MATH 4347 Homework 1

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1.4

$$u_t = a'(t)e^{2x} + b'(t)e^x + c'(t)$$
$$u_{xx} = 4a(t)e^{2x} + b(t)e^x$$

Compare the two derivatives, we get

$$\begin{cases} a' = 4a \\ b' = b \\ c' = 0 \end{cases} \Rightarrow \begin{cases} a(t) = C_1 e^{4t} \\ b(t) = C_2 e^t \\ c(t) = C_3 \end{cases}$$

1.7

The characteristic lines are $x - 4t = m$, so $u = f(x - 4t)$. From the initial condition

$$u(x, 0) = f(x) = 0, \text{ where } x > 0$$

From the boundary condition

$$u(0, t) = f(-4t) = te^{-t} \text{ where } t > 0$$

Let $x = -4t$, then $t = -\frac{1}{4}x$.

$$f(x) = -\frac{1}{4}xe^{\frac{1}{4}x} \text{ where } x < 0$$

Combining above results

$$f(x) = \begin{cases} -\frac{1}{4}xe^{\frac{1}{4}x} \text{ for } x < 0 \\ 0, \text{ for } x > 0 \end{cases}$$

$$u = f(x - 4t) = \begin{cases} -\frac{1}{4}(x - 4t)e^{\frac{1}{4}(x-4t)} \text{ for } 0 < x < 4t \\ 0, \text{ for } x > 4t \end{cases}$$

If the PDE changed to $u_t - 4u_x = 0$, then $u = f(x+4t)$ and the boundary condition implies that

$$f(4t) = te^{-t} \text{ where } t > 0$$

This is a contradiction because the initial condition tells us that $f(x) = 0$ for positive arguments, and therefore the equation has no solution.

2.5

a

Let $u(x,t) = e^{i\xi x + \sigma t}$, then $u_{tt} = \sigma^2 u$, and $u_{xxxx} = \xi^4 u$. So the PDE becomes

$$\sigma^2 = -\xi^4 \Rightarrow \sigma = \pm i\xi^2$$

$$u(x,t) = e^{i\xi x \pm i\xi^2 t} = e^{i\xi(x \pm \xi t)}$$

The wave speed depends on ξ, so the wave is dispersive. The reason for this dependence is that the derivatives in space and time are of different orders, so the resulting dispersion relation $\sigma(\xi)$ is not linear, so there are powers of ξ that cannot be factored out. By contrast, the space and time derivatives in wave equation are both of the second order, therefore $\sigma = \pm c\xi$, and consequently the wave speed does not depend on wave number.

b

$$u_t = \sigma u$$

$$u_x = i\xi u$$

$$u_{xxt} = -\xi^2 \sigma u$$

Dividing the original PDE by u

$$\sigma + ic\xi - \beta\xi^2\sigma = 0 \Rightarrow \sigma = \frac{ic\xi}{\beta\xi^2 - 1}$$

So the solution is

$$u = e^{i\xi x + \frac{ic\xi}{\beta\xi^2 - 1}t} = e^{i\xi(x - \frac{c}{1-\beta\xi^2}t)}$$

which are traveling waves with speed $\frac{c}{1-\beta\xi^2}$. Compared to the solution of linearized KdV equation

$$u(x,t) = e^{i\xi(x-(c-\beta\xi^2)t)}$$

which has wave speed $c - \beta\xi^2$, the wave speed of the solutions of BBM equation is monotonically increasing w.r.t. ξ^2 when $\xi^2 < \frac{1}{\beta}$ and $\xi^2 > \frac{1}{\beta}$, however the wave speed of the solutions of KdV equaitons is monotonically decreasing w.r.t ξ^2

2.6

a

Intuitively, the velocity should be higher when there are less traffic, so $v(u)$ should be monotonically decreasing.

b

Since $v(u)$ is monotonically decreasing and both u and v need to be nonnegative to have physical meanings, $v_{max} = v(u_{min}) = v(0)$ and $v(u_{max}) = v_{min} = 0$

c

From (b), $Q(0) = 0v(0) = 0$, and $Q(u_{max}) = u_{max}v(u_{max}) = u_{max}0 = 0$. If Q doesn't attain maximum at some point in the interval $(0, u_{max})$, then $Q(0) = Q(u_{max}) = 0$ must be the maximum, and for all u in the interval, $Q(u)$ must be strictly less than 0 (otherwise these points will also maximum),which is a contradiction because we required that u and v both be non-negative. Therefore, $Q(u)$ must has a maximum in the interval.

d

Such function can be constructed. Consider

$$Q(u) = u\left(-\frac{1}{4}x^3 + 2x^2 - \frac{11}{2}x + 6\right) = -\frac{1}{4}x^4 + 2x^3 - \frac{11}{2}x^2 + 6x$$

whose derivative is

$$Q'(u) = -(u-1)(u-2)(u-3)$$

The function has maximum at 1 and 3, and we can check that

$$v(u) = -\frac{1}{4}x^3 + 2x^2 - \frac{11}{2}x + 6$$

is positive and monotoniaclly decreasing in interval $(0, 4)$.

