Model Identification and Data Analysis

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Contents

Part I

MIDA 1

Chapter 1

Prediction

1.1 Probability Recall

1.1.1 Random Vectors

Variance $Var[v] = E[(v - E[v])^2]$

Cross-Variance $Var[v, u] = E[(v - E[v])(u - E[u])]$

Variance Matrix $\begin{vmatrix} Var[v_1] & . & . & Var[v_1, v_k] \\ . & . & & . \\ . & & . & . \\ Var[v_k, v_1] & . & . & Var[v_k] \end{vmatrix}$

Covariance coefficient $\delta[i, j] = \frac{Var[i,j]}{\sqrt{Var[i]}\sqrt{Var[j]}}$
$\delta[i, j] = 0 \implies$ i, j uncorrelated
$|\delta[i, j]| = 1 \implies i = \alpha j$

1.1.2 Random processes

$v(t, s)|$ t time instant, s expetiment outcome (generally given)

Mean $m(t) = E[v(t, s)]$

Variance $\lambda^2(t) = Var[v(t)]$

Covariance function $\gamma(t_1, t_2) = E[(v(t_1) - m(t_1))(v(t_2) - m(t_2))] = \gamma(t_2, t_1)$

Normalized Covariance Function $\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}$
$\forall$ stationary processes: $|\rho(\tau)| \leq 1 \quad \forall \tau$

1.1.3 Important process classes

Stationary process

- $m(t) = m$ constant
- $\lambda^2(t) = \lambda^2$ constant
- $\gamma(t_1, t_2) = f(t_2 - t_1) = \gamma(\tau)$ covariance depends only on time difference τ

$|\gamma(\tau)| \leq \gamma(0) \quad \forall\tau$

White noise $\quad \eta(t) \sim WN(m, \lambda^2)$

- Stationary process
- $\gamma(\tau) = 0 \quad \forall\tau \neq 0$

$v(t) = \alpha\eta(t) + \beta \quad \eta(t) \sim WN(0, \lambda^2) \qquad \Longrightarrow \qquad v(t) \sim WN(\beta, \alpha^2\lambda^2)$

1.2 Spectral Analysis

1.2.1 Foundamentals

Spectrum

$$\Gamma(\omega) = \overbrace{F(\gamma(\tau))}^{\text{Fourier transform}} = \sum_{\tau=-\infty}^{+\infty} \gamma(\tau) \cdot e^{-j\omega\tau}$$

Euler formula $\quad \Gamma(\omega) = \gamma(0) + 2cos(\omega)\gamma(1) + 2cos(2\omega)\gamma(2) + \dots$

Spectrum properties

- $\Gamma : \mathbb{R} \to \mathbb{R}$
- Γ is periodic with $T = 2\pi$
- Γ is even $[\Gamma(-\omega) = \Gamma(\omega)]$
- $\Gamma(\omega) \geq 0 \quad \forall\omega$

$\eta(t) \sim WN(0, \lambda^2) \quad \Longrightarrow \quad \Gamma_\eta(\omega) = \gamma(0) = Var[\eta(t)] = \lambda^2$

Anti-Transform

$$\gamma(\tau) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \Gamma(\omega) e^{k\omega\tau} \, dw$$

Complex spectrum

$$\phi(z) = \sum_{\tau=-\infty}^{+\infty} \omega(\tau) z^{-\tau}$$

$$\Gamma(\omega) = \Phi(e^{j\omega})$$

1.2.2 Fundamental theorem of Spectral Analysis

Fundamental theorem of Spectral Analysis allows to derive the (real and/or complex) spectrum of the output from the input and the transfer function of the system

$$\xrightarrow{\;u\;} \boxed{W(z)} \xrightarrow{\;y\;}$$

$$\Gamma_{yy}(\omega) = |W(e^{j\omega})|^2 \quad \cdot \Gamma_{uu}(\omega)$$
$$\Phi_{yy}(z) = W(z)W(z^{-1}) \cdot \Phi_{uu}(z)$$

1.2.3 Canonical representation of a Stationary Process

A stationary process can be represented by an infinite number of transfer functions. The canonical representation is the transfer function $W(z)$ such that:

- Numerator and denominator have same degree
- Numerator and denominator are monic (highest grade coefficient is 1)
- Numerator and denominator are coprime ($W(z)$ cannot be simplified)
- numerator and denominator are stable polynomials (all poles and zeros of $W(z)$ are inside the unit disk)

1.3 Moving Average Processes

Given $\eta(t) \sim WN(0, \lambda^2)$

1.3.1 MA(1):

Model

$$v(t) = c_0 \eta(t) + c_1 \eta(t-1)$$

Mean

$$\begin{aligned} E[v(t)] &= c_0 \cdot E[\eta(t)] + c_1 \cdot E[\eta(t)] \\ &= c_0 \cdot 0 \qquad\quad + c_1 \cdot 0 \end{aligned}$$

$$\boxed{E[v(t)] = 0}$$

Variance

$$\begin{aligned} Var[v(t)] &= E[(v(t) \underbrace{-E[v(t)])^2}_{0}] \\ &= E[(v(t))^2] \\ &= E[(c_0 \cdot \eta(t)^2 \qquad +c_1 \cdot \eta(t-1))^2] \\ &= c_0^2 \cdot E[\eta(t)^2] \qquad +c_1^2 \cdot E[\eta(t-1)^2] + \underbrace{2c_0c_1 \cdot E[\eta(t)\eta(t-1)]}_{0} \\ &= c_0^2 \cdot E[\eta(t)^2] \qquad +c_1^2 \cdot E[\eta(t-1)^2] \\ &= c_0^2\lambda^2 \qquad +c_1^2\lambda^2 \end{aligned}$$

$$\boxed{Var[v(t)] = (c_0^2 + c_1^2)\lambda^2}$$

Covariance

$$\begin{aligned} \gamma(t_1, t_2) &= E[(v(t_1) - E[v(t_1)]) \qquad \cdot(v(t_2) - E[v(t_2)])] \\ &= E[(c_0\eta(t_1) + c_1\eta(t_1 - 1)) \ \cdot(c_0\eta(t_2) + c_1\eta(t_2 - 1))] \\ &= c_0^2 E[\eta(t_1)\eta(t_2)] \qquad +c_1^2 E[\eta(t_1 - 1)\eta(t_2 - 1) \\ &\qquad +c_0c_1E[\eta(t_1)\eta(t_2 - 1)] + c_0c_1E[\eta(t_1 - 1)\eta(t_2)] \end{aligned}$$

$$\boxed{\gamma(\tau) = \begin{cases} c_0^2\lambda^2 + c_1^2\lambda^2 & \text{if } \tau = 0 \\ c_0c_1\lambda^2 & \text{if } \tau = \pm 1 \\ 0 & \text{otherwise} \end{cases}}$$

