MATH 4441 Homework 2

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1.17

By definition, $w = \inf[w_{u(\theta)}] \le Ave[w_{u(\theta)}] = Ave[\frac{1}{2}L_{u(\theta)}]$ where

$$Ave[f(\theta)] := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

By Cauchy's integral formula

$$Ave[\frac{1}{2}L_{u(\theta)}] = \frac{1}{2}Ave[L_{u(\theta)}] = \frac{L}{\pi}$$

Therefore $w \le \frac{L}{\pi}$.

1.18

If the equality holds, then it is necessary that

$$\inf[w_{u(\theta)}] = Ave[w_{u(\theta)}]$$

which is true if and only if $w_{u(\theta)}$ is a constant.

2.7

A circle of radius r with parameterization

$$\alpha(t) = (r\cos t, r\sin t)$$

has arclength of

$$s(t) = \int_0^t \|\alpha'(u)\|du = rt$$

and its inverse

$$s^{-1}(t) = \frac{t}{r}$$

The parameterization with respsect to its arclength is

$$\begin{aligned}\bar{\alpha}(t) &= \alpha \circ s^{-1}(t) = (r\cos\frac{t}{r}, r\sin\frac{t}{r})\\ \bar{\alpha}''(t) &= (-\frac{1}{r}\cos\frac{t}{r}, -\frac{1}{r}\sin\frac{t}{r})\\ \kappa(t) &= \|\bar{\alpha}''(t)\| = \frac{1}{r}\end{aligned}$$

A straight line can be parameterized by

$$\alpha(t) = p + tv$$

Its arclength function then would be

$$s(t) = \int_0^t \|v\| du = \|v\| t, \quad s^{-1}(t) = \frac{t}{\|v\|}$$

and the parameterization with respsect to its arclength is

$$\bar{\alpha}(t) = \alpha \circ s^{-1}(t) = p + t\frac{v}{\|v\|}$$

$$\kappa(t) = \|\bar{\alpha}''(t)\| = 0$$

2.10

Using the parameterization $\alpha(t) = (t, f(t), 0)$,

$$\alpha'(t) = (1, f'(t), 0)$$
$$\alpha''(t) = (0, f''(t), 0)$$
$$\alpha'(t) \times \alpha''(t) = \begin{vmatrix} i & j & k \\ 1 & f' & 0 \\ 0 & f'' & 0 \end{vmatrix} = (0, 0, f'')$$

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3} = \frac{|f''(t)|}{\left(\sqrt{1 + (f'(t))^2}\right)^3}$$

- $f(x) = x$:
$$\kappa = \frac{|0|}{\left(\sqrt{1 + (1)^2}\right)^3} = 0$$
- $f(x) = x^2$:
$$\kappa = \frac{|2|}{\left(\sqrt{1 + (2x)^2}\right)^3} = \frac{2}{(\sqrt{1 + 4x^2})^3}$$

- $f(x) = x^3$:
$$\kappa = \frac{|6x|}{\left(\sqrt{1+(3x^2)^2}\right)^3} = \frac{6|x|}{\left(\sqrt{1+9x^4}\right)^3}$$

- $f(x) = x^4$:
$$\kappa = \frac{|12x^2|}{\left(\sqrt{1+(4x^3)^2}\right)^3} = \frac{12x^2}{\left(\sqrt{1+16x^6}\right)^3}$$

2.11

Suppose $\alpha(t) = (x(t), y(t))$, then it can be reparameterized by its x coordinate:
$$\bar{\alpha}(t) = \alpha \circ x^{-1}(t) = (t, y \circ x^{-1}(t)) = (t, f(t))$$
Since $\bar{\alpha}$ passes through $(0,0)$, $f(0) = 0$, and since $f(t)$ is non-negative, $t = 0$ must be a critical point. So $f'(0) = 0$. Similarly, β can be reparameterized as $(t, g(t))$, where $g(0) = 0$ and $g'(0) = 0$. The taylor expansion of f, g around 0 are:
$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(c_1)t^2 = \frac{1}{2}f''(c_1)t^2 \geq 0$$
$$g(t) = g(0) + g'(0)t + \frac{1}{2}g''(c_2)t^2 = \frac{1}{2}g''(c_2)t^2 \geq 0$$
for some c_1, c_2 between 0 and t. Since β is higher than or at the same height as α for all t, $g''(c_2) \geq f''(c_1)$. Take the limit as $t \to 0$, then $\lim c_1 = \lim c_2 = 0$, and so
$$g''(0) \geq f''(0) \geq 0 \Rightarrow |g''(0)| \geq |f''(0)|$$
which is equivalent to
$$\kappa_\beta(0) \geq \kappa_\alpha(0)$$
because when $f' = 0$,
$$\kappa(t) = \frac{|f''(t)|}{\left(\sqrt{1+(f'(t))^2}\right)^3} = |f''(t)|$$

2.12

We can shrink the circle until it contacts the curve at some point p. Suppose after shrinking the radius becomes $R \leq r$. The closed curve and the circle can be treated as β and α in Ex.11 if we apply a rigid motion to both curves so that the point of contact coincides with the origin and the x axis is tangent to both curves. From the result of Ex.11, at this particular point the curvature of the closed curve is no smaller than that of the circle, which is $1/R$, therefore
$$\kappa_\alpha(\alpha^{-1}(p)) \geq \frac{1}{R} \geq \frac{1}{r}$$

2.13

Suppose the width of the curve is w, then there exist two parallel lines separated by a distance w that contain the curve in between. We can fit a circle of radius w between the lines and slide the circle while keeping it tangent to both lines. Suppose we call the direction of the two lines horizontal and the circle is initially to the right of the closed curve. If we slide the circle to the left, eventually some point must enter the circle from the left, and then touch the right half of the circle from the inside. Then there must be a region around the contact point that is contained in the circle. By the same argument as Ex.12, the curvature at the contact point p must be no less than the curvature of the circle. Therefore,

$$\max \kappa \geq \kappa_p \geq \kappa_{circle} = \frac{1}{w/2} = \frac{2}{w}$$

And since $w \leq \dfrac{L}{\pi}$,

$$\frac{1}{w} \geq \frac{\pi}{L}$$

$$\max \kappa \geq \frac{2\pi}{L}, \text{ or equivalently } L \geq \frac{2\pi}{\max \kappa}$$

MATH 4441 Homework 3

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4.1

Assume $\alpha(t)$ has unit speed, then $\kappa = \|a''\|$. If $\kappa = 0$ then $a'' = 0$,

$$\alpha'(t) = v_0 + \int_0^t \alpha''(s)ds = v_0$$

$$\alpha(t) = x_0 + \int_0^t \alpha'(s)ds = x_0 + v_0 t$$

which is the equation of a straight line.

4.2

By definition $\|T\| = \langle T(t), T(t)\rangle = 1$. Differentiate and then scale both sides:

$$\langle T(t), T(t)\rangle' = 2\langle T(t), T(t)'\rangle = 0$$

$$\left\langle T(t), \frac{T(t)'}{\|T'(t)\|}\right\rangle = N(t) = 0$$

4.4

For a given t, the osculating circle γ can be parameterized by

$$\gamma(s) = p(t) + \frac{1}{\kappa(t)}T(t)\cos s + \frac{1}{\kappa(t)}N(t)\sin s$$

where

$$p(t) := \alpha(t) + \frac{1}{\kappa(t)}N(t)$$

And we can compute the tantrix for the osculating circle:

$$\gamma'(s) = -\frac{1}{\kappa(t)}T(t)\sin s + \frac{1}{\kappa(t)}N(t)\cos s$$

$$\|\gamma'(s)\| = \langle \gamma'(s), \gamma'(s) \rangle^{1/2} = \frac{1}{\kappa(t)}$$

$$\bar{T}(s) = \frac{\gamma'(s)}{\|\gamma'(s)\|} = -T(t) \sin s + N(t) \cos s$$

$$\bar{T}'(s) = -T(t) \cos s - N(t) \sin s$$

$$\|\bar{T}'(s)\| = \langle \bar{T}'(s), \bar{T}'(s) \rangle^{1/2} = 1$$

Circle γ touches α at $s = -\pi/2$, its tantrix at which point is

$$\bar{T}\left(-\frac{\pi}{2}\right) = -T(t) \sin(-\frac{\pi}{2}) + N(t) \cos(-\frac{\pi}{2}) = T(t)$$

so it is indeed tangent to $\alpha(t)$. And its curvature is

$$\kappa_\gamma(s) = \frac{\|\bar{T}'(s)\|}{\|\gamma'(s)\|} = \frac{1}{1/\kappa(t)} = \kappa(t)$$

4.5

i

When α is parameterized by arclength, $\|\alpha'\| = 1$,

$$\begin{aligned} p'(t) &= \alpha'(t) + \frac{1}{\kappa(t)} N'(t) \\ &= \alpha'(t) + \frac{1}{\kappa(t)} (-\kappa(t) T(t)) \\ &= \alpha'(t) - T(t) = \alpha'(t) - \alpha'(t) = 0 \end{aligned}$$

so $p(t)$ is a fixed point.

ii

$$\alpha(t) - p(t) = -\frac{1}{\kappa(t)} N(t) = -\frac{1}{c} N(t)$$

$$\|\alpha(t) - p(t)\| = \frac{1}{c}$$

4.8

$$T = \frac{\gamma'}{\|\gamma'\|} = \frac{\gamma'}{\langle \gamma', \gamma' \rangle^{1/2}}$$

$$T' = \frac{\gamma'' \langle \gamma', \gamma' \rangle + \gamma' \langle \gamma', \gamma'' \rangle}{\|\gamma'\|^3}$$

$$\begin{aligned}
\langle T', iT\rangle &= \left\langle \frac{\gamma''\langle\gamma',\gamma'\rangle + \gamma'\langle\gamma',\gamma''\rangle}{\|\gamma'\|^3}, \frac{i\gamma'}{\|\gamma'\|}\right\rangle \\
&= \frac{\langle\gamma'', i\gamma'\rangle\langle\gamma',\gamma'\rangle + \langle\gamma', i\gamma'\rangle\langle\gamma',\gamma''\rangle}{\|\gamma'\|^4} \\
&= \frac{\langle\gamma'', i\gamma'\rangle\langle\gamma',\gamma'\rangle}{\|\gamma'\|^4} \\
&= \frac{\langle\gamma'', i\gamma'\rangle}{\|\gamma'\|^2}
\end{aligned}$$

