MATH 4441 Homework 10

Wenqi He

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13.3

i

Since differential is a linear map,

$$\begin{aligned}\Big((W+V)f\Big)_p &= (W+V)_p f = df_p(W+V)\\ &= df_p(W) + df_p(V) = W_p f + V_p f\\ &= \Big(Wf + Vf\Big)_p\end{aligned}$$

$$\begin{aligned}\Big((gW)f\Big)_p &= (gW)_p f = (g(p)W_p)f = df_p(g(p)W)\\ &= g(p)\cdot df_p(W) = g(p)W_p f\\ &= \Big(g(Wf)\Big)_p\end{aligned}$$

From the result of Ex.2, $\overline{\nabla}_W V = (WV^1, \cdots, WV^n)$. Therefore,

$$\begin{aligned}\overline{\nabla}_{W+Z} V &= ((W+Z)V^1, \cdots, (W+Z)V^n)\\ &= (WV^1, \cdots, WV^n) + (ZV^1, \cdots, ZV^n)\\ &= \overline{\nabla}_W V + \overline{\nabla}_Z V\end{aligned}$$

$$\begin{aligned}\overline{\nabla}_{fW} V &= ((fW)V^1, \cdots, (fW)V^n)\\ &= (f(WV^1), \cdots, f(WV^n))\\ &= f(WV^1, \cdots, WV^n)\\ &= f\,\overline{\nabla}_W V\end{aligned}$$

ii

Suppose $\gamma(0) = p$, $\gamma'(0) = W_p$. From the linearity of derivatives.

$$\begin{aligned}\Big(W(f+g)\Big)_p &= W_p(f+g) = ((f+g)\circ\gamma)'(0)\\ &= (f\circ\gamma)'(0) + (g\circ\gamma)'(0)\\ &= W_p f + W_p g\\ &= \Big(Wf + Wg\Big)_p\end{aligned}$$

$$
\begin{aligned}
\Big(W(fg)\Big)_p &= W_p(fg) = ((fg)\circ\gamma)'(0) \\
&= D(fg)(p)\gamma'(0) \\
&= \sum_i D_i(fg)(p){\gamma^i}'(0) \\
&= \sum_i D_i f(p)g(p){\gamma^i}'(0) + \sum_i f(p)D_i g(p){\gamma^i}'(0) \\
&= g(p)\sum_i D_i f(p){\gamma^i}'(0) + f(p)\sum_i D_i g(p){\gamma^i}'(0) \\
&= (W_p f)g(p) + f(p)(W_p g) \\
&= \Big((Wf)g + f(Wg)\Big)_p
\end{aligned}
$$

Now applying these identities,

$$
\begin{aligned}
\overline{\nabla}_W(V+Z) &= (W(V^1+Z^1),\cdots,W(V^n+Z^n)) \\
&= (WV^1,\cdots,WV^n) + (WZ^1,\cdots,WZ^n) \\
&= \overline{\nabla}_W V + \overline{\nabla}_W Z
\end{aligned}
$$

$$
\begin{aligned}
\overline{\nabla}_W(fV) &= (W(fV^1),\cdots,W(fV^n)) \\
&= ((Wf)V^1,\cdots,(Wf)V^n) + (f(WV^1),\cdots,f(WV^n)) \\
&= Wf(V^1,\cdots,V^n) + f(WV^1,\cdots,WV^n) \\
&= (Wf)V + f\,\overline{\nabla}_W V
\end{aligned}
$$

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$$
\begin{aligned}
\Big(Z\langle V,W\rangle\Big)_p &= Z_p\langle V,W\rangle = Z_p\sum_i V^iW^i = \sum_i Z_p\left(V^iW^i\right) \\
&= \sum_i (Z_pV^i)W^i + \sum_i V^i(Z_pW^i) \\
&= \sum_i (\overline{\nabla}_{Z_p} V)^iW^i + \sum_i V^i(\overline{\nabla}_{Z_p} W)^i \\
&= \Big\langle \overline{\nabla}_{Z_p} V, W\Big\rangle + \Big\langle V, \overline{\nabla}_{Z_p} W\Big\rangle \\
&= \Big(\Big\langle \overline{\nabla}_Z V, W\Big\rangle + \Big\langle V, \overline{\nabla}_Z W\Big\rangle\Big)_p
\end{aligned}
$$

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From Ex.2, $\overline{\nabla}_W Z = (WZ^1,\cdots,WZ^n)$,

$$\overline{\nabla}_V \overline{\nabla}_W Z = (V(WZ^1),\cdots,V(WZ^n))$$

Similarly,

$$\overline{\nabla}_W \overline{\nabla}_V Z = (W(VZ^1),\cdots,W(VZ^n))$$

And from Ex.5

$$\begin{aligned}\overline{\nabla}_{[V,W]}\, Z &= ([V,W]Z^1, \cdots, [V,W]Z^n) \\ &= (V(WZ^1), \cdots, V(WZ^n)) - (W(VZ^1), \cdots, W(VZ^n))\end{aligned}$$

