MATH 4441 Homework 2

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1.17

By definition, $w = \inf[w_{u(\theta)}] \leq Ave[w_{u(\theta)}] = Ave[\frac{1}{2}L_{u(\theta)}]$ where

$$Ave[f(\theta)] := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

By Cauchy's integral formula

$$Ave[\frac{1}{2}L_{u(\theta)}] = \frac{1}{2}Ave[L_{u(\theta)}] = \frac{L}{\pi}$$

Therefore $w \leq \frac{L}{\pi}$.

1.18

If the equality holds, then it is necessary that

$$\inf[w_{u(\theta)}] = Ave[w_{u(\theta)}]$$

which is true if and only if $w_{u(\theta)}$ is a constant.

2.7

A circle of radius r with parameterization

$$\alpha(t) = (r\cos t, r\sin t)$$

has arclength of

$$s(t) = \int_0^t \|\alpha'(u)\|du = rt$$

and its inverse

$$s^{-1}(t) = \frac{t}{r}$$

The parameterization with respsect to its arclength is

$$\begin{aligned}\bar{\alpha}(t) &= \alpha \circ s^{-1}(t) = (r\cos\frac{t}{r}, r\sin\frac{t}{r})\\ \bar{\alpha}''(t) &= (-\frac{1}{r}\cos\frac{t}{r}, -\frac{1}{r}\sin\frac{t}{r})\\ \kappa(t) &= \|\bar{\alpha}''(t)\| = \frac{1}{r}\end{aligned}$$

A straight line can be parameterized by

$$\alpha(t) = p + tv$$

Its arclength function then would be

$$s(t) = \int_0^t \|v\| du = \|v\|t, \quad s^{-1}(t) = \frac{t}{\|v\|}$$

and the parameterization with respsect to its arclength is

$$\bar{\alpha}(t) = \alpha \circ s^{-1}(t) = p + t\frac{v}{\|v\|}$$

$$\kappa(t) = \|\bar{\alpha}''(t)\| = 0$$

2.10

Using the parameterization $\alpha(t) = (t, f(t), 0)$,

$$\alpha'(t) = (1, f'(t), 0)$$
$$\alpha''(t) = (0, f''(t), 0)$$
$$\alpha'(t) \times \alpha''(t) = \begin{vmatrix} i & j & k \\ 1 & f' & 0 \\ 0 & f'' & 0 \end{vmatrix} = (0, 0, f'')$$

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3} = \frac{|f''(t)|}{\left(\sqrt{1 + (f'(t))^2}\right)^3}$$

- $f(x) = x$:

$$\kappa = \frac{|0|}{\left(\sqrt{1 + (1)^2}\right)^3} = 0$$

- $f(x) = x^2$:

$$\kappa = \frac{|2|}{\left(\sqrt{1 + (2x)^2}\right)^3} = \frac{2}{(\sqrt{1 + 4x^2})^3}$$

- $f(x) = x^3$:

$$\kappa = \frac{|6x|}{\left(\sqrt{1+(3x^2)^2}\right)^3} = \frac{6|x|}{\left(\sqrt{1+9x^4}\right)^3}$$

- $f(x) = x^4$:

$$\kappa = \frac{|12x^2|}{\left(\sqrt{1+(4x^3)^2}\right)^3} = \frac{12x^2}{\left(\sqrt{1+16x^6}\right)^3}$$

2.11

Suppose $\alpha(t) = (x(t), y(t))$, then it can be reparameterized by its x coordinate:

$$\bar{\alpha}(t) = \alpha \circ x^{-1}(t) = (t, y \circ x^{-1}(t)) = (t, f(t))$$

Since $\bar{\alpha}$ passes through $(0,0)$, $f(0) = 0$, and since $f(t)$ is non-negative, $t = 0$ must be a critical point. So $f'(0) = 0$. Similarly, β can be reparameterized as $(t, g(t))$, where $g(0) = 0$ and $g'(0) = 0$. The taylor expansion of f, g around 0 are:

$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(c_1)t^2 = \frac{1}{2}f''(c_1)t^2 \geq 0$$
$$g(t) = g(0) + g'(0)t + \frac{1}{2}g''(c_2)t^2 = \frac{1}{2}g''(c_2)t^2 \geq 0$$

for some c_1, c_2 between 0 and t. Since β is higher than or at the same height as α for all t, $g''(c_2) \geq f''(c_1)$. Take the limit as $t \to 0$, then $\lim c_1 = \lim c_2 = 0$, and so

$$g''(0) \geq f''(0) \geq 0 \Rightarrow |g''(0)| \geq |f''(0)|$$

which is equivalent to

$$\kappa_\beta(0) \geq \kappa_\alpha(0)$$

because when $f' = 0$,

$$\kappa(t) = \frac{|f''(t)|}{\left(\sqrt{1+(f'(t))^2}\right)^3} = |f''(t)|$$

2.12

We can shrink the circle until it contacts the curve at some point p. Suppose after shrinking the radius becomes $R \leq r$. The closed curve and the circle can be treated as β and α in Ex.11 if we apply a rigid motion to both curves so that the point of contact coincides with the origin and the x axis is tangent to both curves. From the result of Ex.11, at this particular point the curvature of the closed curve is no smaller than that of the circle, which is $1/R$, therefore

$$\kappa_\alpha(\alpha^{-1}(p)) \geq \frac{1}{R} \geq \frac{1}{r}$$

2.13

Suppose the width of the curve is w, then there exist two parallel lines separated by a distance w that contain the curve in between. We can fit a circle of radius w between the lines and slide the circle while keeping it tangent to both lines. Suppose we call the direction of the two lines horizontal and the circle is initially to the right of the closed curve. If we slide the circle to the left, eventually some point must enter the circle from the left, and then touch the right half of the circle from the inside. Then there must be a region around the contact point that is contained in the circle. By the same argument as Ex.12, the curvature at the contact point p must be no less than the curvature of the circle. Therefore,

$$\max \kappa \geq \kappa_p \geq \kappa_{circle} = \frac{1}{w/2} = \frac{2}{w}$$

And since $w \leq \dfrac{L}{\pi}$,

$$\frac{1}{w} \geq \frac{\pi}{L}$$

$$\max \kappa \geq \frac{2\pi}{L}, \text{ or equivalently } L \geq \frac{2\pi}{\max \kappa}$$