Suppose $\gamma = (\gamma_1, \gamma_2)$, then

$$\gamma' = (\gamma_1', \gamma_2') = \gamma_1' + i\gamma_2', \quad i\gamma' = -\gamma_2' + i\gamma_1' = (-\gamma_2', \gamma_1')$$

$$\gamma'' = (\gamma_1'', \gamma_2'')$$

$$\langle\gamma'', i\gamma'\rangle = \gamma_1'\gamma_2'' - \gamma_2'\gamma_1'' = (\gamma' \times \gamma'')_z = \langle\gamma' \times \gamma'', (0,0,1)\rangle$$

Substitute it back into the formula:

$$\langle T', iT\rangle = \frac{\langle\gamma' \times \gamma'', (0,0,1)\rangle}{\|\gamma'\|^2}$$

$$\bar\kappa = \frac{\langle T', iT\rangle}{\|\gamma'\|} = \frac{\langle\gamma' \times \gamma'', (0,0,1)\rangle}{\|\gamma'\|^3}$$

4.12

i

A unit circle oriented clockwise can be parameterized as

$$\alpha(t) = (\cos(-t), \sin(-t))$$

for $0 \le t \le 2\pi$.

$$\alpha'(t) = (\sin(-t), -\cos(-t))$$

$$\alpha''(t) = (-\cos(-t), -\sin(-t))$$

$$\langle\alpha' \times \alpha'', (0,0,1)\rangle = -\sin^2(-t) - \cos^2(-t) = -1$$

$$\bar\kappa = \frac{\langle\alpha' \times \alpha'', (0,0,1)\rangle}{\|\alpha'\|^3} = \frac{-1}{1} = -1$$

$$total\ \bar\kappa[\alpha] = \int_0^{2\pi} \bar\kappa dt = -2\pi$$

$$rot[\alpha] = \frac{total\ \bar\kappa[\alpha]}{2\pi} = -1$$

ii

A unit circle oriented counter-clockwise can be parameterized as

$$\alpha(t) = (\cos(t), \sin(t))$$

for $0 \leq t \leq 2\pi$.

$$\alpha'(t) = (-\sin(t), \cos(t))$$

$$\alpha''(t) = (-\cos(t), -\sin(t))$$

$$\langle \alpha' \times \alpha'', (0,0,1) \rangle = \sin^2(t) + \cos^2(t) = 1$$

$$\bar{\kappa} = \frac{\langle \alpha' \times \alpha'', (0,0,1) \rangle}{\|\alpha'\|^3} = \frac{1}{1} = 1$$

$$total\ \bar{\kappa}[\alpha] = \int_0^{2\pi} \bar{\kappa} dt = 2\pi$$

$$rot[\alpha] = \frac{total\ \bar{\kappa}[\alpha]}{2\pi} = 1$$

iii

$$\alpha(t) = (\cos t, \sin 2t)$$

$$\alpha'(t) = (-\sin t, 2\cos 2t)$$

$$\alpha''(t) = (-\cos t, -4\sin 2t)$$

$$\langle \alpha' \times \alpha'', (0,0,1) \rangle = 4\sin t \sin 2t + 2\cos t \cos 2t$$

$$\bar{\kappa}(t) = \frac{\langle \alpha' \times \alpha'', (0,0,1) \rangle}{\|\alpha'\|^3} = \frac{4\sin t \sin 2t + 2\cos t \cos 2t}{\sqrt{\sin^2 t + 4\cos^2 2t}^3}$$

$$total\ \bar{\kappa}[\alpha] = \int_0^{2\pi} \bar{\kappa} dt = \int_0^{2\pi} \frac{4\sin t \sin 2t + 2\cos t \cos 2t}{\sqrt{\sin^2 t + 4\cos^2 2t}^3} dt = 0$$

$$rot[\alpha] = \frac{total\ \bar{\kappa}[\alpha]}{2\pi} = 0$$

MATH 4441 Homework 4

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4.21

Suppose $\theta' = \bar{\kappa} = c$, then

$$\theta(t) = \int_0^t \bar{\kappa}(s)ds + \theta_0 = \int_0^t cds + \theta_0 = ct + \theta_0$$

$$\alpha'(t) = (\cos(ct + \theta_0), \sin(ct + \theta_0))$$

$$\begin{aligned}\alpha(t) &= \left(\int_0^t \cos(cs + \theta_0)ds, \int_0^t \sin(cs + \theta_0)ds\right) + \alpha_0 \\ &= \frac{1}{c}(\sin(ct + \theta_0), -\cos(ct + \theta_0)) + \alpha_0\end{aligned}$$

which a cricle of radius $1/c$ centered at α_0.

5.5

Suppose $\|\alpha'\| = 1$. Then by definition

$$\beta(t) = \alpha(t) + r(t)N(t)$$

$$\beta' = \alpha' + r'N + rN' = T + r'N + r(-\kappa T) = T + r'N - T = r'N$$

$$\|\beta'\| = |r'|$$

$$T_\beta = \frac{\beta'}{\|\beta'\|} = \frac{r'}{|r'|}N = \pm N$$

Since the curvature (and therefore r) is monotone, the sign is always the same.

$$T'_\beta = \pm N' = \mp\kappa T, \quad \|T'_\beta\| = \kappa$$

$$\kappa_\beta(t) = \frac{\|T'_\beta\|}{\|\beta'\|} = \frac{\kappa}{|r'|}$$

which is never zero because κ is never zero.

5.7

If a line is tangent to the curve at two different points, then the osculating circle at either point is not contained in the osculating circle at the other point, because at least its contact point with the curve is not in the other circle. However, by Kneser's Nesting Theorem, if a curve has monotone nonvanishing curvature, then its osculating circles must be nested. Therefore such curves cannot have bitangent lines.

6.1

An ellipse can be parameterized as

$$\alpha(t) = (a\cos(t), b\sin(t))$$

$$\alpha'(t) = (-a\sin(t), b\cos(t))$$

$$\|\alpha'(t)\| = (a^2\sin^2(t) + b^2\cos^2(t))^{\frac{1}{2}}$$

$$\alpha''(t) = (-a\cos(t), -b\sin(t))$$

$$\langle \alpha' \times \alpha'', (0,0,1)\rangle = ab\sin^2(t) + ab\cos^2(t) = ab$$

$$\bar{\kappa} = \frac{\langle \alpha' \times \alpha'', (0,0,1)\rangle}{\|\alpha'\|^3} = \frac{ab}{(a^2\sin^2(t) + b^2\cos^2(t))^{\frac{3}{2}}}$$

$$\bar{\kappa}' = -\frac{3}{2}ab\Big(a^2\sin^2(t) + b^2\cos^2(t)\Big)^{-\frac{5}{2}}\Big(2a^2\sin(t)\cos(t) - 2b^2\cos(t)\sin(t)\Big)$$

The signed curvature has a local extremum when $\bar{\kappa}' = 0$,

$$2(a^2 - b^2)\cos(t)\sin(t) = 0$$

If $a = b$, in which case the ellipse is a circle, the equation is satisfied for all t. If $a \neq b$, then either $\cos(t) = 0$ or $\sin(t) = 0$. The corresponding vertices are located at

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

Therefore an ellipse has exactly 4 vertices unless it's a circle.

6.4

There must be even number of vertices, otherwise there will be two local maxima or minima of $\bar{\kappa}$ next to each other, which is not possible. Therefore, if the curve has fewer than 4 vertices, it must have exactly 2.

MATH 4441 Homework 5

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6.9

First, the number of vertices must be even, otherwise there will be two maxima/minima immediately next to each other. Now suppose there are only 2 vertices, then one must be a maximum and the other must be a minimum. And since they are the only two extrema, curvature must increase monotonically from the minimum to the maximum. Starting from the maximum point and move two points a, b in opposite directions simultaneously, keeping the curvature at both points the same. Then a, b divide the curve into top and bottom halves with

$$\kappa_{bottom} \geq \kappa(a) = \kappa(b) \geq \kappa_{top}$$

Move the points until the arclength of both halves become the same. Then by Shur's arm lemma, the two halves cannot have the same chord length, therefore the assumption cannot be true and so there must be more than 2 vertices. In other words, there must be 4 or more vertices.

6.22

Let $\phi(x) = \sqrt{1+x^2}$, then $\phi''(x) = \dfrac{1}{(1+x^2)^{3/2}} > 0$, which means that ϕ is strictly convex. By Jensen's inequality,

$$\begin{aligned}
\sqrt{1+\bar{f}'^2} &= \phi(\bar{f}') \\
&= \phi(\frac{1}{2}[f' + (-g')]) \\
&< \frac{1}{2}\Big[\phi(f') + \phi(-g')\Big] \\
&= \frac{1}{2}\Big[\sqrt{1+f'^2} + \sqrt{1+(-g')^2}\Big] \\
&= \frac{1}{2}\Big[\sqrt{1+f'^2} + \sqrt{1+g'^2}\Big]
\end{aligned}$$

$$\sqrt{1+\bar{f}'^2} + \sqrt{1+(-\bar{f}')^2} = 2\sqrt{1+\bar{f}'^2} < \sqrt{1+f'^2} + \sqrt{1+g'^2}$$

Therefore the value of the two integrals for $\bar{\alpha}$ is strictly less that of α.

$$\int_a^b \sqrt{1+\bar{f}'^2}dt + \int_a^b \sqrt{1+(-\bar{f}')^2}dt < \int_a^b \sqrt{1+f'^2}dt + \int_a^b \sqrt{1+g'^2}dt$$

The lengths of the two line segments are still the same, because

$$\bar{f} - (-\bar{f}) = 2\bar{f} = f - g$$

Combininig the above results,

$$Length[\bar{\alpha}] < Length[\alpha]$$

7.2

$$\begin{aligned}
\alpha'(t) &= (-r\sin t, r\cos t, h) \\
\|\alpha'\| &= \sqrt{r^2+h^2} \\
s(t) &= \int_0^t \sqrt{r^2+h^2}dz = \sqrt{r^2+h^2}t \\
s^{-1}(t) &= \frac{1}{\sqrt{r^2+h^2}}t = ct
\end{aligned}$$

Reparametrize the helix curve by its arc length

$$\bar{\alpha}(t) = (r\cos(ct), r\sin(ct), hct)$$

Then

$$\begin{aligned}
T &= \bar{\alpha}'(t) = (-cr\sin(ct), cr\cos(ct), ch) \\
T' &= (-c^2r\cos(ct), -c^2r\sin(ct), 0)
\end{aligned}$$

$$\kappa = \|T'\| = c^2r = \frac{r}{r^2+h^2}$$

$$\begin{aligned}
N &= \frac{T'}{\kappa} = (-\cos(ct), -\sin(ct), 0) \\
B = T \times N &= \begin{vmatrix} i & j & k \\ -cr\sin(ct) & cr\cos(ct) & ch \\ -\cos(ct) & -\sin(ct) & 0 \end{vmatrix} \\
&= (ch\sin(ct), -ch\cos(ct), cr) \\
B' &= (c^2h\cos(ct), c^2h\sin(ct), 0)
\end{aligned}$$

$$\tau = -\langle B', N\rangle = c^2h = \frac{h}{r^2+h^2}$$

From the above formulae of κ and τ, κ always decreases as h increases, and τ always decreases as r increases. And taking the partial derivatives:

$$\frac{\partial\kappa}{\partial r} = \frac{h^2 - r^2}{(h^2 + r^2)^2}$$

$$\frac{\partial\tau}{\partial h} = \frac{r^2 - h^2}{(h^2 + r^2)^2}$$

When $r < h$, κ changes in the same direction as r, and τ changes in the opposite direction of h. When $r > h$, κ changes in the opposite direction of r, and τ changes in the same direction as h.

