Model Identification and Data Analysis

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Contents

Part I
Prediction

1 Probability Recall

1.1 Random Vectors

Variance $Var[v] = E[(v - E[v])^2]$

Cross-Variance $Var[v, u] = E[(v - E[v])(u - E[u])]$

Variance Matrix $\begin{vmatrix} Var[v_1] & . & . & Var[v_1, v_k] \\ . & . & & . \\ . & & . & . \\ Var[v_k, v_1] & . & . & Var[v_k] \end{vmatrix}$

Covariance coefficient $\delta[i, j] = \frac{Var[i,j]}{\sqrt{Var[i]}\sqrt{Var[j]}}$
$\delta[i, j] = 0 \implies$ i, j uncorrelated
$|\delta[i, j]| = 1 \implies i = \alpha j$

1.2 Random processes

$v(t, s)|$ t time instant, s expetiment outcome (generally given)

Mean $m(t) = E[v(t, s)]$

Variance $\lambda^2(t) = Var[v(t)]$

Covariance function $\gamma(t_1, t_2) = E[(v(t_1) - m(t_1))(v(t_2) - m(t_2))] = \gamma(t_2, t_1)$

Normalized Covariance Function $\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}$
$\forall$ stationary processes: $|\rho(\tau)| \leq 1 \quad \forall\tau$

1.3 Important process classes

Stationary process

- $m(t) = m$ constant
- $\lambda^2(t) = \lambda^2$ constant
- $\gamma(t_1, t_2) = f(t_2 - t_1) = \gamma(\tau)$ covariance depends only on time difference τ

$|\gamma(\tau)| \leq \gamma(0) \quad \forall\tau$

White noise $\eta(t) \sim WN(m, \lambda^2)$

- Stationary process
- $\gamma(\tau) = 0 \quad \forall \tau \neq 0$

$v(t) = \alpha\eta(t) + \beta \quad \eta(t) \sim WN(0, \lambda^2) \qquad \Longrightarrow \qquad v(t) \sim WN(\beta, \alpha^2\lambda^2)$

2 Spectral Analysis

2.1 Foundamentals

Spectrum

$$\Gamma(\omega) = \overbrace{F(\gamma(\tau))}^{\text{Fourier transform}} = \sum_{\tau=-\infty}^{+\infty} \gamma(\tau) \cdot e^{-j\omega\tau}$$

Euler formula $\Gamma(\omega) = \gamma(0) + 2cos(\omega)\gamma(1) + 2cos(2\omega)\gamma(2) + ...$

Spectrum properties

- $\Gamma : \mathbb{R} \to \mathbb{R}$
- Γ is periodic with $T = 2\pi$
- Γ is even $[\Gamma(-\omega) = \Gamma(\omega)]$
- $\Gamma(\omega) \geq 0 \quad \forall\omega$

$\eta(t) \sim WN(0, \lambda^2) \quad \Longrightarrow \quad \Gamma_\eta(\omega) = \gamma(0) = Var[\eta(t)] = \lambda^2$

Anti-Transform

$$\gamma(\tau) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \Gamma(\omega) e^{k\omega\tau} \, dw$$

Complex spectrum

$$\phi(z) = \sum_{\tau=-\infty}^{+\infty} \omega(\tau) z^{-\tau}$$

$$\Gamma(\omega) = \Phi(e^{j\omega})$$

2.2 Fundamental theorem of Spectral Analysis

Fundamental theorem of Spectral Analysis allows to derive the (real and/or complex) spectrum of the output from the input and the transfer function of the system

$$\xrightarrow{\;u\;} \boxed{W(z)} \xrightarrow{\;y\;}$$

$$\begin{aligned}
\Gamma_{yy}(\omega) &= |W(e^{j\omega})|^2 \quad \cdot \Gamma_{uu}(\omega) \\
\Phi_{yy}(z) &= W(z)W(z^{-1}) \cdot \Phi_{uu}(z)
\end{aligned}$$

2.3 Canonical representation of a Stationary Process

A stationary process can be represented by an infinite number of transfer functions. The canonical representation is the transfer function $W(z)$ such that:

- Numerator and denominator have same degree
- Numerator and denominator are monic (highest grade coefficient is 1)
- Numerator and denominator are coprime ($W(z)$ cannot be simplified)
- numerator and denominator are stable polynomials (all poles and zeros of $W(z)$ are inside the unit disk)

3 Moving Average Processes

Given $\eta(t) \sim WN(0, \lambda^2)$

3.1 MA(1):

Model

$$v(t) = c_0\eta(t) + c_1\eta(t-1)$$

Mean

$$\begin{aligned}
E[v(t)] &= c_0 \cdot E[\eta(t)] + c_1 \cdot E[\eta(t)] \\
&= c_0 \cdot 0 \qquad + c_1 \cdot 0
\end{aligned}$$

$$\boxed{E[v(t)] = 0}$$

Variance

$$\begin{aligned}
Var[v(t)] &= E[(v(t)\underbrace{-E[v(t)])^2}_{0}] \\
&= E[(v(t))^2] \\
&= E[(c_0 \cdot \eta(t)^2 \qquad +c_1 \cdot \eta(t-1))^2] \\
&= c_0^2 \cdot E[\eta(t)^2] \qquad +c_1^2 \cdot E[\eta(t-1)^2] + \underbrace{2c_0c_1 \cdot E[\eta(t)\eta(t-1)]}_{0} \\
&= c_0^2 \cdot E[\eta(t)^2] \qquad +c_1^2 \cdot E[\eta(t-1)^2] \\
&= c_0^2\lambda^2 \qquad +c_1^2\lambda^2
\end{aligned}$$

$$\boxed{Var[v(t)] = (c_0^2 + c_1^2)\lambda^2}$$

Covariance

$$\begin{aligned}
\gamma(t_1, t_2) &= E[(v(t_1) - E[v(t_1)]) \qquad \cdot(v(t_2) - E[v(t_2)])] \\
&= E[(c_0\eta(t_1) + c_1\eta(t_1 - 1)) \ \cdot(c_0\eta(t_2) + c_1\eta(t_2 - 1))] \\
&= c_0^2 E[\eta(t_1)\eta(t_2)] \qquad +c_1^2 E[\eta(t_1 - 1)\eta(t_2 - 1) \\
&\qquad +c_0c_1E[\eta(t_1)\eta(t_2 - 1)] + c_0c_1E[\eta(t_1 - 1)\eta(t_2)]
\end{aligned}$$

$$\boxed{\gamma(\tau) = \begin{cases} c_0^2\lambda^2 + c_1^2\lambda^2 & \text{if } \tau = 0 \\ c_0c_1\lambda^2 & \text{if } \tau = \pm 1 \\ 0 & \text{otherwise} \end{cases}}$$

3.2 MA(n)

Model

$$\begin{aligned}
v(t) &= c_0\eta(t) + c_1\eta(t-1) + ... + c_n\eta(t-n) \\
&= (c_0 + c_1z^{-1} + ... + c_nz^{-n})\eta(t)
\end{aligned}$$

Transfer function

$$W(z) = c_0 + c_1z^{-1} + ... + c_nz^{-n} = \frac{c_0z^n + c_1z^{n-1} + ... + c_n}{z^n}$$

All poles are in the complex origin

Mean

$$E[v(t)] = (c_0 + c_1 + ... + c_n)\underbrace{E[\eta(t)]}_{0}$$

$$\boxed{E[v(t) = 0}$$

Covariance function

$$\boxed{\gamma(\tau) = \begin{cases} \lambda^2 \cdot \sum_{i=0}^{n-\tau} c_i c_{i-\tau} & |\tau| \leq n \\ 0 & \text{otherwise} \end{cases}}$$

example

$$\gamma(0) = (c_0^2 + c_1^2 + ... + c_n^2)\lambda^2$$
$$\gamma(1) = (c_0 c_1 + c_1 c_2 + ... + c_{n-1} c_n)\lambda^2$$
$$\gamma(2) = (c_0 c_2 + c_1 c_3 + ... + c_{n-2} c_n)\lambda^2$$
$$...$$
$$\lambda(n) = (c_0 c_n)\lambda^2$$
$$\lambda(k) = 0 \, \forall k > n$$

3.3 MA(∞)

Model

$$v(t) = c_0\eta(t) + c_1\eta(t-1) + ... + c_k\eta(t-k) + ... = \sum_{i=0}^{\infty} c_i\eta(t-i)$$

Variance

$$\gamma(0) = (c_0^2 + c_1^2 + ... + c_k^2 + ...)\lambda^2 = \lambda^2 \sum_{i=0}^{\infty} c_i^2$$

3.4 Well definition of an MA(∞)

We need to have $|\gamma(\tau)| \leq \gamma(0)$, so we must require that

$$\boxed{\gamma(0) = \lambda^2 \sum_{i=0}^{\infty} c_i^2 \text{ is finite}}$$

4 Auto Regressive Processes

4.1 AR(1)

Model

$$v(t) = av(t-1) + \eta(t)$$

Mean

$$\begin{aligned}
E[v(t)] &= E[av(t-1)] + \overbrace{E[\eta(t)]}^{0} \\
&= aE[v(t-1)] \\
&= aE[v(t)] \\
(1-a)E[v(t)] &= 0 \\
&\boxed{E[v(t)] = 0}
\end{aligned}$$

Covariance

MA(∞) method Observe as an AR(1) can be axpressed as an MA(∞)

$$\begin{aligned}
v(t) &= av(t-1) && +\eta(t) \\
&= a[av(t-2) + \eta(t-1)] && +\eta(t) \\
&= a^2 v(t-2) && +a\eta(t-1) + \eta(t) \\
&= a^2[v(t-3) + \eta(t-2)] && +a\eta(t-1) + \eta(t) \\
&= \underbrace{a^n v(t-n)}_{\to 0} + \underbrace{\sum_{i=0}^{\infty} a^i \eta(t-i)}_{\text{MA}(\infty)}
\end{aligned}$$

