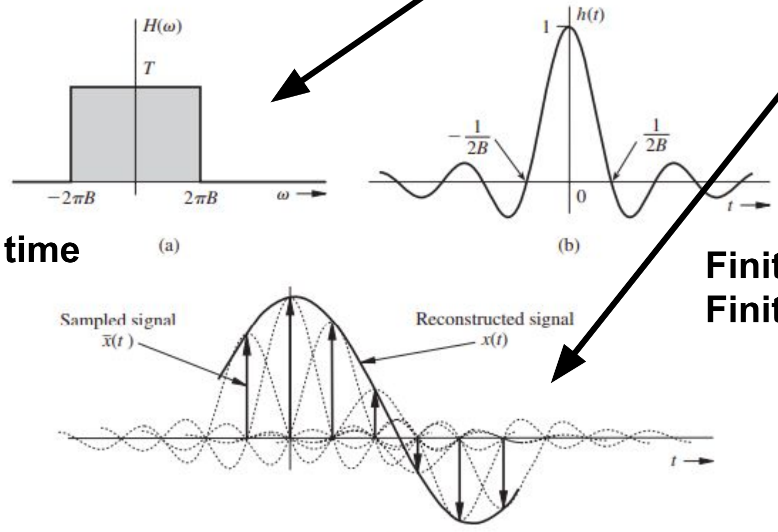
8.2 Signal Reconstruction

Time-Domain View: A Simple Interpolation

op in freq	op in time
conv	mul
mul	conv

Ideal Interpolation

$$\bar{X}(w) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(w - nw_s)$$



(a) (b) (c)

$$h(t) = \mathrm{sinc}\left(\frac{\pi t}{T}\right)$$

Ideal interpolator is infinite in time and hence is impractical

Finit in time <-> infinite in spectrum
Finit in spectrum <-> infinite in time

Sampling: The Bridge from Continuous to Discrete

8.1 The Sampling Theorem

8.2 Signal Reconstruction

8.2-1 Practical Difficulties in Signal Reconstruction

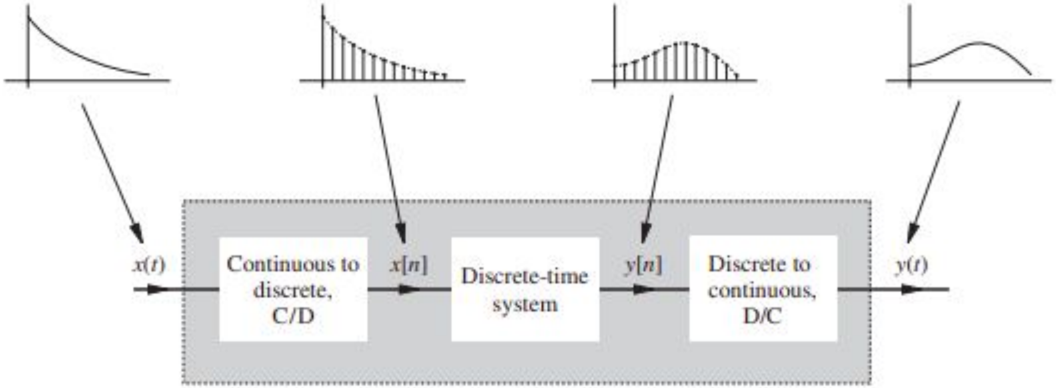


Figure 3.2 Processing a continuous-time signal by means of a discrete-time system.

The solution: Anti Aliasing Filter.
Set the wanted B and use LPF over the continuous signal prior to sampling.
Will remove high frequencies but will prevent aliasing

Lecture Overview

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Fourier Analysis of Discrete-Time Signals

In Chs. 6 and 7, we studied the ways of representing a continuous-time signal as a sum of sinusoids or exponentials. In this chapter we shall discuss similar development for discrete-time signals. Our approach is parallel to that used for continuous-time signals. We first represent a periodic $x[n]$ as a Fourier series formed by a discrete-time exponential (or sinusoid) and its harmonics. Later we extend this representation to an aperiodic signal $x[n]$ by considering $x[n]$ as a limiting case of a periodic signal with the period approaching infinity.

Some Intuition: Periodicity and Sampling

Periodic Continuous Signal $\xrightarrow{\text{CT Fourier Series}}$ Discrete Spectrum

Non-Periodic Continuous Signal $\xrightarrow{\text{CT Fourier Transform}}$ Continuous Spectrum

Discrete Non-Periodic Signal $\xrightarrow{\text{DT Fourier Transform}}$ Periodic Spectrum

Some Intuition: Periodicity and Sampling

Periodic Signal → Discrete Spectrum

Discrete (Sampled) Signal → Periodic Spectrum

Non-Periodic Signal → Continuous Spectrum

Continuous Signal → Non-Periodic Spectrum

Some Intuition: Periodicity and Sampling

Periodic Continuous Signal $\xrightarrow{\text{CT Fourier Series}}$ Discrete Spectrum

Non-Periodic Continuous Signal $\xrightarrow{\text{CT Fourier Transform}}$ Continuous Spectrum

Periodic Discrete Signal $\xrightarrow{\text{DT Fourier Series}}$ Periodic Discrete Spectrum

Non-Periodic Discrete Signal $\xrightarrow{\text{DT Fourier Transform}}$ Periodic Continuous Spectrum

