PHYS 7125 Homework 2

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1

The local flatness property states that for each point p on the manifold there exists a change of coordinates such that the metric $g_{\mu\nu}$ can be transformed into a $g_{\mu'\nu'}$ that satisfies: (i) $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ and (ii) $g_{\mu'\nu',\sigma} = 0$ at point p. This can be shown by a Taylor expansion of $g_{\mu'\nu'}$ to the first order:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$$

$$= \left(x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu} \right)\Big|_p + \left(x^\mu_{,\mu'\lambda} x^\nu_{,\nu'} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'\lambda} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu,\lambda} \right)\Big|_p \epsilon + O(\epsilon^2)$$

The requirement is that

$$\left(x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu} \right)\Big|_p = \eta_{\mu'\nu'}$$

$$\left(x^\mu_{,\mu'\lambda} x^\nu_{,\nu'} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'\lambda} g_{\mu\nu} + x^\mu_{,\mu'} x^\nu_{,\nu'} g_{\mu\nu,\lambda} \right)\Big|_p = 0$$

The first equation has 16 variables in $\partial x^\mu / \partial x^{\mu'}$ and 10 equations, one for each indepedent entry of the metric, and the remaining 6 degrees of freedom exactly matches the dimension of the Lorentz group, under which the metric is preserved. Now that $\partial x^\mu / \partial x^{\mu'}$ is determined, the second equation will only have $4 \cdot 10 = 40$ variables in $\partial^2 x^\mu / \partial x^{\mu'} \partial x^\lambda$ (partial derivatives commute) and coincidentally $10 \cdot 4 = 40$ equations corresponding to the entries of $g_{\mu\nu,\lambda}$ (metric is symmetric by definition), so the system is uniquely determined, which proves that such transformation always exists.

2

a

$$\begin{aligned}
g^{\alpha\beta}{}_{,\gamma} &= \left(g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu} \right)_{,\gamma} \\
&= g^{\alpha\nu}{}_{,\gamma} g^{\beta\mu} g_{\mu\nu} + g^{\alpha\nu} g^{\beta\mu}{}_{,\gamma} g_{\mu\nu} + g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma} \\
&= 2 g^{\alpha\nu} g^{\beta\mu}{}_{,\gamma} g_{\mu\nu} + g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma} \\
&= 2 g^{\alpha\nu} \left(g^{\beta\mu}{}_{,\gamma} g_{\mu\nu} + g^{\beta\mu} g_{\mu\nu,\gamma} \right) - g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma} \\
&= 2 g^{\alpha\nu} \left(g^{\beta\mu} g_{\mu\nu} \right)_{,\gamma} - g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma} \\
&= \cancel{2 g^{\alpha\nu} \delta^\beta_{\nu,\gamma}} - g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma} = -g^{\alpha\nu} g^{\beta\mu} g_{\mu\nu,\gamma}
\end{aligned}$$

b

From the two identities we can derive the formula:

$$
\begin{aligned}
\frac{d}{d\epsilon}\det(A) &= \lim_{\epsilon\to 0}\frac{\det(A+\epsilon\frac{d}{d\epsilon}A+O(\epsilon^2))-\det(A)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\det(A(I+\epsilon A^{-1}\frac{d}{d\epsilon}A))-\det(A)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\det(A)\det(I+\epsilon A^{-1}\frac{d}{d\epsilon}A)-\det(A)}{\epsilon} \\
&= \det(A)\lim_{\epsilon\to 0}\frac{\det(I+\epsilon A^{-1}\frac{d}{d\epsilon}A)-1}{\epsilon} \\
&= \det(A)\lim_{\epsilon\to 0}\frac{1+\epsilon\,\mathrm{tr}(A^{-1}\frac{d}{d\epsilon}A)+O(\epsilon^2)-1}{\epsilon} \\
&= \det(A)\,\mathrm{tr}(A^{-1}\frac{d}{d\epsilon}A)
\end{aligned}
$$

Apply the formula to $g_{\mu\nu}$, replacing $d/d\epsilon$ with ∂_α

$$
g_{,\alpha} = g\cdot\mathrm{tr}(g^{\sigma\mu}g_{\mu\nu,\alpha}) = gg^{\nu\mu}g_{\mu\nu,\alpha}
$$

