PHYS 7125 Homework 1

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1

a

If $\Delta s^2 = 0$ then $\Delta t = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. A particle that travels Δx^α travels at the speed of light. And since the speed of light is constant in all reference frames, in another coordinate system $x^{\alpha'}$, it still travels at the speed of light, that is, $\Delta t' = \sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}$. In other words, $\Delta s'^2 = 0$.

b

Expressing Q in terms of Δx^α gives another quadratic form:

$$Q = \eta_{\alpha'\beta'} \Delta x^{\alpha'} \Delta x^{\beta'} = \eta_{\alpha'\beta'} \Lambda^{\alpha'}{}_\alpha \Delta x^\alpha \Lambda^{\beta'}{}_\beta \Delta x^\beta = \left(\Lambda^{\alpha'}{}_\alpha \Lambda^{\beta'}{}_\beta \eta_{\alpha'\beta'}\right) \Delta x^\alpha \Delta x^\beta = \phi_{\alpha\beta} \Delta x^\alpha \Delta x^\beta$$

On the light cone $\Delta s^2 = 0$. And from the result of (a), $Q = \Delta s'^2 = \Delta s^2 = 0$

c

On the intersection $Q = 0$ everywhere. By spherical symmetry in the spatial components of Δx^α, Q must be invariant under a reverse of sign of any spatial coordinate, therefore all cross terms are eliminated. Q must also be invariant under a permutation of spatial coordinates, so the remaining $\Delta x^2, \Delta y^2, \Delta z^2$ must have the same coefficient. Therefore the general form is

$$Q = c_1 \Delta t^2 + c_2(\Delta x^2 + \Delta y^2 + \Delta z^2)$$

d

On the intersection, which lies on the light cone, $\Delta t^2 = \Delta x^2$,

$$Q = c_1 \Delta t^2 + c_2 \Delta x^2 = (c_1 + c_2)\Delta t^2 = 0 \quad \Rightarrow \quad c_1 = -c_2$$

$$Q = c_2(-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2) = c_2 \eta_{\alpha\beta} \Delta x^\alpha \Delta x^\beta$$

e

The constant c_2 applies to transformations between any two coordinate systems, including the trivial one between one coordinate system and itself, therefore c_2 must equal 1.

2

a

Rename the dummy indices, and apply the definition of symmetric/antisymmetric tensors:

$$A_{\mu\nu}S^{\mu\nu} = A_{\nu\mu}S^{\nu\mu} = (-A_{\mu\nu})S^{\mu\nu} \quad \Rightarrow \quad A_{\mu\nu}S^{\mu\nu} = 0$$

b

Using the same trick as above,

$$V^{\nu\mu}A_{\mu\nu} = V^{\mu\nu}A_{\nu\mu} = -V^{\mu\nu}A_{\mu\nu} \quad \Rightarrow \quad \frac{1}{2}\Big(V^{\mu\nu}A_{\mu\nu} - V^{\nu\mu}A_{\mu\nu}\Big) = V^{\mu\mu}A_{\mu\nu}$$

$$V^{\nu\mu}S_{\mu\nu} = V^{\mu\nu}S_{\nu\mu} = V^{\mu\nu}S_{\mu\nu} \quad \Rightarrow \quad \frac{1}{2}\Big(V^{\mu\nu}S_{\mu\nu} + V^{\nu\mu}S_{\mu\nu}\Big) = V^{\mu\mu}S_{\mu\nu}$$

c

When acting on (co)vectors, the tensor produces a scalar, which is invariant under transformations:

$$\begin{aligned} T^{\alpha'}{}_{\beta'}{}^{\gamma'}u_{\alpha'}v^{\beta'}w_{\gamma'} &= T^{\alpha}{}_{\beta}{}^{\gamma}u_{\alpha}v^{\beta}w_{\gamma} \\ &= T^{\alpha}{}_{\beta}{}^{\gamma}\Lambda^{\alpha'}{}_{\alpha}u_{\alpha'}\Lambda^{\beta}{}_{\beta'}v^{\beta'}\Lambda^{\gamma'}{}_{\gamma}w_{\gamma} \\ &= \Big(\Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta}{}_{\beta'}\Lambda^{\gamma'}{}_{\gamma}T^{\alpha}{}_{\beta}{}^{\gamma}\Big)u_{\alpha'}v^{\beta'}w_{\gamma} \end{aligned}$$

Since this holds true for any (co)vectors, $T^{\alpha'}{}_{\beta'}{}^{\gamma'} = \Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta}{}_{\beta'}\Lambda^{\gamma'}{}_{\gamma}T^{\alpha}{}_{\beta}{}^{\gamma}$.

d

$$\begin{aligned} g_{\alpha\beta}g^{\beta\sigma}g^{\alpha\gamma} &= \delta^{\sigma}_{\alpha}g^{\alpha\gamma} = g^{\sigma\gamma} \\ g_{\sigma\beta}g_{\gamma\alpha}g^{\alpha\beta} &= g_{\sigma\beta}\delta^{\beta}_{\gamma} = g_{\sigma\gamma} \\ g^{\alpha}{}_{\beta} = g^{\alpha\sigma}g_{\sigma\beta} &= g_{\beta\sigma}g^{\sigma\alpha} = \delta^{\alpha}_{\beta} \end{aligned}$$

3

a

$$X^{\mu}{}_{\nu} = X^{\mu\gamma}g_{\gamma\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

b

$$X_{\mu}{}^{\nu} = g_{\mu\gamma}X^{\gamma\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$$

c

$$X^{(\mu\nu)} = \frac{1}{2}\Big(X^{\mu\nu} + X^{\nu\mu}\Big) = \begin{pmatrix} 2 & -1/2 & 0 & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{pmatrix}$$

d

$$X_{\mu\nu} = X_\mu{}^\gamma g_{\gamma\nu} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

$$X_{[\mu\nu]} = \frac{1}{2}\Big(X_{\mu\nu} - X_{\nu\mu}\Big) = \begin{pmatrix} 0 & -1/2 & -1 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1 & -1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{pmatrix}$$

e

$$X^\lambda{}_\lambda = -2 + 0 + 0 - 2 = -4$$

f

$$v^\mu v_\mu = g_{\mu\nu} v^\mu v^\nu = -(-1)^2 + 2^2 + 0^2 + (-2)^2 = 7$$

g

$$v_\mu = g_{\mu\nu} v^\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & -2 \end{pmatrix}$$

$$v_\mu X^{\mu\nu} = \begin{pmatrix} 1 & 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 5 \\ 7 \end{pmatrix}$$