PHYS-7125

Gravity

Homework Assignments

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for Minor in Physics

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This collection is organized in reverse chronological order

PHYS-7125 Term Paper
Overview of Einstein-Cartan Theory of Gravitation

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1 Mathematical Considerations

1.1 Cartan's torsion tensor

On a manifold with connection coefficients Γ^k_{ij}, Cartan's torsion tensor is defined as

$$S_{ij}{}^k := \Gamma^k_{[ij]} = \frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji})$$

which measures the extent to which infinitesimal parallelograms fail to close. A vanishing torsion, as the fundamental theorem of Riemannian geometry states [3], leads to a unique metric-compatible connection on a (pseudo-)Riemanian manifold, which is the familiar Levi-Civita connection:

$$\Gamma^l_{ij} = \{{}^{\,l}_{ij}\} = \frac{1}{2}g^{lk}\Big(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\Big)$$

Here Γ denotes connection coefficients, and $\{\}$ is the historic notation of the Christoffel symbols as appeared in Christoffel's 1869 paper and Einstein's 1915-1916 papers. From a purely mathematical perspective, it is not immediately obvious whether the torsion-free constraint is needed for a successful theory of gravitation. In fact, Einstein originally introduced Christoffel symbols not based on the aforementioned uniqueness theorem, but rather on his postulate that any quantity describing gravitation should depend solely on the metric and its derivatives [1]. However, even Einstein himself considered this assumption unnecessary in his later years. [1]

[1] ...the essential achievement of general relativity, namely to overcome "rigid" space (i.e. the inertial frame), is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the "displacement field" (Γ), which expresses the infinitesimal displacement of vectors. ... This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of "rigid" space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular Γ field can be deduced from a Riemannian metric... (Einstein, 1955)

1.2 Riemann-Cartan spacetime U_4

If torsion is included, the spacetime geometry is non-Riemannian. In terms of connection coefficients, extra terms are added to the Christoffel symbol:

$$\Gamma^l_{ij} = \{^{\,l}_{ij}\} + S_{ij}{}^k - S_j{}^k{}_i + S^k{}_{ij} =: \{^{\,l}_{ij}\} - K_{ij}{}^k$$

The Riemann curvature tensor can be defined as usual to be the measurement of changes in vector components after the vector is parallelly transported around an infinitesimal area back to its starting point:

$$R^l{}_{kij} = 2\partial_{[i}\Gamma^l_{j]k} + 2\Gamma^l_{[i|m}\Gamma^m_{|j]k}$$

Since the definition of Riemann tensor does not depend on the particular form of connection, it is still antisymmetric in the first two and last two indices and still satisfies the Bianchi identity, although this is in general not the case for other identities. By symmetry, the Ricci tensor R_{ij} is still the only meaningful contraction of Riemann tensor, and the Einstein tensor G_{ij} can also be defined in the same way as in general relativity. However, due to Γ being asymmetric, R_{ij} and G_{ij} will be asymmetric in general. Riemann-Cartan spacetime U_4 fits in the hierarchy of manifolds as follows:

$$(L_4, g) \xrightarrow{\nabla g=0} U_4 \xrightarrow{S=0} V_4 \xrightarrow{R=0} R_4$$

In other words, the most general manifold equipped with a metric-preserving connection is the Riemann-Cartan U_4. When torsion vanishes, it reduces to the pseudo-Riemannian V_4 in general relativity. If in addition curvature vanishes, the manifold becomes Minkowskian R_4.

2 Field Equations of Einstein-Cartan Theory

2.1 Motivation

From the perspective of particle physics, all elmentary particles can be classified by the representations of the Poincaré group and be labeled by mass m and spin s. m is associated with the translational part of Poincaré group and s with the rotational part [2]. With spin being such a fundamental notion that is distinct from mass, it is reasonble to hypothesize that spin angular momentum is also a source of a gravitational field. That is, spin can be supposed to be coupled to some geometrical quantity that characterizes the rotational degrees of freedom of spacetime, analogous to how energy-momentum is coupled to metric.

2.2 Canonical tensors

Suppose a matter field $\psi(x^a)$ is distributed over spacetime and its action is

$$S_{matter} = \int \mathfrak{L}(\psi, \partial\psi, g, \partial g, S)d^4x$$

From Noether's theorem, it can be shown that the canonical spin angular momentum tensor, defined as the Noether current for rotations (Lorentz transformations), can be expressed as the variational derivative of the Lagrangian density with respect to contortion, or the non-Christoffel part of the connection [2]:

$$\tau_k{}^{ji} := \frac{1}{\sqrt{-g}} \frac{\delta \mathfrak{L}}{\delta K_{ij}{}^k}$$

Additionally, with the following definitions

$$\sigma^{ij} := \frac{2}{\sqrt{-g}} \frac{\delta \mathfrak{L}}{\delta g_{ij}}, \quad \mu_k{}^{ji} := \frac{1}{\sqrt{-g}} \frac{\delta \mathfrak{L}}{\delta S_{ij}{}^k}, \quad \overset{*}{\nabla}_k := \nabla_k + 2S_{kl}{}^l$$

the canonical energy-momentum tensor, or the Noether current for translations, is [2]:

$$\Sigma^{ij} := \sigma^{ij} - \overset{*}{\nabla}_k \mu^{ijk} = \sigma^{ij} + \overset{*}{\nabla}_k (\tau^{ijk} - \tau^{jki} + \tau^{kij})$$

where σ^{ij}, the variational derivative of the Lagrangian density with respect to metric, is the familiar Hilbert stress-energy tensor, which alone is equivalent to the canonical energy-momentum tensor within the framework of general relativity. In Einstein-Cartan theory, however, the additional divergence term must be included to compensate for a non-zero torsion.

2.3 Variational principles

The field equations can be derived as always by doing a variation of the full action with respect to each independent variable, in this case g_{ij} and $S_{ij}{}^k$. The most natural choice for the Lagrangian density of gravitational field is still the Einstein–Hilbert one:

$$\mathfrak{L}_{gravity} = \frac{1}{2\kappa}\mathfrak{R} = \frac{1}{2\kappa}\sqrt{-g}R$$

where $\kappa = 8\pi G$. The total action is

$$S = S_{matter} + S_{gravity} = \int \Big[\mathfrak{L}(\psi, \partial\psi, g, \partial g, S) + \frac{1}{2\kappa}\sqrt{-g}R\Big] d^4x$$

Varying the action with respect to g_{ij} and $S_{ij}{}^k$ gives:

$$\begin{cases} -\dfrac{\delta(\sqrt{-g}R)}{\delta g_{ij}} = \kappa\sqrt{-g}\left(\dfrac{2}{\sqrt{-g}}\dfrac{\delta \mathfrak{L}}{\delta g_{ij}}\right) = \kappa\sqrt{-g}\sigma^{ij} & (1) \\ -\dfrac{\delta(\sqrt{-g}R)}{\delta S_{ij}{}^k} = 2\kappa\sqrt{-g}\left(\dfrac{1}{\sqrt{-g}}\dfrac{\delta \mathfrak{L}}{\delta S_{ij}{}^k}\right) = 2\kappa\sqrt{-g}\mu_k{}^{ji} & (2) \end{cases}$$

