MATH 4441 Homework 6

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8.4

$$J_{(u^1,u^2)}(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ D_1 f(u^1,u^2) & D_2 f(u^1,u^2) \end{pmatrix}$$

The two columns are always linearly independent, so the Jacobian always has a rank of 2. By definition, the mapping is regular.

8.5

The Jacobian of f

$$J_p(f) = \begin{pmatrix} D_1 f^1(p) & D_2 f^1(p) \\ D_1 f^2(p) & D_2 f^2(p) \\ D_1 f^3(p) & D_2 f^3(p) \end{pmatrix} = \begin{pmatrix} D_1 f(p) & D_2 f(p) \end{pmatrix}$$

The mapping is regular iff the two columns are linearly independent, or equivalently,

$$\|D_1 f(p) \times D_2 f(p)\| \neq 0$$

8.11

Since α is closed, we can extended it to a periodic function $\bar{\alpha} : R \to R^2$. For any point (t, θ) we can take $U = (t - \epsilon, t + \epsilon) \times (\theta - \epsilon, \theta + \epsilon)$, then (U, X) is a regular patch. X is one-to-one because if

$$x(t_1)\cos(\theta_1) = x(t_2)\cos(\theta_2)$$
$$x(t_1)\sin(\theta_1) = x(t_2)\sin(\theta_2)$$
$$y(t_1) = y(t_2)$$

Then from the first two equations, $\tan(\theta_1) = \tan(\theta_2) \Rightarrow \theta_1 = \theta_2 \Rightarrow x(t_1) = x(t_2)$. Then $t_1 = t_2 = \bar{\alpha}^{-1}(x, y)$. X is smooth because $x, y, \sin(x), \cos(x)$ are all C^∞. X is regular because the Jacobian of X is

$$\begin{pmatrix} x'\cos(\theta) & -x\sin(\theta) \\ x'\sin(\theta) & x\cos(\theta) \\ y' & 0 \end{pmatrix}$$

which always has rank 2. X is proper because X maps open sets to open sets and so the preimage of open sets under X^{-1} must be open sets. Therefore, by definition, The surface is a regular embedded surface.

9.5

Define $f(p) := \|p\|^2 = x^2+y^2+z^2$, then $\mathbb{S}^2$ is the level set of $f=1$. Therefore, the gradient $\nabla f = (2x, 2y, 2z) = 2p$ is always orthogonal to $\mathbb{S}^2$. Normalizing the vector gives $n(p) = p$.

9.11

On a sphere of radius r, the Gauss map is $n(p) = \dfrac{p}{r}$. For any $v \in T_pM$, we can find an associated curve $\gamma : (-\epsilon, \epsilon) \to M$, such that $\gamma(0) = p, \gamma'(0) = v$. Then

$$(n \circ \gamma)(t) = n(\gamma(t)) = \frac{\gamma(t)}{r}$$

$$(n \circ \gamma)'(t) = \frac{\gamma'(t)}{r}$$

$$dn_p(v) = (n \circ \gamma)'(0) = \frac{\gamma'(0)}{r} = \frac{v}{r}$$

$$dn_p = \frac{1}{r} \cdot id$$

$$S_p = -dn_p = \begin{pmatrix} -1/r & 0 \\ 0 & -1/r \end{pmatrix}$$

$$K(p) = \det S_p = \frac{1}{r^2}$$

9.13

$$D_1X = (x'\cos\theta, x'\sin\theta, y'), \quad D_2X = (-x\sin\theta, x\cos\theta, 0)$$

$$\langle D_1X, D_1X\rangle = x'^2 + y'^2, \quad \langle D_2X, D_2X\rangle = x^2, \quad \langle D_1X, D_2X\rangle = \langle D_2X, D_1X\rangle = 0$$

$$\det g_{ij} = g_{11}g_{22} - g_{12}^2 = (x'^2+y'^2)x^2$$

$$D_1X \times D_2X = \begin{vmatrix} i & j & k \\ x'\cos\theta & x'\sin\theta & y' \\ -x\sin\theta & x\cos\theta & 0 \end{vmatrix} = (-xy'\cos\theta, -xy'\sin\theta, xx')$$

$$N = \frac{D_1X \times D_2X}{\|D_1X \times D_2X\|} = \frac{1}{\sqrt{x^2(x'^2+y'^2)}}(-xy'\cos\theta, -xy'\sin\theta, xx')$$

$$D_{11}X = (x''\cos\theta, x''\sin\theta, y''), \quad \langle D_{11}X, N\rangle = \frac{-xx''y' + xx'y''}{\sqrt{x^2(x'^2+y'^2)}}$$

$$D_{22}X = (-x\cos\theta, -x\sin\theta, 0), \quad \langle D_{22}X, N\rangle = \frac{x^2y'}{\sqrt{x^2(x'^2+y'^2)}}$$

$$D_{12}X = D_{21}X = (-x'\sin\theta, x'\cos\theta, 0) \quad \langle D_{12}X, N\rangle = \langle D_{21}X, N\rangle = 0$$

$$\det l_{ij} = l_{11}l_{22} - l_{12}^2 = \frac{xy'(x'y'' - x''y')}{x'^2+y'^2}$$

For any point $p = X(t_0, \theta_0)$, we can define $\bar{X}(t,\theta) = X(t+t_0, \theta+\theta_0)$, then $p = \bar{X}(0,0)$. Carrying out the same computation as above, we can get

$$K(p) = \frac{\det \bar{l}_{ij}(0,0)}{\det \bar{g}_{ij}(0,0)} = \frac{\det l_{ij}(t_0,\theta_0)}{\det g_{ij}(t_0,\theta_0)} = \boxed{\left.\frac{y'(x'y''-x''y')}{x(x'^2+y'^2)^2}\right|_{t_0}}$$

For a torus of revolution,

$$x' = -r\sin t, \quad x'' = -r\cos t, \quad y' = r\cos t, \quad y'' = -r\sin t$$

$$K(p) = \left(\frac{y'(x'y''-x''y')}{x(x'^2+y'^2)^2}\right)(t_0) = \boxed{\frac{\cos t_0}{r(R+r\cos t_0)}}$$

Assuming $0 < r < R$, then the denominator is always positive, so the sign of curvature depends on the sign of $\cos t_0$. For $t_0 \in [-\pi, \pi]$, when $|t_0| < \pi/2$, the curvature is positive. When $|t_0| = \pi/2$, the curvature is zero. When $|t_0| > \pi/2$, then curvature is nagative.