Circuits and Transforms

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

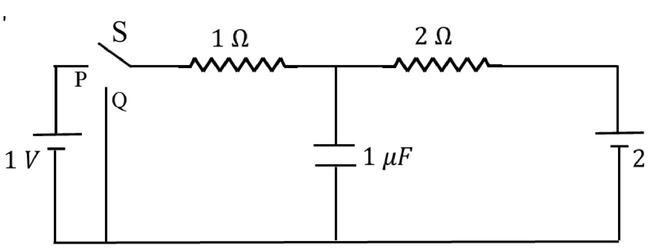
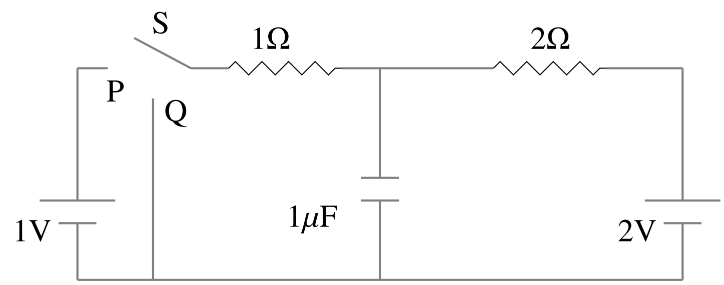


Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution:



3. Find q_1.
Solution: No current will pass through the capacitor, hence

$$i = \frac{2-1}{2+1} = \frac{1}{3}A$$

$$V_c = 2 - 2.i = \frac{4}{3}V$$

$$q_1 = \frac{4}{3}\mu C$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution:

$$U(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt$$

$$= \int_{0}^{\infty} e^{-st}dt = \frac{1}{s}, \quad s > 0$$

5. Show that

$$e^{-at}u(t) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{1}{s+a}, \quad a > 0 \tag{2.1}$$

and find the ROC.
Solution:

$$= \int_{-\infty}^{\infty} u(t)e^{-at}e^{-st}dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$= \frac{1}{s+a}, \quad s + a > 0$$

6. Now consider the following resistive circuit transformed from Fig. **??** where

$$u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_1(s) \tag{2.2}$$

$$2u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_2(s) \tag{2.3}$$

Find the voltage across the capacitor $V_{C_0}(s)$.

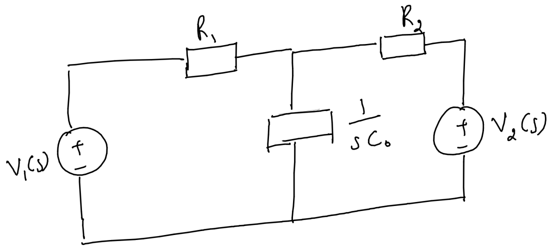


Fig. 2.2

Solution: From Q2.4

$$V_1(s) = \frac{1}{s} \quad V_2(s) = \frac{2}{s}$$

$$\frac{V_1 - V_C}{R_1} + \frac{V_2 - V_C}{R_2} - V_C sC = 0$$

simplifying,

$$\implies V_c(s) = \frac{2R_1 + R_2}{R_1 + R_2}\left(\frac{1}{s} - \frac{1}{\frac{R_1+R_2}{R_1R_2C} + s}\right)$$

7. Find $v_{C_0}(t)$. Plot using python.
   Solution: Using the inverse Laplace transform from the obtained $V_c(s)$ we get,

$$v_c(t) = \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\frac{R_1+R_2}{R_1R_2C}t}\right)$$

$$R_1 = 1\Omega \quad R_2 = 2\Omega \quad C = 1\mu F$$

$$v_c(t) = \frac{4}{3}u(t)\left(1 - e^{-1.5\times10^6 t}\right)$$

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wget https://github.com/Galahad7377/
    EE3900/blob/main/cktsig/codes/2.7.
    py
```

8. Verify your result using ngspice.
   Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/cktsig/codes/2.8.
    cir
```

