PHYS-7125

Gravity

Homework Assignments

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This collection is organized in reverse chronological order

PHYS 7125 Homework 5

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1

The total proper time along a curve γ is

$$\begin{aligned}
\tau_{total} &= \int_\gamma d\tau = \int_\gamma \sqrt{-ds^2} \\
&= \int_\gamma \sqrt{-\Big(\frac{2M}{r} - 1\Big)dt^2 + \Big(\frac{2M}{r} - 1\Big)^{-1} dr^2 - r^2 d\Omega^2}
\end{aligned}$$

Inside the event horizon, $2GM/r > 1$, the first and thrid term in the square root are negative, therefore

$$\begin{aligned}
\tau_{total} &< \int_0^{2M} \sqrt{\Big(\frac{2M}{r} - 1\Big)^{-1}} dr \\
&= \left[-\sqrt{2Mr - r^2} - M\tan^{-1}\left(\frac{M-r}{\sqrt{2Mr - r^2}}\right)\right]\Bigg|_0^{2M} \\
&= M\tan^{-1}\left(\frac{M-r}{\sqrt{2Mr-r^2}}\right)\Bigg|_{2M}^{0} = \pi M
\end{aligned}$$

2

a

[insert diagram here]

b

Yes, because the observer has to be massive. Furthermore, if the worldline is lightlike, then the light signal in (c) will never reach the falling observer because the two worldlines would be parallel straight lines in the Kruskal diagram.

c

At constant $r = R$,

$$T = \frac{1}{2}\sinh\left(\frac{t}{4GM}\right), \quad X = \frac{1}{2}\cosh\left(\frac{t}{4GM}\right), \quad 4X^2 - 4T^2 = 1$$

At $t = 0$ on this worldline,

$$T = \frac{1}{2}\sinh 0 = 0, \quad X = \frac{1}{2}\cosh 0 = \frac{1}{2}$$

At the singularity,

$$T = \cosh\left(\frac{t}{4GM}\right), \quad X = \sinh\left(\frac{t}{4GM}\right), \quad T^2 - X^2 = 1$$

A timelike straight line passing through $(1/2, 0)$ can be expressed as

$$X = kT + \frac{1}{2}$$

where $-1 < k < 1$. When the first observer reaches singularity,

$$(1-k^2)T^2 - kT - \frac{5}{4} = 0 \quad \Rightarrow \quad \boxed{T_s = \frac{k + \sqrt{5-4k^2}}{2(1-k^2)}, \quad X_s = \frac{k^2 + k\sqrt{5-4k^2}}{2(1-k^2)} + \frac{1}{2}}$$

To reach this critical point, the photons emitted by the second observer must follow

$$X = -T + T_s + X_s$$

When the photon's worldline in the past intersects with that of the second observer,

$$\cancel{4T^2} + 4(T_s + X_s)^2 - 8(T_s + X_s)T - \cancel{4T^2} = 1$$

$$\Rightarrow T = \frac{1}{2}\sinh\left(\frac{t}{4GM}\right) = \frac{4(T_s+X_s)^2 - 1}{8(T_s+X_s)}, \quad \boxed{t = 4GM\sinh^{-1}\left(\frac{4(T_s+X_s)^2-1}{4(T_s+X_s)}\right)}$$

which is the latest Schwarzschild time that the second observer should send the signal. When the worldline of the first observer is a vertical line, $k = 0$, $t \approx 4.70GM$

3

A perfect fluid is incompressible, therefore $\nabla_\mu \rho = 0$. From the continuity equation,

$$\nabla_\mu(\rho u^\mu) = \rho \nabla_\mu u^\mu = 0 \quad \Rightarrow \quad \nabla_\mu u^\mu = 0$$

Since the covariant divergence of stress-energy tensor must vanish,

$$\begin{aligned}
\nabla_\mu T^\mu{}_\nu &= (\nabla_\mu p)u^\mu u_\nu + (p+\rho)\cancel{(\nabla_\mu u^\mu)}u_\nu + (p+\rho)u^\mu(\nabla_\mu u_\nu) + \delta^\mu_\nu \nabla_\mu p \\
&= u_\nu(\nabla_{\underset{\sim}{u}} p) + (p+\rho)\nabla_{\underset{\sim}{u}} u_\nu + \nabla_\nu p \\
&= 0
\end{aligned}$$

If the free index ν is dropped,

$$(p+\rho)\nabla_{\underset{\sim}{u}}\underset{\sim}{u} = -\nabla p - \underset{\sim}{u}\nabla_{\underset{\sim}{u}} p$$

PHYS 7125 Homework 4

Wenqi He

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1

For a timelike vector t^μ at any point it is always possible to construct an orthonormal frame $\underset{\sim}{\mathbf{e}}_i^\mu$ where $\underset{\sim}{\mathbf{e}}_0^\mu = t^\mu$. (Without loss of generality, t^μ can be assumed to have unit length (-1). In the general case, the results only differ by a positive factor.) In such coordinates, $t^\mu = (1, 0, 0, 0)$, the metric is locally $\eta_{\mu\nu}$, and the components of the electromagnetic energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{\mu_0}\Big[F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big]$$

Considering t^μ is only non-zero in its time component and $T_{\mu\nu}$ is symmetric, we only need to compute

$$T_{0\nu} = \frac{1}{\mu_0}\Big[F_0{}^\alpha F_{\nu\alpha} - \frac{1}{4}\eta_{0\nu}F_{\alpha\beta}F^{\alpha\beta}\Big] =: A + B$$

