Digital Signal Processing

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Contents

Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
    gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem **??** in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
    import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('
    filter_codes_Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem **??** is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem **??**. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/3.1.py
```

*

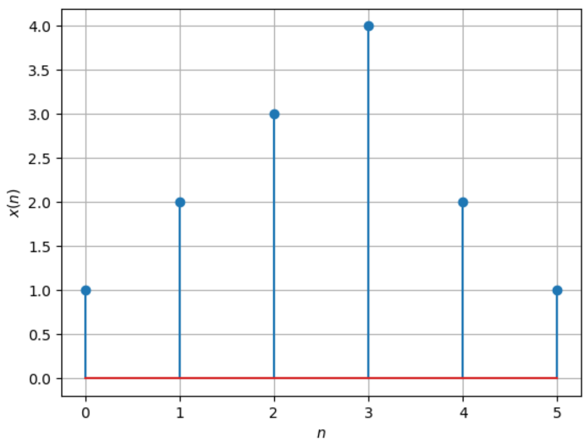


Fig. 3.1

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch $y(n)$.
Solution: The following code yields Fig. **??**.

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/3.1.py
```

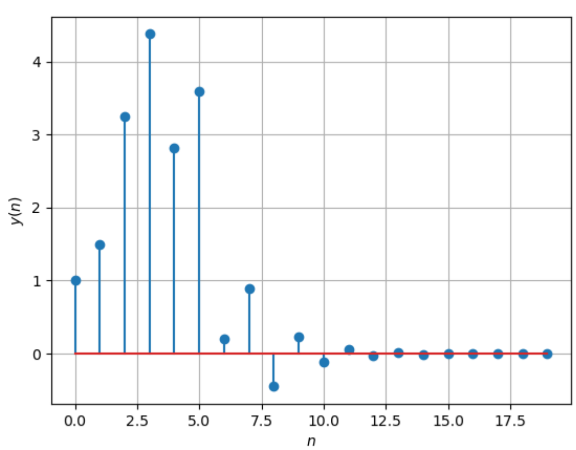


Fig. 3.2

3.3 Repeat the above exercise using a C code.
Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/3.3.c
```

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/3.3.py
```

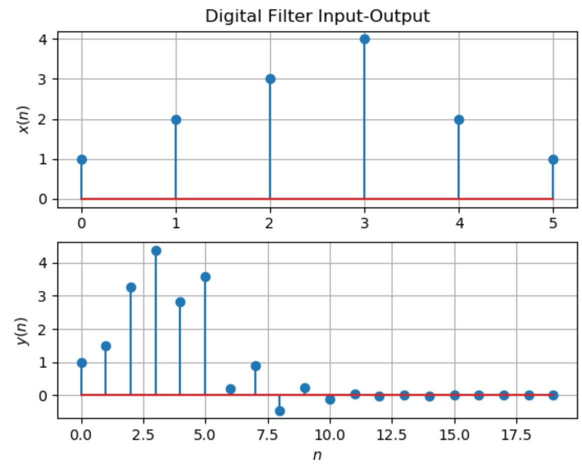


Fig. 3.3

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

Solution: From (**??**),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (4.4)$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.5)$$

resulting in (**??**). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (4.6)$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem **??**.
Solution:

$$x(n) = \{1, 2, 3, 4, 2, 1\} \quad (4.7)$$

$$X(z) = 1 + 2/z + 3/z^2 + 4/z^3 + 2/z^4 + 1/z^5 \quad (4.8)$$

4.3 Find
$$H(z) = \frac{Y(z)}{X(z)} \tag{4.9}$$
from (**??**) assuming that the Z-transform is a linear operation.
Solution: Applying (**??**) in (**??**),
$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.10}$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.11}$$

4.4 Find the Z transform of
$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.12}$$
and show that the Z-transform of
$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$
is
$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.14}$$
Solution: It is easy to show that
$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.15}$$
and from (**??**),
$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.16}$$
$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.17}$$
using the formula for the sum of an infinite geometric progression.

