MATH 4441 Homework 7

Wenqi He

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9.12

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = e_i(p)$. Let $\alpha = X^{-1} \circ \gamma$, then $\gamma(0) = X(\alpha(0)) = X(0,0)$, so $\alpha(0) = (0,0)$. And since

$$\gamma'(0) = (X \circ \alpha)'(0) = J_{(0,0)}(X)\alpha'(0) = e_i(p) = D_i X(0,0)$$

we have $\alpha'(0) = \epsilon_i$, where $\epsilon_1 = (1,0)$, $\epsilon_2 = (0,1)$. Now define $f : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$f(t) := \Big\langle n(\gamma(t)), e_j(\gamma(t)) \Big\rangle \equiv 0$$

Take the derivative at $t = 0$:

$$f'(0) = \Big\langle n(\gamma(t))', e_j(\gamma(t)) \Big\rangle|_{t=0} + \Big\langle n(\gamma(t)), e_j(\gamma(t))' \Big\rangle|_{t=0} = 0 \tag{*}$$

The first term evaluates to

$$\Big\langle (n \circ \gamma)'(0), e_j(\gamma(0)) \Big\rangle = \Big\langle dn_p(e_i(p)), e_j(p) \Big\rangle = \boxed{-\Big\langle S_p(e_i(p)), e_j(p) \Big\rangle}$$

The second terms evaluates to

$$\begin{aligned}
&\Big\langle n(\gamma(0)), (e_j \circ \gamma)'(0) \Big\rangle \\
&= \Big\langle n(X(\alpha(0))), (D_j X \circ X^{-1} \circ X \circ \alpha)'(0) \Big\rangle \\
&= \Big\langle N(\alpha(0)), (D_j X \circ \alpha)'(0) \Big\rangle \\
&= \Big\langle N(0,0), J_{(0,0)}(D_j X)\alpha'(0) \Big\rangle \\
&= \Big\langle N(0,0), J_{(0,0)}(D_j X)\epsilon_i \Big\rangle \\
&= \Big\langle N(0,0), D_i D_j X(0,0) \Big\rangle \\
&\boxed{= \Big\langle N(0,0), D_{ij} X(0,0) \Big\rangle = l_{ij}(0,0)}
\end{aligned}$$

Plug the terms into $(*)$,

$$l_{ij}(0,0) = \Big\langle S_p(e_i(p)), e_j(p) \Big\rangle$$

9.15

$$\det(\text{Hess } f) = \det \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} = \det \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} = 4ab, \quad \text{grad } f(0,0) = \begin{pmatrix} 2ax \\ 2by \end{pmatrix} \Big|_{(0,0)} = 0$$

So the curvature at $(0, 0, f(0, 0))$ is $K = \dfrac{4ab}{(1+0)^2} = 4ab$.

10.3

$$\text{Hess } f = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -6y & -6x \\ -6x & 6y \end{pmatrix}, \quad \det(\text{Hess } f(0,0)) = \det 0_{2,2} = 0$$

Therefore by the curvature formula, $K(p) = 0$. However, the surface is not locally convex at p, because the surface does not lie on one side of the tangent plane $z = 0$. $z > 0$ when $y > 0$ and $|y| > \sqrt{3}|x|$ or $y < 0$ and $|y| < \sqrt{3}|x|$, and $z < 0$ when $y > 0$ and $|y| < \sqrt{3}|x|$ or $y < 0$ and $|y| > \sqrt{3}|x|$,

10.6

Suppose $ac - b^2 > 0$. If $x \neq 0$ but $Q = 0$ at some point, then

$$\frac{Q}{x^2} = a + 2b\frac{y}{x} + c\left(\frac{y}{x}\right)^2 = 0$$

must has a solution, thus the discriminant $4b^2 - 4ac \geq 0 \Rightarrow ac - b^2 \leq 0$, which is a contradiction. Therefore, $ac - b^2 > 0$ implies that Q is definite

Suppose $ac - b^2 < 0$. If $Q \neq 0$ whenever $x \neq 0$, then the above equation cannot have solutions, so the discriminant $4b^2 - 4ac < 0 \Rightarrow ac - b^2 > 0$, which again is a contradiction. Therefore, $ac - b^2 < 0$ implies that Q is not definite.

10.11

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve such that $\gamma(0) = X(u^1, u^2)$ and $\gamma'(0) = D_i X(u^1, u^2)$ and let $\alpha = X^{-1} \circ \gamma$. Since $\gamma'(0) = (X \circ \alpha)'(0) = J_{(u^1,u^2)}(X)\alpha'(0) = D_i X(u^1, u^2)$, we have $\alpha'(0) = \epsilon_i$. Then

$$\begin{aligned} dn(D_i X(u^1, u^2)) &= (n \circ \gamma)'(0) = (n \circ X \circ \alpha)'(0) = ((n \circ X) \circ \alpha)'(0) \\ &= J_{(u^1,u^2)}(n \circ X)\alpha'(0) \\ &= J_{(u^1,u^2)}(n \circ X)\epsilon_i \\ &= D_i(n \circ X)(u^1, u^2) \end{aligned}$$