MATH 4347 Homework 4

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6.5

let $\lambda = \beta^2$, then the eigenfunctions are

$$v = A\cos\beta x + B\sin\beta x, \quad v' = -\beta A\sin\beta x + \beta B\cos\beta x$$

At $x = 0$,

$$\beta B - a_0 A = 0 \Rightarrow B = \frac{a_0 A}{\beta}$$

At $x = l$,

$$-\beta A\sin\beta l + \beta B\cos\beta l + a_l(A\cos\beta l + B\sin\beta l) = 0$$

$$\Rightarrow -\beta A\sin\beta l + a_0 A\cos\beta l + a_l(A\cos\beta l + \frac{a_0 A}{\beta}\sin\beta l) = 0$$

$$\Rightarrow -\beta\tan\beta l + a_0 + a_l + a_l\frac{a_0}{\beta}\tan\beta l = 0$$

$$\Rightarrow \tan\beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_l a_0}$$

a

$(a_0+a_l)\beta/(\beta^2 - a_l a_0)$ decreases to zero continuously as $\beta \to \infty$, therefore it intersects $\tan\beta l$ in every period (except for the first few ones). So there are infinitely many λ_n that satisfies the above equation.

b

Since the two graphs intersect in the positive halves of $\tan\beta l$,

$$\frac{(n-1)\pi}{l} < \beta_n < \frac{(n-1)\pi}{l} + \frac{\pi}{2l} = \frac{(2n-1)\pi}{2l}$$

$$\frac{(n-1)^2\pi^2}{l^2} < \lambda_n < \frac{(2n-1)^2\pi^2}{4l^2}$$

c

Since $(a_0 + a_l)\beta/(\beta^2 - a_l a_0)$ approaches zero as $n \to \infty$, its intersections with $\tan\beta l$ will approach the β-intercepts of $\tan\beta l$, therefore

$$\lim_{n\to\infty}\beta_n = \frac{(n-1)\pi}{l}, \text{ or equivalently, } \lim_{n\to\infty}\lambda_n - \frac{(n-1)^2\pi^2}{l^2} = 0$$

d

$$\tan\left((n-1)\pi+\theta_n l\right)=\frac{(a_0+a_l)\left(\dfrac{(n-1)\pi}{l}+\theta_n\right)}{\left(\dfrac{(n-1)\pi}{l}+\theta_n\right)^2-a_l a_0}$$

$$\tan\theta_n l\left(\frac{(n-1)^2\pi^2}{l^2}+\theta_n^2+2\frac{(n-1)\pi}{l}\theta_n-a_l a_0\right)=(a_0+a_l)\left(\frac{(n-1)\pi}{l}+\theta_n\right)$$

$$\left(\theta_n l+O(\theta_n^3)\right)\left(\frac{(n-1)^2\pi^2}{l^2}-a_l a_0+O(\theta_n)\right)=(a_0+a_l)\left(\frac{(n-1)\pi}{l}+\theta_n\right)$$

$$\left(\frac{(n-1)^2\pi^2}{l}-a_l a_0 l\right)\theta_n+O(\theta_n^2)=(a_0+a_l)\frac{(n-1)\pi}{l}+(a_0+a_l)\theta_n$$

Dropping terms of order higher than the second

$$\left[(n-1)^2\pi^2-(a_0+a_l+a_l a_0 l)l\right]\theta_n=(a_0+a_l)(n-1)\pi$$

$$\theta_n=\frac{(a_0+a_l)(n-1)\pi}{(n-1)^2\pi^2-(a_0+a_l+a_l a_0 l)l}$$

let $x=1/n\to 0$, then

$$\theta_x=\frac{(a_0+a_l)(x-x^2)\pi}{(1-x)^2\pi^2-(a_0+a_l+a_l a_0 l)lx^2}=0+\frac{a_0+a_l}{\pi}x+O(x^2)$$

$$\Rightarrow\boxed{\theta_n=\frac{a_0+a_l}{\pi n}+O(\frac{1}{n^2})}$$

6.7

For each of the two regions,

$$(p_i v')'+\lambda r_i v=0\quad\Rightarrow\quad v''=-\frac{\lambda r_i}{p_i}v$$

$$\Rightarrow v_i=A_i\cos\beta_i x+B_i\sin\beta_i x,\ \text{ where }\beta_i=\sqrt{\frac{\lambda r_i}{p_i}}$$