4.5 Show that
$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.18}$$
Solution:
$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.19}$$
$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \tag{4.20}$$
Since
$$|z| > |a| \tag{4.21}$$
using the formula for the sum of an infinite geometric progression.
$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - az^{-1}} \tag{4.22}$$

4.6 Let
$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.23}$$
Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.
Solution: The following code plots Fig. **??**.

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/4.6.py
```

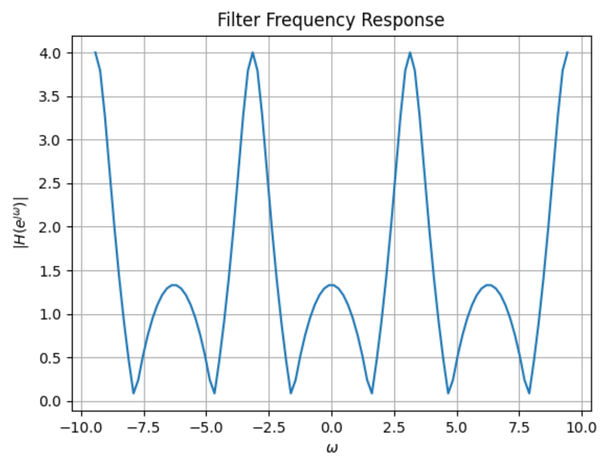


Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

$\left|H\left(e^{\jmath\omega}\right)\right|$ is periodic with a period of 2π.

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$.
Solution:
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h(k)e^{(n-k)\jmath\omega} d\omega$$
$$= \sum_{k=-\infty}^{\infty} h(k)\delta(n-k)$$
$$= h(n)$$

5 Impulse Response

5.1 Using long division, find
$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (??).
Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}}$$

$$\begin{array}{r} 2X-4 \\ \frac{1}{2}X+1 \overline{\smash{\big)}\, X^2 \quad\quad +1} \\ -X^2-2X \\ \hline -2X+1 \\ 2X+4 \\ \hline 5 \end{array}$$

$$H(z) = 2z^{-1} - 4 + \frac{5}{1+\frac{1}{2}z^{-1}} \dots |z| > \frac{1}{2}$$

Using the inverse Z-transform

$$h(n) = 2\delta(n-1) - 4\delta(n) + 5\left(-\frac{1}{2}\right)^n u(n)$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.2}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (??).
Solution: From (??),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.3}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.4}$$

using (??) and (??).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. ??.

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/5.3.py
```

From the figure it clear that h(n) is bounded, also

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \le 1$$

$$h(n) \le 2$$

5.4 Convergent? Justify using the ratio test.
Solution:

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right|$$

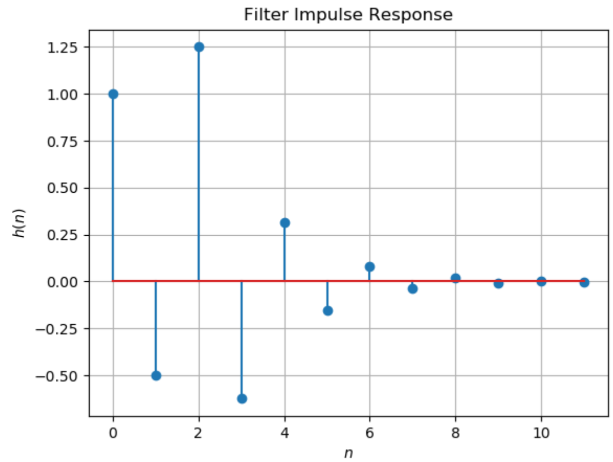


Fig. 5.3: $h(n)$ as the inverse of $H(z)$

$$= \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1)}{\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)}\right|$$

$$= \left|\frac{-\frac{1}{2}-2}{1+4}\right|$$

$$= \frac{1}{2} < 1$$

h(n) is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.5}$$

Is the system defined by (??) stable for the impulse response in (??)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2}$$

$$= \frac{1}{1+\frac{1}{2}} + \frac{1}{1+\frac{1}{2}}$$

$$= \frac{4}{3} < \infty$$

5.6 Verify the above result using a python code.
Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/5.6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \qquad (5.6)$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. **??**. Note that this is the same as Fig. **??**.

