MATH 2106 Homework 4

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1

a

Let σ denote a Hamilton path, and let I_σ be a random vairable that takes value 1 if the tournament contains such a path and 0 otherwise. The number of Hamilton paths is a random variable

$$N = \sum_\sigma I_\sigma$$

The expectation of N is

$$E[N] = E[\sum_\sigma I_\sigma] = \sum_\sigma E[I_\sigma] = \sum_\sigma 1 \cdot P(\sigma) + 0 = \sum_\sigma \left(\frac{1}{2}\right)^2 = 3!\left(\frac{1}{2}\right)^2 = \frac{3}{2}$$

b

Denote the players as a, b, c. The possible outcomes are

$a \to b$	$b \to c$	$c \to a$	3 Hamiltonian paths
$a \to b$	$b \to c$	$a \to c$	1
$a \to b$	$c \to b$	$c \to a$	1
$a \to b$	$c \to b$	$a \to c$	1
$b \to a$	$b \to c$	$c \to a$	1
$b \to a$	$b \to c$	$a \to c$	1
$b \to a$	$c \to b$	$c \to a$	1
$b \to a$	$c \to b$	$a \to c$	3

The average is $\dfrac{12}{8} = \dfrac{3}{2}$

2

Consider K_k. If all edges in K_k are colored red, then there is a red K_k, Otherwise, if not all edges are red, then there must be a blue edge. So $R(k, 2) \leq k$. Now consider K_{k-1}. If all edges are colored red, then the graph contains neither a red K_k nor a blue edge, so $R(k, 2) > k - 1$. Therefore, $R(k, 2) = k$.

3

Consider any vertex v, there are two possibilities:

- Suppose there are at least 6 red edges incident to it. Pick any 6 vertices other than v that are incident to these edges. Since in a clique every pair of vertices are connected by an edge, these 6 vertices form a K_6. Since $R(3,3) = 6$, there exists either a red K_3 or blue K_3 in this K_6. If it's blue then we are done. If it's red, then we can form a red K_4 by adding the 3 red edges connecting each of these vertices to v.
- Suppose there are fewer than 6 red edges incident to v, then there must be more than $9 - 6 = 3$ blue edges. In other words, there are at least 4 blue edges. Any 4 vertices other than v that are incident to these blue edges form a K_4. If it's red then we are done. Otherwise, if any of the edges in this K_4 is blue, since the two edges connecting them to v are also blue, they together form a blue K_3.

Thus we can always find either a red K_4 or a blue K_3 in an arbitrary coloring of K_{10}, which means that $R(4,3) \leq 10$.

4

Consider a random coloring of the elements. The probability that a set is monochromatic is

$$P(monochromatic) = P(red) + P(blue) = 2 \cdot \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k-1}$$

If we define a random variable for each set S

$$I_S = \begin{cases} 1, & \text{if } S \text{ is monochromatic} \\ 0, & \text{otherwise} \end{cases}$$

then the number of monochromatic sets in a collection of m k-sets is $X = \sum_{i=1}^{m} I_{s_i}$, and

$$E[X] = E[\sum_{i=1}^{m} I_{s_i}] = \sum_{i=1}^{m} E[I_{s_i}] = \sum_{i=1}^{m} \left[1 \cdot \left(\frac{1}{2}\right)^{k-1} + 0\right] = m\left(\frac{1}{2}\right)^{k-1}$$

if $m < 2^{k-1}$, then

$$m\left(\frac{1}{2}\right)^{k-1} < 2^{k-1}\left(\frac{1}{2}\right)^{k-1} = 1$$

Since X can only take integer values, it must be zero for at least one Red-Blue coloring, which means that there exists a coloring such that none of the S_i is monochromatic. Therefore by definition, the collection of m k-sets always admits a proper Red-Blue coloring when $m < 2^{k-1}$.

7.6

1. From right to left: If $y = x^2$, $LHS = x^3 + x^4 = RHS$. If $y = -x$, $LHS = x^3 - x^3 = 0 = x^2 - x^2 = RHS$.

2. From left to right: $x^3 + x^2y - y^2 - xy = (x^2 - y)(x + y) = 0$, therefore $y = x^2$ or $y = -x$

7.8

1. From right to left: Suppsoe $a \equiv b \pmod 2$, then $a - b = 2k$. Suppose $a \equiv b \pmod 5$, then $5 \mid a-b$, By Euclid's lemma $5 \mid k$. Rewrite k as $5n$: $a-b = 2 \cdot 5n = 10n$, therefore $a \equiv b \pmod{10}$.

2. From left to right: Suppose $a \equiv b \pmod{10}$, then $a - b = 10k = 2 \cdot 5 \cdot k$. Therefore $2 \mid a - b$ and $5 \mid a - b$, or equivalently, $a \equiv b \pmod 2$ and $a \equiv b \pmod 5$.

7.18

We can simply construct such a set. For example: $X = \mathbb{N} \cup \{\mathbb{N}\}$, or $X = \mathbb{N} \cup \mathcal{P}(\mathbb{N})$.

10.2

Base case: When $n = 1$, the statement is $1^2 = \dfrac{1 \cdot 2 \cdot 3}{6} = 1$, which is true.

Inductive step: Suppose the statement is true for $k \geq 1$, then

$$\begin{aligned}
1 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k+1)\Big[k(2k+1) + 6(k+1)\Big]}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)\Big[(k+1)+1\Big]\Big[2(k+1)+1\Big]}{6}
\end{aligned}$$

So the statement is true for $k + 1$. By induction it's true for all n.

10.10

Base case: When $n = 0$, the statement becomes 3 divdes $5^0 - 1 = 0$. which is true.

Inductive step: Suppose for $k \geq 0$, $3 \mid \left(5^{2k} - 1\right)$, then

$$\begin{aligned}
5^{2k} &= 3m + 1 \\
5^{2(k+1)} - 1 &= 25 \cdot 5^{2k} - 1 \\
&= 25 \cdot (3m + 1) - 1 \\
&= 25 \cdot 3m + 24 \\
&= 3(25m + 8)
\end{aligned}$$

so $3 \mid \left(5^{2(k+1)} - 1\right)$. By induction, the statement is true for all $n \geq 0$.

10.24

Base case: When $n = 1$,

$$LHS = 1\binom{1}{1} = 1 = 1 \cdot 2^{1-1} = 1 = RHS$$

Inductive step: Suppose for $m \geq 1$,

$$\sum_{k=1}^{m} k\binom{m}{k} = m2^{m-1}$$

then

$$\begin{aligned}
\sum_{k=1}^{m+1} k\binom{m+1}{k} &= \sum_{k=1}^{m+1} k\left[\binom{m}{k} + \binom{m}{k-1}\right] \\
&= \sum_{k=1}^{m+1} k\binom{m}{k} + \sum_{k=1}^{m+1} k\binom{m}{k-1} \\
&= \sum_{k=1}^{m+1} k\binom{m}{k} + \sum_{k=0}^{m}(k+1)\binom{m}{k} \\
&= \sum_{k=1}^{m+1} k\binom{m}{k} + \sum_{k=0}^{m} k\binom{m}{k} + \sum_{k=0}^{m}\binom{m}{k} \\
&= 2\sum_{k=1}^{m} k\binom{m}{k} + \sum_{k=0}^{m}\binom{m}{k} \\
&= 2 \cdot m2^{m-1} + 2^m \\
&= (m+1)2^m = (m+1)2^{(m+1)-1}
\end{aligned}$$

So the statement is true for $m + 1$. By induction, the statement is true for all $n \geq 1$.