MATH 4347 Homework 6

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8.3

a

Suppose there exist two distinct solutions u_1 and u_2, let $w = u_1 - u_2$, then w satisfies

$$\Delta w = 0, \quad \frac{\partial w}{\partial n} + \alpha w = 0, x \in \partial U$$

First we can establish the identity:

$$\nabla \cdot (w \nabla w) = \nabla w \cdot \nabla w + w \nabla \cdot \nabla w = |\nabla w|^2 + w \Delta w$$

Now define the energy as

$$\begin{aligned} E_w(t) &= \int_U |\nabla w|^2 dV \\ &= \int_U \nabla \cdot (w \nabla w) dV - \int_U w \Delta w dV \\ &= \int_{\partial U} w \nabla w \cdot \vec{n} dS - \int_U w \Delta w dV \\ &= \int_{\partial U} w \frac{\partial w}{\partial n} dS \\ &= -\alpha \int_{\partial U} w^2 dS \end{aligned}$$

We now have

$$\int_U |\nabla w|^2 dV = -\alpha \int_{\partial U} w^2 dS$$

Since $\alpha > 0$,

$$\int_U |\nabla w|^2 dV \geq 0, \text{ but } -\alpha \int_{\partial U} w^2 dS \leq 0$$

We have

$$\begin{gathered} \int_U |\nabla w|^2 dV = -\alpha \int_{\partial U} w^2 dS = 0 \\ \Rightarrow \quad \nabla w \equiv 0, \quad w|_{\partial U} \equiv 0 \\ \Rightarrow \quad w \equiv 0 \end{gathered}$$

which means that $u_1 = u_2$, so the solution must be unique.

b

Following the same steps as (a),

$$\int_U |\nabla w|^2 dV = -\alpha \int_{\partial U} w^2 dS = 0 \quad \Rightarrow \quad \nabla w \equiv 0 \quad \Rightarrow \quad w = const.$$

Therefore, any two solutions only differ by a constant.

c

Let $n = 1$, then the problem becomes

$$w'' = 0, \quad \begin{cases} -w'(a) + \alpha w(a) = 0 \\ w'(b) + \alpha w(b) = 0 \end{cases} \qquad (a < b)$$

Solving the ODE gives

$$w = Cx + D$$

Plug in the boundary conditions

$$\begin{cases} (\alpha a - 1)C + \alpha D = 0 \\ (\alpha b + 1)C + \alpha D = 0 \end{cases}$$

The linear system does not have a unique solution when

$$\det \begin{pmatrix} \alpha a - 1 & \alpha \\ \alpha b + 1 & \alpha \end{pmatrix} = \alpha^2(a-b) - 2\alpha = 0 \quad \Rightarrow \quad \boxed{\alpha = \frac{2}{a-b} < 0}$$

8.6

A necessary condition for the mean-value property is:

$$\frac{d}{dr} ⨍_{\partial B(0,r)} u(y) dS_y = \frac{d}{dr}\left[\frac{1}{4\pi r^2} \int_{\partial B(0,r)} u(y) dS_y\right] = 0$$

Now suppose $\Delta u(x) = f(x) \not\equiv 0$, then

$$\begin{aligned} \frac{d}{dr}\left[\frac{1}{4\pi r^2} \int_{\partial B(0,r)} u(y) dS_y\right] &= \frac{d}{dr}\left[\frac{1}{4\pi} \int_{\partial B(0,1)} u(ry) dS_y\right] \\ = \frac{1}{4\pi} \int_{\partial B(0,1)} \nabla u(ry) \cdot y \, dS_y &= \frac{1}{4\pi} \int_{\partial B(0,1)} \nabla u(ry) \cdot \vec{n} \, dS_y \\ = \frac{1}{4\pi r^2} \int_{\partial B(0,r)} \nabla u(y) \cdot \vec{n} \, dS_y &= \frac{1}{4\pi r^2} \int_{B(0,r)} \Delta u(y) \, dV_y \\ = \frac{1}{4\pi r^2} \int_{B(0,r)} f(y) \, dV_y &\not\equiv 0 \end{aligned}$$

which means that the mean value of u is not independent of r, which contradicts the assumption that u has the mean-value property. Therefore, we must have $\Delta u = 0$ in U.

6.2.3

Let $u = X(x)Y(y)$, then

$$u_{xx} + u_{yy} = X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

The boundary conditions require that $\lambda \geq 0$.