Using the definition of Σ^{ij} and equation (2), equation (1) can be written as:

$$
\begin{aligned}
-\frac{\delta(\sqrt{-g}R)}{\delta g_{ij}} &= \kappa\sqrt{-g}\left(\Sigma^{ij}+\overset{*}{\nabla}_k\,\mu^{ijk}\right) = \kappa\sqrt{-g}\left[\Sigma^{ij}+\overset{*}{\nabla}_k\left(\frac{g^{li}}{2\kappa\sqrt{-g}}\frac{\delta(\sqrt{-g}R)}{\delta S_{kj}{}^{l}}\right)\right] \\
&= \kappa\sqrt{-g}\left[\Sigma^{ij}+\frac{g^{li}}{2\kappa\sqrt{-g}}\,\overset{*}{\nabla}_k\left(\frac{\delta(\sqrt{-g}R)}{\delta S_{kj}{}^{l}}\right)\right] \\
&= \kappa\sqrt{-g}\Sigma^{ij}+\frac{g^{li}}{2}\,\overset{*}{\nabla}_k\left(\frac{\delta(\sqrt{-g}R)}{\delta S_{kj}{}^{l}}\right)
\end{aligned}
$$

After some tedious calculations, the geometrical terms simplify drastically to G^{ij}:

$$
-\frac{1}{\sqrt{-g}}\left[\frac{\delta(\sqrt{-g}R)}{\delta g_{ij}}+\frac{g^{li}}{2}\,\overset{*}{\nabla}_k\left(\frac{\delta(\sqrt{-g}R)}{\delta S_{kj}{}^{l}}\right)\right] = \kappa\Sigma^{ij} \quad\Rightarrow\quad G^{ij}=\kappa\Sigma^{ij}
$$

Equation (2) can be written in terms of τ^{ijk} by an antisymmetrization:

$$
-\frac{g^{l[j}}{2\sqrt{-g}}\frac{\delta(\sqrt{-g}R)}{\delta S_{i]k}{}^{l}} = \kappa\mu^{[ji]k} = \kappa\tau^{ijk}
$$

The left-hand side evaluates to modified torsion T^{ijk}, defined as torsion plus its trace:

$$
T_{ij}{}^{k} := S_{ij}{}^{k} + 2\delta^{k}_{[i}S_{j]l}{}^{l}
$$

Thus we have obtained the two (sets of) field equations of Einstein-Cartan theory:

$$
\begin{cases}
G^{ij} = \kappa\Sigma^{ij} & (3)\\
T^{ijk} = \kappa\tau^{ijk} & (4)
\end{cases}
$$

3 Physical Implications

3.1 Spin-torsion coupling

The first set of field equations simply reduces to Einstein's equations in the case of zero spin. The second equation, however, reveals the coupling of torsion to the spin angular momentum of matter field. What is significant about (4), apart from the fact that it takes on a form analagous to (3), is that it is a set of simple algebraic equations rather than differential equations, which implies that torsion can only be zero outside of matter distribution and cannot propagate as a wave through vacuum – it is bound to matter.

3.2 Comparison to general relativity

Furthermore, the fact that equation (4) is algebraic makes it possible to pull out the non-Riemannian part of G^{ij} in equation (3) and substitute all T^{ijk} with $\kappa\tau^{ijk}$, resulting in a combined field equation that involves only the Riemannian part of G^{ij} [2]:

$$G^{ij}_{\{\}} = \kappa\left[\sigma^{ij} + \kappa\Big(-4\tau^{ik}{}_{[l}\tau^{jl}{}_{k]} - 2\tau^{ikl}\tau^{j}{}_{kl} + \tau^{kli}\tau_{kl}{}^{j} + \frac{1}{2}g^{ij}\left(4\tau_m{}^{k}{}_{[l}\tau^{ml}{}_{k]} + \tau^{mkl}\tau_{mkl}\right)\Big)\right]$$

Compared to Einstein's field equations, which under this notation is

$$G^{ij}_{\{\}} = \kappa\sigma^{ij}$$

the term consisting of various products of the spin tensor τ manifests itself as the correction term, which obviously is very small due to the additional κ factor. Suppose the matter distribution is made up of polarized elementary particles with particle mass m, spin $\hbar/2$, and number density n. From the combined field equation it is obvious that the mass density $\rho = mn$ receives a correction from spin of order $\kappa s^2 = \kappa(n\hbar/2)^2$. For spin to have an effect as significant as that of mass,

$$mn \sim \kappa n^2\hbar^2 \quad \Rightarrow \quad n \sim \frac{m}{\kappa\hbar^2}$$

In terms of mass density [2],

$$\rho_{critical} \sim \frac{m^2}{\kappa\hbar^2} \approx \begin{cases} 10^{47}\,\mathrm{g\,cm^{-3}} \text{ (electrons)} \\ 10^{54}\,\mathrm{g\,cm^{-3}} \text{ (neutrons)} \end{cases}$$

Thus, Einstein-Cartan theory only produces non-negligible deviations from general relativity under extremely high densities that might in fact be unphysical.

References

[1] Albert Einstein et al. The foundation of the general theory of relativity. *Annalen der Physik*, 49(7):769–822, 1916.

[2] Friedrich W Hehl, Paul Von der Heyde, G David Kerlick, and James M Nester. General relativity with spin and torsion: Foundations and prospects. *Reviews of Modern Physics*, 48(3):393, 1976.

[3] nLab authors. fundamental theorem of Riemannian geometry. `http://ncatlab.org/nlab/show/fundamental%20theorem%20of%20Riemannian%20geometry`, April 2019. Revision 2.

PHYS 7125 Homework 5

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1

The total proper time along any curve γ is

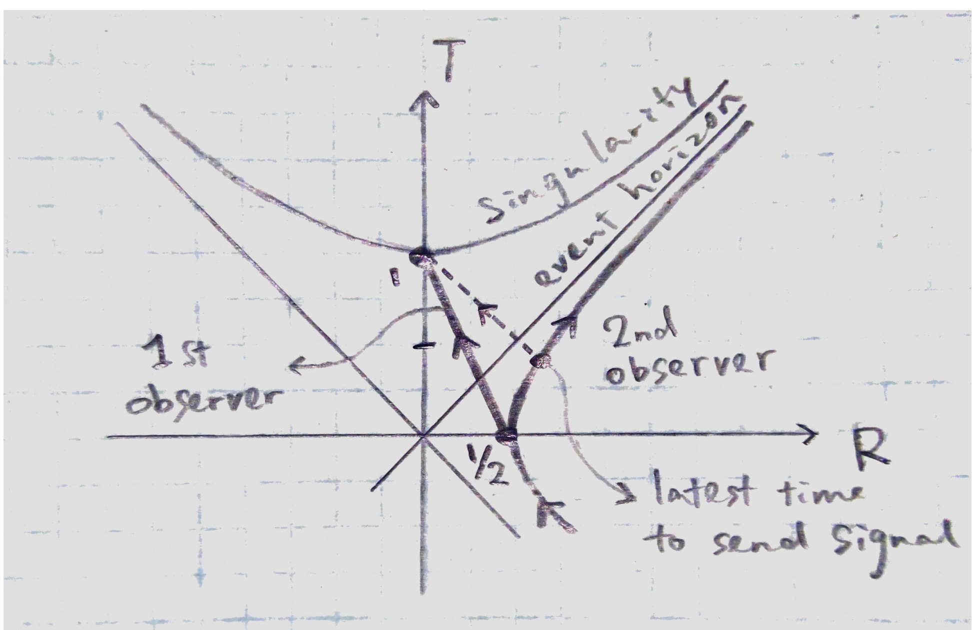
$$\tau_{total} = \int_\gamma d\tau = \int_\gamma \sqrt{-ds^2} = \int_\gamma \sqrt{-\Big(\frac{2M}{r}-1\Big)dt^2 + \Big(\frac{2M}{r}-1\Big)^{-1}dr^2 - r^2 d\Omega^2}$$

