PHYS 7125 Homework 2

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The local flatness property states that for each point p on the manifold there exists a change of coordinates such that the metric $g_{\mu\nu}$ can be transformed into a $g_{\mu'\nu'}$ that satisfies: (i) $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ and (ii) $g_{\mu'\nu',\sigma} = 0$ at point p. This can be shown by a Taylor expansion of $g_{\mu'\nu'}$ to the first order:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$$

$$= \left(x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu} \right)\Big|_p + \left(x^\mu_{,\mu'\lambda} x^\nu_{,\nu'} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'\lambda} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu,\lambda} \right)\Big|_p \epsilon + O(\epsilon^2)$$

The requirement is that

$$\left(x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu} \right)\Big|_p = \eta_{\mu'\nu'}$$

$$\left(x^\mu_{,\mu'\lambda} x^\nu_{,\nu'} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'\lambda} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu,\lambda} \right)\Big|_p = 0$$

The first equation has 16 variables in $\partial x^\mu/\partial x^{\mu'}$ and 10 equations, one for each indepedent entry of the metric. The remaining 6 degrees of freedom exactly matches the dimension of the Lorentz group, under which the metric is preserved. Now that $\partial x^\mu/\partial x^{\mu'}$ is determined, the second equation, since partial derivatives commute, will only have $4 \cdot 10 = 40$ variables in $\partial^2 x^\mu/\partial x^{\mu'}\partial x^\lambda$. Coincidentally, since metric is always symmetric, there are also $10 \cdot 4 = 40$ equations corresponding to the independent entries of $g_{\mu\nu,\lambda}$. The system is uniquely determined, and therefore such transformation always exists.

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a

$$g^{\alpha\beta}{}_{,\gamma} = \left(g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu} \right)_{,\gamma}$$

$$= g^{\alpha\nu}{}_{,\gamma} g^{\beta\mu} g_{\mu\nu} + g^{\alpha\nu} g^{\beta\mu}{}_{,\gamma} g_{\mu\nu} + g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma}$$

$$= 2g^{\alpha\nu} g^{\beta\mu}{}_{,\gamma} g_{\mu\nu} + g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma}$$

$$= 2g^{\alpha\nu} \left(g^{\beta\mu}{}_{,\gamma} g_{\mu\nu} + g^{\beta\mu} g_{\mu\nu,\gamma} \right) - g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma}$$

$$= 2g^{\alpha\nu} \left(g^{\beta\mu} g_{\mu\nu} \right)_{,\gamma} - g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma}$$

$$= 2g^{\alpha\nu} \delta^\beta_{\nu,\gamma} - g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma}$$

$$= -g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma}$$

b

From the two identities we can derive the formula:

$$\frac{d}{d\epsilon}\det(A) = \lim_{\epsilon\to 0}\frac{\det(A + \epsilon\frac{d}{d\epsilon}A + O(\epsilon^2)) - \det(A)}{\epsilon}$$

$$= \lim_{\epsilon\to 0}\frac{\det(A(I + \epsilon A^{-1}\frac{d}{d\epsilon}A)) - \det(A)}{\epsilon}$$

$$= \lim_{\epsilon\to 0}\frac{\det(A)\det(I + \epsilon A^{-1}\frac{d}{d\epsilon}A) - \det(A)}{\epsilon}$$

$$= \det(A)\lim_{\epsilon\to 0}\frac{\det(I + \epsilon A^{-1}\frac{d}{d\epsilon}A) - 1}{\epsilon}$$

$$= \det(A)\lim_{\epsilon\to 0}\frac{1 + \epsilon\,\mathrm{tr}(A^{-1}\frac{d}{d\epsilon}A) + O(\epsilon^2) - 1}{\epsilon}$$

$$= \det(A)\,\mathrm{tr}(A^{-1}\frac{d}{d\epsilon}A)$$

Apply the formula to $g = \det g_{\mu\nu}$, replacing $d/d\epsilon$ with ∂_α

$$g_{,\alpha} = g\cdot\mathrm{tr}(g^{\sigma\mu}g_{\mu\nu,\alpha}) = gg^{\nu\mu}g_{\mu\nu,\alpha}$$