For v_1, the boundary condition at $x=0$ gives $A_1=0$. Therefore

$$v_1=C\sin\beta_1 x,\quad v_1'=\beta_1 C\cos\beta_1 x$$

For v_2, the boundary condition at $x=l$ gives

$$A_2\cos\beta_2 l+B_2\sin\beta_2 l=0$$

$$A_2=-B_2\tan\beta_2 l$$

Plug it into the equation for v,

$$v_2=-B\tan\beta_2 l\cos\beta_2 x+B\sin\beta_2 x,\quad v_2'=\beta_2 B\tan\beta_2 l\sin\beta_2 x+\beta_2 B\cos\beta_2 x$$

Because the eigenfunctions are required to be continuously differentiable, at $x = m$:

$$\begin{cases} C \sin \beta_1 m = -B \tan \beta_2 l \cos \beta_2 m + B \sin \beta_2 m \\ \beta_1 C \cos \beta_1 m = \beta_2 B \tan \beta_2 l \sin \beta_2 m + \beta_2 B \cos \beta_2 m \end{cases}$$

Dividing two equations

$$\boxed{\tan \beta_1 m = \frac{-\beta_1 \tan \beta_2 l \cos \beta_2 m + \beta_1 \sin \beta_2 m}{\beta_2 \tan \beta_2 l \sin \beta_2 m + \beta_2 \cos \beta_2 m}, \quad \beta_1 = \sqrt{\frac{\lambda r_1}{p_1}}, \quad \beta_2 = \sqrt{\frac{\lambda r_2}{p_2}}}$$

6.8

$$v = a \cos \mu x + b \sin \mu x + c \cosh \mu x + d \sinh \mu x$$

$$v' = -a\mu \sin \mu x + b\mu \cos \mu x + c\mu \sinh \mu x + d\mu \cosh \mu x$$

$$v'' = -a\mu^2 \cos \mu x - b\mu^2 \sin \mu x + c\mu^2 \cosh \mu x + d\mu^2 \sinh \mu x$$

$$v''' = a\mu^3 \sin \mu x - b\mu^3 \cos \mu x + c\mu^3 \sinh \mu x + d\mu^3 \cosh \mu x$$

From the boundary conditions

$$v(0) = a + c = 0 \Rightarrow c = -a$$

$$v'(0) = b\mu + d\mu = 0 \Rightarrow d = -b$$

$$v''(l) = -a\mu^2 \cos \mu l - b\mu^2 \sin \mu l - a\mu^2 \cosh \mu l - b\mu^2 \sinh \mu l = 0$$

$$\Rightarrow a \cos \mu l + a \cosh \mu l = -b \sin \mu l - b \sinh \mu l \tag{1}$$

$$v'''(l) = a\mu^3 \sin \mu l - b\mu^3 \cos \mu l - a\mu^3 \sinh \mu l - b\mu^3 \cosh \mu l = 0$$

$$\Rightarrow -a \sin \mu l + a \sinh \mu l = -b \cos \mu l - b \cosh \mu l \tag{2}$$

