MATH 2106 Homework 6

Wenqi He

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11.3.2

The partitions are $\{\{a\},\{b\},\{c\}\}$, $\{\{a,b\},\{c\}\}$, $\{\{a,c\},\{b\}\}$, $\{\{b,c\},\{a\}\}$, and $\{\{a,b,c\}\}$. Each partition corresponds to an equivalence relation on $\{a,b,c\}$.

11.3.4

i

- Reflexivity: $\forall x \in A : \exists X \in P$ such that $x \in X$, therefore by definition of R, xRx
- Symmetry: Suppose xRy, then $\exists X \in P : x \in X \wedge y \in X$, which also implies yRx.
- Transitivity: Suppose xRy, yRz, then $\exists X \in P : x \in X \wedge y \in X$, and $\exists Y \in P : y \in Y \wedge z \in Y$. Furthermore, Y must be the same as X, because $y \in X \cap Y \neq \emptyset$, which cannot be true if $X \neq Y$ since P is a partition. Therefore $z \in Y = X$, and by definition of R, xRz.

Therefore, R is indeed an equivalent relation on A.

ii

Let $S := \{[a], a \in A\}$ be the set of equivalence classes of R. We will show that $S = P$:

For any $X \in P$, we can pick an arbitrary $x \in X$. Now consider its equivalence class $[x]$. For any $a \in [x]$, we have aRx, then by definition of R and the fact that X is the only subset that contains x (because P is a partition), $a \in X$. Since $\forall a \in [x] : a \in X$, we have $[x] \subseteq X$. On the other hand, for any $a \in X$, aRx by definition of R, so $a \in [x]$, and therefore $X \subseteq [x]$. The above results imply that $X = [x]$, which means that X is a equivalence class, or expressed formally, $X \in S$. Since $\forall X \in P : X \in S$, we have shown that $P \subseteq S$.

Now for any equivalence class $Y \in S$, we can pick any $a \in Y$, then by definition of equivalence classes, $Y = [a]$. Since P is a partition, $\exists X \in P$ such that $a \in X$. We can show that $Y = [a] = X \in P$ in the same way as the previous paragraph. Since $\forall Y \in S : Y \in P$, we have shown that $S \subseteq P$.

Therefore, $S = P$. In other words, P is the set of equivalence classes of R.

12.1.6

Domain: $\mathbb{Z}$. Codomain: $\mathbb{Z}$. Range: $\{4n+1 : n \in \mathbb{Z}\}$. $f(10) = 4 \cdot 10 + 5 = 45$.

12.1.8

No. There isn't a $(x, y) \in f$ for all $x \in \mathbb{Z}$. For example, suppose $x = 2$, then there doesn't exist an integer y that satisfies the equation.

12.1.12

Yes. Domain: $\mathbb{R}^2$. Codomain: $\mathbb{R}^3$. Range: $\{(x, y, z) \in \mathbb{R}^3 : z = \frac{x}{3} + \frac{y}{2}\}$

12.2.10

Let $y = \left(\frac{x+1}{x-1}\right)^3$, then $x = \frac{1 + y^{1/3}}{y^{1/3} - 1}$, which means that

$$f^{-1}(x) = \frac{1 + x^{1/3}}{x^{1/3} - 1}$$

Since f is invertible, f must be bijective.

12.2.18

1. Suppose $\frac{(-1)^n(2n-1)+1}{4} = \frac{(-1)^m(2m-1)+1}{4}$, where $n, m \in \mathbb{N}$ then $(-1)^n(2n-1) = (-1)^m(2m-1)$. If n and m have different parities, then

   $$2n - 1 = 1 - 2m \quad \Rightarrow \quad n + m = 1$$

   which is impossible since $n \geq 1$ and $m \geq 1$. Therefore m and n must have the same parity,

   $$2n - 1 = 2m - 1 \quad \Rightarrow \quad n = m$$

   Since $\forall m, n : f(m) = f(n) \Rightarrow m = n$, f is injective.

2. For any $z \in \mathbb{Z}$ and $z > 0$, we have $2z \in \mathbb{N}$, and

   $$f(2z) = \frac{(-1)^{2z}(2 \cdot 2z - 1) + 1}{4} = \frac{4z - 1 + 1}{4} = z$$

   For any $z \in \mathbb{Z}$ and $z \leq 0$, we have $-2z + 1 > 0 \Rightarrow -2z + 1 \in \mathbb{N}$, and

   $$f(-2z+1) = \frac{(-1)^{-2z+1}(2(-2z+1) - 1) + 1}{4} = \frac{-(-4z + 2 - 1) + 1}{4} = z$$

   This shows that $\forall z \in \mathbb{Z} : \exists x \in \mathbb{N} : f(x) = z$, therefore f is surjective.

Since f is both injective and surjective, it's bijective.

12.4.8

$$
\begin{aligned}
(g \circ f)(m,n) &= g(f(m,n)) = g(3m-4n, 2m+n) \\
&= (5(3m-4n) + (2m+n), 3m-4n) \\
&= (17m - 19n, 3m-4n)
\end{aligned}
$$

$$
\begin{aligned}
(f \circ g)(m,n) &= f(g(m,n)) = f(5m+n, m) \\
&= (3(5m+n) - 4(m), 2(5m+n) + (m)) \\
&= (11m + 3n, 11m + 2n)
\end{aligned}
$$

12.4.10

$$
\begin{aligned}
(f \circ f)(x,y) &= f(f(x,y)) = f(xy, x^3) \\
&= (xy \cdot x^3, (xy)^3) \\
&= (x^4 y, x^3 y^3)
\end{aligned}
$$

12.5.4

Let $y = e^{x^3+1}$, then

$$
\begin{aligned}
\log y &= x^3 + 1 \\
x^3 &= \log y - 1 \\
x &= (\log y - 1)^{1/3} = f^{-1}(y)
\end{aligned}
$$

So $f^{-1}(x) = (\log x - 1)^{1/3}$.

12.5.10

From 12.2.8, the inverse $f^{-1} : \mathbb{Z} \to \mathbb{N}$ is:

$$
f^{-1}(z) = \begin{cases} 2z, & z > 0 \\ -2z + 1, & z \leq 0 \end{cases}
$$