1.3.2 MA(n)

Model

$$\begin{aligned} v(t) &= c_0\eta(t) + c_1\eta(t-1) + ... + c_n\eta(t-n) \\ &= (c_0 + c_1z^{-1} + ... + c_nz^{-n})\eta(t) \end{aligned}$$

Transfer function

$$W(z) = c_0 + c_1z^{-1} + ... + c_nz^{-n} = \frac{c_0z^n + c_1z^{n-1} + ... + c_n}{z^n}$$

All poles are in the complex origin

Mean

$$E[v(t)] = (c_0 + c_1 + ... + c_n) \underbrace{E[\eta(t)]}_{0}$$

$$\boxed{E[v(t) = 0}$$

Covariance function

$$\boxed{\gamma(\tau) = \begin{cases} \lambda^2 \cdot \sum_{i=0}^{n-\tau} c_i c_{i-\tau} & |\tau| \leq n \\ 0 & \text{otherwise} \end{cases}}$$

example

$$\begin{aligned} \gamma(0) &= (c_0^2 + c_1^2 + ... + c_n^2)\lambda^2 \\ \gamma(1) &= (c_0 c_1 + c_1 c_2 + ... + c_{n-1} c_n)\lambda^2 \\ \gamma(2) &= (c_0 c_2 + c_1 c_3 + ... + c_{n-2} c_n)\lambda^2 \\ &... \\ \lambda(n) &= (c_0 c_n)\lambda^2 \\ \lambda(k) &= 0 \, \forall k > n \end{aligned}$$

1.3.3 MA(∞)

Model

$$v(t) = c_0\eta(t) + c_1\eta(t-1) + ... + c_k\eta(t-k) + ... = \sum_{i=0}^{\infty} c_i\eta(t-i)$$

Variance

$$\gamma(0) = (c_0^2 + c_1^2 + ... + c_k^2 + ...)\lambda^2 = \lambda^2 \sum_{i=0}^{\infty} c_i^2$$

1.3.4 Well definition of an MA(∞)

We need to have $|\gamma(\tau)| \leq \gamma(0)$, so we must require that

$$\boxed{\gamma(0) = \lambda^2 \sum_{i=0}^{\infty} c_i^2 \text{ is finite}}$$

1.4 Auto Regressive Processes

1.4.1 AR(1)

Model

$$v(t) = av(t-1) + \eta(t)$$

Mean

$$\begin{aligned} E[v(t)] &= E[av(t-1)] + \overbrace{E[\eta(t)]}^{0} \\ &= aE[v(t-1)] \\ &= aE[v(t)] \\ (1-a)E[v(t)] &= 0 \\ &\boxed{E[v(t)] = 0} \end{aligned}$$

Covariance

MA(∞) method Observe as an AR(1) can be axpressed as an MA(∞)

$$\begin{aligned} v(t) &= av(t-1) && +\eta(t) \\ &= a[av(t-2) + \eta(t-1)] && +\eta(t) \\ &= a^2 v(t-2) && +a\eta(t-1) + \eta(t) \\ &= a^2[v(t-3) + \eta(t-2)] && +a\eta(t-1) + \eta(t) \\ &= \underbrace{a^n v(t-n)}_{\to 0} + \underbrace{\sum_{i=0}^{\infty} a^i \eta(t-i)}_{\text{MA}(\infty)} \end{aligned}$$

In particular, the result depends on an $MA(\infty)$ having $\sum_{i=0}^{\infty} c_i = \sum_{i=0}^{\infty} a^i$. To check if the variance is finite we check $\gamma(0) = \lambda^2 \sum_{i=0}^{\infty} a^{2i} < \infty$. The given is a geometric series, convergent for $|a| < 1$. Under this hypothesis its value is

$$\gamma(0) = \lambda^2 \sum_{i=0}^{\infty} a^{2i} = \frac{\lambda^2}{1-a^2}$$

Applying the formula of the variance of MA processes we get

$$\gamma(1) = (c_0c_1 + c_1c_2 + ...)\lambda^2 = (a + aa^2 + ...)\lambda^2 = a(1 + a^2 + a^4 + ...)\lambda^2 = a\lambda^2 \sum_{i=0}^{\infty} a^{2i} = a\frac{\lambda^2}{1-a^2} = a\gamma(0)$$

$$\gamma(2) = (c_0c_2 + c_1c_3 + ...)\lambda^2 = (a^2 + aa^3 + ...)\lambda^2 = a^2(1 + a^2 + a^4 + ...)\lambda^2 = a^2\lambda^2 \sum_{i=0}^{\infty} a^{2i} = a^2\frac{\lambda^2}{1-a^2} = a^2\gamma(0)$$

$$\boxed{\gamma(\tau) = a^{|\tau|}\frac{\lambda^2}{1-a^2}}$$

Yule-Walkler Equations

$$\begin{aligned}
Var[v(t)] &= E[v(t)^2] \\
&= E[(av(t) + \eta(t))^2] \\
&= a^2 \underbrace{E[v(t-1)^2]}_{\substack{=Var[v(t-1)] \\ =Var[v(t)] \\ =\gamma(0)}} + \underbrace{E[\eta(t)^2]}_{=\lambda^2} + 2a \underbrace{E[v(t-1)\eta(t)]}_{\substack{v(t-1) \text{ depends on } \eta(t-2) \\ \eta(t) \text{ independent of } \eta(t-2) \\ \Longrightarrow \\ E[v(t-1)\eta(t)]=0}}
\end{aligned}$$

$$\gamma(0) = a^2\gamma(0) + \lambda^2$$

$$\boxed{\gamma(0) = \frac{\lambda^2}{1-a^2}}$$

To find $\gamma(\tau)$, we start from the model $v(t) = av(t-1) + \eta(t)$.

