CX 4640 Homework 1

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1.5

Suppose the changes in input data are Δx and Δy.

$$\begin{aligned}
\text{relative change in input} &= \frac{||(x+\Delta x, y+\Delta y)-(x,y)||}{||(x,y)||} \\
&= \frac{||(\Delta x, \Delta y)||}{||(x,y)||} \\
&= \frac{|\Delta x| + |\Delta y|}{|x|+|y|} \\
&\approx |\Delta x| + |\Delta y|
\end{aligned}$$

$$\begin{aligned}
\text{relative change in output} &= \left|\frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{f(x,y)}\right| \\
&= \left|\frac{[(x+\Delta x)-(y+\Delta y)]-(x-y)}{x-y}\right| \\
&= \frac{|\Delta x - \Delta y|}{|x-y|} \\
&\approx \frac{|\Delta x - \Delta y|}{\epsilon}
\end{aligned}$$

By definition,

$$\text{cond} = \frac{\text{relative change in output}}{\text{relative change in input}}$$

$$\approx \frac{|\Delta x - \Delta y|/\epsilon}{|\Delta x| + |\Delta y|} = \frac{|\Delta x - \Delta y|}{|\Delta x| + |\Delta y|} \cdot \frac{1}{\epsilon} \leq \frac{|\Delta x| + |-\Delta y|}{|\Delta x| + |\Delta y|} \cdot \frac{1}{\epsilon} = \frac{1}{\epsilon}$$

Thus, subtraction is extremely sensitive when ϵ is close to zero.

1.6

(a)

When $x = 0.1$,

$$\begin{aligned}\text{forward error} &= \hat{f}(0.1) - f(0.1) = 0.1 - \sin(0.1) \approx 1.67 \times 10^{-4} \\ \hat{x} &= \arcsin(\hat{f}(0.1)) = \arcsin(0.1) \\ \text{backward error} &= \hat{x} - x = \arcsin(0.1) - 0.1 \approx 1.67 \times 10^{-4}\end{aligned}$$

When $x = 0.5$,

$$\begin{aligned}\text{forward error} &= \hat{f}(0.5) - f(0.5) = 0.5 - \sin(0.5) \approx 2.06 \times 10^{-2} \\ \hat{x} &= \arcsin(\hat{f}(0.5)) = \arcsin(0.5) \\ \text{backward error} &= \hat{x} - x = \arcsin(0.5) - 0.5 \approx 2.36 \times 10^{-2}\end{aligned}$$

When $x = 1.0$,

$$\begin{aligned}\text{forward error} &= \hat{f}(1.0) - f(1.0) = 1.0 - \sin(1.0) \approx 1.59 \times 10^{-1} \\ \hat{x} &= \arcsin(\hat{f}(1.0)) = \arcsin(1.0) \\ \text{backward error} &= \hat{x} - x = \arcsin(1.0) - 1.0 \approx 5.71 \times 10^{-1}\end{aligned}$$

(b)

When $x = 0.1$,

$$\begin{aligned}\text{forward error} &= \hat{f}(0.1) - f(0.1) = (0.1 - 0.1^3/6) - \sin(0.1) \approx -8.33 \times 10^{-8} \\ \hat{x} &= \arcsin(\hat{f}(0.1)) = \arcsin(0.1 - 0.1^3/6) \\ \text{backward error} &= \hat{x} - x = \arcsin(0.1 - 0.1^3/6) - 0.1 \approx -8.37 \times 10^{-8}\end{aligned}$$

When $x = 0.5$,

$$\begin{aligned}\text{forward error} &= \hat{f}(0.5) - f(0.5) = (0.5 - 0.5^3/6) - \sin(0.5) \approx -2.59 \times 10^{-4} \\ \hat{x} &= \arcsin(\hat{f}(0.5)) = \arcsin(0.5 - 0.5^3/6) \\ \text{backward error} &= \hat{x} - x = \arcsin(0.5 - 0.5^3/6) - 0.5 \approx -2.95 \times 10^{-4}\end{aligned}$$

When $x = 1.0$,

$$\begin{aligned}\text{forward error} &= \hat{f}(1.0) - f(1.0) = (1.0 - 1.0^3/6) - \sin(1.0) \approx -8.14 \times 10^{-3} \\ \hat{x} &= \arcsin(\hat{f}(1.0)) = \arcsin(1.0 - 1.0^3/6) \\ \text{backward error} &= \hat{x} - x = \arcsin(1.0 - 1.0^3/6) - 1.0 \approx -1.49 \times 10^{-2}\end{aligned}$$

1.17

If we express x as

$$\pm\left(d_0 + \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_{p-1}}{\beta^{p-1}}\right)\beta^E,$$

then since y is adjacent to x,

$$y = \pm\left(d_0 + \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_{p-1} \pm 1}{\beta^{p-1}}\right)\beta^E.$$

The spacing between x and y is

$$\frac{1}{\beta^{p-1}} \cdot \beta^E = \beta^{E-p+1},$$

where E is bounded by $[L, U]$.