c

$$RHS = -(-g)^{-1/2}\left[g^{\alpha\beta}(-g)^{1/2}\right]_{,\beta}$$

$$= -(-g)^{-1/2}\left[g^{\alpha\beta}{}_{,\beta}(-g)^{1/2} + g^{\alpha\beta}(-g)^{1/2}_{,\beta}\right]$$

$$= -(-g)^{-1/2}\left[g^{\alpha\beta}{}_{,\beta}(-g)^{1/2} - \frac{1}{2}g^{\alpha\beta}(-g)^{-1/2}g_{,\beta}\right]$$

$$= -g^{\alpha\beta}{}_{,\beta} + \frac{1}{2}g^{\alpha\beta}(-g)^{-1}g_{,\beta}$$

$$= g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta} + \frac{1}{2}g^{\alpha\beta}(-g)^{-1}gg^{\mu\nu}g_{\mu\nu,\beta}$$

$$= g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta} - \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}$$

$$= \frac{1}{2}\left(g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta} + g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta} - g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\right)$$

$$= \frac{1}{2}\left(g^{\mu\nu}g^{\beta\alpha}g_{\mu\beta,\nu} + g^{\nu\beta}g^{\mu\alpha}g_{\nu\mu,\beta} - g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\right)$$

$$= \frac{1}{2}\left(g^{\mu\nu}g^{\beta\alpha}g_{\mu\beta,\nu} + g^{\nu\mu}g^{\beta\alpha}g_{\nu\beta,\mu} - g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\right)$$

$$= g^{\mu\nu}\cdot\frac{1}{2}g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right)$$

$$= g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = LHS$$

d

$$LHS = A^\alpha{}_{,\alpha} + \Gamma^\alpha_{\alpha\lambda}A^\lambda$$

$$= A^\alpha{}_{,\alpha} + \frac{1}{2}g^{\alpha\beta}\Big(g_{\beta\alpha,\lambda} + g_{\beta\lambda,\alpha} - g_{\alpha\lambda,\beta}\Big)A^\lambda$$

$$= A^\alpha{}_{,\alpha} + \frac{1}{2}\Big(g^{\alpha\beta}g_{\beta\alpha,\lambda} + g^{\alpha\beta}g_{\beta\lambda,\alpha} - g^{\alpha\beta}g_{\alpha\lambda,\beta}\Big)A^\lambda$$

$$= A^\alpha{}_{,\alpha} + \frac{1}{2}\Big(g^{\alpha\beta}g_{\beta\alpha,\lambda} + \cancel{g^{\alpha\beta}g_{\beta\lambda,\alpha}} - \cancel{g^{\beta\alpha}g_{\beta\lambda,\alpha}}\Big)A^\lambda$$

$$= A^\alpha{}_{,\alpha} + \frac{1}{2}g^{\alpha\beta}g_{\beta\alpha,\lambda}A^\lambda$$

$$= A^\alpha{}_{,\alpha} + \frac{1}{2}g^{\mu\nu}g_{\mu\nu,\alpha}A^\alpha$$

$$RHS = (-g)^{-1/2}\Big[(-g)^{1/2}A^\alpha\Big]_{,\alpha}$$

$$= (-g)^{-1/2}\Big[(-g)^{1/2}A^\alpha{}_{,\alpha} + (-g)^{1/2}_{,\alpha}A^\alpha\Big]$$

$$= A^\alpha{}_{,\alpha} - \frac{1}{2}(-g)^{-1/2}(-g)^{-1/2}g_{,\alpha}A^\alpha$$

$$= A^\alpha{}_{,\alpha} + \frac{1}{2}(g)^{-1}gg^{\mu\nu}g_{\mu\nu,\alpha}A^\alpha$$

$$= A^\alpha{}_{,\alpha} + \frac{1}{2}g^{\mu\nu}g_{\mu\nu,\alpha}A^\alpha = LHS$$

e

$$\epsilon_{\alpha\beta\gamma\delta;\mu} = \Big((-g)^{1/2}\tilde{\epsilon}_{\alpha\beta\gamma\delta}\Big)_{;\mu} = (-g)^{1/2}_{,\mu}\tilde{\epsilon}_{\alpha\beta\gamma\delta} - (-g)^{1/2}\Big[\Gamma^\lambda_{\alpha\mu}\tilde{\epsilon}_{\lambda\beta\gamma\delta} + \Gamma^\lambda_{\beta\mu}\tilde{\epsilon}_{\alpha\lambda\gamma\delta} + \Gamma^\lambda_{\gamma\mu}\tilde{\epsilon}_{\alpha\beta\lambda\delta} + \Gamma^\lambda_{\delta\mu}\tilde{\epsilon}_{\alpha\beta\gamma\lambda}\Big]$$