Dividing (2) by (1) gives

$$\frac{-\sin \mu l + \sinh \mu l}{\cos \mu l + \cosh \mu l} = \frac{\cos \mu l + \cosh \mu l}{\sin \mu l + \sinh \mu l}$$

$$\sinh^2 \mu l - \sin^2 \mu l = \cos^2 \mu l + \cosh^2 \mu l + 2 \cos \mu l \cosh \mu l$$

$$\boxed{\cos \mu l \cosh \mu l + 1 = 0, \quad \lambda = \mu^4}$$

$$\cos \mu l = -\frac{1}{\cosh \mu l}$$

$\cos \mu l$ oscillates between ± 1, and $-1/\cosh \mu l$ converges to zero from below as μ increases, so there are infinitely many solutions for μ. More precisely, in each cycle of cosine the two functions intersect twice between the trough of cosine and its two x-intercepts. The one on the left of the trough satisfies

$$-\frac{3\pi}{2l} + k\frac{2\pi}{l} < \mu < -\frac{\pi}{l} + k\frac{2\pi}{l}$$

$$(2k - \frac{3}{2})\frac{\pi}{l} < \mu < (2k-1)\frac{\pi}{l}$$

The one on the right of the trough satisfies

$$-\frac{\pi}{l} + k\frac{2\pi}{l} < \mu < -\frac{\pi}{2l} + k\frac{2\pi}{l}$$

$$(2k-1)\frac{\pi}{l} < \mu < (2k - \frac{1}{2})\frac{\pi}{l}$$

As RHS approaches zero, its intersections with LHS also approaches the x-intercepts of LHS:

$$\lim_{n\to\infty} \mu_n \approx -\frac{\pi}{2l} + n\frac{\pi}{l} = \left(n - \frac{1}{2}\right)\frac{\pi}{l}$$

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mu_n^4 \approx \boxed{\left(n - \frac{1}{2}\right)^4 \frac{\pi^4}{l^4} \approx \left(\frac{n\pi}{l}\right)^4}$$

6.9

a

Since f is odd, we have $f(x) = -1, -\pi < x < 0$. The general form of Fourier series is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = -\frac{1}{\pi}\int_{-\pi}^{0} \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx dx = 0$$

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = -\frac{1}{\pi}\int_{-\pi}^{0} \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx dx \\
&= \frac{1}{\pi}\left[\frac{1}{n}\cos nx\Big|_{-\pi}^{0} - \frac{1}{n}\cos nx\Big|_{0}^{\pi}\right] \\
&= \frac{1}{n\pi}\left[(1-\cos n\pi) - (\cos n\pi - 1)\right] \\
&= \frac{2(1-(-1)^n)}{n\pi}
\end{aligned}$$

Thus, only the odd sine terms remain

$$B_{2k+1} = \frac{4}{(2k+1)\pi}$$

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\sin(2k+1)x}$$

b

$$f(\frac{\pi}{4}) = \sum_{odd} \frac{4}{n\pi} \sin\frac{n\pi}{4} = \frac{4}{\pi}\frac{\sqrt{2}}{2} + \frac{4}{3\pi}\frac{\sqrt{2}}{2} - \frac{4}{5\pi}\frac{\sqrt{2}}{2} - \frac{4}{7\pi}\frac{\sqrt{2}}{2} + \cdots$$
$$= \frac{2\sqrt{2}}{\pi}\Big[1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\Big] = 1$$

Therefore,

$$\boxed{1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{2\sqrt{2}}}$$

4.3.4

(i) Let

$$h(\gamma) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_o a_l}$$
$$h'(\gamma) = \frac{(a_0 + a_l)(\gamma^2 - a_0 a_l)}{(\gamma^2 + a_0 a_l)^2} = 0 \Rightarrow \gamma = \sqrt{a_0 a_l}$$

From the graph of $h(\gamma)$ this must be the only maximum. The value of h at this point is

$$h(\sqrt{a_0 a_l}) = \frac{-a_0 - a_l}{2\sqrt{a_0 a_l}} \geq \frac{2\sqrt{(-a_0)(-a_l)}}{2\sqrt{a_0 a_l}} = 1$$

And since $h(0) = 0 < 1$ and $\lim_{\gamma\to\infty} h(\gamma) = 0 < 1$, it must cross $y = 1$ exactly twice.

