MATH 4347 Homework 1

September 9, 2018

1.4

$$
\begin{aligned}
u_t &= a'(t)e^{2x} + b'(t)e^x + c'(t) \\
u_{xx} &= 4a(t)e^{2x} + b(t)e^x
\end{aligned}
$$

Compare the two derivatives, we get

$$
\begin{cases} a' = 4a \\ b' = b \\ c' = 0 \end{cases} \Rightarrow \begin{cases} a(t) = C_1 e^{4t} \\ b(t) = C_2 e^t \\ c(t) = C_3 \end{cases}
$$

1.7

The characteristic lines are $x - 4t = m$, so $u = f(x - 4t)$. From the initial condition

$$u(x, 0) = f(x) = 0, \text{ where } x > 0$$

From the boundary condition

$$u(0, t) = f(-4t) = te^{-t} \text{ where } t > 0$$

Let $x = -4t$, then $t = -\frac{1}{4}x$.

$$f(x) = -\frac{1}{4}xe^{\frac{1}{4}x} \text{ where } x < 0$$

Combining above results

$$f(x) = \begin{cases} -\frac{1}{4}xe^{\frac{1}{4}x} \text{ for } x < 0 \\ 0, \text{ for } x > 0 \end{cases}$$

$$u = f(x - 4t) = \begin{cases} -\frac{1}{4}(x - 4t)e^{\frac{1}{4}(x-4t)} \text{ for } 0 < x < 4t \\ 0, \text{ for } x > 4t \end{cases}$$

If the PDE changed to $u_t - 4u_x = 0$, then $u = f(x+4t)$ and the boundary condition implies that

$$f(4t) = te^{-t} \text{ where } t > 0$$

This is a contradiction because the initial condition tells us that $f(x) = 0$ for positive arguments, and therefore the equation has no solution.

2.5

a

Let $u(x,t) = e^{i\xi x + \sigma t}$, then $u_{tt} = \sigma^2 u$, and $u_{xxxx} = \xi^4 u$. So the PDE becomes

$$\sigma^2 = -\xi^4 \Rightarrow \sigma = \pm i\xi^2$$

$$u(x,t) = e^{i\xi x \pm i\xi^2 t} = e^{i\xi(x \pm \xi t)}$$

The wave speed depends on ξ, so the wave is dispersive. The reason for this dependence is that the derivatives in space and time are of different orders, so the resulting dispersion relation $\sigma(\xi)$ is not linear, so there are powers of ξ that cannot be factored out. By contrast, the space and time derivatives in wave equation are both of the second order, therefore $\sigma = \pm c\xi$, and consequently the wave speed does not depend on wave number.

b

$$u_t = \sigma u$$

$$u_x = i\xi u$$

$$u_{xxt} = -\xi^2 \sigma u$$

Dividing the original PDE by u

$$\sigma + ic\xi - \beta\xi^2\sigma = 0 \Rightarrow \sigma = \frac{ic\xi}{\beta\xi^2 - 1}$$

So the solution is

$$u = e^{i\xi x + \frac{ic\xi}{\beta\xi^2 - 1}t} = e^{i\xi(x - \frac{c}{1-\beta\xi^2}t)}$$

which are traveling waves with speed $\dfrac{c}{1-\beta\xi^2}$. Compared to the solution of linearized KdV equation

$$u(x,t) = e^{i\xi(x-(c-\beta\xi^2)t)}$$

which has wave speed $c - \beta\xi^2$, the wave speed of the solutions of BBM equation is monotonically increasing w.r.t. ξ^2 when $\xi^2 < \dfrac{1}{\beta}$ and $\xi^2 > \dfrac{1}{\beta}$, however the wave speed of the solutions of KdV equaitons is monotonically decreasing w.r.t ξ^2

2.6

a

Intuitively, the velocity should be higher when there are less traffic, so $v(u)$ should be monotonically decreasing.

b

Since $v(u)$ is monotonically decreasing and both u and v need to be nonnegative to have physical meanings, $v_{max} = v(u_{min}) = v(0)$ and $v(u_{max}) = v_{min} = 0$

c

From (b), $Q(0) = 0v(0) = 0$, and $Q(u_{max}) = u_{max}v(u_{max}) = u_{max}0 = 0$. If Q doesn't attain maximum at some point in the interval $(0, u_{max})$, then $Q(0) = Q(u_{max}) = 0$ must be the maximum, and for all u in the interval, $Q(u)$ must be strictly less than 0 (otherwise these points will also maximum),which is a contradiction because we required that u and v both be non-negative. Therefore, $Q(u)$ must has a maximum in the interval.

d

Such function can be constructed. Consider

$$Q(u) = u\left(-\frac{1}{4}x^3 + 2x^2 - \frac{11}{2}x + 6\right) = -\frac{1}{4}x^4 + 2x^3 - \frac{11}{2}x^2 + 6x$$

whose derivative is

$$Q'(u) = -(u-1)(u-2)(u-3)$$

The function has maximum at 1 and 3, and we can check that

$$v(u) = -\frac{1}{4}x^3 + 2x^2 - \frac{11}{2}x + 6$$

is positive and monotoniaclly decreasing in interval $(0, 4)$.

3.2

First solve for the characteristic curves:

$$\frac{dx}{dt} = \frac{1}{1+t^2} \Rightarrow x = \arctan t + m$$

So $u(x,t) = f(m) = f(x - \arctan t)$. Plugin in the initial condition:

$$u(x,0) = f(x - \arctan 0) = f(x) = \sin x$$

Therefore,

$$u(x,t) = f(x - \arctan t) = \sin(x - \arctan t)$$

3.3

Treat the first two terms as the directional derivative of u in $(1,1)$ direction. Consider the characteristic lines $x = t + k$, or $(s, s+k)$ where the equation becomes an ODE:

$$\frac{d}{ds}u + 3u = e^{3s+2k}$$

Solving use integrating factor $\phi(s) = e^{\int 3ds} = e^{3s}$

$$\begin{aligned} u(s) &= \frac{1}{e^{3s}} \int e^{3s} e^{3s+2k} ds = \frac{1}{6} e^{3s+2k} + A(k)e^{-3s} \\ u(x,t) &= \frac{1}{6} e^{2x+t} + A(x-t)e^{-3t} \end{aligned}$$

Plugin the initial condition to solve for $A(x)$:

$$u(x,0) = \frac{1}{6}e^{2x} + A(x) = x \Rightarrow A(x) = x - \frac{1}{6}e^{2x}$$

Thus the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{6}e^{2x+t} + \left((x-t) - \frac{1}{6}e^{2(x-t)}\right) e^{-3t} \\ &= \frac{1}{6}e^{2x+t} - \frac{1}{6}e^{2x-5t} + xe^{-3t} - te^{-3t} \end{aligned}$$

3.8

a

the LHS can be treated as a total derivative w.r.t to t on the characteristic curve

$$\frac{dx}{dt} = t \Rightarrow x = \frac{t^2}{2} + k$$

On the characteristic curve $(s, \frac{s^2}{2} + k)$the PDE becomes ODE

$$\frac{du}{ds} = u^2 \Rightarrow u = -\frac{1}{s + A(k)} = -\frac{1}{t + A(x - t^2/2)}$$

Plug in the initial condition:

$$u(x,0) = -\frac{1}{A(x)} = \frac{1}{1+x^2} \Rightarrow A(x) = -1 - x^2$$

Therefore the solution is

$$u(x,t) = -\frac{1}{t - 1 - (x - t^2/2)^2} = \frac{1}{(x - t^2/2)^2 - t + 1}$$

b

Given t, $u(x,t)$ attains maximum at critical point

$$\frac{\partial u}{\partial x} = -u^2 \cdot 2(x - \frac{t^2}{2}) = 0 \Rightarrow x_c = \frac{t^2}{2}$$

$$\max_x u(x,t) = u(\frac{t^2}{2}, t) = \frac{1}{1-t}$$

which blows up as $t \to 1^-$

1.3. #2

Apply Newton's second law to a segment of the chain from $x = a$ to $x = b$:

$$F(b)\sin\theta - F(a)\sin\theta = \int_a^b \rho u_{tt} \frac{1}{\cos\theta} dx$$

$$\int_b^{L_x} \rho g \frac{\sin\theta}{\cos\theta} dx - \int_a^{L_x} \rho g \frac{\sin\theta}{\cos\theta} dx = -\int_a^b \rho g \frac{\sin\theta}{\cos\theta} dx = \int_a^b \rho u_{tt} \frac{1}{\cos\theta} dx$$

Since this must be true for arbituary a, b, the integrands must be equal.

$$-\rho g \frac{\sin\theta}{\cos\theta} = \rho u_{tt} \frac{1}{\cos\theta}$$

$$u_{tt} = -g \sin\theta$$

where

$$\sin\theta = \frac{u}{\sqrt{u^2 + x^2}} = \frac{u}{x} + \mathcal{O}(u^2) \approx \frac{u}{x}$$

when u is small. Therefore the chains satisfies PDE

$$x u_{tt} + g u = 0$$

1.5 #2

a

No. Suppose there are two different functions u_1 and u_2 that satisfy the equations. Now consider their difference $v = u_1 - u_2$.

$$\begin{cases} u_1''(x) + u_1'(x) = f(x) \\ u_2''(x) + u_2'(x) = f(x) \end{cases} \Rightarrow v''(x) + v'(x) = 0 \Rightarrow v(x) = Ae^{-t} + B$$

and the boundary conditions:

$$v'(0) = v(0) = \frac{1}{2}[v'(l) + v(l)]$$

$$\Rightarrow -A = A + B = \frac{1}{2}B$$

$$\Rightarrow B = -2A, \quad v(x) = Ae^{-t} - 2A$$

Thus if we can find any function u_0 that satisfies the problem, then the family of functions

$$u = u_0 + Ce^{-t} - 2C$$

are all valid solutions.

b

Rearranging the second equation and applying the Fundamental Theorem of Calculus:

$$\begin{aligned} u'(0) + u(0) = u'(l) + u(l) \Rightarrow \Big[u'(x) + u(x)\Big]_0^l = 0 \\ \Rightarrow \int_0^l u''(x) + u'(x)dx = \int_0^l f(x)dx = 0 \end{aligned}$$

Therefore in order for the problem to have a solution, the integral of f from 0 to l must be zero.

MATH 4347 Homework 2

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3.14

If u_0 is strictly increasing, then the characteristic lines $x = u_0(x_0)t + x_0$ will not intersect, because a line that starts with a larger x_0 will also have a greater slope. The fact that u_0 is bounded means that the characteristic lines have a maximum slope and therefore will not approach the x axis. Therefore, the solution is well defined for all $t > 0$.

4.1

Using the d'Alembert's formula:

$$u = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds$$

For any x, when t becomes sufficiently large so that $x - t < 1$ and $x + t > 3$, the above formula becomes

$$u = \frac{1}{2}\int_{1}^{3} \psi(s)ds$$

which is a constant. To make the constant zero, it is both necessary and sufficient that the above definite integral evaluate to zero.

4.2

a

$$\begin{gathered}
e_t = u_t u_{tt} + u_x u_{xt} \\
p_x = u_{tx} u_x + u_t u_{xx} = u_x u_{xt} + u_t u_{tt} = e_t \\
p_t = u_{tt} u_x + u_t u_{xt} \\
e_x = u_t u_{tx} + u_x u_{xx} = u_t u_{xt} + u_x u_{tt} = p_t
\end{gathered}$$

b

From the result of (a),

$$\begin{gathered}
e_{tt} = p_{xt}, \quad e_{xx} = p_{tx} = p_{xt} = e_{tt} \\
p_{tt} = e_{xt}, \quad p_{xx} = e_{tx} = e_{xt} = p_{tt}
\end{gathered}$$

Therefore, they both satisfy the wave equation.

