Private versus central bank issued digital currencies: effects on monetary policy and resulting welfare implications

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Abstract:

This paper looks at the monetary policy and resulting welfare implications of introducing a central bank digital currency (CBDC) to an economy where fiat currency and private digital currency coexist. Adapting the Zhu and Hendry (2018) framework to include a CBDC, I find that as long as the central bank sets the nominal interest rate on CBDC sufficiently low, then CBDC alone will be valued in the economy, and that the optimal monetary policy decision will be to follow the Friedman Rule. This will give greater total welfare than the optimal policy decision when only fiat and private digital currency are used in the economy, thus the introduction of CBDC is welfare enhancing.

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1. Introduction

Since early 2015, the number of Bitcoin wallet users has grown more than 10 times (Statista, 2019), and in 2018 alone the number of verified cryptocurrency users more than doubled (Rauchs et al, 2017). This growing adoption of digital currencies has become a particular concern for central banks, fearing that in competing with fiat currency for adoption, it could interfere with their ability to conduct monetary policy. In a 2018 report on digital currency, the Bank of Canada identified this as a key threat to monetary stability (Davoodalhosseini and Rivadeneyra, 2018), and in particular, they point to a coordination issue in monetary policy, where central bank policy must become reactionary to the policy set on private digital currencies (Zhu and Hendry 2018).

The coordination issue stems from the idea that issuers of private digital currencies may not be welfare maximizing, but instead are more likely profit or adoption maximizing. Zhu and Hendry (2018) find that the welfare maximizing policy in an economy where fiat currency and private digital currency coexist is for both to follow the Friedman Rule, setting their nominal interest rates to zero. This means both currencies are costless to hold and hence agents hold enough real balances to maximize welfare. However, if the private issuer of digital currency does not run a welfare maximizing policy, setting the nominal interest rate on digital currency greater than zero, then the optimal policy decision from the central bank changes. It becomes reactionary, and may in fact be to run some moderate inflation on fiat currency. In deviating from the Friedman Rule, welfare of the economy is diverted from its maximum.

This paper looks at the introduction of a central bank's own digital currency (CBDC) as a potential solution to this problem. Introducing CBDC to an environment where fiat and private digital currency coexist, I investigate at its ability to alleviate this monetary coordination problem, and hence enhance the welfare of the economy. The idea of introducing a CBDC has gained interest from economies all over the world, including Sweden's Riksbank, considering the introduction of an e-krona as a complement to cash, and the Bank of Canada, experimenting with wholesale interbank payments and securities settlements under Project Jasper. Assuming the currency can gain sufficient adoption, then digital currencies would return to the control of the central bank, and the coordination issue on policy should go away.

To investigate this, I adopt the Lagos and Wright (2005) framework, which has become widely adopted by current monetary economists to analyze such monetary issues. Coupling micro foundations with mainstream macro, this framework derives its relationships rather than assuming them, so has the advantage of staying relevant as behaviors change over time. As highlighted by Lagos et al in a 2017 survey, it is also the most realistic of current models, combining the centralized markets from general equilibrium theory with the decentralized markets seen in search theory. This captures the idea that our economic lives are split between times where trade is easy (in a centralized market), and times where it is hard to find a counter party to trade with (in a decentralized market).

Zhu and Hendry (2018) adapt this framework to allow for multiple currencies to coexist in the economy, and it is this adapted variant of the Lagos and Wright (2005) framework that I use in this paper. Looking at an economy where fiat currency, private digital currency and CBDC coexist, I distinguish between each by allowing them to be redeemable in different transaction types: types 1, 2 and 3. Type 1 transactions are for government related services such as paying taxes, that are required to be paid in central bank currency. This restricts agents to spend either fiat currency or CBDC. Type 2 transactions capture the idea that digital currencies are a technological innovation, so may be preferred in, for example, overseas transactions. This means agents spend either private digital currency or CBDC. Type 3 transactions represent the majority of transactions in the economy, where all currencies are spendable and there is no significant advantage to either. Each transaction type represents a proportion α_1, α_2 and α_3 of total transactions respectively.

Focusing on stationary equilibria, I first consider the equilibrium where CBDC is not valued, occurring when the cost of holding CBDC is sufficiently large. The welfare maximizing policy here is for fiat currency and private digital currency to run the Friedman Rule. However, if the private issuer of digital currency sets an interest rate above zero, then the optimal response from the central bank will be to respond with a positive interest rate on fiat currency. This resembles the findings of Zhu and Hendry, and illustrates the policy coordination issue. I then look at the equilibrium where CBDC is valued, occurring when the cost of holding CBDC is sufficiently low, and find that in this case CBDC will be the only currency that is valued in the economy. The optimal monetary policy will be to follow the Friedman Rule. Intuitively this is restoring complete monetary control to the central bank, alleviating the coordination issue, and hence allowing them to set a welfare maximizing policy.

The rest of the paper continues as follows. In section 2 I review the related literature, and in section 3 I provide a background on digital currencies and central bank money. Section 4 outlines the model, augmenting the Zhu and Hendry framework to include CBDC. Section 5 looks at the equilibria, and Section 6 finds the optimal monetary policy decisions and resulting welfare. Section 7 concludes.

2. Literature Review

This concern of the monetary implications of private digital currencies raised by the Bank of Canada is shared by the Bank of England (BoE), who as early as 2014 identified it as a threat to the economy in a quarterly bulletin (Ali et al, 2014). Focusing on the same idea as Zhu and Hendry (2018), they commented that under wide adoption of private digital currencies, there would be erosion of the Monetary Policy Committee's ability to influence aggregate demand through setting interest rates. Admittedly the BoE was doubtful as to whether adoption of private digital currencies would ever be large enough to cause significant erosion; however, with UK Bitcoin holdings at the time representing just 0.003% of broad money, such skepticism is hardly surprising, and it makes sense that the more recent Zhu and Hendry paper addresses the issue with more urgency.

The BoE is also actively researching the potentials of CBDC to benefit monetary policy, highlighting monetary policy implications as one of their primary CBDC research questions ("CBDC research questions", 2017). Looking at CBDC as a potential replacement for cash, they are optimistic about its effect on the zero lower bound of conventional policy (Haldane, 2015). In an economy where cash exists, interest rates are bounded by a natural limit of 0%, where under a negative interest rate policy on deposits, agents would choose to substitute their holdings to cash and receive 0% interest instead. In a CBDC economy, household deposits could potentially be held directly at the central bank, meaning negative rates could feasibly be implemented with no cash alternative, and hence no such zero lower bound would exist. The BoE also identified that, using a universally accessible, interest bearing and account based CBDC, there could be potential monetary benefit through a strengthening of the monetary transmission mechanism, where a given policy change under CBDC would have a larger effect on the real economy than under traditional fiat systems (Meaning et al, 2018). Extending this analysis to focus exclusively on the real economy, Barrdear and Kumhof (2016) adopt a Dynamic Stochastic General Equilibrium model to investigate the macroeconomic consequences of CBDC. They find that, if introduced to the economy through purchase of government bonds, real GDP could increase by as much as 3%.

Davoodalhosseini (2018) parallels this optimism, who similar to Zhu and Hendry adopts the Lagos and Wright framework to investigate the monetary implications of CBDC as a replacement for cash. Adapting the model to allow for the use of both cash and CBDC, he finds that welfare is often lower when both currencies are available to agents, and in fact a welfare enhancing policy would be to entirely remove cash from circulation, allowing only the use of CBDC. Specifically, he suggests that CBDC could lead to as much as a 0.64% increase in welfare. The IMF extend this idea, proposing a decoupling of CBDC and cash, where cash could only be withdrawn from CBDC deposits at a rate below par, allowing a system that supports negative interest rates while CBDC and cash co-exist (Assenmacher and Krogstrup, 2018).