Inside the event horizon, $r < 2M$, the first and third term in the square root are negative, therefore

$$\begin{aligned}\tau_{total} &< \int_0^{2M} \sqrt{\Big(\frac{2M}{r}-1\Big)^{-1}}dr \\ &= \left[-\sqrt{r(2M-r)} + 2M\cot^{-1}\left(\sqrt{\frac{2M}{r}-1}\right)\right]\Bigg|_0^{2M} = \pi M\end{aligned}$$

2

a



b

Yes, because the worldline lies within the 45° light cone at each point.

c

For the second observer at constant r,

$$T = \frac{1}{2}\sinh\left(\frac{t}{4GM}\right), \quad R = \frac{1}{2}\cosh\left(\frac{t}{4GM}\right), \quad R^2 - T^2 = \frac{1}{4}$$

To reach the critical point where the first observer is destroyed, the photon sent by the second observer must follow the straight line $R = 1 - T$, which intersects the second observer's worldline in the past at

$$\cancel{T^2} - 2T + 1 - \cancel{T^2} = \frac{1}{4} \Rightarrow \quad T = \frac{1}{2}\sinh\left(\frac{t}{4GM}\right) = \frac{3}{8}$$

$$t = 4GM\sinh^{-1}\left(\frac{3}{4}\right) \approx 2.77GM$$

3

A perfect fluid is incompressible, therefore $\nabla_\mu \rho = \partial_\mu \rho = 0$. From the continuity equation,

$$\nabla_\mu(\rho u^\mu) = \rho \nabla_\mu u^\mu = 0 \quad \Rightarrow \quad \nabla_\mu u^\mu = 0$$

The conservation law requires the covariant divergence of stress-energy tensor to vanish:

$$\begin{aligned}
\nabla_\mu T^\mu{}_\nu &= (\nabla_\mu p + \cancel{\nabla_\mu \rho})u^\mu u_\nu + (p+\rho)\cancel{(\nabla_\mu u^\mu)}u_\nu + (p+\rho)u^\mu(\nabla_\mu u_\nu) + \delta^\mu_\nu \nabla_\mu p \\
&= u_\nu(u^\mu \nabla_\mu p) + (p+\rho)(u^\mu \nabla_\mu u_\nu) + (\delta^\mu_\nu \nabla_\mu)p \\
&= u_\nu \nabla_{\underset{\sim}{u}} p + (p+\rho)\nabla_{\underset{\sim}{u}} u_\nu + \nabla_\nu p \\
&= 0
\end{aligned}$$

Rearranging the terms (and dropping the free index ν) gives

$$(p+\rho)\nabla_{\underset{\sim}{u}} \underset{\sim}{u} = -\nabla p - \underset{\sim}{u} \nabla_{\underset{\sim}{u}} p$$

PHYS 7125 Homework 4

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1

For a timelike vector t^μ at any point it is always possible to construct an orthonormal frame $\underset{\sim}{\mathbf{e}}_i^\mu$ where $\underset{\sim}{\mathbf{e}}_0^\mu = t^\mu$. (Without loss of generality, t^μ can be assumed to have unit length (-1). In the general case, the results only differ by a positive factor.) In such coordinates, $t^\mu = (1,0,0,0)$, the metric is locally $\eta_{\mu\nu}$, and the components of the electromagnetic energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{\mu_0}\Big[F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big]$$

Considering t^μ is only non-zero in its time component and $T_{\mu\nu}$ is symmetric, we only need to compute

$$T_{0\nu} = \frac{1}{\mu_0}\Big[F_0{}^\alpha F_{\nu\alpha} - \frac{1}{4}\eta_{0\nu}F_{\alpha\beta}F^{\alpha\beta}\Big] =: A + B$$

The first term evaluates to: (Define $\mathbf{S} := \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$)

$$A = \frac{1}{\mu_0}\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 0 \\ E_x/c \\ E_y/c \\ E_z/c \end{pmatrix} = \begin{pmatrix} \epsilon_0 E^2 \\ -\frac{1}{\mu_0}(E_yB_z - E_zB_y)/c \\ -\frac{1}{\mu_0}(E_zB_x - E_xB_z)/c \\ -\frac{1}{\mu_0}(E_xB_y - E_yB_x)/c \end{pmatrix} = \begin{pmatrix} \epsilon_0 E^2 \\ -S_x/c \\ -S_y/c \\ -S_z/c \end{pmatrix}$$

The second term:

$$B = -\frac{1}{4\mu_0}(-2E^2/c^2 + 2B^2)\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\Big(-\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad T_{0\nu} = \begin{pmatrix} \frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) \\ -S_x/c \\ -S_y/c \\ -S_z/c \end{pmatrix}$$

In matrix form, the relevant components of the energy-momentum tensor are:

$$T_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) & -S_x/c & -S_y/c & -S_z/c \\ -S_x/c & \cdots & \cdots & \cdots \\ -S_y/c & \cdots & \cdots & \cdots \\ -S_z/c & \cdots & \cdots & \cdots \end{pmatrix}$$

(i) The weak energy condition is obviously satisfied:

$$T_{\mu\nu}t^\mu t^\nu = T_{00} = \frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) \geq 0$$

(ii)

$$T_{\mu\nu}t^\mu = T_{0\nu} = \Bigg(\frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big), -S_x/c, -S_y/c, -S_z/c\Bigg)$$

$$T^{\nu}{}_{\alpha}t^{\alpha} = T^{\nu}{}_{0} = g^{\nu\nu} \cdot T_{\nu 0} = \left(-\frac{1}{2}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big), -S_x/c, -S_y/c, -S_z/c \right)$$

Using Lagrange's identity for cross products,

$$\begin{aligned}
(T_{\mu\nu}t^{\mu})(T^{\nu}{}_{\alpha}t^{\alpha}) &= -\frac{1}{4}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big)^2 + \frac{\|\mathbf{S}\|^2}{c^2} \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big)^2 + \frac{\epsilon_0}{\mu_0}\|\mathbf{E}\times\mathbf{B}\|^2 \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big)^2 + \frac{\epsilon_0}{\mu_0}\Big(E^2B^2 - (\mathbf{E}\cdot\mathbf{B})^2\Big) \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 - \frac{1}{\mu_0}B^2\Big)^2 - \frac{\epsilon_0}{\mu_0}(\mathbf{E}\cdot\mathbf{B})^2 \le 0
\end{aligned}$$

Thus the dominant energy condition is also satisfied.