4.4

a

By d'Alembert's formula:

$$u(x{+}h,t{+}k) = \frac{1}{2}(\phi(x{+}t{+}(h{+}k)){+}\phi(x{-}t{+}(h{-}k)){+}\frac{1}{2c}\left(\int_{x-t+(h-k)}^{0}\psi(s)ds + \int_{0}^{x+t+(h+k)}\psi(s)ds\right)$$

$$u(x{-}h,t{-}k) = \frac{1}{2}(\phi(x{+}t{-}(h{+}k)){+}\phi(x{-}t{-}(h{-}k)){+}\frac{1}{2c}\left(\int_{x-t-(h-k)}^{0}\psi(s)ds + \int_{0}^{x+t-(h+k)}\psi(s)ds\right)$$

$$u(x{+}k,t{+}h) = \frac{1}{2}(\phi(x{+}t{+}(k{+}h)){+}\phi(x{-}t{+}(k{-}h)){+}\frac{1}{2c}\left(\int_{x-t+(k-h)}^{0}\psi(s)ds + \int_{0}^{x+t+(k+h)}\psi(s)ds\right)$$

$$u(x{-}k,t{-}h) = \frac{1}{2}(\phi(x{+}t{-}(k{+}h)){+}\phi(x{-}t{-}(k{-}h)){+}\frac{1}{2c}\left(\int_{x-t-(k-h)}^{0}\psi(s)ds + \int_{0}^{x+t-(k+h)}\psi(s)ds\right)$$

It can be easily verified that both sides have exactly the same terms.

b

If $c = 2$, the the characteristic coordinates are $x \pm 2t$. Therefore the corresponding identity should be

$$u(x+2h,t+k) + u(x-2h,t-k) = u(x+2k,t+h) + u(x-2k,t-h)$$

4.6

a

The d'Alembert formula

$$u_2(x,t) = \frac{1}{2}(\phi(x+t)+\phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t}\psi(s)ds$$

works for $x > t$, but not $x < t$. From the general solution $u = F(x-t) + G(x+t)$ and initial condition we can still arrive at

$$\begin{cases} F(x) = \frac{1}{2}\phi(x) - \frac{1}{2}\int_0^x \psi(s)ds \\ G(x) = \frac{1}{2}\phi(x) + \frac{1}{2}\int_0^x \psi(s)ds \end{cases}$$

To get $F(x)$ for negative x, we need to apply the boundary condition.

$$u_x(0,t) = G'(t) + F'(-t) = 0$$

$$\int_0^x G'(t)dt + \int_0^x F'(-t)dt = 0$$
$$\int_0^x G'(t)dt - \int_0^{-x} F'(t')dt' = 0$$
$$G(x) - F(-x) = C$$
$$G(0) - F(0) = C$$

From initial condition, $F(0) + G(0) = \phi(0) = 0$. Adding this to the last equation,

$$2G(0) = C = 0$$

Therefore $F(-x) = G(x)$,

$$\begin{aligned} u_1(x,t) &= G(x+t) + F(x-t) = G(x+t) + G(t-x) \\ &= \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds\right) \end{aligned}$$

b

$u_1 \equiv 0$ when $x+t$ and $t-x$ are both outside of $[1,2]$, and the two integrals are both zero regardless of ψ. The only situation where this can be satisfied is

$$x+t < 1, \quad t-x < 1 \quad \Rightarrow \quad t-1 < x < 1-t$$

which is valid only when $t < 1$, and since u_1 is only defined for $0 < x < t$,

$$\begin{cases} 0 < x < 1-t, & \frac{1}{2} < t < 1 \\ 0 < x < t, & t < \frac{1}{2} \end{cases}$$

$u_2 \equiv 0$ when $x+t$ and $x-t$ are on the same side of $[1,2]$:

$$\begin{cases} x+t < 1, \quad x-t < 1 & \Rightarrow \quad t < x < 1-t, \quad 0 < t < \frac{1}{2} \\ x+t > 2, \quad x-t > 2 & \Rightarrow \quad x > t+2 \end{cases}$$

c

$$u(x,t) = \begin{cases} \frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds\right), & 0 < x < t \\ \frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_{x-t}^{0} \psi(s)ds\right), & x > t \end{cases}$$

d

$$u_1 = \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds\right)$$
$$\lim_{x\to t} u_1 = \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_0^{2t} \psi(s)ds$$

$$(u_1)_x = \frac{1}{2}(\phi'(x+t) - \phi'(t-x)) + \frac{1}{2}\left(\psi(x+t) - \psi(t-x)\right)$$

$$\lim_{x\to t}(u_1)_x = \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}\left(\psi(2t) - \psi(0)\right)$$

$$(u_1)_t = \frac{1}{2}(\phi'(x+t) + \phi'(t-x)) + \frac{1}{2}\left(\psi(x+t) + \psi(t-x)\right)$$

$$\lim_{x\to t}(u_1)_t = \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}\left(\psi(2t) + \psi(0)\right)$$

$$u_2 = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t}\psi(s)ds$$

$$\lim_{x\to t} u_2 = \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_{0}^{2t}\psi(s)ds$$

$$(u_2)_x = \frac{1}{2}(\phi'(x+t) + \phi'(x-t)) + \frac{1}{2}(\psi(x+t) - \psi(x-t))$$

$$\lim_{x\to t}(u_2)_x = \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}(\psi(2t) - \psi(0))$$

$$(u_2)_t = \frac{1}{2}(\phi'(x+t) - \phi'(x-t)) + \frac{1}{2}(\psi(x+t) + \psi(x-t))$$

$$\lim_{x\to t}(u_2)_t = \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}(\psi(2t) + \psi(0))$$

i

For u to be continuous,

$$\lim_{x\to t} u_1 = \lim_{x\to t} u_2$$

which is always true.

ii

For u to be C^1,

$$\lim_{x\to t}(u_1)_t = \lim_{x\to t}(u_2)_t, \quad \lim_{x\to t}(u_1)_x = \lim_{x\to t}(u_2)_x$$

$$\phi'(0) = -\phi'(0) \Rightarrow \phi'(0) = 0$$

which means that in order for a smooth wave to be stress-free at one end, there must be no stress applie on that end in the beginning.

4.7

By the same argument as 4.6,

$$\begin{cases} F(x) = \frac{1}{2}\phi(x) - \frac{1}{2}\int_0^x \psi(s)ds \\ G(x) = \frac{1}{2}\phi(x) + \frac{1}{2}\int_0^x \psi(s)ds \end{cases}$$

And for $x > t$,

$$u_2(x,t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds$$

Apply the new boundary condition:

$$u_x(0,t) = G'(t) + F'(-t) = h(t)$$

$$\begin{aligned} \int_0^x G'(t)dt + \int_0^x F'(-t)dt &= \int_0^x h(t)dt + C \\ \int_0^x G'(t)dt - \int_0^{-x} F'(t')dt' &= \int_0^x h(t)dt + C \\ G(x) - F(-x) &= \int_0^x h(t)dt + C \end{aligned}$$

Taking the limit as $x \to 0$, we get $C = 0$, therefore

$$F(-x) = G(x) - \int_0^x h(t)dt$$

Therefore,

$$\begin{aligned} u_1 = G(x+t) + F(x-t) &= -\int_0^{t-x} h(y)dy + G(x+t) + G(t-x) \\ &= -\int_0^{t-x} h(y)dy + \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds\right) \end{aligned}$$

$$\begin{aligned} \lim_{x \to t} u_1 &= \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_0^{2t} \psi(s)ds \\ \lim_{x \to t} u_2 &= \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_0^{2t} \psi(s)ds \end{aligned}$$

So the solution is always continuous.

$$\lim_{x \to t}(u_1)_x = h(0) + \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}(\psi(2t) - \psi(0))$$

$$\lim_{x \to t}(u_1)_t = -h(0) + \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}(\psi(2t) + \psi(0))$$

$$\lim_{x\to t}(u_2)_x = \frac{1}{2}(\phi'(2t)+\phi'(0)) + \frac{1}{2}(\psi(2t)-\psi(0))$$
$$\lim_{x\to t}(u_2)_t = \frac{1}{2}(\phi'(2t)-\phi'(0)) + \frac{1}{2}(\psi(2t)+\psi(0))$$

In order for the derivatives to match, it is necessary that

$$h(0) - \frac{1}{2}\phi'(0) = \frac{1}{2}\phi'(0)$$
$$-h(0) + \frac{1}{2}\phi'(0) = -\frac{1}{2}\phi'(0)$$
$$\phi'(0) = h(0)$$

2.1.1

Using d'Alembert's formula,

$$\begin{aligned} u(x,t) &= \frac{1}{2}(e^{x+ct}+e^{x-ct}) + \frac{1}{2c}\int_{x-ct}^{x+ct}\sin(s)ds \\ &= \frac{1}{2}(e^{x+ct}+e^{x-ct}) - \frac{1}{2c}\big(\cos(x+ct)-\cos(x-ct)\big) \end{aligned}$$

2.1.9

Rewrite the equation using operators

$$(\partial_x^2 - 3\partial_x\partial_t - 4\partial_t^2)u = 0$$
$$(\partial_x - 4\partial_t)(\partial_x + \partial_t)u = 0$$

The first operator is the directional derivative along the line that satisfies $\frac{dx}{dt} = -\frac{1}{4}$. The second one is along the line $\frac{dx}{dt} = 1$. Thus we can choose characteristic coordinates $\xi = x + \frac{1}{4}t$, and $\eta = x - t$.

$$\partial_x = \partial_\xi + \partial_\eta$$
$$\partial_t = \frac{1}{4}\partial_\xi - \partial_\eta$$

Solving the above two equations,

$$\partial_\xi = \frac{4}{5}(\partial_x + \partial_t)$$
$$\partial_\eta = \frac{1}{5}(\partial_x - 4\partial_t)$$

So the original equation becomes

$$(5\partial_\eta)(\frac{5}{4}\partial_\xi)u = \frac{25}{4}u_{\xi\eta} = 0$$

whiche means that the solution has the form

$$u = F(\xi) + G(\eta) = F(x + \frac{1}{4}t) + G(x - t)$$

Initial condition tells us that

$$F(x) + G(x) = x^2$$
$$u_t = \frac{1}{4}F'(x + \frac{1}{4}t) - G'(x - t)$$
$$u_t(x, 0) = \frac{1}{4}F'(x) - G'(x) = e^x$$
$$\int_0^x \frac{1}{4}F'(s) - G'(s)ds = \int_0^x e^s$$
$$\frac{1}{4}F(x) - G(x) = e^x + A$$

From the first and last equations,

$$F(x) = \frac{4}{5}(x^2 + e^x + A) = \frac{1}{5}(4x^2 + 4e^x + 4A)$$

$$G(x) = \frac{1}{5}(x^2 - 4e^x - 4A)$$

$$\begin{aligned} u(x,t) &= F(x + \frac{1}{4}t) + G(x - t) \\ &= \frac{1}{5}\left[4\left(x + \frac{1}{4}t\right)^2 + 4e^{x+\frac{1}{4}t} + 4A\right] + \frac{1}{5}\left[(x-t)^2 - 4e^{x-t} - 4A\right] \\ &= \frac{1}{5}\left[4\left(x + \frac{1}{4}t\right)^2 + 4e^{x+\frac{1}{4}t}\right] + \frac{1}{5}\left[(x-t)^2 - 4e^{x-t}\right] \\ &= \frac{1}{5}\left[4\left(x + \frac{1}{4}t\right)^2 + (x-t)^2\right] + \frac{4}{5}\left[e^{x+\frac{1}{4}t} - e^{x-t}\right] \end{aligned}$$

2.2.3

a

Let $\xi = x - y$, then

$$\partial_x u(x - y, t) = \partial_\xi u(\xi, t)$$
$$\partial_{xx} u(x - y, t) = \partial_{\xi\xi} u(\xi, t)$$

And

$$\partial_{tt} u(x - y, t) = \partial_{tt} u(\xi, t)$$

Therefore

$$\partial_{tt} u(x - y, t) = \partial_{tt} u(\xi, t) = c^2 \partial_{\xi\xi} u(\xi, t) = c^2 \partial_{xx} u(x - y, t)$$

b

$$(u_x)_{tt} = (u_{tt})_x = (c^2 u_{xx})_x = c^2 (u_x)_{xx}$$

Therefore u_x is a solution. The cases for other derivatives can be proved in the same way.

c

Let $\bar{x} = ax$, $\bar{t} = at$,

$$\partial_{tt} u(ax, at) = a^2 \partial_{\bar{t}\bar{t}} u(\bar{x}, \bar{t})$$

$$\partial_{xx} u(ax, at) = a^2 \partial_{\bar{x}\bar{x}} u(\bar{x}, \bar{t})$$

$$\partial_{tt} u(ax, at) - c^2 \partial_{xx} u(ax, at) = a^2 \left[\partial_{\bar{t}\bar{t}} u(\bar{x}, \bar{t}) - c^2 \partial_{\bar{x}\bar{x}} u(\bar{x}, \bar{t}) \right] = 0$$

Therefore $u(ax, at)$ satisfies the equation.

