



MATH-4347

# PARTIAL DIFFERENTIAL EQUATIONS I

## HOMEWORK ASSIGNMENTS

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*This collection is organized in reverse chronological order*

# MATH 4347 Homework 6

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## 8.3

**a**

Suppose there exist two distinct solutions  $u_1$  and  $u_2$ , let  $w = u_1 - u_2$ , then  $w$  satisfies

$$\Delta w = 0, \quad \frac{\partial w}{\partial n} + \alpha w = 0, x \in \partial U$$

First we can establish the identity:

$$\nabla \cdot (w \nabla w) = \nabla w \cdot \nabla w + w \nabla \cdot \nabla w = |\nabla w|^2 + w \Delta w$$

Now define the energy as

$$\begin{aligned} E_w(t) &= \int_U |\nabla w|^2 dV \\ &= \int_U \nabla \cdot (w \nabla w) dV - \int_U w \Delta w dV \\ &= \int_{\partial U} w \nabla w \cdot \vec{n} dS - \int_U w \Delta w dV \\ &= \int_{\partial U} w \frac{\partial w}{\partial n} dS \\ &= -\alpha \int_{\partial U} w^2 dS \end{aligned}$$

We now have

$$\int_U |\nabla w|^2 dV = -\alpha \int_{\partial U} w^2 dS$$

Since  $\alpha > 0$ ,

$$\int_U |\nabla w|^2 dV \geq 0, \text{ but } -\alpha \int_{\partial U} w^2 dS \leq 0$$

We have

$$\begin{aligned} \int_U |\nabla w|^2 dV &= -\alpha \int_{\partial U} w^2 dS = 0 \\ \Rightarrow \quad \nabla w &\equiv 0, \quad w|_{\partial U} \equiv 0 \\ \Rightarrow \quad w &\equiv 0 \end{aligned}$$

which means that  $u_1 = u_2$ , so the solution must be unique.

**b**

Following the same steps as (a),

$$\int_U |\nabla w|^2 dV = -\alpha \int_{\partial U} w^2 dS = 0 \quad \Rightarrow \quad \nabla w \equiv 0 \quad \Rightarrow \quad w = \text{const.}$$

Therefore, any two solutions only differ by a constant.

**c**

Let  $n = 1$ , then the problem becomes

$$w'' = 0, \quad \begin{cases} -w'(a) + \alpha w(a) = 0 \\ w'(b) + \alpha w(b) = 0 \end{cases} \quad (a < b)$$

Solving the ODE gives

$$w = Cx + D$$

Plug in the boundary conditions

$$\begin{cases} (\alpha a - 1)C + \alpha D = 0 \\ (\alpha b + 1)C + \alpha D = 0 \end{cases}$$

The linear system does not have a unique solution when

$$\det \begin{pmatrix} \alpha a - 1 & \alpha \\ \alpha b + 1 & \alpha \end{pmatrix} = \alpha^2(a - b) - 2\alpha = 0 \quad \Rightarrow \quad \boxed{\alpha = \frac{2}{a - b} < 0}$$

## 8.6

A necessary condition for the mean-value property is:

$$\frac{d}{dr} \oint_{\partial B(0,r)} u(y) dS_y = \frac{d}{dr} \left[ \frac{1}{4\pi r^2} \int_{\partial B(0,r)} u(y) dS_y \right] = 0$$

Now suppose  $\Delta u(x) = f(x) \not\equiv 0$ , then

$$\begin{aligned} \frac{d}{dr} \left[ \frac{1}{4\pi r^2} \int_{\partial B(0,r)} u(y) dS_y \right] &= \frac{d}{dr} \left[ \frac{1}{4\pi} \int_{\partial B(0,1)} u(ry) dS_y \right] \\ &= \frac{1}{4\pi} \int_{\partial B(0,1)} \nabla u(ry) \cdot y dS_y = \frac{1}{4\pi} \int_{\partial B(0,1)} \nabla u(ry) \cdot \vec{n} dS_y \\ &= \frac{1}{4\pi r^2} \int_{\partial B(0,r)} \nabla u(y) \cdot \vec{n} dS_y = \frac{1}{4\pi r^2} \int_{B(0,r)} \Delta u(y) dV_y \\ &= \frac{1}{4\pi r^2} \int_{B(0,r)} f(y) dV_y \neq 0 \end{aligned}$$

which means that the mean value of  $u$  is not independent of  $r$ , which contradicts the assumption that  $u$  has the mean-value property. Therefore, we must have  $\Delta u = 0$  in  $U$ .

### 6.2.3

Let  $u = X(x)Y(y)$ , then

$$u_{xx} + u_{yy} = X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

The boundary conditions require that  $\lambda \geq 0$ .

(i) For  $\lambda = 0$ ,

$$Y_0 = C, \quad X_0 = Dx$$

(ii) For  $\lambda > 0$ , let  $\lambda = \beta^2$ .

$$Y_n = \cos ny, \quad \beta_n = n$$

Now solve for corresponding  $X_n$ :

$$X_n'' = n^2 X_n \quad \Rightarrow \quad X_n = A_n \cosh nx + B_n \sinh nx$$

$$X_n(0) = A_n = 0 \quad \Rightarrow \quad X_n = \sinh nx$$

The general solution is

$$u(x, y) = A_0 x + \sum_{i=1}^{\infty} A_n \sinh nx \cos ny$$

$$u(\pi, y) = A_0 \pi + \sum_{i=1}^{\infty} A_n \sinh n\pi \cos ny = \frac{1}{2} + \frac{1}{2} \cos 2y$$

Comparing two sides, we have the non-zero coefficients:

$$A_0 = \frac{1}{2\pi}, \quad A_2 = \frac{1}{2 \sinh 2\pi}$$

Therefore the solution is

$$u(x, y) = \frac{1}{2\pi} x + \frac{1}{2 \sinh 2\pi} \sinh 2x \cos 2y$$

### 6.2.6

Let  $u = X(x)Y(y)Z(z)$ . Separation of variables yields

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Suppose  $X'' = -\beta^2 X$ ,  $Y'' = -\gamma^2 Y$ , then

$$X_n = \cos n\pi x, \quad \beta_n = n\pi, \quad n = 0, 1, \dots$$

$$Y_m = \cos m\pi y, \quad \gamma_m = m\pi, \quad m = 0, 1, \dots$$

The above results already include  $m = 0$  and  $n = 0$  as special cases.

$$Z_{m,n}'' = (\beta_n^2 + \gamma_m^2) Z_{m,n} = (n^2 + m^2) \pi^2 Z_{m,n}$$

For  $n^2 + m^2 \neq 0$ :

$$Z_{m,n} = A \cosh \sqrt{n^2 + m^2} \pi z + B \sinh \sqrt{n^2 + m^2} \pi z$$

$$Z'_{m,n}(0) = 0 \quad \Rightarrow \quad B = 0$$

$$Z_{m,n} = \cosh \sqrt{n^2 + m^2} \pi z$$

For  $n = m = 0$ :

$$Z'' = 0 \quad \Rightarrow \quad Z = A + Bz$$

$$Z'(0) = B = 0 \quad \Rightarrow \quad Z = A$$

which is included in the previous result. The general solution is

$$u = \sum_{m,n=0}^{\infty} A_{m,n} \cos n\pi x \cos m\pi y \cosh \sqrt{n^2 + m^2} \pi z$$

From the last boundary condition:

$$u_z(x, y, 1) = \sum_{m,n=0}^{\infty} \sqrt{n^2 + m^2} \pi A_{m,n} \cos n\pi x \cos m\pi y \sinh \sqrt{n^2 + m^2} \pi = g(x, y)$$

$$A_{m,n} = \frac{4}{\sqrt{n^2 + m^2} \pi \sinh \sqrt{n^2 + m^2} \pi} \int_0^1 \int_0^1 g(x, y) \cos n\pi x \cos m\pi y dx dy, \quad m \neq 0, n \neq 0$$

$$A_{0,n} = \frac{2}{n\pi \sinh n\pi} \int_0^1 g(x, y) \cos n\pi x dx, \quad A_{m,0} = \frac{2}{m\pi \sinh m\pi} \int_0^1 g(x, y) \cos m\pi y dy$$

### 6.3.1

**a**

By the maximum principle of harmonic functions  $\max_{\overline{D}} u = \max_{\partial D} u$ . On the boundary,

$$u = 3 \sin 2\theta + 1 \leq 3 + 1 = 4$$

So the maximum of  $u$  in  $\overline{D}$  is 4.

**b**

By the mean-value property of harmonic functions,

$$\begin{aligned} u(\mathbf{0}) &= \frac{1}{4\pi} \int_0^{2\pi} (3 \sin 2\theta + 1) (2d\theta) \\ &= \frac{1}{2\pi} \left[ -\frac{3 \cos 2\theta}{2} + \theta \right]_0^{2\pi} = \boxed{1} \end{aligned}$$

## 6.4.5

**a**

The steady-state temperature distribution satisfies Laplace equation  $\nabla u = 0$ . Let  $u = X(\theta)R(r)$ , then

$$\frac{R'' + \frac{1}{r}R'}{\frac{1}{r^2}R} = -\frac{X''}{X} = \lambda$$

Since  $\theta$  is not bounded,  $X$  satisfies periodic boundary conditions

$$\begin{aligned} X(0) &= X(2\pi), \quad X'(0) = X'(2\pi) \\ \Rightarrow \quad \begin{cases} X_0 = C, & \lambda = 0 \\ X_n = A \cos n\theta + B \sin n\theta, & \lambda = n^2 \end{cases} \end{aligned}$$

For  $\lambda = 0$ ,

$$R'' + \frac{1}{r}R' = 0 \quad \Rightarrow \quad R_0 = C_1 + C_2 \ln r$$

Since the outer edge is insulated,

$$R'_0(2) = \frac{C_2}{2} = 0 \quad \Rightarrow \quad C_2 = 0$$

Therefore,  $R$  can only be constant

$$\boxed{R_0 = C}$$

For  $\lambda = n^2$

$$\begin{aligned} R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R &= 0 \\ r^2R'' + rR' - n^2R &= 0 \end{aligned}$$

Suppose  $R(r) = r^\alpha$ , then

$$\begin{aligned} r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2r^\alpha &= (\alpha^2 - n^2)r^\alpha = 0 \quad \Rightarrow \quad \alpha = \pm n \\ \Rightarrow \quad R_n &= Cr^n + Dr^{-n} \end{aligned}$$

Since the outer edge is insulated,

$$R'_n(2) = nC2^{n-1} - nD2^{-n-1} = 0 \quad \Rightarrow \quad D = 4^n C$$

So the solution can be rewritten as

$$\boxed{R_n = C[r^n + 4^n r^{-n}]}$$

Combining above results, the general solution is

$$\boxed{u = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n(r^n + 4^n r^{-n}) \cos n\theta + D_n(r^n + 4^n r^{-n}) \sin n\theta}$$

At  $r = 1$ ,

$$u = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n(1 + 4^n) \cos n\theta + D_n(1 + 4^n) \sin n\theta = \sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

Comparing the terms, the non-zero coefficients are

$$C_0 = 1, \quad C_2 = -\frac{1}{34}$$

So the solution is

$$u = \frac{1}{2} - \frac{1}{34} \left( r^2 + \frac{16}{r^2} \right) \cos 2\theta$$

**b**

Following the same steps as (a), for  $\lambda = 0$ ,

$$X_0 = C, \quad R_0 = C_1 + C_2 \ln r$$

And for  $\lambda = n^2$ ,

$$X_n = A \cos n\theta + B \sin n\theta, \quad R_n = Cr^n + Dr^{-n}$$

Now at the outer edge

$$R_0(2) = C_1 + C_2 \ln 2 = 0 \quad \Rightarrow \quad C_2 = -\frac{C_1}{\ln 2}$$

$$R_n(2) = C2^n + D2^{-n} = 0 \quad \Rightarrow \quad D = -4^n C$$

So  $R(r)$  can be rewritten as

$$R_0 = C \left( 1 - \frac{\ln r}{\ln 2} \right), \quad R_n = D (r^n - 4^n r^{-n})$$

Combining the results, the general solution is

$$u = C_0 \left( 1 - \frac{\ln r}{\ln 2} \right) + \sum_{n=1}^{\infty} C_n (r^n - 4^n r^{-n}) \cos n\theta + D_n (r^n - 4^n r^{-n}) \sin n\theta$$

On the inner edge

$$C_0 + \sum_{n=1}^{\infty} C_n (1 - 4^n) \cos n\theta + D_n (1 - 4^n) \sin n\theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

The non-zero coefficients are

$$C_0 = \frac{1}{2}, \quad C_2 = \frac{1}{30}$$

Finally, the solution is

$$u = \frac{1}{2} \left( 1 - \frac{\ln r}{\ln 2} \right) + \frac{1}{30} \left( r^2 - \frac{16}{r^2} \right) \cos 2\theta$$

## 6.4.10

Let  $u = X(\theta)R(r)$ . The boundary condition on  $x = 0$  and  $y = 0$  can be written as

$$X(0) = X(\pi/2) = 0$$

Separation of variables gives

$$\frac{R'' + \frac{1}{r}R'}{\frac{1}{r^2}R} = -\frac{X''}{X} = \lambda$$

For  $\lambda < 0$ , the boundary condition cannot be satisfied.

