An Overview of Einstein's 1916 Paper
The Foundation of the General Theory of Relativity

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1 Philosophical Motivation

Near the end of the first section, Einstein states his guiding principle:

> *The general laws of nature are to be expressed by equations which hold good for all systems of coordinates, that is, are covariant with respect to any substitutions whatever. (generally covariant)*

2 Mathematical Aid

Here he intentionally introduces the mathematics in a simple manner so that *"a special study of the mathematical literature is not required for the understanding of the present paper"*

2.1 Tensor

Following his guiding principle, tensor are very natural objects to investigate, for that

> *...the equations of transformation for their components are linear and homogeneous. Accordingly, all the components in the new system vanish, if they all vanish in the original system. If, therefore, a law of nature is expressed by equating all the components of a tensor to zero, it is generally covariant.*

2.2 Co/Contra-variant Tensor

First he introduced contravariant and covariant tensors and their respective transformation rules under a change of coordinates, which could be generalizes to that of mixed tensors:

$$A'^{\tau}_{\sigma} = \frac{\partial x'_\tau}{\partial x_\nu} \frac{\partial x_\mu}{\partial x'_\sigma} A^{\nu}_{\mu}$$

Then he introduced outer product, contraction, and combining both, inner product.

2.3 Metric

The entire theory was built on top of the metric tensor $g_{\mu\nu}$ (he called it *the covariant fundamental tensor*) which determines the invariant line element of spacetime by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

He further introduces the inverse metric $g^{\mu\nu}$ and demonstrates that one could raise and lower indices by contraction with the (inverse) metric.

2.4 Geodesic

Next, he derived the equation of geodesic by doing variation on arclength

$$S = \int ds = \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

and yields:

$$\frac{d^2 x^\tau}{ds^2} + \Gamma^\tau_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

where $\Gamma^\tau_{\mu\nu}$, the Christoffel symbol, is a shorthand for a mixture of derivatives of $g_{\mu\nu}$.

2.5 Covariant Derivative

In order to write what he called *"generally covariant differential equations"*, he derives the notion of covariant derivatives in the following way: Suppose ϕ is invariant in spacetime. Because ds is invariant, the derivatives of ϕ along the geodesic (so that the above equation can be used to simplify terms) with respect to s

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x^\mu} \frac{\partial x_\mu}{ds} = A_\mu \frac{\partial x^\mu}{ds}$$

$$\frac{d^2\phi}{ds^2} = \left(\frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu} + \Gamma^\tau_{\mu\nu} \frac{\partial \phi}{x^\tau} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \left(\frac{\partial A_\mu}{\partial x^\nu} + \Gamma^\tau_{\mu\nu} A_\tau \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$

must be invariant as well. In order for the derivative to preserves the tensor property,

$$\nabla_\nu A_\mu := \frac{\partial A_\mu}{\partial x^\nu} + \Gamma^\tau_{\mu\nu} A_\tau$$

For rank two tensor $A_{\mu\nu} = A_\mu B_\nu$, requiring that it follows Leibniz rule,

$$\nabla_\sigma A_{\mu\nu} := \nabla_\sigma A_\mu B_\nu + A_\nu \nabla_\sigma B_\mu = \frac{\partial A_{\mu\nu}}{\partial x^\sigma} + \Gamma^\tau_{\sigma\mu} A_{\tau\nu} + \Gamma^\tau_{\sigma\nu} A_{\mu\tau}$$

2.6 Curvature

Einstein's derived the Riemann curvature tensor by taking the commutator of the two covariant derivatives:

$$[\nabla_\sigma, \nabla_\tau] A_\mu = R^\rho_{\mu\sigma\tau} A_\rho$$

where $R^\rho_{\mu\sigma\tau}$ is a shorthand for a mixture of first and second derivatives of $g_{\mu\nu}$. Then he explains the significance of the Riemann tensor by citing (without giving the proof) a fundamental theorem

$$R^\rho_{\mu\sigma\tau} = 0 \Leftrightarrow g_{\mu\nu} = const. \text{ (in appropriate coordinates)}$$

Contracting the Riemann tensor gives the Ricci tensor $R_{\mu\nu}$. In coordinates where $\sqrt{-g} = 1$

$$R_{\mu\nu} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} - \Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\nu\beta}$$

which he considers to be of particular interest, for that

> *...besides $R_{\mu\nu}$ there is no tensor of second rank which is formed from the $g_{\mu\nu}$ and its derivatives, contains no derivations higher than second, and is linear is these derivatives.*

3 Physical Postulates

The famous summary by Wheeler:

> *Spacetime tells matter how to move; matter tells spacetime how to curve.*

3.1 Equations of Motion

The first postulate is that particles follow the geodesic

$$\boxed{\frac{d^2x^\tau}{ds^2} = -\Gamma^\tau_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}$$

Based on this, he calls $\Gamma^\tau_{\mu\nu}$ the *components of the gravitational field*, since