MATH 4347 Homework 3

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5.6

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}+y\right)} H(y)dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}+y\right)} dy \\
&= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_{0}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}} dy \\
&= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \sqrt{4kt} \int_{0}^{\infty} e^{-z^2} dz \\
&= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \sqrt{4kt} \frac{\sqrt{\pi}}{2} = \boxed{\frac{e^{kt-x}}{2}}
\end{aligned}
$$

5.9

a

Let $u = e^{-dt}v$, then

$$u_t = -de^{-dt}v + e^{-dt}v_t, \quad u_{xx} = e^{-dt}v_{xx}$$

The original equation becomes

$$-de^{-dt}v + e^{-dt}v_t + de^{-dt}v = ke^{-dt}v_{xx} \quad \Rightarrow \quad v_t = kv_{xx}$$

$$g(x) = u(x,0) = e^0 v(x,0) = v(x,0)$$

Using the fundamental solution

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)dy \quad \Rightarrow \quad \boxed{u(x,t) = \frac{e^{-dt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)dy}$$

b

It makes the solution decay exponentially at a speed d.

c

Suppose now we let $u = f(t)v(x,t)$, then

$$u_t = f'v + fv_t, \quad u_{xx} = fv_{xx}$$

$$f'v + fv_t + dfv = kfv_{xx}$$

The objective is to let $f'v$ and dfv cancel out, so f should satisfy

$$f' = -df \Rightarrow f = e^{-\int d(t)dt}$$

Therefore the change of variable should be

$$\boxed{u(x,t) = e^{-\int d(t)dt} v(x,t)}$$

6.2

Suppose $u(x,t) = X(x)T(t)$, then the equation becomes

$$XT'' = 9X''T \Rightarrow \frac{T''}{9T} = \frac{X''}{X} = k$$

The initial and boundary conditions imply that

$$X(0) = X(1) = 0$$

First solve for X:

$$X'' = kX$$

In order for X to satisfy the boundary conditions, it cannot be exponential, therefore

$$k = -\beta^2 \Rightarrow X = A\cos\beta x + B\sin\beta x$$

The boundary condition at $x = 0$ implies that $A = 0$. At $x = 1$, $X(1) = B\sin\beta = 0$, so

$$\beta_n = n\pi$$

Let $T(t)$ absorb the constant coefficient, then

$$X_n = \sin(n\pi x)$$

Now solve for each corresponding T_n:

$$T_n'' = 9kT_n = -9\beta_n^2 T_n$$

$$T_n = C_n\cos(3\beta_n t) + D_n\sin(3\beta_n t), \quad T_n' = -3\beta_n C_n\sin(3\beta_n t) + 3\beta_n D_n\cos(3\beta_n t)$$

The general solution is

$$u(x,t) = \sum_{n=0}^{\infty} X_n T_n = \sum_{n=0}^{\infty} \Big[C_n\cos(3n\pi t) + D_n\sin(3n\pi t)\Big]\sin(n\pi x)$$

$$u_t(x,t) = \sum_{n=0}^{\infty} X_n T_n' = \sum_{n=0}^{\infty} \Big[-3n\pi C_n \sin(3n\pi t) + 3n\pi D_n \cos(3n\pi t)\Big] \sin(n\pi x)$$

Apply the initial conditions,

$$u(x,0) = \sum_{n=0}^{\infty} C_n \sin(n\pi x) = 2\sin(\pi x) + 7\sin(3\pi x)$$

$$u_t(x,0) = \sum_{n=0}^{\infty} 3n\pi D_n \sin(n\pi x) = 2\sin(\pi x)$$

Comparing the terms, we can get

$$C_1 = 2, \quad C_3 = 7, \quad D_1 = \frac{2}{3\pi}$$

All other coefficients are zero. Therefore the solution is

$$\boxed{u(x,t) = \Big[2\cos(3\pi t) + \frac{2}{3\pi}\sin(3\pi t)\Big]\sin(\pi x) + 7\cos(9\pi t)\sin(3\pi x)}$$

6.3

a

If $\lambda = 0$, then $v'' = 0$, $\boxed{v = kx + m}$. From the boundary conditions,

$$k - a_0 m = 0, \quad (1 + a_L L)k + a_L m = 0$$

k, m could be any solution of the system of equations.

b

Since the system has non-trivial solutions,

$$\det \begin{pmatrix} 1 & -a_0 \\ 1 + a_L L & a_L \end{pmatrix} = a_0 + a_L + a_0 a_L L = 0$$

c

Determinant being zero is also sufficient for non-trivial solutions to exist, therefore it guarantees that $\lambda = 0$ is an eigenvalue.

2.3.4

a

On the initial line, u attains maximum at $x = 1/2$ and minimum at two end points

$$u(1/2, 0) = 1, \quad u(0,0) = u(1,0) = 0$$

By the maximum principle for heat equation,

$$\max_D u(x,t) = \max_\Gamma u(x,t) = 1, \quad \min_D u(x,t) = \min_\Gamma u(x,t) = 0$$

b

Let $\xi = 1 - x$, then $\bar{u}(x,t) = u(1-x,t) = u(\xi,t)$.

$$\bar{u}_t = u_t, \quad \bar{u}_{xx} = (-1)(-1)u_{\xi\xi} = u_{\xi\xi}$$

Since $u_t = u_{\xi\xi}$, $\bar{u}_t = \bar{u}_{xx}$, which means that $u(1-x,t)$ also satisfies the heat equation. Also,

$$u(1-x,0) = 4(1-x)(1-(1-x)) = 4(1-x)x = u(x,0)$$

The initial data for two functions are the same. By uniqueness of solutions,

$$u(x,t) = u(1-x,t)$$

c

$$\frac{d}{dt}\int_0^1 u^2 dx = 2\int_0^1 u u_t dx = 2\int_0^1 u u_{xx} dx = 2uu_x\Big|_0^1 - 2\int_0^1 u_x^2 dx = -2\int_0^1 u_x^2 dx \leq 0$$

Therefore $\int_0^1 u^2 dx$ is strictly decreasing.

2.4.9

u_{xxx} satisfies the heat equation since

$$(u_{xxx})_t = (u_t)_{xxx} = (ku_{xx})_{xxx} = k(u_{xxx})_{xx}$$

The initial value for u_{xxx} is

$$u_{xxx}(x,0) = (u(x,0))''' = 0$$

Since the zero function is obviously a solution, by uniqueness of solutions, $u_{xxx} \equiv 0$. Integrating yields $u = A(t)x^2 + B(t)x + C(t)$. Plug this into the original problem

$$A'(t)x^2 + B'(t)x + C'(t) = 2kA(t)$$

RHS is a function of t alone, therefore $A' = B' = 0 \Rightarrow A = a, B = b$, where a, b are constants, and $C' = 2kA = 2ka \Rightarrow C = 2kat + c$. Using the initial condition

$$u(x,0) = A(0)x^2 + B(0)x + C(0) = ax^2 + bx + c = x^2 \Rightarrow \begin{cases} a = 1 \\ b = 0 \\ c = 0 \end{cases}$$

Therefore,

$$\boxed{u(x,t) = x^2 + 2kt}$$

3.4.13

Odd-extend u to $\tilde{u}$. The initial and boundary conditions for the extended function are

$$\phi(x) = \tilde{u}(x,0) = x, \quad \psi(x) = \tilde{u}_t(x,0) = 0, \quad h(t) = x(0,t) = t^2$$

The solution for $x > ct$ is

$$u = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0 = \boxed{x}$$

The solution for $x < ct$ is

$$u = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0 + h(t - \frac{x}{c}) = \boxed{x + \left(t - \frac{x}{c}\right)^2}$$

4.1.3

Suppose $u = X(x)T(t)$, then from the equation,

$$\frac{T'}{iT} = \frac{X''}{X} = k$$

First solve for X:

$$X'' = kX, X(0) = X(l) = 0$$

Since X must be zero at two points, we must have $k = -\lambda^2$.

$$X'' = -\lambda^2 X \Rightarrow X = A\cos\lambda x + B\sin\lambda x$$

From the boundary conditions,

$$A = 0, \quad \sin\lambda l = 0 \Rightarrow \lambda_n = \frac{n\pi}{l}$$

Let T absorb the constant, then

$$X_n = \sin\frac{n\pi}{l}x$$

Now solve for each corresponding T_n,

$$T_n' = -i\lambda_n^2 T \quad \Rightarrow \quad T_n = C_n e^{-i\lambda_n^2 t} = C_n e^{-i\left(\frac{n\pi}{l}\right)^2 t}$$

Combining above results, the general solution is

$$\boxed{u = \sum_{n=0}^{\infty} C_n e^{-i\left(\frac{n\pi}{l}\right)^2 t} \sin\frac{n\pi}{l}x|}$$

4.2.1

Suppose $u = X(x)T(t)$, then from the equation,

$$\frac{T'}{\kappa T} = \frac{X''}{X} = k$$

First solve for X:

$$X'' = kX = -\lambda^2 X, \quad X(0) = X'(l) = 0$$

$$X = A\cos\lambda x + B\sin\lambda x$$

From the boundary conditions,

$$A = 0, \quad \cos\lambda l = 0 \Rightarrow \lambda l = \left(n + \frac{1}{2}\right)\pi, \quad \lambda_n = \frac{(2n+1)\pi}{2l}$$

Let T absorb the constant, then

$$X_n = \sin\frac{(2n+1)\pi}{2l}x$$

Now solve for each corresponding T_n,

$$T_n' = -\kappa\lambda_n^2 T \quad \Rightarrow \quad T_n = C_n e^{-\kappa\lambda_n^2 t} = C_n e^{-\kappa((2n+1)\pi/2l)^2 t}$$

Combining above results, the general solution is

$$\boxed{u = \sum_{n=0}^{\infty} C_n e^{-\kappa((2n+1)\pi/2l)^2 t}\sin\frac{(2n+1)\pi}{2l}x}$$

MATH 4347 Homework 4

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6.5

let $\lambda = \beta^2$, then the eigenfunctions are

$$v = A\cos\beta x + B\sin\beta x, \quad v' = -\beta A\sin\beta x + \beta B\cos\beta x$$

At $x = 0$,

$$\beta B - a_0 A = 0 \Rightarrow B = \frac{a_0 A}{\beta}$$

At $x = l$,

$$-\beta A\sin\beta l + \beta B\cos\beta l + a_l(A\cos\beta l + B\sin\beta l) = 0$$
$$\Rightarrow -\beta A\sin\beta l + a_0 A\cos\beta l + a_l(A\cos\beta l + \frac{a_0 A}{\beta}\sin\beta l) = 0$$
$$\Rightarrow -\beta\tan\beta l + a_0 + a_l + a_l\frac{a_0}{\beta}\tan\beta l = 0$$
$$\Rightarrow \tan\beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_l a_0}$$

a

$(a_0+a_l)\beta/(\beta^2-a_l a_0)$ decreases to zero continuously as $\beta \to \infty$, therefore it intersects $\tan\beta l$ in every period (except for the first few ones). So there are infinitely many λ_n that satisfies the above equation.