FOURIER ANALYSIS OF DISCRETE-TIME SIGNALS

9.1 DISCRETE-TIME FOURIER SERIES (DTFS)

A continuous-time sinusoid $\cos \omega t$ is a periodic signal regardless of the value of ω. Such is not the case for the discrete-time sinusoid $\cos \Omega n$ (or exponential $e^{j\Omega n}$). A sinusoid $\cos \Omega n$ is periodic only if $\Omega/2\pi$ is a rational number. This can be proved by observing that if this sinusoid is N_0 periodic, then

$$\cos \Omega(n + N_0) = \cos \Omega n$$

This is possible only if

$$\Omega N_0 = 2\pi m \qquad m \text{ integer}$$

Here, both m and N_0 are integers. Hence, $\Omega/2\pi = m/N_0$ is a rational number. Thus, a sinusoid $\cos \Omega n$ (or exponential $e^{j\Omega n}$) is periodic only if

$$\frac{\Omega}{2\pi} = \frac{m}{N_0} \qquad \text{a rational number}$$

When this condition ($\Omega/2\pi$ a rational number) is satisfied, the period N_0 of the sinusoid $\cos \Omega n$ is given by

$$N_0 = m\left(\frac{2\pi}{\Omega}\right) \tag{9.1}$$

To compute N_0, we must choose the smallest value of m that will make $m(2\pi/\Omega)$ an integer. For example, if $\Omega = 4\pi/17$, then the smallest value of m that will make $m(2\pi/\Omega) = m(17/2)$ an integer is 2. Therefore,

$$N_0 = m\left(\frac{2\pi}{\Omega}\right) = 2\left(\frac{17}{2}\right) = 17$$

However, a sinusoid $\cos(0.8n)$ is not a periodic signal because $0.8/2\pi$ is not a rational number.

Not all discrete sin / cos signals are periodic

FOURIER ANALYSIS OF DISCRETE-TIME SIGNALS

9.1 DISCRETE-TIME FOURIER SERIES (DTFS)

9.1-1 Periodic Signal Representation by Discrete-Time Fourier Series

A continuous-time periodic signal of period T_0 can be represented as a trigonometric Fourier series consisting of a sinusoid of the fundamental frequency $\omega_0 = 2\pi/T_0$, and all its harmonics. The exponential form of the Fourier series consists of exponentials $e^{j0t}, e^{\pm j\omega_0 t}, e^{\pm j2\omega_0 t}, e^{\pm j3\omega_0 t}, \ldots$.

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

A discrete-time periodic signal can be represented by a discrete-time Fourier series using a parallel development. Recall that a periodic signal $x[n]$ with period N_0 is characterized by the fact that

$$x[n] = x[n + N_0]$$

The smallest value of N_0 for which this equation holds is the *fundamental period*. The *fundamental frequency* is $\Omega_0 = 2\pi/N_0$ rad/sample. An N_0-periodic signal $x[n]$ can be represented by a discrete-time Fourier series made up of sinusoids of fundamental frequency $\Omega_0 = 2\pi/N_0$ and its harmonics. As in the continuous-time case, we may use a trigonometric or an exponential form of the Fourier series. Because of its compactness and ease of mathematical manipulations, the exponential form is preferable to the trigonometric. For this reason, we shall bypass the trigonometric form and go directly to the exponential form of the discrete-time Fourier series.

Fourier Analysis of Discrete-Time Signals

9.1 Discrete-Time Fourier Series (DTFS)

9.1-1 Periodic Signal Representation by Discrete-Time Fourier Series

The exponential Fourier series consists of the exponentials $e^{j0n}, e^{\pm j\Omega_0 n}, e^{\pm j2\Omega_0 n}, \ldots, e^{\pm jn\Omega_0 n}$,

$$e^{j(\Omega \pm 2\pi m)n} = e^{j\Omega n} e^{\pm 2\pi mn} = e^{j\Omega n} \qquad m \text{ integer}$$

The consequence of this result is that the rth harmonic is identical to the $(r+N_0)$th harmonic. To demonstrate this, let g_n denote the nth harmonic $e^{jn\Omega_0 n}$. Then

$$g_{r+N_0} = e^{j(r+N_0)\Omega_0 n} = e^{j(r\Omega_0 n + 2\pi n)} = e^{jr\Omega_0 n} = g_r$$

and

$$g_r = g_{r+N_0} = g_{r+2N_0} = \cdots = g_{r+mN_0} \qquad m \text{ integer}$$

Thus, the first harmonic is identical to the (N_0+1)th harmonic, the second harmonic is identical to the (N_0+2)th harmonic, and so on. In other words, there are only N_0 independent harmonics, and their frequencies range over an interval 2π (because the harmonics are separated by $\Omega_0 = 2\pi/N_0$). This means that, unlike the continuous-time counterpart, the discrete-time Fourier series has only a finite number (N_0) of terms. This result is consistent with our observation in Sec. 5.5-1 that all discrete-time signals are bandlimited to a band from $-\pi$ to π. Because the harmonics are separated by $\Omega_0 = 2\pi/N_0$, there can only be N_0 harmonics in this band. We also saw that this band can be taken from 0 to 2π or any other contiguous band of width 2π. This means we may

Fourier Analysis of Discrete-Time Signals

9.1 Discrete-Time Fourier Series (DTFS)

9.1-1 Periodic Signal Representation by Discrete-Time Fourier Series

We now have a discrete-time Fourier series (DTFS) representation of an N_0-periodic signal $x[n]$ as

infinite vs finite # of Dn

the rth harmonic is identical to the $(r+N_0)$th harmonic.

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \longleftrightarrow \qquad x[n] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n} \tag{9.3}$$

Sum goes over r NOT n

where

$$D_n = \frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}\,dt \qquad \longleftrightarrow \qquad \mathcal{D}_r = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jr\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0} \tag{9.4}$$

Fourier Analysis of Discrete-Time Signals

9.1 Discrete-Time Fourier Series (DTFS)

EXAMPLE 9.1 Discrete-Time Fourier Series of a Sinusoid

Find the discrete-time Fourier series (DTFS) for $x[n] = \sin 0.1\pi n$ (Fig. 9.1a). Sketch the amplitude and phase spectra.

