Pingala Series

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/pingala/codes/1.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/pingala/codes/2.2.
    py
```

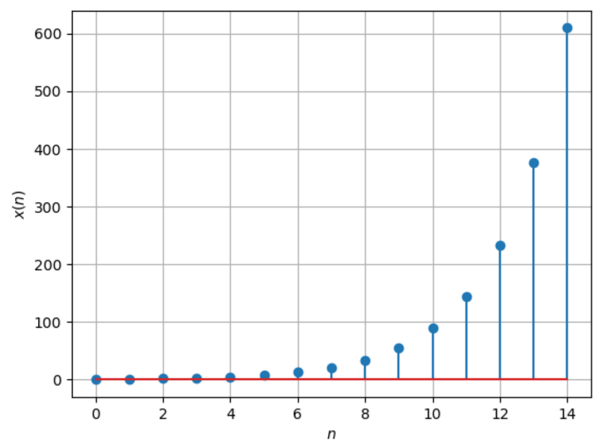

Fig. 2.2: Pingala series

2.3 Find $X^+(z)$.
Solution:

$$x(n+2) = x(n+1) + x(n)$$
$$\implies \mathcal{Z}^+\{x(n+2)\} = \mathcal{Z}^+\{x(n+1)\} + \mathcal{Z}^+\{x(n)\}$$
$$\implies z^2 X^+(z) - z^2 - z = zX^+(z) - z + X^+(z)$$
$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha$$

2.4 Find $x(n)$.
Solution:

$$\frac{1}{1 - z^{-1} - z^{-2}} = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}$$

$$X^+(z) = \left(\frac{z}{\alpha - \beta}\right)\left(\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}}\right)$$

using GP summation

$$= \sum_{n=0}^{\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} z^{-n}$$

$$\implies x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) = a_{n+1} u(n)$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.3}$$

Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/pingala/codes/2.5.
    py
```

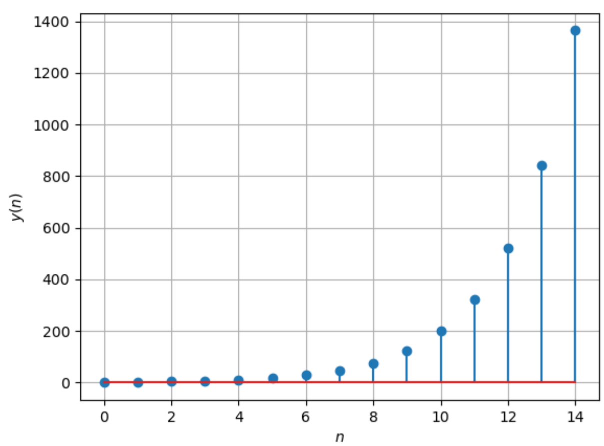

Fig. 2.5: y(n)

2.6 Find $Y^+(z)$.
Solution:

$$\mathcal{Z}^+\{y(n)\} = \mathcal{Z}^+\{x(n+1)\} + \mathcal{Z}^+\{x(n-1)\}$$

$$Y^+(z) = zX^+(z) - z + z^{-1}X^+(z)$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

$$\implies Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha$$

2.7 Find $y(n)$.
Solution:

$$Y^+(z) = X^+(z) + \frac{2}{(z-\alpha)(z-\beta)}$$

$$= X^+(z) + \frac{2}{\alpha - \beta}\left(\sum_{n=0}^{\infty} (\alpha^n + \beta^n) z^{-n}\right)$$

$$\implies y(n) = a_{n+1} u(n) + 2\frac{\alpha^n + \beta^n}{\alpha - \beta} u(n)$$

$$= \frac{\alpha^{n+1} - \beta^{n+1} + 2\alpha^n + 2\beta^n}{\alpha - \beta} u(n)$$

using the fact that $\alpha\beta = 1$ and $\alpha + \beta = -1$

$$\implies y(n) = \alpha^{n+1} + \beta^{n+1}$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.1}$$

Solution:

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1} = \sum_{k=0}^{n-1} x(k)$$

adding an $x(n)$ term

$$= \sum_{k=0}^{n-1} x(k)u(n-1-k) + x(n)u(n-1-n)$$

$$= \sum_{k=0}^{n} x(k)u(n-1-k) = x(n) * u(n-1)$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.2}$$

can be expressed as

$$[x(n+1) - 1]\, u(n) \tag{3.3}$$

Solution: For $n \geq 0$ $x(n) = a_{n+1}$

$$\implies a_{n+2} - 1 = x(n+1) - 1$$

$$= (x(n+1) - 1)u(n)$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.4}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{x(k)}{10^{k+1}} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^{k}} = \frac{1}{10} X^{+}(10)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.5}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.6}$$

and find $W(z)$.

Solution:

$$\alpha^n + \beta^n, \quad n \geq 1$$

is the same as

$$\alpha^{n+1} + \beta^{n+1}, \quad n \geq 0$$

$$\implies w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n)$$

Z-transform of $w(n)$

$$W(z) = \sum_{n=-\infty}^{\infty} w(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \alpha^{n+1} z^{-n} + \beta^{n+1} z^{-n}$$

$$= \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\beta}{1 - \beta z^{-1}}, \quad |z| > \alpha$$

using the fact that $\alpha\beta = 1$ and $\alpha + \beta = -1$

$$\implies W(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^{+}(10) \tag{3.7}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{y(k)}{10^{k+1}} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^{k}} = \frac{1}{10} Y^{+}(10)$$

3.6 Solve the JEE 2019 problem.

Solution:

From 3.1

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1)$$

Positive Z transform of right side gives

$$\implies X^{+}(z) z^{-1} U^{+}(z)$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})}$$

$$= \frac{z}{1 - z^{-1} - z^{-2}} - \frac{z}{1 - z^{-1}}$$

$$= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1}$$

replacing n with $n + 1$

$$= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n}$$

inverse Z transform gives

$$\implies (x(n+1) - 1) u(n) = a_{n+2} - 1$$

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} X^{+}(10) = \frac{1}{10} . \frac{100}{100 - 10 - 1}$$

$$= \frac{10}{89}$$

$$\implies b_n = a_{n+1} + a_{n-1}$$

$$= x(n) + x(n-2) = y(n-1)$$

$$= \alpha^n + \beta^n$$

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} Y^{+}(10) = \frac{1}{10} . \frac{100 + 2.10}{100 - 10 - 1}$$

$$= \frac{12}{89}$$