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/5.7.py
```

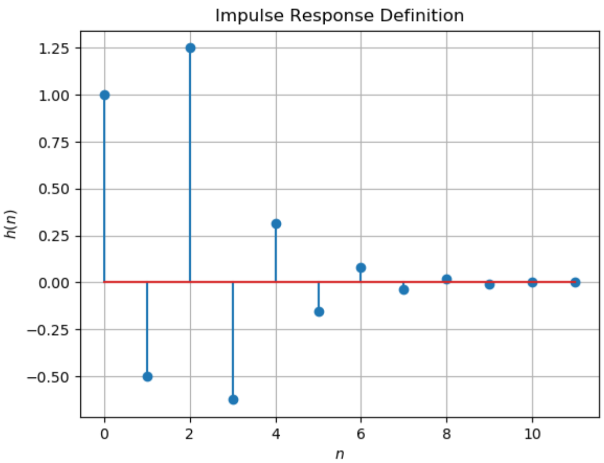


Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \qquad (5.7)$$

Comment. The operation in (**??**) is known as *convolution*.
Solution: The following code plots Fig. **??**. Note that this is the same as $y(n)$ in Fig. **??**.

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/5.8.py
```

5.9 Express the above convolution using a Teoplitz matrix.
Solution: Let x(n) and h(n) be represented by the following matrices. x= $\begin{bmatrix}1\\2\\3\\4\\2\\1\end{bmatrix}$ h= $\begin{bmatrix}1\\-0.5\\1.25\\-0.625\\0.315\\0.15625\end{bmatrix}$ T =

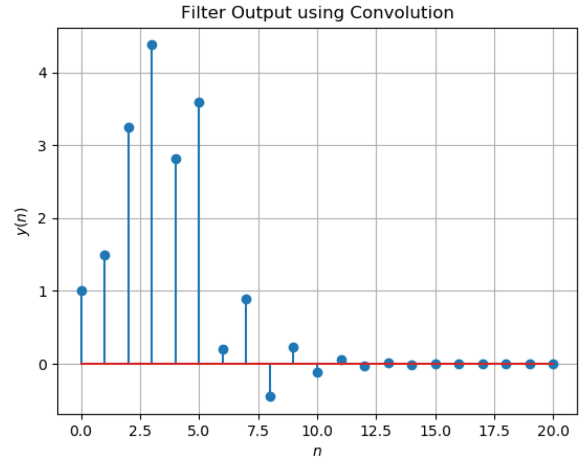


Fig. 5.8: $y(n)$ from the definition of convolution

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ -0.5 & 1 & 0 & 0 & 0\\ 1.25 & -0.5 & 1 & 0 & 0\\ -0.625 & 1.25 & -0.5 & 1 & 0\\ 0.315 & -0.625 & 1.25 & -0.5 & 1\\ 0.156 & 0.315 & -0.625 & 1.25 & -0.5\\ 0 & 0.156 & 0.315 & -0.625 & 1.25\\ 0 & 0 & 0.156 & 0.315 & -0.625\\ 0 & 0 & 0 & 0.156 & 0.315\\ 0 & 0 & 0 & 0 & 0.156\end{bmatrix}$$

We have

$$y = x * h = Tx$$

y= $\begin{bmatrix}1\\1.5\\3.25\\4.38\\2.81\\3.59\\0.12\\0.78\\-0.62\\0\\-0.16\end{bmatrix}$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \qquad (5.8)$$

Solution: We know that,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

replacing n-k with k

$$y(n) = \sum_{n-k=-\infty}^{\infty} x(n-k)h(k)$$

$$\implies y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k)$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/6.1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/6.2.py
```

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. ??. Note that this is the same as $y(n)$ in Fig. ??.