For  $\lambda = 0$ ,

$$\begin{aligned} X'' = 0 &\Rightarrow X = A\theta + B \\ X(0) = B = 0 &\Rightarrow X = A\theta \\ X(\pi/2) = A\pi/2 = 0 &\Rightarrow A = 0, \quad X = 0 \end{aligned}$$

There is no non-trivial solution, so zero is not an eigenvalue.

For  $\lambda > 0$ , let  $\lambda = \beta^2$ , then

$$\begin{aligned} X &= A \cos \beta\theta + B \sin \beta\theta \\ X(0) = 0 &\Rightarrow A = 0, \quad X = \sin \beta\theta \\ X(\pi/2) = \sin \frac{\beta\pi}{2} = 0 &\Rightarrow \beta_n = 2n, \quad \boxed{X_n = \sin 2n\theta} \end{aligned}$$

Now solve for  $R$ :

$$r^2 R'' + rR' - 4n^2 R = 0$$

Suppose  $R = r^\alpha$ ,

$$\alpha^2 - 4n^2 = 0 \Rightarrow \alpha = \pm 2n$$

Since  $\lim_{r \rightarrow 0} r^{-2n} = \infty$ ,  $r^{-2n}$  should be excluded,

$$\boxed{R_n = C_n r^{2n}}$$

The general solution is:

$$\begin{aligned} u &= \sum_{n=1}^{\infty} C_n r^{2n} \sin 2n\theta \\ u_r(a, \theta) &= \sum_{n=1}^{\infty} 2nC_n a^{2n-1} \sin 2n\theta = 1 \\ C_n &= \frac{2}{n\pi a^{2n-1}} \int_0^{\pi/2} \sin 2n\theta d\theta = \frac{1 - (-1)^n}{n^2 \pi a^{2n-1}} = \begin{cases} \frac{2}{n^2 \pi a^{2n-1}}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases} \end{aligned}$$

$$\begin{aligned} u &= \boxed{\sum_{\text{odd}} \frac{2}{n^2 \pi a^{2n-1}} r^{2n} \sin 2n\theta} \\ &= \boxed{\frac{2}{a\pi} r^2 \sin 2\theta + \frac{2}{9a^5\pi} r^6 \sin 6\theta + \dots} \end{aligned}$$



## 7.4.6

**a**

The fundamental solution for Laplace equation in 2 dimension is

$$\Phi(\mathbf{x} - \mathbf{x}_0) = -\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0|$$

Using the reflection method, the corrector function is

$$h^{\mathbf{x}_0}(\mathbf{x}) = \Phi(\mathbf{x} - \hat{\mathbf{x}}_0) = -\frac{1}{2\pi} \ln |\mathbf{x} - \hat{\mathbf{x}}_0|$$

where  $\hat{\mathbf{x}}_0$  is the reflection of  $\mathbf{x}_0$  about the  $x$ -axis. The Green's function is

$$G(\mathbf{x}, \mathbf{x}_0) = \Phi(\mathbf{x} - \mathbf{x}_0) - h^{\mathbf{x}_0}(\mathbf{x}) = \boxed{-\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0| + \frac{1}{2\pi} \ln |\mathbf{x} - \hat{\mathbf{x}}_0|}$$

**b**

Let  $\mathbf{x} = (x, y)$ ,  $\mathbf{x}_0 = (x_0, y_0)$ ,  $\hat{\mathbf{x}}_0 = (x_0, -y_0)$ . On the boundary, which is the  $x$ -axis,

$$\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} = -\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial y} = \frac{1}{2\pi} \frac{y - y_0}{|\mathbf{x} - \mathbf{x}_0|^2} - \frac{1}{2\pi} \frac{y + y_0}{|\mathbf{x} - \hat{\mathbf{x}}_0|^2} = -\frac{y_0}{\pi[(x - x_0)^2 + y_0^2]}$$

The solution is

$$\begin{aligned} u(\mathbf{x}_0) &= -\int_{\partial\mathbb{R}_+^2} \frac{\partial G(\mathbf{x}_0, \mathbf{x})}{\partial n} u(\mathbf{x}) ds \\ \Rightarrow u(x_0, y_0) &= \boxed{\frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x - x_0)^2 + y_0^2} h(x) dx} \end{aligned}$$

**c**

$$\begin{aligned} u(x_0, y_0) &= \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x - x_0)^2 + y_0^2} dx \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\left(\frac{x - x_0}{y_0}\right)^2 + 1} \frac{1}{y_0} dx \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{z^2 + 1} dz = \frac{1}{\pi} \tan^{-1}(z) \Big|_{-\infty}^{\infty} = \boxed{1} \end{aligned}$$

## 7.4.7

**a**

$$\begin{aligned} u_x(x, y) &= \frac{f'(x/y)}{y}, \quad u_{xx}(x, y) = \frac{f''(x/y)}{y^2} \\ u_y(x, y) &= -\frac{f'(x/y)x}{y^2} \end{aligned}$$

$$u_{yy}(x, y) = -\frac{-f''(x/y)x^2 - 2f'(x/y)xy}{y^4} = \frac{f''(x/y)(x/y)^2 + 2f'(x/y)(x/y)}{y^2}$$

$$u_{xx} + u_{yy} = 0 \quad \Rightarrow \quad \boxed{f''(x) + \frac{2x}{x^2 + 1}f'(x) = 0}$$

Let  $g = f'$ , then

$$g'(x) + \frac{2x}{x^2 + 1}g(x) = 0$$

The integrating factor is

$$\int \frac{2x}{x^2 + 1}dx = \ln|x^2 + 1| = \ln(x^2 + 1)$$

$$\phi(x) = e^{\ln(x^2 + 1)} = x^2 + 1$$

$$f'(x) = g(x) = \frac{1}{\phi(x)} \int 0 \cdot \phi(t)dt = \frac{c_1}{x^2 + 1} \quad \Rightarrow \quad \boxed{f(x) = c_1 \tan^{-1}(x) + c_2}$$

**b**

$$u(x, y) = f(x/y) = c_1 \tan^{-1}(x/y) + c_2$$

In polar coordinates (Let  $\theta$  be the angle w.r.t the  $y$ -axis):

$$u(r, \theta) = c_1\theta + c_2 \quad \Rightarrow \quad \frac{\partial u}{\partial r} \equiv 0$$

**c**

If  $\partial u / \partial r \equiv 0$ , then  $u = f(\theta) = (f \circ \tan^{-1})(x/y)$ , where  $\theta$  is the angle w.r.t. the  $y$ -axis.

**d**

$$h(x) = \lim_{y \rightarrow 0} u(x, y) = \lim_{y \rightarrow 0} c_1 \tan^{-1}(x/y) + c_2 = c_1\pi/2 + c_2$$

The boundary value is some constant.

**e**

From parts (c) and (d), if a function  $v(x, y)$  in  $\{y > 0\}$  is harmonic and satisfies  $\partial u / \partial r \equiv 0$ , then its boundary value is a constant.

Using the formula from Ex. 7.4.6:

Suppose  $\partial u / \partial r \equiv 0$ , then  $u$  doesn't depend on  $r$ . In other words, the value of  $u$  does not change if  $x$  and  $y$  are scaled by some constant:

$$\begin{aligned}
u(\lambda x_0, \lambda y_0) &= \frac{\lambda y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x - \lambda x_0)^2 + (\lambda y_0)^2} h(x) dx \\
&= \frac{\lambda y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(\lambda x' - \lambda x_0)^2 + (\lambda y_0)^2} h(\lambda x') \lambda dx' \\
&= \frac{\lambda y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{\lambda^2} \frac{1}{(x' - x_0)^2 + y_0^2} h(\lambda x') \lambda dx' \\
&= \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x' - x_0)^2 + y_0^2} h(\lambda x') dx' \\
&= \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x - x_0)^2 + y_0^2} h(x) dx
\end{aligned}$$

It is necessary that  $h(\lambda x) = h(x)$  for any scaling factor  $\lambda$ , so  $h(x)$  must be a constant function. Thus, the results are consistent.

## 7.4.17

**a**

Suppose  $\mathbf{x}_0 = (x_0, y_0)$ , then the reflection points are  $\mathbf{x}_1 = (-x_0, y_0)$ ,  $\mathbf{x}_2 = (-x_0, -y_0)$ ,  $\mathbf{x}_3 = (x_0, -y_0)$ . The corrector function can be defined as

$$h^{\mathbf{x}_0}(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{x}_1) - \Phi(\mathbf{x} - \mathbf{x}_2) + \Phi(\mathbf{x} - \mathbf{x}_3)$$

$\Delta h^{\mathbf{x}_0}(\mathbf{x}) = 0$  everywhere in  $Q$ , and by symmetry,

$$h^{\mathbf{x}_0}(x, 0) = 0 + \Phi(\mathbf{x} - \mathbf{x}_3) = \Phi(\mathbf{x} - \mathbf{x}_0)$$

$$h^{\mathbf{x}_0}(0, y) = \Phi(\mathbf{x} - \mathbf{x}_1) + 0 = \Phi(\mathbf{x} - \mathbf{x}_0)$$

Therefore, the Green's function is

$$\begin{aligned}
G(\mathbf{x}_0, \mathbf{x}) &= \Phi(\mathbf{x} - \mathbf{x}_0) - \Phi(\mathbf{x} - \mathbf{x}_1) + \Phi(\mathbf{x} - \mathbf{x}_2) - \Phi(\mathbf{x} - \mathbf{x}_3) \\
&= \boxed{-\frac{1}{2\pi} \left( \ln |\mathbf{x} - \mathbf{x}_0| - \ln |\mathbf{x} - \mathbf{x}_1| + \ln |\mathbf{x} - \mathbf{x}_2| - \ln |\mathbf{x} - \mathbf{x}_3| \right)}
\end{aligned}$$

**b**

On the x-axis,

$$\begin{aligned}
\frac{\partial G}{\partial n} &= -\frac{\partial G}{\partial y} = \frac{1}{2\pi} \left[ \frac{y-y_0}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{y-y_1}{|\mathbf{x}-\mathbf{x}_1|^2} + \frac{y-y_2}{|\mathbf{x}-\mathbf{x}_2|^2} - \frac{y-y_3}{|\mathbf{x}-\mathbf{x}_3|^2} \right] \\
&= \frac{y_0}{2\pi} \left[ -\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} + \frac{1}{|\mathbf{x}-\mathbf{x}_1|^2} + \frac{1}{|\mathbf{x}-\mathbf{x}_2|^2} - \frac{1}{|\mathbf{x}-\mathbf{x}_3|^2} \right] \\
&= \frac{y_0}{\pi} \left[ \frac{1}{|\mathbf{x}-\mathbf{x}_1|^2} - \frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} \right] \\
&= \frac{y_0}{\pi} \left[ \frac{1}{(x+x_0)^2+y_0^2} - \frac{1}{(x-x_0)^2+y_0^2} \right]
\end{aligned}$$

On the y-axis,

$$\begin{aligned}
\frac{\partial G}{\partial n} &= -\frac{\partial G}{\partial x} = \frac{1}{2\pi} \left[ \frac{x-x_0}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{x-x_1}{|\mathbf{x}-\mathbf{x}_1|^2} + \frac{x-x_2}{|\mathbf{x}-\mathbf{x}_2|^2} - \frac{x-x_3}{|\mathbf{x}-\mathbf{x}_3|^2} \right] \\
&= \frac{x_0}{2\pi} \left[ -\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{1}{|\mathbf{x}-\mathbf{x}_1|^2} + \frac{1}{|\mathbf{x}-\mathbf{x}_2|^2} + \frac{1}{|\mathbf{x}-\mathbf{x}_3|^2} \right] \\
&= \frac{x_0}{\pi} \left[ \frac{1}{|\mathbf{x}-\mathbf{x}_2|^2} - \frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} \right] \\
&= \frac{x_0}{\pi} \left[ \frac{1}{x_0^2+(y+y_0)^2} - \frac{1}{x_0^2+(y-y_0)^2} \right]
\end{aligned}$$