b

Since the two graphs intersect in the positive halves of $\tan\beta l$,

$$\frac{(n-1)\pi}{l} < \beta_n < \frac{(n-1)\pi}{l} + \frac{\pi}{2l} = \frac{(2n-1)\pi}{2l}$$
$$\frac{(n-1)^2\pi^2}{l^2} < \lambda_n < \frac{(2n-1)^2\pi^2}{4l^2}$$

c

Since $(a_0 + a_l)\beta/(\beta^2 - a_l a_0)$ approaches zero as $n \to \infty$, its intersections with $\tan\beta l$ will approach the β-intercepts of $\tan\beta l$, therefore

$$\lim_{n\to\infty}\beta_n = \frac{(n-1)\pi}{l}, \text{ or equivalently, } \lim_{n\to\infty}\lambda_n - \frac{(n-1)^2\pi^2}{l^2} = 0$$

d

$$\tan\left((n-1)\pi + \theta_n l\right) = \frac{(a_0+a_l)\left(\dfrac{(n-1)\pi}{l}+\theta_n\right)}{\left(\dfrac{(n-1)\pi}{l}+\theta_n\right)^2 - a_l a_0}$$

$$\tan\theta_n l\left(\frac{(n-1)^2\pi^2}{l^2}+\theta_n^2+2\frac{(n-1)\pi}{l}\theta_n - a_l a_0\right) = (a_0+a_l)\left(\frac{(n-1)\pi}{l}+\theta_n\right)$$

$$\Big(\theta_n l + O(\theta_n^3)\Big)\left(\frac{(n-1)^2\pi^2}{l^2} - a_l a_0 + O(\theta_n)\right) = (a_0+a_l)\left(\frac{(n-1)\pi}{l}+\theta_n\right)$$

$$\left(\frac{(n-1)^2\pi^2}{l} - a_l a_0 l\right)\theta_n + O(\theta_n^2) = (a_0+a_l)\frac{(n-1)\pi}{l} + (a_0+a_l)\theta_n$$

Dropping terms of order higher than the second

$$\left[(n-1)^2\pi^2 - (a_0+a_l+a_l a_0 l)l\right]\theta_n = (a_0+a_l)(n-1)\pi$$

$$\theta_n = \frac{(a_0+a_l)(n-1)\pi}{(n-1)^2\pi^2 - (a_0+a_l+a_l a_0 l)l}$$

let $x = 1/n \to 0$, then

$$\theta_x = \frac{(a_0+a_l)(x-x^2)\pi}{(1-x)^2\pi^2 - (a_0+a_l+a_l a_0 l)lx^2} = 0 + \frac{a_0+a_l}{\pi}x + O(x^2)$$

$$\Rightarrow \boxed{\theta_n = \frac{a_0+a_l}{\pi n} + O(\frac{1}{n^2})}$$

6.7

For each of the two regions,

$$(p_i v')' + \lambda r_i v = 0 \quad \Rightarrow \quad v'' = -\frac{\lambda r_i}{p_i}v$$

$$\Rightarrow v_i = A_i\cos\beta_i x + B_i \sin\beta_i x, \text{ where } \beta_i = \sqrt{\frac{\lambda r_i}{p_i}}$$

For v_1, the boundary condition at $x=0$ gives $A_1 = 0$. Therefore

$$v_1 = C\sin\beta_1 x, \quad v_1' = \beta_1 C\cos\beta_1 x$$

For v_2, the boundary condition at $x=l$ gives

$$A_2\cos\beta_2 l + B_2\sin\beta_2 l = 0$$

$$A_2 = -B_2\tan\beta_2 l$$

Plug it into the equation for v,

$$v_2 = -B\tan\beta_2 l\cos\beta_2 x + B\sin\beta_2 x, \quad v_2' = \beta_2 B\tan\beta_2 l\sin\beta_2 x + \beta_2 B\cos\beta_2 x$$

Because the eigenfunctions are required to be continuously differentiable, at $x = m$:

$$\begin{cases} C \sin \beta_1 m = -B \tan \beta_2 l \cos \beta_2 m + B \sin \beta_2 m \\ \beta_1 C \cos \beta_1 m = \beta_2 B \tan \beta_2 l \sin \beta_2 m + \beta_2 B \cos \beta_2 m \end{cases}$$

Dividing two equations

$$\boxed{\tan \beta_1 m = \frac{-\beta_1 \tan \beta_2 l \cos \beta_2 m + \beta_1 \sin \beta_2 m}{\beta_2 \tan \beta_2 l \sin \beta_2 m + \beta_2 \cos \beta_2 m}, \quad \beta_1 = \sqrt{\frac{\lambda r_1}{p_1}}, \quad \beta_2 = \sqrt{\frac{\lambda r_2}{p_2}}}$$

6.8

$$v = a \cos \mu x + b \sin \mu x + c \cosh \mu x + d \sinh \mu x$$
$$v' = -a\mu \sin \mu x + b\mu \cos \mu x + c\mu \sinh \mu x + d\mu \cosh \mu x$$
$$v'' = -a\mu^2 \cos \mu x - b\mu^2 \sin \mu x + c\mu^2 \cosh \mu x + d\mu^2 \sinh \mu x$$
$$v''' = a\mu^3 \sin \mu x - b\mu^3 \cos \mu x + c\mu^3 \sinh \mu x + d\mu^3 \cosh \mu x$$

From the boundary conditions

$$v(0) = a + c = 0 \Rightarrow c = -a$$
$$v'(0) = b\mu + d\mu = 0 \Rightarrow d = -b$$
$$v''(l) = -a\mu^2 \cos \mu l - b\mu^2 \sin \mu l - a\mu^2 \cosh \mu l - b\mu^2 \sinh \mu l = 0$$
$$\Rightarrow a \cos \mu l + a \cosh \mu l = -b \sin \mu l - b \sinh \mu l \tag{1}$$
$$v'''(l) = a\mu^3 \sin \mu l - b\mu^3 \cos \mu l - a\mu^3 \sinh \mu l - b\mu^3 \cosh \mu l = 0$$
$$\Rightarrow -a \sin \mu l + a \sinh \mu l = -b \cos \mu l - b \cosh \mu l \tag{2}$$

Dividing (2) by (1) gives

$$\frac{-\sin \mu l + \sinh \mu l}{\cos \mu l + \cosh \mu l} = \frac{\cos \mu l + \cosh \mu l}{\sin \mu l + \sinh \mu l}$$
$$\sinh^2 \mu l - \sin^2 \mu l = \cos^2 \mu l + \cosh^2 \mu l + 2 \cos \mu l \cosh \mu l$$
$$\boxed{\cos \mu l \cosh \mu l + 1 = 0, \quad \lambda = \mu^4}$$
$$\cos \mu l = -\frac{1}{\cosh \mu l}$$

$\cos \mu l$ oscillates between ± 1, and $-1/\cosh \mu l$ converges to zero from below as μ increases, so there are infinitely many solutions for μ. More precisely, in each cycle of cosine the two functions intersect twice between the trough of cosine and its two x-intercepts. The one on the left of the trough satisfies

$$-\frac{3\pi}{2l} + k\frac{2\pi}{l} < \mu < -\frac{\pi}{l} + k\frac{2\pi}{l}$$

$$(2k - \frac{3}{2})\frac{\pi}{l} < \mu < (2k-1)\frac{\pi}{l}$$

The one on the right of the trough satisfies

$$-\frac{\pi}{l} + k\frac{2\pi}{l} < \mu < -\frac{\pi}{2l} + k\frac{2\pi}{l}$$

$$(2k-1)\frac{\pi}{l} < \mu < (2k - \frac{1}{2})\frac{\pi}{l}$$

As RHS approaches zero, its intersections with LHS also approaches the x-intercepts of LHS:

$$\lim_{n\to\infty} \mu_n \approx -\frac{\pi}{2l} + n\frac{\pi}{l} = \left(n - \frac{1}{2}\right)\frac{\pi}{l}$$

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mu_n^4 \approx \boxed{\left(n - \frac{1}{2}\right)^4 \frac{\pi^4}{l^4} \approx \left(\frac{n\pi}{l}\right)^4}$$

6.9

a

Since f is odd, we have $f(x) = -1, -\pi < x < 0$. The general form of Fourier series is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = -\frac{1}{\pi}\int_{-\pi}^{0} \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx dx = 0$$

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = -\frac{1}{\pi}\int_{-\pi}^{0} \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx dx \\
&= \frac{1}{\pi}\left[\frac{1}{n}\cos nx\Big|_{-\pi}^{0} - \frac{1}{n}\cos nx\Big|_{0}^{\pi}\right] \\
&= \frac{1}{n\pi}\left[(1 - \cos n\pi) - (\cos n\pi - 1)\right] \\
&= \frac{2(1-(-1)^n)}{n\pi}
\end{aligned}$$

Thus, only the odd sine terms remain

$$B_{2k+1} = \frac{4}{(2k+1)\pi}$$

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\sin(2k+1)x}$$

b

$$f(\frac{\pi}{4}) = \sum_{odd} \frac{4}{n\pi}\sin\frac{n\pi}{4} = \frac{4}{\pi}\frac{\sqrt{2}}{2} + \frac{4}{3\pi}\frac{\sqrt{2}}{2} - \frac{4}{5\pi}\frac{\sqrt{2}}{2} - \frac{4}{7\pi}\frac{\sqrt{2}}{2} + \cdots$$
$$= \frac{2\sqrt{2}}{\pi}\Big[1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\Big] = 1$$

Therefore,

$$\boxed{1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{2\sqrt{2}}}$$

4.3.4

(i) Let

$$h(\gamma) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_o a_l}$$
$$h'(\gamma) = \frac{(a_0 + a_l)(\gamma^2 - a_0 a_l)}{(\gamma^2 + a_0 a_l)^2} = 0 \Rightarrow \gamma = \sqrt{a_0 a_l}$$

From the graph of $h(\gamma)$ this must be the only maximum. The value of h at this point is

$$h(\sqrt{a_0 a_l}) = \frac{-a_0 - a_l}{2\sqrt{a_0 a_l}} \geq \frac{2\sqrt{(-a_0)(-a_l)}}{2\sqrt{a_0 a_l}} = 1$$

And since $h(0) = 0 < 1$ and $\lim_{\gamma\to\infty} h(\gamma) = 0 < 1$, it must cross $y = 1$ exactly twice.

(ii) Let $g(\gamma) = \tanh(\gamma l)$, then $\lim_{\gamma\to\infty} g(\gamma) = 1$. Therefore, as $\gamma \to \infty$, $g(\gamma) > h(\gamma)$.

$$g(0) = 0, \quad g'(0) = \frac{l}{\cosh^2(\gamma l)}\Big|_{\gamma=0} = l$$

and from the assumption that $-a_0 - a_l < a_0 a_l l$, we have

$$h'(0) = \frac{(-a_0 - a_l)a_0 a_l}{(a_0 a_l)^2} = \frac{-a_0 - a_l}{a_0 a_l} < \frac{a_0 a_l l}{a_0 a_l} = l = g'(0)$$

Therefore, near $\gamma = 0$, we also have $g(\gamma) > h(\gamma)$. However, at $\gamma = \sqrt{a_0 a_l}$, since $\tanh(x) < 1$,

$$g(\sqrt{a_0 a_l}) < 1 < h(\sqrt{a_0 a_L})$$

From above obsevations, $h(\gamma) - g(\gamma)$ changes sign exactly twice, which means that the two functions intersect exactly twice, which then implies there are two (negative) eigenvalues.

4.3.9

a

$$X'' = 0 \Rightarrow X = ax + b$$

From boundary conditions, we get $b = -a$. Dropping the constant factor,

$$\boxed{X_0(x) = x - 1}$$

b

$$X = A\cos\beta x + B\sin\beta x, \quad X' = -A\beta\sin\beta x + B\beta\cos\beta x$$

From boundary condition at $x = 0$, $A = -\beta B$. Rewriting X as $X = -\beta B\cos\beta x + B\sin\beta x$, and from boundary condition at $x = 1$,

$$-\beta B\cos\beta + B\sin\beta = 0 \Rightarrow \boxed{\tan\beta = \beta}$$

c

From the graph of $f(\beta) = \beta$ and $g(\beta) = \tan\beta$, the two curves intersect infinitely many times, which means that there are infinitely many positive eigenvalues.

d

Suppose there exists a negative eigenvalue, then

$$X'' = -\lambda X = \gamma^2 X$$

$$X = A\cosh\gamma x + B\sinh\gamma x$$

$$X' = A\gamma\sinh\gamma x + B\gamma\cosh\gamma x$$

From the boundary condition at $x = 0$, $A = -\gamma B$. Rewrite X and plug in the boundary condition at $x = 1$:

$$-\gamma B\cosh\gamma + B\sinh\gamma = 0 \Rightarrow \tanh\gamma = \gamma$$

However, since $\tanh(0) = 0$ and $\tanh'(0) = 1/\cosh^2(0) = 1$, γ and $\tanh\gamma$ are tangent at the origin and have no other intersections. Therefore a non-zero γ doesn't exist, which means that there isn't a negative eigenvalue.