2

a

The only non-vanishing components of $\Gamma^{\mu}_{0\nu}$ are

$$\Gamma^{0}_{0i} = \frac{Mx^i}{(1-2M/r)r^3}, \quad \Gamma^{i}_{00} = \frac{Mx^i}{(1+2M/r)r^3} \qquad (i = 1, 2, 3)$$

The geodesic equation can be simplified as

$$\begin{aligned}
\frac{dp_0}{d\lambda} &= \sum_i \left(\frac{Mx^i}{(1-2M/r)r^3}p_0 p^i + \frac{Mx^i}{(1+2M/r)r^3}p^0 p_i \right) \\
&= \sum_i \left(g_{00}\frac{Mx^i}{(1-2M/r)r^3} + g_{ii}\frac{Mx^i}{(1+2M/r)r^3} \right) p^0 p^i \\
&= \sum_i \left(-\frac{Mx^i}{r^3} + \frac{Mx^i}{r^3} \right) p^0 p^i = 0
\end{aligned}$$

b

No. $p^0 = g^{0\nu}p_\nu = g^{00}p_0 = -(1-2M/r)^{-1}p_0$, which is not constant unless r is constant.

c

For the atom at rest on the surface of the sun, $dx^i = 0$, $r = R$,

$$d\tau^2 = -ds^2 = (1-2M/R)dt^2 \quad \Rightarrow \quad u^0 = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-2M/R}}$$

d

For both the atom and the distant observer, $dx^i = 0$,

$$d\tau^2 = -ds^2 = (1-2M/r)dt^2$$

$$\Rightarrow \quad u^\mu = \frac{dx^\mu}{d\tau} = (dt/d\tau, 0, 0, 0) = ((1-2M/r)^{-1/2}, 0, 0, 0)$$

The photon energy observed at both locations can be expressed as

$$E = g_{\mu\nu} p^\mu u^\nu = (1 - 2M/r)^{-1/2} g_{\mu\nu} p^\mu K^\nu$$

where $K^\nu = (1, 0, 0, 0)$ is a Killing vector as the metric has no time dependence; therefore $g_{\mu\nu} p^\mu K^\nu$ is conserved along the photon's world line, which is a geodesic as stated in the problem. Then,

$$\frac{\lambda_r}{\lambda_e} = \frac{hc/\lambda_e}{hc/\lambda_r} = \frac{E_e}{E_r} = \frac{(1 - 2M/R)^{-1/2}}{\lim_{r\to\infty}(1 - \cancel{2M/r})^{-1/2}} = 1 + \frac{M}{R} + \mathcal{O}\left(\frac{M^2}{R^2}\right)$$

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{\lambda_r}{\lambda_e} - 1 = \frac{M}{R} + \mathcal{O}\left(\frac{M^2}{R^2}\right)$$

PHYS 7125 Homework 3

Wenqi He

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1

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

Contracting twice:

$$g^{\mu\lambda} g^{\nu\sigma} \nabla_\lambda R_{\rho\sigma\mu\nu} + g^{\mu\lambda} g^{\nu\sigma} \nabla_\rho R_{\sigma\lambda\mu\nu} + g^{\mu\lambda} g^{\nu\sigma} \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} g^{\nu\sigma} R_{\sigma\rho\nu\mu} \Big) - \nabla_\rho \Big(g^{\mu\lambda} g^{\nu\sigma} R_{\lambda\sigma\mu\nu} \Big) + \nabla_\sigma \Big(g^{\mu\lambda} g^{\nu\sigma} R_{\lambda\rho\mu\nu} \Big) = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) - \nabla_\rho \Big(g^{\nu\sigma} R_{\sigma\nu} \Big) + \nabla_\sigma \Big(g^{\nu\sigma} R_{\rho\nu} \Big) = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) - \nabla_\rho R + \nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) - \nabla_\lambda \Big(\frac{1}{2} \delta^\lambda_\rho R \Big) = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) - \nabla_\lambda \Big(\frac{1}{2} g^{\mu\lambda} g_{\rho\mu} R \Big) = 0$$

$$\nabla_\lambda \Big[g^{\mu\lambda} \Big(R_{\rho\mu} - \frac{1}{2} g_{\rho\mu} R \Big) \Big] = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} G_{\rho\mu} \Big) = \nabla_\lambda \Big(g^{\lambda\mu} G_{\mu\rho} \Big) = 0$$

$$\nabla_\lambda G^\lambda{}_\rho = 0$$

2

The only non-vanishing components of the metric are

$$g_{\psi\psi} = 1, \quad g_{\theta\theta} = \sin^2\psi, \quad g_{\phi\phi} = \sin^2\psi \sin^2\theta$$

Since the matrix is diagonal,

$$g^{\psi\psi} = 1, \quad g^{\theta\theta} = 1/\sin^2\psi, \quad g^{\phi\phi} = 1/\sin^2\psi \sin^2\theta$$

The only non-vanishing first derivatives of the metric components are

$$g_{\theta\theta,\psi} = 2\sin\psi\cos\psi, \quad g_{\phi\phi,\psi} = 2\sin\psi\cos\psi\sin^2\theta, \quad g_{\phi\phi,\theta} = 2\sin^2\psi\sin\theta\cos\theta$$

a

The only non-vanishing Christoffel symbols are

$$\Gamma^{\theta}_{\theta\psi} = \Gamma^{\theta}_{\psi\theta} = \frac{1}{2} g^{\theta\theta} g_{\theta\theta,\psi} = \frac{2\sin\psi\cos\psi}{2\sin^2\psi} = \cot\psi$$

$$\Gamma^{\psi}_{\theta\theta} = -\frac{1}{2} g^{\psi\psi} g_{\theta\theta,\psi} = -\sin\psi\cos\psi$$

$$\Gamma^{\phi}_{\phi\psi} = \Gamma^{\phi}_{\psi\phi} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\psi} = \frac{2\sin\psi\cos\psi\sin^2\theta}{2\sin^2\psi\sin^2\theta} = \cot\psi$$

$$\Gamma^{\psi}_{\phi\phi} = -\frac{1}{2} g^{\psi\psi} g_{\phi\phi,\psi} = -\sin\psi\cos\psi\sin^2\theta$$

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\theta} = \frac{2\sin^2\psi\sin\theta\cos\theta}{2\sin^2\psi\sin^2\theta} = \cot\theta$$