Finally, the solution is

$$\begin{aligned}
u(\mathbf{x}_0) &= -\int_{\partial Q} \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} u(\mathbf{x}) ds \\
&= \frac{y_0}{\pi} \int_0^\infty \left[ \frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2} \right] h(x) dx \\
&\quad + \frac{x_0}{\pi} \int_0^\infty \left[ \frac{1}{x_0^2+(y-y_0)^2} - \frac{1}{x_0^2+(y+y_0)^2} \right] g(y) dy
\end{aligned}$$

# MATH 4347 Homework 5

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## 7.7

The general form of the full Fourier series is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

where

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^0 -\sin x \cos nx dx + \frac{1}{\pi} \int_0^{\pi} \sin x \cos nx dx = \frac{2((-1)^n + 1)}{\pi(1 - n^2)}$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^0 -\sin x \cos nx dx + \frac{1}{\pi} \int_0^{\pi} \sin x \cos nx dx = 0$$

Therefore the series is

$$\begin{aligned} |\sin x| &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2((-1)^n + 1)}{\pi(1 - n^2)} \cos nx = \frac{2}{\pi} + \sum_{\text{even}} \frac{4}{\pi(1 - n^2)} \cos nx \\ &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1 - 4n^2)} \cos 2nx \end{aligned}$$

Since the series converges pointwise, at  $x = 0$

$$|\sin 0| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1 - 4n^2)} \cos 0 \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}}$$

At  $x = \pi/2$

$$|\sin \frac{\pi}{2}| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1 - 4n^2)} \cos n\pi$$

$$1 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1 - 4n^2)}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} - \frac{\pi}{4}}$$

**b**

Let  $g(x)$  be a even 2-periodic function, whose value in interval  $[-1, 1]$  is  $g(x) = x^2$ . The Fourier coefficients are

$$A_0 = \int_{-1}^1 x^2 = \frac{2}{3}, \quad A_n = \int_{-1}^1 x^2 \cos n\pi x = \frac{4(-1)^n}{n^2\pi^2}, \quad B_n = \int_{-1}^1 x^2 \sin n\pi x = 0$$

The full series is

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x$$

Evaluated at  $x = 1$

$$1 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Evaluated at  $x = 0$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

## 8.2

If  $u(x) \equiv c$ , then  $\max_{\bar{U}} u = \max_{\partial U} u \equiv c$ .

Otherwise, by the strong form of the maximum principle,

$$\begin{aligned} \forall x \in U : u(x) < \max_{\partial U} u &\Rightarrow \max_U u < \max_{\partial U} u \\ \max_{\bar{U}} u = \max\{\max_{\partial U} u, \max_U u\} &= \max_{\partial U} u \end{aligned}$$

### 5.2.11

The complex form of full Fourier series is

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} = e^x$$

where the coefficients are

$$\begin{aligned} c_n &= \frac{1}{2l} \int_{-l}^l e^x e^{-in\pi x/l} dx \\ &= \frac{1}{2l} \int_{-l}^l e^{(1-in\pi/l)x} dx \\ &= \frac{1}{2(l-in\pi)} e^{(1-in\pi/l)x} \Big|_{-l}^l \\ &= \frac{e^{l-in\pi} - e^{-l-in\pi}}{2(l-in\pi)} \\ &= (-1)^n \frac{e^l - e^{-l}}{2(l-in\pi)} \end{aligned}$$

So the series is

$$e^x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} e^{in\pi x/l} = \boxed{\sum_{n=-\infty}^{\infty} (-1)^n \sinh l \frac{l + in\pi}{l^2 + n^2\pi^2} e^{in\pi x/l}}$$

And since  $e^{i\theta} = \cos \theta + i \sin \theta$ ,

$$\begin{aligned} e^x &= \frac{e^l - e^{-l}}{2l} + \sum_{n=-1}^{-\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \cos \frac{n\pi x}{l} \\ &\quad + \sum_{n=-1}^{-\infty} (-1)^n \frac{ie^l - ie^{-l}}{2(l - in\pi)} \sin \frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{ie^l - ie^{-l}}{2(l - in\pi)} \sin \frac{n\pi x}{l} \\ &= \frac{e^l - e^{-l}}{2l} + \sum_{n=1}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l + in\pi)} \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \cos \frac{n\pi x}{l} \\ &\quad + \sum_{n=1}^{\infty} (-1)^n \frac{ie^{-l} - ie^l}{2(l + in\pi)} \sin \frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{ie^l - ie^{-l}}{2(l - in\pi)} \sin \frac{n\pi x}{l} \\ &= \frac{e^l - e^{-l}}{2l} + \sum_{n=1}^{\infty} (-1)^n \frac{(e^l - e^{-l})l}{l^2 + n^2\pi^2} \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{n\pi(e^{-l} - e^l)}{l^2 + n^2\pi^2} \sin \frac{n\pi x}{l} \\ &= \boxed{\frac{\sinh l}{l} + 2 \sinh l \sum_{n=1}^{\infty} \frac{(-1)^n}{l^2 + n^2\pi^2} \left[ l \cos \frac{n\pi x}{l} - n\pi \sin \frac{n\pi x}{l} \right]} \end{aligned}$$

### 5.3.4

**a**

Let  $v = u - U$ , then all derivatives of  $v$  is the same as  $u$ , and  $v(0, t) = u(0, t) - U = 0$ . Separation of variables gives

$$\frac{X''}{X} = \frac{T'}{kT} = \lambda, \quad v = X(x)T(t)$$

$\lambda = 0$  under the boundary condition gives only the trivial solution, and for  $\lambda > 0$  the boundary conditions cannot be both satisfied, therefore  $\lambda < 0$ . Let  $\lambda = -\beta^2$ , then

$$X = A \cos \beta x + B \sin \beta x$$

Boundary condition at 0 implies  $A = 0$ , and boundary condition at  $x = l$  implies

$$\cos \beta l = 0 \Rightarrow \beta_n = \left(n - \frac{1}{2}\right) \frac{\pi}{l} = (2n - 1) \frac{\pi}{2l}$$

$$X_n = \sin \beta_n x$$

The corresponding  $T_n$  are

$$T' = -k\beta_n^2 T \Rightarrow T = e^{-k\beta_n^2 t}$$

So the general solution is

$$\begin{aligned}
v &= \sum_{n=1}^{\infty} A_n e^{-k\beta_n^2 t} \sin \beta_n x, \quad \beta_n = (2n-1)\frac{\pi}{2l} \\
v(x, 0) &= \sum_{n=1}^{\infty} A_n \sin \beta_n x = u(x, 0) - U = -U \\
\Rightarrow A_n &= \frac{2}{l} \int_0^l -U \sin \beta_n x = -\frac{2U}{l\beta_n} = -\frac{4U}{(2n-1)\pi} \\
\Rightarrow v &= -\sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-k\pi^2(2n-1)^2 t/4l^2} \sin \frac{(2n-1)\pi}{2l} x \\
u = v + U &= \boxed{U - \sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-k\pi^2(2n-1)^2 t/4l^2} \sin \frac{(2n-1)\pi}{2l} x}
\end{aligned}$$

**b**

Let  $a_n$  be the  $n$ -th term of the series. Apply the ratio test:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2n-1}{2n+1} \frac{\sin \beta_{n+1} x}{\sin \beta_n x} e^{-2k\pi^2 n t/l^2} \right| \\
&= \lim_{n \rightarrow \infty} \frac{2n-1}{2n+1} e^{-2k\pi^2 n t/l^2} \left| \frac{\sin \beta_{n+1} x}{\sin \beta_n x} \right| \\
&\leq \lim_{n \rightarrow \infty} \frac{2n-1}{2n+1} e^{-2k\pi^2 n t/l^2} = 0
\end{aligned}$$

Therefore, the series converges.

**c**

The error is smaller than the second term, which is

$$-\frac{4U}{\pi} e^{-k\pi^2 t/4l^2}$$

In order for the error to be within  $\epsilon$ , it suffices to let

$$\left| \frac{4U}{\pi} e^{-k\pi^2 t/4l^2} \right| = \frac{4|U|}{\pi} e^{-k\pi^2 t/4l^2} < \epsilon$$

$$\boxed{t > -\frac{4l^2}{k\pi^2} \log \frac{\epsilon\pi}{4|U|}}$$



### 5.3.10

**a**

Proof by induction:

Base case: To show that  $(Z_2, Z_1) = 0$ , it's sufficient to show that  $(Y_2, X_1) = 0$ :

$$\begin{aligned}(Y_2, X_1) &= (X_2, X_1) - \left(X_2, \frac{X_1}{\|X_1\|}\right) \left(\frac{X_1}{\|X_1\|}, X_1\right) \\ &= (X_2, X_1) - (X_2, X_1) \frac{(X_1, X_1)}{\|X_1\|^2} \\ &= (X_2, X_1) - (X_2, X_1) = 0\end{aligned}$$

Inductive step: Suppose  $\forall n, m \leq k, n \neq m : (Z_n, Z_m) = 0$ . Then in order to prove that  $\forall n, m \leq k+1, n \neq m : (Z_n, Z_m) = 0$ , we only need to prove that  $(Z_{k+1}, Z_n) = 0$  for all  $n \leq k$ , because all other cases are already proven in the  $k$ -th step. And to that end, it's sufficient to show that  $(Y_{k+1}, Z_n) = 0$ . Since the vector in  $\{Z_n : n \leq k\}$  are orthonormal by inductive hypothesis,

$$\begin{aligned}(Y_{k+1}, Z_n) &= (X_{k+1}, Z_n) - (X_{k+1}, Z_1)(Z_1, Z_n) - \cdots - (X_{k+1}, Z_n)(Z_n, Z_n) - \cdots \\ &= (X_{k+1}, Z_n) - (X_{k+1}, Z_n)(Z_n, Z_n) \\ &= (X_{k+1}, Z_n) - (X_{k+1}, Z_n) = 0\end{aligned}$$

By induction, orthogonality holds for all  $k$ .

**b**

Let  $X_1 = \cos x + \cos 2x$ , and  $X_2 = 3 \cos x - 4 \cos 2x$ , then

$$(X_1, X_1) = (\cos x, \cos x) + (\cos 2x, \cos 2x) = \pi, \quad \|X_1\| = \sqrt{\pi}$$

$$Z_1 = \frac{X_1}{\|X_1\|} = \boxed{\frac{1}{\sqrt{\pi}}(\cos x + \cos 2x)}$$

$$(X_2, Z_1) = \frac{1}{\sqrt{\pi}} \left[ 3(\cos x, \cos x) - 4(\cos 2x, \cos 2x) \right] = -\frac{\sqrt{\pi}}{2}$$

$$\begin{aligned}Y_2 &= X_2 - (X_2, Z_1)Z_1 = 3 \cos x - 4 \cos 2x - \left(-\frac{\sqrt{\pi}}{2}\right) \frac{1}{\sqrt{\pi}}(\cos x + \cos 2x) \\ &= 3 \cos x - 4 \cos 2x + \frac{1}{2}(\cos x + \cos 2x) = \frac{7}{2} \cos x - \frac{7}{2} \cos 2x\end{aligned}$$

$$(Y_2, Y_2) = \frac{49}{4} \frac{\pi}{2} + \frac{49}{4} \frac{\pi}{2} = \frac{49\pi}{4}, \quad \|Y_2\| = \frac{7\sqrt{\pi}}{2}$$

$$Z_2 = \frac{Y_2}{\|Y_2\|} = \boxed{\frac{1}{\sqrt{\pi}}(\cos x - \cos 2x)}$$

### 5.4.7

**a**

The general formula is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x + B_n \sin n\pi x$$

where

$$A_n = \int_{-1}^1 \phi(x) \cos n\pi x dx = \int_{-1}^0 (-1-x) \cos n\pi x dx + \int_0^1 (1-x) \cos n\pi x dx = 0$$

$$B_n = \int_{-1}^1 \phi(x) \sin n\pi x dx = \int_{-1}^0 (-1-x) \sin n\pi x dx + \int_0^1 (1-x) \sin n\pi x dx = \frac{2}{n\pi}$$

Therefore the full series is

$$\phi(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi x$$

**b**

The first three non-zero terms are

$$\frac{2}{\pi} \sin \pi x + \frac{1}{\pi} \sin 2\pi x + \frac{2}{3\pi} \sin 3\pi x$$

**c**

Obviously,

$$\|\phi(x)\|^2 = \int_{-1}^1 \phi^2(x) dx < \infty$$

Therefore, the series converges in the mean square sense.