One of the best suited economies to capitalize on these potential benefits of CBDC is Sweden. Their economy is already almost entirely cashless, seeing a fall in cash as a proportion of GDP from about 10% in 1950 to just 1.5% today (Cecilia Skingsley, 2016), so are actively investigating the potentials of e-krona – a CBDC distributed by the Riksbank, as a digital complement to cash. While they are certainly positive about the potentials of a CBDC to the overall welfare of the economy, they in fact fear that a weakness of CBDC could be its influence on the effective lower bound (ELB) of interest rates (Eva Julin, 2017), in direct contrast to the BoE and Andy Haldane's 2015 speech. Focusing on a non-interest bearing

CBDC to replace cash, they argue that, in being easier to store and hence a stronger substitute to bank deposits, the ELB in a CBDC economy would be much closer to 0% than with cash, which in facing costs associated with managing cash, would see an ELB somewhere further below 0%. Ostensibly this corroborates the opinions of Davoodalhosseini (2018) and Bordo et al (2017), who suggest that the optimal form of a CDBC is for it to be interest bearing. However, in a more recent report the Riksbank argued that this constraint to the ELB can likely be compensated for using other monetary tools (Armelius et al, 2018), so are still undecided on how their e-krona might behave.

One constraint of the current literature is that they only consider two currencies at a time. CBDC co-existing with cash has been explored by Davoodalhosseini (2018), and more broadly the idea of e-money, not necessarily distributed from a Central Bank, competing with cash, has been considered by Chiu and Wong (2015), looking at the welfare implications of a currency that resembles Hong Kong's Octopus cards (similar to London's Oyster cards). Zhu and Hendry (2018) consider an economy with fiat and private digital currencies, and admittedly do touch on the idea of CBDC, briefly commenting that if the private currencies were replaced by CBDC, that a Friedman rule policy could be set over all currency, improving welfare. However, an economy where conventional fiat currency, private digital currencies, and a CBDC all co-exist is yet to be explored. This is where this paper extends the literature, adding CBDC as a third currency to the Zhu and Hendry framework, and exploring its monetary implications.

3. **Background**

3.1. Digital Currencies

The digital currencies I am looking at in this paper do not resemble electronic money, for example deposits in bank accounts or PayPal. Instead they are what are often defined as cryptocurrencies, making use of distributed ledger technology (DLT), which is a self-verifying book-keeping system. The key innovation here is that cryptocurrencies are transferred peer-to-peer, with transactions verified over a decentralized network, rather than through a trusted 3rd party such as a commercial bank (how electronic money transactions are processed). This decentralized network uses an algorithm to verify transactions, requiring a consensus from participants in the network, and records the transactions on a distributed ledger.

There are two key benefits to this, which are captured in my type 2 transactions. First, transactions can be completed much faster than traditional payment systems. Under traditional systems, overseas payments often take multiple days to complete. However, using a cryptocurrency that operates on a permissioned network, restricting access to a limited group and hence allowing faster resolving algorithms, overseas transaction times can be reduced to as little as 4 seconds (Larsen and McCaleb, 2019). Secondly, since there is no centralized verification, the agent can avoid any 3rd party processing fees that might usually be necessary.

Both the private digital currencies and CBDC I consider in my model take this form, so are only differentiated by the issuer. CBDC is distributed by the central bank, which aims to maximize welfare, while the private digital currencies are distributed by a private entity, that is not necessarily welfare maximizing. In practice CBDC would not necessarily have to take this form (it could, for example, resemble the centrally verified currency explored in Chui and Wong (2015)); however, in this paper I focus on a CBDC that captures the same technological benefits as the privately distributed digital currencies, meaning they are perfect substitutes in type 2 transactions.

3.2. Central Bank Currency

The central bank currently issues two forms of money. First is physical cash. This is widely available to all participants in the economy, and is what I term "fiat currency" in this paper. Second is electronic currency. This is restricted to financial institutions, and takes the form of reserves, held in the central bank. These reserves are guaranteed to be redeemable for fiat currency. CBDC would be a third form of money issued by the central bank. Similarly to these, it would offer a risk-free claim to the central bank.

CBDC could either have restricted access, resembling reserves, or be made widely available, resembling cash. The Bank of Canada are investigating a restricted access CBDC to facilitate wholesale interbank payments and securities settlements under Project Jasper (Chapman et al, 2017). In contrast, Sweden's Riksbank are considering the introduction of a widely available "e-krona" to their economy, in response to falling adoption of cash (Cecilia Skingsley, 2016). In this paper I focus on a CBDC that can compete with private digital currencies for adoption, and hence resembles cash, in the sense that it is widely available to all participants in the economy. The key difference here is that CBDC also captures the technological innovations of digital currency, which cash does not. This means it is redeemable in a broader range of transactions, and in particular has a significant advantage over fiat currency in, for example, overseas transactions. This is captured by allowing CBDC to be redeemable in both type 1 and type 2 transactions, while fiat currency is limited to just type 1 transactions.

4. The Environment:

My model is based on the Lagos and Wright (2005) framework, where time is discrete and infinite, indexed by $t = 0, 1, 2 \ldots$ Each period is divided into two sub-periods: a centralized market (CM), and a decentralized market (DM) with frictional matching. There are two types of infinitely lived agents, the buyers and sellers. In the DM sellers can produce the non-storable consumption good y_t for the cost of $c(y_t)$, and in the CM buyers can produce the non-storable general good x_t one-for-one with labor l_t, where x_t is the numeraire. Both agents discount the future at a rate $\beta \epsilon (0,1)$.

Buyers have lifetime utility function:

$$\sum_{t=1}^{\infty} \beta^t [v(x_t) - l_t + u(y_t)]$$

Where I assume $u(0) = 0,\ u'(0) = \infty,\ u'(y) > 0,\ u''(y) < 0\ \ \forall y$. Also, $v'(x) > 0$, $v''(x) < 0\ \ \forall x$, and $\exists\ x^* > 0\ \ s.t.\ v'(x^*) = 1$.

Similarly, sellers have lifetime utility function:

$$\sum_{t=1}^{\infty} \beta^t [v(x_t) - l_t - c(y_t)]$$

Where I assume $c(0) = c'(0) = 0,\ c'(y) > 0,\ c''(y) > 0\ \ \forall y$.

In this model there is no commitment or record-keeping, but instead it uses a medium of exchange. Extending the Zhu and Hendry (2017) model, which used fiat money m_t, and a private issued digital currency h_t, I will also introduce a Central Bank Digital Currency (CBDC), denoted c_t, with prices ϕ_t, φ_t, and μ_t respectively, in terms of the numeraire x_t. All of these currencies are intrinsically worthless. There are three different types of trades that give the currencies different roles in the economy. Type 1 where only central bank issued currencies can be used, meaning agents can choose to spend either fiat currency or CBDC. Type 2 where only digital currencies may be used, meaning agents can choose to spend either private digital currency or CBDC, and type 3 where any currency may be used.

The total supply of each currency is M_t, H_t and C_t for fiat, private digital currency and CBDC respectively, with constant gross growth rates $M_{t+1}/M_t = \gamma^m$, $H_{t+1}/H_t = \gamma^h$ and $C_{t+1}/C_t = \gamma^c$. If any γ^m, γ^h or $\gamma^c > 1$, then there is an injection of the corresponding currency at the beginning of the CM, and if any γ^m, γ^h or $\gamma^c < 1$ then there is a withdrawal of the corresponding currency at the beginning of the CM. This occurs through a lump-sum transfer/tax T_t to the buyers.

