PHYS 7125 Homework 5

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1

The total proper time along a curve γ is

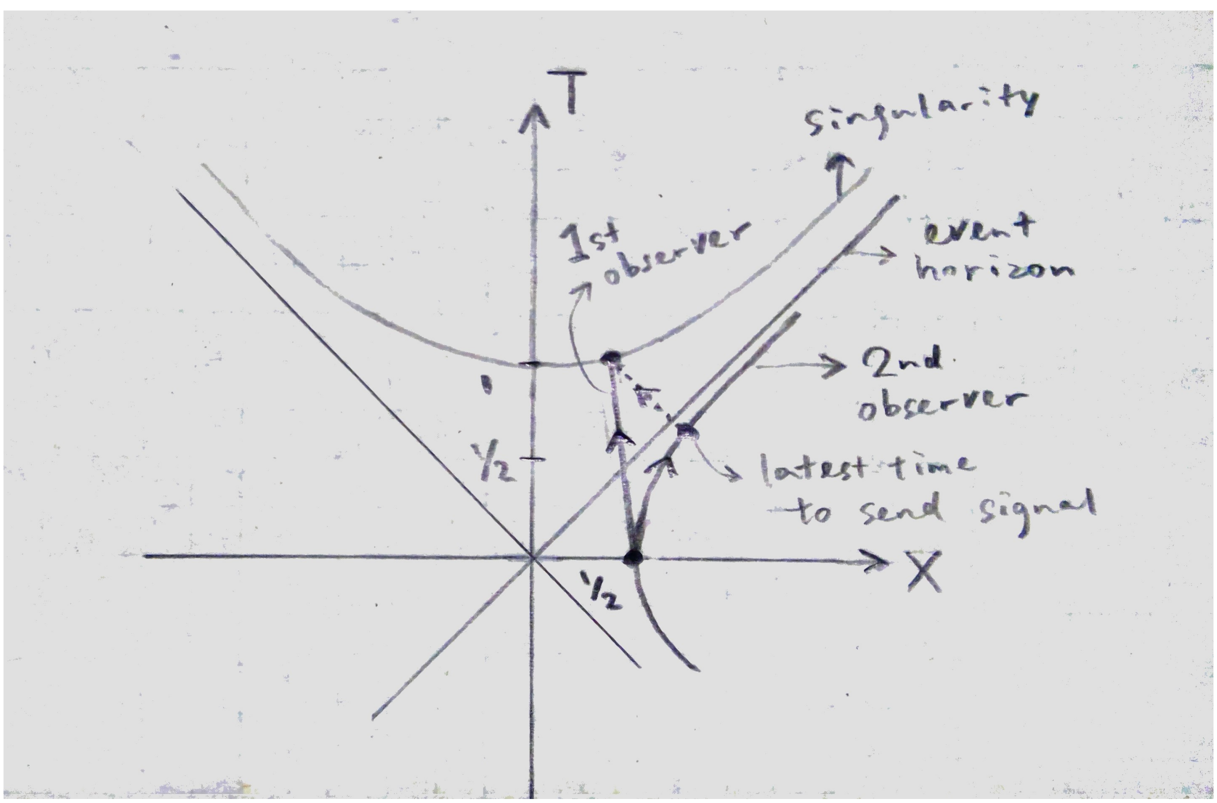
$$\tau_{total} = \int_\gamma d\tau = \int_\gamma \sqrt{-ds^2} = \int_\gamma \sqrt{-\left(\frac{2M}{r} - 1\right)dt^2 + \left(\frac{2M}{r} - 1\right)^{-1} dr^2 - r^2 d\Omega^2}$$

Inside the event horizon, $r < 2M$, the first and third term in the square root are negative, therefore

$$\begin{aligned}\tau_{total} &< \int_0^{2M} \sqrt{\left(\frac{2M}{r} - 1\right)^{-1}} dr \\ &= \left[-\sqrt{r(2M-r)} + 2M \cot^{-1}\left(\sqrt{\frac{2M}{r} - 1}\right)\right]\Bigg|_0^{2M} = \pi M\end{aligned}$$

2

a



b

Yes, because the observer is massive. (If not, the light signal in (c) could never reach the falling observer, because the two worldlines would be parallel straight lines in a Kruskal diagram.)

c

At constant $r = R$,

$$T = \frac{1}{2}\sinh\left(\frac{t}{4GM}\right), \quad X = \frac{1}{2}\cosh\left(\frac{t}{4GM}\right), \quad 4X^2 - 4T^2 = 1$$

At $t = 0$ on this worldline,

$$T = \frac{1}{2}\sinh 0 = 0, \quad X = \frac{1}{2}\cosh 0 = \frac{1}{2}$$

At the singularity,

$$T = \cosh\left(\frac{t}{4GM}\right), \quad X = \sinh\left(\frac{t}{4GM}\right), \quad T^2 - X^2 = 1$$

A timelike straight line passing through $(1/2, 0)$ can be expressed as

$$X = kT + \frac{1}{2}$$

where $-1 < k < 1$. When the first observer reaches singularity,

$$(1-k^2)T^2 - kT - \frac{5}{4} = 0 \quad \Rightarrow \quad \boxed{T_s = \frac{k + \sqrt{5-4k^2}}{2(1-k^2)}, \quad X_s = \frac{k^2 + k\sqrt{5-4k^2}}{2(1-k^2)} + \frac{1}{2}}$$

To reach this critical point, the photon emitted by the second observer must follow the straight line

$$X = -T + T_s + X_s$$

The photon's worldline in the past intersects the second observer's worldline at

$$\cancel{4T^2} + 4(T_s + X_s)^2 - 8(T_s + X_s)T - \cancel{4T^2} = 1$$

$$\Rightarrow T = \frac{1}{2}\sinh\left(\frac{t}{4GM}\right) = \frac{4(T_s + X_s)^2 - 1}{8(T_s + X_s)}, \quad \boxed{t = 4GM \sinh^{-1}\left(\frac{4(T_s + X_s)^2 - 1}{4(T_s + X_s)}\right)}$$

which is the latest Schwarzschild time at which the second observer could send the signal. If the worldline of the first observer is vertical in the Kruskal diagram, i.e. $k = 0$, then $t \approx 4.70GM$

3

A perfect fluid is incompressible, therefore $\nabla_\mu \rho = \partial_\mu \rho = 0$. From the continuity equation,

$$\nabla_\mu(\rho u^\mu) = \rho \nabla_\mu u^\mu = 0 \quad \Rightarrow \quad \nabla_\mu u^\mu = 0$$

Since the covariant divergence of the stress-energy tensor must vanish,

$$\begin{aligned}
\nabla_\mu T^\mu{}_\nu &= (\nabla_\mu p + \cancel{\nabla_\mu \rho})u^\mu u_\nu + (p+\rho)\cancel{(\nabla_\mu u^\mu)}u_\nu + (p+\rho)u^\mu(\nabla_\mu u_\nu) + \delta^\mu_\nu \nabla_\mu p \\
&= u_\nu(\nabla_{\underset{\sim}{u}} p) + (p+\rho)\nabla_{\underset{\sim}{u}} u_\nu + \nabla_\nu p \\
&= 0
\end{aligned}$$

The free index ν can be dropped, yielding

$$(p+\rho)\nabla_{\underset{\sim}{u}} \underset{\sim}{u} = -\nabla p - \underset{\sim}{u} \nabla_{\underset{\sim}{u}} p$$