MATH 4347 Homework 5

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7.7

The general form of the full Fourier series is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin x \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx dx = \frac{2((-1)^n + 1)}{\pi(1-n^2)}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin x \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx dx = 0$$

Therefore the series is

$$\begin{aligned}|\sin x| &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2((-1)^n + 1)}{\pi(1-n^2)}\cos nx = \frac{2}{\pi} + \sum_{even} \frac{4}{\pi(1-n^2)}\cos nx \\ &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)}\cos 2nx\end{aligned}$$

Since the series converges pointwise, at $x = 0$

$$|\sin 0| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)}\cos 0 \quad \Rightarrow \quad \boxed{\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}}$$

At $x = \pi/2$

$$|\sin\frac{\pi}{2}| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)}\cos n\pi$$

$$1 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1-4n^2)}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}}$$

b

Let $g(x)$ be a even 2-periodic function, whose value in interval $[-1,1]$ is $g(x)=x^2$. The Fourier coefficients are

$$A_0 = \int_{-1}^{1} x^2 = \frac{2}{3}, \quad A_n = \int_{-1}^{1} x^2 \cos n\pi x = \frac{4(-1)^n}{n^2\pi^2}, \quad B_n = \int_{-1}^{1} x^2 \sin n\pi x = 0$$

The full series is

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x$$

Evaluated at $x=1$

$$1 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Evaluated at $x=0$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

8.2

If $u(x) \equiv c$, then $\max\limits_{\overline{U}} u = \max\limits_{\partial U} u \equiv c$.

Otherwise, by the strong form of the maximum principle,

$$\forall x \in U : u(x) < \max_{\partial U} u \quad \Rightarrow \quad \max_{U} u < \max_{\partial U} u$$

$$\max_{\overline{U}} u = \max\{\max_{\partial U} u, \max_{U} u\} = \max_{\partial U} u$$

5.2.11

The complex form of full Fourier series is

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} = e^x$$

where the coeeficients are

$$\begin{aligned} c_n &= \frac{1}{2l}\int_{-l}^{l} e^x e^{-in\pi x/l} dx \\ &= \frac{1}{2l}\int_{-l}^{l} e^{(1-in\pi/l)x} dx \\ &= \frac{1}{2(l-in\pi)} e^{(1-in\pi/l)x}\Big|_{-l}^{l} \\ &= \frac{e^{l-in\pi} - e^{in\pi-l}}{2(l-in\pi)} \\ &= (-1)^n \frac{e^l - e^{-l}}{2(l-in\pi)} \end{aligned}$$

So the series is

$$e^x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} e^{in\pi x/l} = \boxed{\sum_{n=-\infty}^{\infty} (-1)^n \sinh l \frac{l + in\pi}{l^2 + n^2\pi^2} e^{in\pi x/l}}$$

And since $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\begin{aligned}
e^x &= \frac{e^l - e^{-l}}{2l} + \sum_{n=-1}^{-\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \cos\frac{n\pi x}{l} \\
&\quad + \sum_{n=-1}^{-\infty} (-1)^n \frac{ie^l - ie^{-l}}{2(l - in\pi)} \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{ie^l - ie^{-l}}{2(l - in\pi)} \sin\frac{n\pi x}{l} \\
&= \frac{e^l - e^{-l}}{2l} + \sum_{n=1}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l + in\pi)} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{e^l - e^{-l}}{2(l - in\pi)} \cos\frac{n\pi x}{l} \\
&\quad + \sum_{n=1}^{\infty} (-1)^n \frac{ie^{-l} - ie^l}{2(l + in\pi)} \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{ie^l - ie^{-l}}{2(l - in\pi)} \sin\frac{n\pi x}{l} \\
&= \frac{e^l - e^{-l}}{2l} + \sum_{n=1}^{\infty} (-1)^n \frac{(e^l - e^{-l})l}{l^2 + n^2\pi^2} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{n\pi(e^{-l} - e^l)}{l^2 + n^2\pi^2} \sin\frac{n\pi x}{l} \\
&= \boxed{\frac{\sinh l}{l} + 2\sinh l \sum_{n=1}^{\infty} \frac{(-1)^n}{l^2 + n^2\pi^2} \left[l\cos\frac{n\pi x}{l} - n\pi \sin\frac{n\pi x}{l} \right]}
\end{aligned}$$