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2} g^{\theta\theta} g_{\phi\phi,\theta} = -\frac{2\sin^2\psi\sin\theta\cos\theta}{2\sin^2\psi} = -\sin\theta\cos\theta$$

b

There are $\frac{1}{12} \cdot 3^2 \cdot (3^2 - 1) = 6$ independent components of the Riemann tensor:

$$\begin{aligned}
R_{\psi\theta\psi\theta} &= g_{\psi\psi} R^{\psi}_{\theta\psi\theta} = \sin^2\psi - \cos^2\psi - 0 + 0 - (-\cos^2\psi) = \sin^2\psi \\
R_{\psi\theta\psi\phi} &= g_{\psi\psi} R^{\psi}_{\theta\psi\phi} = 0 - 0 + 0 - 0 = 0 \\
R_{\psi\theta\theta\phi} &= g_{\psi\psi} R^{\psi}_{\theta\theta\phi} = 0 - 0 + 0 - 0 = 0 \\
R_{\psi\phi\psi\phi} &= g_{\psi\psi} R^{\psi}_{\phi\psi\phi} = (\sin^2\psi - \cos^2\psi)\sin^2\theta - 0 + 0 - (-\cos^2\psi\sin^2\theta) = \sin^2\psi\sin^2\theta \\
R_{\psi\phi\theta\phi} &= g_{\psi\psi} R^{\psi}_{\phi\theta\phi} = -2\sin\psi\cos\psi\sin\theta\cos\theta - 0 + \sin\psi\cos\psi\sin\theta\cos\theta - (-\sin\psi\cos\psi\sin\theta\cos\theta) = 0 \\
R_{\theta\phi\theta\phi} &= g_{\theta\theta} R^{\theta}_{\phi\theta\phi} = \sin^2\psi \Big[\sin^2\theta - \cos^2\theta - 0 + (-\cos^2\psi\sin^2\theta) - (-\cos^2\theta) \Big] = \sin^4\psi\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
R_{\psi\psi} &= g^{\theta\theta} R_{\theta\psi\theta\psi} + g^{\phi\phi} R_{\phi\psi\phi\psi} = 2 \\
R_{\psi\theta} &= g^{\phi\phi} R_{\phi\psi\phi\theta} = 0 \\
R_{\psi\phi} &= g^{\theta\theta} R_{\theta\psi\theta\phi} = 0 \\
R_{\theta\theta} &= g^{\psi\psi} R_{\psi\theta\psi\theta} + g^{\phi\phi} R_{\phi\theta\phi\theta} = \sin^2\psi + \sin^2\psi = 2\sin^2\psi \\
R_{\theta\phi} &= g^{\psi\psi} R_{\psi\theta\psi\phi} = 0 \\
R_{\phi\phi} &= g^{\psi\psi} R_{\psi\phi\psi\phi} + g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta = 2\sin^2\psi\sin^2\theta
\end{aligned}$$

$$R = g^{\psi\psi} R_{\psi\psi} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = 2 + 2 + 2 = 6$$

PHYS 7125 Homework 2

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1

The local flatness property states that for each point p on the manifold there exists a change of coordinates such that the metric $g_{\mu\nu}$ can be transformed into $g_{\mu'\nu'}$ that satisfies: (i) $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ and (ii) $g_{\mu'\nu',\sigma} = 0$ at point p. This can be proven by a Taylor expansion of $g_{\mu'\nu'}$ to the first order:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$$

$$= \left(x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu} \right)\Big|_p + \left(x^\mu_{,\mu'\lambda} x^\nu_{,\nu'} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'\lambda} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu,\lambda} \right)\Big|_p \epsilon + O(\epsilon^2)$$

The requirement translates to:

$$\left(x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu} \right)\Big|_p = \eta_{\mu'\nu'}$$

$$\left(x^\mu_{,\mu'\lambda} x^\nu_{,\nu'} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'\lambda} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu,\lambda} \right)\Big|_p = 0$$

The first equation has 16 variables in $\partial x^\mu / \partial x^{\mu'}$ and 10 equations, one for each indepedent entry of the metric. The remaining 6 degrees of freedom exactly matches the dimension of the Lorentz group, under which the metric is preserved. With $\partial x^\mu / \partial x^{\mu'}$ determined, the second equation will only have $4 \cdot 10 = 40$ variables in $\partial^2 x^\mu / \partial x^{\mu'} \partial x^\lambda$ since partial derivatives commute. Coincidentally, as metric is always symmetric, there are $10 \cdot 4 = 40$ equations corresponding to the independent entries of $g_{\mu\nu,\lambda}$. The system is thus uniquely determined, and therefore such transformation is always possible.

2

a

$$\delta^\alpha_{\sigma,\gamma} = \left(g^{\alpha\beta} g_{\beta\sigma} \right)_{,\gamma} = g^{\alpha\beta}_{,\gamma} g_{\beta\sigma} + g^{\alpha\beta} g_{\beta\sigma,\gamma} = 0 \quad \Rightarrow \quad g^{\alpha\beta}_{,\gamma} g_{\beta\sigma} = -g^{\alpha\beta} g_{\beta\sigma,\gamma}$$

Multiplying by inverse metric,

$$g^{\alpha\beta}_{,\gamma} g_{\beta\sigma} g^{\sigma\lambda} = g^{\alpha\beta}_{,\gamma} \delta^\lambda_\beta = g^{\alpha\lambda}_{,\gamma} = -g^{\sigma\lambda} g^{\alpha\beta} g_{\beta\sigma,\gamma}$$

b

From the two identities we can derive the formula:

$$\begin{aligned}
\frac{d}{d\epsilon}\det(A) &= \lim_{\epsilon\to 0}\frac{\det(A+\epsilon\frac{d}{d\epsilon}A+\cancel{O(\epsilon^2)})-\det(A)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\det(A(I+\epsilon A^{-1}\frac{d}{d\epsilon}A))-\det(A)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\det(A)\det(I+\epsilon A^{-1}\frac{d}{d\epsilon}A)-\det(A)}{\epsilon} \\
&= \det(A)\lim_{\epsilon\to 0}\frac{\det(I+\epsilon A^{-1}\frac{d}{d\epsilon}A)-1}{\epsilon} \\
&= \det(A)\lim_{\epsilon\to 0}\frac{1+\epsilon\,\mathrm{tr}(A^{-1}\frac{d}{d\epsilon}A)+\cancel{O(\epsilon^2)}-1}{\epsilon} \\
&= \det(A)\,\mathrm{tr}(A^{-1}\frac{d}{d\epsilon}A)
\end{aligned}$$

Apply the formula to $g = \det g_{\mu\nu}$, replacing $d/d\epsilon$ with ∂_α

$$g_{,\alpha} = g\cdot\mathrm{tr}(g^{\sigma\mu}g_{\mu\nu,\alpha}) = gg^{\nu\mu}g_{\mu\nu,\alpha}$$

c

$$\begin{aligned}
RHS &= -(-g)^{-1/2}\Big[g^{\alpha\beta}(-g)^{1/2}\Big]_{,\beta} \\
&= -(-g)^{-1/2}\Big[g^{\alpha\beta}{}_{,\beta}(-g)^{1/2}+g^{\alpha\beta}(-g)^{1/2}_{,\beta}\Big] \\
&= -(-g)^{-1/2}\Big[g^{\alpha\beta}{}_{,\beta}(-g)^{1/2}-\frac{1}{2}g^{\alpha\beta}(-g)^{-1/2}g_{,\beta}\Big] \\
&= -g^{\alpha\beta}{}_{,\beta}+\frac{1}{2}g^{\alpha\beta}(-g)^{-1}g_{,\beta} \\
&= g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}+\frac{1}{2}g^{\alpha\beta}(-g)^{-1}gg^{\mu\nu}g_{\mu\nu,\beta} \\
&= g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}-\frac{1}{2}g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta} \\
&= \frac{1}{2}\Big(g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}+g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}-g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\Big) \\
&= \frac{1}{2}\Big(g^{\mu\nu}g^{\beta\alpha}g_{\mu\beta,\nu}+g^{\nu\beta}g^{\mu\alpha}g_{\nu\mu,\beta}-g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\Big) \\
&= \frac{1}{2}\Big(g^{\mu\nu}g^{\beta\alpha}g_{\mu\beta,\nu}+g^{\nu\mu}g^{\beta\alpha}g_{\nu\beta,\mu}-g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\Big) \\
&= g^{\mu\nu}\cdot\frac{1}{2}g^{\alpha\beta}\Big(g_{\beta\mu,\nu}+g_{\beta\nu,\mu}-g_{\mu\nu,\beta}\Big) \\
&= g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = LHS
\end{aligned}$$