In particular, the result depends on an $MA(\infty)$ having $\sum_{i=0}^{\infty} c_i = \sum_{i=0}^{\infty} a^i$.
To check if the variance is finite we check $\gamma(0) = \lambda^2 \sum_{i=0}^{\infty} a^{2i} < \infty$. The given is a geometric series, convergent for $|a| < 1$. Under this hypothesis its value is

$$\gamma(0) = \lambda^2 \sum_{i=0}^{\infty} a^{2i} = \frac{\lambda^2}{1-a^2}$$

Applying the formula of the variance of MA processes we get

$$\gamma(1) = (c_0c_1 + c_1c_2 + ...)\lambda^2 = (a + aa^2 + ...)\lambda^2 = a(1 + a^2 + a^4 + ...)\lambda^2 = a\lambda^2 \sum_{i=0}^{\infty} a^{2i} = a\frac{\lambda^2}{1-a^2} = a\gamma(0)$$

$$\gamma(2) = (c_0c_2 + c_1c_3 + ...)\lambda^2 = (a^2 + aa^3 + ...)\lambda^2 = a^2(1 + a^2 + a^4 + ...)\lambda^2 = a^2\lambda^2 \sum_{i=0}^{\infty} a^{2i} = a^2\frac{\lambda^2}{1-a^2} = a^2\gamma(0)$$

$$\boxed{\gamma(\tau) = a^{|\tau|}\frac{\lambda^2}{1-a^2}}$$

Yule-Walkler Equations

$$
\begin{aligned}
Var[v(t)] &= E[v(t)^2] \\
&= E[(av(t)+\eta(t))^2] \\
&= a^2 \underbrace{E[v(t-1)^2]}_{\substack{=Var[v(t-1)] \\ =Var[v(t)] \\ =\gamma(0)}} + \underbrace{E[\eta(t)^2]}_{=\lambda^2} + 2a \underbrace{E[v(t-1)\eta(t)]}_{\substack{v(t-1) \text{ depends on } \eta(t-2) \\ \eta(t) \text{ independent of } \eta(t-2) \\ \implies \\ E[v(t-1)\eta(t)]=0}}
\end{aligned}
$$

$$\gamma(0) = a^2\gamma(0) + \lambda^2$$

$$\boxed{\gamma(0) = \frac{\lambda^2}{1-a^2}}$$

To find $\gamma(\tau)$, we start from the model $v(t) = av(t-1) + \eta(t)$.

$$
\begin{aligned}
v(t) &= av(t-1) && +\eta(t) \\
v(t)v(t-\tau) &= av(t-1)v(t-\tau) && +\eta(t)v(t-\tau) \\
\underbrace{E[v(t)v(t-\tau)]}_{\gamma(\tau)} &= a\underbrace{E[v(t-1)v(t-\tau)]}_{\gamma(\tau-1)} && + \underbrace{E[\eta(t)v(t-\tau)]}_{0}
\end{aligned}
$$

$$\boxed{\gamma(\tau) = a\gamma(\tau-1)}$$

We can join the two by inductive reasoning, obtaining

$$\boxed{\gamma(\tau) = a^{|\tau|}\frac{\lambda^2}{1-a^2}}$$

Long Division Leads to same result, but is boring

4.2 AR(n)

Model

$$v(t) = a_1v(t-1) + a_2v(t-2) + ... + a_nv(t-n) + \eta(t)$$

Transfer function

$$W(z) = \frac{z^n}{z^n - a_1z_{n-1} - ... - a_n}$$

Mean

$$E[v(t)] = a_1E[v(t-1])+a_2E[v(t-2)]+...+a_nE[v(t-n)]+\underbrace{E[\eta(t)]}_{0}$$

$$m = a_1m \qquad +a_2m \qquad +...+a_nm$$

$$(1-a_1-a_2-...-a_n)m = 0$$

$$\boxed{E[v(t)] = 0}$$

5 ARMA Processes

Model

$$v(t) = a_1 v(t-1) + ... + a_{n_a} v(t-n_a) + c_0\eta(t) + ... + c_{n_c} v(t-n_c)$$

Can also be espressed as $V(t) = \frac{C(z)}{A(z)}\eta(t)$, where

$$\begin{aligned}
C(z) &= c_0 + c_1 z^{-1} + ... + c_{n_c} z^{-n_c} \\
A(z) &= 1 - a_1 z^{-1} - ... - a_{n_a} z^{-n_a}
\end{aligned}$$

Such process is stationary if all the poles of $W(z)$ are inside the unit disk.

6 Prediction problem

We want to predict $v(t+r)$ from $v(t), v(t-1), ...$, where r is called prediction horizon, of the following stationary process:

$$\xrightarrow{\eta} \boxed{W(z)} \xrightarrow{v}$$

6.1 Fake problem

Having a process with transfer function $W(z)$, we can compute it in polynomial form using the long division algorithm

$$W(z) = w_0 + w_1 z^{-1} + w_2 z^{-2} + ...$$

We can calculate

$$v(t{+}r) = W(z)\eta(t{+}r) = \underbrace{w_0\eta(t+r) + w_1\eta(t+r-1) + ... + w_{r-1}\eta(t+1)}_{\alpha(t)\ \textbf{unpredictable:}\ \text{future of } \eta \text{ involved}} + \underbrace{w_r\eta(t) + w_{r+1}\eta(t-1) + ...}_{\beta(t)\ \textbf{predictable}}$$

The optimal fake predictor is then

$$\boxed{v(t+r|t) = w_r\eta(t) + w_{r+1}\eta(t-1) + ...} = \beta(t)$$

And the prediction error is

$$\begin{aligned}
\epsilon(t) &= v(t+r) && -\hat{v}(t+r|t) \\
&= \alpha(t) + \beta(t) && -\beta(t) \\
&= \alpha(t)
\end{aligned}$$

$$\boxed{\epsilon(t) = w_0\eta(t+r) + w_1\eta(t+r-1) + ... + w_{r-1}\eta(t+1)}$$

$$\boxed{Var[\epsilon(t)] = (w_0^2 + w_1^2 + ... + w_{r-1}^2)\lambda^2}$$

6.2 True Problem

We want to estimate $v(t+r)$ form $v(t)$, having transfer function $W(z)$ and $\hat{W}_r(z)$ the solution to the fake problem. We can calculate the transfer function of the real predictor from the process as

$$\boxed{W_r(z) = W(z)^{-1} \cdot \hat{W}_r(z)}$$

For ARMA processes a shortcut exists:

$$\hat{v}_{\text{ARMA}}(t|t-1) = \frac{C(z)A(z)}{C(z)} \qquad \text{having } W(z) = \frac{C(z)}{A(z)}$$

6.3 Prediction with eXogenous variables

An exogenous variable is a <u>deterministic</u> input variable in the system

6.3.1 ARX model

$$v(t) = a_1 v(t-1) + ... + a_{n_a} v(t-n_a) + b_1 u(t-1) + ... + b_{n_b} u(t-n_b) + \eta(t) A(z)v(t) \quad = B(z)u(t-1) +$$

Transfer functions from u and η

$$W_u(z) = \frac{B(z)}{A(z)} \qquad\qquad W_\eta(z) = \frac{1}{A(z)}$$

6.3.2 ARMAX model

$$\begin{aligned} A(z)v(t) &= C(z)\eta(t) + B(z)u(t-1) \\ y(t) &= W(z)\eta(t) + G(z)u(t) \end{aligned}$$

Predictor

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t) + \frac{B(z)}{C(z)} u(t-1)$$

Part II
Identification

Consists of estimating a model from data.

7 Prediction Error Minimization

Aims to minimize $\epsilon(t) = v(t) - \hat{v}(t|t-r)$
Steps:

1. **Data collection:** collect $\vec{u}$ and $\vec{y}$
2. **Family selection:** choose a family of models $M(\theta)$

   MA(1) $\theta = [a]$

   MA(n) $\theta = [a_1, ..., a_n]$

   ARMA(n_a, n_c) $\theta = [a_1, ..., a_{n_a}, c_1, ..., c_{n_c}]$

   ...

3. **Select an optimization criterion**

   Mean Squared error $J(\theta) = \frac{1}{N}\sum_{t=1}^{N} \epsilon_\theta(t)^2$

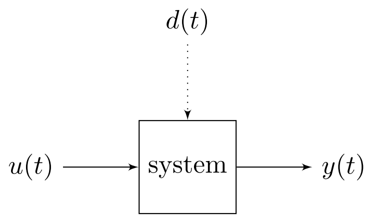
   Mean absolute error $J(\theta) = \frac{1}{N}\sum_{t=1}^{N} |\epsilon_\theta(t)|$

   ...

4. **Optimization** find $\hat{\theta} = argmin\, J(\theta) \implies \frac{dJ(\theta)}{d\theta} = 0$
5. **Validation** verify if the result satisfies the requirements

Part III
Black-Box non-parametric I/O systems

A State-space models



Known (measured) data

$$\begin{aligned} &\{u(1), \dots, u(N)\} && \text{input} \\ &\{y(1), \dots, y(N)\} && \text{output} \end{aligned}$$

A.1 State-space representation

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) & \text{state equations} \\ y(t) = Hx(t) + Du(t) & \text{output equations} \end{cases}$$

Where $F_{n\times n}$, $G_{n\times 1}$, $H_{1\times n}$ and $D_{1\times 1}$ are matrices.

S.S. representation is not unique Given any invertible matrix T, let $F_1 = TFT^{-1}$, $G_1 = TG$, $H_1 = HT^{-1}$, $D_1 = D$. Then the system $\{F, G, H, D\}$ is equivalent to the system $\{F_1, G_1, H_1, D_1\}$.

A.2 Transfer function representation

$$W(z) = \frac{B(z)}{A(z)} z^{-k} = \frac{b_0 + b_1 z^{-1} + \dots + b_p z^{-p}}{a_0 + a_1 z^{-1} + \dots + a_n z^{-n}} z^{-k}$$

$W(z)$ is a rational function of the z operator $\rightarrow$ is a digital filter

Infinite impulse response $W(z) = \frac{z^{-1}}{1+\frac{1}{3}z^{-1}}$

Finite impulse response $W(z) = z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3}$

A.3 Convolution of the input with the inpulse response

Let's call $\omega(1), \omega(2), \dots$ the values of $y(t)$ when $u(t) = \text{impulse}(0)$, and let's measure the values of y at different times: . Then it can be proven that for any $u(t)$

$$y(t) = \sum_{k=0}^{\infty} \omega(k)u(t-k)$$

B Converting representations one to another

B.1 State space to Transfer function

Consider a strictly propter system:

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t+1) = Hx(t) + \cancelto{0}{Du(t)} \end{cases} \Rightarrow \begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

Applying the z operator we get

$$\begin{aligned} zx(t) &= Fx(t) + Gu(t) \\ x(t)(zI - F) &= Gu(t) \\ x(t) &= (zI - F)^{-1}Gu(t) \\ y(t) &= H(zI - F)^{-1}Gu(t) \end{aligned}$$

And we can extract the transfer function:

$$W(z) = H(zI - F)^{-1}G$$

B.2 Transfer Function to State Space

We have the transfer function

$$W(z) = \frac{b_0 z^{n-1} + b_1 z^{n-2} + \dots + b_{n-1}}{z^n + a_0 z^{n-1} + \dots + a_n}$$