The first term evaluates to: (Define $\mathbf{S} := \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$)

$$A = \frac{1}{\mu_0}\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 0 \\ E_x/c \\ E_y/c \\ E_z/c \end{pmatrix} = \begin{pmatrix} \epsilon_0 E^2 \\ -\frac{1}{\mu_0}(E_yB_z - E_zB_y)/c \\ -\frac{1}{\mu_0}(E_zB_x - E_xB_z)/c \\ -\frac{1}{\mu_0}(E_xB_y - E_yB_x)/c \end{pmatrix} = \begin{pmatrix} \epsilon_0 E^2 \\ -S_x/c \\ -S_y/c \\ -S_z/c \end{pmatrix}$$

The second term:

$$B = -\frac{1}{4\mu_0}(-2E^2/c^2 + 2B^2)\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\Big(-\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad\Rightarrow\quad T_{0\nu} = \begin{pmatrix} \frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) \\ -S_x/c \\ -S_y/c \\ -S_z/c \end{pmatrix}$$

In matrix form, the relevant components of the energy-momentum tensor are:

$$T_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) & -S_x/c & -S_y/c & -S_z/c \\ -S_x/c & \cdots & \cdots & \cdots \\ -S_y/c & \cdots & \cdots & \cdots \\ -S_z/c & \cdots & \cdots & \cdots \end{pmatrix}$$

(i) The weak energy condition is obviously satisfied:

$$T_{\mu\nu}t^\mu t^\nu = T_{00} = \frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) \geq 0$$

(ii)

$$T_{\mu\nu}t^\mu = T_{0\nu} = \left(\frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big), -S_x/c, -S_y/c, -S_z/c\right)$$

$$T^\nu{}_\alpha t^\alpha = T^\nu{}_0 = g^{\nu\nu}\cdot T_{\nu 0} = \left(-\frac{1}{2}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big), -S_x/c, -S_y/c, -S_z/c\right)$$

Using Lagrange's identity for cross products,

$$\begin{aligned}
(T_{\mu\nu}t^\mu)(T^\nu{}_\alpha t^\alpha) &= -\frac{1}{4}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big)^2 + \frac{\|\mathbf{S}\|^2}{c^2} \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big)^2 + \frac{\epsilon_0}{\mu_0}\|\mathbf{E}\times\mathbf{B}\|^2 \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big)^2 + \frac{\epsilon_0}{\mu_0}\Big(E^2B^2 - (\mathbf{E}\cdot\mathbf{B})^2\Big) \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 - \frac{1}{\mu_0}B^2\Big)^2 - \frac{\epsilon_0}{\mu_0}(\mathbf{E}\cdot\mathbf{B})^2 \le 0
\end{aligned}$$

Thus the dominant energy condition is also satisfied.

2

a

The only non-vanishing components of $\Gamma^\mu_{0\nu}$ are

$$\Gamma^0_{0i} = \frac{Mx^i}{(1-2M/r)r^3}, \quad \Gamma^i_{00} = \frac{Mx^i}{(1+2M/r)r^3} \qquad (i = 1,2,3)$$

The geodesic equation can be simplified as

$$\begin{aligned}
\frac{dp_0}{d\lambda} &= \sum_i \left(\frac{Mx^i}{(1-2M/r)r^3} p_0 p^i + \frac{Mx^i}{(1+2M/r)r^3} p^0 p_i \right) \\
&= \sum_i \left(g_{00}\frac{Mx^i}{(1-2M/r)r^3} + g_{ii}\frac{Mx^i}{(1+2M/r)r^3} \right) p^0 p^i \\
&= \sum_i \left(-\frac{Mx^i}{r^3} + \frac{Mx^i}{r^3} \right) p^0 p^i = 0
\end{aligned}$$

b

No. $p^0 = g^{0\nu}p_\nu = g^{00}p_0 = -(1-2M/r)^{-1}p_0$, which is not constant unless r is constant.

c

For the atom at rest on the surface of the sun, $dx^i = 0$, $r = R$,

$$d\tau^2 = -ds^2 = (1-2M/R)dt^2 \quad \Rightarrow \quad u^0 = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-2M/R}}$$

d

For both the atom and the distant observer, $dx^i = 0$,

$$d\tau^2 = -ds^2 = (1-2M/r)dt^2$$

$$\Rightarrow \quad u^\mu = \frac{dx^\mu}{d\tau} = (dt/d\tau, 0, 0, 0) = ((1-2M/r)^{-1/2}, 0, 0, 0)$$

The photon energy observed at both locations can be expressed as

$$E = g_{\mu\nu} p^\mu u^\nu = (1 - 2M/r)^{-1/2} g_{\mu\nu} p^\mu K^\nu$$

where $K^\nu = (1, 0, 0, 0)$ is a Killing vector as the metric has no time dependence; therefore $g_{\mu\nu} p^\mu K^\nu$ is conserved along the photon's world line, which is a geodesic as stated in the problem. Then,

$$\frac{\lambda_r}{\lambda_e} = \frac{hc/\lambda_e}{hc/\lambda_r} = \frac{E_e}{E_r} = \frac{(1 - 2M/R)^{-1/2}}{\lim_{r\to\infty}(1 - \cancel{2M/r})^{-1/2}} = 1 + \frac{M}{R} + \mathcal{O}\left(\frac{M^2}{R^2}\right)$$

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{\lambda_r}{\lambda_e} - 1 = \frac{M}{R} + \mathcal{O}\left(\frac{M^2}{R^2}\right)$$