> *If the $\Gamma^\tau_{\mu\nu}$ vanish, then the point moves uniformly in a straight line. These quantities therefore condition the deviation of the motion from uniformity.*

3.2 Field Equations and Laws of Conservation

3.2.1 In the Absence of Matter

Einstein argues that the equation must be less restrictive than, but nonetheless must in special cases be reducible to, the vanishing of the Riemann tensor, for compatibility with special relativity. Thus he requires that Ricci tensor vanish:

$$R_{\mu\nu} = 0$$

which he later compares to $\nabla^2\phi = 0$ in Newtonian gravity. He first verifies that the Lagrangian is

$$L = g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}$$

Then from the Euler-Lagrange equation, a divergence-free quantity arises after some calculation:

$$\frac{\partial}{\partial x^\alpha}\left(\partial_\sigma g^{\mu\nu}\frac{\partial L}{\partial\partial_a g^{\mu\nu}} - \delta^\alpha_\sigma L\right) = 0$$

He argues that

> *This equation expresses the law of conservation of momentum and of energy for the gravitational field*

and gives the definition of the *energy components of the gravitational field* as:

$$t^\alpha_\sigma := -\frac{1}{2\kappa}\left(\partial_\sigma g^{\mu\nu}\frac{\partial L}{\partial\partial_a g^{\mu\nu}} - \delta^\alpha_\sigma L\right) = \frac{1}{2\kappa}\left(\delta^\alpha_\sigma g^{\mu\nu}\Gamma^\lambda_{\mu\beta}\Gamma^\beta_{\nu\lambda} - 2g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\sigma}\right), \quad t = t^\alpha_\alpha$$

Extracting from the field equation all terms expressible in terms of t^α_α and its trace t:

$$\frac{\partial}{\partial x_\alpha}\left(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}\right) = -\kappa\left(t^\sigma_\mu - \frac{1}{2}\delta^\sigma_\mu t\right)$$

which equation, as he immediately points out, *"is particularly useful for a vivid grasp of our subject"*.

3.2.2 General Form

Finally, Einstein argues that the contribution of matter should be analogous to the source term in Poisson's equation $\Delta\phi = 4\pi G\rho$. Thus he modifies the matter-free field equation by introducing the *energy-tensor of matter* T^σ_μ and its trace $T = T^\mu_\mu$ to the right side:

$$\frac{\partial}{\partial x_\alpha}\left(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}\right) = -\kappa\left[(t^\sigma_\mu + T^\sigma_\mu) - \frac{1}{2}\delta^\sigma_\mu(t+T)\right]$$

Absorbing t^σ_μ and t terms back to the left yields the well-known field equation:

$$\boxed{R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)}$$

Lastly, he checks that the conservation laws still hold, that is,

$$\frac{\partial(t^\sigma_\mu + T^\sigma_\mu)}{\partial x_\sigma} = 0$$

3.3 Newtonian Approximation

3.3.1 Equations of Motion

Suppose $\Gamma^\tau_{\mu\nu}$ are small quantities, and the speed of the particle is small compared to the speed of light, namely, for spatial dimensions dx^α/ds are small quantities, and $dt/ds = dx^0/ds \approx 1$. Additionally, suppose the gravitational field to be quasistatic, that is, $\partial g_{\mu\nu}/\partial t = \partial g_{\mu\nu}/\partial x^0 = 0$, then

$$\frac{d^2x^\tau}{ds^2} = -\Gamma^\tau_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \quad\Longrightarrow\quad \frac{d^2x^\alpha}{dt^2} = -\Gamma^\alpha_{00} = -\frac{1}{2}\frac{\partial g_{00}}{\partial x^\alpha}, \quad \alpha = 1,2,3$$

This is exactly the equations of motion in Newton's theory if we set $\phi = g_{00}/2$:

> *What is remarkable in this result is that the component g_{00} of the fundamental tensor alone defines, to a first approximation, the motion of the material point.*

3.3.2 Field Equations

Under the condition that $T_{00} = T = \rho$ and $T_{\mu\nu} = 0$ otherwise, for $\mu = \nu = 0$, we can approximate both sides of the field equation as follows:

$$LHS = R_{00} \approx \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{00} = \frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}\frac{\partial g_{00}}{\partial x^\alpha}\right) = \frac{\partial^2}{\partial {x^\alpha}^2}\left(\frac{g_{00}}{2}\right) = \frac{\partial^2\phi}{\partial {x^\alpha}^2}$$

$$RHS = -\kappa\left(T_{00} - \frac{1}{2}Tg_{00}\right) \approx -\kappa\left(\rho - \frac{1}{2}\rho\right) = -\frac{\kappa\rho}{2}$$

The field equations then reduce to Poisson's equation for Newton's law of gravitation:

$$\frac{\partial^2\phi}{\partial {x^\alpha}^2} = -\frac{\kappa\rho}{2} = \frac{4\pi G\rho}{c^4} \quad\Longrightarrow\quad \kappa = -\frac{8\pi G}{c^4}$$

Thus, the field equations can be rewritten as:

$$\boxed{R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)}$$