(i) For $\lambda = 0$,

$$Y_0 = C, \quad X_0 = Dx$$

(ii) For $\lambda > 0$, let $\lambda = \beta^2$.

$$Y_n = \cos ny, \quad \beta_n = n$$

Now solve for corresponding X_n:

$$X_n'' = n^2 X_n \quad \Rightarrow \quad X_n = A_n \cosh nx + B_n \sinh nx$$

$$X_n(0) = A_n = 0 \quad \Rightarrow \quad X_n = \sinh nx$$

The general solution is

$$u(x, y) = A_0 x + \sum_{i=1}^{\infty} A_n \sinh nx \cos ny$$

$$u(\pi, y) = A_0 \pi + \sum_{i=1}^{\infty} A_n \sinh n\pi \cos ny = \frac{1}{2} + \frac{1}{2} \cos 2y$$

Comparing two sides, we have the non-zero coefficients:

$$A_0 = \frac{1}{2\pi}, \quad A_2 = \frac{1}{2 \sinh 2\pi}$$

Therefore the solution is

$$\boxed{u(x, y) = \frac{1}{2\pi} x + \frac{1}{2 \sinh 2\pi} \sinh 2x \cos 2y}$$

6.2.6

Let $u = X(x)Y(y)Z(z)$. Seperation of variables yields

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Suppose $X'' = -\beta^2 X$, $Y'' = -\gamma^2 Y$, then

$$X_n = \cos n\pi x, \quad \beta_n = n\pi, \quad n = 0, 1, ...$$

$$Y_n = \cos m\pi y, \quad \gamma_m = m\pi, \quad m = 0, 1, ...$$

The above results already include $m = 0$ and $n = 0$ as special cases.

$$Z''_{m,n} = (\beta_n^2 + \gamma_m^2) Z_{m,n} = (n^2 + m^2)\pi^2 Z_{m,n}$$

For $n^2 + m^2 \neq 0$:

$$Z_{m,n} = A\cosh\sqrt{n^2+m^2}\pi z + B\sinh\sqrt{n^2+m^2}\pi z$$

$$Z'_{m,n}(0) = 0 \quad \Rightarrow \quad B = 0$$

$$Z_{m,n} = \cosh\sqrt{n^2+m^2}\pi z$$

For $n = m = 0$:

$$Z'' = 0 \quad \Rightarrow \quad Z = A + Bz$$

$$Z'(0) = B = 0 \quad \Rightarrow \quad Z = A$$

which is included in the previous result. The general solution is

$$\boxed{u = \sum_{m,n=0}^{\infty} A_{m,n}\cos n\pi x\cos m\pi y\cosh\sqrt{n^2+m^2}\pi z}$$

From the last boundary condition:

$$u_z(x,y,1) = \sum_{m,n=0}^{\infty}\sqrt{n^2+m^2}\pi A_{m,n}\cos n\pi x\cos m\pi y\sinh\sqrt{n^2+m^2}\pi = g(x,y)$$

$$\boxed{A_{m,n} = \frac{4}{\sqrt{n^2+m^2}\pi\sinh\sqrt{n^2+m^2}\pi}\int_0^1\int_0^1 g(x,y)\cos n\pi x\cos m\pi y dxdy, \quad m \neq 0, n \neq 0}$$

$$\boxed{A_{0,n} = \frac{2}{n\pi\sinh n\pi}\int_0^1 g(x,y)\cos n\pi x dx, \quad A_{m,0} = \frac{2}{m\pi\sinh m\pi}\int_0^1 g(x,y)\cos m\pi y dy}$$

6.3.1

a

By the maximum principle of harmonic functions $\max_{\overline{D}} u = \max_{\partial D} u$. On the boundary,

$$u = 3\sin 2\theta + 1 \leq 3 + 1 = 4$$

So the maximum of u in $\overline{D}$ is 4.

b

By the mean-value property of harmonic functions,

$$\begin{aligned} u(\mathbf{0}) &= \frac{1}{4\pi}\int_0^{2\pi}(3\sin 2\theta + 1)(2d\theta) \\ &= \frac{1}{2\pi}\Big[-\frac{3\cos 2\theta}{2} + \theta\Big]\Big|_0^{2\pi} = \boxed{1} \end{aligned}$$