5.3.4

a

Let $v = u - U$, then all derivatives of v is the same as u, and $v(0,t) = u(0,t) - U = 0$. Separation of variables gives

$$\frac{X''}{X} = \frac{T'}{kT} = \lambda, \quad v = X(x)T(t)$$

$\lambda = 0$ under the boundary condition gives only the trivial solution, and for $\lambda > 0$ the boundary conditions cannot be both satisfied, therefore $\lambda < 0$. Let $\lambda = -\beta^2$, then

$$X = A\cos\beta x + B\sin\beta x$$

Boundary condition at 0 implies $A = 0$, and boundary condition at $x = l$ implies

$$\cos\beta l = 0 \Rightarrow \beta_n = \left(n - \frac{1}{2}\right)\frac{\pi}{l} = (2n-1)\frac{\pi}{2l}$$

$$X_n = \sin\beta_n x$$

The corresponding T_n are

$$T' = -k\beta_n^2 T \Rightarrow T = e^{-k\beta_n^2 t}$$

So the general solution is

$$v = \sum_{n=1}^{\infty} A_n e^{-k\beta_n^2 t} \sin \beta_n x, \quad \beta_n = (2n-1)\frac{\pi}{2l}$$

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin \beta_n x = u(x,0) - U = -U$$

$$\Rightarrow A_n = \frac{2}{l}\int_0^l -U \sin \beta_n x = -\frac{2U}{l\beta_n} = -\frac{4U}{(2n-1)\pi}$$

$$\Rightarrow v = -\sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-k\pi^2(2n-1)^2 t/4l^2} \sin \frac{(2n-1)\pi}{2l} x$$

$$u = v + U = \boxed{U - \sum_{n=1}^{\infty} \frac{4U}{(2n-1)\pi} e^{-k\pi^2(2n-1)^2 t/4l^2} \sin \frac{(2n-1)\pi}{2l} x}$$

b

Let a_n be the n-th term of the series. Apply the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2n-1}{2n+1} \frac{\sin \beta_{n+1} x}{\sin \beta_n x} e^{-2k\pi^2 nt/l^2} \right| \\
&= \lim_{n\to\infty} \frac{2n-1}{2n+1} e^{-2k\pi^2 nt/l^2} \left| \frac{\sin \beta_{n+1} x}{\sin \beta_n x} \right| \\
&\leq \lim_{n\to\infty} \frac{2n-1}{2n+1} e^{-2k\pi^2 nt/l^2} = 0
\end{aligned}$$

Therefore, the series converges.

c

The error is smaller than the second term, which is

$$-\frac{4U}{\pi} e^{-k\pi^2 t/4l^2}$$

In order for the error to be within ϵ, it suffices to let

$$\left| \frac{4U}{\pi} e^{-k\pi^2 t/4l^2} \right| = \frac{4|U|}{\pi} e^{-k\pi^2 t/4l^2} < \epsilon$$

$$\boxed{t > -\frac{4l^2}{k\pi^2} \log \frac{\epsilon\pi}{4|U|}}$$

5.3.10

a

Proof by induction:
Base case: To show that $(Z_2, Z_1) = 0$, it's sufficient to show that $(Y_2, X_1) = 0$:

$$\begin{aligned}
(Y_2, X_1) &= (X_2, X_1) - \left(X_2, \frac{X_1}{\|X_1\|}\right)\left(\frac{X_1}{\|X_1\|}, X_1\right) \\
&= (X_2, X_1) - (X_2, X_1)\frac{(X_1, X_1)}{\|X_1\|^2} \\
&= (X_2, X_1) - (X_2, X_1) = 0
\end{aligned}$$

Inductive step: Suppose $\forall n, m \leq k, n \neq m : (Z_m, Z_n) = 0$. Then in order to prove that $\forall n, m \leq k+1, n \neq m : (Z_n, Z_m) = 0$, we only need to prove that $(Z_{k+1}, Z_n) = 0$ for all $n \leq k$, because all other cases are already proven in the k-th step. And to that end, it's sufficient to show that $(Y_{k+1}, Z_n) = 0$. Since the vector in $\{Z_n : n \leq k\}$ are orthonormal by inductive hypothesis,

$$\begin{aligned}
(Y_{k+1}, Z_n) &= (X_{k+1}, Z_n) - (X_{k+1}, Z_1)(Z_1, Z_n) - \cdots - (X_{k+1}, Z_n)(Z_n, Z_n) - \cdots \\
&= (X_{k+1}, Z_n) - (X_{k+1}, Z_n)(Z_n, Z_n) \\
&= (X_{k+1}, Z_n) - (X_{k+1}, Z_n) = 0
\end{aligned}$$

By induction, orthogonality holds for all k.

