MATH 4441 Homework 11

Wenqi He

November 6, 2018

14.2

In local coordinates, W, V can be written as

$$V = \sum_j V^j \overline{X}_j, \quad W = \sum_k W^k \overline{X}_k, \quad Z = \sum_i Z^i \overline{X}_i$$

i

$$\begin{aligned}
LHS = R(V,W)Z &= R(\sum_j V^j \overline{X}_j, \sum_k W^k \overline{X}_k) \sum_i Z^i \overline{X}_i \\
&= \sum_{i,j,k} V^j W^k Z^i \Big(R(\overline{X}_j, \overline{X}_k)\overline{X}_i \Big) \\
&= \sum_{i,j,k,l} V^j W^k Z^i R^l_{ijk} \overline{X}_l
\end{aligned}$$

$$\begin{aligned}
RHS &= \langle S(W), Z\rangle S(V) - \langle S(V), Z\rangle S(W) \\
&= \langle S(\sum_k W^k \overline{X}_k), \sum_i Z^i \overline{X}_i \rangle S(\sum_j V^j \overline{X}_j) - \langle S(\sum_j V^j \overline{X}_j), \sum_i Z^i \overline{X}_i \rangle S(\sum_k W^k \overline{X}_k) \\
&= \sum_{i,j,k} V^j W^k Z^i \Big[\langle S(\overline{X}_k), \overline{X}_i \rangle S(\overline{X}_j) - \langle S(\overline{X}_j), \overline{X}_i \rangle S(\overline{X}_k) \Big] \\
&= \sum_{i,j,k} V^j W^k Z^i \Big(l_{ik} l^l_j - l_{ij} l^l_k \Big) \overline{X}_l
\end{aligned}$$

Since the equation holds for arbitrary V, W, Z,

$$R^l_{ijk} = l_{ik} l^l_j - l_{ij} l^l_k$$

which is Gauss's equation in local coordinates.

ii

$$\nabla_V S(W) - \nabla_W S(V) = S([V, W])$$

$$
\begin{aligned}
LHS &= \nabla_V S(W) - \nabla_W S(V) \\
&= \nabla_{\sum_j V^j \overline{X}_j} S(\sum_k W^k \overline{X}_k) - \nabla_{\sum_k W^k \overline{X}_k} S(\sum_j V^j \overline{X}_j) \\
&= \sum_j V^j \nabla_{\overline{X}_j} \sum_k W^k S(\overline{X}_k) - \sum_k W^k \nabla_{\overline{X}_k} \sum_j V^j S(\overline{X}_j) \\
&= \sum_{j,k} V^j \nabla_{\overline{X}_j} \Big(W^k S(\overline{X}_k) \Big) - W^k \nabla_{\overline{X}_k} \Big(V^j S(\overline{X}_j) \Big) \\
&= \sum_{j,k} V^j (\overline{X}_j W^k) S(\overline{X}_k) + V^j W^k \nabla_{\overline{X}_j} S(\overline{X}_k) - W^k (\overline{X}_k V^j) S(\overline{X}_j) - W^k V^j \nabla_{\overline{X}_k} S(\overline{X}_j)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= S([V, W]) \\
&= S(\nabla_V W - \nabla_W V) \\
&= \sum_{j,k} S\Big(V^j \nabla_{\overline{X}_j} \Big(W^k \overline{X}_k \Big) - W^k \nabla_{\overline{X}_k} \Big(V^j \overline{X}_j \Big) \Big) \\
&= \sum_{j,k} S\Big(V^j (\overline{X}_j W^k) \overline{X}_k + V^j W^k \nabla_{\overline{X}_j} \overline{X}_k - W^k (\overline{X}_k V^j) \overline{X}_j - W^k V^j \nabla_{\overline{X}_k} \overline{X}_j \Big) \\
&= \sum_{j,k} V^j (\overline{X}_j W^k) S(\overline{X}_k) + V^j W^k S(\nabla_{\overline{X}_j} \overline{X}_k) - W^k (\overline{X}_k V^j) S(\overline{X}_j) - W^k V^j S(\nabla_{\overline{X}_k} \overline{X}_j)
\end{aligned}
$$

Canceling terms on both sides, the equation can be simplied to

$$
\nabla_{\overline{X}_j} S(\overline{X}_k) - \nabla_{\overline{X}_k} S(\overline{X}_j) = S(\nabla_{\overline{X}_j} \overline{X}_k - \nabla_{\overline{X}_k} \overline{X}_j) = S(0) = 0
$$

Taking inner product with $\overline{X}_i$:

$$
\langle \nabla_{\overline{X}_j} S(\overline{X}_k), \overline{X}_i \rangle - \langle \nabla_{\overline{X}_k} S(\overline{X}_j), \overline{X}_i \rangle = 0
$$

$$
\begin{aligned}
\langle \nabla_{\overline{X}_k} S(\overline{X}_j), \overline{X}_i \rangle &= \overline{X}_k \langle S(\overline{X}_j), \overline{X}_i \rangle - \langle S(\overline{X}_j), \nabla_{\overline{X}_k} \overline{X}_i \rangle \\
&= (l_{ij})_k - \langle S(\overline{X}_j), \sum_l \Gamma^l_{ik} \overline{X}_l \rangle \\
&= \sum_l (l_{ij})_k - \Gamma^l_{ik} \langle S(\overline{X}_j), \overline{X}_l \rangle \\
&= \sum_l (l_{ij})_k - \Gamma^l_{ik} l_{jl}
\end{aligned}
$$