Centralized Market

The CM has value function for the buyer:

$$W_t^b(m_t, h_t, c_t) = \max_{x_t, l_t, \hat{m}_{t+1}, \hat{h}_{t+1}, \hat{c}_{t+1}} v(x_t) - l_t + \beta V_{t+1}\left(\hat{m}_{t+1}, \hat{h}_{t+1}, \hat{c}_{t+1}\right)$$

$$s.t. \qquad x_t + \phi_t \hat{m}_{t+1} + \varphi_t \hat{h}_{t+1} + \mu_t \hat{c}_{t+1} = \phi_t m_t + \varphi_t h_t + \mu_t c_t + l_t + T_t$$

where $\hat{m}_{t+1}, \hat{h}_{t+1}, \hat{c}_{t+1}$ are the amounts of currency brought into the next DM, and T_t is a lump sum transfer/tax injected to the buyer in the CM. This says that a buyer's lifetime utility at the beginning of the CM is equal to the utility gained from the consumption of the general good x_t, plus the continuation value at the beginning of the next DM, minus any disutility from labor. Buyers finance their consumption of x_t, and the balances they bring to the next DM, using labor, the lump sum transfer T_t, and the balances they bring into the CM. Substituting in l_t from the constraint we get:

$$\text{(1)} \qquad \begin{aligned} W_t^b(m_t, h_t, c_t) = \max_{x_t, \hat{m}_{t+1}, \hat{h}_{t+1_t}, \hat{c}_{t+1}} & v(x_t) - x_t - \phi_t \hat{m}_{t+1} - \varphi_t \hat{h}_{t+1} - \mu_t \hat{c}_{t+1} \\ & + \phi_t m_t + \varphi_t h_t + \mu_t c_t + T_t + \beta V_{t+1}(\hat{m}_{t+1}, \hat{h}_{t+1_t}, \hat{c}_{t+1}) \end{aligned}$$

Using this we can see that $W_t^b(m_t, h_t, c_t)$ is linear with respect to balances m_t, h_t, c_t, giving:

$$W_t^b(m_t, h_t, c_t) = W_t(0,0,0) + \phi_t m_t + \varphi_t h_t + \mu_t c_t$$

Taking FOCs of (1) with respect to $\hat{m}_{t+1}, \hat{h}_{t+1}, \hat{c}_{t+1}$ gives:

$$\beta \frac{\partial V_{t+1}^b}{\partial \hat{m}_{t+1}} = \phi_t, \qquad \beta \frac{\partial V_{t+1}^b}{\partial \hat{h}_{t+1}} = \varphi_t \qquad \beta \frac{\partial V_{t+1}^b}{\partial \hat{c}_{t+1}} = \mu_t$$

From this we can see the maximizing choice of money balances in the subsequent DM, $\hat{m}_{t+1}, \hat{h}_{t+1}, \hat{c}_{t+1}$, are independent of balances brought into the CM, m_t, h_t, c_t.

The envelope conditions are:

$$\frac{\partial}{\partial m_t} W_t^b(m_t, h_t, c_t) = \phi_t \qquad \frac{\partial}{\partial h_t} W_t^b(m_t, h_t, c_t) = \varphi_t \qquad \frac{\partial}{\partial c_t} W_t^b(m_t, h_t, c_t) = \mu_t$$

Similarly, the seller has value function in the CM:

$$W_t^s(m_t, h_t, c_t) = \max_{x_t, l_t, \widehat{m}_{t+1}, \widehat{h}_{t+1_t}, \hat{c}_{t+1}} v(x_t) - l_t + \beta V_{t+1}\left(\widehat{m}_{t+1}, \widehat{h}_{t+1_t}, \hat{c}_{t+1}\right)$$

$$s.t. \qquad x_t + \phi_t \widehat{m}_{t+1} + \varphi_t \widehat{h}_{t+1} + \mu_t \hat{c}_{t+1} = \phi_t m_t + \varphi_t h_t + \mu_t c_t + l_t$$

Decentralized Market

In the DM, buyers will be matched with a seller and use transaction type i with probability α_i, $i = 1,2,3$. This gives the following value function for buyers in the DM:

$$\begin{aligned} V_t^b(m_t, h_t, c_t) = {} & \alpha_1[u(y_t^1) + W_t^b(m_t - m_t^1, h_t, c_t - c_t^1)] \\ & + \alpha_2[u(y_t^2) + W_t^b(m_t, h_t - h_t^2, c_t - c_t^2)] \\ & + \alpha_3[u(y_t^3) + W_t^b(m_t - m_t^3, h_t - h_t^3, c_t - c_t^3)] \\ & + (1 - \alpha_1 - \alpha_2 - \alpha_3) W_t^b(m_t, h_t, c_t) \end{aligned}$$

where y_t^i, i=1,2,3 is the consumption in the different types of meetings, $m_t^1 + c_t^1$, $h_t^2 + c_t^2$, $m_t^3 + h_t^3 + c_t^3$ are the corresponding payments, and $W_t^b(m_t, h_t, c_t)$ is the buyer's value function in the CM. This DM value function is saying that, with probability α_1, the buyer will face a type 1 transaction, gaining utility $u_1(y_t^1)$ and will enter the CM with a balance $(m_t - m_t^1, h_t, c_t - c_t^1)$, and similarly for transaction types 2 and 3. With probability $(1 - \alpha_1 - \alpha_2 - \alpha_3)$ the buyer will not match a seller and so will enter the CM with their original balance (m_t, h_t, c_t).

Using the linearity of W_t, this value function can be simplified to the following form:

$$V_t^b(m_t,h_t,c_t)=\alpha_1[u(y_t^1)-\phi_t m_t^1-\mu_t c_t^1]+\alpha_2[u(y_t^2)-\varphi_t h_t^2-\mu_t c_t^2]$$

$$+\alpha_3[u(y_t^3)-\phi_t m_t^3-\varphi_t h_t^3-\mu_t c_t^3]+W_t^b(m_t,h_t,c_t)$$

Similarly, we can find the seller's value function in the DM as follows:

$$V_t^s(m_t,h_t,c_t)=\alpha_1[-c(y_t^1)+\phi_t m_t^1+\mu_t c_t^1]+\alpha_2[-c(y_t^2)+\varphi_t h_t^2+\mu_t c_t^2]$$

$$+\alpha_3[-c(y_t^3)+\phi_t m_t^3+\varphi_t h_t^3+\mu_t c_t^3]+W_t^s(m_t,h_t,c_t)$$

5. Equilibrium

5.1. Deriving Equilibrium Conditions

The total payment in each DM will be some linear combination of the three currencies, so will take the form:

$$\sum_{i=1}^{3} m_t^i+h_t^i+c_t^i$$

As in Rocheteau and Wright (2003), this payment is termed a 'debt'. I denote the total debt in real terms as d, and the buyer's real balance brought into the DM as D. d^* is the debt limit that achieves y^*, the socially efficient quantity of the DM good that maximizes trade surpluses, satisfying $u'(y^*)=c'(y^*)$.

The efficient quantity y^* is illustrated in Figure 1, where we can see it is the quantity maximizing the trade surplus, i.e. the difference between the utility of consuming y and the cost of its production. This occurs where the gradients of the utility and cost function are the same, so $u'(y^*)=c'(y^*)$. Moreover, for all quantities below the optimum $y<y^*$ we have

$u'(y) > c'(y)$, and for all quantities above the optimum $y > y^*$ we have $u'(y) < c'(y)$. This means up to the optimum any increase in y will increase the trade surplus, but beyond y^* and further increases in y will decrease the trade surplus.

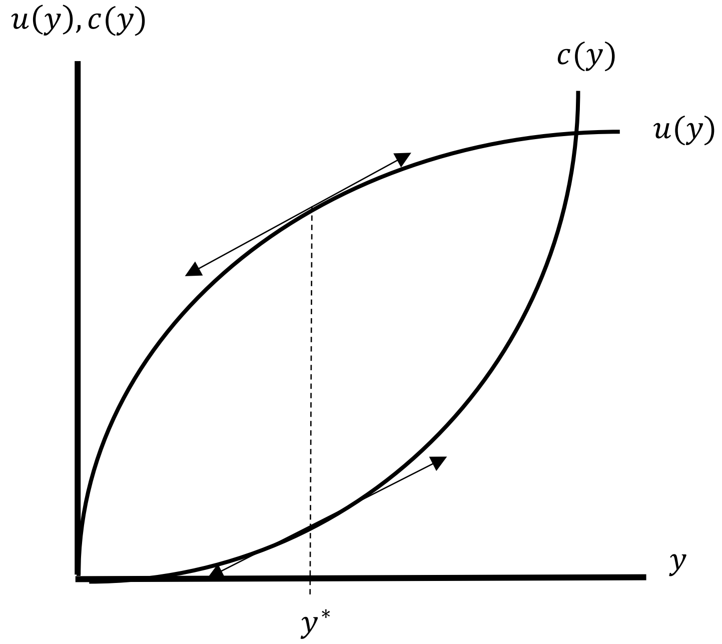


Figure 1: Plot of utility from DM good y against cost of y

If the buyer holds a sufficient real balance D to achieve d^*, then they will achieve the optimal quantity y^*. If not, they will go to the limit $d = D$ and achieve quantity $y = f(d) < y^*$. As functions y and d take the following form:

(2) $y(d) = \begin{cases} f(d), & if\ d < d^* \\ y^*\ otherwise \end{cases}$ and (3) $d(D) = \begin{cases} D, & if\ D < d^* \\ d^*\ otherwise \end{cases}$

These functions are illustrated graphically in Figures 2 and 3. In Figure 2 we have quantities $y(d)$. Up to the optimum y^* this takes the form of a strictly increasing function in the debt paid $f(d)$, with $f'(d) > 0,\ f''(d) < 0$ and $f(0) = 0$. When buyers are no longer constrained in real balances, holding $D \geq d^*$, they achieve the optimum quantity y^*. In

Figure 3 we see the total debt paid, which is simply equal to the buyer's real balanced D up to the optimum, and the efficient d^* for $D \geq d^*$.