If indices are repeated, WLOG, we can suppose $\alpha = \beta$, then $\tilde{\epsilon}_{\alpha\beta\gamma\delta} = 0$; the first term outside the brackets and the last two terms inside the bracket vanish, and also

$$\Gamma^\lambda_{\alpha\mu}\tilde{\epsilon}_{\lambda\beta\gamma\delta} + \Gamma^\lambda_{\beta\mu}\tilde{\epsilon}_{\alpha\lambda\gamma\delta} = \Gamma^\lambda_{\alpha\mu}\tilde{\epsilon}_{\lambda\alpha\gamma\delta} + \Gamma^\lambda_{\alpha\mu}\tilde{\epsilon}_{\alpha\lambda\gamma\delta} = 0$$

So $\epsilon_{\alpha\beta\gamma\delta;\mu} = 0$ in this case. Now consider the case where all indices are distinct, then $\tilde{\epsilon}_{\lambda\beta\gamma\delta}$ (here λ is a free index), vanishes whenever $\lambda \neq \alpha$. Same argument applies to all indices. WLOG, suppose $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3$. The first term on the right-hand side is

$$-\frac{1}{2}(-g)^{-1/2}g_{,\mu}$$

Expand the the second term:

$$\cdots = \frac{1}{2}(-g)^{1/2}g^{\lambda\sigma}\Big(g_{\sigma 0,\mu} + g_{\sigma\mu,0} - g_{0\mu,\sigma}\Big)\tilde{\epsilon}_{\lambda 123} + \frac{1}{2}(-g)^{1/2}g^{\lambda\sigma}\Big(g_{\sigma 1,\mu} + g_{\sigma\mu,1} - g_{1\mu,\sigma}\Big)\tilde{\epsilon}_{0\lambda 23}$$

$$+ \frac{1}{2}(-g)^{1/2}g^{\lambda\sigma}\Big(g_{\sigma 2,\mu} + g_{\sigma\mu,2} - g_{2\mu,\sigma}\Big)\tilde{\epsilon}_{01a\lambda 3} + \frac{1}{2}(-g)^{1/2}g^{\lambda\sigma}\Big(g_{\sigma 3,\mu} + g_{\sigma\mu,3} - g_{3\mu,\sigma}\Big)\tilde{\epsilon}_{012\lambda}$$

$$= \frac{1}{2}(-g)^{1/2}g^{0\sigma}\Big(g_{\sigma 0,\mu} + g_{\sigma\mu,0} - g_{0\mu,\sigma}\Big) + \frac{1}{2}(-g)^{1/2}g^{1\sigma}\Big(g_{\sigma 1,\mu} + g_{\sigma\mu,1} - g_{1\mu,\sigma}\Big)$$

$$+ \frac{1}{2}(-g)^{1/2}g^{2\sigma}\Big(g_{\sigma 2,\mu} + g_{\sigma\mu,2} - g_{2\mu,\sigma}\Big) + \frac{1}{2}(-g)^{1/2}g^{3\sigma}\Big(g_{\sigma 3,\mu} + g_{\sigma\mu,3} - g_{3\mu,\sigma}\Big)$$

$$= \frac{1}{2}(-g)^{1/2}\Big(g^{\lambda\sigma}g_{\lambda\sigma,\mu} + \cancel{g^{\lambda\sigma}g_{\sigma\mu,\lambda}} + \cancel{g^{\lambda\sigma}g_{\lambda\mu,\sigma}}\Big)$$

$$= -\frac{1}{2}(-g)^{-1/2}g_{,\mu}$$

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where the second to last line above regrouped the terms into a summation over dummy index λ and also invoked the result from (b). Two terms on the right-hand side cancel out, therefore $\epsilon_{\alpha\beta\gamma\delta;\mu} = 0$.

