Model Identification and Data Analysis cheatsheet

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1 Probability Recall

1.1 Random Vectors

Variance $Var[v] = E[(v - E[v])^2]$

Cross-Variance $Var[v,u] = E[(v - E[v])(u - E[u])]$

Variance Matrix $\begin{vmatrix} Var[v_1] & . & . & Var[v_1, v_k] \\ . & . & & . \\ . & & . & . \\ Var[v_k, v_1] & . & . & Var[v_k] \end{vmatrix}$

Covariance coefficient $\delta[i,j] = \frac{Var[i,j]}{\sqrt{Var[i]}\sqrt{Var[j]}}$
$\delta[i,j] = 0 \implies$ i, j uncorrelated
$|\delta[i,j]| = 1 \implies i = \alpha j$

1.2 Random processes

$v(t,s)|$ t time instant, s expetiment outcome (generally given)

Mean $m(t) = E[v(t,s)]$

Variance $\lambda^2(t) = Var[v(t)]$

Covariance function $\gamma(t_1, t_2) = E[(v(t_1) - m(t_1))(v(t_2) - m(t_2))] = \gamma(t_2, t_1)$

Normalized Covariance Function $\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}$
$\forall$ stationary processes: $|\rho(\tau)| \leq 1 \quad \forall \tau$

1.3 Important process classes

Stationary process

- $m(t) = m$ constant
- $\lambda^2(t) = \lambda^2$ constant
- $\gamma(t_1, t_2) = f(t_2 - t_1) = \gamma(\tau)$ covariance depends only on time difference τ

$|\gamma(\tau)| \leq \gamma(0) \quad \forall \tau$

White noise $\eta(t) \sim WN(m, \lambda^2)$

- Stationary process
- $\gamma(\tau) = 0 \quad \forall \tau \neq 0$

$v(t) = \alpha\eta(t) + \beta \quad \eta(t) \sim WN(0, \lambda^2) \qquad \implies \qquad v(t) \sim WN(\beta, \alpha^2\lambda^2)$

2 Spectral Analysis

2.1 Foundamentals

Spectrum

$$\Gamma(\omega) = \overbrace{F(\gamma(\tau))}^{\text{Fourier transform}} = \sum_{\tau=-\infty}^{+\infty} \gamma(\tau) \cdot e^{-j\omega\tau}$$

Euler formula $\Gamma(\omega) = \gamma(0) + 2cos(\omega)\gamma(1) + 2cos(2\omega)\gamma(2) + ...$

Spectrum properties

- $\Gamma : \mathbb{R} \to \mathbb{R}$
- Γ is periodic with $T = 2\pi$
- Γ is even $[\Gamma(-\omega) = \Gamma(\omega)]$
- $\Gamma(\omega) \geq 0 \quad \forall\omega$

$\eta(t) \sim WN(0, \lambda^2) \quad \implies \quad \Gamma_\eta(\omega) = \gamma(0) = Var[\eta(t)] = \lambda^2$

Anti-Transform

$$\gamma(\tau) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \Gamma(\omega) e^{k\omega\tau}\, dw$$

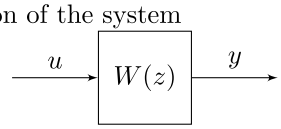
Complex spectrum

$$\phi(z) = \sum_{\tau=-\infty}^{+\infty} \omega(\tau) z^{-\tau}$$

$\Gamma(\omega) = \Phi(e^{j\omega})$

2.2 Fundamental theorem of Spectral Analysis

Fundamental theorem of Spectral Analysis allows to derive the (real and/or complex) spectrum of the output from the input and the transfer function of the system



$$\Gamma_{yy}(\omega) = |W(e^{j\omega})|^2 \quad \cdot \Gamma_{uu}(\omega)$$
$$\Phi_{yy}(z) = W(z)W(z^{-1}) \cdot \Phi_{uu}(z)$$

3 Moving Average Processes

Given $\eta(t) \sim WN(0, \lambda^2)$

MA(1): $v(t) = c_0\eta(t) + c_1\eta(t-1)$

Mean

$$
\begin{aligned}
E[v(t)] &= c_0 \cdot E[\eta(t)] + c_1 \cdot E[\eta(t)] \\
&= c_0 \cdot 0 \qquad + c_1 \cdot 0
\end{aligned}
$$

$$\boxed{E[v(t)] = 0}$$

Variance

$$
\begin{aligned}
Var[v(t)] &= E[(v(t) \underbrace{-E[v(t)])^2}_{0}] \\
&= E[(v(t))^2] \\
&= E[(c_0 \cdot \eta(t)^2 \qquad +c_1 \cdot \eta(t-1))^2] \\
&= c_0^2 \cdot E[\eta(t)^2] \qquad +c_1^2 \cdot E[\eta(t-1)^2] + \underbrace{2c_0c_1 \cdot E[\eta(t)\eta(t-1)]}_{0} \\
&= c_0^2 \cdot E[\eta(t)^2] \qquad +c_1^2 \cdot E[\eta(t-1)^2] \\
&= c_0^2\lambda^2 \qquad +c_1^2\lambda^2
\end{aligned}
$$

$$\boxed{Var[v(t)] = (c_0^2 + c_1^2)\lambda^2}$$

Covariance

$$
\begin{aligned}
\gamma(t_1, t_2) &= E[(v(t_1) - E[v(t_1)]) \qquad \cdot(v(t_2) - E[v(t_2)])] \\
&= E[(c_0\eta(t_1) + c_1\eta(t_1-1)) \ \cdot(c_0\eta(t_2) + c_1\eta(t_2-1))] \\
&= c_0^2E[\eta(t_1)\eta(t_2)] \qquad +c_1^2E[\eta(t_1-1)\eta(t_2-1) \\
&\qquad +c_0c_1E[\eta(t_1)\eta(t_2-1)] + c_0c_1E[\eta(t_1-1)\eta(t_2)]
\end{aligned}
$$

$$\boxed{\gamma(\tau) = \begin{cases} c_0^2\lambda^2 + c_1^2\lambda^2 & \text{if } \tau = 0 \\ c_0c_1\lambda^2 & \text{if } \tau = \pm 1 \\ 0 & \text{otherwise} \end{cases}}$$