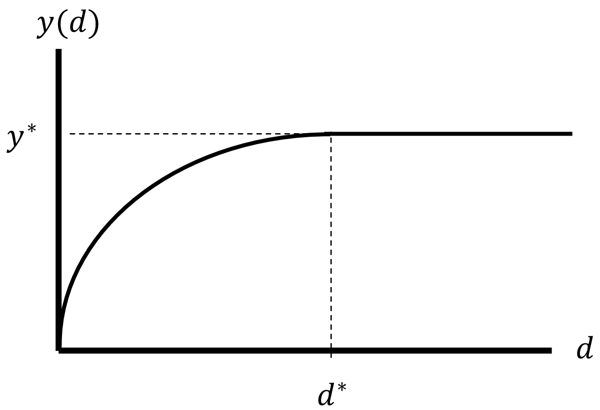


Figure 2: Plot of quantities as a function of debt paid

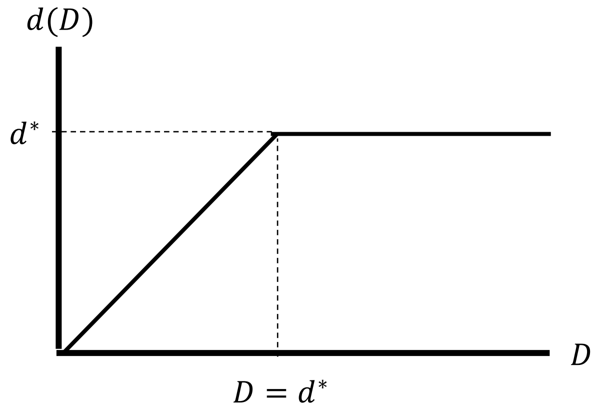


Figure 3: Plot of debt as a function of real balances

Maximization problem

The buyer posts a take it or leave it offer (y, d) to the seller, solving the maximization problem:

$$\max_{y,d}[u(y) + W_t^b(m_t, h_t, c_t) - d]$$

s.t. $$-c(y) + W_t^s(m_t, h_t, c_t) + d \geq W_t^s(m_t, h_t, c_t)$$

and $$d \leq D$$

This says that they will maximize their expected utility, subject to the seller's participation constraints, and that they can afford the real balance to be paid.

The buyers are determining the terms of trade, so will not give the seller any surplus, meaning the seller's participation constraint holds with equality, i.e. $c(y) = d$. For simplicity, I will also take $c(y) = y$, meaning the socially efficient output y^* will be achieved at $u'(y) = c'(y) = 1$. Dropping the terms that are not relevant to the maximization problem, this gives the simplified problem:

$$\max_{y,d}[u(y) - d]$$

$$\Leftrightarrow$$

$$\max_{d}[u \circ y(d) - d]$$

s.t. $$d \leq D \qquad (4)$$

We have Lagrangian:

$$\mathcal{L} = u \circ y(d) - d - \lambda(d - D)$$

Maximizing in terms of real balances d gives FOC:

$$\big(u' \circ y(d)\big)y'(d) - 1 - \lambda = 0$$

Notice we get the same FOC regardless of whether we maximize in nominal or real terms (FOC set equal to 0, so can simply divide by price level to get equivalent expression), which makes the derivation of the Euler Equations simpler.

The Lagrange multiplier λ will take two forms. When the constraint (4) is binding, meaning the buyer does not have sufficient balances to afford d^*, we see a $\lambda > 0$. When the constraint is non-binding, so the buyer can achieve d^*, we see $\lambda = 0$. From (3) we know that under a binding constraint we have $d(D) = D$. Combining this with the FOC, we can see λ as a function of real balances z takes the following form:

$$\lambda(z) = \begin{cases} (u' \circ f(z))f'(z) - 1, & if\ z < d^* \\ 0\ otherwise \end{cases}$$

This can be thought of as a liquidity premium, or the cost of holding money, capturing the marginal benefit of holding real balances in the DM. If the liquidity premium is 0, so money is costless to hold, then buyers will bring enough money into the DM to afford d^*. If the liquidity premium is positive, so money is costly to hold, then they will only bring $z < d^*$. Up to the optimum real balance d^*, λ has derivative $u''\big(f(z)\big)(f'(z))^2 + f''(z)u'\big(f(z)\big) < 0$, so is strictly decreasing in real balances.

Notice that the optimum d^*, achieved at $\lambda = 0$, coincides with the optimum quantity y^*. Writing λ as a function of quantities y we get $u'(y) - 1 = 0$ at the optimum, or equivalently $u'(y) = 1$, which is the condition achieving $y = y^*$.

Deriving Euler Equations

We have the following value function in the DM:

$$V_t^b(m_t, h_t, c_t) = \alpha_1[u(y_t^1) - \phi_t m_t^1 - \mu_t c_t^1] + \alpha_2[u(y_t^2) - \varphi_t h_t^2 - \mu_t c_t^2]$$

$$+\alpha_3[u(y_t^3) - \phi_t m_t^3 - \varphi_t h_t^3 - \mu_t c_t^3] + W_t(m_t, h_t, c_t)$$

Taking the derivative with respect to m_t gives:

$$\begin{aligned} \frac{\partial V_t^b(m_t, h_t, c_t)}{\partial m_t} \\ &= \alpha_1\left[u'(y_t^1)\frac{\partial y_t^1}{\partial m_t} - \frac{\partial(\phi_t m_t^1 + \mu_t c_t^1)}{\partial m_t}\right] \\ &+ \alpha_3\left[u'(y_t^3)\frac{\partial y_t^3}{\partial m_t} - \frac{\partial(\phi_t m_t^3 + \varphi_t h_t^3 + \mu_t c_t^3)}{\partial m_t}\right] \end{aligned} \tag{3}$$

To expand this derivative, we need to find the quantities y_t^1, y_t^3, their corresponding payments $\phi_t m_t^1 + \mu_t c_t^1$, $\phi_t m_t^3 + \varphi_t h_t^3 + \mu_t c_t^3$, and the derivative of these with respect to m_t. Using (2), the function for quantities $y(d)$, we can see that y_t^1, y_t^3, and hence $\partial y_t^1/\partial m_t$ and $\partial y_t^3/\partial m_t$ take the following form:

$$y_t^1 = \begin{cases} f(\phi_t m_t + \mu_t c_t), & if\ \phi_t m_t + \mu_t c_t < d^* \\ y^*\ otherwise \end{cases}$$

$$\Rightarrow \partial y_t^1 / \partial m_t = \begin{cases} \phi_t f'(\phi_t m_t + \mu_t c_t), & if\ \phi_t m_t\ +\ \mu_t c_t < d^* \\ 0\ otherwise \end{cases}$$

$$y_t^3 = \begin{cases} f(\phi_t m_t + \varphi_t h_t + \mu_t c_t), & if\ \phi_t m_t + \varphi_t h_t + \mu_t c_t < d^* \\ y^*\ otherwise \end{cases}$$

$$\Rightarrow\ \ \partial y_t^3 / \partial m_t = \begin{cases} \phi_t f'(\phi_t m_t + \varphi_t h_t + \mu_t c_t), & if\ \phi_t m_t + \varphi_t h_t + \mu_t c_t < d^* \\ 0\ otherwise \end{cases}$$