(ii) Let $g(\gamma) = \tanh(\gamma l)$, then $\lim_{\gamma\to\infty} g(\gamma) = 1$. Therefore, as $\gamma \to \infty$, $g(\gamma) > h(\gamma)$.

$$g(0) = 0, \quad g'(0) = \frac{l}{\cosh^2(\gamma l)}\Big|_{\gamma=0} = l$$

and from the assumption that $-a_0 - a_l < a_0 a_l l$, we have

$$h'(0) = \frac{(-a_0 - a_l)a_0 a_l}{(a_0 a_l)^2} = \frac{-a_0 - a_l}{a_0 a_l} < \frac{a_0 a_l l}{a_0 a_l} = l = g'(0)$$

Therefore, near $\gamma = 0$, we also have $g(\gamma) > h(\gamma)$. However, at $\gamma = \sqrt{a_0 a_l}$, since $\tanh(x) < 1$,

$$g(\sqrt{a_0 a_l}) < 1 < h(\sqrt{a_0 a_L})$$

From above obsevations, $h(\gamma) - g(\gamma)$ changes sign exactly twice, which means that the two functions intersect exactly twice, which then implies there are two (negative) eigenvalues.

4.3.9

a

$$X'' = 0 \Rightarrow X = ax + b$$

From boundary conditions, we get $b = -a$. Dropping the constant factor,

$$\boxed{X_0(x) = x - 1}$$

b

$$X = A\cos\beta x + B\sin\beta x, \quad X' = -A\beta\sin\beta x + B\beta\cos\beta x$$

From boundary condition at $x = 0$, $A = -\beta B$. Rewriting X as $X = -\beta B\cos\beta x + B\sin\beta x$, and from boundary condition at $x = 1$,

$$-\beta B\cos\beta + B\sin\beta = 0 \Rightarrow \boxed{\tan\beta = \beta}$$

c

From the graph of $f(\beta) = \beta$ and $g(\beta) = \tan\beta$, the two curves intersect infinitely many times, which means that there are infinitely many positive eigenvalues.

d

Suppose there exists a negative eigenvalue, then

$$X'' = -\lambda X = \gamma^2 X$$

$$X = A\cosh\gamma x + B\sinh\gamma x$$

$$X' = A\gamma\sinh\gamma x + B\gamma\cosh\gamma x$$

From the boundary condition at $x = 0$, $A = -\gamma B$. Rewrite X and plug in the boundary condition at $x = 1$:

$$-\gamma B\cosh\gamma + B\sinh\gamma = 0 \Rightarrow \tanh\gamma = \gamma$$

However, since $\tanh(0) = 0$ and $\tanh'(0) = 1/\cosh^2(0) = 1$, γ and $\tanh\gamma$ are tangent at the origin and have no other intersections. Therefore a non-zero γ doesn't exist, which means that there isn't a negative eigenvalue.

4.3.18

a

Suppose $u = X(x)T(t)$.

$$XT'' = -c^2X''''T \Rightarrow -\frac{T''}{c^2T} = \frac{X''''}{X} = \lambda \Rightarrow X'''' = \lambda X$$

b

Suppose zero is an eigenvalue, then $X'''' = 0 \Rightarrow X = ax^3 + bx^2 + cx + d$. And its derivatives:

$$X' = 3ax^2 + 2bx + c$$

$$X'' = 6ax + 2b$$

$$X''' = 6a$$

From the boundary conditions, $X(0) = X'(0) = X''(l) = X'''(l) = 0$, thereore a, b, c, d must all be zero, which means that X does not have non-trivial solutions. Therefore, zero is not a eigenvalue.

c

Carrying out the same calculations as in Problem 6.8 above, $\boxed{\cos\beta l \cosh\beta l = -1}$.

d

From Problem 6.8, the frequencies β_n are approximately $\dfrac{(n-1/2)\pi}{l}$ when n is large.

e

Solving the above equation using a computer, the results are

$$\beta_1 \approx \frac{1.875}{l}, \quad \beta_2 \approx \frac{4.694}{l}, \quad \frac{\beta_2^2}{\beta_1^2} \approx 6.267$$

For a vibrating string, $\beta_2^2/\beta_1^2 = 2^2 = 4$. The overtone frequencies of a tunning fork grows faster than a string as n increases.