In this case, the sinusoid $\sin 0.1\pi n$ is periodic because $\Omega/2\pi = 1/20$ is a rational number and the period N_0 is [see Eq. (9.1)]

$$N_0 = m\left(\frac{2\pi}{\Omega}\right) = m\left(\frac{2\pi}{0.1\pi}\right) = 20m$$

The smallest value of m that makes $20m$ an integer is $m = 1$. Therefore, the period $N_0 = 20$ so that $\Omega_0 = 2\pi/N_0 = 0.1\pi$, and from Eq. (9.6),

$$x[n] = \sum_{r=\langle 20 \rangle} \mathcal{D}_r e^{j0.1\pi rn}$$

where the sum is performed over any 20 consecutive values of r. We shall select the range $-10 \le r < 10$ (values of r from -10 to 9). This choice corresponds to synthesizing $x[n]$ using

the spectral components in the fundamental frequency range ($-\pi \le \Omega < \pi$). Thus,

$$x[n] = \sum_{r=-10}^{9} \mathcal{D}_r e^{j0.1\pi rn}$$

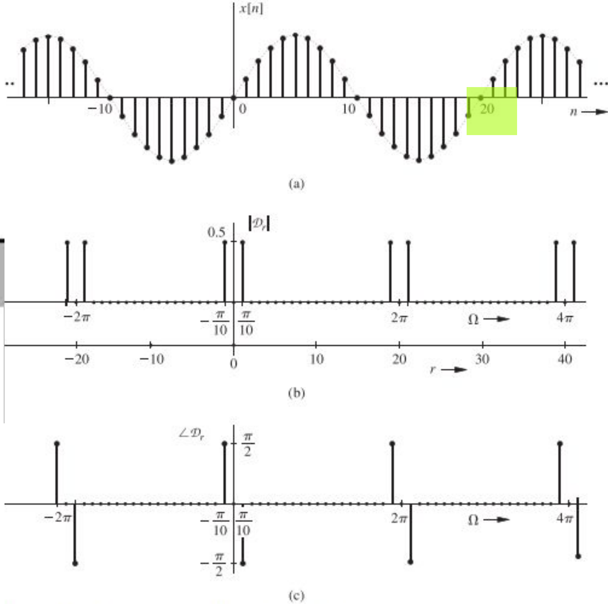


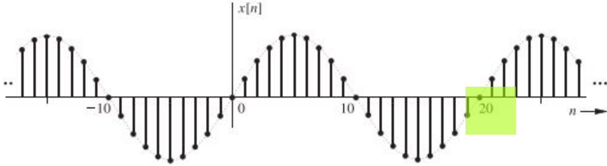
Figure 9.1 Discrete-time sinusoid sin $0.1\pi n$ and its Fourier spectra.

Fourier Analysis of Discrete-Time Signals

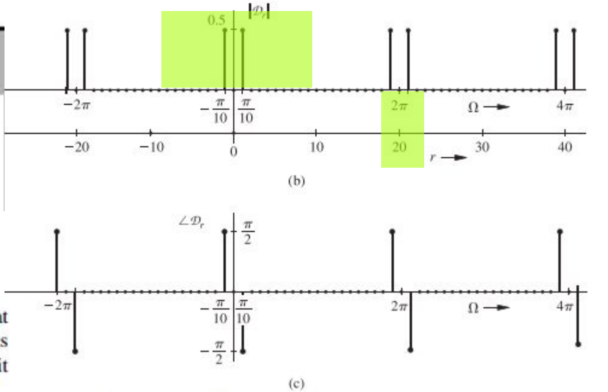
9.1 Discrete-Time Fourier Series (DTFS)

EXAMPLE 9.1 Discrete-Time Fourier Series of a Sinusoid

Find the discrete-time Fourier series (DTFS) for $x[n] = \sin 0.1\pi n$ (Fig. 9.1a). Sketch the amplitude and phase spectra.



This spectrum over the range $-10 \le r < 10$ (or $-\pi \le \Omega < \pi$)



ure 9.1 Discrete-time sinusoid sin $0.1\pi n$ and its Fourier spectra.

$$\mathcal{D}_r = \frac{1}{40j}\left[\sum_{n=-10}^{9} e^{j0.1\pi n(1-r)} - \sum_{n=-10}^{9} e^{-j0.1\pi n(1+r)}\right] \qquad \sum_{n=0}^{N_0-1} e^{jk\Omega_0 n} = \begin{cases} N_0 & k = 0, \pm N_0, \pm 2N_0, \ldots \\ 0 & \text{otherwise} \end{cases}$$

In these sums, r takes on all values between -10 and 9. From Eq. (8.15), it follows that the first sum on the right-hand side is zero for all values of r except $r = 1$, when the sum is equal to $N_0 = 20$. Similarly, the second sum is zero for all values of r except $r = -1$, when it is equal to $N_0 = 20$. Therefore,

$$\mathcal{D}_1 = \frac{1}{2j} \quad \text{and} \quad \mathcal{D}_{-1} = -\frac{1}{2j}$$

and all other coefficients are zero. The corresponding Fourier series is given by

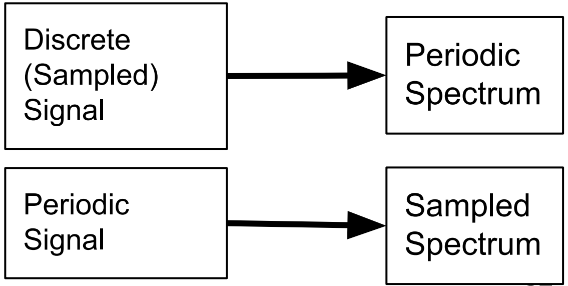
$$x[n] = \sin 0.1\pi n = \frac{1}{2j}(e^{j0.1\pi n} - e^{-j0.1\pi n}) \tag{9.8}$$

Here the fundamental frequency $\Omega_0 = 0.1\pi$, and there are only two nonzero components:

$$\mathcal{D}_1 = \frac{1}{2j} = \frac{1}{2}e^{-j\pi/2} \quad \text{and} \quad \mathcal{D}_{-1} = -\frac{1}{2j} = \frac{1}{2}e^{j\pi/2}$$

