MATH 2106 Homework 5

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1

Let n, m, f be the number of vertices, edges and faces, respectively. Then since each edge is on the boundary of exactly two faces, and each face is enclosed by at least 4 edges (for there are no 3-cycles), we have

$$2m = \sum_{i=1}^{f} deg(F_i) \geq \sum_{i=1}^{f} 4 = 4f \quad \Rightarrow \quad m \geq 2f$$

Then according to Euler's characteristic formula,

$$\begin{aligned} &n - m + f = 2 \\ \Rightarrow\ & 4 = 2n - 2m + 2f \leq 2n - 2m + m = 2n - m \\ \Rightarrow\ & m \leq 2n - 4 \end{aligned}$$

2

K_5 has $5 \cdot 4/2 = 10$ edges, but a planar graph with 5 vertices can have at most $3 \cdot 5 - 6 = 9$ edges, so K_5 is nonplanar. $K_{3,3}$ has $6 \cdot 3/2 = 9$ edges, but a planar graph with 6 vertices and no 3 cycles can have at most $2 \cdot 6 - 4 = 8$ edges, so $K_{3,3}$ is also nonplanar.

8.2

If $x \in \{6n : n \in \mathbb{Z}\}$ then $x = 6k = 2(3k) = 3(2k)$ for some integer k, so $x \in \{2n : n \in \mathbb{Z}\}$ and $x \in \{3n : n \in \mathbb{Z}\}$, therefore

$$\{6n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$$

Now suppose $x \in \{2n : n \in \mathbb{Z}\}$ and $x \in \{3n : n \in \mathbb{Z}\}$, then $x = 2i = 3j$ for some integers i, j. By Euclid's lemma, $3 \mid i$, so we can write i as $3k$. Then $x = 2(3k) = 6k$ for some integer k, and so $x \in \{6n : n \in \mathbb{Z}\}$. Therefore

$$\{2n : n \in \mathbb{Z}\} \cap \{3n : n\, n\mathbb{Z}\} \subseteq \{6n : n \in \mathbb{Z}\}$$

We have shown that both directions hold, so

$$\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} = \{6n : n \in \mathbb{Z}\}$$

8.8

Suppose $x \in A \cup (B \cap C)$, then by definition, $x \in A \vee (x \in B \wedge x \in C)$, then by distributive law, $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$. In terms of sets, $x \in (A \cup B) \cap (A \cup C)$. Therefore by definition,

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

If we follow the same steps but apply the distribution law in the other direction, we will get

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

Since both directions hold,

$$(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$$

8.18

Suppose $(x, y) \in A \times (B - C)$, then by definition of Cartesian products and set differences, $x \in A \wedge (y \in B \wedge y \notin C)$. Since $x \in A \wedge y \in B$, by definition of Cartesian products, $(x, y) \in A \times B$. And since $x \in A$ but $y \notin C$, again by definition of Cartesian products, $(x, y) \notin A \times C$. Then by definition of set differences, $(x, y) \in A \times B - A \times C$. So

$$A \times (B - C) \subseteq A \times B - A \times C$$

Now suppose $(x, y) \in A \times B - A \times C$, then $(x \in A \wedge y \in B) \wedge \neg(x \in A \wedge y \in C)$. From the second statement, $x \notin A \vee y \notin C$, and from from the first statement $x \in A$, in order for both statements to be true, it must be true that $y \notin C$. So now we have $x \in A \wedge (y \in B \wedge y \notin C)$, by definition of Cartesian products and set differences, $(x, y) \in A \times (B - C)$, and therefore

$$A \times B - A \times C \subseteq A \times (B - C)$$

Since both directions hold,

$$A \times (B - C) = A \times B - A \times C$$

11.1.8

For any $x \in \mathbb{Z}$, the only $y \in \mathbb{Z}$ that satisfies $|x - y| < 1$ is x itself. Therefore, we have

- $|x - x| = 0 < 1 \Rightarrow xRx$, so R is reflexive.
- $xRx \Rightarrow xRx$, so R is symmetric.
- $(xRx \wedge xRx) \Rightarrow xRx$, so R is transitive.

R is the identity relation.

11.1.16

- $x^2 = x^2$, therefore $x^2 \equiv x^2 \pmod 4$, so R is reflexive.
- If xRy, then $x^2 \equiv y^2 \pmod 4$. Because congruence relation is symmetric, $y^2 \equiv x^2 \pmod 4$, then by definition yRx. So R is symmetric.
- If xRy, yRz then $x^2 \equiv y^2 \pmod 4$ and $y^2 \equiv z^2 \pmod 4$. Because congruence relation is transitive, $x^2 \equiv z^2 \pmod 4$, then by definition xRz. So R is transitive.

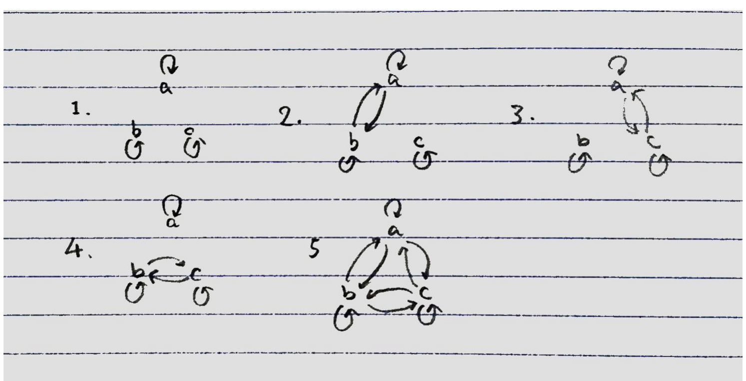
11.2.4

Starting from b,

- $bRc \Rightarrow cRb$.
- $bRc \wedge cRe \Rightarrow bRe \Rightarrow eRb$.
- $bRe \wedge eRa \Rightarrow bRa \Rightarrow aRb$.
- $bRa \wedge aRd \Rightarrow bRd \Rightarrow dRb$.

Therefore $[b] = A$. There is only one equivalence class.

11.2.6



11.2.10

Because R and S are both equivalence relations, for all $x \in A$, $(x, x) \in R$ and $(x, x) \in S$, and therefore $(x, x) \in R \cap S$. so $R \cap S$ is reflexive. If $(x, y) \in R \cap S$, then $(x, y) \in R$ and $(x, y) \in S$. By symmetry, $(y, x) \in R$ and $(y, x) \in S$, therefore $(y, x) \in R \cap S$. So $R \cap S$ is symmetric. Finally, if $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$, then by transitivity

$$\Big((x, y) \in R \wedge (y, z) \in R\Big) \Rightarrow (x, z) \in R, \quad \Big((x, y) \in S \wedge (y, z) \in S\Big) \Rightarrow (x, z) \in S$$

so $(x, z) \in R \cap S$. Therefore $R \cap S$ is also transitive. Since $R \cap S$ has all three properties, it is a equivalence relation.