The three terms cancel out, therefore

$$\overline{R}(V,W)Z = \overline{\nabla}_V\, \overline{\nabla}_W\, Z - \overline{\nabla}_W\, \overline{\nabla}_V\, Z - \overline{\nabla}_{[V,W]}\, Z \equiv \mathbf{0}$$

$$\overline{R}(V,W,Z,Y) = \langle \overline{R}(V,W)Z, Y\rangle \equiv 0$$

13.7

By definition,

$$\nabla_W V = (\overline{\nabla}_W\, V)^\top = \overline{\nabla}_W\, V - (\overline{\nabla}_W\, V)^\perp = \overline{\nabla}_W\, V - \langle \overline{\nabla}_W\, V, n\rangle n$$

i

$$\begin{aligned}\nabla_{W+Z} V &= \overline{\nabla}_{W+Z}\, V - \langle \overline{\nabla}_{W+Z}\, V, n\rangle n \\ &= \overline{\nabla}_W\, V + \overline{\nabla}_Z\, V - \langle \overline{\nabla}_W\, V + \overline{\nabla}_Z\, V, n\rangle n \\ &= \Big[\overline{\nabla}_W\, V - \langle \overline{\nabla}_W\, V, n\rangle n\Big] + \Big[\overline{\nabla}_Z\, V - \langle \overline{\nabla}_Z\, V, n\rangle n\Big] \\ &= \nabla_W V + \nabla_Z V\end{aligned}$$

ii

$$\begin{aligned}\nabla_{fW} V &= \overline{\nabla}_{fW}\, V - \langle \overline{\nabla}_{fW}\, V, n\rangle n \\ &= f\, \overline{\nabla}_W\, V - \langle f\, \overline{\nabla}_W\, V, n\rangle n \\ &= f\, \overline{\nabla}_W\, V - f\langle \overline{\nabla}_W\, V, n\rangle n \\ &= f \nabla_W V\end{aligned}$$

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$$\begin{aligned}\nabla_W (V+Z) &= \overline{\nabla}_W (V+Z) - \langle \overline{\nabla}_W (V+Z), n\rangle n \\ &= \overline{\nabla}_W\, V + \overline{\nabla}_W\, Z - \langle \overline{\nabla}_W\, V + \overline{\nabla}_W\, Z, n\rangle n \\ &= \Big[\overline{\nabla}_W\, V - \langle \overline{\nabla}_W\, V, n\rangle n\Big] + \Big[\overline{\nabla}_W\, Z - \langle \overline{\nabla}_W\, Z, n\rangle n\Big] \\ &= \nabla_W V + \nabla_W Z\end{aligned}$$

iv

$$\begin{aligned}
\nabla_W(fV) &= \overline{\nabla}_W(fV) - \langle \overline{\nabla}_W(fV), n\rangle n \\
&= (Wf)V + f\,\overline{\nabla}_W\,V - \langle (Wf)V + f\,\overline{\nabla}_W\,V, n\rangle n \\
&= \Big[(Wf)V - (Wf)\langle V, n\rangle n\Big] + \Big[f\,\overline{\nabla}_W\,V - f\langle \overline{\nabla}_W\,V, n\rangle n\Big] \\
&= Wf\Big[V - \langle V, n\rangle n\Big] + f\Big[\overline{\nabla}_W\,V - \langle \overline{\nabla}_W\,V, n\rangle n\Big] \\
&= (Wf)V + f\Big[\overline{\nabla}_W\,V - \langle \overline{\nabla}_W\,V, n\rangle n\Big] \\
&= (Wf)V + f\nabla_W V
\end{aligned}$$

v

$$\begin{aligned}
&\langle \nabla_Z V, W\rangle + \langle V, \nabla_Z W\rangle \\
&= \Big\langle \Big(\overline{\nabla}_Z\,V - \langle \overline{\nabla}_Z\,V, n\rangle n\Big), W\Big\rangle + \Big\langle V, \Big(\overline{\nabla}_Z\,W - \langle \overline{\nabla}_Z\,W, n\rangle n\Big)\Big\rangle \\
&= \Big\langle \overline{\nabla}_Z\,V, W\Big\rangle + \Big\langle V, \overline{\nabla}_Z\,W\Big\rangle - \Big\langle \langle \overline{\nabla}_Z\,V, n\rangle n, W\Big\rangle - \Big\langle V, \langle \overline{\nabla}_Z\,W, n\rangle n\Big\rangle \\
&= \Big\langle \overline{\nabla}_Z\,V, W\Big\rangle + \Big\langle V, \overline{\nabla}_Z\,W\Big\rangle = Z\langle V, W\rangle
\end{aligned}$$

The last two terms vanish because W, V are in the tangent space but n is a normal vector.

13.10

The equation becomes Gauss's formula in local coordinates when $W = \overline{X}_j$, $V = \overline{X}_i$. Define $u_j : (-\epsilon, \epsilon) \to \mathbb{R}^2$ as $u_j(t) = tE_j$, then the LHS is

$$\begin{aligned}
\overline{\nabla}_{(\overline{X}_j)_p}\,\overline{X}_i &= \Big(\overline{X}_i \circ (X \circ u_j)\Big)'(0) = \Big(X_i \circ X^{-1} \circ X \circ u_j\Big)'(0) \\
&= \Big(X_i \circ u_j\Big)'(0) = DX_i(0,0)E_j = X_{ij}(0,0)
\end{aligned}$$

and

$$\begin{aligned}
RHS &= \nabla_{(\overline{X}_j)_p}\overline{X}_i + \Big\langle \overline{X}_i(p), S(\overline{X}_j(p))\Big\rangle n(p) \\
&= \sum_k \overline{\Gamma}^k_{ij}(p)\overline{X}_k(p) + \Big\langle \overline{X}_i(p), S(\overline{X}_j(p))\Big\rangle n(p) \\
&= \sum_k \Gamma^k_{ij}(0,0)X_k(0,0) + \Big\langle X_i(0,0), S(X_j(0,0))\Big\rangle N(0,0) \\
&= \sum_k \Gamma^k_{ij}(0,0)X_k(0,0) + l_{ij}(0,0)N(0,0)
\end{aligned}$$

Combining both sides yields Gauss's formulas in local coordinates:

$$X_{ij} = \sum_k \Gamma^k_{ij}X_k + l_{ij}N$$