MATH 4347 Homework 2

Wenqi He

September 23, 2018

3.14

If u_0 is strictly increasing, then the characteristic lines $x = u_0(x_0)t + x_0$ will not intersect, because a line that starts with a larger x_0 will also have a greater slope. The fact that u_0 is bounded means that the characteristic lines have a maximum slope and therefore will not approach the x axis. Therefore, the solution is well defined for all $t > 0$.

4.1

Using the d'Alembert's formula:

$$u = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds$$

For any x, when t becomes sufficiently large so that $x - t < 1$ and $x + t > 3$, the above formula becomes

$$u = \frac{1}{2}\int_{1}^{3} \psi(s)ds$$

which is a constant. To make the constant zero, it is both necessary and sufficient that the above definite integral evaluate to zero.

4.2

a

$$e_t = u_t u_{tt} + u_x u_{xt}$$
$$p_x = u_{tx} u_x + u_t u_{xx} = u_x u_{xt} + u_t u_{tt} = e_t$$
$$p_t = u_{tt} u_x + u_t u_{xt}$$
$$e_x = u_t u_{tx} + u_x u_{xx} = u_t u_{xt} + u_x u_{tt} = p_t$$

b

From the result of (a),

$$e_{tt} = p_{xt}, \quad e_{xx} = p_{tx} = p_{xt} = e_{tt}$$
$$p_{tt} = e_{xt}, \quad p_{xx} = e_{tx} = e_{xt} = p_{tt}$$

Therefore, they both satisfy the wave equation.

4.4

a

By d'Alembert's formula:

$$u(x+h,t+k) = \frac{1}{2}(\phi(x+t+(h+k))+\phi(x-t+(h-k))+\frac{1}{2c}\left(\int_{x-t+(h-k)}^{0}\psi(s)ds+\int_{0}^{x+t+(h+k)}\psi(s)ds\right)$$

$$u(x-h,t-k) = \frac{1}{2}(\phi(x+t-(h+k))+\phi(x-t-(h-k))+\frac{1}{2c}\left(\int_{x-t-(h-k)}^{0}\psi(s)ds+\int_{0}^{x+t-(h+k)}\psi(s)ds\right)$$

$$u(x+k,t+h) = \frac{1}{2}(\phi(x+t+(k+h))+\phi(x-t+(k-h))+\frac{1}{2c}\left(\int_{x-t+(k-h)}^{0}\psi(s)ds+\int_{0}^{x+t+(k+h)}\psi(s)ds\right)$$

$$u(x-k,t-h) = \frac{1}{2}(\phi(x+t-(k+h))+\phi(x-t-(k-h))+\frac{1}{2c}\left(\int_{x-t-(k-h)}^{0}\psi(s)ds+\int_{0}^{x+t-(k+h)}\psi(s)ds\right)$$

It can be easily verified that both sides have exactly the same terms.

b

If $c=2$, the the characteristic coordinates are $x \pm 2t$. Therefore the corresponding identity should be

$$u(x+2h,t+k)+u(x-2h,t-k) = u(x+2k,t+h)+u(x-2k,t-h)$$

4.6

a

The d'Alembert formula

$$u_2(x,t) = \frac{1}{2}(\phi(x+t)+\phi(x-t))+\frac{1}{2}\int_{x-t}^{x+t}\psi(s)ds$$

works for $x > t$, but not $x < t$. From the general solution $u = F(x-t)+G(x+t)$ and initial condition we can still arrive at

$$\begin{cases} F(x) = \frac{1}{2}\phi(x) - \frac{1}{2}\int_0^x \psi(s)ds \\ G(x) = \frac{1}{2}\phi(x) + \frac{1}{2}\int_0^x \psi(s)ds \end{cases}$$

To get $F(x)$ for negative x, we need to apply the boundary condition.

$$u_x(0,t) = G'(t)+F'(-t) = 0$$

$$
\begin{aligned}
\int_0^x G'(t)dt + \int_0^x F'(-t)dt = 0 \\
\int_0^x G'(t)dt - \int_0^{-x} F'(t')dt' = 0 \\
G(x) - F(-x) = C \\
G(0) - F(0) = C
\end{aligned}
$$

From initial condition, $F(0) + G(0) = \phi(0) = 0$. Adding this to the last equation,

$$
2G(0) = C = 0
$$

Therefore $F(-x) = G(x)$,

$$
\begin{aligned}
u_1(x,t) &= G(x+t) + F(x-t) = G(x+t) + G(t-x) \\
&= \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds\right)
\end{aligned}
$$

b

$u_1 \equiv 0$ when $x+t$ and $t-x$ are both outside of $[1,2]$, and the two integrals are both zero regardless of ψ. The only situation where this can be satisfied is

$$
x+t < 1, \quad t-x < 1 \quad \Rightarrow \quad t-1 < x < 1-t
$$

which is valid only when $t < 1$, and since u_1 is only defined for $0 < x < t$,

$$
\begin{cases}
0 < x < 1-t, & \frac{1}{2} < t < 1 \\
0 < x < t, & t < \frac{1}{2}
\end{cases}
$$

