PHYS 7125 Homework 3

Wenqi He

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1

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

Contracting twice:

$$g^{\mu\lambda}g^{\nu\sigma}\nabla_\lambda R_{\rho\sigma\mu\nu} + g^{\mu\lambda}g^{\nu\sigma}\nabla_\rho R_{\sigma\lambda\mu\nu} + g^{\mu\lambda}g^{\nu\sigma}\nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

$$\nabla_\lambda\Big(g^{\mu\lambda}g^{\nu\sigma}R_{\sigma\rho\nu\mu}\Big) - \nabla_\rho\Big(g^{\mu\lambda}g^{\nu\sigma}R_{\lambda\sigma\mu\nu}\Big) + \nabla_\sigma\Big(g^{\mu\lambda}g^{\nu\sigma}R_{\lambda\rho\mu\nu}\Big) = 0$$

$$\nabla_\lambda\Big(g^{\mu\lambda}R_{\rho\mu}\Big) - \nabla_\rho\Big(g^{\nu\sigma}R_{\sigma\nu}\Big) + \nabla_\sigma\Big(g^{\nu\sigma}R_{\rho\nu}\Big) = 0$$

$$\nabla_\lambda\Big(g^{\mu\lambda}R_{\rho\mu}\Big) - \nabla_\rho R + \nabla_\lambda\Big(g^{\mu\lambda}R_{\rho\mu}\Big) = 0$$

$$\nabla_\lambda\Big(g^{\mu\lambda}R_{\rho\mu}\Big) - \nabla_\lambda\Big(\frac{1}{2}\delta^\lambda_\rho R\Big) = 0$$

$$\nabla_\lambda\Big(g^{\mu\lambda}R_{\rho\mu}\Big) - \nabla_\lambda\Big(\frac{1}{2}g^{\mu\lambda}g_{\rho\mu}R\Big) = 0$$

$$\nabla_\lambda\Big[g^{\mu\lambda}\Big(R_{\rho\mu} - \frac{1}{2}g_{\rho\mu}R\Big)\Big] = 0$$

$$\nabla_\lambda\Big(g^{\mu\lambda}G_{\rho\mu}\Big) = \nabla_\lambda\Big(g^{\lambda\mu}G_{\mu\rho}\Big) = 0$$

$$\nabla_\lambda G^\lambda{}_\rho = 0$$

2

The only non-vanishing components of the metric are

$$g_{\psi\psi} = 1, \quad g_{\theta\theta} = \sin^2\psi, \quad g_{\phi\phi} = \sin^2\psi\sin^2\theta$$

Since the matrix is diagonal,

$$g^{\psi\psi} = 1, \quad g^{\theta\theta} = 1/\sin^2\psi, \quad g^{\phi\phi} = 1/\sin^2\psi\sin^2\theta$$

The only non-vanishing first derivatives of the metric components are

$$g_{\theta\theta,\psi} = 2\sin\psi\cos\psi, \quad g_{\phi\phi,\psi} = 2\sin\psi\cos\psi\sin^2\theta, \quad g_{\phi\phi,\theta} = 2\sin^2\psi\sin\theta\cos\theta$$

a

The only non-vanishing Christoffel symbols are

$$\Gamma^{\theta}_{\theta\psi} = \Gamma^{\theta}_{\psi\theta} = \frac{1}{2} g^{\theta\theta} g_{\theta\theta,\psi} = \frac{2\sin\psi\cos\psi}{2\sin^2\psi} = \cot\psi$$

$$\Gamma^{\psi}_{\theta\theta} = -\frac{1}{2} g^{\psi\psi} g_{\theta\theta,\psi} = -\sin\psi\cos\psi$$

$$\Gamma^{\phi}_{\phi\psi} = \Gamma^{\phi}_{\psi\phi} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\psi} = \frac{2\sin\psi\cos\psi\sin^2\theta}{2\sin^2\psi\sin^2\theta} = \cot\psi$$

$$\Gamma^{\psi}_{\phi\phi} = -\frac{1}{2} g^{\psi\psi} g_{\phi\phi,\psi} = -\sin\psi\cos\psi\sin^2\theta$$

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\theta} = \frac{2\sin^2\psi\sin\theta\cos\theta}{2\sin^2\psi\sin^2\theta} = \cot\theta$$

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2} g^{\theta\theta} g_{\phi\phi,\theta} = -\frac{2\sin^2\psi\sin\theta\cos\theta}{2\sin^2\psi} = -\sin\theta\cos\theta$$

b

There are $\frac{1}{12} \cdot 3^2 \cdot (3^2 - 1) = 6$ independent components of the Riemann tensor:

$$\begin{aligned}
R_{\psi\theta\psi\theta} &= g_{\psi\psi} R^{\psi}_{\theta\psi\theta} = \sin^2\psi - \cos^2\psi - 0 + 0 - (-\cos^2\psi) = \sin^2\psi \\
R_{\psi\theta\psi\phi} &= g_{\psi\psi} R^{\psi}_{\theta\psi\phi} = 0 - 0 + 0 - 0 = 0 \\
R_{\psi\theta\theta\phi} &= g_{\psi\psi} R^{\psi}_{\theta\theta\phi} = 0 - 0 + 0 - 0 = 0 \\
R_{\psi\phi\psi\phi} &= g_{\psi\psi} R^{\psi}_{\phi\psi\phi} = (\sin^2\psi - \cos^2\psi)\sin^2\theta - 0 + 0 - (-\cos^2\psi\sin^2\theta) = \sin^2\psi\sin^2\theta \\
R_{\psi\phi\theta\phi} &= g_{\psi\psi} R^{\psi}_{\phi\theta\phi} = -2\sin\psi\cos\psi\sin\theta\cos\theta - 0 + \sin\psi\cos\psi\sin\theta\cos\theta - (-\sin\psi\cos\psi\sin\theta\cos\theta) = 0 \\
R_{\theta\phi\theta\phi} &= g_{\theta\theta} R^{\theta}_{\phi\theta\phi} = \sin^2\psi \Big[\sin^2\theta - \cos^2\theta - 0 + (-\cos^2\psi\sin^2\theta) - (-\cos^2\theta)\Big] = \sin^4\psi\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
R_{\psi\psi} &= g^{\theta\theta} R_{\theta\psi\theta\psi} + g^{\phi\phi} R_{\phi\psi\phi\psi} = 2 \\
R_{\psi\theta} &= g^{\phi\phi} R_{\phi\psi\phi\theta} = 0 \\
R_{\psi\phi} &= g^{\theta\theta} R_{\theta\psi\theta\phi} = 0 \\
R_{\theta\theta} &= g^{\psi\psi} R_{\psi\theta\psi\theta} + g^{\phi\phi} R_{\phi\theta\phi\theta} = \sin^2\psi + \sin^2\psi = 2\sin^2\psi \\
R_{\theta\phi} &= g^{\psi\psi} R_{\psi\theta\psi\phi} = 0 \\
R_{\phi\phi} &= g^{\psi\psi} R_{\psi\phi\psi\phi} + g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta = 2\sin^2\psi\sin^2\theta
\end{aligned}$$

$$R = g^{\psi\psi} R_{\psi\psi} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = 2 + 2 + 2 = 6$$