Therefore,

$$|\mathcal{D}_1| = |\mathcal{D}_{-1}| = \tfrac{1}{2} \quad \text{and} \quad \angle\mathcal{D}_1 = -\frac{\pi}{2},\ \angle\mathcal{D}_{-1} = \frac{\pi}{2}$$



Lecture Overview

1. Analog → Digital → Analog:
   a. Preliminaries: Aliasing
   b. Sampling
   c. Reconstruction
2. Discrete Time Fourier Series (DTFS)
3. **Discrete Time Fourier Transform (DTFT)**
4. Discrete Fourier Transform (DFT)
5. Fast Fourier Transform (FFT)

Reminder ….
Some Intuition: Periodicity and Sampling

$$\text{Periodic Continuous Signal} \xrightarrow{\text{CT Fourier Series}} \text{Discrete Spectrum}$$

$$\text{Non-Periodic Continuous Signal} \xrightarrow{\text{CT Fourier Transform}} \text{Continuous Spectrum}$$

$$\text{Periodic Discrete Signal} \xrightarrow{\text{DT Fourier Series}} \text{Periodic Discrete Spectrum}$$

$$\text{Non-Periodic Discrete Signal} \xrightarrow{\text{DT Fourier Transform}} \text{Periodic Continuous Spectrum}$$

Fourier Analysis of Discrete-Time Signals

9.2 Aperiodic Signal Representation by Fourier Integral

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{jn\Omega} \, d\Omega \tag{9.18}$$

where $\int_{2\pi}$ indicates integration over any continuous interval of 2π. The spectrum $X(\Omega)$ is given by [Eq. (9.13)]

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \tag{9.19}$$

The integral on the right-hand side of Eq. (9.18) is called the *Fourier integral*. We have now succeeded in representing an aperiodic signal $x[n]$ by a Fourier integral (rather than a Fourier series). This integral is basically a Fourier series (in the limit) with fundamental frequency $\Delta\Omega \to 0$, as seen in Eq. (9.17). The amount of the exponential $e^{jr\Delta\Omega n}$ is $X(r\Delta\Omega)\Delta\Omega/2\pi$. Thus, the function $X(\Omega)$ given by Eq. (9.19) acts as a spectral function, which indicates the relative amounts of various exponential components of $x[n]$.

We call $X(\Omega)$ the (direct) discrete-time Fourier transform (DTFT) of $x[n]$, and $x[n]$ the inverse discrete-time Fourier transform (IDTFT) of $X(\Omega)$. This nomenclature can be represented as

$$X(\Omega) = \text{DTFT}\{x[n]\} \qquad \text{and} \qquad x[n] = \text{IDTFT}\{X(\Omega)\}$$

Fourier Analysis of Discrete-Time Signals

9.2 Aperiodic Signal Representation by Fourier Integral

9.2-1 Nature of Fourier Spectra

Fourier Spectra Are Continuous Functions of Ω

Fourier Spectra Are Periodic Functions of Ω with Period 2π

From Eq. (9.19), it follows that

$$X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}e^{-j2\pi n} = X(\Omega) \quad \longrightarrow \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{jn\Omega}\,d\Omega$$

Fourier Analysis of Discrete-Time Signals

9.2 Aperiodic Signal Representation by Fourier Integral

9.2-2 Connection Between the DTFT and the z-Transform

The connection between the (bilateral) z-transform and the DTFT is similar to that between the Laplace transform and the Fourier transform. The z-transform of $x[n]$, according to Eq. (5.1), is

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{9.27}$$

Setting $z = e^{j\Omega}$ in this equation yields

$$X[e^{j\Omega}] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

The right-hand side sum defines $X(\Omega)$, the DTFT of $x[n]$. Does this mean that the DTFT can be

Hence, the general rule is that setting $z = e^{j\Omega}$ in $X[z]$ yields the DTFT $X(\Omega)$ only when the ROC for $X[z]$ includes the unit circle.

Fourier Analysis of Discrete-Time Signals

9.2 Aperiodic Signal Representation by Fourier Integral

EXAMPLE 9.3 DTFT of a Causal Exponential

Find the DTFT of $x[n] = \gamma^n u[n]$.

$$X(\Omega) = \sum_{n=0}^{\infty} \gamma^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (\gamma e^{-j\Omega})^n$$

This is an infinite geometric series with a common ratio $\gamma e^{-j\Omega}$. Therefore (see Sec. B.8-3),

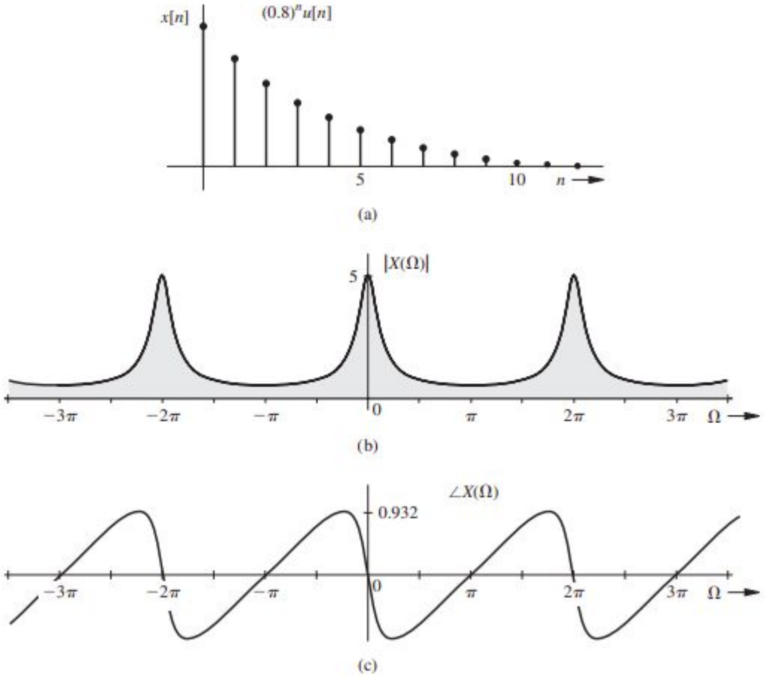
$$X(\Omega) = \frac{1}{1-\gamma e^{-j\Omega}}$$