```
wget https://github.com/Galahad7377/EE3900/
    blob/main/filter/codes/6.3.py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution:

```
wget https://github.com/Galahad7377/EE3900/
    blob/main/filter/codes/6.4.py
```

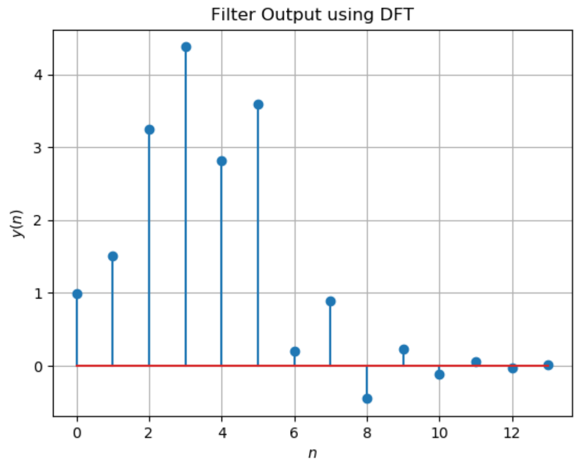
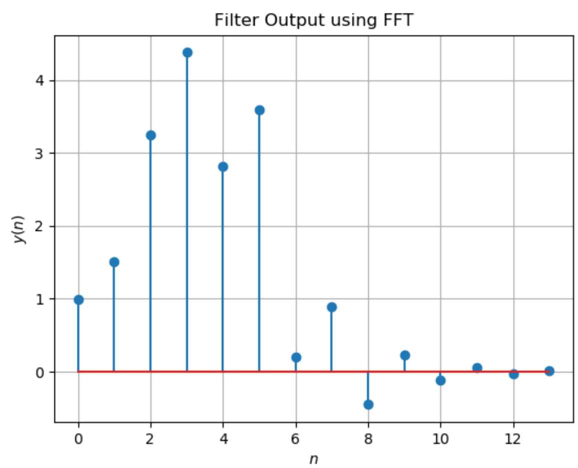


Fig. 6.3: $y(n)$ from the DFT



7 FFT

7.1 The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

7.2 Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\vec{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\vec{F}_N$.

7.3 Let

$$\vec{I}_4 = \vec{e}_4^1 \vec{e}_4^2 \qquad \vec{e}_4^3 \vec{e}_4^4 \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\vec{P}_4 = \vec{e}_4^1 \vec{e}_4^3 \qquad \vec{e}_4^2 \vec{e}_4^4 \tag{7.5}$$

7.4 The 4 point *DFT diagonal matrix* is defined as

$$\vec{D}_4 = diag W_4^0 W_N^1 \qquad W_N^2 W_N^3 \tag{7.6}$$

7.5 Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$W_N^2 = e^{-\jmath 4\pi/N} = e^{\frac{-\jmath 2\pi}{N/2}} = W_{N/2}$$

7.6 Show that

$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.8}$$

Solution:

$$\vec{F}_2 = \begin{bmatrix} W_2^0 & W_1^0 \\ W_2^0 & W_2^1 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix}$$

$$\vec{D}_2 \vec{F}_2 = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix}$$

$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix}$$

$$-\vec{D}_2 \vec{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix}$$

$$\vec{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{bmatrix}$$

$$\vec{W}_4 = \begin{bmatrix} \vec{I}_2 \vec{F}_2 & \vec{D}_2 \vec{F}_2 \\ \vec{I}_2 \vec{F}_2 & -\vec{D}_2 \vec{F}_2 \end{bmatrix}$$

$$\vec{W}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix}$$

Multiplying by $\vec{P}_4$ on both sides

$$\vec{W}_4 \vec{P}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4$$

$$\implies \vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4$$

7.7 Show that

$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.9}$$

Solution: Let $\vec{f}_N^i$ denote the i^{th} column of $\vec{F}_N$.