**d**

$$\phi'(x) = -1, \text{ for } x \in (-1, 0) \cup (0, 1)$$

Since  $\phi$  and  $\phi'$  are both piecewise continuous, the series converges pointwise. [At  $x = 0$ , it converges to  $\frac{1}{2}(-1 + 1) = 0$ .]

**e**

$\phi(x)$  has a discontinuity at 0, so  $\phi \notin C^2[-1, 1]$ , and therefore the series does not converge uniformly.

### 5.5.4

**a**

Separation of variables gives

$$-\frac{X''}{X} = -\frac{T'}{kT} = \lambda$$

For  $\lambda = 0$ ,

$$\begin{aligned} X'' = 0 &\Rightarrow X = A + Bx \\ T' = 0 &\Rightarrow T = \text{const.} \end{aligned}$$

For  $\lambda > 0$ ,

$$\begin{aligned} -X'' = \beta^2 X &\Rightarrow X = C \cos \beta x + D \sin \beta x \\ T' = -k\beta^2 T &\Rightarrow T = e^{-k\beta^2 t} \end{aligned}$$

The general solution is

$$u = A + Bx + \sum_{n=1}^{\infty} e^{-k\beta_n^2 t} [C_n \cos \beta_n x + D_n \sin \beta_n x]$$

**b**

As  $t \rightarrow \infty$ , each term in the sum converges to zero, therefore  $\lim u = A + Bx$ .

**c**

From the boundary condition

$$\begin{aligned} u_x(0, t) = u_x(l, t) &= \frac{u(l, t) - u(0, t)}{l} \\ X'(0)T(t) = X'(l)T(t) &= \frac{X(l)T(t) - X(0)T(t)}{l} \\ X'(0) = X'(l) &= \frac{X(l) - X(0)}{l} \end{aligned}$$

Green's first identity in one dimension is

$$vu' \Big|_0^l = \int_0^l v'u'dx + \int_0^l vu''dx$$

Let  $v = u = X$ , then

$$\begin{aligned} LHS &= XX' \Big|_0^l \\ &= X(l)X'(l) - X(0)X'(0) \\ &= X(l)\frac{X(l) - X(0)}{l} - X(0)\frac{X(l) - X(0)}{l} \\ &= \frac{[X(l) - X(0)]^2}{l} \end{aligned}$$

$$\begin{aligned}
RHS &= \int_0^l (X')^2 dx + \int_0^l X X'' dx \\
&= \int_0^l (X')^2 dx - \int_0^l \lambda X^2 dx
\end{aligned}$$

Multiply both sides by  $l$  and swap both sides,

$$l \int_0^l (X')^2 dx - l \int_0^l \lambda X^2 dx = [X(l) - X(0)]^2$$

If  $\lambda$  is negative, then the second integral is negative, which means that

$$l \int_0^l (X')^2 dx < [X(l) - X(0)]^2$$

This contradicts the inequality in Ex. 3. Therefore, there cannot be negative eigenvalues.

**d**

First we can verify that the boundary condition is symmetric:

$$\begin{aligned}
X'_1(l)X_2(l) - X'_2(l)X_1(l) &= X_2(l) \frac{X_1(l) - X_1(0)}{l} - X_1(l) \frac{X_2(l) - X_2(0)}{l} \\
&= \frac{X_1(l)X_2(0) - X_1(0)X_2(l)}{l}
\end{aligned}$$

$$\begin{aligned}
X'_1(0)X_2(0) - X'_2(0)X_1(0) &= X_2(0) \frac{X_1(l) - X_1(0)}{l} - X_1(0) \frac{X_2(l) - X_2(0)}{l} \\
&= \frac{X_1(l)X_2(0) - X_1(0)X_2(l)}{l}
\end{aligned}$$

$$X'_1 X_2 - X'_2 X_1 \Big|_0^l = 0$$

Therefore eigenfunctions associated with different eigenvalues are orthogonal. At  $t = 0$

$$\phi(x) = u(x, 0) = A + Bx + \sum_{n=1}^{\infty} C \cos \beta_n x + D \sin \beta_n x$$

$$(\phi, 1) = (A + Bx, 1)$$

$$\Rightarrow \int_0^l \phi(x) dx = \int_0^l A + Bx dx = \left[ Ax + \frac{B}{2} x^2 \right]_0^l = lA + \frac{l^2}{2} B$$

$$(\phi, x) = (A + Bx, x)$$

$$\Rightarrow \int_0^l \phi(x) x dx = \int_0^l Ax + Bx^2 dx = \left[ \frac{A}{2} x^2 + \frac{B}{3} x^3 \right]_0^l = \frac{l^2}{2} A + \frac{l^3}{3} B$$

$$\boxed{A = \frac{4}{l} \int_0^l \phi(x) dx - \frac{6}{l^2} \int_0^l \phi(x) x dx}$$

$$\boxed{B = \frac{12}{l^3} \int_0^l \phi(x) x dx - \frac{6}{l^2} \int_0^l \phi(x) dx}$$

### 6.1.2

The equation is  $\Delta u = k^2 u$ . Expressed in spherical coordinates, dropping zero terms:

$$\frac{1}{r} \partial_r^2 (ru) = k^2 u$$

Let  $v = ur$ . Multiply the equation by  $r$

$$v''(r) = k^2 v(r) \quad \Rightarrow \quad v = Ae^{kr} + Be^{-kr}$$

$$u = \frac{v}{r} = \boxed{\frac{1}{r} [Ae^{kr} + Be^{-kr}]}$$

### 6.1.9

**a**

The heat equation is  $u_t = \Delta u$ . In steady state,  $u_t = \Delta u = 0$ . Since  $u$  only depends on  $r$ ,

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (ru) = 0$$

Let  $v(r) = ru(r)$ , then  $v''(r) = 0 \Rightarrow v(r) = Ar + B$ ,

$$u(r) = \frac{v}{r} = A + \frac{B}{r}$$

$$u'(r) = -B \frac{1}{r^2}$$

Plug in the boundary conditions:

$$\begin{cases} u(1) = A + B = 100 \\ u'(2) = -\frac{B}{4} = -\gamma \end{cases} \Rightarrow \begin{cases} A = 100 - 4\gamma \\ B = 4\gamma \end{cases}$$

Therefore, the temperature distribution is

$$\boxed{u(r) = 100 - 4\gamma + \frac{4\gamma}{r}}$$

**b**

$$u'(r) = -\frac{4\gamma}{r^2} < 0$$

Temperature decreases radially, therefore the highest temperature is obtained at  $r = 1$ , which is  $\boxed{100}$ , and the lowest temperature is obtained at  $r = 2$ , which is  $\boxed{100 - 2\gamma}$

**c**

Yes.  $\gamma = 40$ .

# MATH 4347 Homework 4

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## 6.5

let  $\lambda = \beta^2$ , then the eigenfunctions are

$$v = A \cos \beta x + B \sin \beta x, \quad v' = -\beta A \sin \beta x + \beta B \cos \beta x$$

At  $x = 0$ ,

$$\beta B - a_0 A = 0 \Rightarrow B = \frac{a_0 A}{\beta}$$

At  $x = l$ ,

$$\begin{aligned} -\beta A \sin \beta l + \beta B \cos \beta l + a_l(A \cos \beta l + B \sin \beta l) &= 0 \\ \Rightarrow -\beta A \sin \beta l + a_0 A \cos \beta l + a_l(A \cos \beta l + \frac{a_0 A}{\beta} \sin \beta l) &= 0 \\ \Rightarrow -\beta \tan \beta l + a_0 + a_l + a_l \frac{a_0}{\beta} \tan \beta l &= 0 \\ \Rightarrow \tan \beta l &= \frac{(a_0 + a_l)\beta}{\beta^2 - a_l a_0} \end{aligned}$$

**a**

$(a_0 + a_l)\beta/(\beta^2 - a_l a_0)$  decreases to zero continuously as  $\beta \rightarrow \infty$ , therefore it intersects  $\tan \beta l$  in every period (except for the first few ones). So there are infinitely many  $\lambda_n$  that satisfies the above equation.

**b**

Since the two graphs intersect in the positive halves of  $\tan \beta l$ ,

$$\begin{aligned} \frac{(n-1)\pi}{l} < \beta_n < \frac{(n-1)\pi}{l} + \frac{\pi}{2l} &= \frac{(2n-1)\pi}{2l} \\ \frac{(n-1)^2\pi^2}{l^2} < \lambda_n < \frac{(2n-1)^2\pi^2}{4l^2} \end{aligned}$$

**c**

Since  $(a_0 + a_l)\beta/(\beta^2 - a_l a_0)$  approaches zero as  $n \rightarrow \infty$ , its intersections with  $\tan \beta l$  will approach the  $\beta$ -intercepts of  $\tan \beta l$ , therefore

$$\lim_{n \rightarrow \infty} \beta_n = \frac{(n-1)\pi}{l}, \text{ or equivalently, } \lim_{n \rightarrow \infty} \lambda_n - \frac{(n-1)^2\pi^2}{l^2} = 0$$

**d**

$$\begin{aligned}\tan((n-1)\pi + \theta_n l) &= \frac{(a_0 + a_l) \left( \frac{(n-1)\pi}{l} + \theta_n \right)}{\left( \frac{(n-1)\pi}{l} + \theta_n \right)^2 - a_l a_0} \\ \tan \theta_n l \left( \frac{(n-1)^2 \pi^2}{l^2} + \theta_n^2 + 2 \frac{(n-1)\pi}{l} \theta_n - a_l a_0 \right) &= (a_0 + a_l) \left( \frac{(n-1)\pi}{l} + \theta_n \right) \\ \left( \theta_n l + O(\theta_n^3) \right) \left( \frac{(n-1)^2 \pi^2}{l^2} - a_l a_0 + O(\theta_n) \right) &= (a_0 + a_l) \left( \frac{(n-1)\pi}{l} + \theta_n \right) \\ \left( \frac{(n-1)^2 \pi^2}{l} - a_l a_0 l \right) \theta_n + O(\theta_n^2) &= (a_0 + a_l) \frac{(n-1)\pi}{l} + (a_0 + a_l) \theta_n\end{aligned}$$

Dropping terms of order higher than the second

$$\begin{aligned}[(n-1)^2 \pi^2 - (a_0 + a_l + a_l a_0 l)l] \theta_n &= (a_0 + a_l)(n-1)\pi \\ \theta_n &= \frac{(a_0 + a_l)(n-1)\pi}{(n-1)^2 \pi^2 - (a_0 + a_l + a_l a_0 l)l}\end{aligned}$$

let  $x = 1/n \rightarrow 0$ , then

$$\begin{aligned}\theta_x &= \frac{(a_0 + a_l)(x - x^2)\pi}{(1-x)^2 \pi^2 - (a_0 + a_l + a_l a_0 l)lx^2} = 0 + \frac{a_0 + a_l}{\pi} x + O(x^2) \\ \Rightarrow \theta_n &= \frac{a_0 + a_l}{\pi n} + O\left(\frac{1}{n^2}\right)\end{aligned}$$

## 6.7

For each of the two regions,

$$\begin{aligned}(p_i v')' + \lambda r_i v &= 0 \quad \Rightarrow \quad v'' = -\frac{\lambda r_i}{p_i} v \\ \Rightarrow v_i &= A_i \cos \beta_i x + B_i \sin \beta_i x, \text{ where } \beta_i = \sqrt{\frac{\lambda r_i}{p_i}}\end{aligned}$$

For  $v_1$ , the boundary condition at  $x = 0$  gives  $A_1 = 0$ . Therefore

$$v_1 = C \sin \beta_1 x, \quad v_1' = \beta_1 C \cos \beta_1 x$$

For  $v_2$ , the boundary condition at  $x = l$  gives

$$A_2 \cos \beta_2 l + B_2 \sin \beta_2 l = 0$$

$$A_2 = -B_2 \tan \beta_2 l$$

Plug it into the equation for  $v$ ,

$$v_2 = -B \tan \beta_2 l \cos \beta_2 x + B \sin \beta_2 x, \quad v_2' = \beta_2 B \tan \beta_2 l \sin \beta_2 x + \beta_2 B \cos \beta_2 x$$

Because the eigenfunctions are required to be continuously differentiable, at  $x = m$ :

$$\begin{cases} C \sin \beta_1 m = -B \tan \beta_2 l \cos \beta_2 m + B \sin \beta_2 m \\ \beta_1 C \cos \beta_1 m = \beta_2 B \tan \beta_2 l \sin \beta_2 m + \beta_2 B \cos \beta_2 m \end{cases}$$

Dividing two equations

$$\boxed{\tan \beta_1 m = \frac{-\beta_1 \tan \beta_2 l \cos \beta_2 m + \beta_1 \sin \beta_2 m}{\beta_2 \tan \beta_2 l \sin \beta_2 m + \beta_2 \cos \beta_2 m}, \quad \beta_1 = \sqrt{\frac{\lambda r_1}{p_1}}, \quad \beta_2 = \sqrt{\frac{\lambda r_2}{p_2}}}$$