provided $|\gamma e^{-j\Omega}| < 1$. But because $|e^{-j\Omega}| = 1$, this condition implies $|\gamma| < 1$. Therefore,

$$X(\Omega) = \frac{1}{1-\gamma e^{-j\Omega}} \qquad |\gamma| < 1$$

If $|\gamma| > 1$, $X(\Omega)$ does not converge. This result is in conformity with Eqs. (9.21) and (9.22). To determine magnitude and phase responses, we note that

$$X(\Omega) = \frac{1}{1-\gamma\cos\Omega + j\gamma\sin\Omega} \tag{9.23}$$



so

$$|X(\Omega)| = \frac{1}{\sqrt{(1-\gamma\cos\Omega)^2 + (\gamma\sin\Omega)^2}} = \frac{1}{\sqrt{1+\gamma^2-2\gamma\cos\Omega}}$$

and

$$\angle X(\Omega) = -\tan^{-1}\left[\frac{\gamma\sin\Omega}{1-\gamma\cos\Omega}\right]$$

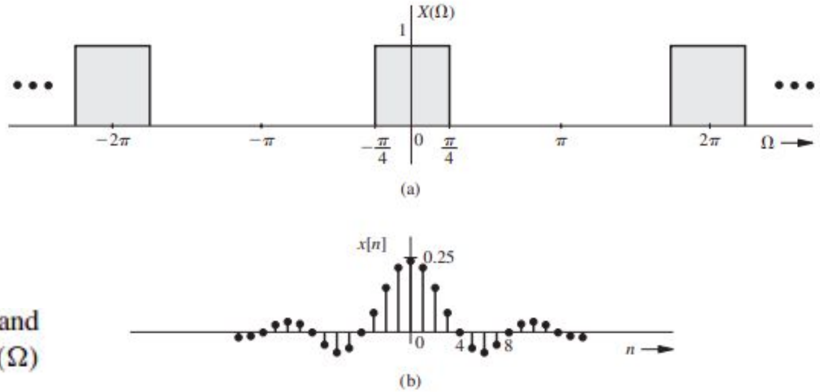
Figure 9.5 shows $x[n] = \gamma^n u[n]$ and its spectra for $\gamma = 0.8$. Observe that the frequency spectra are continuous and periodic functions of Ω with the period 2π. As explained earlier, we need to use the spectrum only over the frequency interval of 2π. We often select this interval to be the fundamental frequency range $(-\pi, \pi)$.

Fourier Analysis of Discrete-Time Signals

9.2 Aperiodic Signal Representation by Fourier Integral

EXAMPLE 9.6 Inverse DTFT of a Rectangular Spectrum

Find the inverse DTFT of the rectangular pulse spectrum described over the fundamental band ($|\Omega| \le \pi$) by $X(\Omega) = \mathrm{rect}(\Omega/2\Omega_c)$ for $\Omega_c \le \pi$. Because of the periodicity property, $X(\Omega)$ repeats at the intervals of 2π, as shown in Fig. 9.9a.



According to Eq. (9.18),

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{jn\Omega}\,d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{jn\Omega}\,d\Omega$$

$$= \frac{1}{j2\pi n}e^{jn\Omega}\Bigg|_{-\Omega_c}^{\Omega_c} = \frac{\sin(\Omega_c n)}{\pi n} = \frac{\Omega_c}{\pi}\,\mathrm{sinc}(\Omega_c n)$$

The signal $x[n]$ is depicted in Fig. 9.9b (for the case $\Omega_c = \pi/4$).

Fourier Analysis of Discrete-Time Signals

9.3 Properties of the DTFT

A close connection exists between the DTFT and the CTFT (continuous-time Fourier transform). For this reason, which Sec. 9.4 discusses, the properties of the DTFT are very similar to those of the CTFT, as the following discussion shows.

Linearity of the DTFT

Time-Shifting Property

If

$$x[n] \Longleftrightarrow X(\Omega)$$

then

$$x[n-k] \Longleftrightarrow X(\Omega)e^{-jk\Omega} \qquad \text{for integer } k \tag{9.31}$$

Frequency-Shifting Property

If

$$x[n] \Longleftrightarrow X(\Omega)$$

then

$$x[n]e^{j\Omega_c n} \Longleftrightarrow X(\Omega - \Omega_c) \tag{9.32}$$

Fourier Analysis of Discrete-Time Signals

9.3 Properties of the DTFT

Time- and Frequency-Convolution Property

If

$$x_1[n] \Longleftrightarrow X_1(\Omega) \qquad \text{and} \qquad x_2[n] \Longleftrightarrow X_2(\Omega)$$

then

$$x_1[n] * x_2[n] \Longleftrightarrow X_1(\Omega)X_2(\Omega) \tag{9.34}$$

and

$$x_1[n]x_2[n] \Longleftrightarrow \frac{1}{2\pi} X_1(\Omega) \circledast X_2(\Omega) \tag{9.35}$$

where

$$x_1[n] * x_2[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

For two continuous, periodic signals, we define the periodic convolution, denoted by symbol $\circledast$ as[†]

$$X_1(\Omega) \circledast X_2(\Omega) = \frac{1}{2\pi} \int_{2\pi} X_1(u)X_2(\Omega - u)\,du$$

The convolution here is not the *linear* convolution used so far. This is a *periodic* (or *circular*) convolution applicable to the convolution of two continuous, periodic functions with the same period. The limit of integration in the convolution extends only to one period.