b

Let $X_1 = \cos x + \cos 2x$, and $X_2 = 3\cos x - 4\cos 2x$, then

$$(X_1, X_1) = (\cos x, \cos x) + (\cos 2x, \cos 2x) = \pi, \quad \|X_1\| = \sqrt{\pi}$$

$$Z_1 = \frac{X_1}{\|X_1\|} = \boxed{\frac{1}{\sqrt{\pi}}(\cos x + \cos 2x)}$$

$$(X_2, Z_1) = \frac{1}{\sqrt{\pi}}\Big[3(\cos x, \cos x) - 4(\cos 2x, \cos 2x)\Big] = -\frac{\sqrt{\pi}}{2}$$

$$\begin{aligned}
Y_2 = X_2 - (X_2, Z_1)Z_1 &= 3\cos x - 4\cos 2x - \left(-\frac{\sqrt{\pi}}{2}\right)\frac{1}{\sqrt{\pi}}(\cos x + \cos 2x) \\
&= 3\cos x - 4\cos 2x + \frac{1}{2}(\cos x + \cos 2x) = \frac{7}{2}\cos x - \frac{7}{2}\cos 2x
\end{aligned}$$

$$(Y_2, Y_2) = \frac{49}{4}\frac{\pi}{2} + \frac{49}{4}\frac{\pi}{2} = \frac{49\pi}{4}, \quad \|Y_2\| = \frac{7\sqrt{\pi}}{2}$$

$$Z_2 = \frac{Y_2}{\|Y_2\|} = \boxed{\frac{1}{\sqrt{\pi}}(\cos x - \cos 2x)}$$

5.4.7

a

The general formula is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x + B_n \sin n\pi x$$

where

$$A_n = \int_{-1}^{1} \phi(x) \cos n\pi x dx = \int_{-1}^{0} (-1-x) \cos n\pi x dx + \int_{0}^{1} (1-x) \cos n\pi x dx = 0$$

$$B_n = \int_{-1}^{1} \phi(x) \sin n\pi x dx = \int_{-1}^{0} (-1-x) \sin n\pi x dx + \int_{0}^{1} (1-x) \sin n\pi x dx = \frac{2}{n\pi}$$

Therefore the full series is

$$\boxed{\phi(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi x}$$

b

The first three non-zero terms are

$$\frac{2}{\pi} \sin \pi x + \frac{1}{\pi} \sin 2\pi x + \frac{2}{3\pi} \sin 3\pi x$$

c

Obviously,

$$\|\phi(x)\|^2 = \int_{-1}^{1} \phi^2(x) dx < \infty$$

Therefore, the series converges in the mean square sense.

d

$$\phi'(x) = -1, \text{ for } x \in (-1,0) \cup (0,1)$$

Since ϕ and ϕ' are both piecewise continuous, the series converges pointwise. [At $x = 0$, it converges to $\frac{1}{2}(-1+1) = 0$.]

e

$\phi(x)$ has a discontinuity at 0, so $\phi \notin C^2[-1,1]$, and therefore the series does not converge uniformly.

5.5.4

a

Separation of variables gives

$$-\frac{X''}{X} = -\frac{T'}{kT} = \lambda$$

For $\lambda = 0$,

$$X'' = 0 \quad \Rightarrow \quad X = A + Bx$$
$$T' = 0 \quad \Rightarrow \quad T = const.$$

For $\lambda > 0$,

$$-X'' = \beta^2 X \quad \Rightarrow \quad X = C\cos\beta x + D\sin\beta x$$
$$T' = -k\beta^2 T \quad \Rightarrow \quad T = e^{-k\beta^2 t}$$

The general solution is

$$\boxed{u = A + Bx + \sum_{n=1}^{\infty} e^{-k\beta_n^2 t}\Big[C_n\cos\beta_n x + D_n\sin\beta_n x\Big]}$$

b

As $t \to \infty$, each term in the sum converges to zero, therefore $\lim u = A + Bx$.

c

From the boundary condition

$$u_x(0,t) = u_x(l,t) = \frac{u(l,t) - u(0,t)}{l}$$
$$X'(0)T(t) = X'(l)T(t) = \frac{X(l)T(t) - X(0)T(t)}{l}$$
$$X'(0) = X'(l) = \frac{X(l) - X(0)}{l}$$