$u_2 \equiv 0$ when $x+t$ and $x-t$ are on the same side of $[1,2]$:

$$
\begin{cases}
x+t < 1, \quad x-t < 1 & \Rightarrow \quad t < x < 1-t, \quad 0 < t < \frac{1}{2} \\
x+t > 2, \quad x-t > 2 & \Rightarrow \quad x > t+2
\end{cases}
$$

c

$$
u(x,t) = \begin{cases}
\frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds\right), & 0 < x < t \\
\frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_{x-t}^{0} \psi(s)ds\right), & x > t
\end{cases}
$$

d

$$
u_1 = \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds\right)
$$

$$
\lim_{x\to t} u_1 = \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_0^{2t} \psi(s)ds
$$

$$(u_1)_x = \frac{1}{2}(\phi'(x+t) - \phi'(t-x)) + \frac{1}{2}\left(\psi(x+t) - \psi(t-x)\right)$$

$$\lim_{x\to t}(u_1)_x = \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}\left(\psi(2t) - \psi(0)\right)$$

$$(u_1)_t = \frac{1}{2}(\phi'(x+t) + \phi'(t-x)) + \frac{1}{2}\left(\psi(x+t) + \psi(t-x)\right)$$

$$\lim_{x\to t}(u_1)_t = \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}\left(\psi(2t) + \psi(0)\right)$$

$$u_2 = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t}\psi(s)ds$$

$$\lim_{x\to t} u_2 = \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_{0}^{2t}\psi(s)ds$$

$$(u_2)_x = \frac{1}{2}(\phi'(x+t) + \phi'(x-t)) + \frac{1}{2}(\psi(x+t) - \psi(x-t))$$

$$\lim_{x\to t}(u_2)_x = \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}(\psi(2t) - \psi(0))$$

$$(u_2)_t = \frac{1}{2}(\phi'(x+t) - \phi'(x-t)) + \frac{1}{2}(\psi(x+t) + \psi(x-t))$$

$$\lim_{x\to t}(u_2)_t = \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}(\psi(2t) + \psi(0))$$

i

For u to be continuous,

$$\lim_{x\to t} u_1 = \lim_{x\to t} u_2$$

which is always true.

ii

For u to be C^1,

$$\lim_{x\to t}(u_1)_t = \lim_{x\to t}(u_2)_t, \quad \lim_{x\to t}(u_1)_x = \lim_{x\to t}(u_2)_x$$

$$\phi'(0) = -\phi'(0) \Rightarrow \phi'(0) = 0$$

which means that in order for a smooth wave to be stress-free at one end, there must be no stress applie on that end in the beginning.

4.7

By the same argument as 4.6,

$$\begin{cases} F(x) = \frac{1}{2}\phi(x) - \frac{1}{2}\int_0^x \psi(s)ds \\ G(x) = \frac{1}{2}\phi(x) + \frac{1}{2}\int_0^x \psi(s)ds \end{cases}$$

And for $x > t$,

$$u_2(x,t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds$$

Apply the new boundary condition:

$$u_x(0,t) = G'(t) + F'(-t) = h(t)$$

$$\begin{aligned} \int_0^x G'(t)dt + \int_0^x F'(-t)dt &= \int_0^x h(t)dt + C \\ \int_0^x G'(t)dt - \int_0^{-x} F'(t')dt' &= \int_0^x h(t)dt + C \\ G(x) - F(-x) &= \int_0^x h(t)dt + C \end{aligned}$$

Taking the limit as $x \to 0$, we get $C = 0$, therefore

$$F(-x) = G(x) - \int_0^x h(t)dt$$

Therefore,

$$\begin{aligned} u_1 &= G(x+t) + F(x-t) = -\int_0^{t-x} h(y)dy + G(x+t) + G(t-x) \\ &= -\int_0^{t-x} h(y)dy + \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2}\left(\int_0^{x+t} \psi(s)ds + \int_0^{t-x} \psi(s)ds\right) \end{aligned}$$

$$\lim_{x\to t} u_1 = \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_0^{2t} \psi(s)ds$$

$$\lim_{x\to t} u_2 = \frac{1}{2}(\phi(2t) + \phi(0)) + \frac{1}{2}\int_0^{2t} \psi(s)ds$$

So the solution is always continuous.

$$\lim_{x\to t} (u_1)_x = h(0) + \frac{1}{2}(\phi'(2t) - \phi'(0)) + \frac{1}{2}\left(\psi(2t) - \psi(0)\right)$$

$$\lim_{x\to t} (u_1)_t = -h(0) + \frac{1}{2}(\phi'(2t) + \phi'(0)) + \frac{1}{2}\left(\psi(2t) + \psi(0)\right)$$

$$\lim_{x\to t}(u_2)_x = \frac{1}{2}(\phi'(2t)+\phi'(0)) + \frac{1}{2}(\psi(2t)-\psi(0))$$
$$\lim_{x\to t}(u_2)_t = \frac{1}{2}(\phi'(2t)-\phi'(0)) + \frac{1}{2}(\psi(2t)+\psi(0))$$