3

a

Since $u_\alpha u^\alpha = -1$,

$$(P_{\alpha\beta}v^\beta)u^\alpha = g_{\alpha\beta}v^\beta u^\alpha + u_\alpha u_\beta v^\beta u^\alpha = v^\beta u_\beta - u_\beta v^\beta = 0$$

b

From (a), $u^\beta v_{\perp_\beta} = u_\beta v_\perp^\beta = 0$, therefore

$$P_{\alpha\beta}v_\perp^\beta = g_{\alpha\beta}v_\perp^\beta + u_\alpha u_\beta v_\perp^\beta = g_{\alpha\beta}v_\perp^\beta + 0 = v_{\perp\alpha}$$

c

$$P_{\alpha\beta} := g_{\alpha\beta} - (q_\lambda q^\lambda)^{-1}q_\alpha q_\beta$$

Proof: Carrying out the same calculations:

$$(P_{\alpha\beta}v^\beta)q^\alpha = g_{\alpha\beta}v^\beta q^\alpha - \cancel{(q_\lambda q^\lambda)^{-1}}\cancel{q_\alpha}q_\beta v^\beta \cancel{q^\alpha} = v^\beta q_\beta - q_\beta v^\beta = 0$$

$$P_{\alpha\beta}v_\perp^\beta = g_{\alpha\beta}v_\perp^\beta - (q_\lambda q^\lambda)^{-1}q_\alpha \cancel{q_\beta v_\perp^\beta} = v_{\perp\alpha}$$

d

Since the norm of a null vector is zero, the orthogonal projection cannot be constructed by substracting a parallel projection which involves a division by the norm, therefore the projection can only be expressed as an expansion w.r.t. the basis vectors orthogonal to $\underline{k}$. Apart from $\underline{k}$, which is orthogonal to itself, there must exist two additional linearly independent vectors to match the dimensionality. It can be shown that (i) orthogonal null vectors are colinear and (ii) null vectors cannot be orthogonal to time-like vectors, therefore the two vectors must be space-like, and we can perform the Gram-Schmidt process like above (plus normalization) to obtain two orthnormal space-like vectors. Finally, $k_\alpha k_\beta$ term must be excluded because if the projection produces a k_μ component, applying the projection twice will then eliminate it, violating $P(v_\perp) = v_\perp$. Therefore the projection tensor is

$$P_{\alpha\beta} = e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}$$

Where $\underline{e}_{(i)}$ are the orthonormal space-like basis vectors. It can be quickly verified that

$$\left(e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}\right)v^\beta k^\alpha = 0$$

and also

$$\left(e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}\right)v_\perp^\beta = \left(e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}\right)\left(e_{(1)}^\beta e_{(1)\sigma} + e_{(2)}^\beta e_{(2)\sigma}\right)v^\sigma$$

$$= \left(e_{(1)\alpha}e_{(1)\sigma} + e_{(2)\alpha}e_{(2)\sigma}\right)v^\sigma = v_{\perp\alpha}$$

However the choice of $\{\underline{e}_{(i)}\}$ is not unique, which in general means that P will not be unique. As an example, suppose the space-time is flat and consider $\underline{k} = (1, 1, 0, 0)$, $\underline{e}_{(1)} = (1, 1, 1, 0)$, $\underline{e}_{(2)} = (1, 1, 0, 1)$, $P_{\alpha\beta} = e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta}$ only has integer components; however, we can also choose $\underline{e}'_{(1)} = (1, 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $\underline{e}'_{(2)} = (1, 1, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, then $P'_{\alpha\beta} = e'_{(1)\alpha}e'_{(1)\beta} + e'_{(2)\alpha}e'_{(2)\beta}$ would have irrational components.