## 6.8

$$\begin{aligned} v &= a \cos \mu x + b \sin \mu x + c \cosh \mu x + d \sinh \mu x \\ v' &= -a\mu \sin \mu x + b\mu \cos \mu x + c\mu \sinh \mu x + d\mu \cosh \mu x \\ v'' &= -a\mu^2 \cos \mu x - b\mu^2 \sin \mu x + c\mu^2 \cosh \mu x + d\mu^2 \sinh \mu x \\ v''' &= a\mu^3 \sin \mu x - b\mu^3 \cos \mu x + c\mu^3 \sinh \mu x + d\mu^3 \cosh \mu x \end{aligned}$$

From the boundary conditions

$$v(0) = a + c = 0 \Rightarrow c = -a$$

$$v'(0) = b\mu + d\mu = 0 \Rightarrow d = -b$$

$$\begin{aligned} v''(l) &= -a\mu^2 \cos \mu l - b\mu^2 \sin \mu l - a\mu^2 \cosh \mu l - b\mu^2 \sinh \mu l = 0 \\ &\Rightarrow a \cos \mu l + a \cosh \mu l = -b \sin \mu l - b \sinh \mu l \end{aligned} \tag{1}$$

$$\begin{aligned} v'''(l) &= a\mu^3 \sin \mu l - b\mu^3 \cos \mu l - a\mu^3 \sinh \mu l - b\mu^3 \cosh \mu l = 0 \\ &\Rightarrow -a \sin \mu l + a \sinh \mu l = -b \cos \mu l - b \cosh \mu l \end{aligned} \tag{2}$$

Dividing (2) by (1) gives

$$\frac{-\sin \mu l + \sinh \mu l}{\cos \mu l + \cosh \mu l} = \frac{\cos \mu l + \cosh \mu l}{\sin \mu l + \sinh \mu l}$$

$$\sinh^2 \mu l - \sin^2 \mu l = \cos^2 \mu l + \cosh^2 \mu l + 2 \cos \mu l \cosh \mu l$$

$$\boxed{\cos \mu l \cosh \mu l + 1 = 0, \quad \lambda = \mu^4}$$

$$\cos \mu l = -\frac{1}{\cosh \mu l}$$

$\cos \mu l$  oscillates between  $\pm 1$ , and  $-1/\cosh \mu l$  converges to zero from below as  $\mu$  increases, so there are infinitely many solutions for  $\mu$ . More precisely, in each cycle of cosine the two functions intersect twice between the trough of cosine and its two x-intercepts. The one on the left of the trough satisfies

$$-\frac{3\pi}{2l} + k\frac{2\pi}{l} < \mu < -\frac{\pi}{l} + k\frac{2\pi}{l}$$



$$(2k - \frac{3}{2})\frac{\pi}{l} < \mu < (2k - 1)\frac{\pi}{l}$$

The one on the right of the trough satisfies

$$-\frac{\pi}{l} + k\frac{2\pi}{l} < \mu < -\frac{\pi}{2l} + k\frac{2\pi}{l}$$

$$(2k - 1)\frac{\pi}{l} < \mu < (2k - \frac{1}{2})\frac{\pi}{l}$$

As RHS approaches zero, its intersections with LHS also approaches the x-intercepts of LHS:

$$\lim_{n \rightarrow \infty} \mu_n \approx -\frac{\pi}{2l} + n\frac{\pi}{l} = \left(n - \frac{1}{2}\right)\frac{\pi}{l}$$

$$\lim_{n \rightarrow \infty} \lambda_n = \lim_{n \rightarrow \infty} \mu_n^4 \approx \boxed{\left(n - \frac{1}{2}\right)^4 \frac{\pi^4}{l^4} \approx \left(\frac{n\pi}{l}\right)^4}$$

## 6.9

**a**

Since  $f$  is odd, we have  $f(x) = -1, -\pi < x < 0$ . The general form of Fourier series is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

where

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = -\frac{1}{\pi} \int_{-\pi}^0 \cos nx dx + \frac{1}{\pi} \int_0^{\pi} \cos nx dx = 0$$

$$\begin{aligned} B_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = -\frac{1}{\pi} \int_{-\pi}^0 \sin nx dx + \frac{1}{\pi} \int_0^{\pi} \sin nx dx \\ &= \frac{1}{\pi} \left[ \frac{1}{n} \cos nx \Big|_{-\pi}^0 - \frac{1}{n} \cos nx \Big|_0^{\pi} \right] \\ &= \frac{1}{n\pi} [(1 - \cos n\pi) - (\cos n\pi - 1)] \\ &= \frac{2(1 - (-1)^n)}{n\pi} \end{aligned}$$

Thus, only the odd sine terms remain

$$B_{2k+1} = \frac{4}{(2k+1)\pi}$$

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin(2k+1)x}$$

**b**

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \sum_{\text{odd}} \frac{4}{n\pi} \sin \frac{n\pi}{4} = \frac{4}{\pi} \frac{\sqrt{2}}{2} + \frac{4}{3\pi} \frac{\sqrt{2}}{2} - \frac{4}{5\pi} \frac{\sqrt{2}}{2} - \frac{4}{7\pi} \frac{\sqrt{2}}{2} + \dots \\ &= \frac{2\sqrt{2}}{\pi} \left[ 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \dots \right] = 1 \end{aligned}$$

Therefore,

$$\boxed{1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{2\sqrt{2}}}$$

### 4.3.4

(i) Let

$$\begin{aligned} h(\gamma) &= -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l} \\ h'(\gamma) &= \frac{(a_0 + a_l)(\gamma^2 - a_0 a_l)}{(\gamma^2 + a_0 a_l)^2} = 0 \Rightarrow \gamma = \sqrt{a_0 a_l} \end{aligned}$$

From the graph of  $h(\gamma)$  this must be the only maximum. The value of  $h$  at this point is

$$h(\sqrt{a_0 a_l}) = \frac{-a_0 - a_l}{2\sqrt{a_0 a_l}} \geq \frac{2\sqrt{(-a_0)(-a_l)}}{2\sqrt{a_0 a_l}} = 1$$

And since  $h(0) = 0 < 1$  and  $\lim_{\gamma \rightarrow \infty} h(\gamma) = 0 < 1$ , it must cross  $y = 1$  exactly twice.

(ii) Let  $g(\gamma) = \tanh(\gamma l)$ , then  $\lim_{\gamma \rightarrow \infty} g(\gamma) = 1$ . Therefore, as  $\gamma \rightarrow \infty$ ,  $g(\gamma) > h(\gamma)$ .

$$g(0) = 0, \quad g'(0) = \frac{l}{\cosh^2(\gamma l)} \Big|_{\gamma=0} = l$$

and from the assumption that  $-a_0 - a_l < a_0 a_l l$ , we have

$$h'(0) = \frac{(-a_0 - a_l)a_0 a_l}{(a_0 a_l)^2} = \frac{-a_0 - a_l}{a_0 a_l} < \frac{a_0 a_l l}{a_0 a_l} = l = g'(0)$$

Therefore, near  $\gamma = 0$ , we also have  $g(\gamma) > h(\gamma)$ . However, at  $\gamma = \sqrt{a_0 a_l}$ , since  $\tanh(x) < 1$ ,

$$g(\sqrt{a_0 a_l}) < 1 < h(\sqrt{a_0 a_l})$$

From above observations,  $h(\gamma) - g(\gamma)$  changes sign exactly twice, which means that the two functions intersect exactly twice, which then implies there are two (negative) eigenvalues.

### 4.3.9

**a**

$$X'' = 0 \Rightarrow X = ax + b$$

From boundary conditions, we get  $b = -a$ . Dropping the constant factor,

$$\boxed{X_0(x) = x - 1}$$

**b**

$$X = A \cos \beta x + B \sin \beta x, \quad X' = -A\beta \sin \beta x + B\beta \cos \beta x$$

From boundary condition at  $x = 0$ ,  $A = -\beta B$ . Rewriting  $X$  as  $X = -\beta B \cos \beta x + B \sin \beta x$ , and from boundary condition at  $x = 1$ ,

$$-\beta B \cos \beta + B \sin \beta = 0 \Rightarrow \boxed{\tan \beta = \beta}$$

**c**

From the graph of  $f(\beta) = \beta$  and  $g(\beta) = \tan \beta$ , the two curves intersect infinitely many times, which means that there are infinitely many positive eigenvalues.

**d**

Suppose there exists a negative eigenvalue, then

$$X'' = -\lambda X = \gamma^2 X$$

$$X = A \cosh \gamma x + B \sinh \gamma x$$

$$X' = A\gamma \sinh \gamma x + B\gamma \cosh \gamma x$$

From the boundary condition at  $x = 0$ ,  $A = -\gamma B$ . Rewrite  $X$  and plug in the boundary condition at  $x = 1$ :

$$-\gamma B \cosh \gamma + B \sinh \gamma = 0 \Rightarrow \tanh \gamma = \gamma$$

However, since  $\tanh(0) = 0$  and  $\tanh'(0) = 1/\cosh^2(0) = 1$ ,  $\gamma$  and  $\tanh \gamma$  are tangent at the origin and have no other intersections. Therefore a non-zero  $\gamma$  doesn't exist, which means that there isn't a negative eigenvalue.

### 4.3.18

**a**

Suppose  $u = X(x)T(t)$ .

$$XT'' = -c^2 X''''T \Rightarrow -\frac{T''}{c^2 T} = \frac{X''''}{X} = \lambda \Rightarrow X'''' = \lambda X$$

**b**

Suppose zero is an eigenvalue, then  $X'''' = 0 \Rightarrow X = ax^3 + bx^2 + cx + d$ . And its derivatives:

$$X' = 3ax^2 + 2bx + c$$

$$X'' = 6ax + 2b$$

$$X''' = 6a$$

From the boundary conditions,  $X(0) = X'(0) = X''(l) = X'''(l) = 0$ , therefore  $a, b, c, d$  must all be zero, which means that  $X$  does not have non-trivial solutions. Therefore, zero is not a eigenvalue.

**c**

Carrying out the same calculations as in Problem 6.8 above,  $\boxed{\cos \beta l \cosh \beta l = -1}$ .

**d**

From Problem 6.8, the frequencies  $\beta_n$  are approximately  $\frac{(n-1/2)\pi}{l}$  when  $n$  is large.

**e**

Solving the above equation using a computer, the results are

$$\beta_1 \approx \frac{1.875}{l}, \quad \beta_2 \approx \frac{4.694}{l}, \quad \frac{\beta_2^2}{\beta_1^2} \approx 6.267$$

For a vibrating string,  $\beta_2^2/\beta_1^2 = 2^2 = 4$ . The overtone frequencies of a tuning fork grows faster than a string as  $n$  increases.

## 5.1.2

**a**

The sine series is

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

where

$$\begin{aligned} A_n &= 2 \int_0^1 x^2 \sin(n\pi x) \\ &= 2 \left[ -\frac{1}{n\pi} x^2 \cos n\pi x + \frac{2}{n^2 \pi^2} x \sin n\pi x + \frac{2}{n^3 \pi^3} \cos n\pi x \right] \Big|_0^1 \\ &= (-1)^n \left( \frac{4}{n^3 \pi^3} - \frac{2}{n\pi} \right) - \frac{4}{n^3 \pi^3} \end{aligned}$$

**b**

The cosine series is

$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

where

$$A_0 = 2 \int_0^1 x^2 = \frac{2}{3}$$

For  $n \geq 1$ ,

$$\begin{aligned}
A_n &= 2 \int_0^1 x^2 \cos(n\pi x) \\
&= 2 \left[ \frac{1}{n\pi} x^2 \sin n\pi x + \frac{2}{n^2 \pi^2} x \cos n\pi x - \frac{2}{n^3 \pi^3} \sin n\pi x \right] \Big|_0^1 \\
&= (-1)^n \frac{4}{n^2 \pi^2}
\end{aligned}$$

Combine the results,

$$\phi(x) = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2 \pi^2} \cos(n\pi x)$$

### 5.1.9

Separate the variables:

$$\frac{X''}{X} = \frac{T''}{c^2 T} = \lambda$$

The boundary conditions and initial conditions then translate to

$$X'(0) = X'(\pi) = 0, \quad T(0) = 0, \quad X(x)T'(0) = \cos^2 x$$

(i) For  $\lambda = 0$ ,  $X'' = 0 \Rightarrow X = Ax + B$ . The boundary condition implies that  $A = 0$ , so

$$X_0 = 1$$

The corresponding  $T_0 = Ct + D$ , using the initial condition  $T_0(0) = D = 0$ . Therefore

$$T_0 = t$$

(ii) For  $\lambda > 0$ , the boundary conditions cannot be satisfied.