Green's first identity in one dimension is

$$vu'\Big|_0^l = \int_0^l v'u'dx + \int_0^l vu''dx$$

Let $v = u = X$, then

$$\begin{aligned} LHS &= XX'\Big|_0^l \\ &= X(l)X'(l) - X(0)X'(0) \\ &= X(l)\frac{X(l) - X(0)}{l} - X(0)\frac{X(l) - X(0)}{l} \\ &= \frac{[X(l) - X(0)]^2}{l} \end{aligned}$$

$$\begin{aligned} RHS &= \int_0^l (X')^2 dx + \int_0^l XX'' dx \\ &= \int_0^l (X')^2 dx - \int_0^l \lambda X^2 dx \end{aligned}$$

Multiply both sides by l and swap both sides,

$$l\int_0^l (X')^2 dx - l\int_0^l \lambda X^2 dx = [X(l) - X(0)]^2$$

If λ is negative, then the second integral is negative, which means that

$$l\int_0^l (X')^2 dx < [X(l) - X(0)]^2$$

This contradicts the inequality in Ex. 3. Therefore, there cannot be negative eigenvalues.

d

First we can verify that the boundary condition is symmetric:

$$\begin{aligned} X_1'(l)X_2(l) - X_2'(l)X_1(l) &= X_2(l)\frac{X_1(l) - X_1(0)}{l} - X_1(l)\frac{X_2(l) - X_2(0)}{l} \\ &= \frac{X_1(l)X_2(0) - X_1(0)X_2(l)}{l} \end{aligned}$$

$$\begin{aligned} X_1'(0)X_2(0) - X_2'(0)X_1(0) &= X_2(0)\frac{X_1(l) - X_1(0)}{l} - X_1(0)\frac{X_2(l) - X_2(0)}{l} \\ &= \frac{X_1(l)X_2(0) - X_1(0)X_2(l)}{l} \end{aligned}$$

$$X_1'X_2 - X_2'X_1\Big|_0^l = 0$$

Therefore eigenfunctions associated with different eigenvalues are orthogonal. At $t = 0$

$$\phi(x) = u(x,0) = A + Bx + \sum_{n=1}^{\infty} C\cos\beta_n x + D\sin\beta_n x$$

$$(\phi, 1) = (A + Bx, 1)$$

$$\Rightarrow \int_0^l \phi(x)dx = \int_0^l A + Bxdx = [Ax + \frac{B}{2}x^2]\Big|_0^l = lA + \frac{l^2}{2}B$$

$$(\phi, x) = (A + Bx, x)$$

$$\Rightarrow \int_0^l \phi(x)xdx = \int_0^l Ax + Bx^2dx = [\frac{A}{2}x^2 + \frac{B}{3}x^3]\Big|_0^l = \frac{l^2}{2}A + \frac{l^3}{3}B$$

$$\boxed{A = \frac{4}{l}\int_0^l \phi(x)dx - \frac{6}{l^2}\int_0^l \phi(x)xdx}$$

$$\boxed{B = \frac{12}{l^3}\int_0^l \phi(x)xdx - \frac{6}{l^2}\int_0^l \phi(x)dx}$$

6.1.2

The equation is $\Delta u = k^2 u$. Expressed in spherical coordinates, dropping zero terms:

$$\frac{1}{r}\partial_r^2(ru) = k^2 u$$

Let $v = ur$. Multiply the equation by r

$$v''(r) = k^2 v(r) \quad \Rightarrow \quad v = Ae^{kr} + Be^{-kr}$$

$$u = \frac{v}{r} = \boxed{\frac{1}{r}\left[Ae^{kr} + Be^{-kr}\right]}$$

6.1.9

a

The heat equation is $u_t = \Delta u$. In steady state, $u_t = \Delta u = 0$. Since u only depends on r,

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = 0$$

Let $v(r) = ru(r)$, then $v''(r) = 0 \quad \Rightarrow \quad v(r) = Ar + B$,

$$u(r) = \frac{v}{r} = A + \frac{B}{r}$$

$$u'(r) = -B\frac{1}{r^2}$$

Plug in the boundary conditions:

$$\begin{cases} u(1) = A + B = 100 \\ u'(2) = -\dfrac{B}{4} = -\gamma \end{cases} \Rightarrow \begin{cases} A = 100 - 4\gamma \\ B = 4\gamma \end{cases}$$

Therefore, the temperature distribution is

$$\boxed{u(r) = 100 - 4\gamma + \frac{4\gamma}{r}}$$

b

$$u'(r) = -\frac{4\gamma}{r^2} < 0$$

Temperature decreases radially, therefore the highest temperature is obtained at $r = 1$, which is $\boxed{100}$, and the lowest temperatue is obtained at $r = 2$, which is $\boxed{100 - 2\gamma}$

c

Yes. $\gamma = 40$.