PHYS 7125 Homework 3

Wenqi He

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1

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

Contracting twice:

$$g^{\mu\lambda} g^{\nu\sigma} \nabla_\lambda R_{\rho\sigma\mu\nu} + g^{\mu\lambda} g^{\nu\sigma} \nabla_\rho R_{\sigma\lambda\mu\nu} + g^{\mu\lambda} g^{\nu\sigma} \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} g^{\nu\sigma} R_{\sigma\rho\nu\mu} \Big) - \nabla_\rho \Big(g^{\mu\lambda} g^{\nu\sigma} R_{\lambda\sigma\mu\nu} \Big) + \nabla_\sigma \Big(g^{\mu\lambda} g^{\nu\sigma} R_{\lambda\rho\mu\nu} \Big) = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) - \nabla_\rho \Big(g^{\nu\sigma} R_{\sigma\nu} \Big) + \nabla_\sigma \Big(g^{\nu\sigma} R_{\rho\nu} \Big) = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) - \nabla_\rho R + \nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) - \nabla_\lambda \Big(\frac{1}{2} \delta^\lambda_\rho R \Big) = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} R_{\rho\mu} \Big) - \nabla_\lambda \Big(\frac{1}{2} g^{\mu\lambda} g_{\rho\mu} R \Big) = 0$$

$$\nabla_\lambda \Big[g^{\mu\lambda} \Big(R_{\rho\mu} - \frac{1}{2} g_{\rho\mu} R \Big) \Big] = 0$$

$$\nabla_\lambda \Big(g^{\mu\lambda} G_{\rho\mu} \Big) = \nabla_\lambda \Big(g^{\lambda\mu} G_{\mu\rho} \Big) = 0$$

$$\nabla_\lambda G^\lambda{}_\rho = 0$$

2

The only non-vanishing components of the metric are

$$g_{\psi\psi} = 1, \quad g_{\theta\theta} = \sin^2\psi, \quad g_{\phi\phi} = \sin^2\psi \sin^2\theta$$

Since the matrix is diagonal,

$$g^{\psi\psi} = 1, \quad g^{\theta\theta} = 1/\sin^2\psi, \quad g^{\phi\phi} = 1/\sin^2\psi \sin^2\theta$$

The only non-vanishing first derivatives of the metric components are

$$g_{\theta\theta,\psi} = 2\sin\psi\cos\psi, \quad g_{\phi\phi,\psi} = 2\sin\psi\cos\psi\sin^2\theta, \quad g_{\phi\phi,\theta} = 2\sin^2\psi\sin\theta\cos\theta$$

a

The only non-vanishing Christoffel symbols are

$$\Gamma^{\theta}_{\theta\psi} = \Gamma^{\theta}_{\psi\theta} = \frac{1}{2} g^{\theta\theta} g_{\theta\theta,\psi} = \frac{2\sin\psi\cos\psi}{2\sin^2\psi} = \cot\psi$$

$$\Gamma^{\psi}_{\theta\theta} = -\frac{1}{2} g^{\psi\psi} g_{\theta\theta,\psi} = -\sin\psi\cos\psi$$

$$\Gamma^{\phi}_{\phi\psi} = \Gamma^{\phi}_{\psi\phi} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\psi} = \frac{2\sin\psi\cos\psi\sin^2\theta}{2\sin^2\psi\sin^2\theta} = \cot\psi$$

$$\Gamma^{\psi}_{\phi\phi} = -\frac{1}{2} g^{\psi\psi} g_{\phi\phi,\psi} = -\sin\psi\cos\psi\sin^2\theta$$

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\theta} = \frac{2\sin^2\psi\sin\theta\cos\theta}{2\sin^2\psi\sin^2\theta} = \cot\theta$$

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2} g^{\theta\theta} g_{\phi\phi,\theta} = -\frac{2\sin^2\psi\sin\theta\cos\theta}{2\sin^2\psi} = -\sin\theta\cos\theta$$

b

There are $\frac{1}{12} \cdot 3^2 \cdot (3^2 - 1) = 6$ independent components of the Riemann tensor:

$$\begin{aligned}
R_{\psi\theta\psi\theta} &= g_{\psi\psi} R^{\psi}_{\theta\psi\theta} = \sin^2\psi - \cos^2\psi - 0 + 0 - (-\cos^2\psi) = \sin^2\psi \\
R_{\psi\theta\psi\phi} &= g_{\psi\psi} R^{\psi}_{\theta\psi\phi} = 0 - 0 + 0 - 0 = 0 \\
R_{\psi\theta\theta\phi} &= g_{\psi\psi} R^{\psi}_{\theta\theta\phi} = 0 - 0 + 0 - 0 = 0 \\
R_{\psi\phi\psi\phi} &= g_{\psi\psi} R^{\psi}_{\phi\psi\phi} = (\sin^2\psi - \cos^2\psi)\sin^2\theta - 0 + 0 - (-\cos^2\psi\sin^2\theta) = \sin^2\psi\sin^2\theta \\
R_{\psi\phi\theta\phi} &= g_{\psi\psi} R^{\psi}_{\phi\theta\phi} = -2\sin\psi\cos\psi\sin\theta\cos\theta - 0 + \sin\psi\cos\psi\sin\theta\cos\theta - (-\sin\psi\cos\psi\sin\theta\cos\theta) = 0 \\
R_{\theta\phi\theta\phi} &= g_{\theta\theta} R^{\theta}_{\phi\theta\phi} = \sin^2\psi\Big[\sin^2\theta - \cos^2\theta - 0 + (-\cos^2\psi\sin^2\theta) - (-\cos^2\theta)\Big] = \sin^4\psi\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
R_{\psi\psi} &= g^{\theta\theta} R_{\theta\psi\theta\psi} + g^{\phi\phi} R_{\phi\psi\phi\psi} = 2 \\
R_{\psi\theta} &= g^{\phi\phi} R_{\phi\psi\phi\theta} = 0 \\
R_{\psi\phi} &= g^{\theta\theta} R_{\theta\psi\theta\phi} = 0 \\
R_{\theta\theta} &= g^{\psi\psi} R_{\psi\theta\psi\theta} + g^{\phi\phi} R_{\phi\theta\phi\theta} = \sin^2\psi + \sin^2\psi = 2\sin^2\psi \\
R_{\theta\phi} &= g^{\psi\psi} R_{\psi\theta\psi\phi} = 0 \\
R_{\phi\phi} &= g^{\psi\psi} R_{\psi\phi\psi\phi} + g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta = 2\sin^2\psi\sin^2\theta
\end{aligned}$$