(iii) For  $\lambda < 0$ , write  $\lambda$  as  $-\beta^2$ , then

$$X = A \cos \beta x + B \sin \beta x, \quad X' = -A\beta \sin \beta x + B\beta \cos \beta x$$

$X'(0) = 0$  implies that  $B = 0$ , and  $X'(\pi) = 0$  implies:

$$-A\beta \sin \beta \pi = 0 \Rightarrow \beta \pi = n\pi \Rightarrow \beta = n$$

$$X_n = \begin{cases} 1, & n = 0 \\ \cos nx, & n > 0 \end{cases}$$

Solving for  $T_n$ :

$$T_n = C_n \cos cnt + D_n \sin cnt$$

The initial condition implies that  $C_n = 0$ , so

$$T_n = \sin cnt, \quad T'_n = cn \cos cnt$$

$$T_n = \begin{cases} t, & n = 0 \\ \sin cnt, & n > 0 \end{cases}$$

Finally

$$u(x, t) = C_0 X_0 T_0 + \sum_{n=1}^{\infty} C_n X_n T_n = C_0 t + \sum_{n=1}^{\infty} C_n \cos nx \sin cnt$$

$$u_t(x, t) = C_0 + \sum_{n=1}^{\infty} cn C_n \cos nx \cos cnt$$

$$u_t(x, 0) = C_0 + \sum_{n=1}^{\infty} cn C_n \cos nx = \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

Comparing the terms, we get

$$C_n = \begin{cases} 1/2, & n = 0 \\ 1/4c, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$u(x, t) = \frac{1}{2}t + \frac{1}{4c} \cos 2x \sin 2ct$$

# MATH 4347 Homework 3

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## 5.6

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt} + y\right)} H(y) dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-\left(\frac{(x-y)^2}{4kt} + y\right)} dy \\ &= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_0^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}} dy \\ &= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \sqrt{4kt} \int_0^{\infty} e^{-z^2} dz \\ &= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \sqrt{4kt} \frac{\sqrt{\pi}}{2} = \boxed{\frac{e^{kt-x}}{2}} \end{aligned}$$

## 5.9

**a**

Let  $u = e^{-dt}v$ , then

$$u_t = -de^{-dt}v + e^{-dt}v_t, \quad u_{xx} = e^{-dt}v_{xx}$$

The original equation becomes

$$-de^{-dt}v + e^{-dt}v_t + de^{-dt}v = ke^{-dt}v_{xx} \quad \Rightarrow \quad v_t = kv_{xx}$$

$$g(x) = u(x, 0) = e^0 v(x, 0) = v(x, 0)$$

Using the fundamental solution

$$v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y) dy \quad \Rightarrow \quad \boxed{u(x, t) = \frac{e^{-dt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y) dy}$$

**b**

It makes the solution decay exponentially at a speed  $d$ .

**c**

Suppose now we let  $u = f(t)v(x, t)$ , then

$$u_t = f'v + fv_t, \quad u_{xx} = fv_{xx}$$

$$f'v + fv_t + dfv = kfv_{xx}$$

The objective is to let  $f'v$  and  $dfv$  cancel out, so  $f$  should satisfy

$$f' = -df \Rightarrow f = e^{-\int d(t)dt}$$

Therefore the change of variable should be

$$\boxed{u(x, t) = e^{-\int d(t)dt}v(x, t)}$$

## 6.2

Suppose  $u(x, t) = X(x)T(t)$ , then the equation becomes

$$XT'' = 9X''T \Rightarrow \frac{T''}{9T} = \frac{X''}{X} = k$$

The initial and boundary conditions imply that

$$X(0) = X(1) = 0$$

First solve for  $X$ :

$$X'' = kX$$

In order for  $X$  to satisfy the boundary conditions, it cannot be exponential, therefore

$$k = -\beta^2 \Rightarrow X = A \cos \beta x + B \sin \beta x$$

The boundary condition at  $x = 0$  implies that  $A = 0$ . At  $x = 1$ ,  $X(1) = B \sin \beta = 0$ , so

$$\beta_n = n\pi$$

Let  $T(t)$  absorb the constant coefficient, then

$$X_n = \sin(n\pi x)$$

Now solve for each corresponding  $T_n$ :

$$T_n'' = 9kT_n = -9\beta_n^2 T_n$$

$$T_n = C_n \cos(3\beta_n t) + D_n \sin(3\beta_n t), \quad T_n' = -3\beta_n C_n \sin(3\beta_n t) + 3\beta_n D_n \cos(3\beta_n t)$$

The general solution is

$$u(x, t) = \sum_{n=0}^{\infty} X_n T_n = \sum_{n=0}^{\infty} [C_n \cos(3n\pi t) + D_n \sin(3n\pi t)] \sin(n\pi x)$$



$$u_t(x, t) = \sum_{n=0}^{\infty} X_n T'_n = \sum_{n=0}^{\infty} \left[ -3n\pi C_n \sin(3n\pi t) + 3n\pi D_n \cos(3n\pi t) \right] \sin(n\pi x)$$

Apply the initial conditions,

$$u(x, 0) = \sum_{n=0}^{\infty} C_n \sin(n\pi x) = 2 \sin(\pi x) + 7 \sin(3\pi x)$$

$$u_t(x, 0) = \sum_{n=0}^{\infty} 3n\pi D_n \sin(n\pi x) = 2 \sin(\pi x)$$

Comparing the terms, we can get

$$C_1 = 2, \quad C_3 = 7, \quad D_1 = \frac{2}{3\pi}$$

All other coefficients are zero. Therefore the solution is

$$u(x, t) = \left[ 2 \cos(3\pi t) + \frac{2}{3\pi} \sin(3\pi t) \right] \sin(\pi x) + 7 \cos(9\pi t) \sin(3\pi x)$$

## 6.3

**a**

If  $\lambda = 0$ , then  $v'' = 0$ ,  $v = kx + m$ . From the boundary conditions,

$$k - a_0 m = 0, \quad (1 + a_L L)k + a_L m = 0$$

$k, m$  could be any solution of the system of equations.

**b**

Since the system has non-trivial solutions,

$$\det \begin{pmatrix} 1 & -a_0 \\ 1 + a_L L & a_L \end{pmatrix} = a_0 + a_L + a_0 a_L L = 0$$

**c**

Determinant being zero is also sufficient for non-trivial solutions to exist, therefore it guarantees that  $\lambda = 0$  is an eigenvalue.

## 2.3.4

**a**

On the initial line,  $u$  attains maximum at  $x = 1/2$  and minimum at two end points

$$u(1/2, 0) = 1, \quad u(0, 0) = u(1, 0) = 0$$

By the maximum principle for heat equation,

$$\max_D u(x, t) = \max_{\Gamma} u(x, t) = 1, \quad \min_D u(x, t) = \min_{\Gamma} u(x, t) = 0$$

**b**

Let  $\xi = 1 - x$ , then  $\bar{u}(x, t) = u(1 - x, t) = u(\xi, t)$ .

$$\bar{u}_t = u_t, \quad \bar{u}_{xx} = (-1)(-1)u_{\xi\xi} = u_{\xi\xi}$$

Since  $u_t = u_{\xi\xi}$ ,  $\bar{u}_t = \bar{u}_{xx}$ , which means that  $u(1 - x, t)$  also satisfies the heat equation. Also,

$$u(1 - x, 0) = 4(1 - x)(1 - (1 - x)) = 4(1 - x)x = u(x, 0)$$

The initial data for two functions are the same. By uniqueness of solutions,

$$u(x, t) = u(1 - x, t)$$

**c**

$$\frac{d}{dt} \int_0^1 u^2 dx = 2 \int_0^1 uu_t dx = 2 \int_0^1 uu_{xx} dx = 2uu_x \Big|_0^1 - 2 \int_0^1 u_x^2 dx = -2 \int_0^1 u_x^2 dx \leq 0$$

Therefore  $\int_0^1 u^2 dx$  is strictly decreasing.

## 2.4.9

$u_{xxx}$  satisfies the heat equation since

$$(u_{xxx})_t = (u_t)_{xxx} = (ku_{xx})_{xxx} = k(u_{xxx})_{xx}$$

The initial value for  $u_{xxx}$  is

$$u_{xxx}(x, 0) = (u(x, 0))''' = 0$$

Since the zero function is obviously a solution, by uniqueness of solutions,  $u_{xxx} \equiv 0$ . Integrating yields  $u = A(t)x^2 + B(t)x + C(t)$ . Plug this into the original problem

$$A'(t)x^2 + B'(t)x + C'(t) = 2kA(t)$$

RHS is a function of  $t$  alone, therefore  $A' = B' = 0 \Rightarrow A = a, B = b$ , where  $a, b$  are constants, and  $C' = 2kA = 2ka \Rightarrow C = 2kat + c$ . Using the initial condition

$$u(x, 0) = A(0)x^2 + B(0)x + C(0) = ax^2 + bx + c = x^2 \Rightarrow \begin{cases} a = 1 \\ b = 0 \\ c = 0 \end{cases}$$

Therefore,

$$\boxed{u(x, t) = x^2 + 2kt}$$

### 3.4.13

Odd-extend  $u$  to  $\tilde{u}$ . The initial and boundary conditions for the extended function are

$$\phi(x) = \tilde{u}(x, 0) = x, \quad \psi(x) = \tilde{u}_t(x, 0) = 0, \quad h(t) = x(0, t) = t^2$$

The solution for  $x > ct$  is

$$u = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} 0 = \boxed{x}$$

The solution for  $x < ct$  is

$$u = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} 0 + h\left(t - \frac{x}{c}\right) = \boxed{x + \left(t - \frac{x}{c}\right)^2}$$

### 4.1.3

Suppose  $u = X(x)T(t)$ , then from the equation,

$$\frac{T'}{iT} = \frac{X''}{X} = k$$

First solve for  $X$ :

$$X'' = kX, X(0) = X(l) = 0$$

Since  $X$  must be zero at two points, we must have  $k = -\lambda^2$ .

$$X'' = -\lambda^2 X \Rightarrow X = A \cos \lambda x + B \sin \lambda x$$

From the boundary conditions,

$$A = 0, \quad \sin \lambda l = 0 \Rightarrow \lambda_n = \frac{n\pi}{l}$$

Let  $T$  absorb the constant, then

$$X_n = \sin \frac{n\pi}{l} x$$

Now solve for each corresponding  $T_n$ ,

$$T'_n = -i\lambda_n^2 T \Rightarrow T_n = C_n e^{-i\lambda_n^2 t} = C_n e^{-i\left(\frac{n\pi}{l}\right)^2 t}$$

Combining above results, the general solution is

$$\boxed{u = \sum_{n=0}^{\infty} C_n e^{-i\left(\frac{n\pi}{l}\right)^2 t} \sin \frac{n\pi}{l} x}$$

### 4.2.1

Suppose  $u = X(x)T(t)$ , then from the equation,

$$\frac{T'}{\kappa T} = \frac{X''}{X} = k$$

First solve for  $X$ :

$$X'' = kX = -\lambda^2 X, \quad X(0) = X'(l) = 0$$

$$X = A \cos \lambda x + B \sin \lambda x$$

From the boundary conditions,

$$A = 0, \quad \cos \lambda l = 0 \Rightarrow \lambda l = \left(n + \frac{1}{2}\right) \pi, \quad \lambda_n = \frac{(2n+1)\pi}{2l}$$

Let  $T$  absorb the constant, then

$$X_n = \sin \frac{(2n+1)\pi}{2l} x$$

Now solve for each corresponding  $T_n$ ,

$$T'_n = -\kappa \lambda_n^2 T \quad \Rightarrow \quad T_n = C_n e^{-\kappa \lambda_n^2 t} = C_n e^{-\kappa ((2n+1)\pi/2l)^2 t}$$

Combining above results, the general solution is

$$u = \sum_{n=0}^{\infty} C_n e^{-\kappa ((2n+1)\pi/2l)^2 t} \sin \frac{(2n+1)\pi}{2l} x$$

# MATH 4347 Homework 2

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## 3.14

If  $u_0$  is strictly increasing, then the characteristic lines  $x = u_0(x_0)t + x_0$  will not intersect, because a line that starts with a larger  $x_0$  will also have a greater slope. The fact that  $u_0$  is bounded means that the characteristic lines have a maximum slope and therefore will not approach the  $x$  axis. Therefore, the solution is well defined for all  $t > 0$ .