4

a

$$\nabla_{\underset{\sim}{u}} u_\mu = \frac{dx^\alpha}{d\tau}\nabla_\alpha u_\mu = \frac{dx^\alpha}{d\tau}\left(\partial_\alpha u_\mu - \Gamma^\beta_{\alpha\mu}u_\beta\right) = \frac{du_\mu}{d\tau} - \Gamma^\beta_{\alpha\mu}u^\alpha u_\beta = 0$$

b

Since the connection is metric-compatible,

$$\nabla_\alpha g^{\mu\nu} = \partial_\alpha g^{\mu\nu} + \Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} + \Gamma^\nu_{\alpha\lambda}g^{\mu\lambda} = 0 \Rightarrow \quad \partial_\alpha g^{\mu\nu} = -\Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} - \Gamma^\nu_{\alpha\lambda}g^{\mu\lambda}$$

$$\Rightarrow \quad \frac{dg^{\mu\nu}}{d\tau} = u^\alpha \partial_\alpha g^{\mu\nu} = -u^\alpha \Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} - u^\alpha \Gamma^\nu_{\alpha\lambda}g^{\mu\lambda}$$

Raising the indices of the equation from (a), we recover the original form of the geodesic equation:

$$g^{\mu\nu}\frac{du_\mu}{d\tau} - g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta$$

$$= \frac{d}{d\tau}\left(g^{\mu\nu}u_\mu\right) - u_\mu \frac{dg^{\mu\nu}}{d\tau} - g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta$$

$$= \frac{d}{d\tau}\left(g^{\mu\nu}u_\mu\right) + u_\mu u^\alpha \Gamma^\mu_{\alpha\lambda}g^{\lambda\nu} + u_\mu u^\alpha \Gamma^\nu_{\alpha\lambda}g^{\mu\lambda} - g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta$$

$$= \frac{d}{d\tau}\left(g^{\mu\nu}u_\mu\right) + \cancel{u_\beta u^\alpha \Gamma^\beta_{\alpha\mu}g^{\mu\nu}} + u_\mu u^\alpha \Gamma^\nu_{\alpha\lambda}g^{\mu\lambda} - \cancel{g^{\mu\nu}\Gamma^\beta_{\alpha\mu}u^\alpha u_\beta}$$

$$= \frac{du^\nu}{d\tau} + \Gamma^\nu_{\alpha\lambda}u^\lambda u^\alpha = 0$$

Without invoking metric compatibility, we can also check that the covariant derivatives of $u_\mu u^\mu$, which is a scalar, reduce to ordinary derivatives:

$$\nabla_{\underset{\sim}{u}}\left(u_\mu u^\mu\right) = u_\mu \nabla_{\underset{\sim}{u}} u^\mu + u^\mu \nabla_{\underset{\sim}{u}} u_\mu = u_\mu\left(\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta\right) + u^\mu\left(\frac{du_\mu}{d\tau} - \Gamma^\beta_{\alpha\mu}u^\alpha u_\beta\right)$$

$$= u_\mu \frac{du^\mu}{d\tau} + \cancel{\Gamma^\mu_{\alpha\beta}u_\mu u^\alpha u^\beta} + u^\mu \frac{du_\mu}{d\tau} - \cancel{\Gamma^\beta_{\alpha\mu}u^\mu u^\alpha u_\beta} = \frac{d}{d\tau}\left(u_\mu u^\mu\right)$$

c

Suppose λ is an affine parameter for a null-geodesic and $\sigma = \sigma(\lambda)$ non-affine:

$$\nabla_{\underset{\sim}{u}} u^\mu = \nabla_{\frac{d}{d\sigma}}\frac{dx^\mu}{d\sigma} = \nabla_{\frac{d\lambda}{d\sigma}\frac{d}{d\lambda}}\left(\frac{d\lambda}{d\sigma}\frac{dx^\mu}{d\lambda}\right) = \frac{d\lambda}{d\sigma}\nabla_{\frac{d}{d\lambda}}\left(\frac{d\lambda}{d\sigma}\frac{dx^\mu}{d\lambda}\right)$$

$$= \frac{d\lambda}{d\sigma}\left(\frac{d}{d\lambda}\frac{d\lambda}{d\sigma}\right)\frac{dx^\mu}{d\lambda} + \frac{d\lambda}{d\sigma}\frac{d\lambda}{d\sigma}\cancel{\nabla_{\frac{d}{d\lambda}}\frac{dx^\mu}{d\lambda}}$$

$$= \left(\frac{d}{d\lambda}\frac{d\lambda}{d\sigma}\right)u^\mu = -\kappa(\lambda)u^\mu$$

where κ is some function of λ. The second term above vanishes because λ is assumed to be an affine.