9. Obtain Fig. **??** using the equivalent differential equation.
   Solution:

$$\frac{dq}{dt} = \frac{V_1 - V_c}{R_1} + \frac{V_2 - V_c}{R_2}$$

$$\frac{dq}{dt} = 1 - \frac{q}{C} + \frac{1}{2}\left(2 - \frac{q}{C}\right)$$

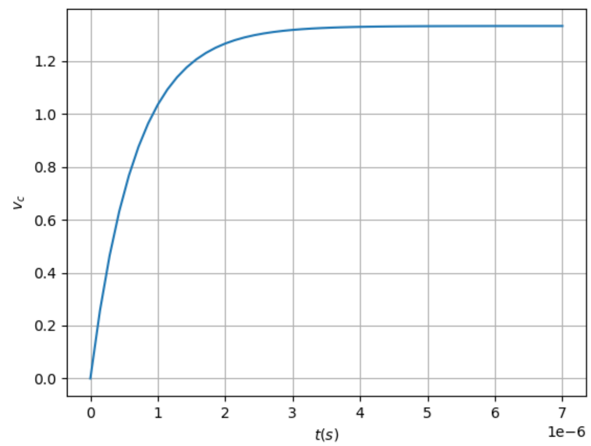


Fig. 2.3: $v_c(t)$

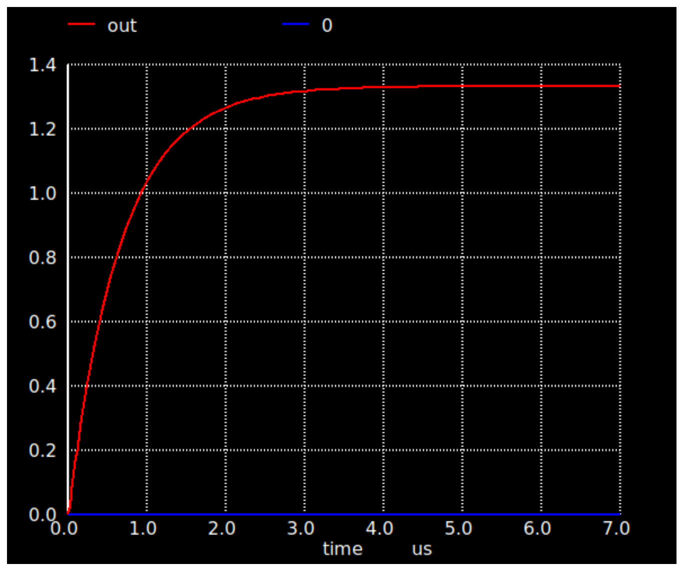


Fig. 2.4

simplifying and integrating,

$$\int_0^{q_1} \frac{dq}{2 - 1.5q/C} = \int_0^t dt$$

$$\ln\frac{2 - 1.5q/C}{2} = 1.5\frac{t}{C}$$

$$\implies q_1 = \frac{4}{3}\left(1 - e^{-1.5\times10^6 t}\right) \times 10^{-6}$$

3 Initial Conditions

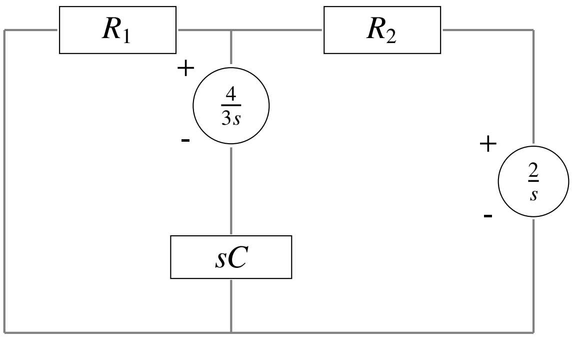
1. Find q_2 in Fig. **??**.
   Solution:

$$i = \frac{2}{1+2} = \frac{2}{3}A$$

$$V_c = 2 - 2.i = \frac{2}{3}V$$

$$\implies q_2 = CV_c = \frac{2}{3}\mu C$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution:



3. $V_{C_0}(s) = ?$
Solution:

$$\frac{V_C - 2/s}{R_2} + \frac{V_C}{R_1} + (V_C - 4/3s)\, sC = 0$$

$$V_C(s) = \frac{\frac{2}{sR_2} + \frac{4}{3}C}{\frac{1}{R_1} + \frac{2}{R_2} + sC}$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: Further simplifying,

$$V_C(s) = \frac{4}{3}\left(\frac{1}{\frac{R_1+R_2}{R_1R_2C} + s}\right) + \frac{2R_1R_2}{R_2\left(R_1+R_2\right)}\left(\frac{1}{s} - \frac{1}{\frac{R_1+R_2}{R_1R_2C} +$$

