MATH 2106 Homework 2

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Problem 1

$GCD(a, b)$ can be written as an integer combination of a and b:

$$g = ax + by$$

If $c \mid a$ and $c \mid b$, then $c \mid RHS$, and therefore $c \mid LHS$

Problem 2

If M is prime, then it's already a contradiction. On the other hand, if it's composite, then it must be divisible by some p_i. Suppose $p_i \mid M$. Since p_i also divides every term on the right side except the i-th term

$$p_i \mid M - \sum_{j \neq i} A_j \Rightarrow p_i \mid A_i$$

where A_i denotes the i-th term. If $p_i \mid A_i$, then p_i must divide some p_j where $j \neq i$, since the prime factorization of the i-th term doesn't contain p_i. However, p_j is only divisible by 1 and itself, so this is a contradiction.

Problem 3

Apply the GCD algorithm;

$$\begin{aligned}
990 &= 11 \cdot 84 + 66 \\
84 &= 1 \cdot 66 + 18 \\
66 &= 3 \cdot 18 + 12 \\
18 &= 1 \cdot 12 + 6 \\
12 &= 2 \cdot 6 + 0
\end{aligned}$$

so $GCD(990, 84) = GCD(12, 6) = 6$.

$$\begin{aligned}
6 &= 18 - 12 \\
&= 18 - (66 - 3 \cdot 18) = -1 \cdot 66 + 4 \cdot 18 \\
&= -1 \cdot 66 + 4 \cdot (84 - 66) = 4 \cdot 84 - 5 \cdot 66 \\
&= 4 \cdot 84 - 5 \cdot (990 - 11 \cdot 84) = -5 \cdot 990 + 59 \cdot 84
\end{aligned}$$

Scale the equation by 4:

$$-20 \cdot 990 + 236 \cdot 84 = 24$$

To find the general solution, we can add and subtract multiples of

$$lcm(a, b) = \frac{ab}{gcd(a, b)} = \frac{990 \cdot 84}{6}$$

$$\begin{aligned}
-20 \cdot 990 + 236 \cdot 84 + \frac{990 \cdot 84}{6}t - \frac{990 \cdot 84}{6}t &= 24 \\
990\left(-20 + \frac{84}{6}t\right) + 84\left(236 - \frac{990}{6}t\right) &= 24 \\
990\left(-20 + 14t\right) + 84\left(236 - 165t\right) &= 24
\end{aligned}$$

Therefore,

$$\begin{cases} x = -20 + 14t \\ y = 236 - 165t \end{cases}$$

1.4.6

$$\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$
$$\mathcal{P}(\{3\}) = \{\emptyset, \{3\}\}$$

$\mathcal{P}(\{1, 2\}) \times \mathcal{P}(\{3\}) =$
$\{(\emptyset, \emptyset), (\emptyset, \{3\}), (\{1\}, \emptyset), (\{1\}, \{3\}), (\{2\}, \emptyset), (\{2\}, \{3\}), (\{1, 2\}, \emptyset), (\{1, 2\}, \{3\})\}$

1.4.18

$$|P(A \times P(B)| = 2^{|A \times P(B)|} = 2^{|A| \cdot |P(B)|} = 2^{|A| \cdot 2^{|B|}} = 2^{m \cdot 2^n}$$

1.8.4

$$\bigcup_{i \in \mathbb{N}} A_i = \{2n : n \in \mathbb{Z}\}, \quad \bigcap_{i \in \mathbb{N}} A_i = \{0\}$$

1.8.8

$$\bigcup_{\alpha\in\mathbb{R}}\{\alpha\}\times[0,1]=\{(x,y)\in\mathbb{R}^2:0\le y\le 1\},\quad \bigcap_{\alpha\in\mathbb{R}}\{\alpha\}\times[0,1]=\emptyset$$

1.8.14

Yes. if $x\in\bigcap_{\alpha\in I}A_\alpha$, then $\forall\alpha\in I: x\in A_\alpha$. Now since $J\subseteq I$,

$$\alpha'\in J\Rightarrow\alpha'\in I\Rightarrow x\in A_{\alpha'}$$

In other words, $\forall\alpha'\in J: x\in A_{\alpha'}$, or equivalently $x\in\bigcap_{\alpha\in J}A_\alpha$. Therefore,

$$x\in\bigcap_{\alpha\in I}A_\alpha\Rightarrow x\in\bigcap_{\alpha\in J}A_\alpha$$

or equivalently,

$$\bigcap_{\alpha\in I}A_\alpha\subseteq\bigcap_{\alpha\in J}A_\alpha$$

2.5.10

The statement is only false when $(P\wedge Q)\vee R$ is true but $R\vee S$ is false. In order for $R\vee S$ to be false, both R and S must be false. And because $(P\wedge Q)\vee R$ is true, $P\wedge Q$ must be true, and so P,Q are both true.

2.6.6

P	Q	R	$P\wedge Q\wedge R$	$\neg(P\wedge Q\wedge R)$	$\neg P$	$\neg Q$	$\neg R$	$(\neg P)\vee(\neg Q)\vee(\neg R)$
T	T	T	T	F	F	F	F	F
T	T	F	F	T	F	F	T	T
T	F	T	F	T	F	T	F	T
T	F	F	F	T	F	T	T	T
F	T	T	F	T	T	F	F	T
F	T	F	F	T	T	F	T	T
F	F	T	F	T	T	T	F	T
F	F	F	F	T	T	T	T	T

Note that the columns corresponding to $\neg(P\wedge Q\wedge R)$ and $(\neg P)\vee(\neg Q)\vee(\neg R)$ have the same truth values.

2.6.10

$$(P\Rightarrow Q)\vee R=(\neg P\vee Q)\vee R=\neg P\vee Q\vee R$$
$$\neg((P\wedge\neg Q)\wedge\neg R)=\neg(P\wedge\neg Q)\vee R=(\neg P\vee Q)\vee R=\neg P\vee Q\vee R$$

Therefore they are equivalent.