6.4.5

a

The steady-state temperature distribution satisfies Laplace equation $\nabla u = 0$. Let $u = X(\theta)R(r)$, then

$$\frac{R'' + \frac{1}{r}R'}{\frac{1}{r^2}R} = -\frac{X''}{X} = \lambda$$

Since θ is not bounded, X satisfies periodic boundary conditions

$$X(0) = X(2\pi), \quad X'(0) = X'(2\pi)$$

$$\Rightarrow \quad \begin{cases} X_0 = C, \quad \lambda = 0 \\ X_n = A\cos n\theta + B\sin n\theta, \quad \lambda = n^2 \end{cases}$$

For $\lambda = 0$,

$$R'' + \frac{1}{r}R' = 0 \quad \Rightarrow \quad R_0 = C_1 + C_2 \ln r$$

Since the outer edge is insulated,

$$R_0'(2) = \frac{C_2}{2} = 0 \quad \Rightarrow \quad C_2 = 0$$

Therefore, R can only be constant

$$\boxed{R_0 = C}$$

For $\lambda = n^2$

$$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = 0$$
$$r^2R'' + rR' - n^2R = 0$$

Suppose $R(r) = r^\alpha$, then

$$r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2r^\alpha = (\alpha^2 - n^2)r^\alpha = 0 \quad \Rightarrow \quad \alpha = \pm n$$

$$\Rightarrow \quad R_n = Cr^n + Dr^{-n}$$

Since the outer edge is insulated,

$$R_n'(2) = nC2^{n-1} - nD2^{-n-1} = 0 \quad \Rightarrow \quad D = 4^nC$$

So the solution can be rewritten as

$$\boxed{R_n = C\left[r^n + 4^nr^{-n}\right]}$$

Combining above results, the general solution is

$$\boxed{u = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n(r^n + 4^nr^{-n})\cos n\theta + D_n(r^n + 4^nr^{-n})\sin n\theta}$$

At $r = 1$,

$$u = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n(1+4^n)\cos n\theta + D_n(1+4^n)\sin n\theta = \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

Comparing the terms, the non-zero coefficients are

$$C_0 = 1, \quad C_2 = -\frac{1}{34}$$

So the solution is

$$\boxed{u = \frac{1}{2} - \frac{1}{34}\left(r^2 + \frac{16}{r^2}\right)\cos 2\theta}$$

b

Following the same steps as (a), for $\lambda = 0$,

$$\boxed{X_0 = C}, \quad R_0 = C_1 + C_2 \ln r$$

And for $\lambda = n^2$,

$$\boxed{X_n = A\cos n\theta + B\sin n\theta}, \quad R_n = Cr^n + Dr^{-n}$$

Now at the outer edge

$$R_0(2) = C_1 + C_2\ln 2 = 0 \quad \Rightarrow \quad C_2 = -\frac{C_1}{\ln 2}$$

$$R_n(2) = C2^n + D2^{-n} = 0 \quad \Rightarrow \quad D = -4^n C$$

So $R(r)$ can be rewritten as

$$\boxed{R_0 = C\left(1 - \frac{\ln r}{\ln 2}\right), \quad R_n = D\left(r^n - 4^n r^{-n}\right)}$$

Combining the results, the general solution is

$$\boxed{u = C_0\left(1 - \frac{\ln r}{\ln 2}\right) + \sum_{n=1}^{\infty} C_n\left(r^n - 4^n r^{-n}\right)\cos n\theta + D_n\left(r^n - 4^n r^{-n}\right)\sin n\theta}$$

On the inner edge

$$C_0 + \sum_{n=1}^{\infty} C_n\left(1 - 4^n\right)\cos n\theta + D_n\left(1 - 4^n\right)\sin n\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

The non-zero coefficients are

$$C_0 = \frac{1}{2}, \quad C_2 = \frac{1}{30}$$

Finally, the solution is

$$\boxed{u = \frac{1}{2}\left(1 - \frac{\ln r}{\ln 2}\right) + \frac{1}{30}\left(r^2 - \frac{16}{r^2}\right)\cos 2\theta}$$

6.4.10

Let $u = X(\theta)R(r)$. The boundary condition on $x = 0$ and $y = 0$ can be written as

$$X(0) = X(\pi/2) = 0$$

Separation of variables gives

$$\frac{R'' + \frac{1}{r}R'}{\frac{1}{r^2}R} = -\frac{X''}{X} = \lambda$$

For $\lambda < 0$, the boundary condition cannot be satisfied.