d

$$
\begin{aligned}
LHS &= A^\alpha{}_{,\alpha} + \Gamma^\alpha_{\alpha\lambda} A^\lambda \\
&= A^\alpha{}_{,\alpha} + \frac{1}{2} g^{\alpha\beta}\Big(g_{\beta\alpha,\lambda} + g_{\beta\lambda,\alpha} - g_{\alpha\lambda,\beta}\Big) A^\lambda \\
&= A^\alpha{}_{,\alpha} + \frac{1}{2}\Big(g^{\alpha\beta} g_{\beta\alpha,\lambda} + g^{\alpha\beta} g_{\beta\lambda,\alpha} - g^{\alpha\beta} g_{\alpha\lambda,\beta}\Big) A^\lambda \\
&= A^\alpha{}_{,\alpha} + \frac{1}{2}\Big(g^{\alpha\beta} g_{\beta\alpha,\lambda} + \cancel{g^{\alpha\beta} g_{\beta\lambda,\alpha}} - \cancel{g^{\beta\alpha} g_{\beta\lambda,\alpha}}\Big) A^\lambda \\
&= A^\alpha{}_{,\alpha} + \frac{1}{2} g^{\alpha\beta} g_{\beta\alpha,\lambda} A^\lambda
\end{aligned}
$$

$$
\begin{aligned}
RHS &= (-g)^{-1/2}\Big[(-g)^{1/2} A^\alpha\Big]_{,\alpha} \\
&= (-g)^{-1/2}\Big[(-g)^{1/2} A^\alpha{}_{,\alpha} + (-g)^{1/2}_{,\alpha} A^\alpha\Big] \\
&= A^\alpha{}_{,\alpha} - \frac{1}{2}(-g)^{-1/2}(-g)^{-1/2} g_{,\alpha} A^\alpha \\
&= A^\alpha{}_{,\alpha} + \frac{1}{2} g^{-1} g g^{\mu\nu} g_{\mu\nu,\alpha} A^\alpha \\
&= A^\alpha{}_{,\alpha} + \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha} A^\alpha = LHS
\end{aligned}
$$

e

$$
\epsilon_{\alpha\beta\gamma\delta;\mu} = \Big((-g)^{1/2}\tilde{\epsilon}_{\alpha\beta\gamma\delta}\Big)_{;\mu} = (-g)^{1/2}_{,\mu}\tilde{\epsilon}_{\alpha\beta\gamma\delta} - (-g)^{1/2}\Big[\Gamma^\lambda_{\alpha\mu}\tilde{\epsilon}_{\lambda\beta\gamma\delta} + \Gamma^\lambda_{\beta\mu}\tilde{\epsilon}_{\alpha\lambda\gamma\delta} + \Gamma^\lambda_{\gamma\mu}\tilde{\epsilon}_{\alpha\beta\lambda\delta} + \Gamma^\lambda_{\delta\mu}\tilde{\epsilon}_{\alpha\beta\gamma\lambda}\Big]
$$

If there are repeated indices, w.l.o.g. we can suppose $\alpha = \beta$, then the first term outside the brackets and the last two terms inside the bracket vanish, and also

$$
\Gamma^\lambda_{\alpha\mu}\tilde{\epsilon}_{\lambda\beta\gamma\delta} + \Gamma^\lambda_{\beta\mu}\tilde{\epsilon}_{\alpha\lambda\gamma\delta} = \Gamma^\lambda_{\alpha\mu}\tilde{\epsilon}_{\lambda\alpha\gamma\delta} + \Gamma^\lambda_{\alpha\mu}\tilde{\epsilon}_{\alpha\lambda\gamma\delta} = 0
$$

So $\epsilon_{\alpha\beta\gamma\delta;\mu} = 0$ in this case. Now consider the case where all indices are distinct, then all $\tilde{\epsilon}_{\lambda\beta\gamma\delta}$ (λ is a free index) vanish unless $\lambda = \alpha$. Same argument applies to other slots as well. W.l.o.g., suppose $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3$. The first term on the right-hand side then evaluates to

$$
-\frac{1}{2}(-g)^{-1/2} g_{,\mu}
$$

Expand the the second term:

$$
\begin{aligned}
\cdots &= \frac{1}{2}(-g)^{1/2} g^{\lambda\sigma}\Big(g_{\sigma0,\mu} + g_{\sigma\mu,0} - g_{0\mu,\sigma}\Big)\tilde{\epsilon}_{\lambda123} + \frac{1}{2}(-g)^{1/2} g^{\lambda\sigma}\Big(g_{\sigma1,\mu} + g_{\sigma\mu,1} - g_{1\mu,\sigma}\Big)\tilde{\epsilon}_{0\lambda23} \\
&\quad + \frac{1}{2}(-g)^{1/2} g^{\lambda\sigma}\Big(g_{\sigma2\mu} + g_{\sigma\mu,2} - g_{2\mu,\sigma}\Big)\tilde{\epsilon}_{01\lambda3} + \frac{1}{2}(-g)^{1/2} g^{\lambda\sigma}\Big(g_{\sigma3,\mu} + g_{\sigma\mu,3} - g_{3\mu,\sigma}\Big)\tilde{\epsilon}_{012\lambda} \\
&= \frac{1}{2}(-g)^{1/2} g^{0\sigma}\Big(g_{\sigma0,\mu} + g_{\sigma\mu,0} - g_{0\mu,\sigma}\Big) + \frac{1}{2}(-g)^{1/2} g^{1\sigma}\Big(g_{\sigma1,\mu} + g_{\sigma\mu,1} - g_{1\mu,\sigma}\Big) \\
&\quad + \frac{1}{2}(-g)^{1/2} g^{2\sigma}\Big(g_{\sigma2,\mu} + g_{\sigma\mu,2} - g_{2\mu,\sigma}\Big) + \frac{1}{2}(-g)^{1/2} g^{3\sigma}\Big(g_{\sigma3,\mu} + g_{\sigma\mu,3} - g_{3\mu,\sigma}\Big) \\
&= \frac{1}{2}(-g)^{1/2}\Big(g^{\lambda\sigma} g_{\lambda\sigma,\mu} + \cancel{g^{\lambda\sigma} g_{\sigma\mu,\lambda}} - \cancel{g^{\lambda\sigma} g_{\lambda\mu,\sigma}}\Big) = -\frac{1}{2}(-g)^{-1/2} g_{,\mu}
\end{aligned}
$$

The last step above reorganized the terms into a summation over dummy λ and invoked the result from (b). Two terms on the right-hand side cancel out, so $\epsilon_{\alpha\beta\gamma\delta;\mu} = 0$.