$$R = g^{\psi\psi} R_{\psi\psi} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = 2 + 2 + 2 = 6$$

PHYS 7125 Homework 2

Wenqi He

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1

The local flatness property states that for each point p on the manifold there exists a change of coordinates such that the metric $g_{\mu\nu}$ can be transformed into $g_{\mu'\nu'}$ that satisfies: (i) $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ and (ii) $g_{\mu'\nu',\sigma} = 0$ at point p. This can be proven by a Taylor expansion of $g_{\mu'\nu'}$ to the first order:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$$

$$= \left(x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu} \right)\Big|_p + \left(x^\mu_{,\mu'\lambda} x^\nu_{,\nu'} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'\lambda} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu,\lambda} \right)\Big|_p \epsilon + O(\epsilon^2)$$

The requirement translates to:

$$\left(x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu} \right)\Big|_p = \eta_{\mu'\nu'}$$

$$\left(x^\mu_{,\mu'\lambda} x^\nu_{,\nu'} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'\lambda} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu,\lambda} \right)\Big|_p = 0$$

The first equation has 16 variables in $\partial x^\mu / \partial x^{\mu'}$ and 10 equations, one for each indepedent entry of the metric. The remaining 6 degrees of freedom exactly matches the dimension of the Lorentz group, under which the metric is preserved. With $\partial x^\mu / \partial x^{\mu'}$ determined, the second equation will only have $4 \cdot 10 = 40$ variables in $\partial^2 x^\mu / \partial x^{\mu'} \partial x^\lambda$ since partial derivatives commute. Coincidentally, as metric is always symmetric, there are $10 \cdot 4 = 40$ equations corresponding to the independent entries of $g_{\mu\nu,\lambda}$. The system is thus uniquely determined, and therefore such transformation is always possible.

2

a

$$\delta^\alpha_{\sigma,\gamma} = \left(g^{\alpha\beta} g_{\beta\sigma} \right)_{,\gamma} = g^{\alpha\beta}_{,\gamma} g_{\beta\sigma} + g^{\alpha\beta} g_{\beta\sigma,\gamma} = 0 \quad \Rightarrow \quad g^{\alpha\beta}_{,\gamma} g_{\beta\sigma} = -g^{\alpha\beta} g_{\beta\sigma,\gamma}$$

Multiplying by inverse metric,

$$g^{\alpha\beta}_{,\gamma} g_{\beta\sigma} g^{\sigma\lambda} = g^{\alpha\beta}_{,\gamma} \delta^\lambda_\beta = g^{\alpha\lambda}_{,\gamma} = -g^{\sigma\lambda} g^{\alpha\beta} g_{\beta\sigma,\gamma}$$

b

From the two identities we can derive the formula:

$$\begin{aligned}
\frac{d}{d\epsilon}\det(A) &= \lim_{\epsilon\to 0}\frac{\det(A+\epsilon\frac{d}{d\epsilon}A+\cancel{O(\epsilon^2)})-\det(A)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\det(A(I+\epsilon A^{-1}\frac{d}{d\epsilon}A))-\det(A)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\det(A)\det(I+\epsilon A^{-1}\frac{d}{d\epsilon}A)-\det(A)}{\epsilon} \\
&= \det(A)\lim_{\epsilon\to 0}\frac{\det(I+\epsilon A^{-1}\frac{d}{d\epsilon}A)-1}{\epsilon} \\
&= \det(A)\lim_{\epsilon\to 0}\frac{1+\epsilon\,\mathrm{tr}(A^{-1}\frac{d}{d\epsilon}A)+\cancel{O(\epsilon^2)}-1}{\epsilon} \\
&= \det(A)\,\mathrm{tr}(A^{-1}\frac{d}{d\epsilon}A)
\end{aligned}$$

Apply the formula to $g=\det g_{\mu\nu}$, replacing $d/d\epsilon$ with ∂_α

$$g_{,\alpha}=g\cdot\mathrm{tr}(g^{\sigma\mu}g_{\mu\nu,\alpha})=gg^{\nu\mu}g_{\mu\nu,\alpha}$$

