MATH 4441 HW 8

Wenqi He

October 17, 2018

10.12

By Leibniz's rule, $D_i \langle F, G \rangle = \langle D_i F, G \rangle + \langle F, D_i G \rangle = 0$, so $\langle D_i F, G \rangle = -\langle F, D_i G \rangle$

10.13

Apply a rigid motion so that p coincides with the origin and $T_p M$ coincides with the x-y plane, then we can use a Monge patch around p, with the mapping

$$X(u^1, u^2) = (u^1, u^2, f(u^1, u^2))$$

Since the surface is tangent to x-y plane at the origin, $D_1 f(0,0) = D_2 f(0,0) = 0$. So

$$D_1 X(0,0) = (1, 0, D_1 f(0,0)) = (1,0,0) := e_1$$

$$D_2 X(0,0) = (0, 1, D_2 f(0,0)) = (1,0,0) := e_2$$

$$g_{ij}(0,0) = \langle D_i X(0,0), D_j X(0,0) \rangle = \langle e_i, e_j \rangle = \delta_{ij}$$

Therefore $[g_{ij}(0,0)] = [\delta_{ij}] = I$

10.15

For any γ, since γ lies on the surfce, $\langle \gamma'(t), n(\gamma(t)) \rangle = 0$. Taking the derivative at $t = 0$,

$$\langle \gamma'(t), n(\gamma(t)) \rangle'|_{t=0} = \langle \gamma''(0), n(p) \rangle + \langle \gamma'(0), (n \circ \gamma)'(0)) \rangle = 0$$

Therefore

$$\langle \gamma''(0), n(p) \rangle = -\langle \gamma'(0), (n \circ \gamma)'(0)) \rangle = \boxed{-\langle v, dn_p(v)) \rangle}$$

which is independent of γ.

10.17

$$\mathrm{II}_p(v, w) = \langle S_p(v), w \rangle = \left\langle \sum_i v_i S_p(e_i), \sum_j w_j e_j \right\rangle = \sum_{i,j} v_i w_j \langle S_p(e_i), e_j \rangle = \sum_{i,j} v_i w_j l_{ij}$$

Similarly

$$\mathrm{II}_p(w,v) = \sum_{i,j} v_i w_j l_{ji}$$

l_{ij} is symmetric since mixed derivatives are interchagable:

$$l_{ij} = \langle\, D_{ij}, N \,\rangle = \langle\, D_{ji}, N \,\rangle = l_{ji}$$

Therefore, $\mathrm{II}_p(v,w) = \mathrm{II}_p(w,v)$.

10.18

1

By bilinearity and symmetry of II_p

$$\begin{aligned}\mathrm{II}_p(v,v) &= \mathrm{II}_p(\cos\theta e_1 + \sin\theta e_2, \cos\theta e_1 + \sin\theta e_2)\\ &= \cos^2\theta\,\mathrm{II}_p(e_1,e_1) + 2\cos\theta\sin\theta\,\mathrm{II}_p(e_1,e_2) + \sin^2\theta\,\mathrm{II}_p(e_2,e_2)\end{aligned}$$

Since we have chosen the unit eigenvectors of S_p to be the basis vectors,

$$\mathrm{II}_p(e_1,e_1) = \langle\, S_p(e_1), e_1 \,\rangle = \langle\, \lambda_1 e_1, e_1 \,\rangle = \lambda_1, \text{ similarly, } \mathrm{II}_p(e_2,e_2) = \lambda_2$$

$$\mathrm{II}_p(e_1,e_2) = \mathrm{II}_p(e_2,e_1) = \langle\, \lambda_1 e_1, e_2 \,\rangle = 0$$

Plug the values into the first equation,

$$\mathrm{II}_p(v,v) = \cos^2\theta\lambda_1 + \sin^2\theta\lambda_2$$

2

$$\frac{d}{d\theta}\,\mathrm{II}_p(v,v) = -2\cos\theta\sin\theta\lambda_1 + 2\sin\theta\cos\theta\lambda_2 = 2(\lambda_2 - \lambda_1)\sin\theta\cos\theta$$

Since $\lambda_1 \neq \lambda_2$, the derivative vanishes when

$$\cos\theta = 0, \sin\theta = \pm 1 \text{ or } \sin\theta = 0, \cos\theta = \pm 1$$

which correspond to the extrema

$$\min \mathrm{II}_p(v,v) = 1\lambda_1 + 0\lambda_2 = \lambda_1$$

$$\max \mathrm{II}_p(v,v) = 0\lambda_1 + 1\lambda_2 = \lambda_2$$