Using the inverse Laplace transform from the obtained Vc(s) we get,

$$v_c(t) = \frac{4}{3}e^{\frac{R_1+R_2}{R_1R_2}\frac{t}{C}} + \frac{2R_1R_2}{R_2\left(R_1+R_2\right)}\left(1 - e^{\frac{R_1+R_2}{R_1R_2}\frac{t}{C}}\right)$$

$$R_1 = 1\Omega \quad R_2 = 2\Omega \quad C = 1\mu F$$

$$\implies v_c(t) = \frac{2}{3}u(t)\left(1 + e^{-1.5\times10^6 t}\right)$$

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/cktsig/codes/3.4.py
```

5. Verify your result using ngspice.
Solution:

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wget https://github.com/Galahad7377/
    EE3900/blob/main/cktsig/codes/3.5.
    cir
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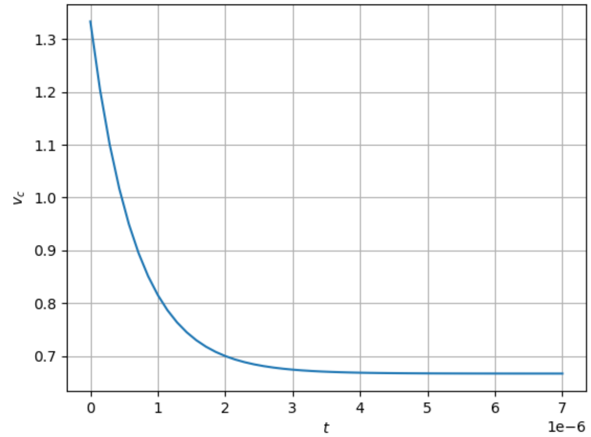

Fig. 3.1: $v_c(t)$

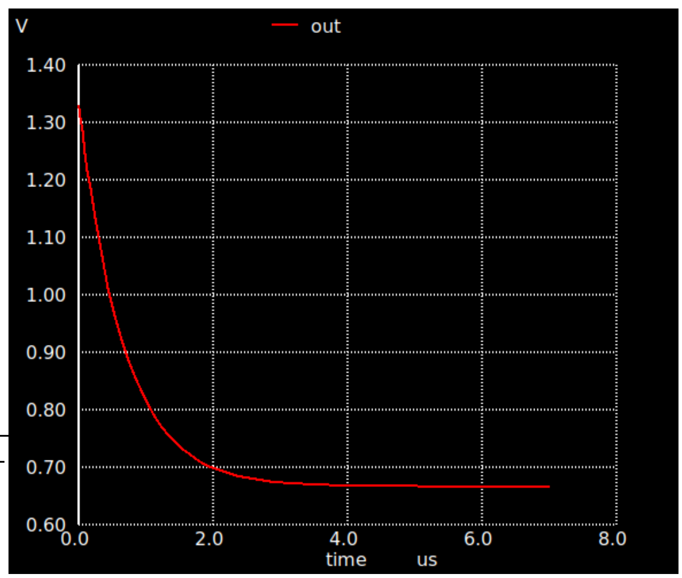

Fig. 3.2

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: from section 2,

$$v_c(0-) = \frac{4}{3}V$$

from Q3.4

$$v_c(0+) = \frac{4}{3}V$$

$$v_c(\infty) = \frac{2}{3}V$$

7. Obtain the Fig. in problem **??** using the equivalent differential equation.

Solution:

$$\frac{dq}{dt} = \frac{2 - V_C}{2} - V_C$$

simplifying and integrating,

$$\int_{4/3\mu C}^{q} \frac{dq}{2 - 3q/C} = 2\int_0^t dt$$

$$\ln\frac{2 - 3q/C}{-2} = -1.5\frac{t}{C}$$

$$q = \frac{2}{3}\left(1 + e^{-1.5\times 10^6 t}\right)\times 10^{-6}$$

4 Bilinear Transform

1. In Fig. **??**, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
2. Find $H(s)$ considering the ouput voltage at the capacitor.
3. Plot $H(s)$. What kind of filter is it?
4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.1}$$

5. Find $H(z)$.
6. How can you obtain $H(z)$ from $H(s)$?