Model Identification and Data Analysis cheatsheet

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Contents

1 Probability Recall **3**
1.1 Random Vectors . . . 3
1.2 Random processes . . . 3
1.3 Important process classes . . . 3

2 Spectral Analysis **4**
2.1 Foundamentals . . . 4
2.2 Fundamental theorem of Spectral Analysis . . . 4

3 Moving Average Processes **5**
3.1 MA(1): . . . 5
3.2 MA(n) . . . 6
3.3 MA(∞) . . . 6
3.4 Well definition of an MA(∞) . . . 7

4 Auto Regressive Processes **7**
4.1 AR(1) . . . 7
4.2 AR(n) . . . 8

5 ARMA Processes **9**

1 Probability Recall

1.1 Random Vectors

Variance $Var[v] = E[(v - E[v])^2]$

Cross-Variance $Var[v, u] = E[(v - E[v])(u - E[u])]$

Variance Matrix $\begin{vmatrix} Var[v_1] & . & . & Var[v_1, v_k] \\ . & . & & . \\ . & & . & . \\ Var[v_k, v_1] & . & . & Var[v_k] \end{vmatrix}$

Covariance coefficient $\delta[i, j] = \frac{Var[i,j]}{\sqrt{Var[i]}\sqrt{Var[j]}}$
$\delta[i, j] = 0 \implies$ i, j uncorrelated
$|\delta[i, j]| = 1 \implies i = \alpha j$

1.2 Random processes

$v(t, s)|$ t time instant, s expetiment outcome (generally given)

Mean $m(t) = E[v(t, s)]$

Variance $\lambda^2(t) = Var[v(t)]$

Covariance function $\gamma(t_1, t_2) = E[(v(t_1) - m(t_1))(v(t_2) - m(t_2))] = \gamma(t_2, t_1)$

Normalized Covariance Function $\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}$
$\forall$ stationary processes: $|\rho(\tau)| \leq 1 \quad \forall \tau$

1.3 Important process classes

Stationary process

- $m(t) = m$ constant
- $\lambda^2(t) = \lambda^2$ constant
- $\gamma(t_1, t_2) = f(t_2 - t_1) = \gamma(\tau)$ covariance depends only on time difference τ

$|\gamma(\tau)| \leq \gamma(0) \quad \forall \tau$

White noise $\eta(t) \sim WN(m, \lambda^2)$

- Stationary process
- $\gamma(\tau) = 0 \quad \forall \tau \neq 0$

$v(t) = \alpha\eta(t) + \beta \quad \eta(t) \sim WN(0, \lambda^2) \qquad \implies \qquad v(t) \sim WN(\beta, \alpha^2\lambda^2)$

2 Spectral Analysis

2.1 Foundamentals

Spectrum

$$\Gamma(\omega) = \overbrace{F(\gamma(\tau))}^{\text{Fourier transform}} = \sum_{\tau=-\infty}^{+\infty} \gamma(\tau) \cdot e^{-j\omega\tau}$$

Euler formula $\Gamma(\omega) = \gamma(0) + 2cos(\omega)\gamma(1) + 2cos(2\omega)\gamma(2) + ...$

Spectrum properties

- $\Gamma : \mathbb{R} \to \mathbb{R}$
- Γ is periodic with $T = 2\pi$
- Γ is even $[\Gamma(-\omega) = \Gamma(\omega)]$
- $\Gamma(\omega) \geq 0 \quad \forall\omega$

$\eta(t) \sim WN(0, \lambda^2) \implies \Gamma_\eta(\omega) = \gamma(0) = Var[\eta(t)] = \lambda^2$

Anti-Transform

$$\gamma(\tau) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \Gamma(\omega) e^{k\omega\tau}\, dw$$

Complex spectrum

$$\phi(z) = \sum_{\tau=-\infty}^{+\infty} \omega(\tau) z^{-\tau}$$

$$\Gamma(\omega) = \Phi(e^{j\omega})$$

2.2 Fundamental theorem of Spectral Analysis

Fundamental theorem of Spectral Analysis allows to derive the (real and/or complex) spectrum of the output from the input and the transfer function of the system

$$\xrightarrow{u} \boxed{W(z)} \xrightarrow{y}$$

$$\Gamma_{yy}(\omega) = |W(e^{j\omega})|^2 \cdot \Gamma_{uu}(\omega)$$
$$\Phi_{yy}(z) = W(z)W(z^{-1}) \cdot \Phi_{uu}(z)$$