For $\lambda = 0$,

$$X'' = 0 \quad \Rightarrow \quad X = A\theta + B$$

$$X(0) = B = 0 \quad \Rightarrow \quad X = A\theta$$

$$X(\pi/2) = A\pi/2 = 0 \quad \Rightarrow \quad A = 0, \quad X = 0$$

There is no non-trivial solution, so zero is not a eigenvalue.

For $\lambda > 0$, let $\lambda = \beta^2$, then

$$X = A\cos\beta\theta + B\sin\beta\theta$$

$$X(0) = 0 \quad \Rightarrow \quad A = 0, \quad X = \sin\beta\theta$$

$$X(\pi/2) = \sin\frac{\beta\pi}{2} = 0 \quad \Rightarrow \quad \beta_n = 2n, \quad \boxed{X_n = \sin 2n\theta}$$

Now solve for R:

$$r^2R'' + rR' - 4n^2R = 0$$

Suppose $R = r^\alpha$,

$$\alpha^2 - 4n^2 = 0 \quad \Rightarrow \quad \alpha = \pm 2n$$

Since $\lim_{r\to 0} r^{-2n} = \infty$, r^{-2n} should be excluded,

$$\boxed{R_n = C_n r^{2n}}$$

The general solution is:

$$u = \sum_{n=1}^{\infty} C_n r^{2n} \sin 2n\theta$$

$$u_r(a,\theta) = \sum_{n=1}^{\infty} 2nC_n a^{2n-1}\sin 2n\theta = 1$$

$$C_n = \frac{2}{n\pi a^{2n-1}}\int_0^{\pi/2}\sin 2n\theta d\theta = \frac{1-(-1)^n}{n^2\pi a^{2n-1}} = \begin{cases} \frac{2}{n^2\pi a^{2n-1}}, & n \text{ is odd} \\ 0, \quad n \text{ is even} \end{cases}$$

$$u = \boxed{\sum_{odd} \frac{2}{n^2\pi a^{2n-1}} r^{2n}\sin 2n\theta}$$

$$= \boxed{\frac{2}{a\pi}r^2\sin 2\theta + \frac{2}{9a^5\pi}r^6\sin 6\theta + \cdots}$$

7.4.6

a

The fundamental solution for Laplace equation in 2 dimension is

$$\Phi(\mathbf{x}-\mathbf{x_0}) = -\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x_0}|$$

Using the reflection method, the corrector function is

$$h^{\mathbf{x_0}}(\mathbf{x}) = \Phi(\mathbf{x}-\hat{\mathbf{x}}_\mathbf{0}) = -\frac{1}{2\pi}\ln|\mathbf{x}-\hat{\mathbf{x}}_\mathbf{0}|$$

where $\hat{\mathbf{x}}_\mathbf{0}$ is the reflection of $\mathbf{x_0}$ about the x-axis. The Green's function is

$$G(\mathbf{x},\mathbf{x_0}) = \Phi(\mathbf{x}-\mathbf{x_0}) - h^{\mathbf{x_0}}(\mathbf{x}) = \boxed{-\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x_0}| + \frac{1}{2\pi}\ln|\mathbf{x}-\hat{\mathbf{x}}_\mathbf{0}|}$$

b

Let $\mathbf{x} = (x,y)$, $\mathbf{x_0} = (x_0,y_0)$, $\hat{\mathbf{x}}_\mathbf{0} = (x_0,-y_0)$. On the boundary, which is the x-axis,

$$\frac{\partial G(\mathbf{x},\mathbf{x_0})}{\partial n} = -\frac{\partial G(\mathbf{x},\mathbf{x_0})}{\partial y} = \frac{1}{2\pi}\frac{y-y_0}{|\mathbf{x}-\mathbf{x_0}|^2} - \frac{1}{2\pi}\frac{y+y_0}{|\mathbf{x}-\hat{\mathbf{x}}_\mathbf{0}|^2} = -\frac{y_0}{\pi[(x-x_0)^2+y_0^2]}$$