Swapping j and k:

$$
\langle \nabla_{\overline{X}_j} S(\overline{X}_k), \overline{X}_i \rangle = \sum_l (l_{ik})_j - \Gamma^l_{ij} l_{kl}
$$

Taking the difference of two terms:

$$
\sum_l \Gamma^l_{ik} l_{jl} - \Gamma^l_{ij} l_{kl} = (l_{ij})_k - (l_{ik})_j
$$

which is Codazzi-Mainardi equation in local coordinates.

14.3

Let $E_1 = V$ and $E_2 = W$, then by Gauss's equation

$$\begin{aligned} R(V,W,W,V) &= \langle R(V,W)W, V\rangle \\ &= \langle S(V), V\rangle\langle S(W), W\rangle - \langle S(W), V\rangle\langle S(V), W\rangle \\ &= l_{11}l_{22} - (l_{12})^2 = \det[l_{ij}] \end{aligned}$$

By Lagrange's identity

$$\begin{aligned} \|V \times W\|^2 &= \|V\|^2\|W\|^2 - \langle V, W\rangle^2 \\ &= \langle V, V\rangle\langle W, W\rangle - \langle V, W\rangle^2 \\ &= g_{11}g_{22} - (g_{12})^2 = \det[g_{ij}] \end{aligned}$$

By definition of Gaussian curvature

$$K = \frac{\det[l_{ij}]}{\det[g_{ij}]} = \frac{R(V,W,W,V)}{\|V \times W\|^2}$$

15.1

i

Without loss of generality, we can consider the equator of a sphere. For a sphere of radius R centered at the origin, a unit speed curve along the equator is

$$\alpha(t) = (R\cos(t/R), R\sin(t/R))$$

$$\alpha'(t) = (-\sin(t/R), \cos(t/R))$$

$$\alpha''(t) = (-\frac{1}{R}\cos(t/R), -\frac{1}{R}\sin(t/R)) = -\frac{1}{R^2}\alpha(t) = -\frac{1}{R}n(t)$$

Therefore,

$$\alpha''^{\top} = 0 \quad \Rightarrow \quad |K_g| = 0$$

ii

For a cylinder of radius R around z-axis, a unit speed helix curve is

$$\alpha(t) = \Big(R\cos\frac{t}{\sqrt{v^2+R^2}}, R\sin\frac{t}{\sqrt{v^2+R^2}}, \frac{v}{\sqrt{v^2+R^2}}t\Big)$$

$$\alpha'(t) = \Big(-\frac{R}{\sqrt{v^2+R^2}}\sin\frac{t}{\sqrt{v^2+R^2}}, \frac{R}{\sqrt{v^2+R^2}}\cos\frac{t}{\sqrt{v^2+R^2}}, \frac{v}{\sqrt{v^2+R^2}}\Big)$$

$$\alpha''(t) = \Big(-\frac{R}{v^2+R^2}\cos\frac{t}{\sqrt{v^2+R^2}}, -\frac{R}{v^2+R^2}\sin\frac{t}{\sqrt{v^2+R^2}}, 0\Big) = -\frac{R}{v^2+R^2}n(t)$$

Therefore,

$$\alpha''^{\top} = 0 \quad \Rightarrow \quad |K_g| = 0$$

15.3

The radius of the circle at $z = h$ is $a = \sqrt{1 - h^2}$. The parametrization of the circle w.r.t. arclength is

$$\alpha(t) = (a\cos(t/a), a\sin(t/a), h) = n(t)$$

$$\alpha'(t) = (-\sin(t/a), \cos(t/a), 0)$$

$$\alpha''(t) = \Big(-\frac{1}{a}\cos(t/a), -\frac{1}{a}\sin(t/a), 0\Big)$$

$$\begin{aligned} J\alpha'(t) &= n(t) \times \alpha'(t) \\ &= (a\cos(t/a), a\sin(t/a), h) \times (-\sin(t/a), \cos(t/a), 0) \\ &= \begin{vmatrix} i & j & k \\ a\cos(t/a) & a\sin(t/a) & h \\ -\sin(t/a) & \cos(t/a) & 0 \end{vmatrix} \\ &= (-h\cos(t/a), -h\sin(t/a), a) \end{aligned}$$

$$\kappa_g = \langle \alpha'', J\alpha' \rangle = \frac{h}{a} = \boxed{\frac{h}{\sqrt{1-h^2}}}$$

15.5

Suppose α is a geodesic, then $\nabla_{\alpha'}\tilde{\alpha}' \equiv 0$.

$$\alpha'(t)\|\tilde{\alpha}'\|^2 = \alpha'(t)\langle \tilde{\alpha}', \tilde{\alpha}' \rangle = 2\langle \nabla_{\alpha'(t)}\tilde{\alpha}', \tilde{\alpha}' \rangle = 0$$

which means that the $\|\tilde{\alpha}'\|$ is constant.