3 Moving Average Processes

Given $\eta(t) \sim WN(0, \lambda^2)$

3.1 MA(1):

Model

$$v(t) = c_0\eta(t) + c_1\eta(t-1)$$

Mean

$$\begin{aligned} E[v(t)] &= c_0 \cdot E[\eta(t)] + c_1 \cdot E[\eta(t)] \\ &= c_0 \cdot 0 \qquad\quad + c_1 \cdot 0 \end{aligned}$$

$$\boxed{E[v(t)] = 0}$$

Variance

$$\begin{aligned} Var[v(t)] &= E[(v(t) \underbrace{-E[v(t)])^2}_{0}] \\ &= E[(v(t))^2] \\ &= E[(c_0 \cdot \eta(t)^2 \qquad +c_1 \cdot \eta(t-1))^2] \\ &= c_0^2 \cdot E[\eta(t)^2] \qquad +c_1^2 \cdot E[\eta(t-1)^2] + \underbrace{2c_0c_1 \cdot E[\eta(t)\eta(t-1)]}_{0} \\ &= c_0^2 \cdot E[\eta(t)^2] \qquad +c_1^2 \cdot E[\eta(t-1)^2] \\ &= c_0^2\lambda^2 \qquad\qquad +c_1^2\lambda^2 \end{aligned}$$

$$\boxed{Var[v(t)] = (c_0^2 + c_1^2)\lambda^2}$$

Covariance

$$\begin{aligned} \gamma(t_1, t_2) &= E[(v(t_1) - E[v(t_1)]) \qquad \cdot(v(t_2) - E[v(t_2)])] \\ &= E[(c_0\eta(t_1) + c_1\eta(t_1 - 1)) \ \cdot(c_0\eta(t_2) + c_1\eta(t_2 - 1))] \\ &= c_0^2E[\eta(t_1)\eta(t_2)] \qquad +c_1^2E[\eta(t_1 - 1)\eta(t_2 - 1) \\ &\qquad +c_0c_1E[\eta(t_1)\eta(t_2 - 1)] + c_0c_1E[\eta(t_1 - 1)\eta(t_2)] \end{aligned}$$

$$\boxed{\gamma(\tau) = \begin{cases} c_0^2\lambda^2 + c_1^2\lambda^2 & \text{if } \tau = 0 \\ c_0c_1\lambda^2 & \text{if } \tau = \pm 1 \\ 0 & \text{otherwise} \end{cases}}$$

3.2 MA(n)

Model

$$\begin{aligned} v(t) &= c_0\eta(t) + c_1\eta(t-1) + ... + c_n\eta(t-n) \\ &= (c_0 + c_1z^{-1} + ... + c_nz^{-n})\eta(t) \end{aligned}$$

Transfer function

$$W(z) = c_0 + c_1z^{-1} + ... + c_nz^{-n} = \frac{c_0z^n + c_1z^{n-1} + ... + c_n}{z^n}$$

All poles are in the complex origin

Mean

$$E[v(t)] = (c_0 + c_1 + ... + c_n)\underbrace{E[\eta(t)]}_{0}$$

$$\boxed{E[v(t) = 0}$$

Covariance function

$$\boxed{\gamma(\tau) = \begin{cases} \lambda^2 \cdot \sum_{i=0}^{n-\tau} c_ic_{i-\tau} & |\tau| \leq n \\ 0 & \text{otherwise} \end{cases}}$$

example

$$\begin{aligned} &\gamma(0) = (c_0^2 + c_1^2 + ... + c_n^2)\lambda^2 \\ &\gamma(1) = (c_0c_1 + c_1c_2 + ... + c_{n-1}c_n)\lambda^2 \\ &\gamma(2) = (c_0c_2 + c_1c_3 + ... + c_{n-2}c_n)\lambda^2 \\ &\quad\quad ... \\ &\lambda(n) = (c_0c_n)\lambda^2 \\ &\lambda(k) = 0 \, \forall k > n \end{aligned}$$

3.3 MA(∞)

Model

$$v(t) = c_0\eta(t) + c_1\eta(t-1) + ... + c_k\eta(t-k) + ... = \sum_{i=0}^{\infty} c_i\eta(t-i)$$

Variance

$$\gamma(0) = (c_0^2 + c_1^2 + ... + c_k^2 + ...)\lambda^2 = \lambda^2\sum_{i=0}^{\infty} c_i^2$$

3.4 Well definition of an MA(∞)

We need to have $|\gamma(\tau)| \leq \gamma(0)$, so we must require that

$$\boxed{\gamma(0) = \lambda^2 \sum_{i=0}^{\infty} c_i^2 \text{ is finite}}$$

4 Auto Regressive Processes

4.1 AR(1)

Model

$$v(t) = av(t-1) + \eta(t)$$

Mean

$$\begin{aligned}
E[v(t)] &= E[av(t-1)] + \overbrace{E[\eta(t)]}^{0} \\
&= aE[v(t-1)] \\
&= aE[v(t)] \\
(1-a)E[v(t)] &= 0 \\
&\boxed{E[v(t)] = 0}
\end{aligned}$$