The formulas for the state space matrices is

$$F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & \dots & \dots & -a_1 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_0 \end{bmatrix} \quad D = 0$$

B.3 Transfer function to Impulse response

Obtained by computing the ∞ long division of $W(z)$

B.4 Impulse response to Tranfer function

Z-transform Given a discrete-time signal $s(t)$ such that $\forall t < 0 : s(t) = 0$, it's Z-transform is

$$\mathcal{Z} = \sum_{t=0}^{\infty} s(t) z^{-t}$$

It can be proven that:

$$W(z) = \mathcal{Z}(\omega(t)) = \sum_{t=0}^{\infty} \omega(t) z^{-1}$$

NB: this works only in theory because of the infinite sum

B.5 State space to Impulse response

Consider the state space model:

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

We have that:

$$\begin{aligned}
x(1) &= \cancelto{0}{Fx(0)} + Gu(0) && = Gu(0) \\
y(1) &= Hx(1) && = HGu(0) \\
\\
x(2) &= Fx(1) + Gu(1) && = FGu(0) + Gu(1) \\
y(2) &= Hx(2) && = HFGu(0) + HG(u1) \\
\\
x(3) &= Fx(2) + Gu(2) && = F^2Gu(0) + FGu(1) + Gu(2) \\
y(3) &= Hx(3) && = HF^2Gu(0) + HFGu(1) + HGu(2) \\
&\vdots \\
y(t) &= 0u(t) + HGu(t-1) + HFGu(t-2) + HF^2Gu(t-3) + \dots
\end{aligned}$$

The IR is:

$$\omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ HF^{t-1}G & \text{if } t > 0 \end{cases}$$

C Controllability and Observability

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

Fully observable system The system is fully observable (from the output) $\iff$ the observability matrix is full rank:

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \qquad rank(O) = n$$

Fully controllable system The system is fully controllable (from the input) $\iff$ the controllability (also called reachability) matrix is full rank:

$$R = \begin{bmatrix} G & FG & \dots & F^{n-1}G \end{bmatrix} \qquad rank(R) = n$$

D Hankel Matrix

Starting from $\omega(1), \omega(2), \dots, \omega(N)$ where $N \geq 2n-1$, we can build the Hankel Matrix of order n:

$$H_n = \begin{bmatrix} \omega(1) & \omega(2) & \omega(3) & \dots & \omega(n) \\ \omega(2) & \omega(3) & \omega(4) & \dots & \omega(n+1) \\ \omega(3) & \omega(4) & \omega(5) & \dots & \omega(n+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega(n) & \omega(n+1) & \omega(n+2) & \dots & \omega(2n-1) \end{bmatrix}$$

Knowing that

$$\omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ HF^{t-1}G & \text{if } t > 0 \end{cases}$$

We can rewrite

$$H_n = \begin{bmatrix} HG & HFG & HF^2G & \dots & HF^{n-1}G \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ HF^{n-1}G & \dots & \dots & \dots & HF^{2n-2}G \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \cdot \begin{bmatrix} G & FG & \dots & F^{n-1}G \end{bmatrix} = O \cdot R$$

E Subspace-based State Space System Identification

Impulse experiment Measure $y(t)$ under the input $u(t) = \text{impulse}(0)(0)$
How to derive F, G, H from $\omega(0), \dots, \omega(n)$?

- Assuming the IR measurement to be noise free $\rightarrow$ easier, not realistic
- Measure $\hat{\omega(t)}$ as a noisy signal and compute $\omega(t) = \eta(t) - \hat{\omega}(t)$

E.1 Obtain F, G, H from a noise-free IR

1. Build the Hankel matrix of increasing order, and conpute the rank until $rank(H_n) = rank(H_{n+1})$. Then, n is the order of the IR

$$H_1 = \begin{bmatrix} \omega(1) \end{bmatrix} \quad H_2 = \begin{bmatrix} \omega(1) & \omega(2) \\ \omega(2) & \omega(3) \end{bmatrix} \quad H_3 = \dots \quad \dots \quad H_n = \dots$$

2. Take H_{n+1} and factorize it in two rectangular matrix of size $(n+1) \times n$ and $n \times (n+1)$: $H_{n+1} = O_{n+1} \cdot R_{n+1}$, where

$$O_{n+1} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^n \end{bmatrix} \qquad R_{n+1} = \begin{bmatrix} G & FG & \dots & F^n G \end{bmatrix}$$

3. Estimate H,F,G:
   - Extract F and G from the first element of O and R
   - Define:

$$O_1 = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \qquad O_2 = \begin{bmatrix} HF \\ \vdots \\ HF^n \end{bmatrix}$$

   - Observe that $O_1 F = O_2$, so $F = O_1^{-1} O_2$

F Obtain F, G, H from a noisy IR

The measurement is of $\hat{\omega}(t) = \omega(t) + \eta(t)$. To identify the process:

1. Build the Hankel matrix from data using all the N data available in one shot:

$$\hat{H}_{q \times d} = \begin{bmatrix} \hat{\omega}(1) & \hat{\omega}(2) & \dots & \hat{\omega}(d) \\ \hat{\omega}(2) & \hat{\omega}(3) & \dots & \hat{\omega}(d+1) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\omega}(q) & \hat{\omega}(q+1) & \dots & \hat{\omega}(q+d+1) \end{bmatrix}$$

   Where $q + d + 1 = N$

2. Calculate the Singular Value Decomposition of $\hat{H}_{q \times d}$:

$$\hat{H}_{q \times d} = \hat{U}_{q \times q} \cdot \hat{S}_{q \times d} \cdot \hat{V}^T_{d \times d}$$

$\hat{U}$ and $\hat{V}$ are unitary matrices: they are invertible and their inverses are equal to their transpose.

$$\hat{S} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{bmatrix}$$

3. Plot the singular values (σ_i)and cut-off the three matrices:
   - Ideally, after a certain n (the order of the IR) there would be a jump dividing the signal (before) from the noise (after)
   - In reality no clear distinction exists, but it's possible to identify an interval of possible values of n. A tradeoff between complexity, precision and oferfitting takes place
4. Split $\hat{U}, \hat{S}, \hat{V}^T$ obtaining $U_{q\times n},\ S_{n\times n},\ V^T_{n\times d}$ and then recreate $H_{qd} = USV^T$
5. H and G are estimated as for the unnoisy case. To estimate F we can build O_1 and O_2 as before, but then the system $O_1 \cdot F = O_2$ cannot be solved directly as O_1 is not square. We can instead compute the approximate least-square solution of the system:

$$F = (O_1^T O_1)^{-1} O_1^T O_2$$

Part IV

Parametric black-box system identification using frequency-domain approach

G Experiment design and data pre-processing

1. Select a set of excitation frequencies $\{\omega_1, \dots, \omega_H\}$. Usually $\omega_i - \omega_{i-1}$ is constant $\forall i \in \{2, \dots, H\}$. ω_H must be selected according to the bandwidth of the system

2. Make H independent experiments:

$$\xrightarrow{u_1(t) = A_1 \sin(\omega_1 t)} \boxed{\text{system}} \xrightarrow{y_1(t)}$$

Experiment 1

$$\xrightarrow{u_2(t) = A_2 \sin(\omega_2 t)} \boxed{\text{system}} \xrightarrow{y_2(t)}$$

Experiment 2

$$\vdots$$

$$\xrightarrow{u_H(t) = A_H \sin(\omega_H t)} \boxed{\text{system}} \xrightarrow{y_H(t)}$$

Experiment H

3. Focusing on experiment i, because of noise the real value of the output will be (unknowns are underlined)

$$\hat{y}_i = \underline{B_i} \sin(\omega_i t + \underline{\phi_i}) = \underline{a_i} \sin(\omega_i t) + \underline{b_i} \cos(\omega_i t)$$

Using the second equation (since it is linear in the unknowns). We want to determine

$$\{\hat{a}_i, \hat{b}_i\} = \arg \min_{\{a_i, b_i\}} J_N(a_i, b_i)$$

$$J_N(a_i, b_i) = \frac{1}{N}\sum_{t=1}^{N}\left(\underbrace{y_i(t)}_{\text{measurement}} \underbrace{-a_i \sin(\omega_i t) - b_i \cos(\omega_i t)}_{\text{model output}}\right)^2$$

This can be solved explicitly

$$\frac{\delta J_N}{\delta a_i} = \frac{2}{N}\sum_{t=1}^{N}\left(-\sin(\omega_i t)\right)\left(y_i(t) - a_i \sin(\omega_i t) - b_i \cos(\omega_i t)\right) \quad = 0$$

$$\frac{\delta J_N}{\delta b_i} = \frac{2}{N}\sum_{t=1}^{N}\left(-\cos(\omega_i t)\right)\left(y_i(t) - a_i \sin(\omega_i t) - b_i \cos(\omega_i t)\right) \quad = 0$$

Which results in the following linear system:

$$\begin{bmatrix} \sum_{t=1}^{N} \sin(\omega_i t)^2 & \sum_{t=1}^{N} \sin(\omega_i t)\cos(\omega_i t) \\ \sum_{t=1}^{N} \sin(\omega_i t)\cos(\omega_i t) & \sum_{t=1}^{N} \cos(\omega_i t)^2 \end{bmatrix}\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^{N} y_i(t)\sin(\omega_i t) \\ \sum_{t=1}^{N} y_i(t)\cos(\omega_i t) \end{bmatrix}$$

4. We want to move back to sin-only form:

$$\hat{\phi}_i = \arctan\left(\frac{\hat{b_i}}{\hat{a_i}}\right)$$

$$\hat{B}_i = \frac{\frac{\hat{a_i}}{\cos\hat{\phi}_i} + \frac{\hat{b_i}}{\sin\hat{\phi}_i}}{2}$$

5. Repeating H experiments we obtain

$$\{\hat{B}_1, \hat{\phi}_1\} \Rightarrow \frac{\hat{B}_1}{A_1}e^{j\hat{\phi}_1}$$

$$\vdots$$

$$\{\hat{B_H}, \hat{\phi_H}\} \Rightarrow \frac{\hat{B_H}}{A_H}e^{j\hat{\phi_H}}$$

So we have H complex numbers representing the frequency response of $W(z)$. These numbers are our dataset

H Model class selection

$$\mathcal{M}(\theta) : W(z, \theta) = \frac{b_0 + b_1 z^{-1} + \cdots + b_p z^{-p}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} z^{-1} \qquad \theta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_0 \\ \vdots \\ b_p \end{bmatrix}$$

I Performance index

$$J_H(\theta) = \frac{1}{H}\sum_{i=1}^{H}\left(W(e^{j\omega_i}, \theta) - \frac{\hat{B}_i}{A_i}e^{j\hat{\phi}_i}\right)^2$$

