From Practice Test

- Every permutation can be decomposed into cycles of *disjoint orbits*. Ex.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix}$$

- **Def.** Projection: $P : V \to V$ s.t. $P^2 = P$ (idempotent).

- **Thm.** Projection matrices are diagnolizable.
  Pf. Suppose rank is r, for any $v = Pu$ in the range, $Pv = P^2u = Pu = v$, therefore the range is the eigenspace associated with eigenvalue 1, and there exists a basis of r eigenvectors for the range. We can find the other $n - r$ eigenvectors from the kernel, which has dim. $n - r$.

- Solve congruence equation by finding multiplicative inverse. Ex. In $\mathbb{Z}/11\mathbb{Z}$, $12 \equiv 1$, so $6a \equiv 9 \Rightarrow 2 \cdot 6a \equiv 2 \cdot 9 \Rightarrow a \equiv 12a \equiv 18$ (2 is the inverse of 6)

- **Fundamental Theorem of Finite Abelian Groups:** A finite Abelian group is isomorphic to a direct sum of cyclic groups of prime-power order, where the decomposition is unique up to the order in which the factors are written.

- **Def.** Ideal of a ring R: A subset of R that (i) forms an *additive subgroup* and (ii) absorbs multiplication with any element from R (Ex. $2\mathbb{Z}$ is an ideal of ring $\mathbb{Z}$)

- **Def.** Boundary of S: The set of points whose neighborhood contains at least one point of S and at least one point not of S.

- The set of rational/irrational numbers is dense in the reals.

- **Fundamental Theorem of Calculus**

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t)dt = f(v(x))v'(x) - f(u(x))u'(x)$$

- Use Lagrange multiplier method under equality constraints: $\nabla f = \lambda \nabla g$

- Gram-Schmidt:

$$u_k = v_k - \sum_{j=1}^{k-1} \mathrm{proj}_{u_j}(v_k)$$

- Trick: Use parity of functions to simplify integrals.

- Trick: Use contrapositive to check *Complicated* $\Rightarrow$ *Simple* statements

From Princeton Review

- To find normal vector, treat surfaces as level sets and take gradient.
- $\det A = \prod \lambda_i,\ \mathrm{Tr}\, A = \sum \lambda_i$
- in $\mathbb{Z}/\mathbb{Z}_n$, $n \cdot 1 = 0$.
- To find the fixed points of recurrence $x_n = f(x_{n-1})$, use $x_{n\to\infty}$
- To find the radius of convergence, use ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$
- $\log x < x$
- To solve the Diophatine equation $ax + by = c$:
  - First solve $ax + by = gcd(a, b)$
  - Scale the solution by $c/gcd(a, b)$
  - Combine with the general solution to $ax + by = 0$:

$$x = x_0 + \frac{b}{gcd(a,b)}t,\ y = y_0 - \frac{a}{gcd(a,b)}t$$

- First three terms of Taylor series: $f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$
- To evaluate limit of $y = f(x)^{g(x)}$, take logarithm first.
- To differentiate $y = f(x)^{g(x)}$, take logarithm of both side first:

$$\log y = \log f(x)^{g(x)} = g(x)\log f(x)$$

$$\frac{y'}{y} = \Big(g(x)\log f(x)\Big)'$$

- $\sin x$ and $\cos x$ have alternating Taylor coefficients.
- Sum of all kth roots of unity equals 0:

$$\sum_0^{k-1} w^k = \frac{1}{w-1}(w-1)\sum_0^{k-1} w^k = \frac{1}{w-1}\left(\sum_1^{k} w^k - \sum_0^{k-1} w^k\right) = \frac{w^k - 1}{w-1} = 0$$

- Evaluate $\left(\frac{\partial \theta}{\partial r}\right)_x$: Let $x = r\cos\theta = c$, take total derivative w.r.t r:

$$\frac{\partial}{\partial r}\Big(r\cos\theta\Big)_\theta \left(\frac{\partial r}{\partial r}\right)_x + \frac{\partial}{\partial \theta}\Big(r\cos\theta\Big)_r \left(\frac{\partial \theta}{\partial r}\right)_x = 0$$

$$\cos\theta - r\sin\theta\left(\frac{\partial \theta}{\partial r}\right)_x = 0$$

- Green's Theorem (Stokes' Theorem)

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$
$$\Rightarrow \int_{\partial\Omega} f dx + g dy = \int_{\Omega} d(f dx + g dy) = \int_{\Omega} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

- Residue Theorem
$$\oint_C f(z)dz = 2\pi i \sum_i Res(f(z), \alpha_i)$$
where $Res(f(z), \alpha_i)$ is a_{-1} of the Laurent series around α_i.

- Cauchy-Riemann equations $\rightarrow$ Cauchy's integral theorem:
$$\oint_C f(z)dz = 0$$