And similarly, using (3), the function for the debt to be paid $d(D)$, we can see $\phi_t m_t^1 + \mu_t c_t^1$ and $\phi_t m_t^3 + \varphi_t h_t^3 + \mu_t c_t^3$, and hence $(\phi_t m_t^1 + \mu_t c_t^1)/\partial m_t$ and $\partial(\phi_t m_t^3 + \varphi_t h_t^3 + \mu_t c_t^3)/\partial m_t$ take the following form:

$$\phi_t m_t^1 + \mu_t c_t^1 = \begin{cases} \phi_t m_t + \mu_t c_t, & if\ \phi_t m_t + \mu_t c_t < d^* \\ d^* \end{cases}$$

$$\Rightarrow \partial(\phi_t m_t^1 + \mu_t c_t^1)/\partial m_t = \begin{cases} \phi_t, & if\ \phi_t m_t + \mu_t c_t < d^* \\ 0\ otherwise \end{cases}$$

$$\phi_t m_t^3 + \varphi_t h_t^3 + \mu_t c_t^3 = \begin{cases} \phi_t m_t + \varphi_t h_t + \mu_t c_t, & if\ \phi_t m_t + \varphi_t h_t + \mu_t c_t < d^* \\ d^* \end{cases}$$

$$\Rightarrow \partial(\phi_t m_t^3 + \varphi_t h_t^3 + \mu_t c_t^3)/\partial m_t = \begin{cases} \phi_t, & if\ \phi_t m_t + \varphi_t h_t + \mu_t c_t < d^* \\ 0\ otherwise \end{cases}$$

Substituting these into (3) and writing in terms of λ, we get the following:

$$\frac{\partial V_t^b(m_t,h_t,c_t)}{\partial m_t} = \alpha_1\phi_t\lambda(\phi_t m_t + \mu_t c_t) + \alpha_3\phi_t\lambda(\phi_t m_t + \varphi_t h_t + \mu_t c_t) + \phi_t$$

Following the same process, we can obtain:

$$\frac{\partial V_t^b(m_t,h_t,c_t)}{\partial h_t} = \alpha_2\varphi_t\lambda(\varphi_t h_t + \mu_t c_t) + \alpha_3\varphi_t\lambda(\phi_t m_t + \varphi_t h_t + \mu_t c_t) + \varphi_t$$

$$\frac{\partial V_t^b(m_t,h_t,c_t)}{\partial c_t} = \alpha_1\mu_t\lambda(\phi_t m_t + \mu_t c_t) + \alpha_2\mu_t\lambda(\varphi_t h_t + \mu_t c_t)$$
$$+\alpha_3\mu_t\lambda(\phi_t m_t + \varphi_t h_t + \mu_t c_t) + \mu_t$$

Finally, taking these in period *t+1*, we can combine with the following CM FOCs outlined in the environment:

$$\beta\frac{\partial V_{t+1}^b}{\partial \widehat{m}_{t+1}} = \phi_t, \qquad \beta\frac{\partial V_{t+1}^b}{\partial \hat{h}_{t+1}} = \varphi_t \qquad \beta\frac{\partial V_{t+1}^b}{\partial \hat{c}_{t+1}} = \mu_t$$

This gives the following Euler Equations:

$$\phi_t = \alpha_1\beta\phi_{t+1}\lambda(\phi_{t+1}\widehat{m}_{t+1} + \mu_{t+1}\hat{c}_{t+1}) + \alpha_3\beta\phi_{t+1}\lambda\left(\phi_{t+1}\widehat{m}_{t+1} + \varphi_{t+1}\hat{h}_{t+1} + \mu_{t+1}\hat{c}_{t+1}\right)$$
$$+ \beta\phi_{t+1}$$

$$\varphi_t = \alpha_2\beta\varphi_{t+1}\lambda\left(\varphi_{t+1}\hat{h}_{t+1} + \mu_{t+1}\hat{c}_{t+1}\right) + \alpha_3\beta\varphi_{t+1}\lambda\left(\phi_{t+1}\widehat{m}_{t+1} + \varphi_{t+1}\hat{h}_{t+1} + \mu_{t+1}\hat{c}_{t+1}\right)$$
$$+ \beta\varphi_{t+1}$$

$$\mu_t = \alpha_1\beta\mu_{t+1}\lambda(\phi_{t+1}\widehat{m}_{t+1} + \mu_{t+1}\hat{c}_{t+1}) + \alpha_2\beta\mu_{t+1}\lambda\left(\varphi_{t+1}\hat{h}_{t+1} + \mu_{t+1}\hat{c}_{t+1}\right)$$
$$+ \alpha_3\beta\mu_{t+1}\lambda\left(\phi_{t+1}\widehat{m}_{t+1} + \varphi_{t+1}\hat{h}_{t+1} + \mu_{t+1}\hat{c}_{t+1}\right) + \beta\varphi_{t+1}$$

These say that the price of each currency in period t is equal to its discounted value in period t+1, plus a liquidity premium for holding real balances in the DM, adjusting for the probability of the relevant transaction types occurring in the DM.

We can also consider these in terms of real balances and growth in money supplies. The money market clears when $\widehat{m}_{t+1} = M_{t+1}$, $\widehat{h}_{t+1} = H_{t+1}$ and $\hat{c}_{t+1} = C_{t+1}$. Substituting in the gross growth rates of money supplies γ^m, γ^h and γ^c, and real balances $z_t^m = \phi_t M_t$, $z_t^h = \varphi_t H_t$ and $z_t^c = \mu_t C_t$, these Euler equations can be written as:

$$z_t^m = \alpha_1\beta\lambda\big(z_{t+1}^m + z_{t+1}^h\big)z_{t+1}^m/\gamma_{t+1}^m + \alpha_3\beta\lambda\big(z_{t+1}^m + z_{t+1}^h + z_{t+1}^c\big)z_{t+1}^m/\gamma_{t+1}^m + \beta z_{t+1}^m/\gamma_{t+1}^m$$

$$z_t^h = \alpha_2\beta\lambda\big(z_{t+1}^h + z_{t+1}^c\big)z_{t+1}^h/\gamma_{t+1}^h + \alpha_3\beta\lambda\big(z_{t+1}^m + z_{t+1}^h + z_{t+1}^c\big)z_{t+1}^h/\gamma_{t+1}^h + \beta z_{t+1}^h/\gamma_{t+1}^h$$

$$\begin{aligned} z_t^c = {} & \alpha_1\beta\lambda\big(z_{t+1}^m + z_{t+1}^h\big)z_{t+1}^c/\gamma_{t+1}^c + \alpha_2\beta\lambda\big(z_{t+1}^h + z_{t+1}^c\big)z_{t+1}^c/\gamma_{t+1}^c \\ & + \alpha_3\beta\lambda\big(z_{t+1}^m + z_{t+1}^h + z_{t+1}^c\big)z_{t+1}^c/\gamma_{t+1}^c + \beta z_{t+1}^c/\gamma_{t+1}^c \end{aligned}$$

An equilibrium is any bounded, non-negative sequence of real balances $\{z_t^m, z_t^h, z_t^c\}_{t=0}^{\infty}$ that satisfies this system of Euler Equations.

5.2. Stationary Equilibrium

In the stationary equilibrium, real balances are constant over time, so $\phi_t M_t = \phi_{t+1}M_{t+1}$, $\varphi_t H_t = \varphi_{t+1}H_{t+1}$ and $\mu_t C_t = \mu_{t+1}C_{t+1}$. Writing in terms of interest rates on each currency, we have equilibrium conditions:

$$i^m z^m = \alpha_1\lambda(z^m + z^c)z^m + \alpha_3\lambda(z^m + z^h + z^c)z^m$$

$$i^h z^h = \alpha_2\lambda(z^h + z^c)z^h + \alpha_3\lambda(z^m + z^h + z^c)z^h$$

$$i^c z^c = \alpha_1\lambda(z^m + z^c)z^c + \alpha_2\lambda(z^h + z^c)z^c + \alpha_3\lambda(z^m + z^h + z^c)z^c$$

where $i = \gamma/\beta - 1$ can be thought of as the nominal interest rate on an illiquid bond in the corresponding currencies (i.e. one that cannot be spent in the DM). This means i can be interpreted as the opportunity cost of holding money, or the forgone interest for holding real balances. In this economy monetary policy is set through changing the money supply growth rates γ, or equivalently the interest rates, with $i = \gamma/\beta - 1$ for each currency. For each currency I assume $i \geq 0$, so money is either costly to hold ($i > 0$), or at minimum costless to hold ($i = 0$). This stops agents demanding infinite real balances.