J Optimization

$$\hat{\theta} = \arg\min_{\theta} J_H(\theta)$$

Part V
Kalman filter

Based on SS representation:

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) + v_1(t) & v_1 \sim WN \\ y(t) = Hx(t) + \cancel{Du(t)} + v_2(t) & v_2 \sim WN \end{cases}$$

K Motivations and Goals

Given a model and noise variances:

- find k-steps ahead predictors of the output y
- find k-steps ahead predictors of the state x
- Find the filter of the state $\hat{x}(t|t)$ to allow software sensing
- Gray box system identification

Usually a dynamic system has m inputs, n states and p outputs

Key problem Usually $p << n$: pysical sensors are much less than system states because:

- Cost
- Cables, power supply
- Maintenance

But we want full state measurements because:

- Control design (using state feedback)
- Monitoring (fault detection, predictive maintenance)

Software sensing determine the internal state using the values measured from input and output

K.1 Kalman on Basic Systems

$$S : \begin{cases} x(t+1) = Fx(t) + \cancel{Gu(t)} + v_1(t) & \text{state equation} \\ y(t) = Hx(t) + v_2(t) & \text{output equation} \end{cases}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \cancel{u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}} \qquad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

v_1 is a vector white noise:

$$v_1 \sim WN(0, V_1) \qquad v_1(t) = \begin{bmatrix} v_1 1(t) \\ \vdots \\ v_1 n(t) \end{bmatrix}$$

- $E[v_1(t)] = \vec{0}$
- $E\left[v_1(t) \cdot v_1(t)^T\right] = V_1$, where V_1 is an $n \times n$ covariance matrix
- $E\left[v_1(t) \cdot v_1(t-\tau)^T\right] = 0 \quad \forall t, \forall \tau \neq \vec{0}$

v_2 is called output/measurement/sensor noise:

- $E[v_2(t)] = \vec{0}$
- $E\left[v_2(t) \cdot v_2(t)^T\right] = V_2$, where V_1 is an $n \times n$ covariance matrix
- $E\left[v_2(t) \cdot v_2(t-\tau)^T\right] = 0 \quad \forall t, \forall \tau \neq \vec{0}$

v_1 and v_2 are assumed to have the following relationships:

$$E\left[v_1(t) \cdot v_2(t-\tau)^T\right] = \underbrace{V_{12}}_{n \times p} = \begin{cases} 0 & \text{if } \tau \neq 0 \\ \text{any} & \text{if } \tau = 0 \end{cases}$$

So they can only be correlated in the same time istant. Since the system is dynamic we need to define its initial conditions:

$$E[x(1)] = \underbrace{X_0}_{n \times 1} \qquad E\left[(x(1) - x(0))(x(1) - x(0))^T\right] = \underbrace{P_0}_{n \times n} \geq 0$$

$P_0 = 0 \iff$ the initial state is perfectly known.
We finally assume that v_1 and v_2 are uncorrelated with the initial state:

$$x(1) \perp v_1(t) \qquad x(1) \perp v_2(t)$$

Solution for Basic Systems

$$\begin{aligned}
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + K(t)e(t) && \text{state equation} \\
\hat{y}(t|t-1) &= H\hat{x}(t|t-1) && \text{output equation} \\
e(t) &= y(t) - \hat{y}(t|t-1) && \text{prediction error} \\
K(t) &= \left(FP(t)H^T + V_{12}\right)\left(HP(t)H^T + V_2\right)^{-1} && \text{gain of the KF} \\
P(t+1) &= \left(FP(t)F^T + V_1\right) \\
&\quad - \left(FP(t)H^T + V12\right)\left(HP(t)H^T + V_2\right)^{-1}\left(FP(t)H^T + V_{12}\right)^T && \text{difference Riccati equation} \\
\hline
\hat{x}(1|0) &= E[x(1)] = X_0 && \text{Initial state} \\
P(1) &= var[x(1)] = P_0 && \text{initial DRE}
\end{aligned}$$

K.2 Exogenous input

$$\hat{x}(t+1|t) = F\hat{x}(t|t-1) + Gu(t) + K(t)e(t) \qquad \text{state equation}$$

$$\text{other equations} = \text{unchanged}$$

K.3 Multi-step prediction

Knowing $\hat{x}(t+1|T)$ from the basic solution we can derive

$$\begin{aligned}
\hat{x}(t+2|t) &= F\hat{x}(t+1|t) \\
\hat{x}(t+2|t) &= F^2\hat{x}(t+1|t) \\
&\vdots \\
\hat{x}(t+k|t) &= F^{k-1}\hat{x}(t+1|t) \\
\hline
\hat{y}(t+k|t) &= H\hat{x}(t+k|t)
\end{aligned}$$

K.4 Filter ($\hat{x}(t|t)$

F invertible

$$\hat{x}(t+1|t) = F\hat{x}(t|t) \qquad \Longrightarrow \qquad \hat{x}(t|t) = F^{-1}\hat{x}(t+1|t)$$

F not invertible assuming $V_{12} = 0$, then we can re-formulate the K.F. solutions:

$$\begin{aligned}
\hat{x}(t|t) &= F\hat{x}(t-1|t-1) + Gu(t-1) + K_0(t)e(t) \\
\hat{y}(t|t-1) &= H\hat{x}(t|t-1) \\
e(t) &= y(t) - \hat{y}(t|t-1) \\
K_0(t) &= \left(P(t)H^T\right)\left(HP(t)H^T + V_2\right)^{-1} \\
P(t+1) &= \text{unchanged}
\end{aligned}$$

K.5 Time-varying systems

$$S : \begin{cases} x(t+1) = F(t)x(t) + G(t)u(t) + v_1(t) \\ y(t) = H(t)x(t + v_2(t) \end{cases}$$

K.F. equations are unchanged

K.6 Non linear system

Much more complicated extension. Look for Extended Kalman Filter if interested (I'm not)

L Asymptotic solution of K.F.

KF is time variant, because the gain $K(t)$ is time varying. This causes 2 problems:

- It is difficult to check the stability of the system
- $K(t)$ must be computed at each sampling time, including the inversion of $\left(HP(t)H^T\right)_{p\times p} \Rightarrow$ computationally intensive

Because of this, the asymptotic version of KF is preferred

L.1 Basic idea

If $P(t)$ converges to constant $\overline{P}$, then also $K(t)$ will converge to some constant $\overline{K}$. Using $\overline{K}$ instead of $K(t)$ the KF becomes time-invariant:

$$\begin{aligned}
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + Gu(t) + \overline{K}e(t) \\
&= F\hat{x}(t|t-1) + Gu(t) + \overline{K}\left(y(t) - \hat{y}(t|t-1)\right) \\
&= F\hat{x}(t|t-1) + Gu(t) + \overline{K}\left(y(t) - H\hat{x}(t|t-1)\right) \\
&= \underbrace{\left(F - \overline{K}H\right)}_{\text{new state matrix}} \hat{x}(t|t-1) + Gu(t) + \overline{K}y(t)
\end{aligned}$$

If $\overline{K}$ exists, thes the KF is asymptotically stable $\iff$ all the eigenvalues of $F - \overline{K}H$ are stricly inside the unit circle

L.2 Existance of $\overline{K}$

$$\overline{K} = \left(F\overline{P}H^T + V_{12}\right) + \left(H\overline{P}H^T + V_2\right)^{-1}$$

$\overline{K}$ exists if $\overline{P}$ exists. DRE is an autonomous discrete time system $x(t+1) = f(x(t))$, in equilibrium when $x(t+1) = x(t) \Rightarrow f(\overline{x}) = \overline{x}$. Applyed to P this leads to the following Algebraic Riccardi Equation:

$$\overline{P} = f(\overline{P}) \iff \overline{P} = \left(F\overline{P}F^T + V_1\right) - \left(F\overline{P}H^T + V_{12}\right)\left(H\overline{P}H^T + V_2\right)^{-1}\left(F\overline{P}H^T + V_{12}\right)^T$$

L.2.1 First asymptotic theorem

Assuming $V_{12} = 0$ and the system is asymptotically stable (all eigenvalues of F strictly inside the unit circle), then:

- $\exists!$ semi-definite positive solution of ARE: $\overline{P} \geq 0$
- DRE converges to $\overline{P} \quad \forall P_0 \geq 0$
- The corresponding $\overline{K}$ will make the KF asymptotically stable

L.2.2 Second asymptotic theorem

Assuming $V_{12} = 0$, (F, H) is observable, (F, Γ) is controllable. Then:

- $\exists!$ semi-definite positive solution of ARE: $\overline{P} \geq 0$
- DRE converges to $\overline{P} \quad \forall P_0 \geq 0$
- The corresponding $\overline{K}$ will make the KF asymptotically stable

M Extension to non-linear systems

$$S : \begin{cases} x(t+1) = f(x(t), u(t)) + v_1(t) \\ y(t) = h(x(t)) + v_2(t) \end{cases}$$

Where f and h are non-linear functions.
For the gain block of the KF we have 2 types of solutions:

- The gain is a non linear function of $e(t)$
- The gain is a linear time-varying function

The second solution is preferred, as it allows us to reuse the formulae with just little tweaks. In particular, F and H are computed as follows:

$$F(t) = \left.\frac{\delta f(x(t), u(t))}{\delta x(t)}\right|_{x(t)=\hat{x}(t|t-1)}$$
$$H(t) = \left.\frac{\delta h(x(t))}{\delta x(t)}\right|_{x(t)=\hat{x}(t|t-1)}$$

EKF is the time-verying solution of KF, where F and H are computed around the last available state prediction $\hat{x}(t|t-1)$

Algorithm

1. Take the last available state prediction $\hat{x}(t|t-1)$
2. Use $\hat{x}(t|t-1)$ to compute $F(t)$ and $H(t)$
3. Compute $K(t)$ and update the DRE equations
4. Compute $\hat{x}(t+1|t)$

N Optimization of gain K

$$S : \begin{cases} x(t+1) = 2x(t) \\ y(t) = x(t) + v(t) \quad v \sim WN(0,1) \end{cases}$$

N.1 Direct solution

Starting from the standard observer structure:

$$\begin{cases} \hat{x}(t+1|t) = 2\hat{x}(t|t-1) + K(y(t) - \hat{y}(t|t-1)) \\ \hat{y}(t|t-1) = \hat{x}(t|t-1) \end{cases}$$