4.3.18

a

Suppose $u = X(x)T(t)$.

$$XT'' = -c^2X''''T \Rightarrow -\frac{T''}{c^2T} = \frac{X''''}{X} = \lambda \Rightarrow X'''' = \lambda X$$

b

Suppose zero is an eigenvalue, then $X'''' = 0 \Rightarrow X = ax^3 + bx^2 + cx + d$. And its derivatives:

$$X' = 3ax^2 + 2bx + c$$

$$X'' = 6ax + 2b$$

$$X''' = 6a$$

From the boundary conditions, $X(0) = X'(0) = X''(l) = X'''(l) = 0$, thereore a, b, c, d must all be zero, which means that X does not have non-trivial solutions. Therefore, zero is not a eigenvalue.

c

Carrying out the same calculations as in Problem 6.8 above, $\boxed{\cos\beta l \cosh\beta l = -1}$.

d

From Problem 6.8, the frequencies β_n are approximately $\dfrac{(n-1/2)\pi}{l}$ when n is large.

e

Solving the above equation using a computer, the results are

$$\beta_1 \approx \frac{1.875}{l}, \quad \beta_2 \approx \frac{4.694}{l}, \quad \frac{\beta_2^2}{\beta_1^2} \approx 6.267$$

For a vibrating string, $\beta_2^2/\beta_1^2 = 2^2 = 4$. The overtone frequencies of a tunning fork grows faster than a string as n increases.

5.1.2

a

The sine series is

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

where

$$\begin{aligned} A_n &= 2\int_0^1 x^2 \sin(n\pi x) \\ &= 2\Big[-\frac{1}{n\pi}x^2\cos n\pi x + \frac{2}{n^2\pi^2}x\sin n\pi x + \frac{2}{n^3\pi^3}\cos n\pi x\Big]\Big|_0^1 \\ &= (-1)^n\left(\frac{4}{n^3\pi^3} - \frac{2}{n\pi}\right) - \frac{4}{n^3\pi^3} \end{aligned}$$

b

The cosine series is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

where

$$A_0 = 2\int_0^1 x^2 = \frac{2}{3}$$

For $n \geq 1$,

$$\begin{aligned}
A_n &= 2\int_0^1 x^2\cos(n\pi x) \\
&= 2\Big[\frac{1}{n\pi}x^2\sin n\pi x + \frac{2}{n^2\pi^2}x\cos n\pi x - \frac{2}{n^3\pi^3}\sin n\pi x\Big]\Big|_0^1 \\
&= (-1)^n\frac{4}{n^2\pi^2}
\end{aligned}$$

Combine the results,

$$\phi(x) = \frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2\pi^2}\cos(n\pi x)$$

5.1.9

Separate the variables:

$$\frac{X''}{X} = \frac{T''}{c^2T} = \lambda$$

The boundary conditions and initial conditions then translate to

$$X'(0) = X'(\pi) = 0, \quad T(0) = 0, \quad X(x)T'(0) = \cos^2 x$$

(i) For $\lambda = 0$, $X'' = 0 \Rightarrow X = Ax + B$. The boundary condition implies that $A = 0$, so

$$X_0 = 1$$

The corresponding $T_0 = Ct + D$, using the initial condition $T_0(0) = D = 0$. Therefore

$$T_0 = t$$

(ii) For $\lambda > 0$, the boundary conditions cannot be satisfied.

(iii) For $\lambda < 0$, write λ as $-\beta^2$, then

$$X = A\cos\beta x + B\sin\beta x, \quad X' = -A\beta\sin\beta x + B\beta\cos\beta x$$

$X'(0) = 0$ implies that $B = 0$, and $X'(\pi) = 0$ implies:

$$-A\beta\sin\beta\pi = 0 \Rightarrow \beta\pi = n\pi \Rightarrow \beta = n$$

$$X_n = \begin{cases} 1, & n = 0 \\ \cos nx, & n > 0 \end{cases}$$

Solving for T_n:

$$T_n = C_n\cos cnt + D_n\sin cnt$$

The initial condition implies that $C_n = 0$, so

$$T_n = \sin cnt, \quad T_n' = cn\cos cnt$$

$$T_n = \begin{cases} t, & n = 0 \\ \sin cnt, & n > 0 \end{cases}$$

Finally

$$u(x,t) = C_0X_0T_0 + \sum_{n=1}^{\infty} C_nX_nT_n = C_0t + \sum_{n=1}^{\infty} C_n \cos nx \sin cnt$$

$$u_t(x,t) = C_0 + \sum_{n=1}^{\infty} cnC_n \cos nx \cos cnt$$

$$u_t(x,0) = C_0 + \sum_{n=1}^{\infty} cnC_n \cos nx = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

Comparing the terms, we get

$$C_n = \begin{cases} 1/2, & n = 0 \\ 1/4c, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\boxed{u(x,t) = \frac{1}{2}t + \frac{1}{4c}\cos 2x \sin 2ct}$$

MATH 4347 Homework 5

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7.7

The general form of the full Fourier series is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin x \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx dx = \frac{2((-1)^n + 1)}{\pi(1-n^2)}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin x \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx dx = 0$$

Therefore the series is

$$\begin{aligned} |\sin x| &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2((-1)^n + 1)}{\pi(1-n^2)} \cos nx = \frac{2}{\pi} + \sum_{even} \frac{4}{\pi(1-n^2)} \cos nx \\ &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos 2nx \end{aligned}$$

Since the series converges pointwise, at $x = 0$

$$|\sin 0| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos 0 \quad \Rightarrow \quad \boxed{\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}}$$

At $x = \pi/2$

$$|\sin \frac{\pi}{2}| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos n\pi$$

$$1 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1-4n^2)}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}}$$

b

Let $g(x)$ be a even 2-periodic function, whose value in interval $[-1, 1]$ is $g(x) = x^2$. The Fourier coefficients are

$$A_0 = \int_{-1}^{1} x^2 = \frac{2}{3}, \quad A_n = \int_{-1}^{1} x^2 \cos n\pi x = \frac{4(-1)^n}{n^2\pi^2}, \quad B_n = \int_{-1}^{1} x^2 \sin n\pi x = 0$$

The full series is

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x$$

Evaluated at $x = 1$

$$1 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Evaluated at $x = 0$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

8.2

If $u(x) \equiv c$, then $\max\limits_{\overline{U}} u = \max\limits_{\partial U} u \equiv c$.

Otherwise, by the strong form of the maximum principle,

$$\forall x \in U : u(x) < \max_{\partial U} u \quad \Rightarrow \quad \max_{U} u < \max_{\partial U} u$$

$$\max_{\overline{U}} u = \max\{\max_{\partial U} u, \max_{U} u\} = \max_{\partial U} u$$

5.2.11

The complex form of full Fourier series is

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} = e^x$$

where the coeeficients are

$$\begin{aligned} c_n &= \frac{1}{2l} \int_{-l}^{l} e^x e^{-in\pi x/l} dx \\ &= \frac{1}{2l} \int_{-l}^{l} e^{(1-in\pi/l)x} dx \\ &= \frac{1}{2(l - in\pi)} e^{(1-in\pi/l)x} \Big|_{-l}^{l} \\ &= \frac{e^{l-in\pi} - e^{in\pi-l}}{2(l - in\pi)} \\ &= (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \end{aligned}$$

So the series is

$$e^x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} e^{in\pi x/l} = \boxed{\sum_{n=-\infty}^{\infty} (-1)^n \sinh l \frac{l + in\pi}{l^2 + n^2\pi^2} e^{in\pi x/l}}$$

And since $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\begin{aligned}
e^x &= \frac{e^l - e^{-l}}{2l} + \sum_{n=-1}^{-\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \cos\frac{n\pi x}{l} \\
&\quad + \sum_{n=-1}^{-\infty} (-1)^n \frac{ie^l - ie^{-l}}{2(l - in\pi)} \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{ie^l - ie^{-l}}{2(l - in\pi)} \sin\frac{n\pi x}{l} \\
&= \frac{e^l - e^{-l}}{2l} + \sum_{n=1}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l + in\pi)} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \cos\frac{n\pi x}{l} \\
&\quad + \sum_{n=1}^{\infty} (-1)^n \frac{ie^{-l} - ie^{l}}{2(l + in\pi)} \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{ie^l - ie^{-l}}{2(l - in\pi)} \sin\frac{n\pi x}{l} \\
&= \frac{e^l - e^{-l}}{2l} + \sum_{n=1}^{\infty} (-1)^n \frac{(e^l - e^{-l})l}{l^2 + n^2\pi^2} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{n\pi(e^{-l} - e^{l})}{l^2 + n^2\pi^2} \sin\frac{n\pi x}{l} \\
&= \boxed{\frac{\sinh l}{l} + 2\sinh l \sum_{n=1}^{\infty} \frac{(-1)^n}{l^2 + n^2\pi^2} \left[l\cos\frac{n\pi x}{l} - n\pi \sin\frac{n\pi x}{l} \right]}
\end{aligned}$$

5.3.4

a

Let $v = u - U$, then all derivatives of v is the same as u, and $v(0,t) = u(0,t) - U = 0$. Separation of variables gives

$$\frac{X''}{X} = \frac{T'}{kT} = \lambda, \quad v = X(x)T(t)$$

$\lambda = 0$ under the boundary condition gives only the trivial solution, and for $\lambda > 0$ the boundary conditions cannot be both satisfied, therefore $\lambda < 0$. Let $\lambda = -\beta^2$, then

$$X = A\cos\beta x + B\sin\beta x$$

Boundary condition at 0 implies $A = 0$, and boundary condition at $x = l$ implies

$$\cos\beta l = 0 \Rightarrow \beta_n = \left(n - \frac{1}{2}\right)\frac{\pi}{l} = (2n-1)\frac{\pi}{2l}$$

$$X_n = \sin\beta_n x$$

The corresponding T_n are

$$T' = -k\beta_n^2 T \Rightarrow T = e^{-k\beta_n^2 t}$$

So the general solution is

$$v = \sum_{n=1}^{\infty} A_n e^{-k\beta_n^2 t} \sin \beta_n x, \quad \beta_n = (2n-1)\frac{\pi}{2l}$$

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin \beta_n x = u(x,0) - U = -U$$

$$\Rightarrow A_n = \frac{2}{l}\int_0^l -U \sin \beta_n x = -\frac{2U}{l\beta_n} = -\frac{4U}{(2n-1)\pi}$$

$$\Rightarrow v = -\sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-k\pi^2(2n-1)^2 t/4l^2} \sin \frac{(2n-1)\pi}{2l} x$$

$$u = v + U = \boxed{U - \sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-k\pi^2(2n-1)^2 t/4l^2} \sin \frac{(2n-1)\pi}{2l} x}$$

b

Let a_n be the n-th term of the series. Apply the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2n-1}{2n+1} \frac{\sin \beta_{n+1} x}{\sin \beta_n x} e^{-2k\pi^2 nt/l^2} \right| \\
&= \lim_{n\to\infty} \frac{2n-1}{2n+1} e^{-2k\pi^2 nt/l^2} \left| \frac{\sin \beta_{n+1} x}{\sin \beta_n x} \right| \\
&\leq \lim_{n\to\infty} \frac{2n-1}{2n+1} e^{-2k\pi^2 nt/l^2} = 0
\end{aligned}$$

Therefore, the series converges.