7.5

$$\begin{aligned}
\langle v, T\rangle' &= a' = 0 = \langle v', T\rangle + \langle v, T'\rangle = 0 + \langle v, \kappa N\rangle = \kappa b \\
&\Rightarrow b = 0 \\
\langle v, N\rangle' &= b' = 0 = \langle v', N\rangle + \langle v, N'\rangle = 0 + \langle v, -\kappa T + \tau B\rangle = -\kappa + c\tau \\
&\Rightarrow c = \frac{\kappa}{\tau}
\end{aligned}$$

Therefore

$$v = T + \frac{\kappa}{\tau} B$$

Check:

$$\langle T, v\rangle = 1$$

$$\begin{aligned}
\langle v, B\rangle' = \left(\frac{\kappa}{\tau}\right)' &= \langle v', B\rangle + \langle v, B'\rangle = 0 + \langle v, -\tau N\rangle = 0 \\
v' = T' + \left(\frac{\kappa}{\tau}\right)' B + \frac{\kappa}{\tau} B' &= \kappa N + 0 + \frac{\kappa}{\tau}(-\tau N) = 0
\end{aligned}$$

7.7

Suppose the curve indeed lies on some sphere, we can describe the center of the sphere using the moving frame:

$$p = \alpha + aT + bN + cB$$

Since α lies on the sphere, T must be tangent to the sphere, whereas $p - \alpha$ must be in (the opposite of) the normal direction. Therefore $a = \langle p - \alpha, T\rangle = 0$. If such sphere exists, then it must have a fixed center, so $p' = 0$. Differentiate the above equation (suppose α has unit speed):

$$\begin{aligned}
\langle p-\alpha, T\rangle' &= \langle -T, T\rangle + \langle p-\alpha, T'\rangle \\
&= -1 + \langle p-\alpha, \kappa N\rangle \\
&= -1 + \kappa b = 0 \\
b &= \frac{1}{\kappa}
\end{aligned}$$

$$\begin{aligned}
\langle p-\alpha, N\rangle' = b' &= \left(\frac{1}{\kappa}\right)' = -\frac{\kappa'}{\kappa^2} \\
&= \langle -T, N\rangle + \langle p-\alpha, N'\rangle \\
&= 0 + \langle p-\alpha, -\kappa T + \tau B\rangle = c\tau \\
c &= -\frac{\kappa'}{\tau\kappa^2}
\end{aligned}$$

If the curve satisfies the equation concerning κ and τ, then

$$r^2 = \left(\frac{\kappa'}{\kappa^2\tau}\right)^2 + \frac{1}{\kappa^2} = c^2 + b^2 = \|p-\alpha\|^2$$

wihch means that the distance between p and $\alpha(t)$ is a constant r. Therefore, the curve indeed lies on a sphere.

MATH 4441 Homework 6

Wenqi He

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8.4

$$J_{(u^1,u^2)}(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ D_1 f(u^1,u^2) & D_2 f(u^1,u^2) \end{pmatrix}$$

The two columns are always linearly independent, so the Jacobian always has a rank of 2. By definition, the mapping is regular.

8.5

The Jacobian of f

$$J_p(f) = \begin{pmatrix} D_1 f^1(p) & D_2 f^1(p) \\ D_1 f^2(p) & D_2 f^2(p) \\ D_1 f^3(p) & D_2 f^3(p) \end{pmatrix} = \begin{pmatrix} D_1 f(p) & D_2 f(p) \end{pmatrix}$$

The mapping is regular iff the two columns are linearly independent, or equivalently,

$$\|D_1 f(p) \times D_2 f(p)\| \neq 0$$

8.11

Since α is closed, we can extended it to a periodic function $\bar{\alpha} : R \to R^2$. For any point (t, θ) we can take $U = (t-\epsilon, t+\epsilon) \times (\theta - \epsilon, \theta + \epsilon)$, then (U, X) is a regular patch. X is one-to-one because if

$$\begin{aligned} x(t_1)\cos(\theta_1) &= x(t_2)\cos(\theta_2) \\ x(t_1)\sin(\theta_1) &= x(t_2)\sin(\theta_2) \\ y(t_1) &= y(t_2) \end{aligned}$$

Then from the first two equations, $\tan(\theta_1) = \tan(\theta_2) \Rightarrow \theta_1 = \theta_2 \Rightarrow x(t_1) = x(t_2)$. Then $t_1 = t_2 = \bar{\alpha}^{-1}(x, y)$. X is smooth because $x, y, \sin(x), \cos(x)$ are all C^∞. X is regular because the Jacobian of X is

$$\begin{pmatrix} x'\cos(\theta) & -x\sin(\theta) \\ x'\sin(\theta) & x\cos(\theta) \\ y' & 0 \end{pmatrix}$$

which always has rank 2. X is proper because X maps open sets to open sets and so the preimage of open sets under X^{-1} must be open sets. Therefore, by definition, The surface is a regular embedded surface.

9.5

Define $f(p) := \|p\|^2 = x^2 + y^2 + z^2$, then $\mathbb{S}^2$ is the level set of $f = 1$. Therefore, the gradient $\nabla f = (2x, 2y, 2z) = 2p$ is always orthogonal to $\mathbb{S}^2$. Normalizing the vector gives $n(p) = p$.

9.11

On a sphere of radius r, the Gauss map is $n(p) = \dfrac{p}{r}$. For any $v \in T_pM$, we can find an associated curve $\gamma : (-\epsilon, \epsilon) \to M$, such that $\gamma(0) = p, \gamma'(0) = v$. Then

$$(n \circ \gamma)(t) = n(\gamma(t)) = \frac{\gamma(t)}{r}$$

$$(n \circ \gamma)'(t) = \frac{\gamma'(t)}{r}$$

$$dn_p(v) = (n \circ \gamma)'(0) = \frac{\gamma'(0)}{r} = \frac{v}{r}$$

$$dn_p = \frac{1}{r} \cdot id$$

$$S_p = -dn_p = \begin{pmatrix} -1/r & 0 \\ 0 & -1/r \end{pmatrix}$$

$$K(p) = \det S_p = \frac{1}{r^2}$$

9.13

$$D_1X = (x'\cos\theta, x'\sin\theta, y'), \quad D_2X = (-x\sin\theta, x\cos\theta, 0)$$

$$\langle D_1X, D_1X\rangle = x'^2 + y'^2, \quad \langle D_2X, D_2X\rangle = x^2, \quad \langle D_1X, D_2X\rangle = \langle D_2X, D_1X\rangle = 0$$

$$\det g_{ij} = g_{11}g_{22} - g_{12}^2 = (x'^2 + y'^2)x^2$$

$$D_1X \times D_2X = \begin{vmatrix} i & j & k \\ x'\cos\theta & x'\sin\theta & y' \\ -x\sin\theta & x\cos\theta & 0 \end{vmatrix} = (-xy'\cos\theta, -xy'\sin\theta, xx')$$

$$N = \frac{D_1X \times D_2X}{\|D_1X \times D_2X\|} = \frac{1}{\sqrt{x^2(x'^2 + y'^2)}}(-xy'\cos\theta, -xy'\sin\theta, xx')$$

$$D_{11}X = (x''\cos\theta, x''\sin\theta, y''), \quad \langle D_{11}X, N\rangle = \frac{-xx''y' + xx'y''}{\sqrt{x^2(x'^2 + y'^2)}}$$

$$D_{22}X = (-x\cos\theta, -x\sin\theta, 0), \quad \langle D_{22}X, N\rangle = \frac{x^2y'}{\sqrt{x^2(x'^2 + y'^2)}}$$

$$D_{12}X = D_{21}X = (-x'\sin\theta, x'\cos\theta, 0) \quad \langle D_{12}X, N\rangle = \langle D_{21}X, N\rangle = 0$$

$$\det l_{ij} = l_{11}l_{22} - l_{12}^2 = \frac{xy'(x'y'' - x''y')}{x'^2 + y'^2}$$

For any point $p = X(t_0, \theta_0)$, we can define $\bar{X}(t, \theta) = X(t + t_0, \theta + \theta_0)$, then $p = \bar{X}(0, 0)$. Carrying out the same computation as above, we can get

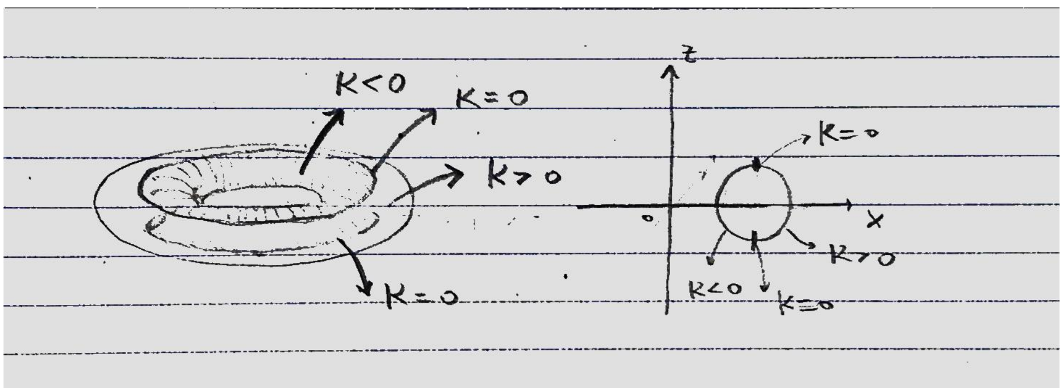
$$K(p) = \frac{\det \bar{l}_{ij}(0,0)}{\det \bar{g}_{ij}(0,0)} = \frac{\det l_{ij}(t_0, \theta_0)}{\det g_{ij}(t_0, \theta_0)} = \boxed{\left.\frac{y'(x'y'' - x''y')}{x(x'^2 + y'^2)^2}\right|_{t_0}}$$

For a torus of revolution,

$$x' = -r \sin t, \quad x'' = -r \cos t, \quad y' = r \cos t, \quad y'' = -r \sin t$$

$$K(p) = \left(\frac{y'(x'y'' - x''y')}{x(x'^2 + y'^2)^2}\right)(t_0) = \boxed{\frac{\cos t_0}{r(R + r \cos t_0)}}$$

Assuming $0 < r < R$, then the denominator is always positive, so the sign of curvature depends on the sign of $\cos t_0$. For $t_0 \in [-\pi, \pi]$, when $|t_0| < \pi/2$, the curvature is positive. When $|t_0| = \pi/2$, the curvature is zero. When $|t_0| > \pi/2$, then curvature is nagative.