Using the stationary conditions we can write the growth rates for each currency in terms of their prices:

$$\gamma^m = M_{t+1}/M_t = \phi_t/\phi_{t+1}$$

$$\gamma^h = H_{t+1}/H_t = \varphi_t/\varphi_{t+1}$$

$$\gamma^c = C_{t+1}/C_t = \mu_t/\mu_{t+1}$$

This means i can also be thought of as an indicator of the inflation rates on each currency, $\phi_t/\phi_{t+1} - 1$, $\varphi_t/\varphi_{t+1} - 1$ and $\mu_t/\mu_{t+1} - 1$, so in this sense too shows the cost of holding money.

5.3. When currencies are valued

Potential solutions to the equilibrium conditions include any combination of z^m, z^h, z^c being 0. This implies that there are equilibria where not all currencies are valued (or in fact none are). There are two main equilibria that are of interest. First, where CBDC is not valued. This gives rise to the economy where fiat and private digital currency are competing, and results in the coordination issue of policy explored by Zhu and Hendry (2018). Second is when CBDC is valued. This gives rise to an economy where CBDC is the only currency valued in the economy, and the coordination issue is resolved.

5.3.1. Two-currency economy

Consider first the equilibrium where CBDC is not valued. The coordination issue occurs when both fiat and private digital currency are valued in the economy. Using the equilibrium conditions, we can establish a boundary for when both fiat and private digital currency are valued in the economy.

The relevant equilibrium conditions here are:

$$i^m = \alpha_1 \lambda(z^m) + \alpha_3 \lambda(z^m + z^h)$$

$$i^h = \alpha_2 \lambda(z^h) + \alpha_3 \lambda(z^m + z^h)$$

Taking differences we get:

$$i^m = \alpha_1 \lambda(z^m) - \alpha_2 \lambda(z^h) + i^h$$

Let $\bar{z}^h$ be the real balance of private digital currency satisfying:

$$i^h = \alpha_2 \lambda(\bar{z}^h) + \alpha_3 \lambda(\bar{z}^h) \tag{4}$$

i.e. the balance held when private digital currency alone is valued in the economy. If agents substitute holdings towards fiat currency, then holdings of private digital currency will fall, meaning if fiat currency is valued, i.e. $z^m > 0$, we have $z^h < \bar{z}^h$. λ is a decreasing function in real balances, so substituting this into our difference equation, along with $0 < z^m$, we get the following constraint:

$$i^m < \alpha_1 \lambda(0) - \alpha_2 \lambda(\bar{z}^h) + i^h$$

Using $i^h = \alpha_2 \lambda(\bar{z}^h) + \alpha_3 \lambda(\bar{z}^h)$ we get the boundary:

$$i^m < \alpha_1 \lambda(0) + \alpha_3 \lambda(\bar{z}^h) \equiv \mathcal{M} \qquad (5)$$

Fiat currency will be valued when this constraint is satisfied. Clearly this requires the cost of holding fiat currency, i^m, to be sufficiently low. Changes in each of α_1, α_2, α_3, and the interest rate on private digital currency i^h will also impact the boundary $\mathcal{M}$.

$\underline{i^h}$:

An increase in the cost of holding private digital currency, i^h, will cause a fall in $\bar{z}^h$ through (4), so $\mathcal{M}$ increases.

$\underline{\alpha_1}$:

An increase in the proportion of government transactions, α_1, will directly increase $\mathcal{M}$ through (5), and not change $\bar{z}^h$, so $\mathcal{M}$ increases.

$\underline{\alpha_2}$:

An increase in the proportion of overseas transactions, α_2, will increase $\bar{z}^h$, so $\mathcal{M}$ decreases. We can see this by rearranging the equilibrium condition for i^h to obtain:

$$\bar{z}^h = \lambda^{-1}(i^h/(\alpha_2 + \alpha_3))$$

The inverse of a decreasing function is also decreasing, so λ^{-1} is decreasing. This means an increase in α_2 will increase $\bar{z}^h$, and hence decrease $\mathcal{M}$.

$\underline{\alpha_3}$:

An increase in α_3 will directly increase $\mathcal{M}$ through (5), but also increase $\bar{z}^h$ through (4), which decreases $\mathcal{M}$. The net effect is an increase in $\mathcal{M}$, which can be seen by considering the derivative:

$$\frac{d\alpha_3 \lambda(\bar{z}^h)}{d\alpha_3} = \lambda(\bar{z}^h) - \frac{\alpha_3}{(\alpha_3 + \alpha_2)} \lambda(\bar{z}^h) > 0$$

Real Balances

By applying the implicit-function rule to the equilibrium conditions, we can extend this analysis to find the comparative statics on real balances. The implicit-function rule gives the following system to solve:

$$\underbrace{\begin{bmatrix} \alpha_1 \frac{\partial \lambda(z^m)}{\partial z^m} + \alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^m} & \alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^h} \\ \alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^m} & \alpha_2 \frac{\partial \lambda(z^h)}{\partial z^h} + \alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^h} \end{bmatrix}}_{\text{Jacobian } \mathbf{J}} \begin{bmatrix} \frac{dz^m}{di^m} \\ \frac{dz^h}{di^m} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where the determinant of the Jacobian $|\mathbf{J}|$ takes the following form:

$$|\mathbf{J}| = \alpha_1 \alpha_2 \frac{\partial \lambda(z^m)}{\partial z^m} \frac{\partial \lambda(z^h)}{\partial z^h} + \alpha_1 \alpha_3 \frac{\partial \lambda(z^m)}{\partial z^m} \frac{\partial \lambda(z^m + z^h)}{\partial z^h} + \alpha_2 \alpha_3 \frac{\partial \lambda(z^h)}{\partial z^h} \frac{\partial \lambda(z^m + z^h)}{\partial z^m} > 0$$

$|\mathbf{J}|$ is greater than 0 since λ is a decreasing function, so its derivative is negative, and hence each product in $|\mathbf{J}|$ is positive. Solving using Cramer's rule we get the following expressions for dz^m/di^m *and* dz^h/di^m*:*

$$\frac{dz^m}{di^m} = \frac{\begin{vmatrix} 1 & \alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^h} \\ 0 & \alpha_2 \frac{\partial \lambda(z^h)}{\partial z^h} + \alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^h} \end{vmatrix}}{|\mathbf{J}|} = \frac{\alpha_2 \frac{\partial \lambda(z^h)}{\partial z^h} + \alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^h}}{|\mathbf{J}|} < 0$$

$$\frac{dz^h}{di^m} = \frac{\begin{vmatrix} \alpha_1 \frac{\partial \lambda(z^m)}{\partial z^m} + \alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^m} & 1 \\ \alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^m} & 0 \end{vmatrix}}{|\mathbf{J}|} = -\frac{\alpha_3 \frac{\partial \lambda(z^m + z^h)}{\partial z^m}}{|\mathbf{J}|} > 0$$

Similarly, we can apply this method to find the impacts of i^h *and each* $\alpha_1, \alpha_2, \alpha_3$ on real balances, which are summarised in Table 1 (see Appendix 9.1 for full derivatives). A *"*$+$*"* indicates a positive derivative, and a "$-$" indicates a negative derivative. *"*0" indicates there is no effect.

Table 1: Comparative Statics in economy where CBDC is not valued

	i^m	i^h	α_1	α_2	α_3
$\mathcal{M}$	0	$+$	$+$	$-$	$+$
z^m	$-$	$+$	$+$	$-$	$+$
z^h	$+$	$-$	$-$	$+$	$+$

The intuition follows nicely. Fiat currency will be valued if the cost of holding it, i^m, is sufficiently low, and the cost of holding private digital currency, i^h, is sufficiently high. The proportion of transactions fiat currency can be spent in, α_1 and α_3, must also be sufficiently large. If the proportion of government transactions increases (α_1 increases), then fiat currency becomes more valuable, and hence agents substitute their holdings away from digital currency and towards fiat currency. Similarly, if the proportion of overseas transactions increases (α_2 increases), then fiat currency becomes less valuable, and agents choose to substitute their holdings towards private digital currency.