c

The error is smaller than the second term, which is

$$-\frac{4U}{\pi} e^{-k\pi^2 t/4l^2}$$

In order for the error to be within ϵ, it suffices to let

$$\left| \frac{4U}{\pi} e^{-k\pi^2 t/4l^2} \right| = \frac{4|U|}{\pi} e^{-k\pi^2 t/4l^2} < \epsilon$$

$$\boxed{t > -\frac{4l^2}{k\pi^2} \log \frac{\epsilon\pi}{4|U|}}$$

5.3.10

a

Proof by induction:
Base case: To show that $(Z_2, Z_1) = 0$, it's sufficient to show that $(Y_2, X_1) = 0$:

$$\begin{aligned}(Y_2, X_1) &= (X_2, X_1) - \left(X_2, \frac{X_1}{\|X_1\|}\right)\left(\frac{X_1}{\|X_1\|}, X_1\right) \\ &= (X_2, X_1) - (X_2, X_1)\frac{(X_1, X_1)}{\|X_1\|^2} \\ &= (X_2, X_1) - (X_2, X_1) = 0\end{aligned}$$

Inductive step: Suppose $\forall n, m \leq k, n \neq m : (Z_m, Z_n) = 0$. Then in order to prove that $\forall n, m \leq k+1, n \neq m : (Z_n, Z_m) = 0$, we only need to prove that $(Z_{k+1}, Z_n) = 0$ for all $n \leq k$, because all other cases are already proven in the k-th step. And to that end, it's sufficient to show that $(Y_{k+1}, Z_n) = 0$. Since the vector in $\{Z_n : n \leq k\}$ are orthonormal by inductive hypothesis,

$$\begin{aligned}(Y_{k+1}, Z_n) &= (X_{k+1}, Z_n) - (X_{k+1}, Z_1)(Z_1, Z_n) - \cdots - (X_{k+1}, Z_n)(Z_n, Z_n) - \cdots \\ &= (X_{k+1}, Z_n) - (X_{k+1}, Z_n)(Z_n, Z_n) \\ &= (X_{k+1}, Z_n) - (X_{k+1}, Z_n) = 0\end{aligned}$$

By induction, orthogonality holds for all k.

b

Let $X_1 = \cos x + \cos 2x$, and $X_2 = 3\cos x - 4\cos 2x$, then

$$(X_1, X_1) = (\cos x, \cos x) + (\cos 2x, \cos 2x) = \pi, \quad \|X_1\| = \sqrt{\pi}$$

$$Z_1 = \frac{X_1}{\|X_1\|} = \boxed{\frac{1}{\sqrt{\pi}}(\cos x + \cos 2x)}$$

$$(X_2, Z_1) = \frac{1}{\sqrt{\pi}}\Big[3(\cos x, \cos x) - 4(\cos 2x, \cos 2x)\Big] = -\frac{\sqrt{\pi}}{2}$$

$$\begin{aligned}Y_2 = X_2 - (X_2, Z_1)Z_1 &= 3\cos x - 4\cos 2x - \left(-\frac{\sqrt{\pi}}{2}\right)\frac{1}{\sqrt{\pi}}(\cos x + \cos 2x) \\ &= 3\cos x - 4\cos 2x + \frac{1}{2}(\cos x + \cos 2x) = \frac{7}{2}\cos x - \frac{7}{2}\cos 2x\end{aligned}$$

$$(Y_2, Y_2) = \frac{49}{4}\frac{\pi}{2} + \frac{49}{4}\frac{\pi}{2} = \frac{49\pi}{4}, \quad \|Y_2\| = \frac{7\sqrt{\pi}}{2}$$

$$Z_2 = \frac{Y_2}{\|Y_2\|} = \boxed{\frac{1}{\sqrt{\pi}}(\cos x - \cos 2x)}$$

5.4.7

a

The general formula is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x + B_n \sin n\pi x$$

where

$$A_n = \int_{-1}^{1} \phi(x)\cos n\pi x dx = \int_{-1}^{0} (-1-x)\cos n\pi x dx + \int_{0}^{1} (1-x)\cos n\pi x dx = 0$$

$$B_n = \int_{-1}^{1} \phi(x)\sin n\pi x dx = \int_{-1}^{0} (-1-x)\sin n\pi x dx + \int_{0}^{1} (1-x)\sin n\pi x dx = \frac{2}{n\pi}$$

Therefore the full series is

$$\boxed{\phi(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin n\pi x}$$

b

The first three non-zero terms are

$$\frac{2}{\pi}\sin \pi x + \frac{1}{\pi}\sin 2\pi x + \frac{2}{3\pi}\sin 3\pi x$$

c

Obviously,

$$\|\phi(x)\|^2 = \int_{-1}^{1} \phi^2(x)dx < \infty$$

Therefore, the series converges in the mean square sense.

d

$$\phi'(x) = -1, \text{ for } x \in (-1,0) \cup (0,1)$$

Since ϕ and ϕ' are both piecewise continuous, the series converges pointwise. [At $x = 0$, it converges to $\frac{1}{2}(-1+1) = 0$.]

e

$\phi(x)$ has a discontinuity at 0, so $\phi \notin C^2[-1,1]$, and therefore the series does not converge uniformly.

5.5.4

a

Separation of variables gives

$$-\frac{X''}{X} = -\frac{T'}{kT} = \lambda$$

For $\lambda = 0$,

$$X'' = 0 \quad \Rightarrow \quad X = A + Bx$$
$$T' = 0 \quad \Rightarrow \quad T = const.$$

For $\lambda > 0$,

$$-X'' = \beta^2 X \quad \Rightarrow \quad X = C\cos\beta x + D\sin\beta x$$
$$T' = -k\beta^2 T \quad \Rightarrow \quad T = e^{-k\beta^2 t}$$

The general solution is

$$\boxed{u = A + Bx + \sum_{n=1}^{\infty} e^{-k\beta_n^2 t}\Big[C_n\cos\beta_n x + D_n\sin\beta_n x\Big]}$$

b

As $t \to \infty$, each term in the sum converges to zero, therefore $\lim u = A + Bx$.

c

From the boundary condition

$$u_x(0,t) = u_x(l,t) = \frac{u(l,t) - u(0,t)}{l}$$
$$X'(0)T(t) = X'(l)T(t) = \frac{X(l)T(t) - X(0)T(t)}{l}$$
$$X'(0) = X'(l) = \frac{X(l) - X(0)}{l}$$

Green's first identity in one dimension is

$$vu'\Big|_0^l = \int_0^l v'u'dx + \int_0^l vu''dx$$

Let $v = u = X$, then

$$\begin{aligned} LHS &= XX'\Big|_0^l \\ &= X(l)X'(l) - X(0)X'(0) \\ &= X(l)\frac{X(l) - X(0)}{l} - X(0)\frac{X(l) - X(0)}{l} \\ &= \frac{[X(l) - X(0)]^2}{l} \end{aligned}$$

$$\begin{aligned} RHS &= \int_0^l (X')^2 dx + \int_0^l XX'' dx \\ &= \int_0^l (X')^2 dx - \int_0^l \lambda X^2 dx \end{aligned}$$

Multiply both sides by l and swap both sides,

$$l\int_0^l (X')^2 dx - l\int_0^l \lambda X^2 dx = [X(l) - X(0)]^2$$

If λ is negative, then the second integral is negative, which means that

$$l\int_0^l (X')^2 dx < [X(l) - X(0)]^2$$

This contradicts the inequality in Ex. 3. Therefore, there cannot be negative eigenvalues.

d

First we can verify that the boundary condition is symmetric:

$$\begin{aligned} X_1'(l)X_2(l) - X_2'(l)X_1(l) &= X_2(l)\frac{X_1(l) - X_1(0)}{l} - X_1(l)\frac{X_2(l) - X_2(0)}{l} \\ &= \frac{X_1(l)X_2(0) - X_1(0)X_2(l)}{l} \end{aligned}$$

$$\begin{aligned} X_1'(0)X_2(0) - X_2'(0)X_1(0) &= X_2(0)\frac{X_1(l) - X_1(0)}{l} - X_1(0)\frac{X_2(l) - X_2(0)}{l} \\ &= \frac{X_1(l)X_2(0) - X_1(0)X_2(l)}{l} \end{aligned}$$

$$X_1'X_2 - X_2'X_1\Big|_0^l = 0$$

Therefore eigenfunctions associated with different eigenvalues are orthogonal. At $t = 0$

$$\phi(x) = u(x,0) = A + Bx + \sum_{n=1}^{\infty} C\cos\beta_n x + D\sin\beta_n x$$

$$(\phi, 1) = (A + Bx, 1)$$

$$\Rightarrow \int_0^l \phi(x)dx = \int_0^l A + Bxdx = [Ax + \frac{B}{2}x^2]\Big|_0^l = lA + \frac{l^2}{2}B$$

$$(\phi, x) = (A + Bx, x)$$

$$\Rightarrow \int_0^l \phi(x)xdx = \int_0^l Ax + Bx^2dx = [\frac{A}{2}x^2 + \frac{B}{3}x^3]\Big|_0^l = \frac{l^2}{2}A + \frac{l^3}{3}B$$

$$\boxed{A = \frac{4}{l}\int_0^l \phi(x)dx - \frac{6}{l^2}\int_0^l \phi(x)xdx}$$

$$\boxed{B = \frac{12}{l^3}\int_0^l \phi(x)xdx - \frac{6}{l^2}\int_0^l \phi(x)dx}$$

6.1.2

The equation is $\Delta u = k^2 u$. Expressed in spherical coordinates, dropping zero terms:

$$\frac{1}{r}\partial_r^2(ru) = k^2 u$$

Let $v = ur$. Multiply the equation by r

$$v''(r) = k^2 v(r) \quad \Rightarrow \quad v = Ae^{kr} + Be^{-kr}$$

$$u = \frac{v}{r} = \boxed{\frac{1}{r}\left[Ae^{kr} + Be^{-kr}\right]}$$

6.1.9

a

The heat equation is $u_t = \Delta u$. In steady state, $u_t = \Delta u = 0$. Since u only depends on r,

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = 0$$

Let $v(r) = ru(r)$, then $v''(r) = 0 \quad \Rightarrow \quad v(r) = Ar + B$,

$$u(r) = \frac{v}{r} = A + \frac{B}{r}$$

$$u'(r) = -B\frac{1}{r^2}$$

Plug in the boundary conditions:

$$\begin{cases} u(1) = A + B = 100 \\ u'(2) = -\dfrac{B}{4} = -\gamma \end{cases} \Rightarrow \begin{cases} A = 100 - 4\gamma \\ B = 4\gamma \end{cases}$$

Therefore, the temperature distribution is

$$\boxed{u(r) = 100 - 4\gamma + \frac{4\gamma}{r}}$$

b

$$u'(r) = -\frac{4\gamma}{r^2} < 0$$

Temperature decreases radially, therefore the highest temperature is obtained at $r = 1$, which is $\boxed{100}$, and the lowest temperatue is obtained at $r = 2$, which is $\boxed{100 - 2\gamma}$

c

Yes. $\gamma = 40$.