3

a

Since $u_\alpha u^\alpha = -1$,

$$(P_{\alpha\beta}v^\beta)u^\alpha = g_{\alpha\beta}v^\beta u^\alpha + u_\alpha u_\beta v^\beta u^\alpha = v^\beta u_\beta - u_\beta v^\beta = 0$$

b

From (a), $u^\beta v_{\perp_\beta} = u_\beta v_\perp^\beta = 0$, therefore

$$P_{\alpha\beta}v_\perp^\beta = g_{\alpha\beta}v_\perp^\beta + u_\alpha u_\beta v_\perp^\beta = g_{\alpha\beta}v_\perp^\beta + 0 = v_{\perp\alpha}$$

c

$$P_{\alpha\beta} := g_{\alpha\beta} - (q_\lambda q^\lambda)^{-1} q_\alpha q_\beta$$

Proof: Carrying out the same calculations:

$$(P_{\alpha\beta}v^\beta)q^\alpha = g_{\alpha\beta}v^\beta q^\alpha - \cancel{(q_\lambda q^\lambda)^{-1}}\cancel{q_\alpha} q_\beta v^\beta \cancel{q^\alpha} = v^\beta q_\beta - q_\beta v^\beta = 0$$

$$P_{\alpha\beta}v_\perp^\beta = g_{\alpha\beta}v_\perp^\beta - (q_\lambda q^\lambda)^{-1} q_\alpha \cancel{q_\beta v_\perp^\beta} = v_{\perp\alpha}$$

d

Since the norm of a null vector is zero, the orthogonal projection cannot be constructed by substracting a parallel projection which involves a division by the norm, therefore the projection can only be expressed explicitly as an expansion w.r.t. the basis vectors orthogonal to $\underset{\sim}{k}$. Apart from $\underset{\sim}{k}$, which is orthogonal to itself, there must exist two additional linearly independent vectors to match the dimensionality. It can be shown that (i) orthogonal null vectors are colinear and (ii) null vectors cannot be orthogonal to time-like vectors, therefore the two vectors must be space-like, and we can perform the Gram-Schmidt process like above to obtain two orthonormal space-like vectors. Finally, $k_\alpha k_\beta$ term must be excluded because if the projection produces a k_μ component, applying the projection twice will then eliminate it, violating $P(v_\perp) = v_\perp$. Therefore the projection tensor is

$$P_{\alpha\beta} = e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}$$

Where $\underset{\sim}{e}_{(i)}$ are the orthonormal space-like basis vectors. It can be quickly verified that

$$\Big(e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}\Big)v^\beta k^\alpha = 0$$

and also

$$\begin{aligned}\Big(e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}\Big)v_\perp^\beta &= \Big(e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}\Big)\Big(e_{(1)}^\beta e_{(1)\sigma} + e_{(2)}^\beta e_{(2)\sigma}\Big)v^\sigma \\ &= \Big(e_{(1)\alpha}e_{(1)\sigma} + e_{(2)\alpha}e_{(2)\sigma}\Big)v^\sigma = v_{\perp\alpha}\end{aligned}$$

However the choice of $\{\underset{\sim}{e}_{(i)}\}$ is not unique, which means that P in general will not be unique. As an example, suppose coordinates are chosen such that space-time is locally flat and consider $\underset{\sim}{k} = (1,1,0,0)$, $\underset{\sim}{e}_{(1)} = (1,1,1,0)$, $\underset{\sim}{e}_{(2)} = (1,1,0,1)$, $P_{\alpha\beta} = e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}$ only has integer components; however, we can also choose $\underset{\sim}{e}'_{(1)} = (1,1,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, $\underset{\sim}{e}'_{(2)} = (1,1,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$, then $P'_{\alpha\beta} = e'_{(1)\alpha}e'_{(1)\beta} + e'_{(2)\alpha}e'_{(2)\beta}$ would have irrational components.

4

a

$$\nabla_{\underset{\sim}{u}} u_\mu = \frac{dx^\alpha}{d\tau}\nabla_\alpha u_\mu = \frac{dx^\alpha}{d\tau}\left(\partial_\alpha u_\mu - \Gamma^\beta_{\alpha\mu}u_\beta\right) = \frac{du_\mu}{d\tau} - \Gamma^\beta_{\alpha\mu}u^\alpha u_\beta = 0$$

b

Since the connection is metric-compatible,

$$\nabla_\alpha g^{\mu\nu} = \partial_\alpha g^{\mu\nu} + \Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} + \Gamma^\nu_{\alpha\lambda}g^{\mu\lambda} = 0 \Rightarrow \quad \partial_\alpha g^{\mu\nu} = -\Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} - \Gamma^\nu_{\alpha\lambda}g^{\mu\lambda}$$
$$\Rightarrow \quad \frac{dg^{\mu\nu}}{d\tau} = u^\alpha\partial_\alpha g^{\mu\nu} = -u^\alpha\Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} - u^\alpha\Gamma^\nu_{\alpha\lambda}g^{\mu\lambda}$$

Raising the indices of the equation from (a) recovers the original form of geodesic equation:

$$\begin{aligned}
&g^{\mu\nu}\frac{du_\mu}{d\tau} - g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta \\
=&\frac{d}{d\tau}\left(g^{\mu\nu}u_\mu\right) - u_\mu\frac{dg^{\mu\nu}}{d\tau} - g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta \\
=&\frac{d}{d\tau}\left(g^{\mu\nu}u_\mu\right) + u_\mu u^\alpha\Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} + u_\mu u^\alpha\Gamma^\nu_{\alpha\lambda}g^{\mu\lambda} - g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta \\
=&\frac{d}{d\tau}\left(g^{\mu\nu}u_\mu\right) + \cancel{u_\beta u^\alpha\Gamma^\beta_{\alpha\mu}g^{\mu\nu}} + u_\mu u^\alpha\Gamma^\nu_{\alpha\lambda}g^{\mu\lambda} - \cancel{g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta} \\
=&\frac{du^\nu}{d\tau} + \Gamma^\nu_{\alpha\lambda}u^\lambda u^\alpha = 0
\end{aligned}$$

Without invoking metric compatibility, we can also check that the covariant derivatives of $u_\mu u^\mu$, which is a scalar, reduce to ordinary derivatives:

$$\begin{aligned}
\nabla_{\underset{\sim}{u}}\left(u_\mu u^\mu\right) &= u_\mu\nabla_{\underset{\sim}{u}}u^\mu + u^\mu\nabla_{\underset{\sim}{u}}u_\mu = u_\mu\left(\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta\right) + u^\mu\left(\frac{du_\mu}{d\tau} - \Gamma^\beta_{\alpha\mu}u^\alpha u_\beta\right) \\
&= u_\mu\frac{du^\mu}{d\tau} + \cancel{\Gamma^\mu_{\alpha\beta}u_\mu u^\alpha u^\beta} + u^\mu\frac{du_\mu}{d\tau} - \cancel{\Gamma^\beta_{\alpha\mu}u^\mu u^\alpha u_\beta} = \frac{d}{d\tau}\left(u_\mu u^\mu\right)
\end{aligned}$$

c

Suppose λ is an affine parameter for a null-geodesic and $\sigma(\lambda)$ non-affine:

$$\begin{aligned}
\nabla_{\underset{\sim}{u}}u^\mu &= \nabla_{\frac{d}{d\sigma}}\frac{dx^\mu}{d\sigma} = \nabla_{\frac{d\lambda}{d\sigma}\frac{d}{d\lambda}}\left(\frac{d\lambda}{d\sigma}\frac{dx^\mu}{d\lambda}\right) = \frac{d\lambda}{d\sigma}\nabla_{\frac{d}{d\lambda}}\left(\frac{d\lambda}{d\sigma}\frac{dx^\mu}{d\lambda}\right) \\
&= \frac{d\lambda}{d\sigma}\left(\frac{d}{d\lambda}\frac{d\lambda}{d\sigma}\right)\frac{dx^\mu}{d\lambda} + \frac{d\lambda}{d\sigma}\frac{d\lambda}{d\sigma}\cancel{\nabla_{\frac{d}{d\lambda}}\frac{dx^\mu}{d\lambda}} \\
&= \left(\frac{d}{d\lambda}\frac{d\lambda}{d\sigma}\right)u^\mu = -\kappa(\lambda)u^\mu
\end{aligned}$$

where κ is some function of λ. The second term above vanishes because λ is affine.