5.3.2. CBDC valued in the economy

When the central bank sets the interest rate on CBDC sufficiently low, it will be the only currency valued in the economy. To see this, consider the equilibrium conditions:

$$i^m = \alpha_1 \lambda(z^m + z^c) + \alpha_3 \lambda(z^m + z^h + z^c)$$

$$i^h = \alpha_2 \lambda(z^h + z^c) + \alpha_3 \lambda(z^m + z^h + z^c)$$

$$i^c = \alpha_1 \lambda(z^m + z^c) + \alpha_2 \lambda(z^h + z^c) + \alpha_3 \lambda(z^m + z^h + z^c)$$

Taking differences we obtain:

$$i^m = i^c - \alpha_2 \lambda(z^h + z^c)$$

$$i^h = i^c - \alpha_1 \lambda(z^m + z^c)$$

Let $\bar{z}^c$ be the real balance of CBDC satisfying $i^c = \alpha_1\lambda(\bar{z}^c) + \alpha_2\lambda(\bar{z}^c) + \alpha_3\lambda(\bar{z}^c)$, i.e. the balance of CBDC held when CBDC alone is valued in the economy. If agents substitute holdings towards fiat or private digital currency, then holdings of CBDC will fall, meaning for a given $z^m, z^h > 0$ we have $z^c < \bar{z}^c$. Using this we can see that for fiat currency and private digital currency to be valued, occurring when $z^m, z^h > 0$, we get the following constraints:

$$i^m < i^c - \alpha_2 \lambda(z^h + \bar{z}^c)$$

$$i^h < i^c - \alpha_1 \lambda(z^m + \bar{z}^c)$$

This means for fiat and private digital currency to be valued, the cost of holding CBDC, i^c, must be sufficiently large. In contrast, if the central bank sets the i^c sufficiently low, it can ensure CBDC is the only currency valued in the economy. Suppose for example the central bank follows the Friedman Rule on CBDC, setting $i^c = 0$. The constraints reduce to:

$$i^m < -\alpha_2 \lambda(z^h + \bar{z}^c)$$

$$i^h < -\alpha_1 \lambda(z^m + \bar{z}^c)$$

The liquidity premium λ is non-negative, meaning the right-hand side of these constraints is less than or equal to 0. Even if both issuers minimized the cost of holding currency, setting i^m and i^h to 0, these constraints would not be satisfied. Even with the best response from the private issuers, the central bank can ensure CBDC alone is valued.

6. Optimal Monetary Policy

6.1. The Friedman Rule

In this economy monetary policy is set through changing the money supply growth rates γ, or equivalently the interest rates, with $i = \gamma/\beta - 1$ for each currency. The optimal policy rule is the one that maximizes welfare of the economy, which in this framework is equal to the sum of the trade surpluses over all transaction types in the DM plus a constant (Zhu and Hendry 2018), giving welfare function:

$$\begin{aligned}\mathcal{W}(i^m, i^h, i^c) = \alpha_1[u \circ y(z^m + z^c) - c \circ y(z^m + z^c)] \\ + \alpha_2[u \circ y(z^h + z^c) - c \circ (z^h + z^c)] \\ + \alpha_3[u \circ y(z^m + z^h + z^c) - c \circ y(z^m + z^h + z^c)] \\ + constant\end{aligned}$$

Trade surpluses, and hence welfare, are maximized when the efficient y* quantity of the DM good is achieved, occurring when $u'(y) = c'(y) = 1$, or equivalently when the liquidity premium on holding real balances λ is 0. If we consider an equilibrium where only a single currency is valued, say fiat currency, we have the following equilibrium condition:

$$i^m = \alpha_1 \lambda(z^m) + \alpha_3 \lambda(z^m)$$

which can be rearranged to give:

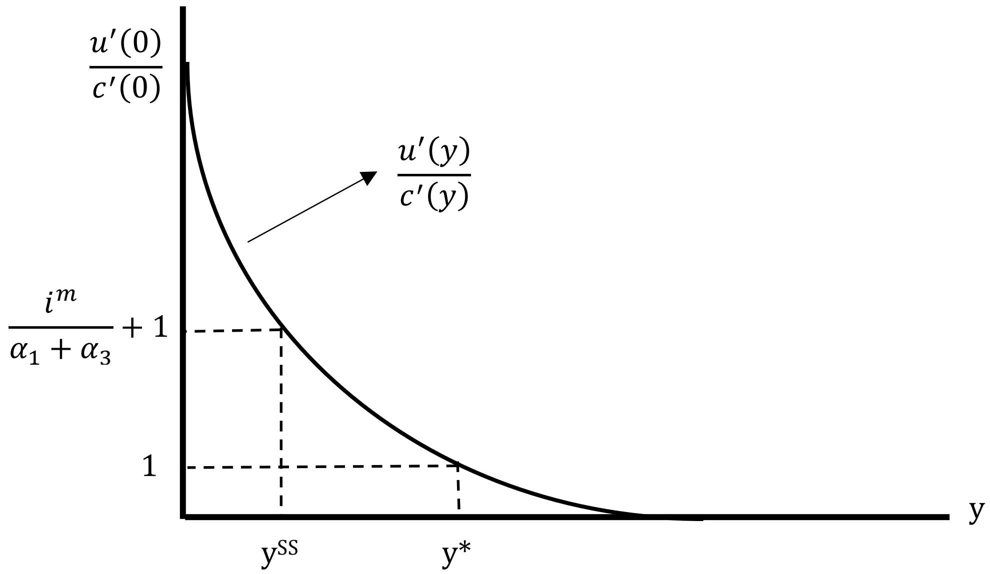
$$\lambda(z^m) = \frac{i^m}{\alpha_1 + \alpha_3}$$

From this we can see the optimum $\lambda = 0$ is achieved when i^m, the opportunity cost of holding money, is set to zero, meaning buyers hold sufficient real balances to maximize their trade surpluses and achieve y*. Using the equation for the nominal interest rate $i = \gamma/\beta - 1$ we can see that a rate of zero is achieved when $\gamma = \beta$, so when the money supply is contracted at approximately the rate of time preference. This is known as the Friedman Rule.

Considering this in terms of quantities, we get:

$$\frac{u'(y)}{c'(y)} = \frac{i^m}{\alpha_1 + \alpha_3} + 1$$

Which is interpreted graphically below. This shows that the optimum y* is achieved at $i^m = 0$ under the Friedman Rule, and that when there is inflation, and hence a positive i^m on fiat currency, then output is inefficiently low in the DM, achieving y^{ss}.



<u>Figure 4:</u> Plot of equilibrium condition as function of quantities.

<u>6.2</u> Coordination Issue of Monetary Policy

When there are competing currencies in the economy, welfare is maximized when all currencies follow the Friedman Rule. However, if one currency deviates from the Friedman Rule, then the optimal response for the other currency might be to respond with some moderate inflation. To see this, consider the equilibrium when CBDC is not valued, but fiat currency and private digital currency are, giving equilibrium conditions:

$$i^m = \alpha_1 \lambda(z^m) + \alpha_3 \lambda(z^m + z^h)$$

$$i^h = \alpha_2 \lambda(z^h) + \alpha_3 \lambda(z^m + z^h)$$

The optimal policy here occurs when the liquidity premium on all real balances z^m, z^h and $z^m + z^h$ is 0, which we can see from the equilibrium conditions coincides with both i^m and i^h being set to 0, meaning both currencies follow the Friedman Rule.

