MATH 4441 Homework 12

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15.2

Let $E_1 = (1, 0, 0), \quad E_2 = (0, 1, 0)$, then

$$JE_1 = n \times E_1 = (0, 0, 1) \times (1, 0, 0) = (0, 1, 0)$$

$$JE_2 = n \times E_1 = (0, 0, 1) \times (0, 1, 0) = (-1, 0, 0)$$

Therefore, in $\mathbb{R}^2$, w.r.t. E_1, E_2,

$$J = \begin{pmatrix} JE_1 & JE_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R(\pi/2)$$

15.4

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$$\begin{aligned}
\Big(\frac{1}{\|\alpha'(s^{-1}(t))\|}\Big)' &= \Big(\Big\langle \alpha'(s^{-1}(t)), \alpha'(s^{-1}(t)) \Big\rangle^{-1/2}\Big)' \\
&= -\frac{1}{2}\Big\langle \alpha'(s^{-1}(t)), \alpha'(s^{-1}(t)) \Big\rangle^{-3/2} \Big\langle \alpha'(s^{-1}(t)), \alpha'(s^{-1}(t)) \Big\rangle' \\
&= -\Big\langle \alpha'(s^{-1}(t)), \alpha'(s^{-1}(t)) \Big\rangle^{-3/2} \Big\langle \Big(\alpha'(s^{-1}(t))\Big)', \alpha'(s^{-1}(t)) \Big\rangle \\
&= -\frac{1}{\|\alpha'(s^{-1})\|^3} \Big\langle \frac{\alpha''(s^{-1})}{\|\alpha'(s^{-1})\|}, \alpha'(s^{-1}) \Big\rangle \\
&= -\frac{1}{\|\alpha'(s^{-1})\|^3} \Big\langle \frac{\alpha''(s^{-1})}{\|\alpha'(s^{-1})\|}, \alpha'(s^{-1}) \Big\rangle \\
&= \frac{-\Big\langle \alpha''(s^{-1}), \alpha'(s^{-1}) \Big\rangle}{\|\alpha'(s^{-1})\|^4}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{\alpha}'' &= \Big(\alpha'(s^{-1}(t))\Big)' \cdot \frac{1}{\|\alpha'(s^{-1}(t))\|} + \alpha'(s^{-1}(t)) \cdot \Big(\frac{1}{\|\alpha'(s^{-1}(t))\|}\Big)' \\
&= \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'(s^{-1}) \cdot \frac{-\Big\langle \alpha''(s^{-1}), \alpha'(s^{-1}) \Big\rangle}{\|\alpha'(s^{-1})\|^4}
\end{aligned}$$

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$$\kappa_g = \overline{\kappa}_g = \langle \overline{\alpha}'', J\overline{\alpha}' \rangle = \left\langle \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \lambda\alpha'(s^{-1}),\ J\alpha'(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|} \right\rangle$$

Since $\langle \alpha', J\alpha' \rangle = 0$,

$$\kappa_g = \left\langle \alpha'' \cdot \frac{1}{\|\alpha'\|^2},\ J\alpha' \cdot \frac{1}{\|\alpha'\|} \right\rangle = \frac{\langle \alpha'', J\alpha' \rangle}{\|\alpha'\|^3}$$

15.6

Suppose α is parametrized by arclength, then $\alpha' = T$, $\|\alpha'\| = \|J\alpha'\| = 1$, $\kappa_g = \langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \rangle$. Expanding $\nabla_{\alpha'}\tilde{\alpha}'$ in the orthonormal frame $\{\tilde{\alpha}', J\tilde{\alpha}'\}$:

$$\begin{aligned}\nabla_{\alpha'}\tilde{\alpha}' &= \langle \nabla_{\alpha'}\tilde{\alpha}', \tilde{\alpha}' \rangle \tilde{\alpha}' + \langle \nabla_{\alpha'}\tilde{\alpha}', J\tilde{\alpha}' \rangle J\tilde{\alpha}' \\ &= \langle \nabla_{\alpha'}\tilde{\alpha}', \tilde{\alpha}' \rangle \tilde{\alpha}' + \kappa_g J\tilde{\alpha}'\end{aligned}$$