## 4.1

Using the d'Alembert's formula:

$$u = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds$$

For any  $x$ , when  $t$  becomes sufficiently large so that  $x-t < 1$  and  $x+t > 3$ , the above formula becomes

$$u = \frac{1}{2} \int_1^3 \psi(s) ds$$

which is a constant. To make the constant zero, it is both necessary and sufficient that the above definite integral evaluate to zero.

## 4.2

**a**

$$\begin{aligned} e_t &= u_t u_{tt} + u_x u_{xt} \\ p_x &= u_{tx} u_x + u_t u_{xx} = u_x u_{xt} + u_t u_{tt} = e_t \\ p_t &= u_{tt} u_x + u_t u_{xt} \\ e_x &= u_t u_{tx} + u_x u_{xx} = u_t u_{xt} + u_x u_{tt} = p_t \end{aligned}$$

**b**

From the result of (a),

$$\begin{aligned} e_{tt} &= p_{xt}, & e_{xx} &= p_{tx} = p_{xt} = e_{tt} \\ p_{tt} &= e_{xt}, & p_{xx} &= e_{tx} = e_{xt} = p_{tt} \end{aligned}$$

Therefore, they both satisfy the wave equation.

## 4.4

**a**

By d'Alembert's formula:

$$u(x+h, t+k) = \frac{1}{2}(\phi(x+t+(h+k)) + \phi(x-t+(h-k))) + \frac{1}{2c} \left( \int_{x-t+(h-k)}^0 \psi(s) ds + \int_0^{x+t+(h+k)} \psi(s) ds \right)$$

$$u(x-h, t-k) = \frac{1}{2}(\phi(x+t-(h+k)) + \phi(x-t-(h-k))) + \frac{1}{2c} \left( \int_{x-t-(h-k)}^0 \psi(s) ds + \int_0^{x+t-(h+k)} \psi(s) ds \right)$$

$$u(x+k, t+h) = \frac{1}{2}(\phi(x+t+(k+h)) + \phi(x-t+(k-h))) + \frac{1}{2c} \left( \int_{x-t+(k-h)}^0 \psi(s) ds + \int_0^{x+t+(k+h)} \psi(s) ds \right)$$

$$u(x-k, t-h) = \frac{1}{2}(\phi(x+t-(k+h)) + \phi(x-t-(k-h))) + \frac{1}{2c} \left( \int_{x-t-(k-h)}^0 \psi(s) ds + \int_0^{x+t-(k+h)} \psi(s) ds \right)$$

It can be easily verified that both sides have exactly the same terms.

**b**

If  $c = 2$ , the the characteristic coordinates are  $x \pm 2t$ . Therefore the corresponding identity should be

$$u(x + 2h, t + k) + u(x - 2h, t - k) = u(x + 2k, t + h) + u(x - 2k, t - h)$$

## 4.6

**a**

The d'Alembert formula

$$u_2(x, t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds$$

works for  $x > t$ , but not  $x < t$ . From the general solution  $u = F(x-t) + G(x+t)$  and initial condition we can still arrive at

$$\begin{cases} F(x) = \frac{1}{2}\phi(x) - \frac{1}{2} \int_0^x \psi(s) ds \\ G(x) = \frac{1}{2}\phi(x) + \frac{1}{2} \int_0^x \psi(s) ds \end{cases}$$

To get  $F(x)$  for negative  $x$ , we need to apply the boundary condition.

$$u_x(0, t) = G'(t) + F'(-t) = 0$$

$$\begin{aligned}
\int_0^x G'(t)dt + \int_0^x F'(-t)dt &= 0 \\
\int_0^x G'(t)dt - \int_0^{-x} F'(t')dt' &= 0 \\
G(x) - F(-x) &= C \\
G(0) - F(0) &= C
\end{aligned}$$

From initial condition,  $F(0) + G(0) = \phi(0) = 0$ . Adding this to the last equation,

$$2G(0) = C = 0$$

Therefore  $F(-x) = G(x)$ ,

$$\begin{aligned}
u_1(x, t) &= G(x+t) + F(x-t) = G(x+t) + G(t-x) \\
&= \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2} \left( \int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds \right)
\end{aligned}$$

**b**

$u_1 \equiv 0$  when  $x+t$  and  $t-x$  are both outside of  $[1, 2]$ , and the two integrals are both zero regardless of  $\psi$ . The only situation where this can be satisfied is

$$x+t < 1, \quad t-x < 1 \quad \Rightarrow \quad t-1 < x < 1-t$$

which is valid only when  $t < 1$ , and since  $u_1$  is only defined for  $0 < x < t$ ,

$$\begin{cases} 0 < x < 1-t, & \frac{1}{2} < t < 1 \\ 0 < x < t, & t < \frac{1}{2} \end{cases}$$

$u_2 \equiv 0$  when  $x+t$  and  $x-t$  are on the same side of  $[1, 2]$ :

$$\begin{cases} x+t < 1, & x-t < 1 & \Rightarrow & t < x < 1-t, & 0 < t < \frac{1}{2} \\ x+t > 2, & x-t > 2 & \Rightarrow & x > t+2 \end{cases}$$

**c**

$$u(x, t) = \begin{cases} \frac{1}{2} \left( \int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds \right), & 0 < x < t \\ \frac{1}{2} \left( \int_0^{x+t} \psi(s)ds + \int_{x-t}^0 \psi(s)ds \right), & x > t \end{cases}$$

**d**

$$\begin{aligned}
u_1 &= \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2} \left( \int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds \right) \\
\lim_{x \rightarrow t} u_1 &= \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2} \int_0^{2t} \psi(s)ds
\end{aligned}$$

$$\begin{aligned}
(u_1)_x &= \frac{1}{2}(\phi'(x+t) - \phi'(t-x)) + \frac{1}{2}(\psi(x+t) - \psi(t-x)) \\
\lim_{x \rightarrow t} (u_1)_x &= \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}(\psi(2t) - \psi(0)) \\
(u_1)_t &= \frac{1}{2}(\phi'(x+t) + \phi'(t-x)) + \frac{1}{2}(\psi(x+t) + \psi(t-x)) \\
\lim_{x \rightarrow t} (u_1)_t &= \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}(\psi(2t) + \psi(0)) \\
u_2 &= \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds \\
\lim_{x \rightarrow t} u_2 &= \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2} \int_0^{2t} \psi(s) ds \\
(u_2)_x &= \frac{1}{2}(\phi'(x+t) + \phi'(x-t)) + \frac{1}{2}(\psi(x+t) - \psi(x-t)) \\
\lim_{x \rightarrow t} (u_2)_x &= \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}(\psi(2t) - \psi(0)) \\
(u_2)_t &= \frac{1}{2}(\phi'(x+t) - \phi'(x-t)) + \frac{1}{2}(\psi(x+t) + \psi(x-t)) \\
\lim_{x \rightarrow t} (u_2)_t &= \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}(\psi(2t) + \psi(0))
\end{aligned}$$

**i**

For  $u$  to be continuous,

$$\lim_{x \rightarrow t} u_1 = \lim_{x \rightarrow t} u_2$$

which is always true.

**ii**

For  $u$  to be  $C^1$ ,

$$\begin{aligned}
\lim_{x \rightarrow t} (u_1)_t &= \lim_{x \rightarrow t} (u_2)_t, \quad \lim_{x \rightarrow t} (u_1)_x = \lim_{x \rightarrow t} (u_2)_x \\
\phi'(0) &= -\phi'(0) \Rightarrow \phi'(0) = 0
\end{aligned}$$

which means that in order for a smooth wave to be stress-free at one end, there must be no stress applied on that end in the beginning.



## 4.7

By the same argument as 4.6,

$$\begin{cases} F(x) = \frac{1}{2}\phi(x) - \frac{1}{2}\int_0^x \psi(s)ds \\ G(x) = \frac{1}{2}\phi(x) + \frac{1}{2}\int_0^x \psi(s)ds \end{cases}$$

And for  $x > t$ ,

$$u_2(x, t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds$$

Apply the new boundary condition:

$$u_x(0, t) = G'(t) + F'(-t) = h(t)$$

$$\begin{aligned} \int_0^x G'(t)dt + \int_0^x F'(-t)dt &= \int_0^x h(t)dt + C \\ \int_0^x G'(t)dt - \int_0^{-x} F'(t')dt' &= \int_0^x h(t)dt + C \\ G(x) - F(-x) &= \int_0^x h(t)dt + C \end{aligned}$$

Taking the limit as  $x \rightarrow 0$ , we get  $C = 0$ , therefore

$$F(-x) = G(x) - \int_0^x h(t)dt$$

Therefore,

$$\begin{aligned} u_1 &= G(x+t) + F(x-t) = -\int_0^{t-x} h(y)dy + G(x+t) + G(t-x) \\ &= -\int_0^{t-x} h(y)dy + \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds\right) \\ \lim_{x \rightarrow t} u_1 &= \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_0^{2t} \psi(s)ds \\ \lim_{x \rightarrow t} u_2 &= \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_0^{2t} \psi(s)ds \end{aligned}$$

So the solution is always continuous.

$$\begin{aligned} \lim_{x \rightarrow t} (u_1)_x &= h(0) + \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}(\psi(2t) - \psi(0)) \\ \lim_{x \rightarrow t} (u_1)_t &= -h(0) + \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}(\psi(2t) + \psi(0)) \end{aligned}$$

$$\lim_{x \rightarrow t} (u_2)_x = \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}(\psi(2t) - \psi(0))$$

$$\lim_{x \rightarrow t} (u_2)_t = \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}(\psi(2t) + \psi(0))$$

In order for the derivatives to match, it is necessary that

$$h(0) - \frac{1}{2}\phi'(0) = \frac{1}{2}\phi'(0)$$

$$-h(0) + \frac{1}{2}\phi'(0) = -\frac{1}{2}\phi'(0)$$

$$\phi'(0) = h(0)$$

### 2.1.1

Using d'Alembert's formula,

$$\begin{aligned} u(x, t) &= \frac{1}{2}(e^{x+ct} + e^{x-ct}) + \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(s) ds \\ &= \frac{1}{2}(e^{x+ct} + e^{x-ct}) - \frac{1}{2c}(\cos(x+ct) - \cos(x-ct)) \end{aligned}$$

### 2.1.9

Rewrite the equation using operators

$$(\partial_x^2 - 3\partial_x\partial_t - 4\partial_t^2)u = 0$$

$$(\partial_x - 4\partial_t)(\partial_x + \partial_t)u = 0$$

The first operator is the directional derivative along the line that satisfies  $\frac{dx}{dt} = -\frac{1}{4}$ . The second one is along the line  $\frac{dx}{dt} = 1$ . Thus we can choose characteristic coordinates  $\xi = x + \frac{1}{4}t$ , and  $\eta = x - t$ .

$$\partial_x = \partial_\xi + \partial_\eta$$

$$\partial_t = \frac{1}{4}\partial_\xi - \partial_\eta$$

Solving the above two equations,

$$\partial_\xi = \frac{4}{5}(\partial_x + \partial_t)$$

$$\partial_\eta = \frac{1}{5}(\partial_x - 4\partial_t)$$

So the original equation becomes

$$(5\partial_\eta)(\frac{5}{4}\partial_\xi)u = \frac{25}{4}u_{\xi\eta} = 0$$

whiche means that the solution has the form

$$u = F(\xi) + G(\eta) = F(x + \frac{1}{4}t) + G(x - t)$$

Initial condition tells us that

$$\begin{aligned} F(x) + G(x) &= x^2 \\ u_t &= \frac{1}{4}F'(x + \frac{1}{4}t) - G'(x - t) \\ u_t(x, 0) &= \frac{1}{4}F'(x) - G'(x) = e^x \\ \int_0^x \frac{1}{4}F'(s) - G'(s)ds &= \int_0^x e^s \\ \frac{1}{4}F(x) - G(x) &= e^x + A \end{aligned}$$

From the first and last equations,

$$\begin{aligned} F(x) &= \frac{4}{5}(x^2 + e^x + A) = \frac{1}{5}(4x^2 + 4e^x + 4A) \\ G(x) &= \frac{1}{5}(x^2 - 4e^x - 4A) \end{aligned}$$

$$\begin{aligned} u(x, t) &= F(x + \frac{1}{4}t) + G(x - t) \\ &= \frac{1}{5} \left[ 4 \left( x + \frac{1}{4}t \right)^2 + 4e^{x+\frac{1}{4}t} + 4A \right] + \frac{1}{5} [(x - t)^2 - 4e^{x-t} - 4A] \\ &= \frac{1}{5} \left[ 4 \left( x + \frac{1}{4}t \right)^2 + 4e^{x+\frac{1}{4}t} \right] + \frac{1}{5} [(x - t)^2 - 4e^{x-t}] \\ &= \frac{1}{5} \left[ 4 \left( x + \frac{1}{4}t \right)^2 + (x - t)^2 \right] + \frac{4}{5} [e^{x+\frac{1}{4}t} - e^{x-t}] \end{aligned}$$

### 2.2.3

**a**

Let  $\xi = x - y$ , then

$$\begin{aligned} \partial_x u(x - y, t) &= \partial_\xi u(\xi, t) \\ \partial_{xx} u(x - y, t) &= \partial_{\xi\xi} u(\xi, t) \end{aligned}$$

And

$$\partial_{tt} u(x - y, t) = \partial_{tt} u(\xi, t)$$

Therefore

$$\partial_{tt} u(x - y, t) = \partial_{tt} u(\xi, t) = c^2 \partial_{\xi\xi} u(\xi, t) = c^2 \partial_{xx} u(x - y, t)$$

**b**

$$(u_x)_{tt} = (u_{tt})_x = (c^2 u_{xx})_x = c^2 (u_x)_{xx}$$

Therefore  $u_x$  is a solution. The cases for other derivatives can be proved in the same way.

