PHYS 7125 Homework 4

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March 8, 2019

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For any time-like vector t^μ we can always choose a local coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu}$ and $t^\mu = (1, 0, 0, 0)$. The electromagnetic energy-momentum tensor expressed in such coordinates is

$$T_{\mu\nu} = \begin{pmatrix} \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) & -S_x/c & -S_y/c & -S_z/c \\ -S_x/c & \cdots & \cdots & \cdots \\ -S_y/c & \cdots & \cdots & \cdots \\ -S_z/c & \cdots & \cdots & \cdots \end{pmatrix}$$

where

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

(i) First, the weak energy condition is satisfied:

$$T_{\mu\nu}t^\nu t^\nu = \frac{1}{2}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big) \geq 0$$

(ii)

$$T_{\mu\nu}t^\mu = \Big(\frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2), -S_x/c, -S_y/c, -S_z/c\Big)$$

Raising one index,

$$T^\nu{}_\alpha t^\alpha = \Big(-\frac{1}{2}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big), -S_x/c, -S_y/c, -S_z/c\Big)$$

Using Lagrange's identity for cross products,

$$\begin{aligned}
(T_{\mu\nu}t^\mu)(T^\nu{}_\alpha t^\alpha) &= -\frac{1}{4}\Big(\epsilon_0^2 E^4 + \frac{1}{\mu_0^2}B^4 + 2\frac{\epsilon_0}{\mu_0}E^2B^2\Big) + \frac{\|\mathbf{S}\|^2}{c^2} \\
&= -\frac{1}{4}\Big(\epsilon_0^2 E^4 + \frac{1}{\mu_0^2}B^4 + 2\frac{\epsilon_0}{\mu_0}E^2B^2\Big) + \frac{\epsilon_0}{\mu_0}(E^2B^2 - (\mathbf{E}\cdot\mathbf{B})^2) \\
&= -\frac{1}{4}\Big(\epsilon_0 E^2 - \frac{1}{\mu_0}B^2\Big)^2 - \frac{\epsilon_0}{\mu_0}(\mathbf{E}\cdot\mathbf{B})^2 \leq 0
\end{aligned}$$

Thus the dominant energy condition is satisfied.

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a

Since the metric is diagonal, indices can be lowered by a simple multiplication

$$V_\mu = \sum_\nu V^\nu g_{\mu\nu} = V^\mu \cdot g_{\mu\mu}$$

The non-vanishing components of $\Gamma^{\mu}_{0\nu}$ are $(i = 1, 2, 3)$

$$\Gamma^0_{0i} = \frac{Mx^i}{(1 - 2M/r)r^3}, \quad \Gamma^i_{00} = \frac{Mx^i}{(1 + 2M/r)r^3}$$

The geodesic equation can be simplified as

$$\begin{aligned}
\frac{dp_0}{d\lambda} &= \sum_i \Big[\frac{Mx^i}{(1 - 2M/r)r^3} p_0 p^i + \frac{Mx^i}{(1 + 2M/r)r^3} p^0 p_i\Big] \\
&= \sum_i \Big[g_{00} \frac{Mx^i}{(1 - 2M/r)r^3} p^0 p^i + g_{ii} \frac{Mx^i}{(1 + 2M/r)r^3} p^0 p^i\Big] \\
&= \sum_i \Big[-\frac{Mx^i}{r^3} p^0 p^i + \frac{Mx^i}{r^3} p^0 p^i\Big] = 0
\end{aligned}$$

b

The geodesic equation for p^0 is

$$\frac{dp^0}{d\lambda} = \Gamma^0_{\mu\nu} p^{\mu} p^{\nu}$$

The only non-vanishing $\Gamma^0_{\mu\nu}$ are $(i = 1, 2, 3)$

$$\Gamma^0_{0i} = \Gamma^0_{i0} = \frac{Mx^i}{(1 - 2M/r)r^3}$$

The geodesic equation shows that $dp^0/d\lambda$ is generally non-zero:

$$\frac{dp^0}{d\lambda} = \sum_i \Big[\frac{2Mx^i}{(1 - 2M/r)r^3} p^0 p^i\Big] \not\equiv 0$$

c

Since the atom is at rest, $u^i = 0$. The geodesic equation is

$$\frac{du^0}{d\lambda} = \sum_i \Big[\frac{2Mx^i}{(1 - 2M/r)r^3} u^0 u^i\Big] = 0$$

Therefore $u^0 = dt/d\lambda$ could be any constant, which means that $\lambda = at + b$.

d

Since both the atom and the oberver are at rest, $dx^i = 0$,

$$ds^2 = -(1 - 2M/r)dt^2 = -d\tau^2 \quad \Rightarrow \quad d\tau = (1 - 2M/r)^{1/2} dt$$

On the surface of the sun $d\tau_e = (1 - 2M/R)^{1/2} dt$ and far away from the sun $d\tau_r \approx dt$, so the time dilation is $d\tau_e/d\tau_r = (1 - 2M/R)^{1/2}$. And since the speed of light is constant everywhere,

$$\frac{\lambda_e}{\lambda_r} = \frac{c\Delta\tau_e}{c\Delta\tau_r} = (1 - 2M/R)^{1/2} = 1 - \frac{M}{R} + \mathcal{O}\left((M/R)^2\right)$$

$$\frac{\lambda_r - \lambda_e}{\lambda_r} = \frac{M}{R} + \mathcal{O}\left((M/R)^2\right)$$