In order for the derivatives to match, it is necessary that

$$h(0) - \frac{1}{2}\phi'(0) = \frac{1}{2}\phi'(0)$$
$$-h(0) + \frac{1}{2}\phi'(0) = -\frac{1}{2}\phi'(0)$$
$$\phi'(0) = h(0)$$

2.1.1

Using d'Alembert's formula,

$$\begin{aligned} u(x,t) &= \frac{1}{2}(e^{x+ct}+e^{x-ct}) + \frac{1}{2c}\int_{x-ct}^{x+ct}\sin(s)ds \\ &= \frac{1}{2}(e^{x+ct}+e^{x-ct}) - \frac{1}{2c}\big(\cos(x+ct)-\cos(x-ct)\big) \end{aligned}$$

2.1.9

Rewrite the equation using operators

$$(\partial_x^2 - 3\partial_x\partial_t - 4\partial_t^2)u = 0$$
$$(\partial_x - 4\partial_t)(\partial_x+\partial_t)u = 0$$

The first operator is the directional derivative along the line that satisfies $\dfrac{dx}{dt} = -\dfrac{1}{4}$. The second one is along the line $\dfrac{dx}{dt} = 1$. Thus we can choose characteristic coordinates $\xi = x + \dfrac{1}{4}t$, and $\eta = x - t$.

$$\partial_x = \partial_\xi + \partial_\eta$$
$$\partial_t = \frac{1}{4}\partial_\xi - \partial_\eta$$

Solving the above two equations,

$$\partial_\xi = \frac{4}{5}(\partial_x + \partial_t)$$
$$\partial_\eta = \frac{1}{5}(\partial_x - 4\partial_t)$$

So the original equation becomes

$$(5\partial_\eta)(\frac{5}{4}\partial_\xi)u = \frac{25}{4}u_{\xi\eta} = 0$$

whiche means that the solution has the form

$$u = F(\xi) + G(\eta) = F(x + \frac{1}{4}t) + G(x - t)$$

Initial condition tells us that

$$F(x) + G(x) = x^2$$

$$u_t = \frac{1}{4}F'(x + \frac{1}{4}t) - G'(x - t)$$

$$u_t(x, 0) = \frac{1}{4}F'(x) - G'(x) = e^x$$

$$\int_0^x \frac{1}{4}F'(s) - G'(s)ds = \int_0^x e^s$$

$$\frac{1}{4}F(x) - G(x) = e^x + A$$

From the first and last equations,

$$F(x) = \frac{4}{5}(x^2 + e^x + A) = \frac{1}{5}(4x^2 + 4e^x + 4A)$$

$$G(x) = \frac{1}{5}(x^2 - 4e^x - 4A)$$

$$\begin{aligned}
u(x,t) &= F(x + \frac{1}{4}t) + G(x - t) \\
&= \frac{1}{5}\left[4\left(x + \frac{1}{4}t\right)^2 + 4e^{x+\frac{1}{4}t} + 4A\right] + \frac{1}{5}\left[(x-t)^2 - 4e^{x-t} - 4A\right] \\
&= \frac{1}{5}\left[4\left(x + \frac{1}{4}t\right)^2 + 4e^{x+\frac{1}{4}t}\right] + \frac{1}{5}\left[(x-t)^2 - 4e^{x-t}\right] \\
&= \frac{1}{5}\left[4\left(x + \frac{1}{4}t\right)^2 + (x-t)^2\right] + \frac{4}{5}\left[e^{x+\frac{1}{4}t} - e^{x-t}\right]
\end{aligned}$$

2.2.3

a

Let $\xi = x - y$, then

$$\partial_x u(x - y, t) = \partial_\xi u(\xi, t)$$

$$\partial_{xx} u(x - y, t) = \partial_{\xi\xi} u(\xi, t)$$

And

$$\partial_{tt} u(x - y, t) = \partial_{tt} u(\xi, t)$$

Therefore

$$\partial_{tt} u(x - y, t) = \partial_{tt} u(\xi, t) = c^2 \partial_{\xi\xi} u(\xi, t) = c^2 \partial_{xx} u(x - y, t)$$

b

$$(u_x)_{tt} = (u_{tt})_x = (c^2 u_{xx})_x = c^2 (u_x)_{xx}$$

Therefore u_x is a solution. The cases for other derivatives can be proved in the same way.

c

Let $\bar{x} = ax$, $\bar{t} = at$,

$$\partial_{tt} u(ax, at) = a^2 \partial_{\bar{t}\bar{t}} u(\bar{x}, \bar{t})$$

$$\partial_{xx} u(ax, at) = a^2 \partial_{\bar{x}\bar{x}} u(\bar{x}, \bar{t})$$

$$\partial_{tt} u(ax, at) - c^2 \partial_{xx} u(ax, at) = a^2 \left[\partial_{\bar{t}\bar{t}} u(\bar{x}, \bar{t}) - c^2 \partial_{\bar{x}\bar{x}} u(\bar{x}, \bar{t}) \right] = 0$$

Therefore $u(ax, at)$ satisfies the equation.