c

$$\begin{aligned}
RHS &= -(-g)^{-1/2}\Big[g^{\alpha\beta}(-g)^{1/2}\Big]_{,\beta} \\
&= -(-g)^{-1/2}\Big[g^{\alpha\beta}{}_{,\beta}(-g)^{1/2}+g^{\alpha\beta}(-g)^{1/2}_{,\beta}\Big] \\
&= -(-g)^{-1/2}\Big[g^{\alpha\beta}{}_{,\beta}(-g)^{1/2}-\frac{1}{2}g^{\alpha\beta}(-g)^{-1/2}g_{,\beta}\Big] \\
&= -g^{\alpha\beta}{}_{,\beta}+\frac{1}{2}g^{\alpha\beta}(-g)^{-1}g_{,\beta} \\
&= g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}+\frac{1}{2}g^{\alpha\beta}(-g)^{-1}gg^{\mu\nu}g_{\mu\nu,\beta} \\
&= g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}-\frac{1}{2}g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta} \\
&= \frac{1}{2}\Big(g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}+g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}-g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\Big) \\
&= \frac{1}{2}\Big(g^{\mu\nu}g^{\beta\alpha}g_{\mu\beta,\nu}+g^{\nu\beta}g^{\mu\alpha}g_{\nu\mu,\beta}-g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\Big) \\
&= \frac{1}{2}\Big(g^{\mu\nu}g^{\beta\alpha}g_{\mu\beta,\nu}+g^{\nu\mu}g^{\beta\alpha}g_{\nu\beta,\mu}-g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\Big) \\
&= g^{\mu\nu}\cdot\frac{1}{2}g^{\alpha\beta}\Big(g_{\beta\mu,\nu}+g_{\beta\nu,\mu}-g_{\mu\nu,\beta}\Big) \\
&= g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}=LHS
\end{aligned}$$

d

$$\begin{aligned}
LHS &= A^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}_{\alpha\lambda} A^{\lambda} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} g^{\alpha\beta} \Big(g_{\beta\alpha,\lambda} + g_{\beta\lambda,\alpha} - g_{\alpha\lambda,\beta} \Big) A^{\lambda} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} \Big(g^{\alpha\beta} g_{\beta\alpha,\lambda} + g^{\alpha\beta} g_{\beta\lambda,\alpha} - g^{\alpha\beta} g_{\alpha\lambda,\beta} \Big) A^{\lambda} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} \Big(g^{\alpha\beta} g_{\beta\alpha,\lambda} + \cancel{g^{\alpha\beta} g_{\beta\lambda,\alpha}} - \cancel{g^{\beta\alpha} g_{\beta\lambda,\alpha}} \Big) A^{\lambda} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} g^{\alpha\beta} g_{\beta\alpha,\lambda} A^{\lambda}
\end{aligned}$$

$$\begin{aligned}
RHS &= (-g)^{-1/2} \Big[(-g)^{1/2} A^{\alpha} \Big]_{,\alpha} \\
&= (-g)^{-1/2} \Big[(-g)^{1/2} A^{\alpha}{}_{,\alpha} + (-g)^{1/2}_{,\alpha} A^{\alpha} \Big] \\
&= A^{\alpha}{}_{,\alpha} - \frac{1}{2} (-g)^{-1/2} (-g)^{-1/2} g_{,\alpha} A^{\alpha} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} g^{-1} g g^{\mu\nu} g_{\mu\nu,\alpha} A^{\alpha} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha} A^{\alpha} = LHS
\end{aligned}$$

e

$$\epsilon_{\alpha\beta\gamma\delta;\mu} = \Big((-g)^{1/2} \tilde{\epsilon}_{\alpha\beta\gamma\delta} \Big)_{;\mu} = (-g)^{1/2}_{,\mu} \tilde{\epsilon}_{\alpha\beta\gamma\delta} - (-g)^{1/2} \Big[\Gamma^{\lambda}_{\alpha\mu} \tilde{\epsilon}_{\lambda\beta\gamma\delta} + \Gamma^{\lambda}_{\beta\mu} \tilde{\epsilon}_{\alpha\lambda\gamma\delta} + \Gamma^{\lambda}_{\gamma\mu} \tilde{\epsilon}_{\alpha\beta\lambda\delta} + \Gamma^{\lambda}_{\delta\mu} \tilde{\epsilon}_{\alpha\beta\gamma\lambda} \Big]$$

If there are repeated indices, w.l.o.g. we can suppose $\alpha = \beta$, then the first term outside the brackets and the last two terms inside the bracket vanish, and also

$$\Gamma^{\lambda}_{\alpha\mu} \tilde{\epsilon}_{\lambda\beta\gamma\delta} + \Gamma^{\lambda}_{\beta\mu} \tilde{\epsilon}_{\alpha\lambda\gamma\delta} = \Gamma^{\lambda}_{\alpha\mu} \tilde{\epsilon}_{\lambda\alpha\gamma\delta} + \Gamma^{\lambda}_{\alpha\mu} \tilde{\epsilon}_{\alpha\lambda\gamma\delta} = 0$$

So $\epsilon_{\alpha\beta\gamma\delta;\mu} = 0$ in this case. Now consider the case where all indices are distinct, then all $\tilde{\epsilon}_{\lambda\beta\gamma\delta}$ (λ is a free index) vanish unless $\lambda = \alpha$. Same argument applies to other slots as well. W.l.o.g., suppose $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3$. The first term on the right-hand side then evaluates to

$$-\frac{1}{2} (-g)^{-1/2} g_{,\mu}$$

Expand the the second term:

$$\begin{aligned}
\cdots &= \frac{1}{2} (-g)^{1/2} g^{\lambda\sigma} \Big(g_{\sigma 0,\mu} + g_{\sigma\mu,0} - g_{0\mu,\sigma} \Big) \tilde{\epsilon}_{\lambda 123} + \frac{1}{2} (-g)^{1/2} g^{\lambda\sigma} \Big(g_{\sigma 1,\mu} + g_{\sigma\mu,1} - g_{1\mu,\sigma} \Big) \tilde{\epsilon}_{0\lambda 23} \\
&\quad + \frac{1}{2} (-g)^{1/2} g^{\lambda\sigma} \Big(g_{\sigma 2\mu} + g_{\sigma\mu,2} - g_{2\mu,\sigma} \Big) \tilde{\epsilon}_{01\lambda 3} + \frac{1}{2} (-g)^{1/2} g^{\lambda\sigma} \Big(g_{\sigma 3,\mu} + g_{\sigma\mu,3} - g_{3\mu,\sigma} \Big) \tilde{\epsilon}_{012\lambda} \\
&= \frac{1}{2} (-g)^{1/2} g^{0\sigma} \Big(g_{\sigma 0,\mu} + g_{\sigma\mu,0} - g_{0\mu,\sigma} \Big) + \frac{1}{2} (-g)^{1/2} g^{1\sigma} \Big(g_{\sigma 1,\mu} + g_{\sigma\mu,1} - g_{1\mu,\sigma} \Big) \\
&\quad + \frac{1}{2} (-g)^{1/2} g^{2\sigma} \Big(g_{\sigma 2,\mu} + g_{\sigma\mu,2} - g_{2\mu,\sigma} \Big) + \frac{1}{2} (-g)^{1/2} g^{3\sigma} \Big(g_{\sigma 3,\mu} + g_{\sigma\mu,3} - g_{3\mu,\sigma} \Big) \\
&= \frac{1}{2} (-g)^{1/2} \Big(g^{\lambda\sigma} g_{\lambda\sigma,\mu} + \cancel{g^{\lambda\sigma} g_{\sigma\mu,\lambda}} - \cancel{g^{\lambda\sigma} g_{\lambda\mu,\sigma}} \Big) = -\frac{1}{2} (-g)^{-1/2} g_{,\mu}
\end{aligned}$$

The last step above reorganized the terms into a summation over dummy λ and invoked the result from (b). Two terms on the right-hand side cancel out, so $\epsilon_{\alpha\beta\gamma\delta;\mu} = 0$.

3

a

Since $u_\alpha u^\alpha = -1$,

$$(P_{\alpha\beta}v^\beta)u^\alpha = g_{\alpha\beta}v^\beta u^\alpha + u_\alpha u_\beta v^\beta u^\alpha = v^\beta u_\beta - u_\beta v^\beta = 0$$

b

From (a), $u^\beta v_{\perp_\beta} = u_\beta v_\perp^\beta = 0$, therefore

$$P_{\alpha\beta}v_\perp^\beta = g_{\alpha\beta}v_\perp^\beta + u_\alpha u_\beta v_\perp^\beta = g_{\alpha\beta}v_\perp^\beta + 0 = v_{\perp\alpha}$$

c

$$P_{\alpha\beta} := g_{\alpha\beta} - (q_\lambda q^\lambda)^{-1} q_\alpha q_\beta$$

Proof: Carrying out the same calculations:

$$(P_{\alpha\beta}v^\beta)q^\alpha = g_{\alpha\beta}v^\beta q^\alpha - \cancel{(q_\lambda q^\lambda)^{-1}}\cancel{q_\alpha} q_\beta v^\beta \cancel{q^\alpha} = v^\beta q_\beta - q_\beta v^\beta = 0$$

$$P_{\alpha\beta}v_\perp^\beta = g_{\alpha\beta}v_\perp^\beta - (q_\lambda q^\lambda)^{-1} q_\alpha \cancel{q_\beta v_\perp^\beta} = v_{\perp\alpha}$$

d

Since the norm of a null vector is zero, the orthogonal projection cannot be constructed by substracting a parallel projection which involves a division by the norm, therefore the projection can only be expressed explicitly as an expansion w.r.t. the basis vectors orthogonal to $\underset{\sim}{k}$. Apart from $\underset{\sim}{k}$, which is orthogonal to itself, there must exist two additional linearly independent vectors to match the dimensionality. It can be shown that (i) orthogonal null vectors are colinear and (ii) null vectors cannot be orthogonal to time-like vectors, therefore the two vectors must be space-like, and we can perform the Gram-Schmidt process like above to obtain two orthonormal space-like vectors. Finally, $k_\alpha k_\beta$ term must be excluded because if the projection produces a k_μ component, applying the projection twice will then eliminate it, violating $P(v_\perp) = v_\perp$. Therefore the projection tensor is

$$P_{\alpha\beta} = e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}$$

Where $\underset{\sim}{e}_{(i)}$ are the orthonormal space-like basis vectors. It can be quickly verified that

$$\Big(e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}\Big)v^\beta k^\alpha = 0$$

and also

$$\begin{aligned}\Big(e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}\Big)v_\perp^\beta &= \Big(e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}\Big)\Big(e_{(1)}^\beta e_{(1)\sigma} + e_{(2)}^\beta e_{(2)\sigma}\Big)v^\sigma \\ &= \Big(e_{(1)\alpha}e_{(1)\sigma} + e_{(2)\alpha}e_{(2)\sigma}\Big)v^\sigma = v_{\perp\alpha}\end{aligned}$$

However the choice of $\{\underset{\sim}{e}_{(i)}\}$ is not unique, which means that P in general will not be unique. As an example, suppose coordinates are chosen such that space-time is locally flat and consider $\underset{\sim}{k} = (1,1,0,0)$, $\underset{\sim}{e}_{(1)} = (1,1,1,0)$, $\underset{\sim}{e}_{(2)} = (1,1,0,1)$, $P_{\alpha\beta} = e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}$ only has integer components; however, we can also choose $\underset{\sim}{e}'_{(1)} = (1,1,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, $\underset{\sim}{e}'_{(2)} = (1,1,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$, then $P'_{\alpha\beta} = e'_{(1)\alpha}e'_{(1)\beta} + e'_{(2)\alpha}e'_{(2)\beta}$ would have irrational components.