Since α is a unit speed curve, $\|\tilde{\alpha}'\|^2 \equiv const$,

$$\alpha'\|\tilde{\alpha}'\|^2 = \alpha'\langle \tilde{\alpha}', \tilde{\alpha}' \rangle = 2\langle \nabla_{\alpha'}\tilde{\alpha}', \tilde{\alpha}' \rangle = 0$$

Therefore,

$$\nabla_{\alpha'}\tilde{\alpha}' = \kappa_g J\tilde{\alpha}' \quad \Rightarrow \quad \|\nabla_{\alpha'}\tilde{\alpha}'\| = \|\kappa_g J\tilde{\alpha}'\| = |\kappa_g|$$

15.7

In the backward direction: If $\nabla_{\alpha'}\tilde{\alpha}' \equiv 0$, then obviously $\langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \rangle \equiv 0 \Rightarrow \kappa_g \equiv 0$, so the curve is a geodesic. In the forward direction: If the curve is a geodesic, then

$$\kappa_g \equiv 0 \Rightarrow \langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \rangle = \|\alpha'\|\langle \nabla_{\alpha'}\tilde{\alpha}', JT \rangle \equiv 0$$

$$\Rightarrow \langle \nabla_{\alpha'}\tilde{\alpha}', JT \rangle \equiv 0$$

From Ex.5, α must have constant speed, which means that

$$\alpha'\|\tilde{\alpha}'\|^2 = \alpha'\langle \tilde{\alpha}', \tilde{\alpha}' \rangle = 2\langle \nabla_{\alpha'}\tilde{\alpha}', \tilde{\alpha}' \rangle = 2\|\tilde{\alpha}'\|\langle \nabla_{\alpha'}\tilde{\alpha}', T \rangle \equiv 0$$

$$\Rightarrow \langle \nabla_{\alpha'}\tilde{\alpha}', T \rangle \equiv 0$$

Since $\nabla_{\alpha'}\tilde{\alpha}'$ lies in the tangent space, using the orthonormal frame $\{T, JT\}$

$$\begin{aligned}\nabla_{\alpha'}\tilde{\alpha}' &= \langle \nabla_{\alpha'}\tilde{\alpha}', T \rangle T + \langle \nabla_{\alpha'}\tilde{\alpha}', JT \rangle JT \\ &= 0 + 0 = 0\end{aligned}$$

15.8

From Ex.12.3, the only non-vanishing Christoffel symbols for a surface of revolution with patch X defined as $X(t,\theta) = (x(t)\cos\theta, x(t)\sin\theta, y(t))$ are

$$\Gamma^1_{11} = \frac{x'x'' + y'y''}{x'^2 + y'^2}, \quad \Gamma^1_{22} = -\frac{xx'}{x'^2 + y'^2}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{x'}{x}$$

Because the first coordinate is named t, and the prime notation is already used to denote derivatives w.r.t variable t, here I use dot notation to denote derivatives w.r.t parameter λ for curve $\alpha(\lambda)$.

$$\ddot{t} + \frac{x'x'' + y'y''}{x'^2 + y'^2}\dot{t}^2 - \frac{xx'}{x'^2 + y'^2}\dot{\theta}^2 = 0 \tag{1}$$

$$\ddot{\theta} + 2\frac{x'}{x}\dot{t}\dot{\theta} = 0 \tag{2}$$

If the surface is a sphere of radius R, then

$$x(t) = R\cos t, \quad y(t) = R\sin t, \quad X(t,\theta) = (R\cos t\cos\theta, R\cos t\sin\theta, R\sin t)$$

Along the equator, $t \equiv 0$, so $\dot{t} = \ddot{t} = 0$, and $x' = -R\sin t \equiv 0$. Each term on the LHS of equation (1) vanishes, so equation (1) is satisfied. A unit-speed curve along the equator can be parametrized as

$$\alpha(\lambda) = (R\cos\theta(\lambda), R\sin\theta(\lambda), 0), \text{ where } \theta(\lambda) = \frac{\lambda}{R} \quad \Rightarrow \quad \ddot{\theta} = 0$$

From above results, each term of equation (2) vanishes, so equation (2) is also satisfied, and therefore the equator is a geodesic. Since all great circles on a sphere can be moved to the equator by a rotation, and κ_g is invariant under isometry, all great circles are geodesics.