MATH 4347 Homework 6

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8.3

a

Suppose there exist two distinct solutions u_1 and u_2, let $w = u_1 - u_2$, then w satisfies

$$\Delta w = 0, \quad \frac{\partial w}{\partial n} + \alpha w = 0, x \in \partial U$$

First we can establish the identity:

$$\nabla \cdot (w \nabla w) = \nabla w \cdot \nabla w + w \nabla \cdot \nabla w = |\nabla w|^2 + w \Delta w$$

Now define the energy as

$$\begin{aligned}
E_w(t) &= \int_U |\nabla w|^2 dV \\
&= \int_U \nabla \cdot (w \nabla w) dV - \int_U w \Delta w dV \\
&= \int_{\partial U} w \nabla w \cdot \vec{n} dS - \int_U w \Delta w dV \\
&= \int_{\partial U} w \frac{\partial w}{\partial n} dS \\
&= -\alpha \int_{\partial U} w^2 dS
\end{aligned}$$

We now have

$$\int_U |\nabla w|^2 dV = -\alpha \int_{\partial U} w^2 dS$$

Since $\alpha > 0$,

$$\int_U |\nabla w|^2 dV \geq 0, \text{ but } -\alpha \int_{\partial U} w^2 dS \leq 0$$

We have

$$\begin{gathered}
\int_U |\nabla w|^2 dV = -\alpha \int_{\partial U} w^2 dS = 0 \\
\Rightarrow \quad \nabla w \equiv 0, \quad w|_{\partial U} \equiv 0 \\
\Rightarrow \quad w \equiv 0
\end{gathered}$$

which means that $u_1 = u_2$, so the solution must be unique.

b

Following the same steps as (a),

$$\int_U |\nabla w|^2 dV = -\alpha \int_{\partial U} w^2 dS = 0 \quad \Rightarrow \quad \nabla w \equiv 0 \quad \Rightarrow \quad w = const.$$

Therefore, any two solutions only differ by a constant.

c

Let $n = 1$, then the problem becomes

$$w'' = 0, \quad \begin{cases} -w'(a) + \alpha w(a) = 0 \\ w'(b) + \alpha w(b) = 0 \end{cases} \qquad (a < b)$$

Solving the ODE gives

$$w = Cx + D$$

Plug in the boundary conditions

$$\begin{cases} (\alpha a - 1)C + \alpha D = 0 \\ (\alpha b + 1)C + \alpha D = 0 \end{cases}$$

The linear system does not have a unique solution when

$$\det \begin{pmatrix} \alpha a - 1 & \alpha \\ \alpha b + 1 & \alpha \end{pmatrix} = \alpha^2(a-b) - 2\alpha = 0 \quad \Rightarrow \quad \boxed{\alpha = \frac{2}{a-b} < 0}$$

8.6

A necessary condition for the mean-value property is:

$$\frac{d}{dr} \fint_{\partial B(0,r)} u(y) dS_y = \frac{d}{dr} \left[\frac{1}{4\pi r^2} \int_{\partial B(0,r)} u(y) dS_y \right] = 0$$

Now suppose $\Delta u(x) = f(x) \not\equiv 0$, then

$$\begin{aligned} \frac{d}{dr} \left[\frac{1}{4\pi r^2} \int_{\partial B(0,r)} u(y) dS_y \right] &= \frac{d}{dr} \left[\frac{1}{4\pi} \int_{\partial B(0,1)} u(ry) dS_y \right] \\ = \frac{1}{4\pi} \int_{\partial B(0,1)} \nabla u(ry) \cdot y \, dS_y &= \frac{1}{4\pi} \int_{\partial B(0,1)} \nabla u(ry) \cdot \vec{n} \, dS_y \\ = \frac{1}{4\pi r^2} \int_{\partial B(0,r)} \nabla u(y) \cdot \vec{n} \, dS_y &= \frac{1}{4\pi r^2} \int_{B(0,r)} \Delta u(y) \, dV_y \\ = \frac{1}{4\pi r^2} \int_{B(0,r)} f(y) \, dV_y &\not\equiv 0 \end{aligned}$$

which means that the mean value of u is not independent of r, which contradicts the assumption that u has the mean-value property. Therefore, we must have $\Delta u = 0$ in U.

6.2.3

Let $u = X(x)Y(y)$, then

$$u_{xx} + u_{yy} = X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

The boundary conditions require that $\lambda \geq 0$.

(i) For $\lambda = 0$,

$$Y_0 = C, \quad X_0 = Dx$$

(ii) For $\lambda > 0$, let $\lambda = \beta^2$.

$$Y_n = \cos ny, \quad \beta_n = n$$

Now solve for corresponding X_n:

$$X_n'' = n^2 X_n \quad \Rightarrow \quad X_n = A_n \cosh nx + B_n \sinh nx$$

$$X_n(0) = A_n = 0 \quad \Rightarrow \quad X_n = \sinh nx$$

The general solution is

$$u(x, y) = A_0 x + \sum_{i=1}^{\infty} A_n \sinh nx \cos ny$$

$$u(\pi, y) = A_0 \pi + \sum_{i=1}^{\infty} A_n \sinh n\pi \cos ny = \frac{1}{2} + \frac{1}{2} \cos 2y$$

Comparing two sides, we have the non-zero coefficients:

$$A_0 = \frac{1}{2\pi}, \quad A_2 = \frac{1}{2 \sinh 2\pi}$$

Therefore the solution is

$$\boxed{u(x, y) = \frac{1}{2\pi} x + \frac{1}{2 \sinh 2\pi} \sinh 2x \cos 2y}$$

6.2.6

Let $u = X(x)Y(y)Z(z)$. Seperation of variables yields

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Suppose $X'' = -\beta^2 X$, $Y'' = -\gamma^2 Y$, then

$$X_n = \cos n\pi x, \quad \beta_n = n\pi, \quad n = 0, 1, ...$$

$$Y_n = \cos m\pi y, \quad \gamma_m = m\pi, \quad m = 0, 1, ...$$

The above results already include $m = 0$ and $n = 0$ as special cases.

$$Z''_{m,n} = (\beta_n^2 + \gamma_m^2) Z_{m,n} = (n^2 + m^2)\pi^2 Z_{m,n}$$

For $n^2 + m^2 \neq 0$:

$$Z_{m,n} = A \cosh \sqrt{n^2 + m^2}\pi z + B \sinh \sqrt{n^2 + m^2}\pi z$$

$$Z'_{m,n}(0) = 0 \quad \Rightarrow \quad B = 0$$

$$Z_{m,n} = \cosh \sqrt{n^2 + m^2}\pi z$$

For $n = m = 0$:

$$Z'' = 0 \quad \Rightarrow \quad Z = A + Bz$$

$$Z'(0) = B = 0 \quad \Rightarrow \quad Z = A$$

which is included in the previous result. The general solution is

$$\boxed{u = \sum_{m,n=0}^{\infty} A_{m,n} \cos n\pi x \cos m\pi y \cosh \sqrt{n^2 + m^2}\pi z}$$

From the last boundary condition:

$$u_z(x, y, 1) = \sum_{m,n=0}^{\infty} \sqrt{n^2 + m^2}\pi A_{m,n} \cos n\pi x \cos m\pi y \sinh \sqrt{n^2 + m^2}\pi = g(x, y)$$

$$\boxed{A_{m,n} = \frac{4}{\sqrt{n^2 + m^2}\pi \sinh \sqrt{n^2 + m^2}\pi} \int_0^1 \int_0^1 g(x, y) \cos n\pi x \cos m\pi y dx dy, \quad m \neq 0, n \neq 0}$$

$$\boxed{A_{0,n} = \frac{2}{n\pi \sinh n\pi} \int_0^1 g(x, y) \cos n\pi x dx, \quad A_{m,0} = \frac{2}{m\pi \sinh m\pi} \int_0^1 g(x, y) \cos m\pi y dy}$$

6.3.1

a

By the maximum principle of harmonic functions $\max\limits_{\overline{D}} u = \max\limits_{\partial D} u$. On the boundary,

$$u = 3 \sin 2\theta + 1 \leq 3 + 1 = 4$$

So the maximum of u in $\overline{D}$ is 4.

b

By the mean-value property of harmonic functions,

$$\begin{aligned} u(\mathbf{0}) &= \frac{1}{4\pi} \int_0^{2\pi} (3 \sin 2\theta + 1)(2d\theta) \\ &= \frac{1}{2\pi} \Big[-\frac{3 \cos 2\theta}{2} + \theta \Big] \Big|_0^{2\pi} = \boxed{1} \end{aligned}$$

6.4.5

a

The steady-state temperature distribution satisfies Laplace equation $\nabla u = 0$. Let $u = X(\theta)R(r)$, then

$$\frac{R'' + \frac{1}{r}R'}{\frac{1}{r^2}R} = -\frac{X''}{X} = \lambda$$

Since θ is not bounded, X satisfies periodic boundary conditions

$$X(0) = X(2\pi), \quad X'(0) = X'(2\pi)$$

$$\Rightarrow \quad \begin{cases} X_0 = C, & \lambda = 0 \\ X_n = A\cos n\theta + B\sin n\theta, & \lambda = n^2 \end{cases}$$

For $\lambda = 0$,

$$R'' + \frac{1}{r}R' = 0 \quad \Rightarrow \quad R_0 = C_1 + C_2 \ln r$$

Since the outer edge is insulated,

$$R_0'(2) = \frac{C_2}{2} = 0 \quad \Rightarrow \quad C_2 = 0$$

Therefore, R can only be constant

$$\boxed{R_0 = C}$$

For $\lambda = n^2$

$$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = 0$$

$$r^2R'' + rR' - n^2R = 0$$

Suppose $R(r) = r^\alpha$, then

$$r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2r^\alpha = (\alpha^2 - n^2)r^\alpha = 0 \quad \Rightarrow \quad \alpha = \pm n$$

$$\Rightarrow \quad R_n = Cr^n + Dr^{-n}$$

Since the outer edge is insulated,

$$R_n'(2) = nC2^{n-1} - nD2^{-n-1} = 0 \quad \Rightarrow \quad D = 4^nC$$

So the solution can be rewritten as

$$\boxed{R_n = C\Big[r^n + 4^nr^{-n}\Big]}$$

Combining above results, the general solution is

$$\boxed{u = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n(r^n + 4^nr^{-n})\cos n\theta + D_n(r^n + 4^nr^{-n})\sin n\theta}$$

At $r = 1$,

$$u = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n(1+4^n)\cos n\theta + D_n(1+4^n)\sin n\theta = \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

Comparing the terms, the non-zero coefficients are

$$C_0 = 1, \quad C_2 = -\frac{1}{34}$$

So the solution is

$$\boxed{u = \frac{1}{2} - \frac{1}{34}\left(r^2 + \frac{16}{r^2}\right)\cos 2\theta}$$

b

Following the same steps as (a), for $\lambda = 0$,

$$\boxed{X_0 = C}, \quad R_0 = C_1 + C_2 \ln r$$

And for $\lambda = n^2$,

$$\boxed{X_n = A\cos n\theta + B\sin n\theta}, \quad R_n = Cr^n + Dr^{-n}$$

Now at the outer edge

$$R_0(2) = C_1 + C_2\ln 2 = 0 \quad \Rightarrow \quad C_2 = -\frac{C_1}{\ln 2}$$

$$R_n(2) = C2^n + D2^{-n} = 0 \quad \Rightarrow \quad D = -4^n C$$

So $R(r)$ can be rewritten as

$$\boxed{R_0 = C\left(1 - \frac{\ln r}{\ln 2}\right), \quad R_n = D\left(r^n - 4^n r^{-n}\right)}$$

Combining the results, the general solution is

$$\boxed{u = C_0\left(1 - \frac{\ln r}{\ln 2}\right) + \sum_{n=1}^{\infty} C_n\left(r^n - 4^n r^{-n}\right)\cos n\theta + D_n\left(r^n - 4^n r^{-n}\right)\sin n\theta}$$

On the inner edge

$$C_0 + \sum_{n=1}^{\infty} C_n\left(1 - 4^n\right)\cos n\theta + D_n\left(1 - 4^n\right)\sin n\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

The non-zero coefficients are

$$C_0 = \frac{1}{2}, \quad C_2 = \frac{1}{30}$$

Finally, the solution is

$$\boxed{u = \frac{1}{2}\left(1 - \frac{\ln r}{\ln 2}\right) + \frac{1}{30}\left(r^2 - \frac{16}{r^2}\right)\cos 2\theta}$$

6.4.10

Let $u = X(\theta)R(r)$. The boundary condition on $x = 0$ and $y = 0$ can be written as

$$X(0) = X(\pi/2) = 0$$

Separation of variables gives

$$\frac{R'' + \frac{1}{r}R'}{\frac{1}{r^2}R} = -\frac{X''}{X} = \lambda$$

For $\lambda < 0$, the boundary condition cannot be satisfied.

For $\lambda = 0$,

$$X'' = 0 \quad \Rightarrow \quad X = A\theta + B$$

$$X(0) = B = 0 \quad \Rightarrow \quad X = A\theta$$

$$X(\pi/2) = A\pi/2 = 0 \quad \Rightarrow \quad A = 0, \quad X = 0$$

There is no non-trivial solution, so zero is not a eigenvalue.