c

From right to left

$$
\begin{aligned}
&-(-g)^{-1/2}\Big[g^{\alpha\beta}(-g)^{1/2}\Big]_{,\beta} \\
&= -(-g)^{-1/2}\Big[g^{\alpha\beta}{}_{,\beta}(-g)^{1/2}+g^{\alpha\beta}(-g)^{1/2}_{,\beta}\Big] \\
&= -(-g)^{-1/2}\Big[g^{\alpha\beta}{}_{,\beta}(-g)^{1/2}-\frac{1}{2}g^{\alpha\beta}(-g)^{-1/2}g_{,\beta}\Big] \\
&= -g^{\alpha\beta}{}_{,\beta}+\frac{1}{2}g^{\alpha\beta}(-g)^{-1}g_{,\beta} \\
&= g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}+\frac{1}{2}g^{\alpha\beta}(-g)^{-1}gg^{\mu\nu}g_{\mu\nu,\beta} \\
&= g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}-\frac{1}{2}g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta} \\
&= \frac{1}{2}\Big(g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}+g^{\mu\beta}g^{\nu\alpha}g_{\mu\nu,\beta}-g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\Big) \\
&= \frac{1}{2}\Big(g^{\mu\nu}g^{\beta\alpha}g_{\mu\beta,\nu}+g^{\nu\beta}g^{\mu\alpha}g_{\nu\mu,\beta}-g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\Big) \\
&= \frac{1}{2}\Big(g^{\mu\nu}g^{\beta\alpha}g_{\mu\beta,\nu}+g^{\nu\mu}g^{\beta\alpha}g_{\nu\beta,\mu}-g^{\alpha\beta}g^{\mu\nu}g_{\mu\nu,\beta}\Big) \\
&= g^{\mu\nu}\cdot\frac{1}{2}g^{\alpha\beta}\Big(g_{\beta\mu,\nu}+g_{\beta\nu,\mu}-g_{\mu\nu,\beta}\Big) = g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}
\end{aligned}
$$

d

$$
\begin{aligned}
LHS &= A^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}_{\alpha\lambda} A^{\lambda} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} g^{\alpha\beta} \Big(g_{\beta\alpha,\lambda} + g_{\beta\lambda,\alpha} - g_{\alpha\lambda,\beta} \Big) A^{\lambda} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} \Big(g^{\alpha\beta} g_{\beta\alpha,\lambda} + g^{\alpha\beta} g_{\beta\lambda,\alpha} - g^{\alpha\beta} g_{\alpha\lambda,\beta} \Big) A^{\lambda} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} \Big(g^{\alpha\beta} g_{\beta\alpha,\lambda} + \cancel{g^{\alpha\beta} g_{\beta\lambda,\alpha}} - \cancel{g^{\beta\alpha} g_{\beta\lambda,\alpha}} \Big) A^{\lambda} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} g^{\alpha\beta} g_{\beta\alpha,\lambda} A^{\lambda} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha} A^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
RHS &= (-g)^{-1/2} \Big[(-g)^{1/2} A^{\alpha} \Big]_{,\alpha} \\
&= (-g)^{-1/2} \Big[(-g)^{1/2} A^{\alpha}{}_{,\alpha} + (-g)^{1/2}_{,\alpha} A^{\alpha} \Big] \\
&= A^{\alpha}{}_{,\alpha} - \frac{1}{2} (-g)^{-1/2} (-g)^{-1/2} g_{,\alpha} A^{\alpha} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} (g)^{-1} g g^{\mu\nu} g_{\mu\nu,\alpha} A^{\alpha} \\
&= A^{\alpha}{}_{,\alpha} + \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha} A^{\alpha} = LHS
\end{aligned}
$$

e

Since $g = \det(g_{\mu\nu}) = \prod \lambda_i = -1 \cdot 1 \cdot 1 \cdot 1 = -1$ is constant,

$$
\epsilon_{\alpha\beta\gamma\delta;\mu} = \Big((-g)^{1/2} \tilde{\epsilon}_{\alpha\beta\gamma\delta} \Big)_{;\mu} = (-g)^{1/2}_{;u} \tilde{\epsilon}_{\alpha\beta\gamma\delta} = 0
$$

3

a

Since $u_{\alpha} u^{\alpha} = -1$,

$$
P_{\alpha\beta} v^{\beta} u^{\alpha} = g_{\alpha\beta} v^{\beta} u^{\alpha} + u_{\alpha} u_{\beta} v^{\beta} u^{\alpha} = v_{\alpha} u^{\alpha} - u_{\beta} v^{\beta} = 0
$$

b

From (a), $u^{\beta} v_{\perp_{\beta}} = u_{\beta} v_{\perp}^{\beta} = 0$, therefore

$$
P_{\alpha\beta} v_{\perp}^{\beta} = g_{\alpha\beta} v_{\perp}^{\beta} + u_{\alpha} u_{\beta} v_{\perp}^{\beta} = g_{\alpha\beta} v_{\perp}^{\beta} + 0 = v_{\perp\alpha}
$$

c

$$
P_{\alpha\beta} := g_{\alpha\beta} - (q_{\lambda} q^{\lambda})^{-1} q_{\alpha} q_{\beta}
$$

