# MATH 4441 Homework 3

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## 4.1

Assume $\alpha(t)$ has unit speed, then $\kappa = \|a''\|$. If $\kappa = 0$ then $a'' = 0$,

$$\alpha'(t) = v_0 + \int_0^t \alpha''(s)ds = v_0$$

$$\alpha(t) = x_0 + \int_0^t \alpha'(s)ds = x_0 + v_0 t$$

which is the equation of a straight line.

## 4.2

By definition $\|T\| = \langle T(t), T(t)\rangle = 1$. Differentiate and then scale both sides:

$$\langle T(t), T(t)\rangle' = 2\langle T(t), T(t)'\rangle = 0$$

$$\left\langle T(t), \frac{T(t)'}{\|T'(t)\|}\right\rangle = N(t) = 0$$

## 4.4

For a given $t$, the osculating circle $\gamma$ can be parameterized by

$$\gamma(s) = p(t) + \frac{1}{\kappa(t)}T(t)\cos s + \frac{1}{\kappa(t)}N(t)\sin s$$

where

$$p(t) := \alpha(t) + \frac{1}{\kappa(t)}N(t)$$

And we can compute the tantrix for the osculating circle:

$$\gamma'(s) = -\frac{1}{\kappa(t)}T(t)\sin s + \frac{1}{\kappa(t)}N(t)\cos s$$

$$\|\gamma'(s)\| = \langle \gamma'(s), \gamma'(s) \rangle^{1/2} = \frac{1}{\kappa(t)}$$

$$\bar{T}(s) = \frac{\gamma'(s)}{\|\gamma'(s)\|} = -T(t)\sin s + N(t)\cos s$$

$$\bar{T}'(s) = -T(t)\cos s - N(t)\sin s$$

$$\|\bar{T}'(s)\| = \langle \bar{T}'(s), \bar{T}'(s) \rangle^{1/2} = 1$$

Circle $\gamma$ touches $\alpha$ at $s = -\pi/2$, its tantrix at which point is

$$\bar{T}\left(-\frac{\pi}{2}\right) = -T(t)\sin(-\frac{\pi}{2}) + N(t)\cos(-\frac{\pi}{2}) = T(t)$$

so it is indeed tangent to $\alpha(t)$. And its curvature is

$$\kappa_\gamma(s) = \frac{\|\bar{T}'(s)\|}{\|\gamma'(s)\|} = \frac{1}{1/\kappa(t)} = \kappa(t)$$

# 4.5

## i

When $\alpha$ is parameterized by arclength, $\|\alpha'\| = 1$,

$$\begin{aligned} p'(t) &= \alpha'(t) + \frac{1}{\kappa(t)} N'(t) \\ &= \alpha'(t) + \frac{1}{\kappa(t)}(-\kappa(t)T(t)) \\ &= \alpha'(t) - T(t) = \alpha'(t) - \alpha'(t) = 0 \end{aligned}$$

so $p(t)$ is a fixed point.

## ii

$$\alpha(t) - p(t) = -\frac{1}{\kappa(t)} N(t) = -\frac{1}{c} N(t)$$

$$\|\alpha(t) - p(t)\| = \frac{1}{c}$$

# 4.8

$$T = \frac{\gamma'}{\|\gamma'\|} = \frac{\gamma'}{\langle \gamma', \gamma' \rangle^{1/2}}$$

$$T' = \frac{\gamma''\langle \gamma', \gamma' \rangle + \gamma'\langle \gamma', \gamma'' \rangle}{\|\gamma'\|^3}$$

$$
\begin{aligned}
\langle T', iT\rangle &= \left\langle \frac{\gamma''\langle\gamma',\gamma'\rangle + \gamma'\langle\gamma',\gamma''\rangle}{\|\gamma'\|^3}, \frac{i\gamma'}{\|\gamma'\|} \right\rangle \\
&= \frac{\langle\gamma'', i\gamma'\rangle\langle\gamma',\gamma'\rangle + \langle\gamma', i\gamma'\rangle\langle\gamma',\gamma''\rangle}{\|\gamma'\|^4} \\
&= \frac{\langle\gamma'', i\gamma'\rangle\langle\gamma',\gamma'\rangle}{\|\gamma'\|^4} \\
&= \frac{\langle\gamma'', i\gamma'\rangle}{\|\gamma'\|^2}
\end{aligned}
$$