3.2

First solve for the characteristic curves:

$$\frac{dx}{dt} = \frac{1}{1+t^2} \Rightarrow x = \arctan t + m$$

So $u(x,t) = f(m) = f(x - \arctan t)$. Plugin in the initial condition:

$$u(x,0) = f(x - \arctan 0) = f(x) = \sin x$$

Therefore,

$$u(x,t) = f(x - \arctan t) = \sin(x - \arctan t)$$

3.3

Treat the first two terms as the directional derivative of u in $(1,1)$ direction. Consider the characteristic lines $x = t + k$, or $(s, s+k)$ where the equation becomes an ODE:

$$\frac{d}{ds}u + 3u = e^{3s+2k}$$

Solving use integrating factor $\phi(s) = e^{\int 3ds} = e^{3s}$

$$\begin{aligned} u(s) &= \frac{1}{e^{3s}}\int e^{3s}e^{3s+2k}ds = \frac{1}{6}e^{3s+2k} + A(k)e^{-3s} \\ u(x,t) &= \frac{1}{6}e^{2x+t} + A(x-t)e^{-3t} \end{aligned}$$

Plugin the initial condition to solve for $A(x)$:

$$u(x,0) = \frac{1}{6}e^{2x} + A(x) = x \Rightarrow A(x) = x - \frac{1}{6}e^{2x}$$

Thus the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{6}e^{2x+t} + \left((x-t) - \frac{1}{6}e^{2(x-t)}\right)e^{-3t} \\ &= \frac{1}{6}e^{2x+t} - \frac{1}{6}e^{2x-5t} + xe^{-3t} - te^{-3t} \end{aligned}$$

3.8

a

the LHS can be treated as a total derivative w.r.t to t on the characteristic curve

$$\frac{dx}{dt} = t \Rightarrow x = \frac{t^2}{2} + k$$

On the characteristic curve $(s, \frac{s^2}{2} + k)$the PDE becomes ODE

$$\frac{du}{ds} = u^2 \Rightarrow u = -\frac{1}{s + A(k)} = -\frac{1}{t + A(x - t^2/2)}$$

Plug in the initial condition:

$$u(x,0) = -\frac{1}{A(x)} = \frac{1}{1+x^2} \Rightarrow A(x) = -1 - x^2$$

Therefore the solution is

$$u(x,t) = -\frac{1}{t - 1 - (x - t^2/2)^2} = \frac{1}{(x - t^2/2)^2 - t + 1}$$

b

Given t, $u(x,t)$ attains maximum at critical point

$$\frac{\partial u}{\partial x} = -u^2 \cdot 2(x - \frac{t^2}{2}) = 0 \Rightarrow x_c = \frac{t^2}{2}$$

$$\max_x u(x,t) = u(\frac{t^2}{2}, t) = \frac{1}{1-t}$$

which blows up as $t \to 1^-$

1.3. #2

Apply Newton's second law to a segment of the chain from $x = a$ to $x = b$:

$$F(b)\sin\theta - F(a)\sin\theta = \int_a^b \rho u_{tt} \frac{1}{\cos\theta} dx$$

$$\int_b^{L_x} \rho g \frac{\sin\theta}{\cos\theta} dx - \int_a^{L_x} \rho g \frac{\sin\theta}{\cos\theta} dx = -\int_a^b \rho g \frac{\sin\theta}{\cos\theta} dx = \int_a^b \rho u_{tt} \frac{1}{\cos\theta} dx$$

Since this must be true for arbituary a, b, the integrands must be equal.

$$-\rho g \frac{\sin\theta}{\cos\theta} = \rho u_{tt} \frac{1}{\cos\theta}$$
$$u_{tt} = -g\sin\theta$$

where

$$\sin\theta = \frac{u}{\sqrt{u^2 + x^2}} = \frac{u}{x} + \mathcal{O}(u^2) \approx \frac{u}{x}$$

when u is small. Therefore the chains satisfies PDE

$$x u_{tt} + g u = 0$$

1.5 #2

a

No. Suppose there are two different functions u_1 and u_2 that satisfy the equations. Now consider their difference $v = u_1 - u_2$.

$$\begin{cases} u_1''(x) + u_1'(x) = f(x) \\ u_2''(x) + u_2'(x) = f(x) \end{cases} \Rightarrow v''(x) + v'(x) = 0 \Rightarrow v(x) = Ae^{-t} + B$$

and the boundary conditions:

$$v'(0) = v(0) = \frac{1}{2}[v'(l) + v(l)]$$
$$\Rightarrow -A = A + B = \frac{1}{2}B$$
$$\Rightarrow B = -2A, \quad v(x) = Ae^{-t} - 2A$$

Thus if we can find any function u_0 that satisfies the problem, then the family of functions

$$u = u_0 + Ce^{-t} - 2C$$

are all valid solutions.

b

Rearranging the second equation and applying the Fundamental Theorem of Calculus:

$$\begin{aligned} u'(0) + u(0) = u'(l) + u(l) \Rightarrow \Big[u'(x) + u(x)\Big]_0^l &= 0 \\ \Rightarrow \int_0^l u''(x) + u'(x)dx = \int_0^l f(x)dx &= 0 \end{aligned}$$

Therefore in order for the problem to have a solution, the integral of f from 0 to l must be zero.