**c**

Let  $\bar{x} = ax$ ,  $\bar{t} = at$ ,

$$\partial_{tt}u(ax, at) = a^2 \partial_{\bar{t}\bar{t}}u(\bar{x}, \bar{t})$$

$$\partial_{xx}u(ax, at) = a^2 \partial_{\bar{x}\bar{x}}u(\bar{x}, \bar{t})$$

$$\partial_{tt}u(ax, at) - c^2 \partial_{xx}u(ax, at) = a^2 \left[ \partial_{\bar{t}\bar{t}}u(\bar{x}, \bar{t}) - c^2 \partial_{\bar{x}\bar{x}}u(\bar{x}, \bar{t}) \right] = 0$$

Therefore  $u(ax, at)$  satisfies the equation.

# MATH 4347 Homework 1

September 9, 2018

## 1.4

$$\begin{aligned}u_t &= a'(t)e^{2x} + b'(t)e^x + c'(t) \\u_{xx} &= 4a(t)e^{2x} + b(t)e^x\end{aligned}$$

Compare the two derivatives, we get

$$\begin{cases} a' = 4a \\ b' = b \\ c' = 0 \end{cases} \Rightarrow \begin{cases} a(t) = C_1 e^{4t} \\ b(t) = C_2 e^t \\ c(t) = C_3 \end{cases}$$

## 1.7

The characteristic lines are  $x - 4t = m$ , so  $u = f(x - 4t)$ . From the initial condition

$$u(x, 0) = f(x) = 0, \text{ where } x > 0$$

From the boundary condition

$$u(0, t) = f(-4t) = te^{-t} \text{ where } t > 0$$

Let  $x = -4t$ , then  $t = -\frac{1}{4}x$ .

$$f(x) = -\frac{1}{4}xe^{\frac{1}{4}x} \text{ where } x < 0$$

Combining above results

$$f(x) = \begin{cases} -\frac{1}{4}xe^{\frac{1}{4}x} & \text{for } x < 0 \\ 0, & \text{for } x > 0 \end{cases}$$

$$u = f(x - 4t) = \begin{cases} -\frac{1}{4}(x - 4t)e^{\frac{1}{4}(x - 4t)} & \text{for } 0 < x < 4t \\ 0, & \text{for } x > 4t \end{cases}$$

If the PDE changed to  $u_t - 4u_x = 0$ , then  $u = f(x + 4t)$  and the boundary condition implies that

$$f(4t) = te^{-t} \text{ where } t > 0$$

This is a contradiction because the initial condition tells us that  $f(x) = 0$  for positive arguments, and therefore the equation has no solution.

## 2.5

**a**

Let  $u(x, t) = e^{i\xi x + \sigma t}$ , then  $u_{tt} = \sigma^2 u$ , and  $u_{xxx} = \xi^3 u$ . So the PDE becomes

$$\sigma^2 = -\xi^4 \Rightarrow \sigma = \pm i\xi^2$$

$$u(x, t) = e^{i\xi x \pm i\xi^2 t} = e^{i\xi(x \pm \xi t)}$$

The wave speed depends on  $\xi$ , so the wave is dispersive. The reason for this dependence is that the derivatives in space and time are of different orders, so the resulting dispersion relation  $\sigma(\xi)$  is not linear, so there are powers of  $\xi$  that cannot be factored out. By contrast, the space and time derivatives in wave equation are both of the second order, therefore  $\sigma = \pm c\xi$ , and consequently the wave speed does not depend on wave number.

**b**

$$u_t = \sigma u$$

$$u_x = i\xi u$$

$$u_{xxt} = -\xi^2 \sigma u$$

Dividing the original PDE by  $u$

$$\sigma + ic\xi - \beta\xi^2\sigma = 0 \Rightarrow \sigma = \frac{ic\xi}{\beta\xi^2 - 1}$$

So the solution is

$$u = e^{i\xi x + \frac{ic\xi}{\beta\xi^2 - 1}t} = e^{i\xi(x - \frac{c}{1 - \beta\xi^2}t)}$$

which are traveling waves with speed  $\frac{c}{1 - \beta\xi^2}$ . Compared to the solution of linearized KdV equation

$$u(x, t) = e^{i\xi(x - (c - \beta\xi^2)t)}$$

which has wave speed  $c - \beta\xi^2$ , the wave speed of the solutions of BBM equation is monotonically increasing w.r.t.  $\xi^2$  when  $\xi^2 < \frac{1}{\beta}$  and  $\xi^2 > \frac{1}{\beta}$ , however the wave speed of the solutions of KdV equation is monotonically decreasing w.r.t.  $\xi^2$

## 2.6

**a**

Intuitively, the velocity should be higher when there are less traffic, so  $v(u)$  should be monotonically decreasing.

**b**

Since  $v(u)$  is monotonically decreasing and both  $u$  and  $v$  need to be nonnegative to have physical meanings,  $v_{max} = v(u_{min}) = v(0)$  and  $v(u_{max}) = v_{min} = 0$

**c**

From (b),  $Q(0) = 0v(0) = 0$ , and  $Q(u_{max}) = u_{max}v(u_{max}) = u_{max}0 = 0$ . If  $Q$  doesn't attain maximum at some point in the interval  $(0, u_{max})$ , then  $Q(0) = Q(u_{max}) = 0$  must be the maximum, and for all  $u$  in the interval,  $Q(u)$  must be strictly less than 0 (otherwise these points will also maximum), which is a contradiction because we required that  $u$  and  $v$  both be non-negative. Therefore,  $Q(u)$  must have a maximum in the interval.

**d**

Such function can be constructed. Consider

$$Q(u) = u \left( -\frac{1}{4}u^3 + 2u^2 - \frac{11}{2}u + 6 \right) = -\frac{1}{4}u^4 + 2u^3 - \frac{11}{2}u^2 + 6u$$

whose derivative is

$$Q'(u) = -(u-1)(u-2)(u-3)$$

The function has maximum at 1 and 3, and we can check that

$$v(u) = -\frac{1}{4}u^3 + 2u^2 - \frac{11}{2}u + 6$$

is positive and monotonically decreasing in interval  $(0, 4)$ .

## 3.2

First solve for the characteristic curves:

$$\frac{dx}{dt} = \frac{1}{1+t^2} \Rightarrow x = \arctan t + m$$

So  $u(x, t) = f(m) = f(x - \arctan t)$ . Plug in the initial condition:

$$u(x, 0) = f(x - \arctan 0) = f(x) = \sin x$$

Therefore,

$$u(x, t) = f(x - \arctan t) = \sin(x - \arctan t)$$

### 3.3

Treat the first two terms as the directional derivative of  $u$  in  $(1, 1)$  direction. Consider the characteristic lines  $x = t + k$ , or  $(s, s + k)$  where the equation becomes an ODE:

$$\frac{d}{ds}u + 3u = e^{3s+2k}$$

Solving use integrating factor  $\phi(s) = e^{\int 3ds} = e^{3s}$

$$\begin{aligned} u(s) &= \frac{1}{e^{3s}} \int e^{3s} e^{3s+2k} ds = \frac{1}{6} e^{3s+2k} + A(k) e^{-3s} \\ u(x, t) &= \frac{1}{6} e^{2x+t} + A(x-t) e^{-3t} \end{aligned}$$

Plugin the initial condition to solve for  $A(x)$ :

$$u(x, 0) = \frac{1}{6} e^{2x} + A(x) = x \Rightarrow A(x) = x - \frac{1}{6} e^{2x}$$

Thus the solution is

$$\begin{aligned} u(x, t) &= \frac{1}{6} e^{2x+t} + \left( (x-t) - \frac{1}{6} e^{2(x-t)} \right) e^{-3t} \\ &= \frac{1}{6} e^{2x+t} - \frac{1}{6} e^{2x-5t} + x e^{-3t} - t e^{-3t} \end{aligned}$$

### 3.8

**a**

the LHS can be treated as a total derivative w.r.t to  $t$  on the characteristic curve

$$\frac{dx}{dt} = t \Rightarrow x = \frac{t^2}{2} + k$$

On the characteristic curve  $(s, \frac{s^2}{2} + k)$  the PDE becomes ODE

$$\frac{du}{ds} = u^2 \Rightarrow u = -\frac{1}{s + A(k)} = -\frac{1}{t + A(x - t^2/2)}$$

Plug in the initial condition:

$$u(x, 0) = -\frac{1}{A(x)} = \frac{1}{1+x^2} \Rightarrow A(x) = -1 - x^2$$

Therefore the solution is

$$u(x, t) = -\frac{1}{t - 1 - (x - t^2/2)^2} = \frac{1}{(x - t^2/2)^2 - t + 1}$$



**b**

Given  $t$ ,  $u(x, t)$  attains maximum at critical point

$$\frac{\partial u}{\partial x} = -u^2 \cdot 2(x - \frac{t^2}{2}) = 0 \Rightarrow x_c = \frac{t^2}{2}$$

$$\max_x u(x, t) = u(\frac{t^2}{2}, t) = \frac{1}{1-t}$$

which blows up as  $t \rightarrow 1^-$

### 1.3. #2

Apply Newton's second law to a segment of the chain from  $x = a$  to  $x = b$ :

$$F(b) \sin \theta - F(a) \sin \theta = \int_a^b \rho u_{tt} \frac{1}{\cos \theta} dx$$

$$\int_b^{L_x} \rho g \frac{\sin \theta}{\cos \theta} dx - \int_a^{L_x} \rho g \frac{\sin \theta}{\cos \theta} dx = - \int_a^b \rho g \frac{\sin \theta}{\cos \theta} dx = \int_a^b \rho u_{tt} \frac{1}{\cos \theta} dx$$

Since this must be true for arbitrary  $a, b$ , the integrands must be equal.

$$-\rho g \frac{\sin \theta}{\cos \theta} = \rho u_{tt} \frac{1}{\cos \theta}$$

$$u_{tt} = -g \sin \theta$$

where

$$\sin \theta = \frac{u}{\sqrt{u^2 + x^2}} = \frac{u}{x} + \mathcal{O}(u^2) \approx \frac{u}{x}$$

when  $u$  is small. Therefore the chain satisfies PDE

$$xu_{tt} + gu = 0$$

### 1.5 #2

**a**

No. Suppose there are two different functions  $u_1$  and  $u_2$  that satisfy the equations. Now consider their difference  $v = u_1 - u_2$ .

$$\begin{cases} u_1''(x) + u_1'(x) = f(x) \\ u_2''(x) + u_2'(x) = f(x) \end{cases} \Rightarrow v''(x) + v'(x) = 0 \Rightarrow v(x) = Ae^{-x} + B$$

and the boundary conditions:

$$v'(0) = v(0) = \frac{1}{2}[v'(l) + v(l)]$$

$$\Rightarrow -A = A + B = \frac{1}{2}B$$

$$\Rightarrow B = -2A, \quad v(x) = Ae^{-x} - 2A$$

Thus if we can find any function  $u_0$  that satisfies the problem, then the family of functions

$$u = u_0 + Ce^{-t} - 2C$$

are all valid solutions.

**b**

Rearranging the second equation and applying the Fundamental Theorem of Calculus:

$$\begin{aligned} u'(0) + u(0) &= u'(l) + u(l) \Rightarrow \left[ u'(x) + u(x) \right]_0^l = 0 \\ \Rightarrow \int_0^l u''(x) + u'(x) dx &= \int_0^l f(x) dx = 0 \end{aligned}$$

Therefore in order for the problem to have a solution, the integral of  $f$  from 0 to  $l$  must be zero.