MATH 4441 Homework 7

Wenqi He

October 10, 2018

9.12

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = e_i(p)$. Let $\alpha = X^{-1} \circ \gamma$, then $\gamma(0) = X(\alpha(0)) = X(0,0)$, so $\alpha(0) = (0,0)$. And since

$$\gamma'(0) = (X \circ \alpha)'(0) = J_{(0,0)}(X)\alpha'(0) = e_i(p) = D_i X(0,0)$$

we have $\alpha'(0) = \epsilon_i$, where $\epsilon_1 = (1,0)$, $\epsilon_2 = (0,1)$. Now define $f : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$f(t) := \Big\langle n(\gamma(t)), e_j(\gamma(t)) \Big\rangle \equiv 0$$

Take the derivative at $t = 0$:

$$f'(0) = \Big\langle n(\gamma(t))', e_j(\gamma(t)) \Big\rangle|_{t=0} + \Big\langle n(\gamma(t)), e_j(\gamma(t))' \Big\rangle|_{t=0} = 0 \tag{*}$$

The first term evaluates to

$$\Big\langle (n \circ \gamma)'(0), e_j(\gamma(0)) \Big\rangle = \Big\langle dn_p(e_i(p)), e_j(p) \Big\rangle = \boxed{-\Big\langle S_p(e_i(p)), e_j(p) \Big\rangle}$$

The second terms evaluates to

$$\begin{aligned}
&\Big\langle n(\gamma(0)), (e_j \circ \gamma)'(0) \Big\rangle \\
&= \Big\langle n(X(\alpha(0))), (D_j X \circ X^{-1} \circ X \circ \alpha)'(0) \Big\rangle \\
&= \Big\langle N(\alpha(0)), (D_j X \circ \alpha)'(0) \Big\rangle \\
&= \Big\langle N(0,0), J_{(0,0)}(D_j X)\alpha'(0) \Big\rangle \\
&= \Big\langle N(0,0), J_{(0,0)}(D_j X)\epsilon_i \Big\rangle \\
&= \Big\langle N(0,0), D_i D_j X(0,0) \Big\rangle \\
&\boxed{= \Big\langle N(0,0), D_{ij} X(0,0) \Big\rangle = l_{ij}(0,0)}
\end{aligned}$$

Plug the terms into $(*)$,

$$l_{ij}(0,0) = \Big\langle S_p(e_i(p)), e_j(p) \Big\rangle$$

9.15

$$\det(\operatorname{Hess} f) = \det \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} = \det \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} = 4ab, \quad \operatorname{grad} f(0,0) = \begin{pmatrix} 2ax \\ 2by \end{pmatrix} \Big|_{(0,0)} = 0$$

So the curvature at $(0, 0, f(0, 0))$ is $K = \dfrac{4ab}{(1+0)^2} = 4ab$.

10.3

$$\operatorname{Hess} f = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -6y & -6x \\ -6x & 6y \end{pmatrix}, \quad \det(\operatorname{Hess} f(0,0)) = \det 0_{2,2} = 0$$

Therefore by the curvature formula, $K(p) = 0$. However, the surface is not locally convex at p, because the surface does not lie on one side of the tangent plane $z = 0$. $z > 0$ when $y > 0$ and $|y| > \sqrt{3}|x|$ or $y < 0$ and $|y| < \sqrt{3}|x|$, and $z < 0$ when $y > 0$ and $|y| < \sqrt{3}|x|$ or $y < 0$ and $|y| > \sqrt{3}|x|$,

10.6

Suppose $ac - b^2 > 0$. If $x \neq 0$ but $Q = 0$ at some point, then

$$\frac{Q}{x^2} = a + 2b\frac{y}{x} + c\left(\frac{y}{x}\right)^2 = 0$$

must has a solution, thus the discriminant $4b^2 - 4ac \geq 0 \Rightarrow ac - b^2 \leq 0$, which is a contradiction. Therefore, $ac - b^2 > 0$ implies that Q is definite

Suppose $ac - b^2 < 0$. If $Q \neq 0$ whenever $x \neq 0$, then the above equation cannot have solutions, so the discriminant $4b^2 - 4ac < 0 \Rightarrow ac - b^2 > 0$, which again is a contradiction. Therefore, $ac - b^2 < 0$ implies that Q is not definite.

10.11

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve such that $\gamma(0) = X(u^1, u^2)$ and $\gamma'(0) = D_i X(u^1, u^2)$ and let $\alpha = X^{-1} \circ \gamma$. Since $\gamma'(0) = (X \circ \alpha)'(0) = J_{(u^1,u^2)}(X)\alpha'(0) = D_i X(u^1, u^2)$, we have $\alpha'(0) = \epsilon_i$. Then

$$\begin{aligned} dn(D_i X(u^1, u^2)) &= (n \circ \gamma)'(0) = (n \circ X \circ \alpha)'(0) = ((n \circ X) \circ \alpha)'(0) \\ &= J_{(u^1,u^2)}(n \circ X)\alpha'(0) \\ &= J_{(u^1,u^2)}(n \circ X)\epsilon_i \\ &= D_i(n \circ X)(u^1, u^2) \end{aligned}$$

MATH 4441 HW 8

Wenqi He

October 17, 2018

10.12

By Leibniz's rule, $D_i \langle F, G \rangle = \langle D_i F, G \rangle + \langle F, D_i G \rangle = 0$, so $\langle D_i F, G \rangle = -\langle F, D_i G \rangle$

10.13

Apply a rigid motion so that p coincides with the origin and T_pM coincides with the x-y plane, then we can use a Monge patch around p, with the mapping

$$X(u^1, u^2) = (u^1, u^2, f(u^1, u^2))$$

Since the surface is tangent to x-y plane at the origin, $D_1 f(0,0) = D_2 f(0,0) = 0$. So

$$D_1 X(0,0) = (1, 0, D_1 f(0,0)) = (1,0,0) := e_1$$

$$D_2 X(0,0) = (0, 1, D_2 f(0,0)) = (1,0,0) := e_2$$

$$g_{ij}(0,0) = \langle D_i X(0,0), D_j X(0,0) \rangle = \langle e_i, e_j \rangle = \delta_{ij}$$

Therefore $[g_{ij}(0,0)] = [\delta_{ij}] = I$

10.15

For any γ, since γ lies on the surfce, $\langle \gamma'(t), n(\gamma(t)) \rangle = 0$. Taking the derivative at $t = 0$,

$$\langle \gamma'(t), n(\gamma(t)) \rangle' |_{t=0} = \langle \gamma''(0), n(p) \rangle + \langle \gamma'(0), (n \circ \gamma)'(0)) \rangle = 0$$

Therefore

$$\langle \gamma''(0), n(p) \rangle = -\langle \gamma'(0), (n \circ \gamma)'(0)) \rangle = \boxed{-\langle v, dn_p(v)) \rangle}$$

which is independent of γ.

10.17

$$\mathrm{II}_p(v, w) = \langle S_p(v), w \rangle = \left\langle \sum_i v_i S_p(e_i), \sum_j w_j e_j \right\rangle = \sum_{i,j} v_i w_j \langle S_p(e_i), e_j \rangle = \sum_{i,j} v_i w_j l_{ij}$$

Similarly

$$\mathrm{II}_p(w,v) = \sum_{i,j} v_i w_j l_{ji}$$

l_{ij} is symmetric since mixed derivatives are interchagable:

$$l_{ij} = \langle D_{ij}, N \rangle = \langle D_{ji}, N \rangle = l_{ji}$$

Therefore, $\mathrm{II}_p(v,w) = \mathrm{II}_p(w,v)$.

10.18

1

By bilinearity and symmetry of II_p

$$\begin{aligned}\mathrm{II}_p(v,v) &= \mathrm{II}_p(\cos\theta e_1 + \sin\theta e_2, \cos\theta e_1 + \sin\theta e_2) \\ &= \cos^2\theta\, \mathrm{II}_p(e_1,e_1) + 2\cos\theta\sin\theta\, \mathrm{II}_p(e_1,e_2) + \sin^2\theta\, \mathrm{II}_p(e_2,e_2)\end{aligned}$$

Since we have chosen the unit eigenvectors of S_p to be the basis vectors,

$$\mathrm{II}_p(e_1,e_1) = \langle S_p(e_1), e_1 \rangle = \langle \lambda_1 e_1, e_1 \rangle = \lambda_1, \text{ similarly, } \mathrm{II}_p(e_2,e_2) = \lambda_2$$

$$\mathrm{II}_p(e_1,e_2) = \mathrm{II}_p(e_2,e_1) = \langle \lambda_1 e_1, e_2 \rangle = 0$$

Plug the values into the first equation,

$$\mathrm{II}_p(v,v) = \cos^2\theta\lambda_1 + \sin^2\theta\lambda_2$$

2

$$\frac{d}{d\theta}\mathrm{II}_p(v,v) = -2\cos\theta\sin\theta\lambda_1 + 2\sin\theta\cos\theta\lambda_2 = 2(\lambda_2 - \lambda_1)\sin\theta\cos\theta$$

Since $\lambda_1 \neq \lambda_2$, the derivative vanishes when

$$\cos\theta = 0, \sin\theta = \pm 1 \text{ or } \sin\theta = 0, \cos\theta = \pm 1$$

which correspond to the extrema

$$\min \mathrm{II}_p(v,v) = 1\lambda_1 + 0\lambda_2 = \lambda_1$$

$$\max \mathrm{II}_p(v,v) = 0\lambda_1 + 1\lambda_2 = \lambda_2$$

MATH 4441 Homework 9

Wenqi He

October 25, 2018

12.1

Since $[g_{ij}][g^{ij}] = I$, $\det[g_{ij}]\det[g^{ij}] = \det I = 1$

$$\det[g^{ij}] = \frac{1}{\det[g_{ij}]}$$

And since $[l_i^j] = [l_{ij}][g^{ij}]$,

$$\det[l_i^j] = \det[l_{ij}]\det[g^{ij}] = \frac{\det[l_{ij}]}{\det[g_{ij}]} = K$$

which is the Gaussian curvature.