5.1.2

a

The sine series is

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

where

$$\begin{aligned} A_n &= 2\int_0^1 x^2 \sin(n\pi x) \\ &= 2\Big[-\frac{1}{n\pi}x^2\cos n\pi x + \frac{2}{n^2\pi^2}x\sin n\pi x + \frac{2}{n^3\pi^3}\cos n\pi x\Big]\Big|_0^1 \\ &= (-1)^n\left(\frac{4}{n^3\pi^3} - \frac{2}{n\pi}\right) - \frac{4}{n^3\pi^3} \end{aligned}$$

b

The cosine series is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

where

$$A_0 = 2\int_0^1 x^2 = \frac{2}{3}$$

For $n \geq 1$,

$$\begin{aligned} A_n &= 2\int_0^1 x^2 \cos(n\pi x) \\ &= 2\Big[\frac{1}{n\pi}x^2 \sin n\pi x + \frac{2}{n^2\pi^2} x\cos n\pi x - \frac{2}{n^3\pi^3}\sin n\pi x\Big]\Big|_0^1 \\ &= (-1)^n \frac{4}{n^2\pi^2} \end{aligned}$$

Combine the results,

$$\phi(x) = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2}\cos(n\pi x)$$

5.1.9

Separate the variables:

$$\frac{X''}{X} = \frac{T''}{c^2 T} = \lambda$$

The boundary conditions and initial conditions then translate to

$$X'(0) = X'(\pi) = 0, \quad T(0) = 0, \quad X(x)T'(0) = \cos^2 x$$

(i) For $\lambda = 0$, $X'' = 0 \Rightarrow X = Ax + B$. The boundary condition implies that $A = 0$, so

$$X_0 = 1$$

The corresponding $T_0 = Ct + D$, using the initial condition $T_0(0) = D = 0$. Therefore

$$T_0 = t$$

(ii) For $\lambda > 0$, the boundary conditions cannot be satisfied.

(iii) For $\lambda < 0$, write λ as $-\beta^2$, then

$$X = A\cos\beta x + B\sin\beta x, \quad X' = -A\beta\sin\beta x + B\beta\cos\beta x$$

$X'(0) = 0$ implies that $B = 0$, and $X'(\pi) = 0$ implies:

$$-A\beta\sin\beta\pi = 0 \Rightarrow \beta\pi = n\pi \Rightarrow \beta = n$$

$$X_n = \begin{cases} 1, & n = 0 \\ \cos nx, & n > 0 \end{cases}$$

Solving for T_n:

$$T_n = C_n \cos cnt + D_n \sin cnt$$

The initial condition implies that $C_n = 0$, so

$$T_n = \sin cnt, \quad T_n' = cn\cos cnt$$

$$T_n = \begin{cases} t, & n = 0 \\ \sin cnt, & n > 0 \end{cases}$$

Finally

$$u(x,t) = C_0X_0T_0 + \sum_{n=1}^{\infty} C_nX_nT_n = C_0t + \sum_{n=1}^{\infty} C_n \cos nx \sin cnt$$

$$u_t(x,t) = C_0 + \sum_{n=1}^{\infty} cnC_n \cos nx \cos cnt$$

$$u_t(x,0) = C_0 + \sum_{n=1}^{\infty} cnC_n \cos nx = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

Comparing the terms, we get

$$C_n = \begin{cases} 1/2, & n = 0 \\ 1/4c, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\boxed{u(x,t) = \frac{1}{2}t + \frac{1}{4c}\cos 2x \sin 2ct}$$