MATH 4441 Homework 4

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4.21

Suppose $\theta' = \bar{\kappa} = c$, then

$$\theta(t) = \int_0^t \bar{\kappa}(s)ds + \theta_0 = \int_0^t cds + \theta_0 = ct + \theta_0$$

$$\alpha'(t) = (\cos(ct + \theta_0), \sin(ct + \theta_0))$$

$$\begin{aligned}\alpha(t) &= \left(\int_0^t \cos(cs + \theta_0)ds, \int_0^t \sin(cs + \theta_0)ds\right) + \alpha_0 \\ &= \frac{1}{c}(\sin(ct + \theta_0), -\cos(ct + \theta_0)) + \alpha_0\end{aligned}$$

which a cricle of radius $1/c$ centered at α_0.

5.5

Suppose $\|\alpha'\| = 1$. Then by definition

$$\beta(t) = \alpha(t) + r(t)N(t)$$

$$\beta' = \alpha' + r'N + rN' = T + r'N + r(-\kappa T) = T + r'N - T = r'N$$

$$\|\beta'\| = |r'|$$

$$T_\beta = \frac{\beta'}{\|\beta'\|} = \frac{r'}{|r'|}N = \pm N$$

Since the curvature (and therefore r) is monotone, the sign is always the same.

$$T_\beta' = \pm N' = \mp \kappa T, \quad \|T_\beta'\| = \kappa$$

$$\kappa_\beta(t) = \frac{\|T_\beta'\|}{\|\beta'\|} = \frac{\kappa}{|r'|}$$

which is never zero because κ is never zero.

5.7

If a line is tangent to the curve at two different points, then the osculating circle at either point is not contained in the osculating circle at the other point, because at least its contact point with the curve is not in the other circle. However, by Kneser's Nesting Theorem, if a curve has monotone nonvanishing curvature, then its osculating circles must be nested. Therefore such curves cannot have bitangent lines.

6.1

An ellipse can be parameterized as

$$\alpha(t) = (a\cos(t), b\sin(t))$$

$$\alpha'(t) = (-a\sin(t), b\cos(t))$$

$$\|\alpha'(t)\| = (a^2\sin^2(t) + b^2\cos^2(t))^{\frac{1}{2}}$$

$$\alpha''(t) = (-a\cos(t), -b\sin(t))$$

$$\langle\alpha' \times \alpha'', (0,0,1)\rangle = ab\sin^2(t) + ab\cos^2(t) = ab$$

$$\bar{\kappa} = \frac{\langle\alpha' \times \alpha'', (0,0,1)\rangle}{\|\alpha'\|^3} = \frac{ab}{(a^2\sin^2(t) + b^2\cos^2(t))^{\frac{3}{2}}}$$

$$\bar{\kappa}' = -\frac{3}{2}ab\Big(a^2\sin^2(t) + b^2\cos^2(t)\Big)^{-\frac{5}{2}}\Big(2a^2\sin(t)\cos(t) - 2b^2\cos(t)\sin(t)\Big)$$

The signed curvature has a local extremum when $\bar{\kappa}' = 0$,

$$2(a^2 - b^2)\cos(t)\sin(t) = 0$$

If $a = b$, in which case the ellipse is a circle, the equation is satisfied for all t. If $a \neq b$, then either $\cos(t) = 0$ or $\sin(t) = 0$. The corresponding vertices are located at

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

Therefore an ellipse has exactly 4 vertices unless it's a circle.

6.4

There must be even number of vertices, otherwise there will be two local maxima or minima of $\bar{\kappa}$ next to each other, which is not possible. Therefore, if the curve has fewer than 4 vertices, it must have exactly 2.