Minimizing the variance of the prediction error $var[\eta(t)] \Rightarrow$ minimizing $var[x(t)-\hat{x}(t|t-1]$

$$\begin{aligned} \eta(t) &= 2x(t) - [2\hat{x}(t|t-1) + K(y(t) - \hat{y}(t|t-1)] \\ &= 2x(t) - 2\hat{x}(t|t-1) - K(x(T) + v(t) - \hat{x}(t|t-1)) \\ &= (2-K)(x(t) - \hat{x}(t|t-1)) - Kv(t) \end{aligned}$$

$$\eta(t+1) = (2-K)\eta(t) - Kv(t) \qquad v \sim WN(0,1)$$

This is an AR(1) process:

$$\eta(t) = \frac{1}{1-(2-K)z^{-1}}e(t) \qquad e(t) = -Kv(t) \qquad e \sim WN(0,K^2)$$

The variance of η is

$$\gamma_\eta(0) = \frac{K^2}{1-(2-K)^2}$$

Minimizing wrt K:

$$\frac{\delta\gamma_\eta(0)}{\delta K} = 0 \qquad \Rightarrow \qquad \begin{cases} K_1 = 0 \\ K_2 = \frac{3}{2} \end{cases}$$

N.2 KF theory solution

From S we can derive:

$$\begin{cases} F = 2 \\ H = 1 \\ V_1 = 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} \Gamma = 0 \\ V_2 = 1 \\ V_{12} = 0 \end{cases}$$

- F is not asymptotically stable $\Rightarrow$ cannot use theorem 1
- (F,Γ) is not fully reachable $\Rightarrow$ cannot use theorem 2

$$DRE = 4P(t) - \frac{(2P(t))^2}{P(t)+1}\dots$$

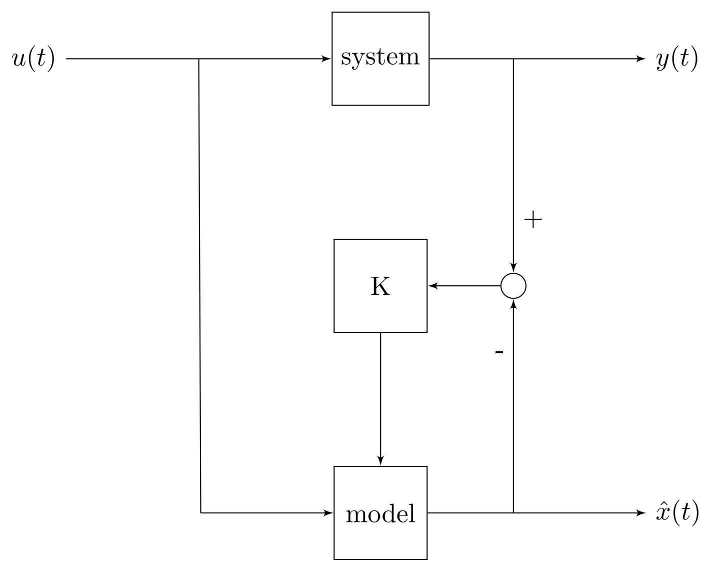
$$P(t+1) = \frac{4P(t)}{P(t)+1}$$

Solving ARE:

$$\overline{P} = \frac{4\overline{P}}{\overline{P}+1} \qquad \Rightarrow \qquad \begin{cases} \overline{P_1} = 1 \\ \overline{P_2} = 3 \end{cases} \qquad \Rightarrow \qquad \begin{cases} K_1 = 0 \\ K_2 = \frac{3}{2} \end{cases}$$

Part VI

Software-sensing with Blabk box Methods



Features

- A white-box model is required
- No need of a training dataset
- Works by feedback estimation
- Constructive method
- Can be used to estimate unmeasurable states

O Linear Time Invariant Systems

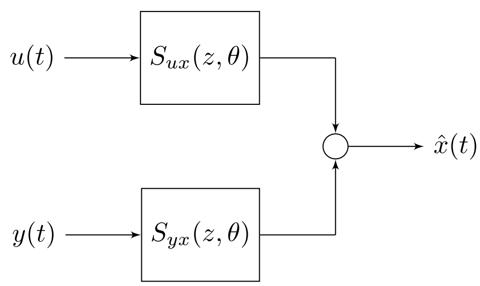
Known white-box model of the system Draw the block diagram of the system and the KF, then (from the diagram) calculate $\hat{x}(t) = f(u(t), y(t))$. Done.

Unknown model for the system A BB estimation is possible iff all the states are measurable.

Dataset

$$\{u(1), \ldots, u(N)\}$$
$$\{y(1), \ldots, y(N)\}$$
$$\{x(1), \ldots, x(N)\}$$

Model to be optimized fot θ



Performance index

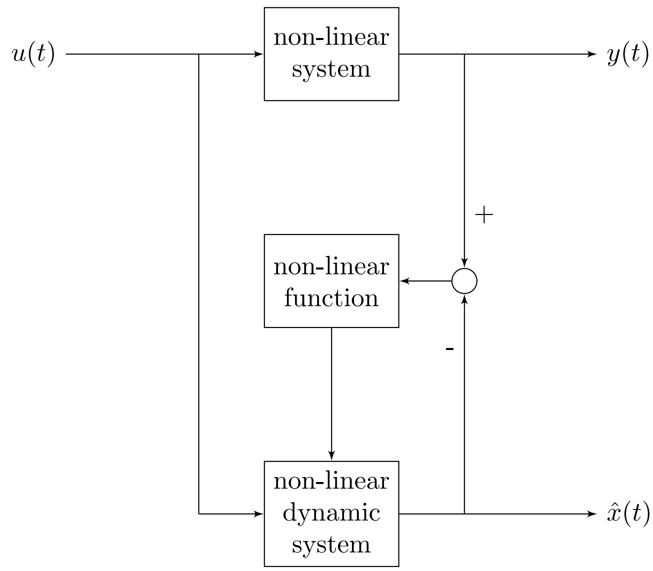
$$J_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} \left(x(t) - (S_{ux}(z,\theta)u(t) + S_{yx}(z,\theta)y(t))\right)^2$$

Optimization

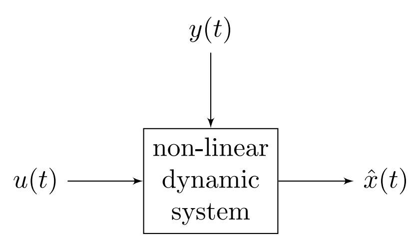
$$\hat{\theta}_N = \arg\min_{\theta} J_N(\theta)$$

We get $S_{ux}(z, \hat{\theta}_N)$ and $S_{yx}(z, \hat{\theta}_N)$, the transfer functions for our software sensors