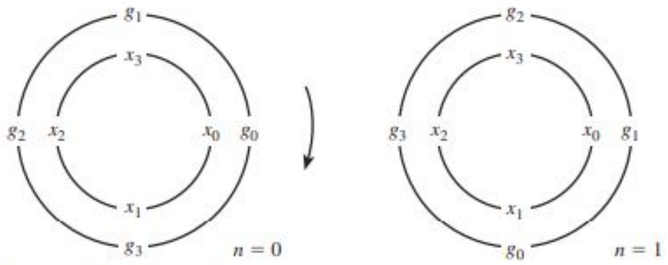


Figure 8.21 Graphical depictions of circular convolution.

Circular convolution can be implemented by convolving 2 cyclic buffers

Fourier Analysis of Discrete-Time Signals

	Continuous Signal	Discrete Signal
Periodic Signal		
Aperiodic Signal		

	Continuous Spectrum	Discrete Spectrum
Periodic Spectrum		
Aperiodic Spectrum		

Fourier Analysis of Discrete-Time Signals

	Continuous Signal	**Discrete Signal**
Periodic Signal	Discrete & Aperiodic Spectrum (CTFS)	Discrete & Periodic Spectrum (DTFS)
Aperiodic Signal	Continuous & Aperiodic Spectrum (CTFT)	Continuous & Periodic Spectrum (DTFT)

	Continuous Spectrum	**Discrete Spectrum**
Periodic Spectrum	Discrete & Aperiodic Signal (DTFT)	Discrete & Periodic Signal (DTFS)
Aperiodic Spectrum	Continuous & Aperiodic Signal (CTFT)	Continuous & Periodic Signal (CTFS)

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8.5 NUMERICAL COMPUTATION OF THE FOURIER TRANSFORM: THE DISCRETE FOURIER TRANSFORM

In practice, because the processing of the signals is done by a computer, <u>both x(t) and X(w) must be discrete in time.</u>

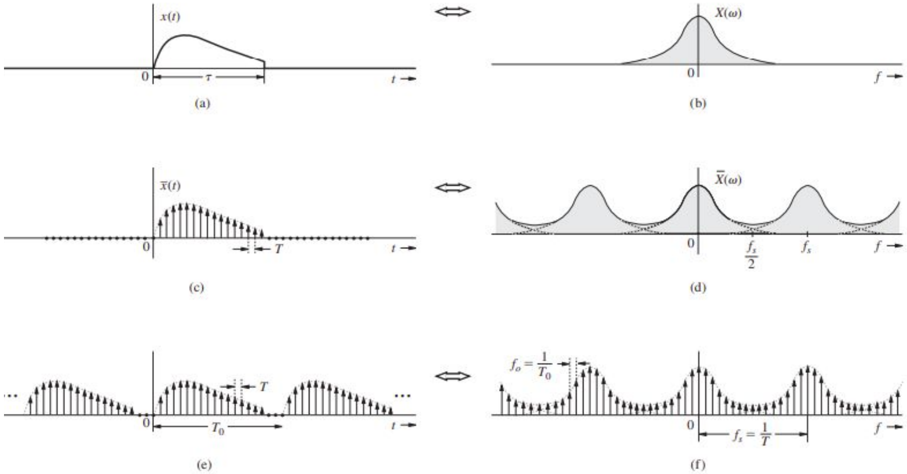
Source 1, 2

8.5 Numerical Computation of the Fourier Transform: The Discrete Fourier Transform

In practice, because the processing of the signals is done by a computer, both x(t) and X(w) must be discrete in time.

Numerical computation of the Fourier transform of $x(t)$ requires sample values of $x(t)$ because a digital computer can work only with discrete data (sequence of numbers). Moreover, a computer can compute $X(\omega)$ only at some discrete values of ω [samples of $X(\omega)$]. We therefore need to

8.5 Numerical Computation of the Fourier Transform: The Discrete Fourier Transform



We begin with a timelimited signal $x(t)$ (Fig. 8.16a) and its spectrum $X(\omega)$ (Fig. 8.16b). Since $x(t)$ is timelimited, $X(\omega)$ is nonbandlimited. For convenience, we shall show all spectra as functions of the frequency variable f (in hertz) rather than ω. According to the sampling theorem, the spectrum $\overline{X}(\omega)$ of the sampled signal $\bar{x}(t)$ consists of $X(\omega)$ repeating every f_s Hz, where $f_s = 1/T$, as depicted in Fig. 8.16d.† In the next step, the sampled signal in Fig. 8.16c is repeated periodically every T_0 seconds, as illustrated in Fig. 8.16e. According to the spectral sampling theorem, such an operation results in sampling the spectrum at a rate of T_0 samples/Hz. This sampling rate means that the samples are spaced at $f_0 = 1/T_0$ Hz, as depicted in Fig. 8.16f.

The foregoing discussion shows that when a signal $x(t)$ is sampled and then periodically repeated, the corresponding spectrum is also sampled and periodically repeated. Our goal is to relate the samples of $x(t)$ to the samples of $X(\omega)$.