The central bank only controls the interest rate on fiat currency however, and it is possible that that the private issuer of digital currency is not welfare maximizing. Assume that instead of setting the welfare maximizing policy $i^h = 0$, the private issuer sets $i^h > 0$. Looking at the case where $\alpha_1 = 0$, the equilibrium conditions reduce to:

$$i^m = \alpha_3 \lambda(z^m + z^h)$$

$$i^h = \alpha_2 \lambda(z^h) + \alpha_3 \lambda(z^m + z^h)$$

And the derivative of the welfare function with respect to i^m takes the form:

$$\frac{\partial \mathcal{W}(i^m, i^h)}{\partial i^m} = \alpha_2 \, [u' \circ y(z^h) - c' \circ y(z^h)] y'(z^h) \frac{\partial z^h}{\partial i^m}$$
$$+ \alpha_3 [u' \circ y(z^m + z^h) - c' \circ y(z^m + z^h)] \, y'(z^h + z^m) \frac{\partial (z^h + z^m)}{\partial i^m}$$

Rearranging the equilibrium conditions, we get $z^h = \lambda^{-1}(i^h - i^m/\alpha_2)$, and hence derivative:

$$\frac{\partial z^h}{\partial i^m} = -\frac{1}{\alpha_2 \lambda' \circ \lambda^{-1}(\frac{1}{\alpha_2}(i^h - i^m))}$$

$$= -\frac{1}{\alpha_2 \lambda'(\frac{1}{\alpha_2}(i^h - i^m))} = -\frac{1}{\alpha_2 \lambda'(\lambda(z^h))}$$

$$-\frac{1}{\alpha_2 \lambda'(z^h)}$$

Since λ is a decreasing function, we get $\lambda' < 0$, and hence $\partial z^h / \partial i^m > 0$.

If the Central Bank responds by following the Friedman Rule, setting $i^m = 0$, then we get $\alpha_3 \lambda(z^m + z^h) = 0$ and $i^h/\alpha_2 = \lambda(z^h)$ from the equilibrium conditions. From this we can see for $i^h > 0$, the liquidity premium $\lambda(z^h)$ will be non-zero, and hence agents hold $z^h < d^*$. This means the optimum quantity y^* is not achieved, so $y < y^*$, and hence $u'(y) - c'(y) > 0$ (from figure 1 we know this holds for all $y < y^*$). Also, $y(z)$ is an increasing function, and hence $y'(z) > 0$. Combining these, we can see the welfare function is increasing in i^m:

$$\left.\frac{\partial \mathcal{W}(i^m, i^h)}{\partial i^m}\right|_{i^m=0} = \alpha_2 [u' \circ y(z^h) - c' \circ y(z^h)] y'(z^h) \frac{\partial z^h}{\partial i^m} > 0$$

This means the optimal policy is to run moderate inflation on fiat currency, $i^m > 0$. Moreover, this is a composition of continuous functions, and hence is continuous. Using this continuity, we can see for an α_1 sufficiently close to 0, the optimal policy will still be to run $i^m > 0$.

This illustrates the coordination issue, where the optimal policy decision of the central bank is now reactive to the rate on private digital currency. The intuition here is that inflation on fiat currency will cause agent to substitute their holdings towards private digital currency, which will increase its value and decreases the value on fiat currency. This increases the amount they can spend in type 2 transactions, but decrease the amount they can spend in type 1 transactions. However, as long the proportion of transactions requiring fiat currency (α_1) is sufficiently small, then the benefit of increased spending in type 2 transactions will outweigh the loss in type 1, and hence welfare improves.

6.3 CBDC is valued in the economy

When the central bank sets the interest rate on CBDC sufficiently low, it will be the only currency valued in the economy (from section 4.3.2). In this case, welfare can be maximized by following the Friedman Rule. To see this, consider the equilibrium condition when CBDC alone is valued in the economy:

$$i^c = \alpha_1\lambda(z^c) + \alpha_2\lambda(z^c) + \alpha_3\lambda(z^c)$$

Rearranging for λ we get:

$$\lambda(z^c) = \frac{i^c}{\alpha_1 + \alpha_2 + \alpha_3}$$

From this we can see the welfare maximizing y*, achieved at $\lambda = 0$, can be reached by following the Friedman Rule.

The intuition here is that full monetary control is being returned to the central bank, and hence their ability to set a welfare maximizing policy is restored. CBDC is a perfect substitute for private digital currencies in type 2 transactions, but offers the additional benefit that it is redeemable in type 1 transactions, meaning as long as the cost of holding it is sufficiently low, buyers will choose to hold this instead. The same idea applies to fiat currency, leaving CBDC as the only currency valued in the economy. There is no valued currency from a private distributer, and hence no coordination issue. This gives the central bank full monetary control over the economy, and hence the ability to maximize welfare by setting the Friedman Rule.

7. Conclusion

This paper finds that the introduction of a CBDC to an economy where fiat currency and private digital currency coexist can be welfare enhancing, providing its nominal interest rate is set sufficiently low. Using a modified version of the Lagos and Wright (2005) framework, that allows for the coexistence of CBDC, fiat and private digital currency, I establish a boundary for when CBDC will be value in the economy. If the cost of holding CBDC is too large, then it will not be valued. This results in a coordination issue in monetary policy, where the optimal decision by the central bank becomes reactionary to the policy set on private digital currency, and may in fact be to run some moderate inflation on fiat currency. However, if the cost of holding CBDC is sufficiently small, then it will be the only currency valued in the economy. This restores full monetary control to the central bank, alleviating the coordination issue, and hence allows welfare to be maximized by following the Friedman Rule on CBDC. This paper supports the case for distribution of digital currency by central banks, and highlights its potential to alleviate the monetary constraints that might be imposed by the growing adoption of private digital currencies.

A shortcoming of this paper is that CBDC and private digital currencies will not necessarily use the same technology, and so would not be perfect substitutes in type 2 transactions. Even when interest rates on CBDC are set sufficiently low, this paper potentially overestimates the extent to which agents substitute their holdings of private digital currency to CBDC, and hence the extent to which CBDC alleviates the monetary coordination issue. Further research could model transactions differently, and more explicitly capture the technological innovations that private digital currency and CBDC are able to provide. This would allow for a more realistic substitution between the currencies. Additionally, this paper only considers non-negative interest rates. If CBDC deposits were held directly at the central bank, then they could feasibly support negative rates. Further research could consider the implications of this on the coordination issue, and CBDC’s ability to alleviate this.

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9. Appendix

9.1 Comparative Statics

$$\frac{dz^m}{di^h} = -\frac{\alpha_3 \frac{\partial\lambda\left(z^m + z^h\right)}{\partial z^h}}{\left||\mathbf{J}|\right|} > 0$$

$$\frac{dz^h}{di^h} = \frac{\alpha_1 \frac{\partial\lambda(z^m)}{\partial z^m} + \alpha_3 \frac{\partial\lambda\left(z^m + z^h\right)}{\partial z^m}}{|\mathbf{J}|} < 0$$

$$\frac{dz^m}{d\alpha_1} = -\frac{\lambda(z^m)\left(\alpha_2 \frac{\partial\lambda\left(z^h\right)}{\partial z^h} + \alpha_3 \frac{\partial\lambda\left(z^m + z^h\right)}{\partial z^h}\right)}{|\mathbf{J}|} > 0$$

$$\frac{dz^h}{d\alpha_1} = \frac{\alpha_3 \lambda(z^m) \frac{\partial\lambda\left(z^m + z^h\right)}{\partial z^m}}{|\mathbf{J}|} < 0$$

$$\frac{dz^m}{d\alpha_2} = \frac{\alpha_3 \lambda\left(z^h\right) \frac{\partial\lambda\left(z^m + z^h\right)}{\partial z^h}}{|\mathbf{J}|} < 0$$

$$\frac{dz^h}{d\alpha_2} = -\frac{\lambda\left(z^h\right)\left(\alpha_1 \frac{\partial\lambda(z^m)}{\partial z^m} + \alpha_3 \frac{\partial\lambda\left(z^m + z^h\right)}{\partial z^m}\right)}{|\mathbf{J}|} > 0$$

$$\frac{dz^m}{d\alpha_3} = -\frac{\alpha_2 \lambda\left(z^m + z^h\right) \frac{\partial\lambda\left(z^h\right)}{\partial z^h}}{|\mathbf{J}|} > 0$$

$$\frac{dz^h}{d\alpha_3} = -\frac{\alpha_2 \lambda\left(z^m + z^h\right) \frac{\partial\lambda\left(z^h\right)}{\partial z^h}}{|\mathbf{J}|} > 0$$