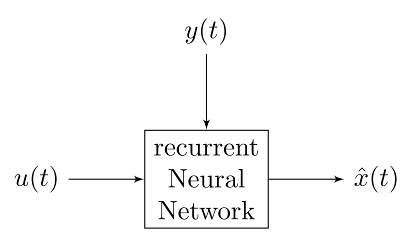
P Non-linear systems



Model

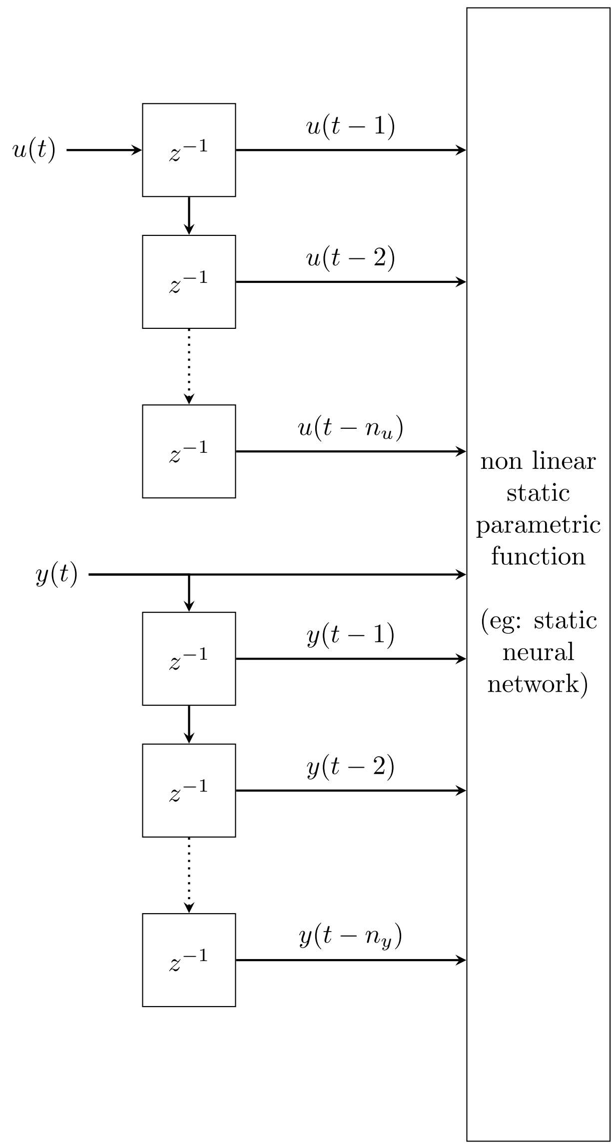


P.1 Recurrent neural network

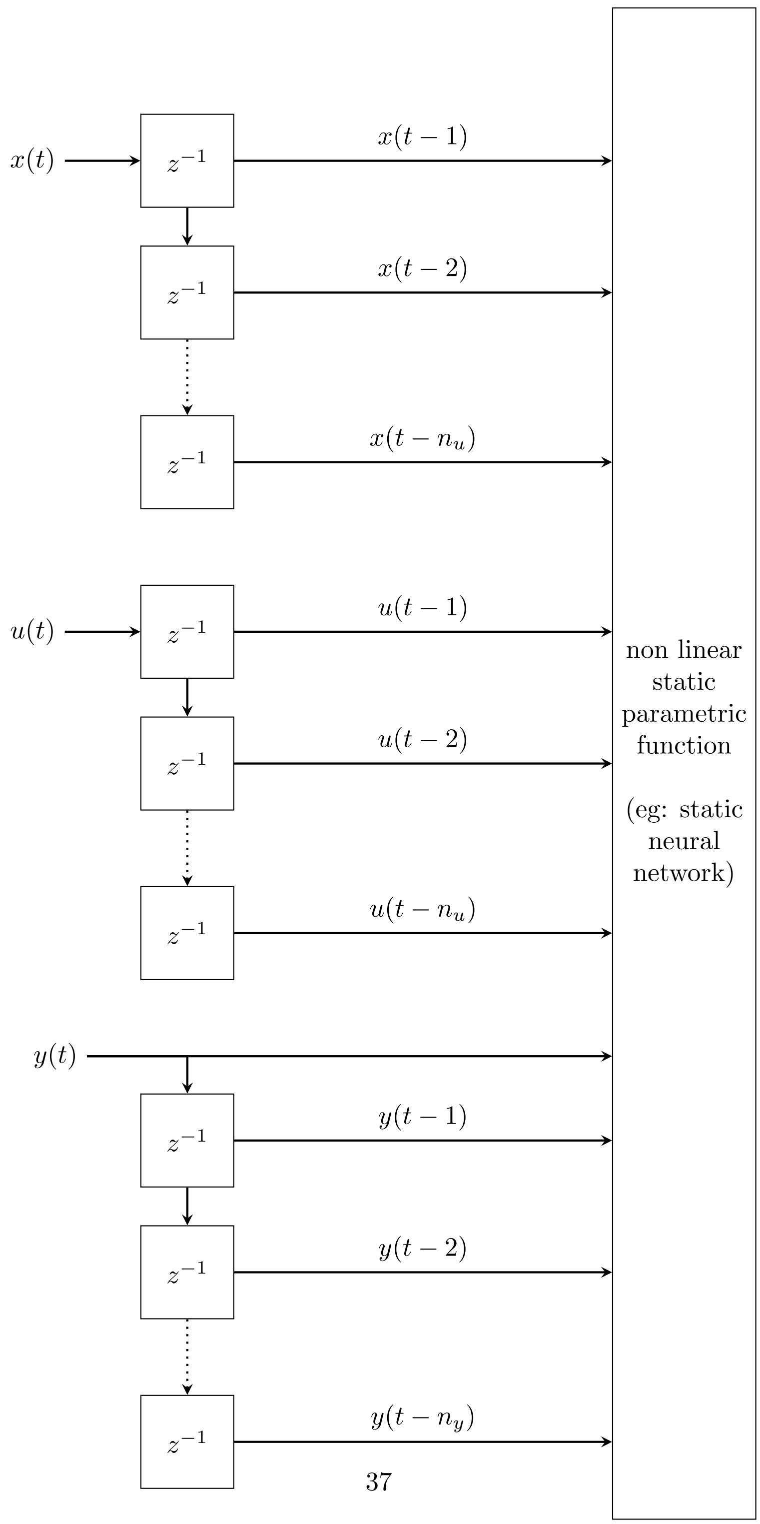


P.2 FIR architecture

split the system into a static non-linear system and linear dynamics

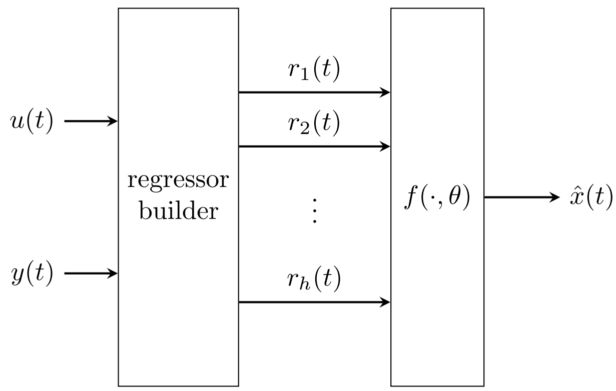

$u(t)$
z^{-1}
$u(t-1)$
z^{-1}
$u(t-2)$
z^{-1}
$u(t-n_u)$
$y(t)$
z^{-1}
$y(t-1)$
z^{-1}
$y(t-2)$
z^{-1}
$y(t-n_y)$
non linear static parametric function
(eg: static neural network)

P.3 IRR scheme



P.4 Physical regressors

Using the physical knowledge of the system, provide a set of regressors (ie: smaller and more meaningful set of signals) elaborated by a static non-linear system



Part VII
Grey-box System Identification

Q Using Kalman Filter

Q.1 Problem definition

- We have a model:

$$S: \begin{cases} x(t+1) = f(x(t), u(t), \theta) + v_1(t) \\ y(t) = h(x(t), \theta) + v_2(t) \end{cases}$$

- f and h are functions (linear or not) depending on some unknown parameter θ carrying physical meaning (mass, resistance,...)
- We want to estimate $\hat{\theta}$
- This is archieved by managing the unknown parameters as extended states

Q.2 State extension

$$S: \begin{cases} x(t+1) = f(x(t), u(t), \theta(t)) + v_1(t) \\ \theta(t+1) = \theta(t) + v_\theta(t) \\ y(t) = h(x(t), \theta(t)) + v_2(t) \end{cases}$$

And the extended state vector is $x_E = \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix}$
The noise in the equation of θ is added to prevent the KF form getting stuck on the initial conditions.

Q.3 Design choice

The choice of the covariance matrix of $v_\theta \sim WN(0, V_\theta)$

- Assume $v_1 \perp v_\theta$ and $v_2 \perp v_\theta$:

$$V_\theta = \begin{bmatrix} \lambda_{1\theta}^2 & & & \\ & \lambda_{2\theta}^2 & & \\ & & \ddots & \\ & & & \lambda_{n_\theta \theta}^2 \end{bmatrix}_{n_\theta \times n_\theta}$$

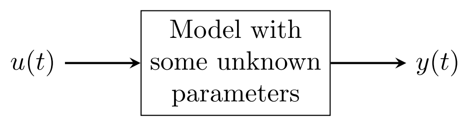
- Usually, it is assumed that $\lambda_{i\theta} = \lambda_{j\theta} \quad \forall i \forall j$
- Assume that $v_\theta(t)$ is a set of <u>independent</u> WN all with the same variance λ_θ^2

- Bigger values of λ_θ^2 lead to a quicker convergence, but less stable (stronger obscillations around the steady-state)
- The selection of λ_θ^2 is leaded by application-specific constraints

Q.4 Applicability

In theory, this trick can work with any number of sensors, states, and parameters. In practice it works well only on a limited number of parameters (∼3 sensors, 5 states, 2 parameters)

R Simulation Error Method



R.1 Dataset

from an experiment, collect:

$$\{\overline{u}(1), \overline{u}(2), \ldots, \overline{u}(N)$$
$$\{\overline{y}(1), \overline{y}(2), \ldots, \overline{y}(N)$$

R.2 Model

$$y(t) = \mathcal{M}(u(t), \overline{\theta}, \theta)$$

$\overline{\theta}$ is the set of known parameters, θ the set of unknown parameters

R.3 Performance index

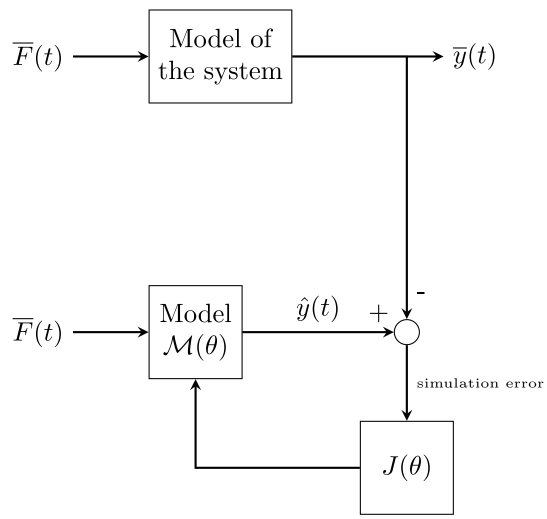
$$J_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} \left(\overline{y}(t) - \mathcal{M}(\overline{u}(t), \overline{\theta}, \theta)\right)^2$$

R.4 Optimization

$$\hat{\theta}_N = \arg\min_\theta J_N(\theta)$$

R.5 Limitations

- Usually J_N has no analytic expression
- Computing the value of J_N requires an entire simulation from $t = 1$ to $t = N$
- Usually J_N is non-quadratic and non-convex $\rightarrow$ iterative and randomizad optimization must be used
- Computationally demanding



Part VIII
Minimum Variance Control

The goal is to design a feedback system

- Control design is the ain motivation of system identification and software sensing
- MVC is based on prediction theory

S Setup the problem

Consider a generic ARMAX model:

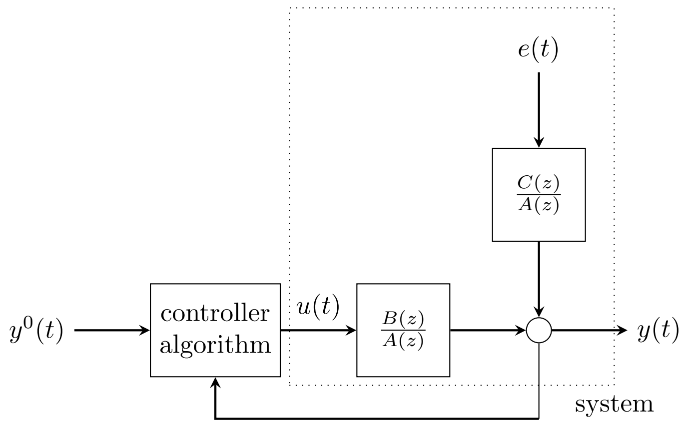
$$y(t) = \frac{B(z)}{A(z)}u(t-k) + \frac{C(z)}{A(z)}e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

$$\begin{aligned}
B(z) &= b_0 + b_1 z^{-1} + \cdots + b_p z^{-p} \\
A(z) &= 1 + a_1 z^{-1} + \cdots + a_m z^{-m} \\
C(z) &= 1 + c_1 z^{-1} + \cdots + c_n z^{-n}
\end{aligned}$$

Assumptions

- $\frac{C(z)}{A(z)}$ is in canonical form
- $b_0 \neq 0 \rightarrow$ k is the actual delay
- $\frac{B(z)}{A(z)}$ is minimum phase $\iff$ all the roots of $B(z)$ are strictly inside the unit circle
- $y^0(t) \perp e(t)$
- $y^0(t)$ is known only up to the present time ($y^0(t)$ is totally unpredictable)

We want to determine the optimal tracking of the desired output behaviour:



Formally, MVC tries to minimize the following performance index:

$$J = E\left[(y(t) - y^0(t))^2\right)$$

Which is the variance of the tracking error

S.1 Simplified problem 1

$$S : y(t) = ay(t-1) + b_0u(t-1) + b_1u(t-2) \qquad y(t) = \frac{b_0 + b_1z^{-1}}{1 - az^{-1}}u(t-1)$$

Assuming:

- $y^0(t) = \overline{y}^0$
- no noise
- $b_0 \neq 0$
- Root of numerator inside the unit circle

$$\begin{aligned}
J &= \left(y(t) - y^0(y)\right)^2 \\
&= \left(y(t) - \overline{y}^0\right)^2 \\
&= \left(ay(t-1) + b_0u(t-1) + b_1u(t-2) - \overline{y^0}\right)^2 \\
&= \left(ay(t) + b_0u(t) + b_1u(t-1) - \overline{y^0}\right)^2 \\
\frac{\delta J}{\delta u(t)} &= 2\left(ay(t) + b_0u(t) + b_1\underbrace{u(t-1)}_{\substack{\text{number}\\\text{not variable}}} - \overline{y^0}\right)(b_0)
\end{aligned}$$

$$\frac{\delta J}{\delta u(t)} = 0 \implies ay(t)+b_0u(t)+b_1u(t-1)-\overline{y^0} = 0 \implies u(t) = (\overline{y}^0 - ay(t))\frac{1}{b_0 + b_1z^{-1}}$$

S.2 Simplified problem 2

$$S : y(t) = ay(t-1) + b_0u(t-1) + b_1u(t-2) + e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

Reference variable $y^0(t)$

Performance index $E\left[(y(t) - y^0(t))^2\right]$

Trick rewrite $y(t)$ as

$$y(t) = \hat{y}(t|t-1) + \epsilon(t)$$

$$k = 1 \implies \epsilon(t) = e(t) \implies y(t) = \hat{y}(t|t-1) + e(t)$$

$$\begin{aligned}
J &= E\left[\left(\hat{y}(t|t-1) + e(t) - y^0(t)\right)^2\right] \\
&= E\left[\left((\hat{y}(t|t-1) - y^0(t)) + e(t)\right)^2\right] \\
&= E\left[\left(\hat{y}(t|t-1) - y^0(t)\right)^2\right] + E\left[e(t)^2\right] + \cancel{2E\left[e(t)(\hat{y}(t|t-1) - y^0(t))\right]}
\end{aligned}$$

$$\arg\min_{y^0(t)} E\left[\left(\hat{y}(t|t-1) - y^0(t)\right)^2\right] + E\left[e(t)^2\right] = \arg\min_{y^0(t)} E\left[\left(\hat{y}(t|t-1) - y^0(t)\right)^2\right]$$

The best result is when $\hat{y}(t|t-1) = y^0(t)$. Writing the system in terms of transfer functions we get:

$$S : y(t) = \frac{b_0 + b_1z^{-1}}{1 - az^{-1}}u(t-1) + \frac{1}{1 - az^{-1}}e(t)$$

Applying the general 1-step predictor for $ARMAX$:

$$\hat{y}(t|t-1) = \frac{b_0 + b_1z^{-1}}{1}u(t-1) + \frac{1 - 1 + az^{-1}}{1}y(t) = (b_0b_1z^{-1})u(t-1) + ay(t-1)$$

Imposing the optimality condition we get:

$$\begin{aligned}
b_0u(t-1) + b_1u(t-2) + ay(t-1) &= y^0(t) \\
b_0u(t) + b_1u(t-1) + ay(t) &= y^0(t+1) \\
u(t) &= \left(y^0(t+1) - ay(t)\right)\frac{1}{b_0b_1z^{-1}}
\end{aligned}$$

Since we cannot predict the future we must approximate:

$$u(t) \approx \left(y^0(t) - ay(t)\right)\frac{1}{b_0 + b_1z^{-1}}$$

S.3 General solution

$$S: y(t) = \frac{B(z)}{A(z)} u(t-k) + \frac{C(z)}{A(z)} e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

Assumptions

- $b_0 \neq 0$
- $B(z)$ has all roots inside the unit circle
- $\frac{C(z)}{A(z)}$ is in canonical form
- $y^0(t) \perp e(t)$
- $y^0(t)$ is unpredictable

Trick rewrite $y(t) = \hat{y}(t|t-k) + \epsilon(t)$

$$\begin{aligned}
J &= E\left[\left(\hat{y}(t|t-k) + \epsilon(t) - y^0(t)\right)^2\right] \\
&= E\left[\left((\hat{y}(t|t-k) - y^0(t)) + \epsilon(t)\right)^2\right] \\
&= E\left[\left(\hat{y}(t|t-k) - y^0(t)\right)^2\right] + E\left[\epsilon(t)^2\right] + \cancel{2E\left[\epsilon(t)\left(\hat{y}(t|t-k) - y^0(t)\right)\right]} \\
&= E\left[\left(\hat{y}(t|t-k) - y^0(t)\right)^2\right] + \textbf{constant}
\end{aligned}$$

Part IX
Recursive Identification

T Least square

$$\hat{\theta}_N = \arg\min_\theta \left\{ J_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}\left(y(t) - \hat{y}(t|t-1,\theta)\right)^2 \right\}$$

We want to find the predictor model $\hat{y}(t|t-1,\theta)$

$$y(t) = \phi(t)^T\theta + e(t)$$

Since $e(t)$ is unpredictable, the best possible 1-step predictor is $\hat{y}(t|t-1,\theta) = \phi(t)^T\theta$. Substituting in J:

$$J_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}\left(y(t) - \phi(t)^T\theta\right)^2$$

Analytically we can find the minimum:

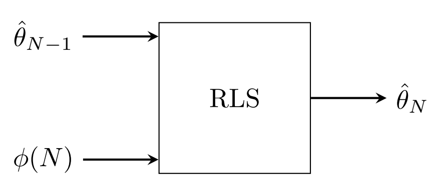
$$\frac{\delta J_N(\theta)}{\delta\theta} = 0$$

$$\hat{\theta}_N = S(N)^{-1}\sum_{t=1}^{N}\phi(t)y(t)$$

$$S(N) = \sum_{t=1}^{N}\phi(t)\phi(t)^T$$

But the procedure must be repeated at each t step

T.1 Recursive Least Square



T.2 First form

$$
\begin{aligned}
\hat{\theta}_N &= S(N)^{-1} \sum_{t=1}^{N} \phi(t) y(t) \\
S(N)\hat{\theta}_N &= \sum_{t=1}^{N} \phi(t) y(t) \\
S(N-1)\hat{\theta}_{N-1} &= \sum_{t=1}^{N-1} \phi(t) y(t) \\
\sum_{t=1}^{N} \phi(t) y(t) &= \sum_{t=1}^{N-1} \phi(t) y(t) + \phi(N) y(N) \\
\sum_{t=1}^{N} \phi(t) y(t) &= S(N-1)\hat{\theta}_{N-1} + \phi(N) y(N) \\
&\boxed{\hat{\theta}_N = \hat{\theta}_{N-1} + S(N)^{-1}\phi(N)\left[y(N) - \phi(N)^T \hat{\theta}_{N-1}\right]} \\
S(N) &= S(N-1) + \phi(N)\phi(N)^T
\end{aligned}
$$

However, this way $S(N) \to \infty$, so the domputing unit saturates.

T.3 Second form

Normalize wrt. N

$$
\begin{aligned}
S(N) &= S(N-1) + \phi(N)\phi(N)^T \\
R(N) &= \frac{1}{N} S(N) \\
R(N) &= \frac{N-1}{N} R(N-1) + \frac{1}{N}\phi(N)\phi(N)^T \\
K(N) &= \frac{1}{N} R(N)^{-1}\phi(N) \\
\epsilon(N) &= y(N) - \phi(N)^T \hat{\theta}_{N-1} \\
&\boxed{\hat{\theta}_N = \hat{\theta}_{N-1} + K(N)\epsilon(N)}
\end{aligned}
$$

This however reqires matrix inversion at each iteration, which is expensive

T.4 Third form

It can be proven that the following works

$$\epsilon(N) = y(N) - \phi(N)^T \hat{\theta}_{N-1}$$

$$\boxed{V(N) = V(N-1) - \frac{V(N-1)\phi(N)\phi(N)^T V(N-1)}{1 + \phi(N)^T V(N-1)\phi(N)}}$$

$$\boxed{K(N) = V(N)\phi(N)}$$

$$\hat{\theta}_N = \hat{\theta}_{N-1} + K(N)\epsilon(N)$$

This does not require matrix inversion

T.5 Forgetting factor

If theta contains some parameter changing over time, the objective function should "forget" old data. The new objective function becomes:

$$J_N(\theta) = \frac{1}{N}\rho^{N-t}\left(\hat{y}(t|t-1,\theta)\right)^2$$

Where $\rho \in [0,1]$ is the forgetting factor (the smaller, the more forgetful). The new equations become:

$$\epsilon(N) = y(N) - \phi(N)^T \hat{\theta}_{N-1}$$

$$\boxed{S(N) = \rho S(N-1) + \phi(N)\phi(N)^T}$$

$$K(N) = S(N)^{-1}\phi(N)$$

$$\hat{\theta}_N = \hat{\theta}_{N-1} + K(N)\epsilon(N)$$

Part X
Cheatsheet

A Probability Recall

Cross-Variance $Var[v,u] = E[(v - E[v])(u - E[u])]$

Variance Matrix $\begin{vmatrix} Var[v_1] & . & . & Var[v_1, v_k] \\ . & . & & . \\ . & & . & . \\ Var[v_k, v_1] & . & . & Var[v_k] \end{vmatrix}$

Covariance coefficient $\delta[i,j] = \frac{Var[i,j]}{\sqrt{Var[i]}\sqrt{Var[j]}}$

Stationary process

- m constant
- λ^2 constant
- covariance $\gamma(\tau)$ depends only on time difference
- $|\gamma(\tau)| \leq \gamma(0) \quad \forall\tau$

White noise $\eta(t) \sim WN(m, \lambda^2)$

- Stationary process
- $\gamma(\tau) = 0 \quad \forall\tau \neq 0$
- $v(t) = \alpha\eta(t) + \beta \implies v(t) \sim WN(\beta, \alpha^2\lambda^2)$

Canonical representation

- Monic
- Same degree
- Coprime
- Poles and zeros in unit disk

B Spectral analysis

Spectrum

- $\Gamma(\omega) = \gamma(0) + 2cos(\omega)\gamma(1) + 2cos(2\omega)\gamma(2) + ...$
- Periodic $T = 2\pi$
- Even
- $\Gamma_\eta(\omega) = \gamma_\eta(0) = \lambda^2$

Complex spectrum

- $\Phi(z) = \sum_{\tau=-\infty}^{+\infty} \omega(\tau) z^{-\tau}$
- $\Gamma(\omega) = \Phi(e^{j\omega})$

Fundamental theorem of spectral analysis

- $\Gamma_{\text{out}}(\omega) = |W(e^{j\omega})|^2 \cdot \Gamma_{\text{in}}(\omega)$
- $\Phi_{\text{out}}(z) = W(z)W(z^{-1}) \cdot \Phi_{\text{in}}(z)$

C Moving Average MA(n)

- $W(z) = \frac{c_0 z^n + c_1 z^{n-1} + ... + c_n}{z_n}$
- $m = 0$
- $\gamma(\tau) = \begin{cases} (c_0 c_\tau + c_1 c_{1+\tau} + ... + c_{n-\tau} c_\tau)\lambda^2 & |\tau| \leq n \\ 0 & \text{otherwise} \end{cases}$