$$\begin{bmatrix} \vec{I}_{N/2} \vec{F}_{N/2} \\ \vec{I}_{N/2} \vec{F}_{N/2} \end{bmatrix} = \begin{bmatrix} \vec{f}_N^1 & \vec{f}_N^3 \dots \vec{f}_N^N \end{bmatrix}$$

$$\begin{bmatrix} \vec{D}_{N/2} \vec{F}_{N/2} \\ -\vec{D}_{N/2} \vec{F}_{N/2} \end{bmatrix} = \begin{bmatrix} \vec{f}_N^2 & \vec{f}_N^4 \dots \vec{f}_N^N \end{bmatrix}$$

$$\begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} = \begin{bmatrix} \vec{I}_{N/2} \vec{F}_{N/2} & \vec{D}_{N/2} \vec{F}_{N/2} \\ \vec{I}_{N/2} \vec{F}_{N/2} & -\vec{D}_{N/2} \vec{F}_{N/2} \end{bmatrix}$$

$$= \begin{bmatrix} \vec{f}_N^1 \dots \vec{f}_N^{N-1} & \vec{f}_N^2 \dots \vec{f}_N^N \end{bmatrix}$$

Multiplying both sides by $\vec{P}_N$

$$\begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N$$

$$= \begin{bmatrix} \vec{f}_N^1 \dots \vec{f}_N^{N-1} & \vec{f}_N^2 \dots \vec{f}_N^N \end{bmatrix} \vec{P}_N$$

$$= \begin{bmatrix} \vec{f}_N^1 \dots \vec{f}_N^N \end{bmatrix}$$

$$\implies \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N = \vec{F}_N$$

7.8 Find

$$\vec{P}_4 \vec{x} \tag{7.10}$$

Solution:

$$\vec{P}_4 \vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_2 \\ x_1 \\ x_3 \end{bmatrix}$$

7.9 Show that

$$\vec{X} = \vec{F}_N \vec{x} \tag{7.11}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution:

$$X(k) \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1$$

$$X(N-1) = x(0) + x(1) e^{-\jmath 2\pi (N-1).1/N} + \dots$$
$$+ x(N-1) e^{-\jmath 2\pi (N-1)(N-1)/N}$$

So,

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \vec{F} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

where

$$F_{mn} = e^{-\jmath 2mn\pi/N}$$

$$\implies \vec{X} = \vec{F}_N \vec{x}$$

7.10 Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.12}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.13}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.14}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.15}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.16}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.17}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.18}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.19}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.20}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.21}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.22}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.23}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.24}$$

Solution: For an 8 point FFT

$$X(k) = \sum_{n=0}^{7} x(n) e^{-\jmath 2kn\pi/8}$$

$$= \sum_{n=0}^{3} \left(x(2n) e^{-jkn\pi/4} + x(2n+1) e^{-jk\pi/4} e^{-jkn\pi/2} \right)$$

$$= X_1(k) + X_2(k) e^{-jk\pi/4}$$

X_1 and X_2 are the 4 point FFTs of the even and odd terms of $X(k)$ respectively. Splitting X_1 in a similar manner,

$$X_1(k) = \sum_{n=0}^{3} x(n) e^{-\jmath 2kn\pi/4}$$

$$= \sum_{n=0}^{1} \left(x_1(2n) e^{-jkn\pi/2} + x_2(2n+1) e^{-jk\pi/2} e^{-jkn\pi} \right)$$

$$= X_3(k) + X_4(k) e^{-jk\pi/2}$$

$x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$
We can write

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix}$$

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix}$$

proceeding in a similar fashion for X_2 we come to

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix}$$

where $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_4(k) = x(4k+1)$, $x_6(k) = x(4k+3)$

7.11 For

$$\vec{x} = 1 \quad (7.25)$$
$$2 \quad (7.26)$$
$$3 \quad (7.27)$$
$$4 \quad (7.28)$$
$$2 \quad (7.29)$$
$$1 \quad (7.30)$$

compute the DFT using (**??**)
Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/7.11.
    py
```

7.12 Repeat the above exercise using the FFT after zero padding $\vec{x}$.
Solution: Check 7.13

7.13 Write a C program to compute the 8-point FFT.
Solution:

```
wget https://github.com/Galahad7377/
    EE3900/blob/main/filter/codes/7.13.
    py
```

8 Exercises

Answer the following questions by looking at the python code in Problem **??**.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem **??** is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

8.2 Repeat all the exercises in the previous sections for the above a and b.

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.