8.5 Numerical Computation of the Fourier Transform: The Discrete Fourier Transform

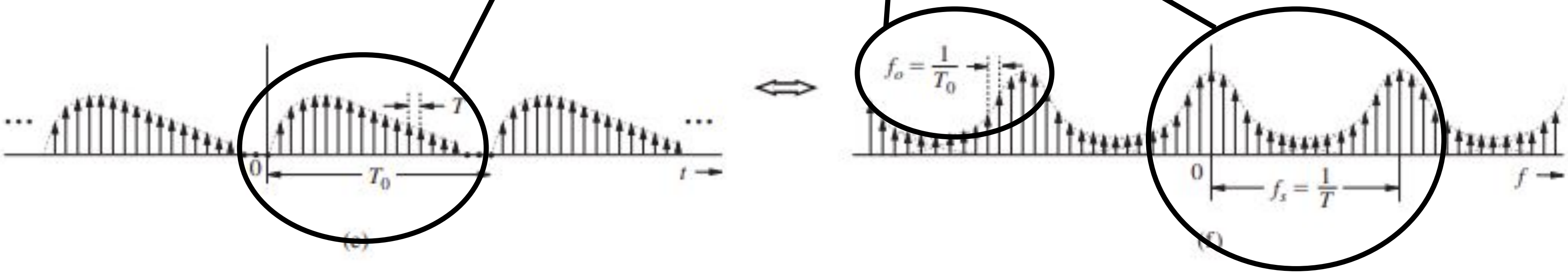
Number of Samples

One interesting observation from Figs. 8.16e and 8.16f is that N_0, the number of samples of the signal in Fig. 8.16e in one period T_0, is identical to N_0', the number of samples of the spectrum in Fig. 8.16f in one period f_s. To see this, we notice that

$$N_0 = \frac{T_0}{T} \quad N_0' = \frac{f_s}{f_0} \quad f_s = \frac{1}{T} \quad \text{and} \quad f_0 = \frac{1}{T_0} \tag{8.10}$$

Using these relations, we see that

$$N_0 = \frac{T_0}{T} = \frac{f_s}{f_0} = N_0'$$

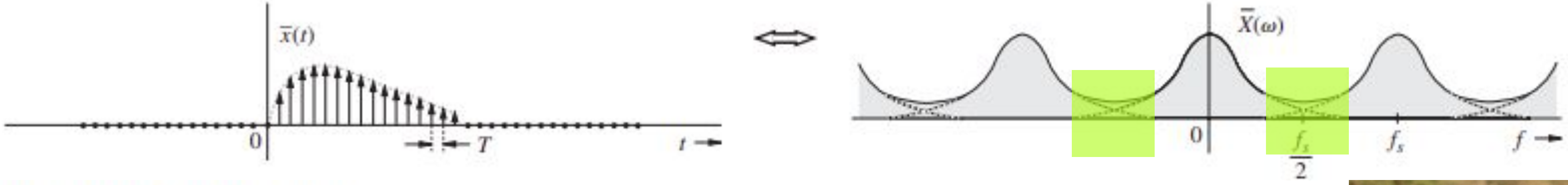


8.5 Numerical Computation of the Fourier Transform: The Discrete Fourier Transform

There are 2 phenomenons that occur: Leakage & Picket Fence Effect

Aliasing and Leakage in Numerical Computation

Figure 8.16f shows the presence of aliasing in the samples of the spectrum $X(\omega)$. This aliasing error can be reduced as much as desired by increasing the sampling frequency f_s (decreasing the sampling interval $T = 1/f_s$). The aliasing can never be eliminated for timelimited $x(t)$, however, because its spectrum $X(\omega)$ is nonbandlimited. Had we started with a signal having a bandlimited



Picket Fence Effect

The numerical computation method yields only the uniform sample values of $X(\omega)$. The major peaks or valleys of $X(\omega)$ can lie between two samples and may remain hidden, giving a false picture of reality. Viewing samples is like viewing the signal and its spectrum through a "picket fence" with upright posts that are very wide and placed close together. What is hidden behind the pickets is much more than what we can see.

8.5 Numerical Computation of the Fourier Transform: The Discrete Fourier Transform

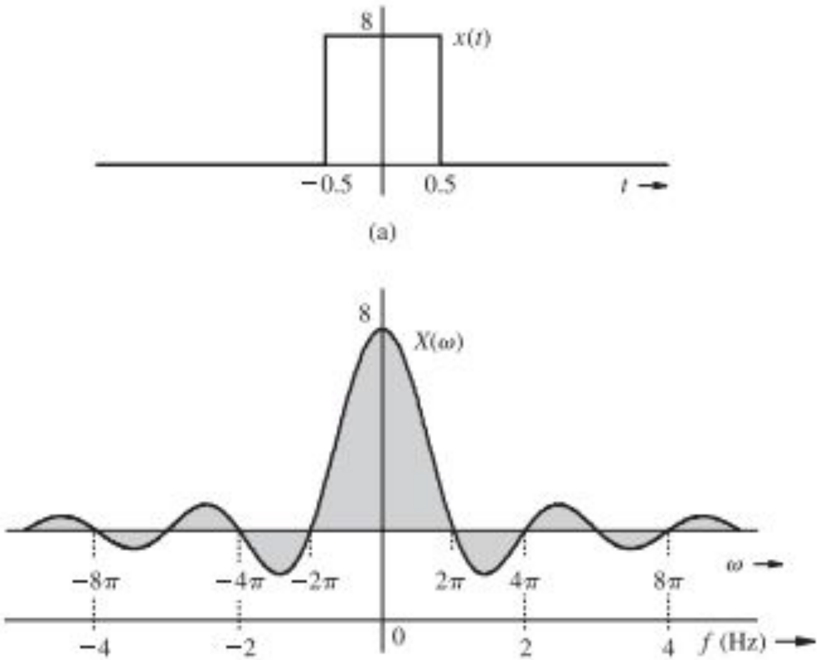
These equations define the direct and the inverse *discrete Fourier transforms,* with X_r the direct discrete Fourier transform (DFT) of x_n, and x_n the inverse discrete Fourier transform (IDFT) of X_r. The notation