12.2

$$\begin{aligned} N_i &= -\sum_j l_i^j X_j = -\sum_{j,k} l_{ik} g^{kj} X_j = -\sum_{j,k,l} S_{il} g_{lk} g^{kj} X_j = -\sum_{j,l} S_{il}\delta_l^j X_j \\ &= -\sum_j S_{ij} X_j = -S(X_i) = dn(X_i) \end{aligned}$$

12.3

The map for a surface of revolution is

$$X(t,\theta) = (x(t)\cos\theta, x(t)\sin\theta, y(t))$$

$$D_1X(t,\theta) = (x'(t)\cos\theta, x'(t)\sin\theta, y'(t)), \quad D_2X(t,\theta) = (-x(t)\sin\theta, x(t)\cos\theta, 0)$$

$$g_{11} = \langle D_1X, D_1X\rangle = x'^2 + y'^2, \quad g_{12} = g_{21} = \langle D_1X, D_2X\rangle = 0, \quad g_{22} = \langle D_2X, D_2X\rangle = x^2$$

$$[g_{ij}] = \begin{pmatrix} x'^2 + y'^2 & 0 \\ 0 & x^2 \end{pmatrix}, \quad [g^{ij}] = \begin{pmatrix} 1/(x'^2 + y'^2) & 0 \\ 0 & 1/x^2 \end{pmatrix}$$

Only the diagonal g^{ij} are non-zero, so in the formula for Γ, the term where $l \neq k$ vanishes:

$$\Gamma_{ij}^k = \frac{1}{2}\Big((g_{ki})_j + (g_{jk})_i - (g_{ij})_k\Big) g^{kk}$$

The only non-zero derivatives of g_{ij} are

$$(g_{11})_1 = 2x'x'' + 2y'y'', \quad (g_{22})_1 = 2xx'$$

Therefore, the only non-zero Christoffel symbols are

$$\Gamma^1_{11} = \frac{1}{2}(g_{11})_1 g^{11} = \frac{x'x'' + y'y''}{x'^2 + y'^2}$$

$$\Gamma^1_{22} = -\frac{1}{2}(g_{22})_1 g^{11} = -\frac{xx'}{x'^2 + y'^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}(g_{22})_1 g^{22} = \frac{x'}{x}$$

12.4

If $M = \mathbb{R}^2$, then we can use a trivial patch $(\mathbb{R}^2, id)$ for the surface, which means that

$$X(u^1, u^2) = (u^1, u^2)$$

$$D_1 X = (1, 0), \quad D_2 X = (0, 1)$$

$$[g_{ij}] = \begin{pmatrix} \langle D_1X, D_1X\rangle & \langle D_1X, D_2X\rangle \\ \langle D_2X, D_1X\rangle & \langle D_2X, D_2X\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since g_{ij} are all constants, all derivatives vanish, therefore all Γ^k_{ij} are zero. Since Γ^k_{ij} are zero, their derivatives must also be zero. Thus, by the intrinsic definition of Riemann curvature tensor, $R^l_{ijk} = 0$. Extrinsically, since $D_i X$ are all constants, $D_{ij}X = 0$, therefore $l_{ij} = \langle D_{ij}X, N\rangle = 0$, then according to Gauss's equation, we also have $R^l_{ijk} = 0$.

12.6

Let $U = (0, 2\pi) \times (0, \pi)$, and define the mapping $X : U \to \mathbb{S}^2$ as

$$X(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$D_1X(\theta,\phi) = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0), \quad D_2X(\theta,\phi) = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$$

$$g_{11} = \langle D_1X, D_1X\rangle = \sin^2\phi, \quad g_{12} = g_{21} = \langle D_1X, D_2X\rangle = 0, \quad g_{22} = \langle D_2X, D_2X\rangle = 1$$

$$[g_{ij}] = \begin{pmatrix} \sin^2\phi & 0 \\ 0 & 1 \end{pmatrix}, \quad [g^{ij}] = \begin{pmatrix} 1/\sin^2\phi & 0 \\ 0 & 1 \end{pmatrix}$$

The only non-zero derivative of g_{ij} is

$$(g_{11})_2 = 2\sin\phi\cos\phi$$

Therefore,

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = \Gamma^2_{22} = 0$$

$$\Gamma^2_{11} = -\frac{1}{2}(g_{11})_2 g^{22} = -\sin\phi\cos\phi$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}(g_{11})_2 g^{11} = \frac{\cos\phi}{\sin\phi}$$

The R^l_{ijk} where $j \neq k$ are:

$$R^1_{112} = R^1_{121} = R^2_{212} = R^2_{221} = 0$$

$$R^2_{112} = -(\Gamma^2_{11})_2 + \Gamma^1_{12}\Gamma^2_{11} = -\sin^2\phi, \quad R^2_{121} = -R^2_{112} = \sin^2\phi$$
$$R^1_{212} = -(\Gamma^1_{21})_2 - \Gamma^1_{21}\Gamma^1_{12} = 1, \quad R^1_{221} = -R^1_{212} = -1$$

Combining above results:

$$\boxed{[R^2_{1ij}] = \begin{pmatrix} 0 & -\sin^2\phi \\ \sin^2\phi & 0 \end{pmatrix}, \quad [R^1_{2ij}] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

Now compute extrinsically. On $\mathbb{S}^2$, $n = id$, so $N = n \circ X = X$,

$$N(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$D_{11}X(\theta,\phi) = (-\sin\phi\cos\theta, -\sin\phi\sin\theta, 0)$$
$$D_{12}X(\theta,\phi) = D_{21}X(\theta,\phi) = (-\cos\phi\sin\theta, \cos\phi\cos\theta, 0)$$
$$D_{22}X(\theta,\phi) = (-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi)$$
$$l_{11} = \langle D_{11}X, N\rangle = -\sin^2\phi, \quad l_{12} = l_{21} = \langle D_{12}X, N\rangle = 0, \quad l_{22} = \langle D_{22}X, N\rangle = -1$$
$$[l^j_i] = [l_{ij}][g^{ij}] = \begin{pmatrix} -\sin^2\phi & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/\sin^2\phi & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since l_{ij} and l^j_i are both diagonal, $l_{ik}l^l_j - l_{ij}l^l_k$ is only non-zero when either (i) $i = k$ and $l = j$ or (ii) $i = j$ and $l = k$. The corresponding components are:

$$R^1_{221} = -l_{22}l^1_1 = -(-1)(-1) = -1, \quad R^1_{212} = 1$$

$$R^2_{112} = -l_{11}l^2_2 = -(-\sin^2\phi)(-1) = -\sin^2\phi, \quad R^2_{121} = \sin^2\phi$$

which match previous results.

MATH 4441 Homework 10

Wenqi He

October 31, 2018

13.3

i

Since differential is a linear map,

$$\begin{aligned}\Big((W+V)f\Big)_p &= (W+V)_p f = df_p(W+V)\\ &= df_p(W) + df_p(V) = W_p f + V_p f\\ &= \Big(Wf + Vf\Big)_p\end{aligned}$$

$$\begin{aligned}\Big((gW)f\Big)_p &= (gW)_p f = (g(p)W_p)f = df_p(g(p)W)\\ &= g(p)\cdot df_p(W) = g(p)W_p f\\ &= \Big(g(Wf)\Big)_p\end{aligned}$$

From the result of Ex.2, $\overline{\nabla}_W V = (WV^1, \cdots, WV^n)$. Therefore,

$$\begin{aligned}\overline{\nabla}_{W+Z} V &= ((W+Z)V^1, \cdots, (W+Z)V^n)\\ &= (WV^1, \cdots, WV^n) + (ZV^1, \cdots, ZV^n)\\ &= \overline{\nabla}_W V + \overline{\nabla}_Z V\end{aligned}$$

$$\begin{aligned}\overline{\nabla}_{fW} V &= ((fW)V^1, \cdots, (fW)V^n)\\ &= (f(WV^1), \cdots, f(WV^n))\\ &= f(WV^1, \cdots, WV^n)\\ &= f\,\overline{\nabla}_W V\end{aligned}$$

ii

Suppose $\gamma(0) = p$, $\gamma'(0) = W_p$. From the linearity of derivatives.

$$\begin{aligned}\Big(W(f+g)\Big)_p &= W_p(f+g) = ((f+g)\circ\gamma)'(0)\\ &= (f\circ\gamma)'(0) + (g\circ\gamma)'(0)\\ &= W_p f + W_p g\\ &= \Big(Wf + Wg\Big)_p\end{aligned}$$

$$
\begin{aligned}
\Big(W(fg)\Big)_p &= W_p(fg) = ((fg)\circ\gamma)'(0) \\
&= D(fg)(p)\gamma'(0) \\
&= \sum_i D_i(fg)(p){\gamma^i}'(0) \\
&= \sum_i D_i f(p) g(p){\gamma^i}'(0) + \sum_i f(p) D_i g(p){\gamma^i}'(0) \\
&= g(p)\sum_i D_i f(p){\gamma^i}'(0) + f(p)\sum_i D_i g(p){\gamma^i}'(0) \\
&= (W_p f)g(p) + f(p)(W_p g) \\
&= \Big((Wf)g + f(Wg)\Big)_p
\end{aligned}
$$

Now applying these identities,

$$
\begin{aligned}
\overline{\nabla}_W(V+Z) &= (W(V^1+Z^1),\cdots,W(V^n+Z^n)) \\
&= (WV^1,\cdots,WV^n) + (WZ^1,\cdots,WZ^n) \\
&= \overline{\nabla}_W V + \overline{\nabla}_W Z
\end{aligned}
$$

$$
\begin{aligned}
\overline{\nabla}_W(fV) &= (W(fV^1),\cdots,W(fV^n)) \\
&= ((Wf)V^1,\cdots,(Wf)V^n) + (f(WV^1),\cdots,f(WV^n)) \\
&= Wf(V^1,\cdots,V^n) + f(WV^1,\cdots,WV^n) \\
&= (Wf)V + f\,\overline{\nabla}_W V
\end{aligned}
$$

13.4

$$
\begin{aligned}
\Big(Z\langle V,W\rangle\Big)_p &= Z_p\langle V,W\rangle = Z_p\sum_i V^iW^i = \sum_i Z_p\left(V^iW^i\right) \\
&= \sum_i (Z_pV^i)W^i + \sum_i V^i(Z_pW^i) \\
&= \sum_i (\overline{\nabla}_{Z_p} V)^iW^i + \sum_i V^i(\overline{\nabla}_{Z_p} W)^i \\
&= \Big\langle \overline{\nabla}_{Z_p} V, W\Big\rangle + \Big\langle V, \overline{\nabla}_{Z_p} W\Big\rangle \\
&= \Big(\Big\langle \overline{\nabla}_Z V, W\Big\rangle + \Big\langle V, \overline{\nabla}_Z W\Big\rangle\Big)_p
\end{aligned}
$$