Suppose $\gamma = (\gamma_1, \gamma_2)$, then

$$\gamma' = (\gamma_1', \gamma_2') = \gamma_1' + i\gamma_2', \quad i\gamma' = -\gamma_2' + i\gamma_1' = (-\gamma_2', \gamma_1')$$

$$\gamma'' = (\gamma_1'', \gamma_2'')$$

$$\langle\gamma'', i\gamma'\rangle = \gamma_1'\gamma_2'' - \gamma_2'\gamma_1'' = (\gamma' \times \gamma'')_z = \langle\gamma' \times \gamma'', (0,0,1)\rangle$$

Substitute it back into the formula:

$$\langle T', iT\rangle = \frac{\langle\gamma' \times \gamma'', (0,0,1)\rangle}{\|\gamma'\|^2}$$

$$\bar{\kappa} = \frac{\langle T', iT\rangle}{\|\gamma'\|} = \frac{\langle\gamma' \times \gamma'', (0,0,1)\rangle}{\|\gamma'\|^3}$$

## 4.12

### i

A unit circle oriented clockwise can be parameterized as

$$\alpha(t) = (\cos(-t), \sin(-t))$$

for $0 \le t \le 2\pi$.

$$\alpha'(t) = (\sin(-t), -\cos(-t))$$

$$\alpha''(t) = (-\cos(-t), -\sin(-t))$$

$$\langle\alpha' \times \alpha'', (0,0,1)\rangle = -\sin^2(-t) - \cos^2(-t) = -1$$

$$\bar{\kappa} = \frac{\langle\alpha' \times \alpha'', (0,0,1)\rangle}{\|\alpha'\|^3} = \frac{-1}{1} = -1$$

$$\mathit{total}\ \bar{\kappa}[\alpha] = \int_0^{2\pi} \bar{\kappa} dt = -2\pi$$

$$\mathit{rot}[\alpha] = \frac{\mathit{total}\ \bar{\kappa}[\alpha]}{2\pi} = -1$$

### ii

A unit circle oriented counter-clockwise can be parameterized as

$$\alpha(t) = (\cos(t), \sin(t))$$

for $0 \leq t \leq 2\pi$.

$$\alpha'(t) = (-\sin(t), \cos(t))$$

$$\alpha''(t) = (-\cos(t), -\sin(t))$$

$$\langle \alpha' \times \alpha'', (0,0,1) \rangle = \sin^2(t) + \cos^2(t) = 1$$

$$\bar{\kappa} = \frac{\langle \alpha' \times \alpha'', (0,0,1) \rangle}{\|\alpha'\|^3} = \frac{1}{1} = 1$$

$$total\ \bar{\kappa}[\alpha] = \int_0^{2\pi} \bar{\kappa} dt = 2\pi$$

$$rot[\alpha] = \frac{total\ \bar{\kappa}[\alpha]}{2\pi} = 1$$

### iii

$$\alpha(t) = (\cos t, \sin 2t)$$

$$\alpha'(t) = (-\sin t, 2\cos 2t)$$

$$\alpha''(t) = (-\cos t, -4\sin 2t)$$

$$\langle \alpha' \times \alpha'', (0,0,1) \rangle = 4 \sin t \sin 2t + 2 \cos t \cos 2t$$

$$\bar{\kappa}(t) = \frac{\langle \alpha' \times \alpha'', (0,0,1) \rangle}{\|\alpha'\|^3} = \frac{4 \sin t \sin 2t + 2 \cos t \cos 2t}{\sqrt{\sin^2 t + 4\cos^2 2t}^3}$$

$$total\ \bar{\kappa}[\alpha] = \int_0^{2\pi} \bar{\kappa} dt = \int_0^{2\pi} \frac{4 \sin t \sin 2t + 2 \cos t \cos 2t}{\sqrt{\sin^2 t + 4\cos^2 2t}^3} dt = 0$$

$$rot[\alpha] = \frac{total\ \bar{\kappa}[\alpha]}{2\pi} = 0$$