MATH 4441 Homework 13

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16.1

Let M be the planar region bounded by a triangle, and let α_i be the i-th interior angle, then $K = \kappa_g = 0$, $\theta_i = \pi - \alpha_i$, and $\chi(M) = 1$. By the Gauss-Bonnet theorem,

$$0 + 0 + \sum_{i=1}^{3}(\pi - \alpha_i) = 2\pi \quad \Rightarrow \quad \sum_{i=1}^{3}\alpha_i = \pi$$

16.2

Let M be the region enclosed by the curve, then $\chi(M) = 1$, and $K = 0$ everywhere on M. By the Gauss-Bonnet theorem,

$$0 + \int_{\partial M}\kappa_g ds + 0 = 2\pi \cdot 1 \quad \Rightarrow \quad \int_{\partial M}\kappa_g ds = 2\pi$$

16.3

Since the surface M is homeomorphic to the torus, there is no boundary ∂M and $\chi(M) = 0$. Therefore, by the Gauss-Bonnet theorem,

$$\int_M K dA = 0 \quad \Rightarrow \quad \frac{1}{Area(M)}\int_M K dA = 0$$

By the mean value theorem for integrals, there always exists a point p on M where $K(p) = 0$

16.4

Let M be either one of the two regions enclosed by the curve. By the Gauss-Bonnet theorem,

$$2\pi = \int_M K dA + \int_{\partial M}\kappa_g ds + 0 = \int_M \frac{1}{R^2} dA + 0 + 0 = \frac{1}{R^2}\int_M dA$$

$$Area(M) = \int_M dA = 2\pi R^2$$

which is exactly one half of the area of the sphere.

16.5

If there exist more than one closed geodesics, then consider the region M enclosed by two of them. M is homeomorphic to a cylinder, so $\chi(M) = 0$. By the Gauss-Bonnet theorem,

$$\int_M K dA + 0 + 0 = 0, \qquad \text{where } K < 0$$

Since the two geodesics are distinct by assumption, M must be nonempty, and therefore the integral must be strictly negative, which is a contradiction. Therefore, there cannot be more than one closed geodesic.

16.6

On a sphere of radius 1, $K = 1$. Let α_i be the interior angles, then by the Gauss-Bonnet theorem,

$$\int_M dA + 0 + \sum_{i=1}^{k} (\pi - \alpha_i) = Area(M) + k\pi - \sum_{i=1}^{k} \alpha_i = 2\pi$$

$$\Rightarrow \quad Area(M) = \sum_{i=1}^{k} \alpha_i - (k-2)\pi$$

16.7

By the Gauss-Bonnet theorem,

$$\int_T K dA + 0 + \Big[(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma)\Big] = 2\pi \quad \Rightarrow \quad \int_T K dA = \alpha + \beta + \gamma - \pi$$

$$\Rightarrow \quad Ave_T[K] = \frac{1}{Area(T)} \int_T K dA = \frac{\alpha + \beta + \gamma - \pi}{Area(T)}$$

By the mean value theorem, there exists a point p_0 within T where $K(p_0)$ equals the average of K. As T shrinks to $\{p\}$, p_0 must coincide with p eventually, therefore,

$$K(p) = \lim_{T \to p} K(p_0) = \lim_{T \to p} \frac{\alpha + \beta + \gamma - \pi}{Area(T)}$$

16.8

On geodesics $\kappa_g = 0$. Let α_i be the interior angles, then by the Gauss-Bonnet theorem,

$$\int_M K dA + 0 + \sum_{i=1}^{3} (\pi - \alpha_i) = 2\pi$$

$$\Rightarrow \quad \sum_{i=1}^{3} \alpha_i = \pi + \int_M K dA \begin{cases} > \pi, \text{ if } K > 0 \\ < \pi, \text{ if } K < 0 \end{cases}$$

16.9

Suppose the two geodesics meet. Let M be the region bounded by the two geodesics. M is homeomorphic to a disk, so $\chi(M) = 1$

$$\int_M K dA + 0 + \sum_{i=1}^{2} (\pi - \alpha_i) = 2\pi$$

$$\int_M K dA + \sum_{i=1}^{2} (-\alpha_i) = 0$$

However, since $K < 0$, all terms on the LHS are negative, which means that the equation cannot hold. Therefore, the geodesics will never meet.

16.10

For each of the two regions, $\chi(M) = 1$. By the Gauss-Bonnet theorem,

$$\begin{aligned} 2\pi &= \int_M K dA + 0 + 0 \\ &= \int_M \det(S) dA = \int_M \det(dn) dA = \int_M |\det(dn)| dA \\ &= \int_{n(M)} dA' = Area(n(M)) \end{aligned}$$

The area of each of the two regions on $\mathbb{S}^2$ is exactly half of the area of $\mathbb{S}^2$, which is 4π. In other words, the areas of the two regions under the Gauss map are equal.

16.11

The pseudo-sphere can be parametrized as

$$X(t, \theta) = \left(\frac{1}{\cosh t} \cos\theta, \frac{1}{\cosh t} \sin\theta, t - \tanh t \right)$$

Consider the top half, whose boundary is the equator $(\cos\lambda, \sin\lambda, 0)$. For a curve along the equator, the principal normal vector is $N = (-\cos\lambda, -\sin\lambda, 0)$, and JT is along the direction of X_t:

$$X_t = \left(-\frac{\tanh t}{\cosh t} \cos\theta, -\frac{\tanh t}{\cosh t} \sin\theta, 1 - \frac{1}{\cosh^2 t} \right), \quad \|X_t\| = \tanh t$$

$$\Rightarrow \quad JT = \frac{X_t}{\|X_t\|} = \left(-\frac{\cos\theta}{\cosh t}, -\frac{\sin\theta}{\cosh t}, \frac{1}{\tanh t} - \frac{1}{\cosh t \sinh t} \right)$$

On the equator, which is a circle of radius 1, the geodesic curvatue is

$$\kappa_g = \kappa \langle N, JT \rangle = \frac{1}{1} \cdot \frac{1}{\cosh t} = 1$$

The top half is homeomorphic to a cylinder, so $\chi(M) = 0$. From the previous results, we also have $K = -1$ everywhere on M, and $\kappa_g = 1$ along the boundary ∂M, therefore by the Gauss-Bonnet theorem,

$$-\int_M dA + \int_{\partial M} ds + 0 = -Area(M) + 2\pi = 0 \quad \Rightarrow \quad Area(M) = 2\pi$$

The total area is

$$2 \cdot Area(M) = \boxed{4\pi}$$