

# PHYS 7125 Homework 3

Wenqi He

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## 1

$$\begin{aligned}
 \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} &= 0 \\
 g^{\mu\lambda} g^{\nu\sigma} \nabla_\lambda R_{\rho\sigma\mu\nu} + g^{\mu\lambda} g^{\nu\sigma} \nabla_\rho R_{\sigma\lambda\mu\nu} + g^{\mu\lambda} g^{\nu\sigma} \nabla_\sigma R_{\lambda\rho\mu\nu} &= 0 \\
 \nabla_\lambda (g^{\mu\lambda} g^{\nu\sigma} R_{\sigma\rho\nu\mu}) - \nabla_\rho (g^{\mu\lambda} g^{\nu\sigma} R_{\lambda\sigma\mu\nu}) + \nabla_\sigma (g^{\mu\lambda} g^{\nu\sigma} R_{\lambda\rho\mu\nu}) &= 0 \\
 \nabla_\lambda (g^{\mu\lambda} R_{\rho\mu}) - \nabla_\rho (g^{\nu\sigma} R_{\sigma\nu}) + \nabla_\sigma (g^{\nu\sigma} R_{\rho\nu}) &= 0 \\
 \nabla_\lambda (g^{\mu\lambda} R_{\rho\mu}) - \nabla_\rho R + \nabla_\lambda (g^{\mu\lambda} R_{\rho\mu}) &= 0 \\
 \nabla_\lambda (g^{\mu\lambda} R_{\rho\mu}) - \nabla_\lambda \left( \frac{1}{2} \delta_\rho^\lambda R \right) &= 0 \\
 \nabla_\lambda (g^{\mu\lambda} R_{\rho\mu}) - \nabla_\lambda \left( \frac{1}{2} g^{\mu\lambda} g_{\rho\mu} R \right) &= 0 \\
 \nabla_\lambda \left[ g^{\mu\lambda} \left( R_{\rho\mu} - \frac{1}{2} g_{\rho\mu} R \right) \right] &= 0 \\
 \nabla_\lambda (g^{\mu\lambda} G_{\rho\mu}) &= 0 \\
 \nabla_\lambda G^\lambda{}_\rho &= 0
 \end{aligned}$$

## 2

The only non-vanishing components of the metric are

$$g_{\psi\psi} = 1, \quad g_{\theta\theta} = \sin^2 \psi, \quad g_{\phi\phi} = \sin^2 \psi \sin^2 \theta$$

Since the matrix is diagonal, the corresponding components of the inverse metric are simply

$$g^{\psi\psi} = 1, \quad g^{\theta\theta} = 1/\sin^2 \psi, \quad g^{\phi\phi} = 1/\sin^2 \psi \sin^2 \theta$$

The only non-vanishing first derivatives of the metric components are

$$g_{\theta\theta,\psi} = 2 \sin \psi \cos \psi, \quad g_{\phi\phi,\psi} = 2 \sin \psi \cos \psi \sin^2 \theta, \quad g_{\phi\phi,\theta} = 2 \sin^2 \psi \sin \theta \cos \theta$$

**a**

The only non-vanishing Christoffel symbols are

$$\begin{aligned}
\Gamma_{\theta\psi}^{\theta} &= \Gamma_{\psi\theta}^{\theta} = \frac{1}{2}g^{\theta\theta}g_{\theta\theta,\psi} = \frac{2\sin\psi\cos\psi}{2\sin^2\psi} = \cot\psi \\
\Gamma_{\theta\theta}^{\psi} &= -\frac{1}{2}g^{\psi\psi}g_{\theta\theta,\psi} = -\sin\psi\cos\psi \\
\Gamma_{\phi\psi}^{\phi} &= \Gamma_{\psi\phi}^{\phi} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\psi} = \frac{2\sin\psi\cos\psi\sin^2\theta}{2\sin^2\psi\sin^2\theta} = \cot\psi \\
\Gamma_{\phi\phi}^{\psi} &= -\frac{1}{2}g^{\psi\psi}g_{\phi\phi,\psi} = -\sin\psi\cos\psi\sin^2\theta \\
\Gamma_{\phi\theta}^{\phi} &= \Gamma_{\theta\phi}^{\phi} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\theta} = \frac{2\sin^2\psi\sin\theta\cos\theta}{2\sin^2\psi\sin^2\theta} = \cot\theta \\
\Gamma_{\phi\phi}^{\theta} &= -\frac{1}{2}g^{\theta\theta}g_{\phi\phi,\theta} = -\frac{2\sin^2\psi\sin\theta\cos\theta}{2\sin^2\psi} = -\sin\theta\cos\theta
\end{aligned}$$

**b**

There are  $\frac{1}{12} \cdot 3^2 \cdot (3^2 - 1) = 6$  independent components (others can be obtained by symmetry)

$$\begin{aligned}
R_{\psi\theta\psi\theta} &= g_{\psi\psi}R_{\theta\psi\theta}^{\psi} = \sin^2\psi - \cos^2\psi - 0 + 0 - (-\cos^2\psi) = \sin^2\psi \\
R_{\psi\theta\psi\phi} &= g_{\psi\psi}R_{\theta\psi\phi}^{\psi} = 0 - 0 + 0 - 0 = 0 \\
R_{\psi\theta\theta\phi} &= g_{\psi\psi}R_{\theta\theta\phi}^{\psi} = 0 - 0 + 0 - 0 = 0 \\
R_{\psi\phi\psi\phi} &= g_{\psi\psi}R_{\phi\psi\phi}^{\psi} = (\sin^2\psi - \cos^2\psi)\sin^2\theta - 0 + 0 - (-\cos^2\psi\sin^2\theta) = \sin^2\psi\sin^2\theta \\
R_{\psi\phi\theta\phi} &= g_{\psi\psi}R_{\phi\theta\phi}^{\psi} = -2\sin\psi\cos\psi\sin\theta\cos\theta - 0 + \sin\psi\cos\psi\sin\theta\cos\theta - (-\sin\psi\cos\psi\sin\theta\cos\theta) = 0 \\
R_{\theta\phi\theta\phi} &= g_{\theta\theta}R_{\phi\theta\phi}^{\theta} = \sin^2\psi \left[ \sin^2\theta - \cos^2\theta - 0 + (-\cos^2\psi\sin^2\theta) - (-\cos^2\theta) \right] = \sin^4\psi\sin^2\theta \\
R_{\psi\psi} &= g^{\theta\theta}R_{\theta\psi\theta} + g^{\phi\phi}R_{\phi\psi\phi} = 2 \\
R_{\psi\theta} &= g^{\phi\phi}R_{\phi\psi\phi} = 0 \\
R_{\psi\phi} &= g^{\theta\theta}R_{\theta\psi\theta} = 0 \\
R_{\theta\theta} &= g^{\psi\psi}R_{\psi\theta\psi} + g^{\phi\phi}R_{\phi\theta\phi} = \sin^2\psi + \sin^2\psi = 2\sin^2\psi \\
R_{\theta\phi} &= g^{\psi\psi}R_{\psi\theta\phi} = 0 \\
R_{\phi\phi} &= g^{\psi\psi}R_{\psi\phi\psi} + g^{\theta\theta}R_{\theta\phi\theta} = \sin^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta = 2\sin^2\psi\sin^2\theta \\
R &= g^{\psi\psi}R_{\psi\psi} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 2 + 2 + 2 = 6
\end{aligned}$$

**3**

Proposed topic: alternative theories of gravity, specifically, Einstein-Cartan and Brans-Dicke