Proof: Carrying out the same calculation as above,

$$
\begin{aligned}
P_{\alpha\beta} v^{\beta} q^{\alpha} &= g_{\alpha\beta} v^{\beta} q^{\alpha} - (q_{\lambda} q^{\lambda})^{-1} q_{\alpha} q_{\beta} v^{\beta} q^{\alpha} = v_{\alpha} q^{\alpha} - q_{\beta} v^{\beta} = 0 \\
P_{\alpha\beta} v_{\perp}^{\beta} &= g_{\alpha\beta} v_{\perp}^{\beta} - (q_{\lambda} q^{\lambda})^{-1} q_{\alpha} q_{\beta} v_{\perp}^{\beta} = g_{\alpha\beta} v_{\perp}^{\beta} - 0 = v_{\perp\alpha}
\end{aligned}
$$

d

The candidates for the projection tensor should take the form $Ag_{\alpha\beta} + Bk_\alpha k_\beta$. In order for the projection to be orthogonal,

$$\Big(Ag_{\alpha\beta} + Bk_\alpha k_\beta\Big)v^\beta k^\alpha = Av_\alpha k^\alpha + 0 = 0$$

Since $v_\alpha k^\alpha \neq 0$ in general, A must be zero. However, in order that $P_{\alpha\beta}v_\perp^\beta = v_{\perp\alpha}$,

$$\Big(Ag_{\alpha\beta} + Bk_\alpha k_\beta\Big)v_\perp^\beta = Av_{\perp\alpha} + 0 = v_{\perp\alpha}$$

A must equal 1, therefore there is no unique projection tensor, which must satisfy both conditions.

4

a

$$\begin{aligned}\nabla_{\underset{\sim}{u}} w_\mu &= \nabla_{\frac{d}{d\tau}} w_\mu = \nabla_{\frac{dx^\alpha}{d\tau}\partial_\alpha} w_\mu = \frac{dx^\alpha}{d\tau}\nabla_{\partial_\alpha} w_\mu = \frac{dx^\alpha}{d\tau}\Big(\partial_\alpha w_\mu - \Gamma^\beta_{\alpha\mu} w_\beta\Big) \\ &= \frac{dw_\mu}{d\tau} - \Gamma^\beta_{\alpha\mu} u^\alpha w_\beta = 0\end{aligned}$$

b

Since $u_\mu u^\mu \equiv -1$, it must be true that $\nabla_{\underset{\sim}{u}}\Big(u_\mu u^\mu\Big) \equiv 0$. Using the answer above, indeed

$$\begin{aligned}\nabla_{\underset{\sim}{u}}\Big(u_\mu u^\mu\Big) &= u_\mu \nabla_{\underset{\sim}{u}} u^\mu + u^\mu \nabla_{\underset{\sim}{u}} u_\mu = u_\mu\Big(\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta\Big) + u^\mu\Big(\frac{du_\mu}{d\tau} - \Gamma^\beta_{\alpha\mu}u^\alpha u_\beta\Big) \\ &= u_\mu\frac{du^\mu}{d\tau} + \cancel{\Gamma^\mu_{\alpha\beta}u_\mu u^\alpha u^\beta} + u^\mu\frac{du_\mu}{d\tau} - \cancel{\Gamma^\beta_{\alpha\mu}u^\mu u^\alpha u_\beta} \\ &= \frac{d}{d\tau}\Big(u_\mu u^\mu\Big) = 0\end{aligned}$$

c

Suppose λ is an affine parameter for a null-geodesic, and σ non-affine:

$$\begin{aligned}\nabla_{\underset{\sim}{u}} u^\mu &= \nabla_{\frac{d}{d\sigma}}\frac{dx^\mu}{d\sigma} = \nabla_{\frac{d\lambda}{d\sigma}\frac{d}{d\lambda}}\Big(\frac{d\lambda}{d\sigma}\frac{dx^\mu}{d\lambda}\Big) = \frac{d\lambda}{d\sigma}\nabla_{\frac{d}{d\lambda}}\Big(\frac{d\lambda}{d\sigma}\frac{dx^\mu}{d\lambda}\Big) \\ &= \frac{d\lambda}{d\sigma}\frac{d\lambda}{d\sigma}\cancel{\nabla_{\frac{d}{d\lambda}}\frac{dx^\mu}{d\lambda}} + \frac{d\lambda}{d\sigma}\frac{dx^\mu}{d\lambda}\frac{d}{d\lambda}\frac{d\lambda}{d\sigma} \\ &= \frac{d}{d\lambda}\frac{d\lambda}{d\sigma}u^\mu =: -\kappa u^\mu\end{aligned}$$

where $\nabla_{\frac{d}{d\lambda}}\frac{dx^\mu}{d\lambda} = 0$ by the definition of affineness.