$$\begin{aligned}
v(t) &= av(t-1) && +\eta(t) \\
v(t)v(t-\tau) &= av(t-1)v(t-\tau) && +\eta(t)v(t-\tau) \\
\underbrace{E[v(t)v(t-\tau)]}_{\gamma(\tau)} &= a \underbrace{E[v(t-1)v(t-\tau)]}_{\gamma(\tau-1)} + \underbrace{E[\eta(t)v(t-\tau)]}_{0}
\end{aligned}$$

$$\boxed{\gamma(\tau) = a\gamma(\tau-1)}$$

We can join the two by inductive reasoning, obtaining

$$\boxed{\gamma(\tau) = a^{|\tau|}\frac{\lambda^2}{1-a^2}}$$

Long Division Leads to same result, but is boring

1.4.2 AR(n)

Model

$$v(t) = a_1v(t-1) + a_2v(t-2) + ... + a_nv(t-n) + \eta(t)$$

Transfer function

$$W(z) = \frac{z^n}{z^n - a_1 z_{n-1} - ... - a_n}$$

Mean

$$E[v(t)] = a_1 E[v(t-1]) + a_2 E[v(t-2)] + ... + a_n E[v(t-n)] + \underbrace{E[\eta(t)]}_{0}$$

$$m = a_1 m \qquad +a_2 m \qquad +...+a_n m$$

$$(1 - a_1 - a_2 - ... - a_n)m = 0$$

$$\boxed{E[v(t)] = 0}$$

1.5 ARMA Processes

Model

$$v(t) = a_1 v(t-1) + ... + a_{n_a} v(t - n_a) + c_0 \eta(t) + ... + c_{n_c} v(t - n_c)$$

Can also be espressed as $V(t) = \frac{C(z)}{A(z)} \eta(t)$, where

$$C(z) = c_0 + c_1 z^{-1} + ... + c_{n_c} z^{-n_c}$$
$$A(z) = 1 - a_1 z^{-1} - ... - a_{n_a} z^{-n_a}$$

Such process is stationary if all the poles of $W(z)$ are inside the unit disk.

1.6 Prediction problem

We want to predict $v(t+r)$ from $v(t), v(t-1), ...$, where r is called prediction horizon, of the following stationary process:

$$\xrightarrow{\ \eta\ } \boxed{W(z)} \xrightarrow{\ v\ }$$

1.6.1 Fake problem

Having a process with transfer function $W(z)$, we can compute it in polynomial form using the long division algorithm

$$W(z) = w_0 + w_1 z^{-1} + w_2 z^{-2} + ...$$

We can calculate

$$v(t{+}r) = W(z)\eta(t{+}r) = \underbrace{w_0 \eta(t+r) + w_1 \eta(t+r-1) + ... + w_{r-1}\eta(t+1)}_{\alpha(t) \ \textbf{unpredictable:} \text{ future of } \eta \text{ involved}} + \underbrace{w_r \eta(t) + w_{r+1}\eta(t-1) + ...}_{\beta(t) \ \textbf{predictable}}$$

The optimal fake predictor is then

$$\boxed{v(t+r|t) = w_r\eta(t) + w_{r+1}\eta(t-1) + ...} = \beta(t)$$

And the prediction error is

$$\begin{aligned}
\epsilon(t) &= v(t+r) & -\hat{v}(t+r|t) \\
&= \alpha(t) + \beta(t) & -\beta(t) \\
&= \alpha(t)
\end{aligned}$$

$$\boxed{\epsilon(t) = w_0\eta(t+r) + w_1\eta(t+r-1) + ... + w_{r-1}\eta(t+1)}$$

$$\boxed{Var[\epsilon(t)] = (w_0^2 + w_1^2 + ... + w_{r-1}^2)\lambda^2}$$

1.6.2 True Problem

We want to estimate $v(t+r)$ form $v(t)$, having transfer function $W(z)$ and $\hat{W}_r(z)$ the solution to the fake problem. We can calculate the transfer function of the real predictor from the process as

$$\boxed{W_r(z) = W(z)^{-1} \cdot \hat{W}_r(z)}$$

For ARMA processes a shortcut exists:

$$\hat{v}_{\text{ARMA}}(t|t-1) = \frac{C(z)A(z)}{C(z)} \qquad \text{having } W(z) = \frac{C(z)}{A(z)}$$

1.6.3 Prediction with eXogenous variables

An exogenous variable is a <u>deterministic</u> input variable in the system

ARX model

$$v(t) = a_1 v(t-1) + ... + a_{n_a} v(t-n_a) + b_1 u(t-1) + ... + b_{n_b} u(t-n_b) + \eta(t) A(z)v(t) \quad = B(z)u(t-1) +$$

Transfer functions from u and η

$$W_u(z) = \frac{B(z)}{A(z)} \qquad W_\eta(z) = \frac{1}{A(z)}$$

ARMAX model

$$\begin{aligned}
A(z)v(t) &= C(z)\eta(t) + B(z)u(t-1) \\
y(t) &= W(z)\eta(t) + G(z)u(t)
\end{aligned}$$

Predictor

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t) + \frac{B(z)}{C(z)} u(t-1)$$

Chapter 2

Identification

Consists of estimating a model from data.

2.1 Prediction Error Minimization

Aims to minimize $\epsilon(t) = v(t) - \hat{v}(t|t-r)$
Steps:

1. **Data collection:** collect $\vec{u}$ and $\vec{y}$

2. **Family selection:** choose a family of models $M(\theta)$

   MA(1) $\theta = [a]$

   MA(n) $\theta = [a_1, ..., a_n]$

   ARMA(n_a, n_c) $\theta = [a_1, ..., a_{n_a}, c_1, ..., c_{n_c}]$

   ...

3. **Select an optimization criterion**

   Mean Squared error $J(\theta) = \frac{1}{N} \sum_{t=1}^{N} \epsilon_\theta(t)^2$

   Mean absolute error $J(\theta) = \frac{1}{N} \sum_{t=1}^{N} |\epsilon_\theta(t)|$

   ...

4. **Optimization** find $\hat{\theta} = argmin\, J(\theta) \implies \frac{dJ(\theta)}{d\theta} = 0$

5. **Validation** verify if the result satisfies the requirements

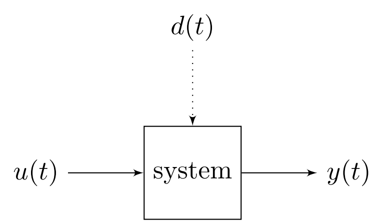
Part II

MIDA 2

Chapter 3

Black-Box non-parametric I/O systems

3.1 State-space models



Known (measured) data

$$
\begin{aligned}
&\{u(1), \ldots, u(N)\} && \text{input} \\
&\{y(1), \ldots, y(N)\} && \text{output}
\end{aligned}
$$

3.1.1 State-space representation

$$
\begin{cases}
x(t+1) = Fx(t) + Gu(t) & \text{state equations} \\
y(t) = Hx(t) + Du(t) & \text{output equations}
\end{cases}
$$

Where $F_{n\times n}$, $G_{n\times 1}$, $H_{1\times n}$ and $D_{1\times 1}$ are matrices.