PHYS 7125 Homework 1

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1

a

If $\Delta s^2 = 0$ for a particle then $\Delta t^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, which means that it travels at the speed of light in x^α coordinates. Since the speed of light is constant in all reference frames, under another coordinate system $x^{\alpha'}$ it still travels at the speed of light, that is, $\Delta t'^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2$ and therefore $\Delta s'^2 = 0$.

b

Expressing Q in terms of Δx^α:

$$Q = \eta_{\alpha'\beta'} \Delta x^{\alpha'} \Delta x^{\beta'} = \eta_{\alpha'\beta'} \Lambda^{\alpha'}{}_{\alpha} \Delta x^\alpha \Lambda^{\beta'}{}_{\beta} \Delta x^\beta = \left(\Lambda^{\alpha'}{}_{\alpha} \Lambda^{\beta'}{}_{\beta} \eta_{\alpha'\beta'}\right) \Delta x^\alpha \Delta x^\beta = \phi_{\alpha\beta} \Delta x^\alpha \Delta x^\beta$$

c

On the intersection $Q = 0$ identically. By spherical symmetry, Q must be invariant under (spatial) reflections, therefore all cross terms, which would change signs under a reflection, are eliminated; Q must also be invariant under (spatial) rotations, so the remaining $\Delta x^2, \Delta y^2, \Delta z^2$ must have the same coefficient as they are indistinguishable. The general form satisfying these requirements is

$$Q = c_1 \Delta t^2 + c_2 (\Delta x^2 + \Delta y^2 + \Delta z^2)$$

d

Since the intersection lies on the light cone, $\Delta t^2 = \Delta x^2 + 0 + 0$,

$$Q = c_1 \Delta t^2 + c_2 \Delta x^2 = (c_1 + c_2) \Delta t^2 = 0 \quad \Rightarrow \quad c_1 = -c_2$$
$$\Rightarrow \quad Q = c_2(-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2) = c_2 \eta_{\alpha\beta} \Delta x^\alpha \Delta x^\beta$$

e

The constant c_2 applies to transformations between any two frames, which certainly include the trivial transformation from one frame to itself, therefore c_2 must equal 1.

2

a

Renaming the dummy indices,

$$A_{\mu\nu} S^{\mu\nu} = A_{\nu\mu} S^{\nu\mu} = (-A_{\mu\nu}) S^{\mu\nu} \quad \Rightarrow \quad A_{\mu\nu} S^{\mu\nu} = 0$$

b

Using the same trick as above,

$$V^{\nu\mu}A_{\mu\nu} = V^{\mu\nu}A_{\nu\mu} = -V^{\mu\nu}A_{\mu\nu} \quad \Rightarrow \quad \frac{1}{2}\Big(V^{\mu\nu}A_{\mu\nu} - V^{\nu\mu}A_{\mu\nu}\Big) = V^{\mu\nu}A_{\mu\nu}$$

$$V^{\nu\mu}S_{\mu\nu} = V^{\mu\nu}S_{\nu\mu} = V^{\mu\nu}S_{\mu\nu} \quad \Rightarrow \quad \frac{1}{2}\Big(V^{\mu\nu}S_{\mu\nu} + V^{\nu\mu}S_{\mu\nu}\Big) = V^{\mu\nu}S_{\mu\nu}$$

c

A tensor acting on vectors and covectors produces a scalar, which is invariant under transformations:

$$\begin{aligned} T^{\alpha'}{}_{\beta'}{}^{\gamma'}u_{\alpha'}v^{\beta'}w_{\gamma'} &= T^{\alpha}{}_{\beta}{}^{\gamma}u_{\alpha}v^{\beta}w_{\gamma} \\ &= T^{\alpha}{}_{\beta}{}^{\gamma}\Lambda^{\alpha'}{}_{\alpha}u_{\alpha'}\Lambda^{\beta}{}_{\beta'}v^{\beta'}\Lambda^{\gamma'}{}_{\gamma}w_{\gamma'} \\ &= \Big(\Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta}{}_{\beta'}\Lambda^{\gamma'}{}_{\gamma}T^{\alpha}{}_{\beta}{}^{\gamma}\Big)u_{\alpha'}v^{\beta'}w_{\gamma'} \end{aligned}$$

Since this holds for any vectors and covectors, $T^{\alpha'}{}_{\beta'}{}^{\gamma'} = \Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta}{}_{\beta'}\Lambda^{\gamma'}{}_{\gamma}T^{\alpha}{}_{\beta}{}^{\gamma}$.

d

$$\begin{aligned} g_{\alpha\beta}g^{\beta\sigma}g^{\alpha\gamma} &= \delta^{\sigma}_{\alpha}g^{\alpha\gamma} = g^{\sigma\gamma} \\ g_{\sigma\beta}g_{\gamma\alpha}g^{\alpha\beta} &= g_{\sigma\beta}\delta^{\beta}_{\gamma} = g_{\sigma\gamma} \\ g^{\alpha}{}_{\beta} = g^{\alpha\sigma}g_{\sigma\beta} &= g_{\beta\sigma}g^{\sigma\alpha} = \delta^{\alpha}_{\beta} \end{aligned}$$

3

a

$$X^{\mu}{}_{\nu} = X^{\mu\gamma}g_{\gamma\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

b

$$X_{\mu}{}^{\nu} = g_{\mu\gamma}X^{\gamma\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$$

c

$$X^{(\mu\nu)} = \frac{1}{2}\Big(X^{\mu\nu} + X^{\nu\mu}\Big) = \begin{pmatrix} 2 & -1/2 & 0 & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{pmatrix}$$

d

$$X_{\mu\nu} = X_\mu{}^\gamma g_{\gamma\nu} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

$$X_{[\mu\nu]} = \frac{1}{2}\Big(X_{\mu\nu} - X_{\nu\mu}\Big) = \begin{pmatrix} 0 & -1/2 & -1 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1 & -1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{pmatrix}$$

e

$$X^\lambda{}_\lambda = -2 + 0 + 0 - 2 = -4$$

f

$$v^\mu v_\mu = g_{\mu\nu} v^\mu v^\nu = -(-1)^2 + 2^2 + 0^2 + (-2)^2 = 7$$

g

$$v_\mu = g_{\mu\nu} v^\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & -2 \end{pmatrix}$$

$$v_\mu X^{\mu\nu} = \begin{pmatrix} 1 & 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 5 \\ 7 \end{pmatrix}$$