MATH 4441 Homework 9

Wenqi He

October 25, 2018

12.1

Since $[g_{ij}][g^{ij}] = I$, $\det[g_{ij}]\det[g^{ij}] = \det I = 1$

$$\det[g^{ij}] = \frac{1}{\det[g_{ij}]}$$

And since $[l_i^j] = [l_{ij}][g^{ij}]$,

$$\det[l_i^j] = \det[l_{ij}]\det[g^{ij}] = \frac{\det[l_{ij}]}{\det[g_{ij}]} = K$$

which is the Gaussian curvature.

12.2

$$\begin{aligned} N_i &= -\sum_j l_i^j X_j = -\sum_{j,k} l_{ik} g^{kj} X_j = -\sum_{j,k,l} S_{il} g_{lk} g^{kj} X_j = -\sum_{j,l} S_{il}\delta_l^j X_j \\ &= -\sum_j S_{ij} X_j = -S(X_i) = dn(X_i) \end{aligned}$$

12.3

The map for a surface of revolution is

$$X(t,\theta) = (x(t)\cos\theta, x(t)\sin\theta, y(t))$$

$$D_1X(t,\theta) = (x'(t)\cos\theta, x'(t)\sin\theta, y'(t)), \quad D_2X(t,\theta) = (-x(t)\sin\theta, x(t)\cos\theta, 0)$$

$$g_{11} = \langle D_1X, D_1X\rangle = x'^2 + y'^2, \quad g_{12} = g_{21} = \langle D_1X, D_2X\rangle = 0, \quad g_{22} = \langle D_2X, D_2X\rangle = x^2$$

$$[g_{ij}] = \begin{pmatrix} x'^2 + y'^2 & 0 \\ 0 & x^2 \end{pmatrix}, \quad [g^{ij}] = \begin{pmatrix} 1/(x'^2 + y'^2) & 0 \\ 0 & 1/x^2 \end{pmatrix}$$

Only the diagonal g^{ij} are non-zero, so in the formula for Γ, the term where $l \neq k$ vanishes:

$$\Gamma_{ij}^k = \frac{1}{2}\Big((g_{ki})_j + (g_{jk})_i - (g_{ij})_k\Big)g^{kk}$$

The only non-zero derivatives of g_{ij} are

$$(g_{11})_1 = 2x'x'' + 2y'y'', \quad (g_{22})_1 = 2xx'$$

Therefore, the only non-zero Christoffel symbols are

$$\Gamma^1_{11} = \frac{1}{2}(g_{11})_1 g^{11} = \frac{x'x'' + y'y''}{x'^2 + y'^2}$$

$$\Gamma^1_{22} = -\frac{1}{2}(g_{22})_1 g^{11} = -\frac{xx'}{x'^2 + y'^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}(g_{22})_1 g^{22} = \frac{x'}{x}$$

12.4

If $M = \mathbb{R}^2$, then we can use a trivial patch $(\mathbb{R}^2, id)$ for the surface, which means that

$$X(u^1, u^2) = (u^1, u^2)$$

$$D_1 X = (1, 0), \quad D_2 X = (0, 1)$$

$$[g_{ij}] = \begin{pmatrix} \langle D_1 X, D_1 X\rangle & \langle D_1 X, D_2 X\rangle \\ \langle D_2 X, D_1 X\rangle & \langle D_2 X, D_2 X\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since g_{ij} are all constants, all derivatives vanish, therefore all Γ^k_{ij} are zero. Since Γ^k_{ij} are zero, their derivatives must also be zero. Thus, by the intrinsic definition of Riemann curvature tensor, $R^l_{ijk} = 0$. Extrinsically, since $D_i X$ are all constants, $D_{ij} X = 0$, therefore $l_{ij} = \langle D_{ij} X, N\rangle = 0$, then according to Gauss's equation, we also have $R^l_{ijk} = 0$.

12.6

Let $U = (0, 2\pi) \times (0, \pi)$, and define the mapping $X : U \to \mathbb{S}^2$ as

$$X(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$D_1 X(\theta, \phi) = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0), \quad D_2 X(\theta, \phi) = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$$

$$g_{11} = \langle D_1 X, D_1 X\rangle = \sin^2\phi, \quad g_{12} = g_{21} = \langle D_1 X, D_2 X\rangle = 0, \quad g_{22} = \langle D_2 X, D_2 X\rangle = 1$$

$$[g_{ij}] = \begin{pmatrix} \sin^2\phi & 0 \\ 0 & 1 \end{pmatrix}, \quad [g^{ij}] = \begin{pmatrix} 1/\sin^2\phi & 0 \\ 0 & 1 \end{pmatrix}$$

The only non-zero derivative of g_{ij} is

$$(g_{11})_2 = 2\sin\phi\cos\phi$$

Therefore,

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = \Gamma^2_{22} = 0$$

$$\Gamma^2_{11} = -\frac{1}{2}(g_{11})_2 g^{22} = -\sin\phi\cos\phi$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}(g_{11})_2 g^{11} = \frac{\cos\phi}{\sin\phi}$$

The R^l_{ijk} where $j \neq k$ are:

$$R^1_{112} = R^1_{121} = R^2_{212} = R^2_{221} = 0$$

$$R^2_{112} = -(\Gamma^2_{11})_2 + \Gamma^1_{12}\Gamma^2_{11} = -\sin^2\phi, \quad R^2_{121} = -R^2_{112} = \sin^2\phi$$
$$R^1_{212} = -(\Gamma^1_{21})_2 - \Gamma^1_{21}\Gamma^1_{12} = 1, \quad R^1_{221} = -R^1_{212} = -1$$

Combining above results:

$$\boxed{[R^2_{1ij}] = \begin{pmatrix} 0 & -\sin^2\phi \\ \sin^2\phi & 0 \end{pmatrix}, \quad [R^1_{2ij}] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

Now compute extrinsically. On $\mathbb{S}^2$, $n = id$, so $N = n \circ X = X$,

$$N(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$D_{11}X(\theta, \phi) = (-\sin\phi\cos\theta, -\sin\phi\sin\theta, 0)$$
$$D_{12}X(\theta, \phi) = D_{21}X(\theta, \phi) = (-\cos\phi\sin\theta, \cos\phi\cos\theta, 0)$$
$$D_{22}X(\theta, \phi) = (-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi)$$
$$l_{11} = \langle D_{11}X, N\rangle = -\sin^2\phi, \quad l_{12} = l_{21} = \langle D_{12}X, N\rangle = 0, \quad l_{22} = \langle D_{22}X, N\rangle = -1$$
$$[l^j_i] = [l_{ij}][g^{ij}] = \begin{pmatrix} -\sin^2\phi & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/\sin^2\phi & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since l_{ij} and l^j_i are both diagonal, $l_{ik}l^l_j - l_{ij}l^l_k$ is only non-zero when either (i) $i = k$ and $l = j$ or (ii) $i = j$ and $l = k$. The corresponding components are:

$$R^1_{221} = -l_{22}l^1_1 = -(-1)(-1) = -1, \quad R^1_{212} = 1$$

$$R^2_{112} = -l_{11}l^2_2 = -(-\sin^2\phi)(-1) = -\sin^2\phi, \quad R^2_{121} = \sin^2\phi$$

which match previous results.