Covariance

MA(∞) method Observe as an AR(1) can be axpressed as an MA(∞)

$$\begin{aligned}
v(t) &= av(t-1) && +\eta(t) \\
&= a[av(t-2) + \eta(t-1)] && +\eta(t) \\
&= a^2 v(t-2) && +a\eta(t-1) + \eta(t) \\
&= a^2[v(t-3) + \eta(t-2)] && +a\eta(t-1) + \eta(t) \\
&= \underbrace{a^n v(t-n)}_{\to 0} + \underbrace{\sum_{i=0}^{\infty} a^i \eta(t-i)}_{\text{MA}(\infty)}
\end{aligned}$$

In particular, the result depends on an $MA(\infty)$ having $\sum_{i=0}^{\infty} c_i = \sum_{i=0}^{\infty} a^i$. To check if the variance is finite we check $\gamma(0) = \lambda^2 \sum_{i=0}^{\infty} a^{2i} < \infty$. The given is a geometric series, convergent for $|a| < 1$. Under this hypothesis its value is

$$\gamma(0) = \lambda^2 \sum_{i=0}^{\infty} a^{2i} = \frac{\lambda^2}{1-a^2}$$

Applying the formula of the variance of MA processes we get

$$\gamma(1) = (c_0c_1 + c_1c_2 + ...)\lambda^2 = (a + aa^2 + ...)\lambda^2 = a(1 + a^2 + a^4 + ...)\lambda^2 = a\lambda^2 \sum_{i=0}^{\infty} a^{2i} = a\frac{\lambda^2}{1-a^2} = a\gamma(0)$$

$$\gamma(2) = (c_0c_2 + c_1c_3 + ...)\lambda^2 = (a^2 + aa^3 + ...)\lambda^2 = a^2(1 + a^2 + a^4 + ...)\lambda^2 = a^2\lambda^2 \sum_{i=0}^{\infty} a^{2i} = a^2\frac{\lambda^2}{1-a^2} = a^2\gamma(0)$$

$$\boxed{\gamma(\tau) = a^{|\tau|}\frac{\lambda^2}{1-a^2}}$$

Yule-Walkler Equations

$$\begin{aligned} Var[v(t)] &= E[v(t)^2] \\ &= E[(av(t) + \eta(t))^2] \\ &= a^2 \underbrace{E[v(t-1)^2]}_{\substack{=Var[v(t-1)] \\ =Var[v(t)] \\ =\gamma(0)}} + \underbrace{E[\eta(t)^2]}_{=\lambda^2} + 2a \underbrace{E[v(t-1)\eta(t)]}_{\substack{v(t-1) \text{ depends on } \eta(t-2) \\ \eta(t) \text{ independent of } \eta(t-2) \\ \implies \\ E[v(t-1)\eta(t)]=0}} \end{aligned}$$

$$\gamma(0) = a^2\gamma(0) + \lambda^2$$

$$\boxed{\gamma(0) = \frac{\lambda^2}{1-a^2}}$$

To find $\gamma(\tau)$, we start from the model $v(t) = av(t-1) + \eta(t)$.

$$\begin{aligned} v(t) &= av(t-1) & &+\eta(t) \\ v(t)v(t-\tau) &= av(t-1)v(t-\tau) & &+\eta(t)v(t-\tau) \\ \underbrace{E[v(t)v(t-\tau)]}_{\gamma(\tau)} &= a\,\underbrace{E[v(t-1)v(t-\tau)]}_{\gamma(\tau-1)} & &+ \underbrace{E[\eta(t)v(t-\tau)]}_{0} \end{aligned}$$

$$\boxed{\gamma(\tau) = a\gamma(\tau-1)}$$

We can join the two by inductive reasoning, obtaining

$$\boxed{\gamma(\tau) = a^{|\tau|}\frac{\lambda^2}{1-a^2}}$$

Long Division Leads to same result, but is boring

4.2 AR(n)

Model

$$v(t) = a_1v(t-1) + a_2v(t-2) + ... + a_nv(t-n) + \eta(t)$$

Transfer function

$$W(z) = \frac{z^n}{z^n - a_1 z_{n-1} - ... - a_n}$$

Mean

$$E[v(t)] = a_1 E[v(t-1])+a_2 E[v(t-2)]+...+a_n E[v(t-n)]+\underbrace{E[\eta(t)]}_{0}$$

$$m = a_1 m \qquad +a_2 m \qquad +...+a_n m$$

$$(1 - a_1 - a_2 - ... - a_n)m = 0$$

$$\boxed{E[v(t)] = 0}$$

5 ARMA Processes

Model

$$v(t) = a_1 v(t-1) + ... + a_{n_a} v(t - n_a) + c_0 \eta(t) + ... + c_{n_c} v(t - n_c)$$

Can also be espressed as $V(t) = \frac{C(z)}{A(z)}\eta(t)$, where

$$C(z) = c_0 + c_1 z^{-1} + ... + c_{n_c} z^{-n_c}$$
$$A(z) = 1 - a_1 z^{-1} - ... - a_{n_a} z^{-n_a}$$