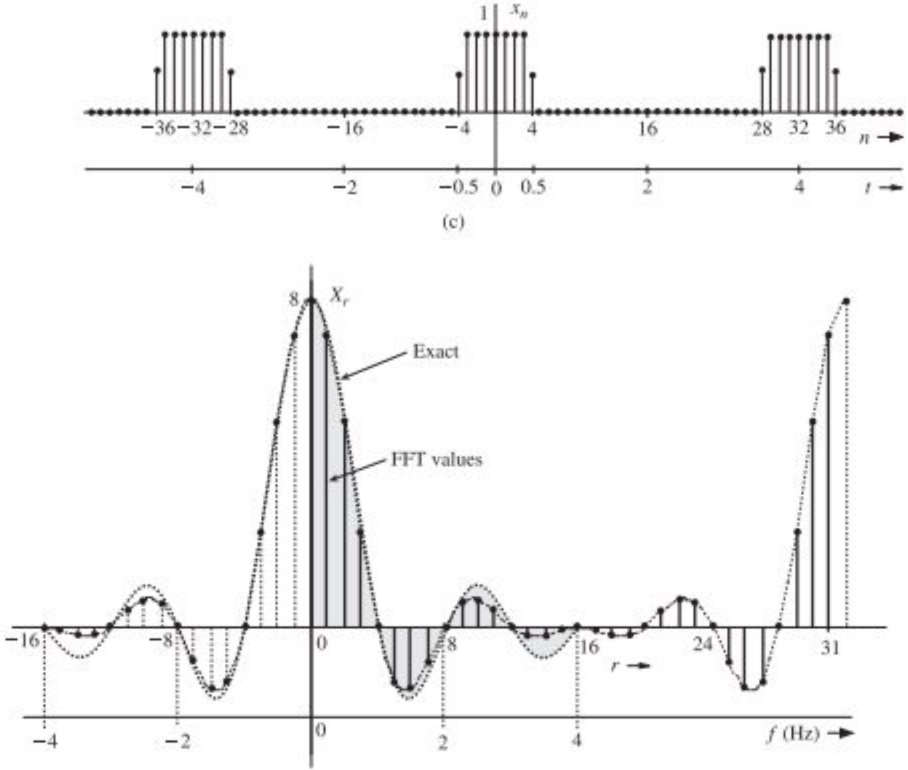
$$x_n \Longleftrightarrow X_r$$

$$X_r = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n} \qquad \Omega_0 = \omega_0 T = \frac{2\pi}{N_0}$$

$$x_n = \frac{1}{N_0} \sum_{r=0}^{N_0-1} X_r e^{jr\Omega_0 n}$$

8.5 Numerical Computation of the Fourier Transform: The Discrete Fourier Transform





8.5 NUMERICAL COMPUTATION OF THE FOURIER TRANSFORM: THE DISCRETE FOURIER TRANSFORM

$$X_r = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n} \qquad \Omega_0 = \omega_0 T = \frac{2\pi}{N_0}$$

$$x_n = \frac{1}{N_0} \sum_{r=0}^{N_0-1} X_r e^{jr\Omega_0 n}$$

ZERO PADDING

Recall that observing X_r is like observing the spectrum $X(\omega)$ through a picket fence. If the frequency sampling interval f_0 is not sufficiently small, we could miss out on some significant details and obtain a misleading picture. To obtain a higher number of samples, we need to reduce f_0. Because $f_0 = 1/T_0$, a higher number of samples requires us to increase the value of T_0, the period of repetition for $x(t)$. This option increases N_0, the number of samples of $x(t)$, by adding dummy samples of 0 value. This addition of dummy samples is known as *zero padding*. Thus, zero padding increases the number of samples and may help in getting a better idea of the spectrum $X(\omega)$ from its samples X_r. To continue with our picket fence analogy, zero padding is like using more, and narrower, pickets.

Since No=N, zero padding will improve visualization but will not add information (will sample the frequency response more frequent). This is because no additional information is given by zero padding the signal

8.5 Numerical Computation of the Fourier Transform: The Discrete Fourier Transform

$$X_r = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n} \qquad \Omega_0 = \omega_0 T = \frac{2\pi}{N_0}$$

$$x_n = \frac{1}{N_0} \sum_{r=0}^{N_0-1} X_r e^{jr\Omega_0 n}$$

$$x[m] = \begin{cases} x[n] & \text{if } n = 0, 1, ..., N-1 \\ 0 & \text{if } n = N, N+1, ...M-1 \end{cases}$$

$$X_r = \sum_{m=0}^{M-1} x_m e^{-j\Omega_0 m} = \sum_{m=0}^{M-1} x_m e^{-\frac{j2\pi mr}{M_0}}$$

$$X_r = \sum_{n=0}^{N-1} x_n e^{-\frac{j2\pi nr}{M_0}}, r = 0, ..., M_0 - 1$$

We now sample Mo>No samples from the DTFT's envelope, but the envelope remains the same (no additional information is given because of the zero padding).

Lecture Overview

1. Analog → Digital → Analog:
   a. Preliminaries: Aliasing
   b. Sampling
   c. Reconstruction
2. Discrete Time Fourier Series (DTFS)
3. Discrete Time Fourier Transform (DTFT)
4. Discrete Fourier Transform (DFT)
5. **Fast Fourier Transform (FFT)**

8.6 The Fast Fourier Transform (FFT)

The number of computations required in performing the DFT was dramatically reduced by an algorithm developed by Cooley and Tukey in 1965 [5]. This algorithm, known as the *fast Fourier transform* (FFT), reduces the number of computations from something on the order of N_0^2 to $N_0 \log N_0$. To compute one sample X_r from Eq. (8.12), we require N_0 complex multiplications and $N_0 - 1$ complex additions. To compute N_0 such values (X_r for $r = 0, 1, \ldots, N_0 - 1$), we require a total of N_0^2 complex multiplications and $N_0(N_0 - 1)$ complex additions. For a large N_0, these computations can be prohibitively time-consuming, even for a high-speed computer. The FFT algorithm is what made the use of Fourier transform accessible for digital signal processing.

Lecture Goals

1. Grasp some intuition on the process of Analog → Digital → Analog
2. Get familiar with Discrete Time Fourier Series (for discrete and periodic signals)
3. Get familiar with Discrete Time Fourier Transform (for discrete and non-periodic signals) and some intuition on it’s properties
4. Get familiar with Discrete Fourier Transform (for practical signals)