# PHYS 7125 Homework 4

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## 1

For a timelike vector $t^\mu$ at any point it is always possible to construct an orthonormal frame $\underset{\sim}{\mathbf{e}}_i^\mu$ where $\underset{\sim}{\mathbf{e}}_0^\mu = t^\mu$. (Without loss of generality, $t^\mu$ can be assumed to have unit length (-1). In the general case, the results only differ by a positive factor.) In such coordinates, $t^\mu = (1,0,0,0)$, the metric is locally $\eta_{\mu\nu}$, and the components of the electromagnetic energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{\mu_0}\Big[F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big]$$

Considering $t^\mu$ is only non-zero in its time component and $T_{\mu\nu}$ is symmetric, we only need to compute

$$T_{0\nu} = \frac{1}{\mu_0}\Big[F_0{}^\alpha F_{\nu\alpha} - \frac{1}{4}\eta_{0\nu}F_{\alpha\beta}F^{\alpha\beta}\Big] =: A + B$$

The first term evaluates to: (Define $\mathbf{S} := \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$)

$$A = \frac{1}{\mu_0}\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 0 \\ E_x/c \\ E_y/c \\ E_z/c \end{pmatrix} = \begin{pmatrix} \epsilon_0 E^2 \\ -\frac{1}{\mu_0}(E_yB_z - E_zB_y)/c \\ -\frac{1}{\mu_0}(E_zB_x - E_xB_z)/c \\ -\frac{1}{\mu_0}(E_xB_y - E_yB_x)/c \end{pmatrix} = \begin{pmatrix} \epsilon_0 E^2 \\ -S_x/c \\ -S_y/c \\ -S_z/c \end{pmatrix}$$

The second term:

$$B = -\frac{1}{4\mu_0}(-2E^2/c^2 + 2B^2)\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\Big(-\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad\Rightarrow\quad T_{0\nu} = \begin{pmatrix} \frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) \\ -S_x/c \\ -S_y/c \\ -S_z/c \end{pmatrix}$$

In matrix form, the relevant components of the energy-momentum tensor are:

$$T_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) & -S_x/c & -S_y/c & -S_z/c \\ -S_x/c & \cdots & \cdots & \cdots \\ -S_y/c & \cdots & \cdots & \cdots \\ -S_z/c & \cdots & \cdots & \cdots \end{pmatrix}$$

(i) The weak energy condition is obviously satisfied:

$$T_{\mu\nu}t^\mu t^\nu = T_{00} = \frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big) \geq 0$$

(ii)

$$T_{\mu\nu}t^\mu = T_{0\nu} = \left(\frac{1}{2}\Big(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\Big), -S_x/c, -S_y/c, -S_z/c\right)$$

$$T^\nu{}_\alpha t^\alpha = T^\nu{}_0 = g^{\nu\nu} \cdot T_{\nu 0} = \left( -\frac{1}{2}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\Big), -S_x/c, -S_y/c, -S_z/c \right)$$

Using Lagrange's identity for cross products,

$$\begin{aligned}
(T_{\mu\nu} t^\mu)(T^\nu{}_\alpha t^\alpha) &= -\frac{1}{4}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\Big)^2 + \frac{\|\mathbf{S}\|^2}{c^2} \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\Big)^2 + \frac{\epsilon_0}{\mu_0}\|\mathbf{E} \times \mathbf{B}\|^2 \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\Big)^2 + \frac{\epsilon_0}{\mu_0}\Big(E^2 B^2 - (\mathbf{E} \cdot \mathbf{B})^2\Big) \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 - \frac{1}{\mu_0} B^2\Big)^2 - \frac{\epsilon_0}{\mu_0}(\mathbf{E} \cdot \mathbf{B})^2 \le 0
\end{aligned}$$

Thus the dominant energy condition is also satisfied.

# 2

## a

The only non-vanishing components of $\Gamma^\mu_{0\nu}$ are

$$\Gamma^0_{0i} = \frac{Mx^i}{(1-2M/r)r^3}, \quad \Gamma^i_{00} = \frac{Mx^i}{(1+2M/r)r^3} \qquad (i = 1, 2, 3)$$

The geodesic equation can be simplified as

$$\begin{aligned}
\frac{dp_0}{d\lambda} &= \sum_i \left( \frac{Mx^i}{(1-2M/r)r^3} p_0 p^i + \frac{Mx^i}{(1+2M/r)r^3} p^0 p_i \right) \\
&= \sum_i \left( g_{00} \frac{Mx^i}{(1-2M/r)r^3} + g_{ii} \frac{Mx^i}{(1+2M/r)r^3} \right) p^0 p^i \\
&= \sum_i \left( -\frac{Mx^i}{r^3} + \frac{Mx^i}{r^3} \right) p^0 p^i = 0
\end{aligned}$$

## b

No. $p^0 = g^{0\nu} p_\nu = g^{00} p_0 = -(1-2M/r)^{-1} p_0$, which is not constant unless $r$ is constant.

## c

For the atom at rest on the surface of the sun, $dx^i = 0$, $r = R$,

$$d\tau^2 = -ds^2 = (1-2M/R)dt^2 \quad \Rightarrow \quad u^0 = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-2M/R}}$$

## d

For both the atom and the distant observer, $dx^i = 0$,

$$d\tau^2 = -ds^2 = (1-2M/r)dt^2$$

$$\Rightarrow \quad u^\mu = \frac{dx^\mu}{d\tau} = (dt/d\tau, 0, 0, 0) = ((1-2M/r)^{-1/2}, 0, 0, 0)$$

The photon energy observed at both locations can be expressed as

$$E = g_{\mu\nu} p^\mu u^\nu = (1 - 2M/r)^{-1/2} g_{\mu\nu} p^\mu K^\nu$$

where $K^\nu = (1, 0, 0, 0)$ is a Killing vector as the metric has no time dependence; therefore $g_{\mu\nu} p^\mu K^\nu$ is conserved along the photon's world line, which is a geodesic as stated in the problem. Then,

$$\frac{\lambda_r}{\lambda_e} = \frac{hc/\lambda_e}{hc/\lambda_r} = \frac{E_e}{E_r} = \frac{(1 - 2M/R)^{-1/2}}{\lim_{r\to\infty}(1 - \cancel{2M/r})^{-1/2}} = 1 + \frac{M}{R} + \mathcal{O}\left(\frac{M^2}{R^2}\right)$$

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{\lambda_r}{\lambda_e} - 1 = \frac{M}{R} + \mathcal{O}\left(\frac{M^2}{R^2}\right)$$