S.S. representation is not unique Given any invertible matrix T, let $F_1 = TFT^{-1}$, $G_1 = TG$, $H_1 = HT^{-1}$, $D_1 = D$. Then the system $\{F, G, H, D\}$ is equivalent to the system $\{F_1, G_1, H_1, D_1\}$.

3.1.2 Transfer function representation

$$W(z) = \frac{B(z)}{A(z)} z^{-k} = \frac{b_0 + b_1 z^{-1} + \cdots + b_p z^{-p}}{a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}} z^{-k}$$

$W(z)$ is a rational function of the z operator $\rightarrow$ is a digital filter

Infinite impulse response $W(z) = \frac{z^{-1}}{1+\frac{1}{3}z^{-1}}$

Finite impulse response $W(z) = z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3}$

3.1.3 Convolution of the input with the inpulse response

Let's call $\omega(1), \omega(2), \ldots$ the values of $y(t)$ when $u(t) = \text{impulse}(0)$, and let's measure the values of y at different times: . Then it can be proven that for any $u(t)$

$$y(t) = \sum_{k=0}^{\infty} \omega(k) u(t-k)$$

3.2 Converting representations one to another

3.2.1 State space to Transfer function

Consider a strictly propter system:

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t+1) = Hx(t) + \cancelto{0}{Du(t)} \end{cases} \Rightarrow \begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

Applying the z operator we get

$$\begin{aligned} zx(t) &= Fx(t) + Gu(t) \\ x(t)(zI - F) &= Gu(t) \\ x(t) &= (zI - F)^{-1} Gu(t) \\ y(t) &= H(zI - F)^{-1} Gu(t) \end{aligned}$$

And we can extract the transfer function:

$$W(z) = H(zI - F)^{-1} G$$

3.2.2 Transfer Function to State Space

We have the transfer function

$$W(z) = \frac{b_0 z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1}}{z^n + a_0 z^{n-1} + \cdots + a_n}$$

The formulas for the state space matrices is

$$F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & \dots & \dots & -a_1 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_0 \end{bmatrix} \quad D = 0$$

3.2.3 Transfer function to Impulse response

Obtained by computing the ∞ long division of $W(z)$

3.2.4 Impulse response to Tranfer function

Z-transform Given a discrete-time signal $s(t)$ such that $\forall t < 0 : s(t) = 0$, it's Z-transform is

$$\mathcal{Z} = \sum_{t=0}^{\infty} s(t) z^{-t}$$

It can be proven that:

$$W(z) = \mathcal{Z}(\omega(t)) = \sum_{t=0}^{\infty} \omega(t) z^{-1}$$

NB: this works only in theory because of the infinite sum

3.2.5 State space to Impulse response

Consider the state space model:

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

We have that:

$$\begin{aligned}
x(1) &= \cancelto{0}{Fx(0)} + Gu(0) & &= Gu(0) \\
y(1) &= Hx(1) & &= HGu(0) \\
\\
x(2) &= Fx(1) + Gu(1) & &= FGu(0) + Gu(1) \\
y(2) &= Hx(2) & &= HFGu(0) + HG(u1) \\
\\
x(3) &= Fx(2) + Gu(2) & &= F^2Gu(0) + FGu(1) + Gu(2) \\
y(3) &= Hx(3) & &= HF^2Gu(0) + HFGu(1) + HGu(2) \\
&\vdots
\end{aligned}$$

$$y(t) = 0u(t) + HGu(t-1) + HFGu(t-2) + HF^2Gu(t-3) + \dots$$

The IR is:

$$\omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ HF^{t-1}G & \text{if } t > 0 \end{cases}$$

3.3 Controllability and Observability

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

Fully observable system The system is fully observable (from the output) $\iff$ the observability matrix is full rank:

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \qquad rank(O) = n$$

Fully controllable system The system is fully controllable (from the input) $\iff$ the controllability (also called reachability) matrix is full rank:

$$R = \begin{bmatrix} G & FG & \dots & F^{n-1}G \end{bmatrix} \qquad rank(R) = n$$

3.4 Hankel Matrix

Starting from $\omega(1), \omega(2), \dots, \omega(N)$ where $N \geq 2n - 1$, we can build the Hankel Matrix of order n:

$$H_n = \begin{bmatrix} \omega(1) & \omega(2) & \omega(3) & \dots & \omega(n) \\ \omega(2) & \omega(3) & \omega(4) & \dots & \omega(n+1) \\ \omega(3) & \omega(4) & \omega(5) & \dots & \omega(n+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega(n) & \omega(n+1) & \omega(n+2) & \dots & \omega(2n-1) \end{bmatrix}$$

Knowing that

$$\omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ HF^{t-1}G & \text{if } t > 0 \end{cases}$$

We can rewrite

$$H_n = \begin{bmatrix} HG & HFG & HF^2G & \dots & HF^{n-1}G \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ HF^{n-1}G & \dots & \dots & \dots & HF^{2n-2}G \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \cdot \begin{bmatrix} G & FG & \dots & F^{n-1}G \end{bmatrix} = O \cdot R$$

3.5 Subspace-based State Space System Identification

Impulse experiment Measure $y(t)$ under the input $u(t) = \text{impulse}(0)(0)$
How to derive F, G, H from $\omega(0), \dots, \omega(n)$?

- Assuming the IR measurement to be noise free $\rightarrow$ easier, not realistic
- Measure $\hat{\omega(t)}$ as a noisy signal and compute $\omega(t) = \eta(t) - \hat{\omega}(t)$

3.5.1 Obtain F, G, H from a noise-free IR

1. Build the Hankel matrix of increasing order, and conpute the rank until $rank(H_n) = rank(H_{n+1})$. Then, n is the order of the IR

$$H_1 = \begin{bmatrix} \omega(1) \end{bmatrix} \quad H_2 = \begin{bmatrix} \omega(1) & \omega(2) \\ \omega(2) & \omega(3) \end{bmatrix} \quad H_3 = \dots \quad \dots \quad H_n = \dots$$

2. Take H_{n+1} and factorize it in two rectangular matrix of size $(n+1) \times n$ and $n \times (n+1)$: $H_{n+1} = O_{n+1} \cdot R_{n+1}$, where

$$O_{n+1} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^n \end{bmatrix} \qquad R_{n+1} = \begin{bmatrix} G & FG & \dots & F^nG \end{bmatrix}$$

3. Estimate H,F,G:
   - Extract F and G from the first element of O and R
   - Define:

$$O_1 = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \qquad O_2 = \begin{bmatrix} HF \\ \vdots \\ HF^n \end{bmatrix}$$

   - Observe that $O_1 F = O_2$, so $F = O_1^{-1} O_2$

3.6 Obtain F, G, H from a noisy IR

The measurement is of $\hat{\omega}(t) = \omega(t) + \eta(t)$. To identify the process:

1. Build the Hankel matrix from data using all the N data available in one shot:

$$\hat{H}_{q\times d} = \begin{bmatrix} \hat{\omega}(1) & \hat{\omega}(2) & \dots & \hat{\omega}(d) \\ \hat{\omega}(2) & \hat{\omega}(3) & \dots & \hat{\omega}(d+1) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\omega}(q) & \hat{\omega}(q+1) & \dots & \hat{\omega}(q+d+1) \end{bmatrix}$$

   Where $q + d + 1 = N$

2. Calculate the Singular Value Decomposition of $\hat{H}_{q\times d}$:

$$\hat{H}_{q\times d} = \hat{U}_{q\times q} \cdot \hat{S}_{q\times d} \cdot \hat{V}^T_{d\times d}$$

   $\hat{U}$ and $\hat{V}$ are unitary matrices: they are invertible and their inverses are equal to their transpose.

$$\hat{S} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{bmatrix}$$

3. Plot the singular values (σ_i)and cut-off the three matrices:

   - Ideally, after a certain n (the order of the IR) there would be a jump dividing the signal (before) from the noise (after)
   - In reality no clear distinction exists, but it's possible to identify an interval of possible values of n. A tradeoff between complexity, precision and oferfitting takes place

4. Split $\hat{U}, \hat{S}, \hat{V}^T$ obtaining $U_{q\times n}$, $S_{n\times n}$, $V^T_{n\times d}$ and then recreate $H_{qd} = USV^T$

5. H and G are estimated as for the unnoisy case. To estimate F we can build O_1 and O_2 as before, but then the system $O_1 \cdot F = O_2$ cannot be solved directly as O_1 is not square. We can instead compute the approximate least-square solution of the system:

$$F = (O_1^T O_1)^{-1} O_1^T O_2$$

Chapter 4

Parametric black-box system identification using frequency-domain approach

4.1 Experiment design and data pre-processing

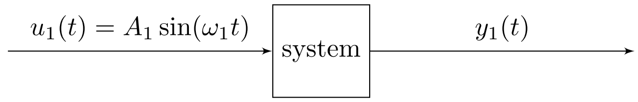
1. Select a set of excitation frequencies $\{\omega_1, \dots, \omega_H\}$. Usually $\omega_i - \omega_{i-1}$ is constant $\forall i \in \{2, \dots, H\}$. ω_H must be selected according to the bandwidth of the system
2. Make H independent experiments:
3. Focusing on experiment i, because of noise the real value of the output will be (unknowns are underlined)

$$\hat{y}_i = \underline{B_i} \sin(\omega_i t + \underline{\phi_i}) = \underline{a_i} \sin(\omega_i t) + \underline{b_i} \cos(\omega_i t)$$

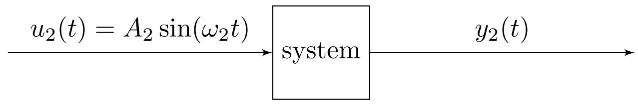
   Using the second equation (since it is linear in the unknowns). We want to determine

$$\{\hat{a}_i, \hat{b}_i\} = \arg \min_{\{a_i, b_i\}} J_N(a_i, b_i)$$

$$J_N(a_i, b_i) = \frac{1}{N} \sum_{t=1}^{N} \left(\underbrace{y_i(t)}_{\text{measurement}} \underbrace{- a_i \sin(\omega_i t) - b_i \cos(\omega_i t)}_{\text{model output}} \right)^2$$

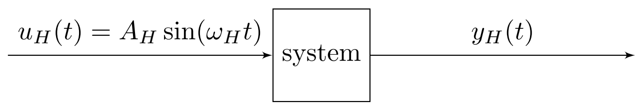


Experiment 1



Experiment 2

$$\vdots$$



Experiment H

This can be solved explicitly

$$\begin{aligned}
\frac{\delta J_N}{\delta a_i} &= \frac{2}{N} \sum_{t=1}^{N} \left(-\sin(\omega_i t)\right) \left(y_i(t) - a_i \sin(\omega_i t) - b_i \cos(\omega_i t)\right) &&= 0 \\
\frac{\delta J_N}{\delta b_i} &= \frac{2}{N} \sum_{t=1}^{N} \left(-\cos(\omega_i t)\right) \left(y_i(t) - a_i \sin(\omega_i t) - b_i \cos(\omega_i t)\right) &&= 0
\end{aligned}$$

Which results in the following linear system:

$$\begin{bmatrix} \sum_{t=1}^{N} \sin(\omega_i t)^2 & \sum_{t=1}^{N} \sin(\omega_i t)\cos(\omega_i t) \\ \sum_{t=1}^{N} \sin(\omega_i t)\cos(\omega_i t) & \sum_{t=1}^{N} \cos(\omega_i t)^2 \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^{N} y_i(t)\sin(\omega_i t) \\ \sum_{t=1}^{N} y_i(t)\cos(\omega_i t) \end{bmatrix}$$

4. We want to move back to sin-only form:

$$\hat{\phi}_i = \arctan\left(\frac{\hat{b}_i}{\hat{a}_i}\right)$$

$$\hat{B}_i = \frac{\frac{\hat{a}_i}{\cos \hat{\phi}_i} + \frac{\hat{b}_i}{\sin \hat{\phi}_i}}{2}$$

5. Repeating H experiments we obtain

$$\{\hat{B}_1, \hat{\phi_1}\} \Rightarrow \frac{\hat{B}_1}{A_1} e^{j\hat{\phi_1}}$$

$$\vdots$$

$$\{\hat{B_H}, \hat{\phi_H}\} \Rightarrow \frac{\hat{B_H}}{A_H} e^{j\hat{\phi_H}}$$

So we have H complex numbers representing the frequency response of $W(z)$. These numbers are our dataset

4.2 Model class selection

$$\mathcal{M}(\theta) : W(z,\theta) = \frac{b_0 + b_1 z^{-1} + \cdots + b_p z^{-p}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} z^{-1} \qquad \theta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_0 \\ \vdots \\ b_p \end{bmatrix}$$

4.3 Performance index

$$J_H(\theta) = \frac{1}{H} \sum_{i=1}^{H} \left(W(e^{j\omega_i}, \theta) - \frac{\hat{B}_i}{A_i} e^{j\hat{\phi_i}} \right)^2$$

$$\hat{\theta} = \arg\min_{\theta} J_H(\theta)$$

Chapter 5

Kalman filter

Based on SS representation:

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) + v_1(t) & v_1 \sim WN \\ y(t) = Hx(t) + \cancel{Du(t)} + v_2(t) & v_2 \sim WN \end{cases}$$

5.1 Motivations and Goals

Given a model and noise variances:

- find k-steps ahead predictors of the output y
- find k-steps ahead predictors of the state x
- Find the filter of the state $\hat{x}(t|t)$ to allow software sensing
- Gray box system identification

Usually a dynamic system has m inputs, n states and p outputs

Key problem Usually $p << n$: pysical sensors are much less than system states because:

- Cost
- Cables, power supply
- Maintenance

But we want full state measurements because:

- Control design (using state feedback)
- Monitoring (fault detection, predictive maintenance)

Software sensing determine the internal state using the values measured from input and output

5.1.1 Kalman on Basic Systems

$$S : \begin{cases} x(t+1) = Fx(t) + \cancel{Gu(t)} + v_1(t) & \text{state equation} \\ y(t) = Hx(t) + v_2(t) & \text{output equation} \end{cases}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \cancel{u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}} \qquad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

v_1 is a vector white noise:

$$v_1 \sim WN(0, V_1) \qquad v_1(t) = \begin{bmatrix} v_1 1(t) \\ \vdots \\ v_1 n(t) \end{bmatrix}$$

- $E\left[v_1(t)\right] = \vec{0}$
- $E\left[v_1(t) \cdot v_1(t)^T\right] = V_1$, where V_1 is an $n \times n$ covariance matrix
- $E\left[v_1(t) \cdot v_1(t-\tau)^T\right] = 0 \quad \forall t, \forall \tau \neq \vec{0}$

v_2 is called output/measurement/sensor noise:

- $E\left[v_2(t)\right] = \vec{0}$
- $E\left[v_2(t) \cdot v_2(t)^T\right] = V_2$, where V_1 is an $n \times n$ covariance matrix
- $E\left[v_2(t) \cdot v_2(t-\tau)^T\right] = 0 \quad \forall t, \forall \tau \neq \vec{0}$

v_1 and v_2 are assumed to have the following relationships:

$$E\left[v_1(t) \cdot v_2(t-\tau)^T\right] = \underbrace{V_{12}}_{n \times p} = \begin{cases} 0 & \text{if } \tau \neq 0 \\ \text{any} & \text{if } \tau = 0 \end{cases}$$

So they can only be correlated in the same time istant. Since the system is dynamic we need to define its initial conditions:

$$E\left[x(1)\right] = \underbrace{X_0}_{n \times 1} \qquad E\left[(x(1) - x(0))(x(1) - x(0))^T\right] = \underbrace{P_0}_{n \times n} \geq 0$$

$P_0 = 0 \iff$ the initial state is perfectly known.
We finally assume that v_1 and v_2 are uncorrelated with the initial state:

$$x(1) \perp v_1(t) \qquad x(1) \perp v_2(t)$$

Solution for Basic Systems

$$
\begin{array}{rlr}
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + K(t)e(t) & \text{state equation} \\
\hat{y}(t|t-1) &= H\hat{x}(t|t-1) & \text{output equation} \\
e(t) &= y(t) - \hat{y}(t|t-1) & \text{prediction error} \\
K(t) &= \left(FP(t)H^T + V_{12}\right)\left(HP(t)H^T + V_2\right)^{-1} & \text{gain of the KF} \\
P(t+1) &= \left(FP(t)F^T + V_1\right) & \\
&\quad - \left(FP(t)H^T + V12\right)\left(HP(t)H^T + V_2\right)^{-1}\left(FP(t)H^T + V_{12}\right)^T & \text{difference Riccati equation} \\
\hline
\hat{x}(1|0) &= E\left[x(1)\right] = X_0 & \text{Initial state} \\
P(1) &= var\left[x(1)\right] = P_0 & \text{initial DRE}
\end{array}
$$

5.1.2 Exogenous input

$$
\begin{array}{rlr}
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + Gu(t) + K(t)e(t) & \text{state equation} \\
\text{other equations} &= \text{unchanged} &
\end{array}
$$

5.1.3 Multi-step prediction

Knowing $\hat{x}(t+1|T)$ from the basic solution we can derive

$$
\begin{array}{rl}
\hat{x}(t+2|t) &= F\hat{x}(t+1|t) \\
\hat{x}(t+2|t) &= F^2\hat{x}(t+1|t) \\
&\vdots \\
\hat{x}(t+k|t) &= F^{k-1}\hat{x}(t+1|t) \\
\hline
\hat{y}(t+k|t) &= H\hat{x}(t+k|t)
\end{array}
$$

5.1.4 Filter $(\hat{x}(t|t)$

F invertible

$$
\hat{x}(t+1|t) = F\hat{x}(t|t) \qquad \Longrightarrow \qquad \hat{x}(t|t) = F^{-1}\hat{x}(t+1|t)
$$

F not invertible assuming $V_{12} = 0$, then we can re-formulate the K.F. solutions:

$$
\begin{aligned}
\hat{x}(t|t) &= F\hat{x}(t-1|t-1) + Gu(t-1) + K_0(t)e(t) \\
\hat{y}(t|t-1) &= H\hat{x}(t|t-1) \\
e(t) &= y(t) - \hat{y}(t|t-1) \\
K_0(t) &= \left(P(t)H^T\right)\left(HP(t)H^T + V_2\right)^{-1} \\
P(t+1) &= \text{unchanged}
\end{aligned}
$$

5.1.5 Time-varying systems

$$S : \begin{cases} x(t+1) = F(t)x(t) + G(t)u(t) + v_1(t) \\ y(t) = H(t)x(t + v_2(t) \end{cases}$$

K.F. equations are unchanged

5.1.6 Non linear system

Much more complicated extension. Look for Extended Kalman Filter if interested (I'm not)

5.2 Asymptotic solution of K.F.

KF is time variant, because the gain $K(t)$ is time varying. This causes 2 problems:

- It is difficult to check the stability of the system
- $K(t)$ must be computed at each sampling time, including the inversion of $\left(HP(t)H^T\right)_{p\times p} \Rightarrow$ computationally intensive

Because of this, the asymptotic version of KF is preferred

5.2.1 Basic idea

If $P(t)$ converges to constant $\overline{P}$, then also $K(t)$ will converge to some constant $\overline{K}$. Using $\overline{K}$ instead of $K(t)$ the KF becomes time-invariant:

$$\begin{aligned} \hat{x}(t+1|t) &= F\hat{x}(t|t-1) + Gu(t) + \overline{K}e(t) \\ &= F\hat{x}(t|t-1) + Gu(t) + \overline{K}\left(y(t) - \hat{y}(t|t-1)\right) \\ &= F\hat{x}(t|t-1) + Gu(t) + \overline{K}\left(y(t) - H\hat{x}(t|t-1)\right) \\ &= \underbrace{\left(F - \overline{K}H\right)}_{\text{new state matrix}} \hat{x}(t|t-1) + Gu(t) + \overline{K}y(t) \end{aligned}$$

