MATH 4441 Homework 5

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September 26, 2018

6.9

First, the number of vertices must be even, otherwise there will be two maxima/minima immediately next to each other. Now suppose there are only 2 vertices, then one must be a maximum and the other must be a minimum. And since they are the only two extrema, curvature must increase monotonically from the minimum to the maximum. Starting from the maximum point and move two points a, b in opposite directions simultaneously, keeping the curvature at both points the same. Then a, b divide the curve into top and bottom halves with

$$\kappa_{bottom} \geq \kappa(a) = \kappa(b) \geq \kappa_{top}$$

Move the points until the arclength of both halves become the same. Then by Shur's arm lemma, the two halves cannot have the same chord length, therefore the assumption cannot be true and so there must be more than 2 vertices. In other words, there must be 4 or more vertices.

6.22

Let $\phi(x) = \sqrt{1+x^2}$, then $\phi''(x) = \dfrac{1}{(1+x^2)^{3/2}} > 0$, which means that ϕ is strictly convex. By Jensen's inequality,

$$\begin{aligned}
\sqrt{1+\bar{f}'^2} &= \phi(\bar{f}') \\
&= \phi(\frac{1}{2}[f' + (-g')]) \\
&< \frac{1}{2}\Big[\phi(f') + \phi(-g')\Big] \\
&= \frac{1}{2}\Big[\sqrt{1+f'^2} + \sqrt{1+(-g')^2}\Big] \\
&= \frac{1}{2}\Big[\sqrt{1+f'^2} + \sqrt{1+g'^2}\Big]
\end{aligned}$$

$$\sqrt{1+\bar{f}'^2} + \sqrt{1+(-\bar{f}')^2} = 2\sqrt{1+\bar{f}'^2} < \sqrt{1+f'^2} + \sqrt{1+g'^2}$$

Therefore the value of the two integrals for $\bar{\alpha}$ is strictly less that of α.

$$\int_a^b \sqrt{1+\bar{f}'^2}dt + \int_a^b \sqrt{1+(-\bar{f}')^2}dt < \int_a^b \sqrt{1+f'^2}dt + \int_a^b \sqrt{1+g'^2}dt$$

The lengths of the two line segments are still the same, because

$$\bar{f} - (-\bar{f}) = 2\bar{f} = f - g$$

Combininig the above results,

$$Length[\bar{\alpha}] < Length[\alpha]$$

7.2

$$\begin{aligned}
\alpha'(t) &= (-r\sin t, r\cos t, h) \\
\|\alpha'\| &= \sqrt{r^2+h^2} \\
s(t) &= \int_0^t \sqrt{r^2+h^2}dz = \sqrt{r^2+h^2}t \\
s^{-1}(t) &= \frac{1}{\sqrt{r^2+h^2}}t = ct
\end{aligned}$$

Reparametrize the helix curve by its arc length

$$\bar{\alpha}(t) = (r\cos(ct), r\sin(ct), hct)$$

Then

$$\begin{aligned}
T &= \bar{\alpha}'(t) = (-cr\sin(ct), cr\cos(ct), ch) \\
T' &= (-c^2 r\cos(ct), -c^2 r\sin(ct), 0)
\end{aligned}$$

$$\kappa = \|T'\| = c^2 r = \frac{r}{r^2+h^2}$$

$$\begin{aligned}
N &= \frac{T'}{\kappa} = (-\cos(ct), -\sin(ct), 0) \\
B = T \times N &= \begin{vmatrix} i & j & k \\ -cr\sin(ct) & cr\cos(ct) & ch \\ -\cos(ct) & -\sin(ct) & 0 \end{vmatrix} \\
&= (ch\sin(ct), -ch\cos(ct), cr) \\
B' &= (c^2 h\cos(ct), c^2 h\sin(ct), 0)
\end{aligned}$$

$$\tau = -\langle B', N\rangle = c^2 h = \frac{h}{r^2+h^2}$$

From the above formulae of κ and τ, κ always decreases as h increases, and τ always decreases as r increases. And taking the partial derivatives:

$$\frac{\partial \kappa}{\partial r} = \frac{h^2 - r^2}{(h^2 + r^2)^2}$$

$$\frac{\partial \tau}{\partial h} = \frac{r^2 - h^2}{(h^2 + r^2)^2}$$

When $r < h$, κ changes in the same direction as r, and τ changes in the opposite direction of h. When $r > h$, κ changes in the opposite direction of r, and τ changes in the same direction as h.

7.5

$$\begin{aligned}
\langle v, T\rangle' = a' = 0 &= \langle v', T\rangle + \langle v, T'\rangle = 0 + \langle v, \kappa N\rangle = \kappa b \\
\Rightarrow b = 0 & \\
\langle v, N\rangle' = b' = 0 &= \langle v', N\rangle + \langle v, N'\rangle = 0 + \langle v, -\kappa T + \tau B\rangle = -\kappa + c\tau \\
\Rightarrow c = \frac{\kappa}{\tau} &
\end{aligned}$$

Therefore

$$v = T + \frac{\kappa}{\tau} B$$

Check:

$$\langle T, v\rangle = 1$$

$$\begin{aligned}
\langle v, B\rangle' = \left(\frac{\kappa}{\tau}\right)' &= \langle v', B\rangle + \langle v, B'\rangle = 0 + \langle v, -\tau N\rangle = 0 \\
v' = T' + \left(\frac{\kappa}{\tau}\right)' B + \frac{\kappa}{\tau} B' &= \kappa N + 0 + \frac{\kappa}{\tau}(-\tau N) = 0
\end{aligned}$$

7.7

Suppose the curve indeed lies on some sphere, we can describe the center of the sphere using the moving frame:

$$p = \alpha + aT + bN + cB$$

Since α lies on the sphere, T must be tangent to the sphere, whereas $p - \alpha$ must be in (the opposite of) the normal direction. Therefore $a = \langle p - \alpha, T\rangle = 0$. If such sphere exists, then it must have a fixed center, so $p' = 0$. Differentiate the above equation (suppose α has unit speed):

$$\begin{aligned}
\langle p-\alpha, T\rangle' &= \langle -T, T\rangle + \langle p-\alpha, T'\rangle \\
&= -1 + \langle p-\alpha, \kappa N\rangle \\
&= -1 + \kappa b = 0 \\
b &= \frac{1}{\kappa}
\end{aligned}$$

$$\begin{aligned}
\langle p-\alpha, N\rangle' = b' &= \left(\frac{1}{\kappa}\right)' = -\frac{\kappa'}{\kappa^2} \\
&= \langle -T, N\rangle + \langle p-\alpha, N'\rangle \\
&= 0 + \langle p-\alpha, -\kappa T + \tau B\rangle = c\tau \\
c &= -\frac{\kappa'}{\tau\kappa^2}
\end{aligned}$$

If the curve satisfies the equation concerning κ and τ, then

$$r^2 = \left(\frac{\kappa'}{\kappa^2\tau}\right)^2 + \frac{1}{\kappa^2} = c^2 + b^2 = \|p-\alpha\|^2$$

wihch means that the distance between p and $\alpha(t)$ is a constant r. Therefore, the curve indeed lies on a sphere.