Lecture 12

1 Grassmann Algebra

Definition. A *(differential) n-form* on a smooth manifold M is a $(0,n)$-tensor field ω that is totally antisymmetric, i.e.

$$\omega(X_1,\cdots,X_n)=\operatorname{sgn}(\pi)\cdot\omega(X_{\pi(1)},\cdots,X_{\pi(n)}),\qquad \text{where } X_1,\cdots,X_m\in\Gamma(TM)$$

The set of all n-forms is denoted $\Omega^n(M)$, which is a $C^\infty(M)$-module.

Example. $\Omega^0=C^\infty(M)$ and $\Omega^1(M)=\Gamma(T*M)$

Definition. The wedge product $\wedge:\Omega^n\times\Omega^m\xrightarrow{\sim}\Omega^{n+m}$ is defined as

$$(\omega\wedge\sigma)(X_1,\cdots,X_{n+m}):=\frac{1}{n!\cdot m!}\sum_{\pi}\operatorname{sgn}(\pi)(\omega\otimes\sigma)(X_{\pi(1)},\cdots,X_{\pi(n+m)})$$

Definition. Suppose $h:M\to N$ be a smooth map and $\omega\in\Omega^nN$, then the *pull-back* h^* is defined in terms of the push-forward h_* of vector fields:

$$(h^*\omega)(X_1,\cdots X_n):=\omega(h_*(X_1),\cdots,h_*(X_n))$$

Theorem. The pull-back distributes over wedge product:

$$h^*(\omega\wedge\sigma)=h^*(\omega)\wedge h^*(\sigma)$$

Definition. Define $\Omega(M)\equiv GR(M)$ as

$$\Omega(M):=\Omega^0(M)\oplus\Omega^1(M)\cdots\oplus\Omega^{\dim M}(M)$$

Then $(\Omega(M),+,\cdot)$ is a $C^\infty(M)$-module, and $(\Omega(M),+,\cdot,\wedge)$ is called the *Grassmann algebra* on manifold M, where $\wedge:\Omega(M)\times\Omega(M)\to\Omega(M)$ is defined by linear continuation of the wedge product, i.e.

$$(\omega+\sigma)\wedge\psi=\omega\wedge\psi+\sigma\wedge\psi$$

where $\omega\in\Omega^a(M)$, $\sigma\in\Omega^b(M)$ and $\psi\in\Omega^c(M)$

Theorem. Suppose $\omega\in\Omega^m(M)$, $\sigma\in\Omega^n(M)$, then

$$\omega\wedge\sigma=(-1)^{m+n}\sigma\wedge\omega$$

Remark. The above theorem does not apply to Grassman algebra in general.

Definition. The *exterior derivative* operator $d : \Omega^n(M) \to \Omega^{n+1}(M)$ is defined as

$$(d\omega)(X_1, \cdots, X_{n+1}) := \sum_{i=1}^{n+1} (-1)^{i+1} X_i \Big(\omega(X_1, \cdots, \cancel{X_i} \cdots, X_{n+1}) \Big) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \cancel{X_i}, \cdots, \cancel{X_j}, \cdots, X_{n+1})$$

It could be verified that $d\omega$ is totally antisymmetric and $C^\infty(M)$-multilinear. The action of the operator extends by linear continuation to $\Omega(M)$.

Example. Suppose $\omega \in \Omega^1(M)$, then

$$(d\omega)(X, Y) = X(\omega(Y) - Y(\omega(X)) - \omega([X, Y])$$

Theorem. Suppose $\omega \in \Omega^m(M)$, $\psi \in \Omega^n(M)$, then

$$d(\omega \wedge \psi) = d\omega \wedge \psi + (-1)^n \omega \wedge d\psi$$

Theorem.

$$h^*(d\omega) = d(h^*\omega)$$

2 de Rham Cohomology

Definition. The *antisymmetric part* of a tensor is defined as

$$A_{[m_1, \cdots m_f]} = \frac{1}{f!} \sum_\pi \operatorname{sgn}(\pi) A_{\pi(1), \cdots, \pi(f)}$$

And the *symmetric part* is defined as

$$A_{(m_1, \cdots m_f)} = \frac{1}{f!} \sum_\pi A_{\pi(1), \cdots, \pi(f)}$$

Lemma.

$$A_{ab} B^{[ab]} = A_{[ab]} B^{ab}$$

Lemma.

$$A_{(ab)} B^{[ab]} = 0$$

Theorem.

$$d^2\omega = d(d\omega) = 0$$

Proof. Suppose $\omega \in \Omega^n(M)$. In local coordinates,

$$\omega = \omega_{a_1 \cdots a_n} dx^{a_1} \wedge \cdots \wedge dx^{a_n}$$

It can be shown that

$$d\omega = (\partial_b \omega_{a_1 \cdots a_n}) dx^b \wedge dx^{a_1} \wedge \cdots \wedge dx^{a_n}$$
$$d^2\omega = (\partial_c \partial_b \omega_{a_1 \cdots a_n}) dx^c \wedge dx^b \wedge dx^{a_1} \wedge \cdots \wedge dx^{a_n}$$

Since $\partial_c \partial_b \omega_{a_1 \cdots a_n}$ is symmetric in b and c, and $dx^c \wedge dx^b \wedge dx^{a_1} \wedge \cdots \wedge dx^{a_n}$ is antisymmetric in b and c, the product is zero.

Definition. Consider the sequence of maps

$$0 \xrightarrow{d} \Omega(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim M}(M) \xrightarrow{d} 0$$

The set of *exact forms* B^n is defined as

$$B^n := \mathrm{im}(d) \subseteq \Omega^n(M)$$

The set of *closed forms* Z^n is defined as

$$Z^n := \ker(d) \subseteq \Omega^n(M)$$

Remark. From the theorem above,

$$B^n \subseteq Z^n$$

Theorem (Poincare). If $M = \mathbb{R}^m$ then $B^n = Z^n$.

Definition. The n-th *de Rham cohomology group* is the quotient vector space

$$H^n(M) := Z^n / B^n$$

Theorem (de Rham). $H^n(M)$ only depends on the topology of M.

Example.

$$H^0(M) \cong \mathbb{R}^{\#\text{ of connected pieces of M}}$$