For $\lambda > 0$, let $\lambda = \beta^2$, then

$$X = A\cos\beta\theta + B\sin\beta\theta$$

$$X(0) = 0 \quad \Rightarrow \quad A = 0, \quad X = \sin\beta\theta$$

$$X(\pi/2) = \sin\frac{\beta\pi}{2} = 0 \quad \Rightarrow \quad \beta_n = 2n, \quad \boxed{X_n = \sin 2n\theta}$$

Now solve for R:

$$r^2R'' + rR' - 4n^2R = 0$$

Suppose $R = r^\alpha$,

$$\alpha^2 - 4n^2 = 0 \quad \Rightarrow \quad \alpha = \pm 2n$$

Since $\lim_{r\to 0} r^{-2n} = \infty$, r^{-2n} should be excluded,

$$\boxed{R_n = C_n r^{2n}}$$

The general solution is:

$$u = \sum_{n=1}^{\infty} C_n r^{2n} \sin 2n\theta$$

$$u_r(a,\theta) = \sum_{n=1}^{\infty} 2nC_n a^{2n-1} \sin 2n\theta = 1$$

$$C_n = \frac{2}{n\pi a^{2n-1}} \int_0^{\pi/2} \sin 2n\theta d\theta = \frac{1-(-1)^n}{n^2\pi a^{2n-1}} = \begin{cases} \dfrac{2}{n^2\pi a^{2n-1}}, & n \text{ is odd} \\ 0, \quad n \text{ is even} \end{cases}$$

$$u = \boxed{\sum_{odd} \frac{2}{n^2\pi a^{2n-1}} r^{2n} \sin 2n\theta}$$

$$= \boxed{\frac{2}{a\pi} r^2 \sin 2\theta + \frac{2}{9a^5\pi} r^6 \sin 6\theta + \cdots}$$

7.4.6

a

The fundamental solution for Laplace equation in 2 dimension is

$$\Phi(\mathbf{x}-\mathbf{x_0}) = -\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x_0}|$$

Using the reflection method, the corrector function is

$$h^{\mathbf{x_0}}(\mathbf{x}) = \Phi(\mathbf{x}-\hat{\mathbf{x}}_\mathbf{0}) = -\frac{1}{2\pi}\ln|\mathbf{x}-\hat{\mathbf{x}}_\mathbf{0}|$$

where $\hat{\mathbf{x}}_\mathbf{0}$ is the reflection of $\mathbf{x_0}$ about the x-axis. The Green's function is

$$G(\mathbf{x},\mathbf{x_0}) = \Phi(\mathbf{x}-\mathbf{x_0}) - h^{\mathbf{x_0}}(\mathbf{x}) = \boxed{-\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x_0}| + \frac{1}{2\pi}\ln|\mathbf{x}-\hat{\mathbf{x}}_\mathbf{0}|}$$

b

Let $\mathbf{x}=(x,y)$, $\mathbf{x_0}=(x_0,y_0)$, $\hat{\mathbf{x}}_\mathbf{0}=(x_0,-y_0)$. On the boundary, which is the x-axis,

$$\frac{\partial G(\mathbf{x},\mathbf{x_0})}{\partial n} = -\frac{\partial G(\mathbf{x},\mathbf{x_0})}{\partial y} = \frac{1}{2\pi}\frac{y-y_0}{|\mathbf{x}-\mathbf{x_0}|^2} - \frac{1}{2\pi}\frac{y+y_0}{|\mathbf{x}-\hat{\mathbf{x}}_\mathbf{0}|^2} = -\frac{y_0}{\pi[(x-x_0)^2+y_0^2]}$$

The solution is

$$\begin{aligned} u(\mathbf{x_0}) &= -\int_{\partial\mathbb{R}^2_+}\frac{\partial G(\mathbf{x_0},\mathbf{x})}{\partial n}u(\mathbf{x})ds \\ \Rightarrow\quad u(x_0,y_0) &= \boxed{\frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-x_0)^2+y_0^2}h(x)dx} \end{aligned}$$

c

$$\begin{aligned} u(x_0,y_0) &= \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-x_0)^2+y_0^2}dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{(\frac{x-x_0}{y_0})^2+1}\frac{1}{y_0}dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{z^2+1}dz = \frac{1}{\pi}\tan^{-1}(z)\Big|_{-\infty}^{\infty} = \boxed{1} \end{aligned}$$

7.4.7

a

$$u_x(x,y) = \frac{f'(x/y)}{y}, \quad u_{xx}(x,y) = \frac{f''(x/y)}{y^2}$$

$$u_y(x,y) = -\frac{f'(x/y)x}{y^2}$$

$$u_{yy}(x,y) = -\frac{-f''(x/y)x^2 - 2f'(x/y)xy}{y^4} = \frac{f''(x/y)(x/y)^2 + 2f'(x/y)(x/y)}{y^2}$$

$$u_{xx} + u_{yy} = 0 \quad \Rightarrow \quad \boxed{f''(x) + \frac{2x}{x^2+1}f'(x) = 0}$$

Let $g = f'$, then

$$g'(x) + \frac{2x}{x^2+1}g(x) = 0$$

The integrating factor is

$$\int \frac{2x}{x^2+1}dx = \ln|x^2+1| = \ln(x^2+1)$$

$$\phi(x) = e^{\ln(x^2+1)} = x^2+1$$

$$f'(x) = g(x) = \frac{1}{\phi(x)}\int 0\cdot\phi(t)dt = \frac{c_1}{x^2+1} \quad \Rightarrow \quad \boxed{f(x) = c_1\tan^{-1}(x) + c_2}$$

b

$$u(x,y) = f(x/y) = c_1\tan^{-1}(x/y) + c_2$$

In polar coordinates (Let θ be the angle w.r.t the y-axis):

$$u(r,\theta) = c_1\theta + c_2 \quad \Rightarrow \quad \frac{\partial u}{\partial r} \equiv 0$$

c

If $\partial u/\partial r \equiv 0$, then $u = f(\theta) = (f\circ\tan^{-1})(x/y)$, where θ is the angle w.r.t. the y-axis.

d

$$h(x) = \lim_{y\to 0} u(x,y) = \lim_{y\to 0} c_1\tan^{-1}(x/y) + c_2 = c_1\pi/2 + c_2$$

The boundary value is some constant.

e

From parts (c) and (d), if a function $v(x,y)$ in $\{y>0\}$ is harmonic and satisfies $\partial u/\partial r \equiv 0$, then its boundary value is a constant.

Using the formula from Ex. 7.4.6:

Suppose $\partial u/\partial r \equiv 0$, then u doesn't depend on r. In other words, the value of u does not change if x and y are scaled by some constant:

$$
\begin{aligned}
u(\lambda x_0, \lambda y_0) &= \frac{\lambda y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x-\lambda x_0)^2 + (\lambda y_0)^2} h(x) dx \\
&= \frac{\lambda y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(\lambda x' - \lambda x_0)^2 + (\lambda y_0)^2} h(\lambda x') \lambda dx' \\
&= \frac{\lambda y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{\lambda^2} \frac{1}{(x' - x_0)^2 + y_0^2} h(\lambda x') \lambda dx' \\
&= \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x' - x_0)^2 + y_0^2} h(\lambda x') dx' \\
&= \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x - x_0)^2 + y_0^2} h(x) dx
\end{aligned}
$$

It is necessary that $h(\lambda x) = h(x)$ for any scaling factor λ, so $h(x)$ must be a constant function. Thus, the results are consistent.

7.4.17

a

Suppose $\mathbf{x_0} = (x_0, y_0)$, then the reflection points are $\mathbf{x_1} = (-x_0, y_0)$, $\mathbf{x_2} = (-x_0, -y_0)$, $\mathbf{x_3} = (x_0, -y_0)$. The corrector function can be defined as

$$h^{\mathbf{x_0}}(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{x_1}) - \Phi(\mathbf{x} - \mathbf{x_2}) + \Phi(\mathbf{x} - \mathbf{x_3})$$

$\Delta h^{\mathbf{x_0}}(\mathbf{x}) = 0$ everywhere in Q, and by symmetry,

$$h^{\mathbf{x_0}}(x, 0) = 0 + \Phi(\mathbf{x} - \mathbf{x_3}) = \Phi(\mathbf{x} - \mathbf{x_0})$$

$$h^{\mathbf{x_0}}(0, y) = \Phi(\mathbf{x} - \mathbf{x_1}) + 0 = \Phi(\mathbf{x} - \mathbf{x_0})$$

Therefore, the Green's function is

$$
\begin{aligned}
G(\mathbf{x_0}, \mathbf{x}) &= \Phi(\mathbf{x} - \mathbf{x_0}) - \Phi(\mathbf{x} - \mathbf{x_1}) + \Phi(\mathbf{x} - \mathbf{x_2}) - \Phi(\mathbf{x} - \mathbf{x_3}) \\
&= \boxed{-\frac{1}{2\pi}\Big(\ln|\mathbf{x} - \mathbf{x_0}| - \ln|\mathbf{x} - \mathbf{x_1}| + \ln|\mathbf{x} - \mathbf{x_2}| - \ln|\mathbf{x} - \mathbf{x_3}| \Big)}
\end{aligned}
$$

b

On the x-axis,

$$\begin{aligned}
\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial y} &= \frac{1}{2\pi}\left[\frac{y-y_0}{|\mathbf{x}-\mathbf{x_0}|^2} - \frac{y-y_1}{|\mathbf{x}-\mathbf{x_1}|^2} + \frac{y-y_2}{|\mathbf{x}-\mathbf{x_2}|^2} - \frac{y-y_3}{|\mathbf{x}-\mathbf{x_3}|^2}\right] \\
&= \frac{y_0}{2\pi}\left[-\frac{1}{|\mathbf{x}-\mathbf{x_0}|^2} + \frac{1}{|\mathbf{x}-\mathbf{x_1}|^2} + \frac{1}{|\mathbf{x}-\mathbf{x_2}|^2} - \frac{1}{|\mathbf{x}-\mathbf{x_3}|^2}\right] \\
&= \frac{y_0}{\pi}\left[\frac{1}{|\mathbf{x}-\mathbf{x_1}|^2} - \frac{1}{|\mathbf{x}-\mathbf{x_0}|^2}\right] \\
&= \frac{y_0}{\pi}\left[\frac{1}{(x+x_0)^2+y_0^2} - \frac{1}{(x-x_0)^2+y_0^2}\right]
\end{aligned}$$

On the y-axis,

$$\begin{aligned}
\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial x} &= \frac{1}{2\pi}\left[\frac{x-x_0}{|\mathbf{x}-\mathbf{x_0}|^2} - \frac{x-x_1}{|\mathbf{x}-\mathbf{x_1}|^2} + \frac{x-x_2}{|\mathbf{x}-\mathbf{x_2}|^2} - \frac{x-x_3}{|\mathbf{x}-\mathbf{x_3}|^2}\right] \\
&= \frac{x_0}{2\pi}\left[-\frac{1}{|\mathbf{x}-\mathbf{x_0}|^2} - \frac{1}{|\mathbf{x}-\mathbf{x_1}|^2} + \frac{1}{|\mathbf{x}-\mathbf{x_2}|^2} + \frac{1}{|\mathbf{x}-\mathbf{x_3}|^2}\right] \\
&= \frac{x_0}{\pi}\left[\frac{1}{|\mathbf{x}-\mathbf{x_2}|^2} - \frac{1}{|\mathbf{x}-\mathbf{x_0}|^2}\right] \\
&= \frac{x_0}{\pi}\left[\frac{1}{x_0^2+(y+y_0)^2} - \frac{1}{x_0^2+(y-y_0)^2}\right]
\end{aligned}$$

Finally, the solution is

$$\begin{aligned}
u(\mathbf{x_0}) &= -\int_{\partial Q} \frac{\partial G(\mathbf{x},\mathbf{x_0})}{\partial n} u(\mathbf{x}) ds \\
&= \frac{y_0}{\pi}\int_0^\infty \left[\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2}\right] h(x) dx \\
&+ \frac{x_0}{\pi}\int_0^\infty \left[\frac{1}{x_0^2+(y-y_0)^2} - \frac{1}{x_0^2+(y+y_0)^2}\right] g(y) dy
\end{aligned}$$