C.1 MA(∞)

- $\gamma(0) = (c_0^2 + c_1^2 + ... + c_k^2 + ...)\lambda^2$
- $\gamma(0)$ must converge to a finite value

D Auto Regressive AR(n)

- $m = 0$
- $W(z) = \frac{z^n}{z^n - a_1 z_{n-1} - ... - a_n}$
- Covariance calculated by its definition

E Known predictors

AR(1) $\hat{v}(t|t-r) = a^r v(t-r)$

MA(1) $\hat{v}(t|t-1) = v(t-1) - c\hat{v}(t-1|t-2)$

MA(n) $\hat{v}(t|t-\mathbf{k}) = 0 \quad \forall k > n$

ARMA(n_a, n_b) $\hat{v}(t|t-1) = \frac{C(z)-A(z)}{C(z)} v(t)$

ARMAX(n_a, n_b) $\hat{y}(t|t-1) = \frac{C(z)-A(z)}{C(z)} y(t) + \frac{B(z)}{C(z)} u(t-1)$

F State-space representation

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) & \text{state equations} \\ y(t) = Hx(t) + Du(t) & \text{output equations} \end{cases}$$

G BB representations shift

G.1 State space to Transfer function

$$W(z) = H(zI - F)^{-1}G$$

G.2 Transfer Function to State Space

$$W(z) = \frac{b_0 z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1}}{z^n + a_0 z^{n-1} + \cdots + a_n}$$

$$\Longrightarrow$$

$$F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & \dots & \dots & -a_1 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_0 \end{bmatrix} \quad D = 0$$

G.3 Transfer function to Impulse response

∞ long division of $W(z)$

G.4 Impulse response to Tranfer function

Z-transform

$$\mathcal{Z} = \sum_{t=0}^{\infty} s(t) z^{-t}$$

$$W(z) = \mathcal{Z}(\omega(t)) = \sum_{t=0}^{\infty} \omega(t) z^{-1}$$

G.5 State space to Impulse response

$$\omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ HF^{t-1}G & \text{if } t > 0 \end{cases}$$

H Controllability and Observability

H.1 Fully observable system

$\iff$ O is full rank:

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \qquad rank(O) = n$$

H.2 Fully controllable system

Fully controllable $\iff$ R is full rank:

$$R = \begin{bmatrix} G & FG & \dots & F^{n-1}G \end{bmatrix} \qquad rank(R) = n$$

H.3 Hankel Matrix

Starting from $\omega(1), \omega(2), \dots, \omega(N)$ where $N \geq 2n - 1$, we can build the Hankel Matrix of order n:

$$H_n = \begin{bmatrix} \omega(1) & \omega(2) & \omega(3) & \dots & \omega(n) \\ \omega(2) & \omega(3) & \omega(4) & \dots & \omega(n+1) \\ \omega(3) & \omega(4) & \omega(5) & \dots & \omega(n+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega(n) & \omega(n+1) & \omega(n+2) & \dots & \omega(2n-1) \end{bmatrix}$$

$$H_n = \begin{bmatrix} HG & HFG & HF^2G & \dots & HF^{n-1}G \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ HF^{n-1}G & \dots & \dots & \dots & HF^{2n-2}G \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \cdot \begin{bmatrix} G & FG & \dots & F^{n-1}G \end{bmatrix} = O \cdot R$$

Part XI
System Identification

Impulse experiment Measure $y(t)$ under the input $u(t) = \text{impulse}(0)(0)$

I F, G, H from noise-free IR

1.

$$H_1 = \begin{bmatrix}\omega(1)\end{bmatrix} \quad H_2 = \begin{bmatrix}\omega(1) & \omega(2) \\ \omega(2) & \omega(3)\end{bmatrix} \quad H_3 = \dots \quad \dots \quad H_n = \dots$$

   If $rank(H_n) = rank(H_{n+1})$, then n is the order of the system

2. Factorize H_{n+1} as $H_{n+1} = O_{n+1[(n+1)\times n]} \cdot R_{n+1[n\times(n+1)]}$

$$O_{n+1} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^n \end{bmatrix} \qquad R_{n+1} = \begin{bmatrix} G & FG & \dots & F^n G \end{bmatrix}$$

3. $H = O[0]$, $G = R[0]$

4. Define

$$O_1 = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \qquad O_2 = \begin{bmatrix} HF \\ \vdots \\ HF^n \end{bmatrix}$$

5. $F = O_1^{-1} O_2$

J Obtain F, G, H from a noisy IR (TODO: not well understood)

The measurement is of $\hat{\omega}(t) = \omega(t) + \eta(t)$. To identify the process:

1. Build the Hankel matrix using all the N data:

$$\hat{H}_{q\times d} = \begin{bmatrix} \hat{\omega}(1) & \hat{\omega}(2) & \dots & \hat{\omega}(d) \\ \hat{\omega}(2) & \hat{\omega}(3) & \dots & \hat{\omega}(d+1) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\omega}(q) & \hat{\omega}(q+1) & \dots & \hat{\omega}(q+d+1=N) \end{bmatrix}$$

2. Calculate the Singular Value Decomposition of $\hat{H}_{q\times d}$:

   $$\hat{H}_{q\times d} = \hat{U}_{q\times q} \cdot \hat{S}_{q\times d} \cdot \hat{V}^T_{d\times d}$$

   $\hat{U}$ and $\hat{V}$ are unitary matrices: they are invertible and their inverses are equal to their transpose.

   $$\hat{S} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{bmatrix}$$

3. Plot the singular values (σ_i)and cut-off the three matrices:

   - Ideally, after a certain n (the order of the IR) there would be a jump dividing the signal (before) from the noise (after)
   - In reality no clear distinction exists, but it's possible to identify an interval of possible values of n. A tradeoff between complexity, precision and oferfitting takes place

4. Split $\hat{U}, \hat{S}, \hat{V}^T$ obtaining $U_{q\times n}$, $S_{n\times n}$, $V^T_{n\times d}$ and then recreate $H_{qd} = USV^T$

5. H and G are estimated as for the unnoisy case. To estimate F we can build O_1 and O_2 as before, but then the system $O_1 \cdot F = O_2$ cannot be solved directly as O_1 is not square. We can instead compute the approximate least-square solution of the system:

   $$F = (O_1^T O_1)^{-1} O_1^T O_2$$

Part XII
Kalman Filter

K Representations

K.1 For basic systems

$$
\begin{aligned}
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + K(t)e(t) && \text{state equation} \\
\hat{y}(t|t-1) &= H\hat{x}(t|t-1) && \text{output equation} \\
e(t) &= y(t) - \hat{y}(t|t-1) && \text{prediction error} \\
K(t) &= \left(FP(t)H^T + V_{12}\right)\left(HP(t)H^T + V_2\right)^{-1} && \text{gain of the KF} \\
P(t+1) &= \left(FP(t)F^T + V_1\right) \\
&\quad - \left(FP(t)H^T + V12\right)\left(HP(t)H^T + V_2\right)^{-1}\left(FP(t)H^T + V_{12}\right)^T && \text{difference Riccati equation} \\
\hline
\hat{x}(1|0) &= E\left[x(1)\right] = X_0 && \text{Initial state} \\
P(1) &= var\left[x(1)\right] = P_0 && \text{initial DRE}
\end{aligned}
$$

K.2 With input

$$
\begin{aligned}
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + Gu(t) + K(t)e(t) \qquad \text{state equation} \\
\text{other equations} &= \text{unchanged}
\end{aligned}
$$

K.3 Multi-step prediction

Knowing $\hat{x}(t+1|T)$ from the basic solution we can derive

$$
\begin{aligned}
\hat{x}(t+k|t) &= F^{k-1}\hat{x}(t+1|t) \\
\hat{y}(t+k|t) &= H\hat{x}(t+k|t)
\end{aligned}
$$

K.4 Filter $(\hat{x}(t|t))$

F invertible

$$
\hat{x}(t+1|t) = F\hat{x}(t|t) \qquad \Longrightarrow \qquad \hat{x}(t|t) = F^{-1}\hat{x}(t+1|t)
$$

F not invertible assuming $V_{12} = 0$:

$$
\begin{aligned}
\hat{x}(t|t) &= F\hat{x}(t-1|t-1) + Gu(t-1) + K_0(t)e(t) \\
\hat{y}(t|t-1) &= H\hat{x}(t|t-1) \\
e(t) &= y(t) - \hat{y}(t|t-1) \\
K_0(t) &= \left(P(t)H^T\right)\left(HP(t)H^T + V_2\right)^{-1} \\
P(t+1) &= \text{unchanged}
\end{aligned}
$$

K.5 Time-varying systems

$$S : \begin{cases} x(t+1) = F(t)x(t) + G(t)u(t) + v_1(t) \\ y(t) = H(t)x(t + v_2(t)) \end{cases}$$

K.F. equations are unchanged

L Asymptotic KF

L.1 ARE

$$\overline{P} = \left(F\overline{P}F^T + V_1\right) - \left(F\overline{P}H^T + V_{12}\right)\left(H\overline{P}H^T + V_2\right)^{-1}\left(F\overline{P}H^T + V_{12}\right)^T$$

L.1.1 First asymptotic theorem

$V_{12} = 0$ and the system is asymptotically stable (all eigenvalues of F strictly inside the unit circle) $\implies$:

- $\exists!$ solution of ARE: $\overline{P} \geq 0$
- DRE converges to $\overline{P} \quad \forall P_0 \geq 0$
- The corresponding $\overline{K}$ will make the KF asymptotically stable

L.1.2 Second asymptotic theorem

$V_{12} = 0$, (F, H) is observable, (F, Γ) is controllable $\implies$

- $\exists!$ solution of ARE: $\overline{P} \geq 0$
- DRE converges to $\overline{P} \quad \forall P_0 \geq 0$
- The corresponding $\overline{K}$ will make the KF asymptotically stable

M Non-linear systems

$$F(t) = \left.\frac{\delta f(x(t), u(t))}{\delta x(t)}\right|_{x(t)=\hat{x}(t|t-1)}$$
$$H(t) = \left.\frac{\delta h(x(t))}{\delta x(t)}\right|_{x(t)=\hat{x}(t|t-1)}$$

N Optimizing K

Part XIII

Minimum Variance Control

$$y(t) = ay(t-1) + b_0 u(t-1) + b_1 u(t-2) + e(t) \implies u(t) \approx \left(y^0(t) - ay(t)\right) \frac{1}{b_0 + b_1 z^{-1}}$$