MATH 4347 Homework 3

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5.6

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}+y\right)} H(y)dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\left(\frac{(x-y)^2}{4kt}+y\right)} dy \\
&= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_{0}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}} dy \\
&= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \sqrt{4kt} \int_{0}^{\infty} e^{-z^2} dz \\
&= \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \sqrt{4kt} \frac{\sqrt{\pi}}{2} = \boxed{\frac{e^{kt-x}}{2}}
\end{aligned}$$

5.9

a

Let $u = e^{-dt}v$, then

$$u_t = -de^{-dt}v + e^{-dt}v_t, \quad u_{xx} = e^{-dt}v_{xx}$$

The original equation becomes

$$-de^{-dt}v + e^{-dt}v_t + de^{-dt}v = ke^{-dt}v_{xx} \quad \Rightarrow \quad v_t = kv_{xx}$$

$$g(x) = u(x,0) = e^0 v(x,0) = v(x,0)$$

Using the fundamental solution

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)dy \quad \Rightarrow \quad \boxed{u(x,t) = \frac{e^{-dt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)dy}$$

b

It makes the solution decay exponentially at a speed d.

c

Suppose now we let $u = f(t)v(x,t)$, then

$$u_t = f'v + fv_t, \quad u_{xx} = fv_{xx}$$

$$f'v + fv_t + dfv = kfv_{xx}$$

The objective is to let $f'v$ and dfv cancel out, so f should satisfy

$$f' = -df \Rightarrow f = e^{-\int d(t)dt}$$

Therefore the change of variable should be

$$\boxed{u(x,t) = e^{-\int d(t)dt}v(x,t)}$$

6.2

Suppose $u(x,t) = X(x)T(t)$, then the equation becomes

$$XT'' = 9X''T \Rightarrow \frac{T''}{9T} = \frac{X''}{X} = k$$

The initial and boundary conditions imply that

$$X(0) = X(1) = 0$$

First solve for X:

$$X'' = kX$$

In order for X to satisfy the boundary conditions, it cannot be exponential, therefore

$$k = -\beta^2 \Rightarrow X = A\cos\beta x + B\sin\beta x$$

The boundary condition at $x = 0$ implies that $A = 0$. At $x = 1$, $X(1) = B\sin\beta = 0$, so

$$\beta_n = n\pi$$

Let $T(t)$ absorb the constant coefficient, then

$$X_n = \sin(n\pi x)$$

Now solve for each corresponding T_n:

$$T_n'' = 9kT_n = -9\beta_n^2 T_n$$

$$T_n = C_n\cos(3\beta_n t) + D_n\sin(3\beta_n t), \quad T_n' = -3\beta_n C_n\sin(3\beta_n t) + 3\beta_n D_n\cos(3\beta_n t)$$

The general solution is

$$u(x,t) = \sum_{n=0}^{\infty} X_n T_n = \sum_{n=0}^{\infty} \Big[C_n\cos(3n\pi t) + D_n\sin(3n\pi t)\Big]\sin(n\pi x)$$

$$u_t(x,t) = \sum_{n=0}^{\infty} X_n T_n' = \sum_{n=0}^{\infty} \Big[-3n\pi C_n \sin(3n\pi t) + 3n\pi D_n \cos(3n\pi t) \Big] \sin(n\pi x)$$

Apply the initial conditions,

$$u(x,0) = \sum_{n=0}^{\infty} C_n \sin(n\pi x) = 2\sin(\pi x) + 7\sin(3\pi x)$$

$$u_t(x,0) = \sum_{n=0}^{\infty} 3n\pi D_n \sin(n\pi x) = 2\sin(\pi x)$$

Comparing the terms, we can get

$$C_1 = 2, \quad C_3 = 7, \quad D_1 = \frac{2}{3\pi}$$

All other coefficients are zero. Therefore the solution is

$$\boxed{u(x,t) = \Big[2\cos(3\pi t) + \frac{2}{3\pi}\sin(3\pi t)\Big] \sin(\pi x) + 7\cos(9\pi t)\sin(3\pi x)}$$

6.3

a

If $\lambda = 0$, then $v'' = 0$, $\boxed{v = kx + m}$. From the boundary conditions,

$$k - a_0 m = 0, \quad (1 + a_L L)k + a_L m = 0$$

k, m could be any solution of the system of equations.

b

Since the system has non-trivial solutions,

$$\det \begin{pmatrix} 1 & -a_0 \\ 1 + a_L L & a_L \end{pmatrix} = a_0 + a_L + a_0 a_L L = 0$$

c

Determinant being zero is also sufficient for non-trivial solutions to exist, therefore it guarantees that $\lambda = 0$ is an eigenvalue.