13.6

From Ex.2, $\overline{\nabla}_W Z = (WZ^1,\cdots,WZ^n)$,

$$\overline{\nabla}_V \overline{\nabla}_W Z = (V(WZ^1),\cdots,V(WZ^n))$$

Similarly,

$$\overline{\nabla}_W \overline{\nabla}_V Z = (W(VZ^1),\cdots,W(VZ^n))$$

And from Ex.5

$$\begin{aligned}\overline{\nabla}_{[V,W]}\, Z &= ([V,W]Z^1, \cdots, [V,W]Z^n) \\ &= (V(WZ^1), \cdots, V(WZ^n)) - (W(VZ^1), \cdots, W(VZ^n))\end{aligned}$$

The three terms cancel out, therefore

$$\overline{R}(V,W)Z = \overline{\nabla}_V\, \overline{\nabla}_W\, Z - \overline{\nabla}_W\, \overline{\nabla}_V\, Z - \overline{\nabla}_{[V,W]}\, Z \equiv \mathbf{0}$$

$$\overline{R}(V,W,Z,Y) = \langle \overline{R}(V,W)Z, Y\rangle \equiv 0$$

13.7

By definition,

$$\nabla_W V = (\overline{\nabla}_W\, V)^\top = \overline{\nabla}_W\, V - (\overline{\nabla}_W\, V)^\perp = \overline{\nabla}_W\, V - \langle \overline{\nabla}_W\, V, n\rangle n$$

i

$$\begin{aligned}\nabla_{W+Z} V &= \overline{\nabla}_{W+Z}\, V - \langle \overline{\nabla}_{W+Z}\, V, n\rangle n \\ &= \overline{\nabla}_W\, V + \overline{\nabla}_Z\, V - \langle \overline{\nabla}_W\, V + \overline{\nabla}_Z\, V, n\rangle n \\ &= \left[\overline{\nabla}_W\, V - \langle \overline{\nabla}_W\, V, n\rangle n\right] + \left[\overline{\nabla}_Z\, V - \langle \overline{\nabla}_Z\, V, n\rangle n\right] \\ &= \nabla_W V + \nabla_Z V\end{aligned}$$

ii

$$\begin{aligned}\nabla_{fW} V &= \overline{\nabla}_{fW}\, V - \langle \overline{\nabla}_{fW}\, V, n\rangle n \\ &= f\, \overline{\nabla}_W\, V - \langle f\, \overline{\nabla}_W\, V, n\rangle n \\ &= f\, \overline{\nabla}_W\, V - f\langle \overline{\nabla}_W\, V, n\rangle n \\ &= f\nabla_W V\end{aligned}$$

iii

$$\begin{aligned}\nabla_W (V+Z) &= \overline{\nabla}_W (V+Z) - \langle \overline{\nabla}_W (V+Z), n\rangle n \\ &= \overline{\nabla}_W\, V + \overline{\nabla}_W\, Z - \langle \overline{\nabla}_W\, V + \overline{\nabla}_W\, Z, n\rangle n \\ &= \left[\overline{\nabla}_W\, V - \langle \overline{\nabla}_W\, V, n\rangle n\right] + \left[\overline{\nabla}_W\, Z - \langle \overline{\nabla}_W\, Z, n\rangle n\right] \\ &= \nabla_W V + \nabla_W Z\end{aligned}$$

iv

$$\begin{aligned}
\nabla_W(fV) &= \overline{\nabla}_W(fV) - \langle \overline{\nabla}_W(fV), n\rangle n \\
&= (Wf)V + f\,\overline{\nabla}_W\,V - \langle (Wf)V + f\,\overline{\nabla}_W\,V, n\rangle n \\
&= \Big[(Wf)V - (Wf)\langle V, n\rangle n\Big] + \Big[f\,\overline{\nabla}_W\,V - f\langle \overline{\nabla}_W\,V, n\rangle n\Big] \\
&= Wf\Big[V - \langle V, n\rangle n\Big] + f\Big[\overline{\nabla}_W\,V - \langle \overline{\nabla}_W\,V, n\rangle n\Big] \\
&= (Wf)V + f\Big[\overline{\nabla}_W\,V - \langle \overline{\nabla}_W\,V, n\rangle n\Big] \\
&= (Wf)V + f\nabla_W V
\end{aligned}$$

v

$$\begin{aligned}
&\langle \nabla_Z V, W\rangle + \langle V, \nabla_Z W\rangle \\
&= \Big\langle \Big(\overline{\nabla}_Z\,V - \langle \overline{\nabla}_Z\,V, n\rangle n\Big), W\Big\rangle + \Big\langle V, \Big(\overline{\nabla}_Z\,W - \langle \overline{\nabla}_Z\,W, n\rangle n\Big)\Big\rangle \\
&= \Big\langle \overline{\nabla}_Z\,V, W\Big\rangle + \Big\langle V, \overline{\nabla}_Z\,W\Big\rangle - \Big\langle \langle \overline{\nabla}_Z\,V, n\rangle n, W\Big\rangle - \Big\langle V, \langle \overline{\nabla}_Z\,W, n\rangle n\Big\rangle \\
&= \Big\langle \overline{\nabla}_Z\,V, W\Big\rangle + \Big\langle V, \overline{\nabla}_Z\,W\Big\rangle = Z\langle V, W\rangle
\end{aligned}$$

The last two terms vanish because W, V are in the tangent space but n is a normal vector.

13.10

The equation becomes Gauss's formula in local coordinates when $W = \overline{X}_j$, $V = \overline{X}_i$. Define $u_j : (-\epsilon, \epsilon) \to \mathbb{R}^2$ as $u_j(t) = tE_j$, then the LHS is

$$\begin{aligned}
\overline{\nabla}_{(\overline{X}_j)_p}\,\overline{X}_i &= \Big(\overline{X}_i \circ (X \circ u_j)\Big)'(0) = \Big(X_i \circ X^{-1} \circ X \circ u_j\Big)'(0) \\
&= \Big(X_i \circ u_j\Big)'(0) = DX_i(0,0)E_j = X_{ij}(0,0)
\end{aligned}$$

and

$$\begin{aligned}
RHS &= \nabla_{(\overline{X}_j)_p}\overline{X}_i + \Big\langle \overline{X}_i(p), S(\overline{X}_j(p))\Big\rangle n(p) \\
&= \sum_k \overline{\Gamma}^k_{ij}(p)\overline{X}_k(p) + \Big\langle \overline{X}_i(p), S(\overline{X}_j(p))\Big\rangle n(p) \\
&= \sum_k \Gamma^k_{ij}(0,0)X_k(0,0) + \Big\langle X_i(0,0), S(X_j(0,0))\Big\rangle N(0,0) \\
&= \sum_k \Gamma^k_{ij}(0,0)X_k(0,0) + l_{ij}(0,0)N(0,0)
\end{aligned}$$

Combining both sides yields Gauss's formulas in local coordinates:

$$X_{ij} = \sum_k \Gamma^k_{ij}X_k + l_{ij}N$$

MATH 4441 Homework 11

Wenqi He

November 6, 2018

14.2

In local coordinates, W, V can be written as

$$V = \sum_j V^j \overline{X}_j, \quad W = \sum_k W^k \overline{X}_k, \quad Z = \sum_i Z^i \overline{X}_i$$

i

$$\begin{aligned}
LHS = R(V,W)Z &= R(\sum_j V^j \overline{X}_j, \sum_k W^k \overline{X}_k) \sum_i Z^i \overline{X}_i \\
&= \sum_{i,j,k} V^j W^k Z^i \Big(R(\overline{X}_j, \overline{X}_k)\overline{X}_i \Big) \\
&= \sum_{i,j,k,l} V^j W^k Z^i R^l_{ijk} \overline{X}_l
\end{aligned}$$

$$\begin{aligned}
RHS &= \langle S(W), Z\rangle S(V) - \langle S(V), Z\rangle S(W) \\
&= \langle S(\sum_k W^k \overline{X}_k), \sum_i Z^i \overline{X}_i\rangle S(\sum_j V^j \overline{X}_j) - \langle S(\sum_j V^j \overline{X}_j), \sum_i Z^i \overline{X}_i\rangle S(\sum_k W^k \overline{X}_k) \\
&= \sum_{i,j,k} V^j W^k Z^i \Big[\langle S(\overline{X}_k), \overline{X}_i\rangle S(\overline{X}_j) - \langle S(\overline{X}_j), \overline{X}_i\rangle S(\overline{X}_k) \Big] \\
&= \sum_{i,j,k} V^j W^k Z^i \Big(l_{ik} l^l_j - l_{ij} l^l_k \Big) \overline{X}_l
\end{aligned}$$

Since the equation holds for arbitrary V, W, Z,

$$R^l_{ijk} = l_{ik} l^l_j - l_{ij} l^l_k$$

which is Gauss's equation in local coordinates.

ii

$$\nabla_V S(W) - \nabla_W S(V) = S([V, W])$$

$$
\begin{aligned}
LHS &= \nabla_V S(W) - \nabla_W S(V) \\
&= \nabla_{\sum_j V^j \overline{X}_j} S(\sum_k W^k \overline{X}_k) - \nabla_{\sum_k W^k \overline{X}_k} S(\sum_j V^j \overline{X}_j) \\
&= \sum_j V^j \nabla_{\overline{X}_j} \sum_k W^k S(\overline{X}_k) - \sum_k W^k \nabla_{\overline{X}_k} \sum_j V^j S(\overline{X}_j) \\
&= \sum_{j,k} V^j \nabla_{\overline{X}_j} \Big(W^k S(\overline{X}_k) \Big) - W^k \nabla_{\overline{X}_k} \Big(V^j S(\overline{X}_j) \Big) \\
&= \sum_{j,k} V^j (\overline{X}_j W^k) S(\overline{X}_k) + V^j W^k \nabla_{\overline{X}_j} S(\overline{X}_k) - W^k (\overline{X}_k V^j) S(\overline{X}_j) - W^k V^j \nabla_{\overline{X}_k} S(\overline{X}_j)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= S([V,W]) \\
&= S(\nabla_V W - \nabla_W V) \\
&= \sum_{j,k} S\Big(V^j \nabla_{\overline{X}_j} \Big(W^k \overline{X}_k \Big) - W^k \nabla_{\overline{X}_k} \Big(V^j \overline{X}_j \Big) \Big) \\
&= \sum_{j,k} S\Big(V^j (\overline{X}_j W^k) \overline{X}_k + V^j W^k \nabla_{\overline{X}_j} \overline{X}_k - W^k (\overline{X}_k V^j) \overline{X}_j - W^k V^j \nabla_{\overline{X}_k} \overline{X}_j \Big) \\
&= \sum_{j,k} V^j (\overline{X}_j W^k) S(\overline{X}_k) + V^j W^k S(\nabla_{\overline{X}_j} \overline{X}_k) - W^k (\overline{X}_k V^j) S(\overline{X}_j) - W^k V^j S(\nabla_{\overline{X}_k} \overline{X}_j)
\end{aligned}
$$

