Lecture 13

1 Lie Group

Definition. A *Lie group* $(G, \cdot)$ is a group and a smooth manifold that satisfies

(i) The map $\mu : (g_1, g_2) \mapsto g_1 \cdot g_2$ is smooth

(ii) The map $i : g \mapsto g^{-1}$ is smooth.

Example.

- $(\mathbb{R}^n, +_{\mathbb{R}^n})$ is a commutative Lie group called the n-dimensional *translation group*.
- $U(1) := (S^1, \cdot_{\mathbb{C}})$, where $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$
- The *general linear group* $GL(n, \mathbb{R}) := \{\phi : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \mid \det \phi \neq 0\}$ with operation $\circ$.
- Suppose V is a $\mathbb{R}$-vector space. Let

$$O(p, q) := \{\psi : V \xrightarrow{\sim} V \mid (\psi(v), \psi(w)) = (w, v)\} \subseteq GL(p + q, \mathbb{R})$$

$(O(p, q), \circ)$ is called the *orthogonal group* w.r.t. pseudo-inner product $(\cdot, \cdot)$.

Definition. A *psuedo-inner product* $(\cdot, \cdot)$ satisfies the following properties:

(i) It is bilinear

(ii) It is symmetric

(iii) It is non-degenerate: $\forall w \in V : (v, w) = 0 \Rightarrow v = 0$

Theorem. There always exists a basis $\{e_i\}$ of V such that the matrix (e_i, e_j) has ± 1 on the diagonal and 0 in all other entiries.

Definition. Suppose the basis is chosen so that (e_i, e_j) takes the above form, and there are p 1's and q -1's on the diagnoal. (p, q) is called the *signature* of the metric.

Theorem. There are (up to isomorphism) only as many pseudo-inner products on V as there are different signatures.

2 The Lie algebra of a Lie group

Definition. Suppose $(G, \cdot)$ is a Lie group, then for any $g \in G$, the *left translation* $l_g : G \to G$ is defined as

$$l_g(h) := g \cdot h$$

It can be easily verified that l_g is bijective. Furthermore, l_g and its inverse are smooth by definition of a Lie group, therefore it is a diffeomorphism.

Remark. The push-forward of a vector in general cannot be extended to a vector field, unless the underlying map is a diffeomorphism.

Definition. The push-forward of a vector field X on G is defined as:

$$\Big(l_{g*}X\Big)_{gh} := l_{g*}(X_h)$$

Definition. A vector field X on G is called *left-invariant* if for any $g \in G$,

$$l_{g*}X = X \tag{1}$$

or equivalently,

$$\forall h \in G : l_{g*}X_h = X_{gh} \tag{2}$$

Proposition. If X is left-invariant, then

$$X(f \circ l_g) = Xf \circ l_g \tag{3}$$

Proof. By definition of left-invariance, $(l_{g*}X_h)f = X_{gh}f$,

$$LHS = X_h(f \circ l_g) = [X(f \circ l_g)](h)$$

$$RHS = (Xf)(gh) = [Xf \circ l_g](h)$$

Definition. The set of all left-invariant vector fields on a Lie group G is denoted $L(G)$. It can be verified that $L(G)$ is a $C^\infty(G)$-submodule of $\Gamma(TG)$, and hence also a $\mathbb{R}$-vector space.

Definition. A *Lie algebra* $(L, +, \cdot, [\![\cdot, \cdot]\!])$ is a k-vector space $(L, +, \cdot)$ equiqqed with a *Lie bracket* $[\![\cdot, \cdot]\!]$ that satisfies:

(i) It is bilinear.

(ii) It is antisymmetric: $[\![x, y]\!] = -[\![y, x]\!]$

(iii) Jacobi identity: $[\![x, [\![y, z]\!]]\!] + [\![y, [\![z, x]\!]]\!] + [\![z, [\![x, y]\!]]\!] = 0$

Example. $(\Gamma(TM), +, \cdot, [\cdot, \cdot])$ is an infinite-dimensional Lie algebra.

Theorem. $(L(G), [\cdot, \cdot])$ is a Lie subalgebra of $\Gamma(TG), [\cdot, \cdot]$.

Proof. It remains to be shown that $[\cdot, \cdot] : L(G) \times L(G) \to L(G)$ is closed, or equivalently, $\forall X, Y \in L(G) : [X, Y] \in L(G)$. Using (3), it suffices to show that

$$[X, Y](f \circ l_g) = [X, Y]f \circ l_g$$

which follows immediately from the definition of $L(G)$.

Theorem.

$$L(G) \cong_{\text{vector space}} T_eG$$

Corollary.

$$\dim L(G) = \dim T_eG = \dim G$$

Proof. Define a map $j : T_eG \xrightarrow{\sim} L(G)$ as

$$j(A)_g := l_{g*}A, \quad \forall g \in G$$

It can be verified that j is an isomorphism of the vector spaces. If we further define a Lie bracket on T_eG as $[\![A, B]\!] := j^{-1}[j(A), j(B)]$, then

$$(T_eG, [\![\cdot, \cdot]\!]) \cong_{\text{Lie algebra}} (L(G), [\cdot, \cdot])$$