(a)

The minimum possible spacing is β^{L-p+1}. For single-precision, it's

$$2^{-126-24+1} \approx 1.40 \times 10^{-45}$$

For double-precision, it's

$$2^{-1022-53+1} \approx 4.94 \times 10^{-324}$$

(b)

The maximum possible spacing is β^{U-p+1}. For single-precision, it's

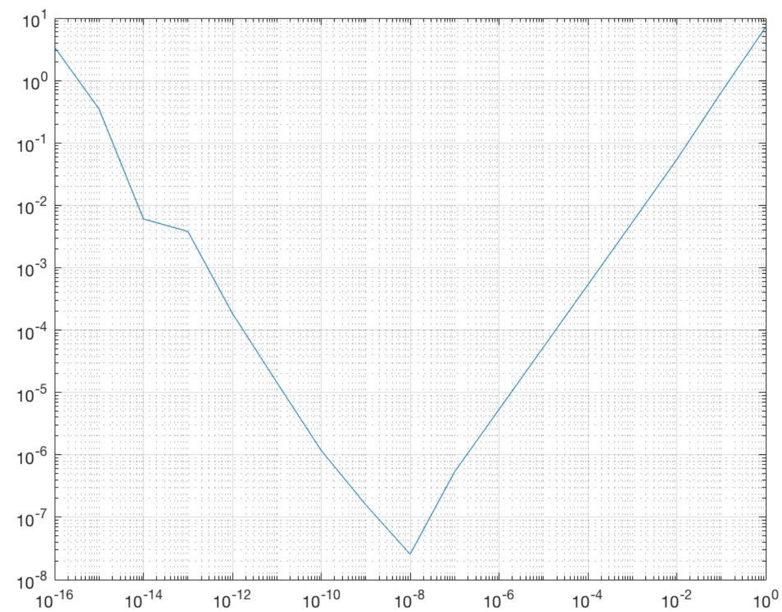
$$2^{127-24+1} \approx 2.03 \times 10^{31}$$

For double-precision, it's

$$2^{1023-53+1} \approx 2.00 \times 10^{292}$$

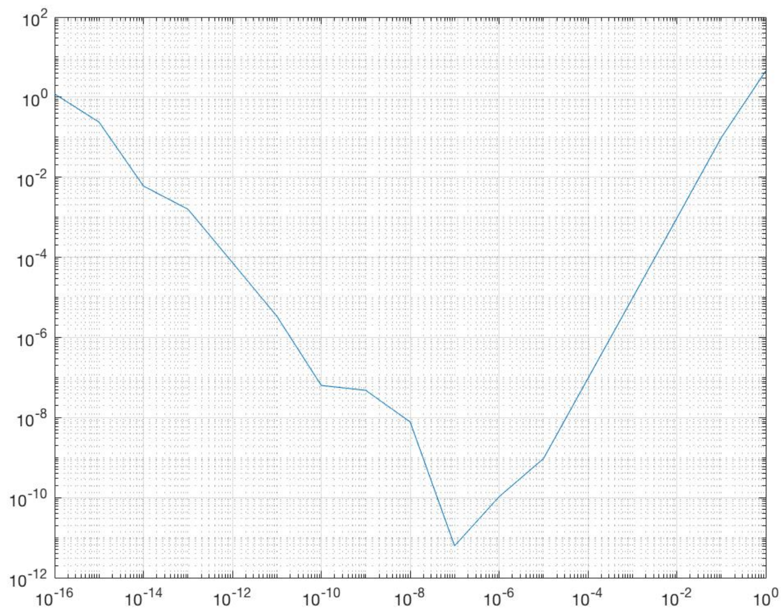
1.7

(a)



The minimum value of the magnitude of the error is approximately 2.5×10^{-8}, and the corresponding h is approximately $10^{-8} = \sqrt{10^{-16}} \approx \sqrt{\epsilon_{mach}}$

(b)



The minimum value of the magnitude of the error is approximately 6×10^{-12}, and the corresponding h is approximately 10^{-7}

Using Taylor expansion,

$$f(x+h) = f(x) + f'(x) \cdot h + \frac{f''(x)}{2} \cdot h^2 + \frac{f'''(c_1)}{3!} \cdot h^3 \quad \text{, for some } c_1 \in [x, x+h]$$

$$\begin{aligned} f(x-h) &= f(x) + f'(x) \cdot (-h) + \frac{f''(x)}{2} \cdot (-h)^2 + \frac{f'''(c_2)}{3!} \cdot (-h)^3 \\ &= f(x) - f'(x) \cdot h + \frac{f''(x)}{2} \cdot h^2 - \frac{f'''(c_2)}{3!} \cdot h^3 \quad \text{, for some } c_2 \in [x-h, x], \end{aligned}$$

$$\begin{aligned} f(x+h) - f(x-h) &= \left(f(x) + f'(x) \cdot h + \frac{f''(x)}{2} \cdot h^2 + \frac{f'''(c_1)}{3!} \cdot h^3 \right) \\ &\quad - \left(f(x) - f'(x) \cdot h + \frac{f''(x)}{2} \cdot h^2 - \frac{f'''(c_2)}{3!} \cdot h^3 \right) \\ &= 2f'(x) \cdot h + \frac{f'''(c_1) + f'''(c_2)}{6} \cdot h^3 \end{aligned}$$