4

a

$$\nabla_{\underset{\sim}{u}} u_\mu = \frac{dx^\alpha}{d\tau}\nabla_\alpha u_\mu = \frac{dx^\alpha}{d\tau}\left(\partial_\alpha u_\mu - \Gamma^\beta_{\alpha\mu}u_\beta\right) = \frac{du_\mu}{d\tau} - \Gamma^\beta_{\alpha\mu}u^\alpha u_\beta = 0$$

b

Since the connection is metric-compatible,

$$\nabla_\alpha g^{\mu\nu} = \partial_\alpha g^{\mu\nu} + \Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} + \Gamma^\nu_{\alpha\lambda}g^{\mu\lambda} = 0 \Rightarrow \quad \partial_\alpha g^{\mu\nu} = -\Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} - \Gamma^\nu_{\alpha\lambda}g^{\mu\lambda}$$
$$\Rightarrow \quad \frac{dg^{\mu\nu}}{d\tau} = u^\alpha\partial_\alpha g^{\mu\nu} = -u^\alpha\Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} - u^\alpha\Gamma^\nu_{\alpha\lambda}g^{\mu\lambda}$$

Raising the indices of the equation from (a) recovers the original form of geodesic equation:

$$\begin{aligned}
&g^{\mu\nu}\frac{du_\mu}{d\tau} - g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta \\
=& \frac{d}{d\tau}\left(g^{\mu\nu}u_\mu\right) - u_\mu\frac{dg^{\mu\nu}}{d\tau} - g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta \\
=& \frac{d}{d\tau}\left(g^{\mu\nu}u_\mu\right) + u_\mu u^\alpha\Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} + u_\mu u^\alpha\Gamma^\nu_{\alpha\lambda}g^{\mu\lambda} - g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta \\
=& \frac{d}{d\tau}\left(g^{\mu\nu}u_\mu\right) + \cancel{u_\beta u^\alpha\Gamma^\beta_{\alpha\mu}g^{\mu\nu}} + u_\mu u^\alpha\Gamma^\nu_{\alpha\lambda}g^{\mu\lambda} - \cancel{g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta} \\
=& \frac{du^\nu}{d\tau} + \Gamma^\nu_{\alpha\lambda}u^\lambda u^\alpha = 0
\end{aligned}$$

Without invoking metric compatibility, we can also check that the covariant derivatives of $u_\mu u^\mu$, which is a scalar, reduce to ordinary derivatives:

$$\begin{aligned}
\nabla_{\underset{\sim}{u}}\left(u_\mu u^\mu\right) &= u_\mu\nabla_{\underset{\sim}{u}}u^\mu + u^\mu\nabla_{\underset{\sim}{u}}u_\mu = u_\mu\left(\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta\right) + u^\mu\left(\frac{du_\mu}{d\tau} - \Gamma^\beta_{\alpha\mu}u^\alpha u_\beta\right) \\
&= u_\mu\frac{du^\mu}{d\tau} + \cancel{\Gamma^\mu_{\alpha\beta}u_\mu u^\alpha u^\beta} + u^\mu\frac{du_\mu}{d\tau} - \cancel{\Gamma^\beta_{\alpha\mu}u^\mu u^\alpha u_\beta} = \frac{d}{d\tau}\left(u_\mu u^\mu\right)
\end{aligned}$$

c

Suppose λ is an affine parameter for a null-geodesic and $\sigma(\lambda)$ non-affine:

$$\begin{aligned}
\nabla_{\underset{\sim}{u}}u^\mu &= \nabla_{\frac{d}{d\sigma}}\frac{dx^\mu}{d\sigma} = \nabla_{\frac{d\lambda}{d\sigma}\frac{d}{d\lambda}}\left(\frac{d\lambda}{d\sigma}\frac{dx^\mu}{d\lambda}\right) = \frac{d\lambda}{d\sigma}\nabla_{\frac{d}{d\lambda}}\left(\frac{d\lambda}{d\sigma}\frac{dx^\mu}{d\lambda}\right) \\
&= \frac{d\lambda}{d\sigma}\left(\frac{d}{d\lambda}\frac{d\lambda}{d\sigma}\right)\frac{dx^\mu}{d\lambda} + \frac{d\lambda}{d\sigma}\frac{d\lambda}{d\sigma}\cancel{\nabla_{\frac{d}{d\lambda}}\frac{dx^\mu}{d\lambda}} \\
&= \left(\frac{d}{d\lambda}\frac{d\lambda}{d\sigma}\right)u^\mu = -\kappa(\lambda)u^\mu
\end{aligned}$$

where κ is some function of λ. The second term above vanishes because λ is affine.