Canceling terms on both sides, the equation can be simplied to

$$
\nabla_{\overline{X}_j} S(\overline{X}_k) - \nabla_{\overline{X}_k} S(\overline{X}_j) = S(\nabla_{\overline{X}_j} \overline{X}_k - \nabla_{\overline{X}_k} \overline{X}_j) = S(0) = 0
$$

Taking inner product with $\overline{X}_i$:

$$
\langle \nabla_{\overline{X}_j} S(\overline{X}_k), \overline{X}_i \rangle - \langle \nabla_{\overline{X}_k} S(\overline{X}_j), \overline{X}_i \rangle = 0
$$

$$
\begin{aligned}
\langle \nabla_{\overline{X}_k} S(\overline{X}_j), \overline{X}_i \rangle &= \overline{X}_k \langle S(\overline{X}_j), \overline{X}_i \rangle - \langle S(\overline{X}_j), \nabla_{\overline{X}_k} \overline{X}_i \rangle \\
&= (l_{ij})_k - \langle S(\overline{X}_j), \sum_l \Gamma^l_{ik} \overline{X}_l \rangle \\
&= \sum_l (l_{ij})_k - \Gamma^l_{ik} \langle S(\overline{X}_j), \overline{X}_l \rangle \\
&= \sum_l (l_{ij})_k - \Gamma^l_{ik} l_{jl}
\end{aligned}
$$

Swapping j and k:

$$
\langle \nabla_{\overline{X}_j} S(\overline{X}_k), \overline{X}_i \rangle = \sum_l (l_{ik})_j - \Gamma^l_{ij} l_{kl}
$$

Taking the difference of two terms:

$$
\sum_l \Gamma^l_{ik} l_{jl} - \Gamma^l_{ij} l_{kl} = (l_{ij})_k - (l_{ik})_j
$$

which is Codazzi-Mainardi equation in local coordinates.

14.3

Let $E_1 = V$ and $E_2 = W$, then by Gauss's equation

$$\begin{aligned} R(V,W,W,V) &= \langle R(V,W)W, V\rangle \\ &= \langle S(V), V\rangle\langle S(W), W\rangle - \langle S(W), V\rangle\langle S(V), W\rangle \\ &= l_{11}l_{22} - (l_{12})^2 = \det[l_{ij}] \end{aligned}$$

By Lagrange's identity

$$\begin{aligned} \|V \times W\|^2 &= \|V\|^2\|W\|^2 - \langle V, W\rangle^2 \\ &= \langle V, V\rangle\langle W, W\rangle - \langle V, W\rangle^2 \\ &= g_{11}g_{22} - (g_{12})^2 = \det[g_{ij}] \end{aligned}$$

By definition of Gaussian curvature

$$K = \frac{\det[l_{ij}]}{\det[g_{ij}]} = \frac{R(V,W,W,V)}{\|V \times W\|^2}$$

15.1

i

Without loss of generality, we can consider the equator of a sphere. For a sphere of radius R centered at the origin, a unit speed curve along the equator is

$$\alpha(t) = (R\cos(t/R), R\sin(t/R))$$

$$\alpha'(t) = (-\sin(t/R), \cos(t/R))$$

$$\alpha''(t) = (-\frac{1}{R}\cos(t/R), -\frac{1}{R}\sin(t/R)) = -\frac{1}{R^2}\alpha(t) = -\frac{1}{R}n(t)$$

Therefore,

$$\alpha''^{\top} = 0 \quad \Rightarrow \quad |K_g| = 0$$

ii

For a cylinder of radius R around z-axis, a unit speed helix curve is

$$\alpha(t) = \Big(R\cos\frac{t}{\sqrt{v^2+R^2}}, R\sin\frac{t}{\sqrt{v^2+R^2}}, \frac{v}{\sqrt{v^2+R^2}}t\Big)$$

$$\alpha'(t) = \Big(-\frac{R}{\sqrt{v^2+R^2}}\sin\frac{t}{\sqrt{v^2+R^2}}, \frac{R}{\sqrt{v^2+R^2}}\cos\frac{t}{\sqrt{v^2+R^2}}, \frac{v}{\sqrt{v^2+R^2}}\Big)$$

$$\alpha''(t) = \Big(-\frac{R}{v^2+R^2}\cos\frac{t}{\sqrt{v^2+R^2}}, -\frac{R}{v^2+R^2}\sin\frac{t}{\sqrt{v^2+R^2}}, 0\Big) = -\frac{R}{v^2+R^2}n(t)$$

Therefore,

$$\alpha''^{\top} = 0 \quad \Rightarrow \quad |K_g| = 0$$

15.3

The radius of the circle at $z = h$ is $a = \sqrt{1-h^2}$. The parametrization of the circle w.r.t. arclength is

$$\alpha(t) = (a\cos(t/a), a\sin(t/a), h) = n(t)$$

$$\alpha'(t) = (-\sin(t/a), \cos(t/a), 0)$$

$$\alpha''(t) = \Big(-\frac{1}{a}\cos(t/a), -\frac{1}{a}\sin(t/a), 0\Big)$$

$$\begin{aligned} J\alpha'(t) &= n(t) \times \alpha'(t) \\ &= (a\cos(t/a), a\sin(t/a), h) \times (-\sin(t/a), \cos(t/a), 0) \\ &= \begin{vmatrix} i & j & k \\ a\cos(t/a) & a\sin(t/a) & h \\ -\sin(t/a) & \cos(t/a) & 0 \end{vmatrix} \\ &= (-h\cos(t/a), -h\sin(t/a), a) \end{aligned}$$

$$\kappa_g = \langle \alpha'', J\alpha' \rangle = \frac{h}{a} = \boxed{\frac{h}{\sqrt{1-h^2}}}$$

15.5

Suppose α is a geodesic, then $\nabla_{\alpha'}\tilde{\alpha}' \equiv 0$.

$$\alpha'(t)\|\tilde{\alpha}'\|^2 = \alpha'(t)\langle \tilde{\alpha}', \tilde{\alpha}' \rangle = 2\langle \nabla_{\alpha'(t)}\tilde{\alpha}', \tilde{\alpha}' \rangle = 0$$

which means that the $\|\tilde{\alpha}'\|$ is constant.

MATH 4441 Homework 12

Wenqi He

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15.2

Let $E_1 = (1, 0, 0), \quad E_2 = (0, 1, 0)$, then

$$JE_1 = n \times E_1 = (0, 0, 1) \times (1, 0, 0) = (0, 1, 0)$$

$$JE_2 = n \times E_1 = (0, 0, 1) \times (0, 1, 0) = (-1, 0, 0)$$

Therefore, in $\mathbb{R}^2$, w.r.t. E_1, E_2,

$$J = \begin{pmatrix} JE_1 & JE_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R(\pi/2)$$

15.4

i

$$\begin{aligned}
\Big(\frac{1}{\|\alpha'(s^{-1}(t))\|}\Big)' &= \Big(\Big\langle \alpha'(s^{-1}(t)), \alpha'(s^{-1}(t))\Big\rangle^{-1/2}\Big)' \\
&= -\frac{1}{2}\Big\langle \alpha'(s^{-1}(t)), \alpha'(s^{-1}(t))\Big\rangle^{-3/2}\Big\langle \alpha'(s^{-1}(t)), \alpha'(s^{-1}(t))\Big\rangle' \\
&= -\Big\langle \alpha'(s^{-1}(t)), \alpha'(s^{-1}(t))\Big\rangle^{-3/2}\Big\langle \Big(\alpha'(s^{-1}(t))\Big)', \alpha'(s^{-1}(t))\Big\rangle \\
&= -\frac{1}{\|\alpha'(s^{-1})\|^3}\Big\langle \frac{\alpha''(s^{-1})}{\|\alpha'(s^{-1})\|}, \alpha'(s^{-1})\Big\rangle \\
&= -\frac{1}{\|\alpha'(s^{-1})\|^3}\Big\langle \frac{\alpha''(s^{-1})}{\|\alpha'(s^{-1})\|}, \alpha'(s^{-1})\Big\rangle \\
&= \frac{-\Big\langle \alpha''(s^{-1}), \alpha'(s^{-1})\Big\rangle}{\|\alpha'(s^{-1})\|^4}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{\alpha}'' &= \Big(\alpha'(s^{-1}(t))\Big)' \cdot \frac{1}{\|\alpha'(s^{-1}(t))\|} + \alpha'(s^{-1}(t)) \cdot \Big(\frac{1}{\|\alpha'(s^{-1}(t))\|}\Big)' \\
&= \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'(s^{-1}) \cdot \frac{-\Big\langle \alpha''(s^{-1}), \alpha'(s^{-1})\Big\rangle}{\|\alpha'(s^{-1})\|^4}
\end{aligned}$$

ii

$$\kappa_g = \overline{\kappa}_g = \langle \overline{\alpha}'', J\overline{\alpha}' \rangle = \left\langle \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \lambda\alpha'(s^{-1}),\ J\alpha'(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|} \right\rangle$$

Since $\langle \alpha', J\alpha' \rangle = 0$,

$$\kappa_g = \left\langle \alpha'' \cdot \frac{1}{\|\alpha'\|^2},\ J\alpha' \cdot \frac{1}{\|\alpha'\|} \right\rangle = \frac{\langle \alpha'', J\alpha' \rangle}{\|\alpha'\|^3}$$

15.6

Suppose α is parametrized by arclength, then $\alpha' = T$, $\|\alpha'\| = \|J\alpha'\| = 1$, $\kappa_g = \langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \rangle$. Expanding $\nabla_{\alpha'}\tilde{\alpha}'$ in the orthonormal frame $\{\tilde{\alpha}', J\tilde{\alpha}'\}$:

$$\begin{aligned} \nabla_{\alpha'}\tilde{\alpha}' &= \langle \nabla_{\alpha'}\tilde{\alpha}', \tilde{\alpha}' \rangle \tilde{\alpha}' + \langle \nabla_{\alpha'}\tilde{\alpha}', J\tilde{\alpha}' \rangle J\tilde{\alpha}' \\ &= \langle \nabla_{\alpha'}\tilde{\alpha}', \tilde{\alpha}' \rangle \tilde{\alpha}' + \kappa_g J\tilde{\alpha}' \end{aligned}$$