2.3.4

a

On the initial line, u attains maximum at $x = 1/2$ and minimum at two end points

$$u(1/2, 0) = 1, \quad u(0,0) = u(1,0) = 0$$

By the maximum principle for heat equation,

$$\max_D u(x,t) = \max_\Gamma u(x,t) = 1, \quad \min_D u(x,t) = \min_\Gamma u(x,t) = 0$$

b

Let $\xi = 1 - x$, then $\bar{u}(x,t) = u(1-x,t) = u(\xi,t)$.

$$\bar{u}_t = u_t, \quad \bar{u}_{xx} = (-1)(-1)u_{\xi\xi} = u_{\xi\xi}$$

Since $u_t = u_{\xi\xi}$, $\bar{u}_t = \bar{u}_{xx}$, which means that $u(1-x,t)$ also satisfies the heat equation. Also,

$$u(1-x,0) = 4(1-x)(1-(1-x)) = 4(1-x)x = u(x,0)$$

The initial data for two functions are the same. By uniqueness of solutions,

$$u(x,t) = u(1-x,t)$$

c

$$\frac{d}{dt}\int_0^1 u^2 dx = 2\int_0^1 uu_t dx = 2\int_0^1 uu_{xx} dx = 2uu_x\Big|_0^1 - 2\int_0^1 u_x^2 dx = -2\int_0^1 u_x^2 dx \le 0$$

Therefore $\displaystyle\int_0^1 u^2 dx$ is strictly decreasing.

2.4.9

u_{xxx} satisfies the heat equation since

$$(u_{xxx})_t = (u_t)_{xxx} = (ku_{xx})_{xxx} = k(u_{xxx})_{xx}$$

The initial value for u_{xxx} is

$$u_{xxx}(x,0) = (u(x,0))''' = 0$$

Since the zero function is obviously a solution, by uniqueness of solutions, $u_{xxx} \equiv 0$. Integrating yields $u = A(t)x^2 + B(t)x + C(t)$. Plug this into the original problem

$$A'(t)x^2 + B'(t)x + C'(t) = 2kA(t)$$

RHS is a function of t alone, therefore $A' = B' = 0 \Rightarrow A = a, B = b$, where a, b are constants, and $C' = 2kA = 2ka \Rightarrow C = 2kat + c$. Using the initial condition

$$u(x,0) = A(0)x^2 + B(0)x + C(0) = ax^2 + bx + c = x^2 \Rightarrow \begin{cases} a = 1 \\ b = 0 \\ c = 0 \end{cases}$$

Therefore,

$$\boxed{u(x,t) = x^2 + 2kt}$$

3.4.13

Odd-extend u to $\tilde{u}$. The initial and boundary conditions for the extended function are

$$\phi(x) = \tilde{u}(x,0) = x, \quad \psi(x) = \tilde{u}_t(x,0) = 0, \quad h(t) = x(0,t) = t^2$$

The solution for $x > ct$ is

$$u = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0 = \boxed{x}$$

The solution for $x < ct$ is

$$u = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0 + h(t - \frac{x}{c}) = \boxed{x + \left(t - \frac{x}{c}\right)^2}$$

4.1.3

Suppose $u = X(x)T(t)$, then from the equation,

$$\frac{T'}{iT} = \frac{X''}{X} = k$$

First solve for X:

$$X'' = kX, X(0) = X(l) = 0$$

Since X must be zero at two points, we must have $k = -\lambda^2$.

$$X'' = -\lambda^2 X \Rightarrow X = A\cos\lambda x + B\sin\lambda x$$

From the boundary conditions,

$$A = 0, \quad \sin\lambda l = 0 \Rightarrow \lambda_n = \frac{n\pi}{l}$$

Let T absorb the constant, then

$$X_n = \sin\frac{n\pi}{l}x$$

Now solve for each corresponding T_n,

$$T_n' = -i\lambda_n^2 T \quad \Rightarrow \quad T_n = C_n e^{-i\lambda_n^2 t} = C_n e^{-i\left(\frac{n\pi}{l}\right)^2 t}$$

Combining above results, the general solution is

$$\boxed{u = \sum_{n=0}^{\infty} C_n e^{-i\left(\frac{n\pi}{l}\right)^2 t} \sin\frac{n\pi}{l}x|}$$

4.2.1

Suppose $u = X(x)T(t)$, then from the equation,

$$\frac{T'}{\kappa T} = \frac{X''}{X} = k$$

First solve for X:

$$X'' = kX = -\lambda^2 X, \quad X(0) = X'(l) = 0$$

$$X = A\cos\lambda x + B\sin\lambda x$$

From the boundary conditions,

$$A = 0, \quad \cos\lambda l = 0 \Rightarrow \lambda l = \left(n + \frac{1}{2}\right)\pi, \quad \lambda_n = \frac{(2n+1)\pi}{2l}$$

Let T absorb the constant, then

$$X_n = \sin\frac{(2n+1)\pi}{2l}x$$

Now solve for each corresponding T_n,

$$T_n' = -\kappa\lambda_n^2 T \quad \Rightarrow \quad T_n = C_n e^{-\kappa\lambda_n^2 t} = C_n e^{-\kappa((2n+1)\pi/2l)^2 t}$$

Combining above results, the general solution is

$$\boxed{u = \sum_{n=0}^{\infty} C_n e^{-\kappa((2n+1)\pi/2l)^2 t} \sin\frac{(2n+1)\pi}{2l}x}$$