PHYS 7125 Homework 5

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1

The total proper time along any curve γ is

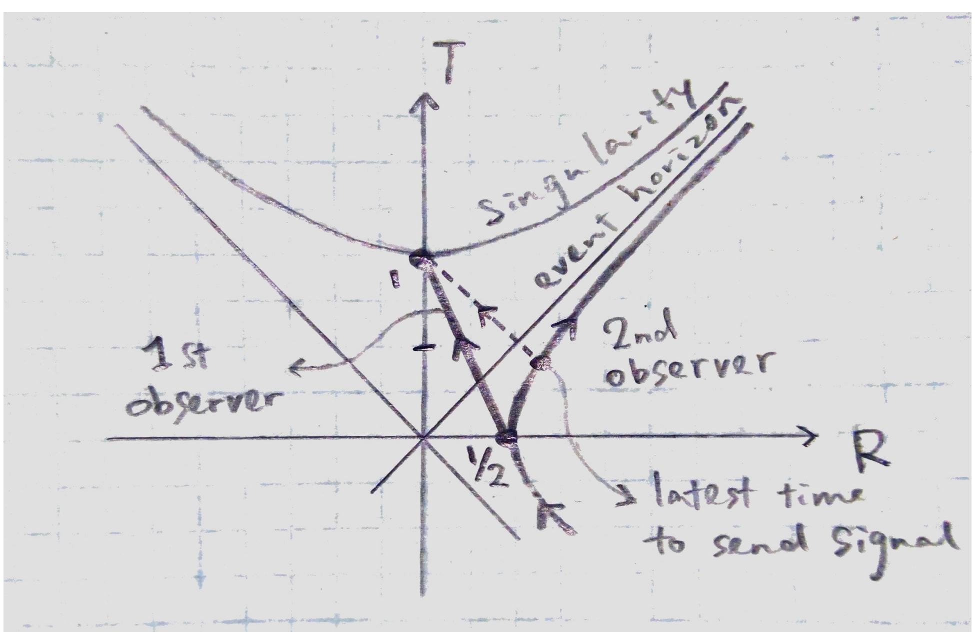
$$\tau_{total} = \int_\gamma d\tau = \int_\gamma \sqrt{-ds^2} = \int_\gamma \sqrt{-\Big(\frac{2M}{r} - 1\Big)dt^2 + \Big(\frac{2M}{r} - 1\Big)^{-1} dr^2 - r^2 d\Omega^2}$$

Inside the event horizon, $r < 2M$, the first and third term in the square root are negative, therefore

$$\begin{aligned}\tau_{total} &< \int_0^{2M} \sqrt{\Big(\frac{2M}{r} - 1\Big)^{-1}} dr \\ &= \left[-\sqrt{r(2M-r)} + 2M\cot^{-1}\left(\sqrt{\frac{2M}{r} - 1}\right)\right]\Bigg|_0^{2M} = \pi M\end{aligned}$$

2

a



b

Yes, because the worldline lies within the 45° light cone at each point.

c

For the second observer at constant r,

$$T = \frac{1}{2}\sinh\left(\frac{t}{4GM}\right), \quad R = \frac{1}{2}\cosh\left(\frac{t}{4GM}\right), \quad R^2 - T^2 = \frac{1}{4}$$

To reach the critical point where the first observer is destroyed, the photon sent by the second observer must follow the straight line $R = 1 - T$, which intersects the second observer's worldline in the past at

$$\cancel{T^2} - 2T + 1 - \cancel{T^2} = \frac{1}{4} \Rightarrow \quad T = \frac{1}{2}\sinh\left(\frac{t}{4GM}\right) = \frac{3}{8}$$

$$t = 4GM\sinh^{-1}\left(\frac{3}{4}\right) \approx 2.77GM$$

3

A perfect fluid is incompressible, therefore $\nabla_\mu \rho = \partial_\mu \rho = 0$. From the continuity equation,

$$\nabla_\mu(\rho u^\mu) = \rho \nabla_\mu u^\mu = 0 \quad \Rightarrow \quad \nabla_\mu u^\mu = 0$$

The conservation law requires the covariant divergence of stress-energy tensor to vanish:

$$\begin{aligned}
\nabla_\mu T^\mu{}_\nu &= (\nabla_\mu p + \cancel{\nabla_\mu \rho}) u^\mu u_\nu + (p+\rho)\cancel{(\nabla_\mu u^\mu)} u_\nu + (p+\rho) u^\mu (\nabla_\mu u_\nu) + \delta^\mu_\nu \nabla_\mu p \\
&= u_\nu (u^\mu \nabla_\mu p) + (p+\rho)(u^\mu \nabla_\mu u_\nu) + (\delta^\mu_\nu \nabla_\mu) p \\
&= u_\nu \nabla_{\underset{\sim}{u}} p + (p+\rho) \nabla_{\underset{\sim}{u}} u_\nu + \nabla_\nu p \\
&= 0
\end{aligned}$$

Rearranging the terms (and dropping the free index ν) gives

$$(p+\rho)\nabla_{\underset{\sim}{u}} \underset{\sim}{u} = -\nabla p - \underset{\sim}{u} \nabla_{\underset{\sim}{u}} p$$