The solution is

$$u(\mathbf{x_0}) = -\int_{\partial\mathbb{R}^2_+}\frac{\partial G(\mathbf{x_0},\mathbf{x})}{\partial n}u(\mathbf{x})ds$$

$$\Rightarrow \quad u(x_0,y_0) = \boxed{\frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-x_0)^2+y_0^2}h(x)dx}$$

c

$$\begin{aligned} u(x_0,y_0) &= \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-x_0)^2+y_0^2}dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{(\frac{x-x_0}{y_0})^2+1}\frac{1}{y_0}dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{z^2+1}dz = \frac{1}{\pi}\tan^{-1}(z)\Big|_{-\infty}^{\infty} = \boxed{1} \end{aligned}$$

7.4.7

a

$$u_x(x,y) = \frac{f'(x/y)}{y}, \quad u_{xx}(x,y) = \frac{f''(x/y)}{y^2}$$

$$u_y(x,y) = -\frac{f'(x/y)x}{y^2}$$

$$u_{yy}(x,y) = -\frac{-f''(x/y)x^2 - 2f'(x/y)xy}{y^4} = \frac{f''(x/y)(x/y)^2 + 2f'(x/y)(x/y)}{y^2}$$

$$u_{xx} + u_{yy} = 0 \quad \Rightarrow \quad \boxed{f''(x) + \frac{2x}{x^2+1}f'(x) = 0}$$

Let $g = f'$, then

$$g'(x) + \frac{2x}{x^2+1}g(x) = 0$$

The integrating factor is

$$\int \frac{2x}{x^2+1}dx = \ln|x^2+1| = \ln(x^2+1)$$

$$\phi(x) = e^{\ln(x^2+1)} = x^2+1$$

$$f'(x) = g(x) = \frac{1}{\phi(x)}\int 0 \cdot \phi(t)dt = \frac{c_1}{x^2+1} \quad \Rightarrow \quad \boxed{f(x) = c_1 \tan^{-1}(x) + c_2}$$

b

$$u(x,y) = f(x/y) = c_1 \tan^{-1}(x/y) + c_2$$

In polar coordinates (Let θ be the angle w.r.t the y-axis):

$$u(r,\theta) = c_1\theta + c_2 \quad \Rightarrow \quad \frac{\partial u}{\partial r} \equiv 0$$

c

If $\partial u/\partial r \equiv 0$, then $u = f(\theta) = (f \circ \tan^{-1})(x/y)$, where θ is the angle w.r.t. the y-axis.

d

$$h(x) = \lim_{y\to 0} u(x,y) = \lim_{y\to 0} c_1 \tan^{-1}(x/y) + c_2 = c_1\pi/2 + c_2$$

The boundary value is some constant.

e

From parts (c) and (d), if a function $v(x,y)$ in $\{y > 0\}$ is harmonic and satisfies $\partial u/\partial r \equiv 0$, then its boundary value is a constant.

Using the formula from Ex. 7.4.6:

Suppose $\partial u/\partial r \equiv 0$, then u doesn't depend on r. In other words, the value of u does not change if x and y are scaled by some constant:

$$\begin{aligned}
u(\lambda x_0, \lambda y_0) &= \frac{\lambda y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-\lambda x_0)^2+(\lambda y_0)^2}h(x)dx \\
&= \frac{\lambda y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(\lambda x'-\lambda x_0)^2+(\lambda y_0)^2}h(\lambda x')\lambda dx' \\
&= \frac{\lambda y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{\lambda^2}\frac{1}{(x'-x_0)^2+y_0^2}h(\lambda x')\lambda dx' \\
&= \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x'-x_0)^2+y_0^2}h(\lambda x')dx' \\
&= \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{1}{(x-x_0)^2+y_0^2}h(x)dx
\end{aligned}$$

It is necessary that $h(\lambda x) = h(x)$ for any scaling factor λ, so $h(x)$ must be a constant function. Thus, the results are consistent.

7.4.17

a

Suppose $\mathbf{x_0} = (x_0, y_0)$, then the reflection points are $\mathbf{x_1} = (-x_0, y_0)$, $\mathbf{x_2} = (-x_0, -y_0)$, $\mathbf{x_3} = (x_0, -y_0)$. The corrector function can be defined as

$$h^{\mathbf{x_0}}(\mathbf{x}) = \Phi(\mathbf{x}-\mathbf{x_1}) - \Phi(\mathbf{x}-\mathbf{x_2}) + \Phi(\mathbf{x}-\mathbf{x_3})$$