PHYS 7125 Homework 1

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1

a

If $\Delta s^2 = 0$ for a particle then $\Delta t^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, which means that it travels at the speed of light in x^α coordinates. Since the speed of light is constant in all reference frames, under another coordinate system $x^{\alpha'}$ it still travels at the speed of light, that is, $\Delta t'^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2$ and therefore $\Delta s'^2 = 0$.

b

Expressing Q in terms of Δx^α:

$$Q = \eta_{\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'} = \eta_{\alpha'\beta'}\Lambda^{\alpha'}{}_\alpha \Delta x^\alpha \Lambda^{\beta'}{}_\beta \Delta x^\beta = \left(\Lambda^{\alpha'}{}_\alpha \Lambda^{\beta'}{}_\beta \eta_{\alpha'\beta'}\right)\Delta x^\alpha \Delta x^\beta = \phi_{\alpha\beta}\Delta x^\alpha \Delta x^\beta$$

c

On the intersection $Q = 0$ identically. By spherical symmetry, Q must be invariant under (spatial) reflections, therefore all cross terms, which would change signs under a reflection, are eliminated; Q must also be invariant under (spatial) rotations, so the remaining $\Delta x^2, \Delta y^2, \Delta z^2$ must have the same coefficient as they are indistinguishable. The general form satisfying these requirements is

$$Q = c_1 \Delta t^2 + c_2(\Delta x^2 + \Delta y^2 + \Delta z^2)$$

d

Since the intersection lies on the light cone, $\Delta t^2 = \Delta x^2 + 0 + 0$,

$$Q = c_1\Delta t^2 + c_2 \Delta x^2 = (c_1 + c_2)\Delta t^2 = 0 \quad \Rightarrow \quad c_1 = -c_2$$
$$\Rightarrow \quad Q = c_2(-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2) = c_2 \eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta$$

e

The constant c_2 applies to transformations between any two frames, which certainly include the trivial transformation from one frame to itself, therefore c_2 must equal 1.

2

a

Renaming the dummy indices,

$$A_{\mu\nu}S^{\mu\nu} = A_{\nu\mu}S^{\nu\mu} = (-A_{\mu\nu})S^{\mu\nu} \quad \Rightarrow \quad A_{\mu\nu}S^{\mu\nu} = 0$$

b

Using the same trick as above,

$$V^{\nu\mu}A_{\mu\nu} = V^{\mu\nu}A_{\nu\mu} = -V^{\mu\nu}A_{\mu\nu} \quad \Rightarrow \quad \frac{1}{2}\Big(V^{\mu\nu}A_{\mu\nu} - V^{\nu\mu}A_{\mu\nu}\Big) = V^{\mu\nu}A_{\mu\nu}$$

$$V^{\nu\mu}S_{\mu\nu} = V^{\mu\nu}S_{\nu\mu} = V^{\mu\nu}S_{\mu\nu} \quad \Rightarrow \quad \frac{1}{2}\Big(V^{\mu\nu}S_{\mu\nu} + V^{\nu\mu}S_{\mu\nu}\Big) = V^{\mu\nu}S_{\mu\nu}$$

c

A tensor acting on vectors and covectors produces a scalar, which is invariant under transformations:

$$\begin{aligned} T^{\alpha'}{}_{\beta'}{}^{\gamma'}u_{\alpha'}v^{\beta'}w_{\gamma'} &= T^{\alpha}{}_{\beta}{}^{\gamma}u_{\alpha}v^{\beta}w_{\gamma} \\ &= T^{\alpha}{}_{\beta}{}^{\gamma}\Lambda^{\alpha'}{}_{\alpha}u_{\alpha'}\Lambda^{\beta}{}_{\beta'}v^{\beta'}\Lambda^{\gamma'}{}_{\gamma}w_{\gamma'} \\ &= \Big(\Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta}{}_{\beta'}\Lambda^{\gamma'}{}_{\gamma}T^{\alpha}{}_{\beta}{}^{\gamma}\Big)u_{\alpha'}v^{\beta'}w_{\gamma'} \end{aligned}$$

Since this holds for any vectors and covectors, $T^{\alpha'}{}_{\beta'}{}^{\gamma'} = \Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta}{}_{\beta'}\Lambda^{\gamma'}{}_{\gamma}T^{\alpha}{}_{\beta}{}^{\gamma}$.

d

$$g_{\alpha\beta}g^{\beta\sigma}g^{\alpha\gamma} = \delta^{\sigma}_{\alpha}g^{\alpha\gamma} = g^{\sigma\gamma}$$

$$g_{\sigma\beta}g_{\gamma\alpha}g^{\alpha\beta} = g_{\sigma\beta}\delta^{\beta}_{\gamma} = g_{\sigma\gamma}$$

$$g^{\alpha}{}_{\beta} = g^{\alpha\sigma}g_{\sigma\beta} = g_{\beta\sigma}g^{\sigma\alpha} = \delta^{\alpha}_{\beta}$$

3

a

$$X^{\mu}{}_{\nu} = X^{\mu\gamma}g_{\gamma\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

b

$$X_{\mu}{}^{\nu} = g_{\mu\gamma}X^{\gamma\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$$

c

$$X^{(\mu\nu)} = \frac{1}{2}\Big(X^{\mu\nu} + X^{\nu\mu}\Big) = \begin{pmatrix} 2 & -1/2 & 0 & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{pmatrix}$$

d

$$X_{\mu\nu} = X_\mu{}^\gamma g_{\gamma\nu} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

$$X_{[\mu\nu]} = \frac{1}{2}\Big(X_{\mu\nu} - X_{\nu\mu}\Big) = \begin{pmatrix} 0 & -1/2 & -1 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1 & -1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{pmatrix}$$

e

$$X^\lambda{}_\lambda = -2 + 0 + 0 - 2 = -4$$

f

$$v^\mu v_\mu = g_{\mu\nu} v^\mu v^\nu = -(-1)^2 + 2^2 + 0^2 + (-2)^2 = 7$$

g

$$v_\mu = g_{\mu\nu} v^\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & -2 \end{pmatrix}$$

$$v_\mu X^{\mu\nu} = \begin{pmatrix} 1 & 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 5 \\ 7 \end{pmatrix}$$