If $\overline{K}$ exists, thes the KF is asymptotically stable $\iff$ all the eigenvalues of $F - \overline{K}H$ are stricly inside the unit circle

5.2.2 Existance of $\overline{K}$

$$\overline{K} = \left(F\overline{P}H^T + V_{12}\right) + \left(H\overline{P}H^T + V_2\right)^{-1}$$

$\overline{K}$ exists if $\overline{P}$ exists. DRE is an autonomous discrete time system $x(t+1) = f(x(t))$, in equilibrium when $x(t+1) = x(t) \Rightarrow f(\overline{x}) = \overline{x}$. Applyed to P this leads to the following Algebraic Riccardi Equation:

$$\overline{P} = f(\overline{P}) \iff \overline{P} = \left(F\overline{P}F^T + V_1\right) - \left(F\overline{P}H^T + V_{12}\right)\left(H\overline{P}H^T + V_2\right)^{-1}\left(F\overline{P}H^T + V_{12}\right)^T$$

First asymptotic theorem

Assuming $V_{12} = 0$ and the system is asymptotically stable (all eigenvalues of F strictly inside the unit circle), then:

- $\exists!$ semi-definite positive solution of ARE: $\overline{P} \geq 0$
- DRE converges to $\overline{P} \quad \forall P_0 \geq 0$
- The corresponding $\overline{K}$ will make the KF asymptotically stable

Second asymptotic theorem

Assuming $V_{12} = 0$, (F, H) is observable, (F, Γ) is controllable. Then:

- $\exists!$ semi-definite positive solution of ARE: $\overline{P} \geq 0$
- DRE converges to $\overline{P} \quad \forall P_0 \geq 0$
- The corresponding $\overline{K}$ will make the KF asymptotically stable

5.3 Extension to non-linear systems

$$S: \begin{cases} x(t+1) = f(x(t), u(t)) + v_1(t) \\ y(t) = h(x(t)) + v_2(t) \end{cases}$$

Where f and h are non-linear functions.
For the gain block of the KF we have 2 types of solutions:

- The gain is a non linear function of $e(t)$
- The gain is a linear time-varying function

The second solution is preferred, as it allows us to reuse the formulae with just little tweaks. In particular, F and H are computed as follows:

$$F(t) = \left. \frac{\delta f(x(t), u(t))}{\delta x(t)} \right|_{x(t)=\hat{x}(t|t-1)}$$

$$H(t) = \left. \frac{\delta h(x(t))}{\delta x(t)} \right|_{x(t)=\hat{x}(t|t-1)}$$

EKF is the time-verying solution of KF, where F and H are computed around the last available state prediction $\hat{x}(t|t-1)$

Algorithm

1. Take the last available state prediction $\hat{x}(t|t-1)$
2. Use $\hat{x}(t|t-1)$ to compute $F(t)$ and $H(t)$
3. Compute $K(t)$ and update the DRE equations
4. Compute $\hat{x}(t+1|t)$

5.4 Optimization of gain K

$$S : \begin{cases} x(t+1) = 2x(t) \\ y(t) = x(t) + v(t) \quad v \sim WN(0,1) \end{cases}$$

5.4.1 Direct solution

Starting from the standard observer structure:

$$\begin{cases} \hat{x}(t+1|t) = 2\hat{x}(t|t-1) + K(y(t) - \hat{y}(t|t-1)) \\ \hat{y}(t|t-1) = \hat{x}(t|t-1) \end{cases}$$

Minimizing the variance of the prediction error $var[\eta(t)] \Rightarrow$ minimizing $var[x(t) - \hat{x}(t|t-1]$

$$\begin{aligned} \eta(t) &= 2x(t) - [2\hat{x}(t|t-1) + K(y(t) - \hat{y}(t|t-1)] \\ &= 2x(t) - 2\hat{x}(t|t-1) - K(x(T) + v(t) - \hat{x}(t|t-1)) \\ &= (2-K)(x(t) - \hat{x}(t|t-1)) - Kv(t) \\ \eta(t+1) &= (2-K)\eta(t) - Kv(t) \qquad\qquad v \sim WN(0,1) \end{aligned}$$

This is an AR(1) process:

$$\eta(t) = \frac{1}{1-(2-K)z^{-1}} e(t) \qquad e(t) = -Kv(t) \qquad e \sim WN(0, K^2)$$

The variance of η is

$$\gamma_\eta(0) = \frac{K^2}{1-(2-K)^2}$$

Minimizing wrt K:

$$\frac{\delta\gamma_\eta(0)}{\delta K} = 0 \qquad \Rightarrow \qquad \begin{cases} K_1 = 0 \\ K_2 = \frac{3}{2} \end{cases}$$

5.4.2 KF theory solution

From S we can derive:

$$\begin{cases} F = 2 \\ H = 1 \\ V_1 = 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} \Gamma = 0 \\ V_2 = 1 \\ V_{12} = 0 \end{cases}$$

- F is not asymptotically stable $\Rightarrow$ cannot use theorem 1
- (F, Γ) is not fully reachable $\Rightarrow$ cannot use theorem 2

$$DRE = 4P(t) - \frac{(2P(t))^2}{P(t)+1} \dots$$

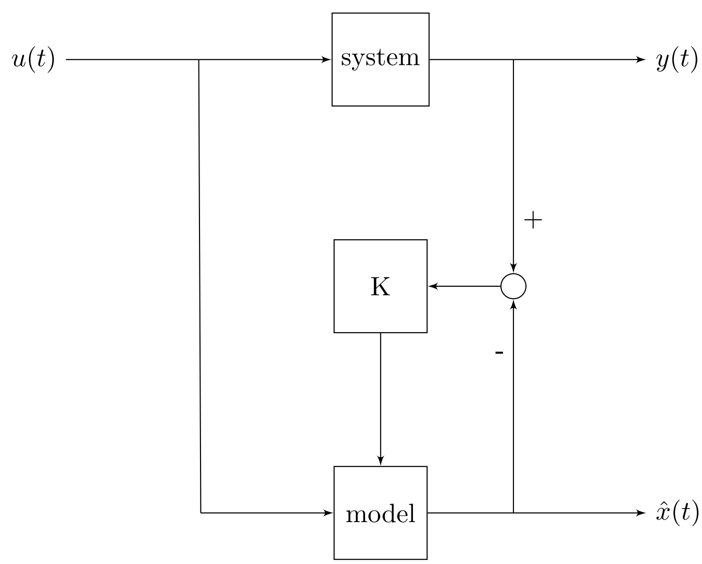
$$P(t+1) = \frac{4P(t)}{P(t)+1}$$

Solving ARE:

$$\overline{P} = \frac{4\overline{P}}{\overline{P}+1} \quad \Rightarrow \quad \begin{cases} \overline{P_1} = 1 \\ \overline{P_2} = 3 \end{cases} \quad \Rightarrow \quad \begin{cases} K_1 = 0 \\ K_2 = \frac{3}{2} \end{cases}$$