$\Delta h^{\mathbf{x_0}}(\mathbf{x}) = 0$ everywhere in Q, and by symmetry,

$$h^{\mathbf{x_0}}(x, 0) = 0 + \Phi(\mathbf{x}-\mathbf{x_3}) = \Phi(\mathbf{x}-\mathbf{x_0})$$

$$h^{\mathbf{x_0}}(0, y) = \Phi(\mathbf{x}-\mathbf{x_1}) + 0 = \Phi(\mathbf{x}-\mathbf{x_0})$$

Therefore, the Green's function is

$$\begin{aligned}
G(\mathbf{x_0}, \mathbf{x}) &= \Phi(\mathbf{x}-\mathbf{x_0}) - \Phi(\mathbf{x}-\mathbf{x_1}) + \Phi(\mathbf{x}-\mathbf{x_2}) - \Phi(\mathbf{x}-\mathbf{x_3}) \\
&= \boxed{-\frac{1}{2\pi}\Big(\ln|\mathbf{x}-\mathbf{x_0}| - \ln|\mathbf{x}-\mathbf{x_1}| + \ln|\mathbf{x}-\mathbf{x_2}| - \ln|\mathbf{x}-\mathbf{x_3}|\Big)}
\end{aligned}$$

b

On the x-axis,

$$\begin{aligned}
\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial y} &= \frac{1}{2\pi}\left[\frac{y-y_0}{|\mathbf{x}-\mathbf{x_0}|^2} - \frac{y-y_1}{|\mathbf{x}-\mathbf{x_1}|^2} + \frac{y-y_2}{|\mathbf{x}-\mathbf{x_2}|^2} - \frac{y-y_3}{|\mathbf{x}-\mathbf{x_3}|^2}\right] \\
&= \frac{y_0}{2\pi}\left[-\frac{1}{|\mathbf{x}-\mathbf{x_0}|^2} + \frac{1}{|\mathbf{x}-\mathbf{x_1}|^2} + \frac{1}{|\mathbf{x}-\mathbf{x_2}|^2} - \frac{1}{|\mathbf{x}-\mathbf{x_3}|^2}\right] \\
&= \frac{y_0}{\pi}\left[\frac{1}{|\mathbf{x}-\mathbf{x_1}|^2} - \frac{1}{|\mathbf{x}-\mathbf{x_0}|^2}\right] \\
&= \frac{y_0}{\pi}\left[\frac{1}{(x+x_0)^2+y_0^2} - \frac{1}{(x-x_0)^2+y_0^2}\right]
\end{aligned}$$

On the y-axis,

$$\begin{aligned}
\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial x} &= \frac{1}{2\pi}\left[\frac{x-x_0}{|\mathbf{x}-\mathbf{x_0}|^2} - \frac{x-x_1}{|\mathbf{x}-\mathbf{x_1}|^2} + \frac{x-x_2}{|\mathbf{x}-\mathbf{x_2}|^2} - \frac{x-x_3}{|\mathbf{x}-\mathbf{x_3}|^2}\right] \\
&= \frac{x_0}{2\pi}\left[-\frac{1}{|\mathbf{x}-\mathbf{x_0}|^2} - \frac{1}{|\mathbf{x}-\mathbf{x_1}|^2} + \frac{1}{|\mathbf{x}-\mathbf{x_2}|^2} + \frac{1}{|\mathbf{x}-\mathbf{x_3}|^2}\right] \\
&= \frac{x_0}{\pi}\left[\frac{1}{|\mathbf{x}-\mathbf{x_2}|^2} - \frac{1}{|\mathbf{x}-\mathbf{x_0}|^2}\right] \\
&= \frac{x_0}{\pi}\left[\frac{1}{x_0^2+(y+y_0)^2} - \frac{1}{x_0^2+(y-y_0)^2}\right]
\end{aligned}$$

Finally, the solution is

$$\begin{aligned}
u(\mathbf{x_0}) &= -\int_{\partial Q}\frac{\partial G(\mathbf{x},\mathbf{x_0})}{\partial n}u(\mathbf{x})ds \\
&= \frac{y_0}{\pi}\int_0^\infty\left[\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2}\right]h(x)dx \\
&+ \frac{x_0}{\pi}\int_0^\infty\left[\frac{1}{x_0^2+(y-y_0)^2} - \frac{1}{x_0^2+(y+y_0)^2}\right]g(y)dy
\end{aligned}$$