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + \frac{f'''(c_1)+f'''(c_2)}{12}\cdot h^2$$
$$\frac{f(x+h)-f(x-h)}{2h} - f'(x) = \frac{f'''(c_1)+f'''(c_2)}{12}\cdot h^2$$

Suppose $f'''(x) \leq M$ for $x \in [x-h, x+h]$, then:

$$\left|\frac{f(x+h)-f(x-h)}{2h} - f'(x)\right| = \frac{|f'''(c_1)+f'''(c_2)|}{12}\cdot h^2 \leq \frac{|f'''(c_1)|+|f'''(c_2)|}{12}\cdot h^2 \leq \frac{2M}{12}\cdot h^2 = \frac{Mh^2}{6}$$

The upperbound for truncation error is $\frac{Mh^2}{6}$.
Suppose the errors in function values are bounded by ϵ, that is

$$\left|\hat{f}(x) - f(x)\right| = \delta \leq \epsilon, \text{ for all } x$$

Then,

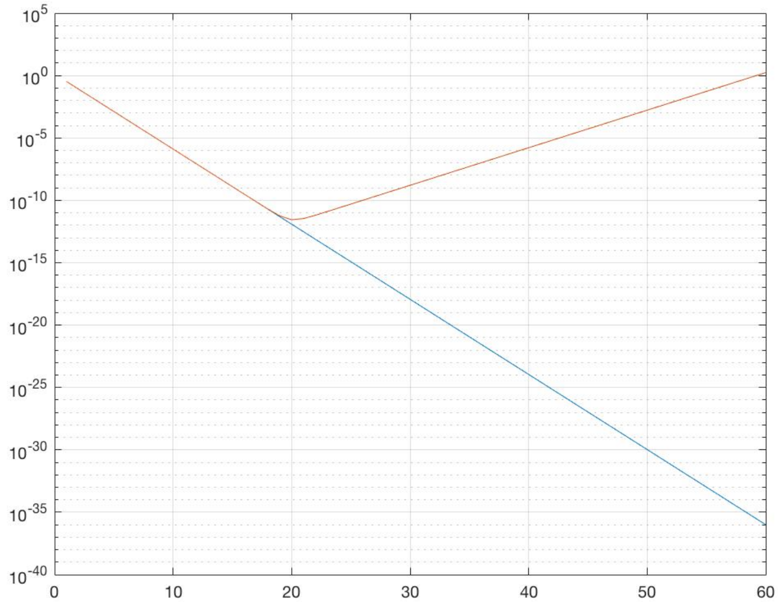
$$\begin{aligned}
&\left|\frac{\hat{f}(x+h)-\hat{f}(x-h)}{2h} - \frac{f(x+h)-f(x-h)}{2h}\right| \\
&= \frac{\left|\Big(\hat{f}(x+h)-f(x+h)\Big) - \Big(\hat{f}(x-h)-f(x-h)\Big)\right|}{2h} \\
&\leq \frac{\left|\hat{f}(x+h)-f(x+h)\right| + \left|-\Big(\hat{f}(x-h)-f(x-h)\Big)\right|}{2h} \\
&= \frac{\left|\hat{f}(x+h)-f(x+h)\right| + \left|\hat{f}(x-h)-f(x-h)\right|}{2h} \\
&= \frac{\delta_1+\delta_2}{2h} \leq \frac{2\epsilon}{2h} = \frac{\epsilon}{h}
\end{aligned}$$

The upperbound for rounding error is $\frac{\epsilon}{h}$.
The total computational error bound is therefore $\frac{Mh^2}{6} + \frac{\epsilon}{h}$
The minimum magnitude of error occurs when

$$\left(\frac{Mh^2}{6} + \frac{\epsilon}{h}\right)' = \frac{Mh}{3} - \frac{\epsilon}{h^2} = 0$$

$$h = \sqrt[3]{\frac{3\epsilon}{M}}$$

1.17



The graph exhibits expected behavior for small k's, however, after k reaches 20, the sequence suddenly starts to increase.

Let A be the advance operator, then the equation can be written as

$$A^2 x_k - 2.25 A x_k + 0.5 x_k = 0$$

The characteristic equation is

$$\lambda^2 - 2.25\lambda + 0.5 = 0$$

$$\lambda_1 = 2, \lambda_2 = \frac{1}{4}$$

The general solution to the difference equation is

$$x_k = c_1 \cdot 2^k + c_2 \cdot \left(\frac{1}{4}\right)^k$$

The absolute value of the first term increases and the second term decreases as k grows larger. For the particular initial condition specified in this problem,

$$c_1 = 0, \quad c_2 = \frac{4}{3},$$

there is no contribution from the first term, therefore the sequence converges to 0. However this initial condition is very unstable, as c_1 would become non-zero even for the slightest perturbations resulted from machine errors. Then the sequence no longer converges to 0, and the first term would explode when k grows larger, which explains the unexpected behavior described above.