Since α is a unit speed curve, $\|\tilde{\alpha}'\|^2 \equiv const$,

$$\alpha'\|\tilde{\alpha}'\|^2 = \alpha'\langle \tilde{\alpha}', \tilde{\alpha}' \rangle = 2\langle \nabla_{\alpha'}\tilde{\alpha}', \tilde{\alpha}' \rangle = 0$$

Therefore,

$$\nabla_{\alpha'}\tilde{\alpha}' = \kappa_g J\tilde{\alpha}' \quad \Rightarrow \quad \|\nabla_{\alpha'}\tilde{\alpha}'\| = \|\kappa_g J\tilde{\alpha}'\| = |\kappa_g|$$

15.7

In the backward direction: If $\nabla_{\alpha'}\tilde{\alpha}' \equiv 0$, then obviously $\langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \rangle \equiv 0 \Rightarrow \kappa_g \equiv 0$, so the curve is a geodesic. In the forward direction: If the curve is a geodesic, then

$$\kappa_g \equiv 0 \Rightarrow \langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \rangle = \|\alpha'\|\langle \nabla_{\alpha'}\tilde{\alpha}', JT \rangle \equiv 0$$

$$\Rightarrow \langle \nabla_{\alpha'}\tilde{\alpha}', JT \rangle \equiv 0$$

From Ex.5, α must have constant speed, which means that

$$\alpha'\|\tilde{\alpha}'\|^2 = \alpha'\langle \tilde{\alpha}', \tilde{\alpha}' \rangle = 2\langle \nabla_{\alpha'}\tilde{\alpha}', \tilde{\alpha}' \rangle = 2\|\tilde{\alpha}'\|\langle \nabla_{\alpha'}\tilde{\alpha}', T \rangle \equiv 0$$

$$\Rightarrow \langle \nabla_{\alpha'}\tilde{\alpha}', T \rangle \equiv 0$$

Since $\nabla_{\alpha'}\tilde{\alpha}'$ lies in the tangent space, using the orthonormal frame $\{T, JT\}$

$$\begin{aligned} \nabla_{\alpha'}\tilde{\alpha}' &= \langle \nabla_{\alpha'}\tilde{\alpha}', T \rangle T + \langle \nabla_{\alpha'}\tilde{\alpha}', JT \rangle JT \\ &= 0 + 0 = 0 \end{aligned}$$

15.8

From Ex.12.3, the only non-vanishing Christoffel symbols for a surface of revolution with patch X defined as $X(t, \theta) = (x(t)\cos\theta, x(t)\sin\theta, y(t))$ are

$$\Gamma^1_{11} = \frac{x'x'' + y'y''}{x'^2 + y'^2}, \quad \Gamma^1_{22} = -\frac{xx'}{x'^2 + y'^2}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{x'}{x}$$

Because the first coordinate is named t, and the prime notation is already used to denote derivatives w.r.t variable t, here I use dot notation to denote derivatives w.r.t parameter λ for curve $\alpha(\lambda)$.

$$\ddot{t} + \frac{x'x'' + y'y''}{x'^2 + y'^2}\dot{t}^2 - \frac{xx'}{x'^2 + y'^2}\dot{\theta}^2 = 0 \tag{1}$$

$$\ddot{\theta} + 2\frac{x'}{x}\dot{t}\dot{\theta} = 0 \tag{2}$$

If the surface is a sphere of radius R, then

$$x(t) = R\cos t, \quad y(t) = R\sin t, \quad X(t, \theta) = (R\cos t\cos\theta, R\cos t\sin\theta, R\sin t)$$

Along the equator, $t \equiv 0$, so $\dot{t} = \ddot{t} = 0$, and $x' = -R\sin t \equiv 0$. Each term on the LHS of equation (1) vanishes, so equation (1) is satisfied. A unit-speed curve along the equator can be parametrized as

$$\alpha(\lambda) = (R\cos\theta(\lambda), R\sin\theta(\lambda), 0), \text{ where } \theta(\lambda) = \frac{\lambda}{R} \quad \Rightarrow \quad \ddot{\theta} = 0$$

From above results, each term of equation (2) vanishes, so equation (2) is also satisfied, and therefore the equator is a geodesic. Since all great circles on a sphere can be moved to the equator by a rotation, and κ_g is invariant under isometry, all great circles are geodesics.

MATH 4441 Homework 13

Wenqi He

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16.1

Let M be the planar region bounded by a triangle, and let α_i be the i-th interior angle, then $K = \kappa_g = 0$, $\theta_i = \pi - \alpha_i$, and $\chi(M) = 1$. By the Gauss-Bonnet theorem,

$$0 + 0 + \sum_{i=1}^{3}(\pi - \alpha_i) = 2\pi \quad \Rightarrow \quad \sum_{i=1}^{3}\alpha_i = \pi$$

16.2

Let M be the region enclosed by the curve, then $\chi(M) = 1$, and $K = 0$ everywhere on M. By the Gauss-Bonnet theorem,

$$0 + \int_{\partial M} \kappa_g ds + 0 = 2\pi \cdot 1 \quad \Rightarrow \quad \int_{\partial M} \kappa_g ds = 2\pi$$

16.3

Since the surface M is homeomorphic to the torus, there is no boundary ∂M and $\chi(M) = 0$. Therefore, by the Gauss-Bonnet theorem,

$$\int_M K dA = 0 \quad \Rightarrow \quad \frac{1}{Area(M)} \int_M K dA = 0$$

By the mean value theorem for integrals, there always exists a point p on M where $K(p) = 0$

16.4

Let M be either one of the two regions enclosed by the curve. By the Gauss-Bonnet theorem,

$$2\pi = \int_M K dA + \int_{\partial M} \kappa_g ds + 0 = \int_M \frac{1}{R^2} dA + 0 + 0 = \frac{1}{R^2}\int_M dA$$

$$Area(M) = \int_M dA = 2\pi R^2$$

which is exactly one half of the area of the sphere.

16.5

If there exist more than one closed geodesics, then consider the region M enclosed by two of them. M is homeomorphic to a cylinder, so $\chi(M) = 0$. By the Gauss-Bonnet theorem,

$$\int_M K dA + 0 + 0 = 0, \qquad \text{where } K < 0$$

Since the two geodesics are distinct by assumption, M must be nonempty, and therefore the integral must be strictly negative, which is a contradiction. Therefore, there cannot be more than one closed geodesic.

16.6

On a sphere of radius 1, $K = 1$. Let α_i be the interior angles, then by the Gauss-Bonnet theorem,

$$\int_M dA + 0 + \sum_{i=1}^{k} (\pi - \alpha_i) = Area(M) + k\pi - \sum_{i=1}^{k} \alpha_i = 2\pi$$

$$\Rightarrow \quad Area(M) = \sum_{i=1}^{k} \alpha_i - (k-2)\pi$$

16.7

By the Gauss-Bonnet theorem,

$$\int_T K dA + 0 + \Big[(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma)\Big] = 2\pi \quad \Rightarrow \quad \int_T K dA = \alpha + \beta + \gamma - \pi$$

$$\Rightarrow \quad Ave_T[K] = \frac{1}{Area(T)} \int_T K dA = \frac{\alpha + \beta + \gamma - \pi}{Area(T)}$$

By the mean value theorem, there exists a point p_0 within T where $K(p_0)$ equals the average of K. As T shrinks to $\{p\}$, p_0 must coincide with p eventually, therefore,

$$K(p) = \lim_{T \to p} K(p_0) = \lim_{T \to p} \frac{\alpha + \beta + \gamma - \pi}{Area(T)}$$

16.8

On geodesics $\kappa_g = 0$. Let α_i be the interior angles, then by the Gauss-Bonnet theorem,

$$\int_M K dA + 0 + \sum_{i=1}^{3} (\pi - \alpha_i) = 2\pi$$

$$\Rightarrow \quad \sum_{i=1}^{3} \alpha_i = \pi + \int_M K dA \begin{cases} > \pi, \text{ if } K > 0 \\ < \pi, \text{ if } K < 0 \end{cases}$$

16.9

Suppose the two geodesics meet. Let M be the region bounded by the two geodesics. M is homeomorphic to a disk, so $\chi(M) = 1$

$$\int_M K dA + 0 + \sum_{i=1}^{2} (\pi - \alpha_i) = 2\pi$$

$$\int_M K dA + \sum_{i=1}^{2} (-\alpha_i) = 0$$

However, since $K < 0$, all terms on the LHS are negative, which means that the equation cannot hold. Therefore, the geodesics will never meet.

16.10

For each of the two regions, $\chi(M) = 1$. By the Gauss-Bonnet theorem,

$$\begin{aligned} 2\pi &= \int_M K dA + 0 + 0 \\ &= \int_M \det(S) dA = \int_M \det(dn) dA = \int_M |\det(dn)| dA \\ &= \int_{n(M)} dA' = Area(n(M)) \end{aligned}$$

The area of each of the two regions on $\mathbb{S}^2$ is exactly half of the area of $\mathbb{S}^2$, which is 4π. In other words, the areas of the two regions under the Gauss map are equal.

16.11

The pseudo-sphere can be parametrized as

$$X(t, \theta) = \left(\frac{1}{\cosh t} \cos\theta, \frac{1}{\cosh t} \sin\theta, t - \tanh t \right)$$

Consider the top half, whose boundary is the equator $(\cos\lambda, \sin\lambda, 0)$. For a curve along the equator, the principal normal vector is $N = (-\cos\lambda, -\sin\lambda, 0)$, and JT is along the direction of X_t:

$$X_t = \left(-\frac{\tanh t}{\cosh t} \cos\theta, -\frac{\tanh t}{\cosh t} \sin\theta, 1 - \frac{1}{\cosh^2 t} \right), \quad \|X_t\| = \tanh t$$

$$\Rightarrow \quad JT = \frac{X_t}{\|X_t\|} = \left(-\frac{\cos\theta}{\cosh t}, -\frac{\sin\theta}{\cosh t}, \frac{1}{\tanh t} - \frac{1}{\cosh t \sinh t} \right)$$

On the equator, which is a circle of radius 1, the geodesic curvatue is

$$\kappa_g = \kappa \langle N, JT \rangle = \frac{1}{1} \cdot \frac{1}{\cosh t} = 1$$

The top half is homeomorphic to a cylinder, so $\chi(M) = 0$. From the previous results, we also have $K = -1$ everywhere on M, and $\kappa_g = 1$ along the boundary ∂M, therefore by the Gauss-Bonnet theorem,

$$-\int_M dA + \int_{\partial M} ds + 0 = -Area(M) + 2\pi = 0 \quad \Rightarrow \quad Area(M) = 2\pi$$

The total area is

$$2 \cdot Area(M) = \boxed{4\pi}$$