Chapter 6

Software-sensing with Blabk box Methods



Features

- A white-box model is required
- No need of a training dataset
- Works by feedback estimation
- Constructive method
- Can be used to estimate unmeasurable states

6.1 Linear Time Invariant Systems

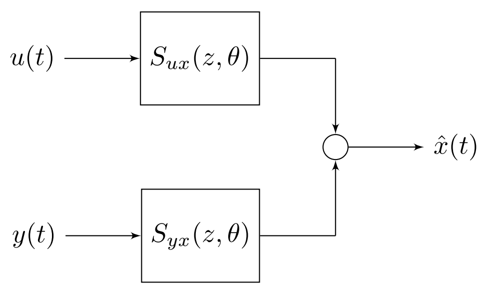
Known white-box model of the system Draw the block diagram of the system and the KF, then (from the diagram) calculate $\hat{x}(t) = f(u(t), y(t))$. Done.

Unknown model for the system A BB estimation is possible iff all the states are measurable.

Dataset

$$\begin{aligned}&\{u(1), \dots, u(N)\}\\&\{y(1), \dots, y(N)\}\\&\{x(1), \dots, x(N)\}\end{aligned}$$

Model to be optimized fot θ



Performance index

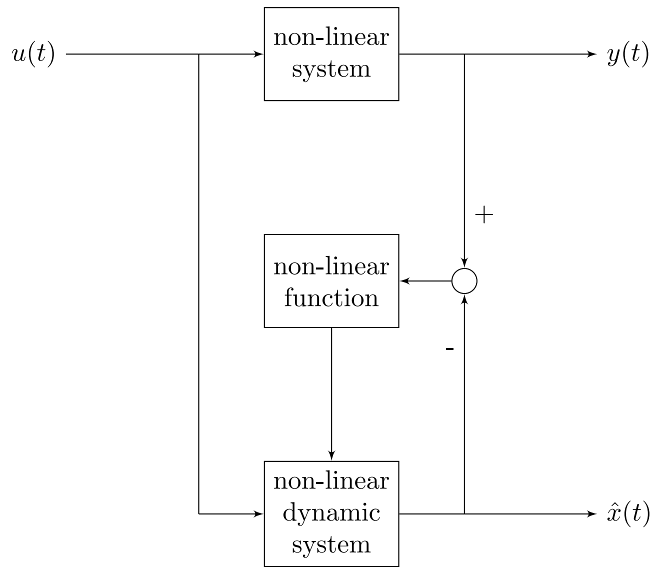
$$J_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} (x(t) - (S_{ux}(z, \theta)u(t) + S_{yx}(z, \theta)y(t)))^2$$

Optimization

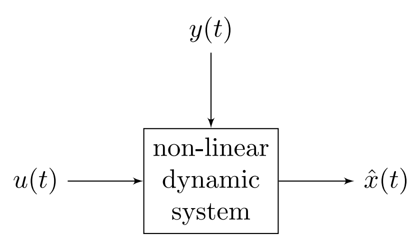
$$\hat{\theta}_N = \arg\min_{\theta} J_N(\theta)$$

We get $S_{ux}(z, \hat{\theta}_N)$ and $S_{yx}(z, \hat{\theta}_N)$, the transfer functions for our software sensors

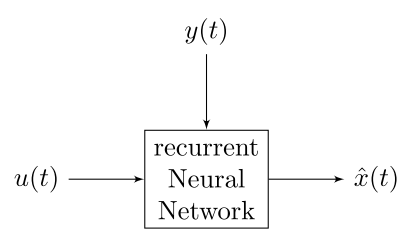
6.2 Non-linear systems



Model

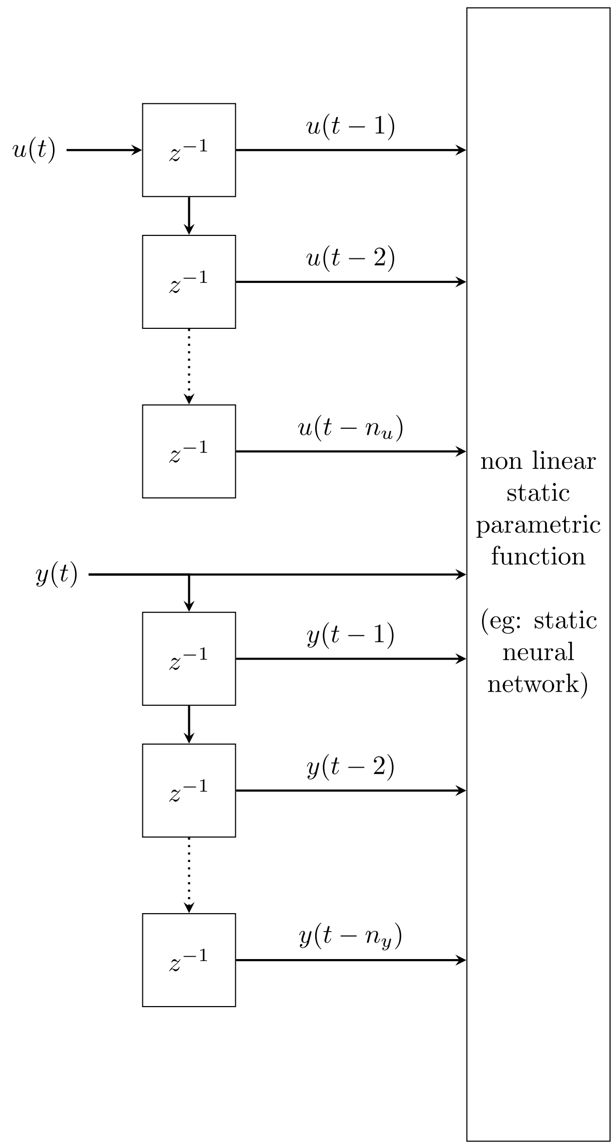


6.2.1 Recurrent neural network



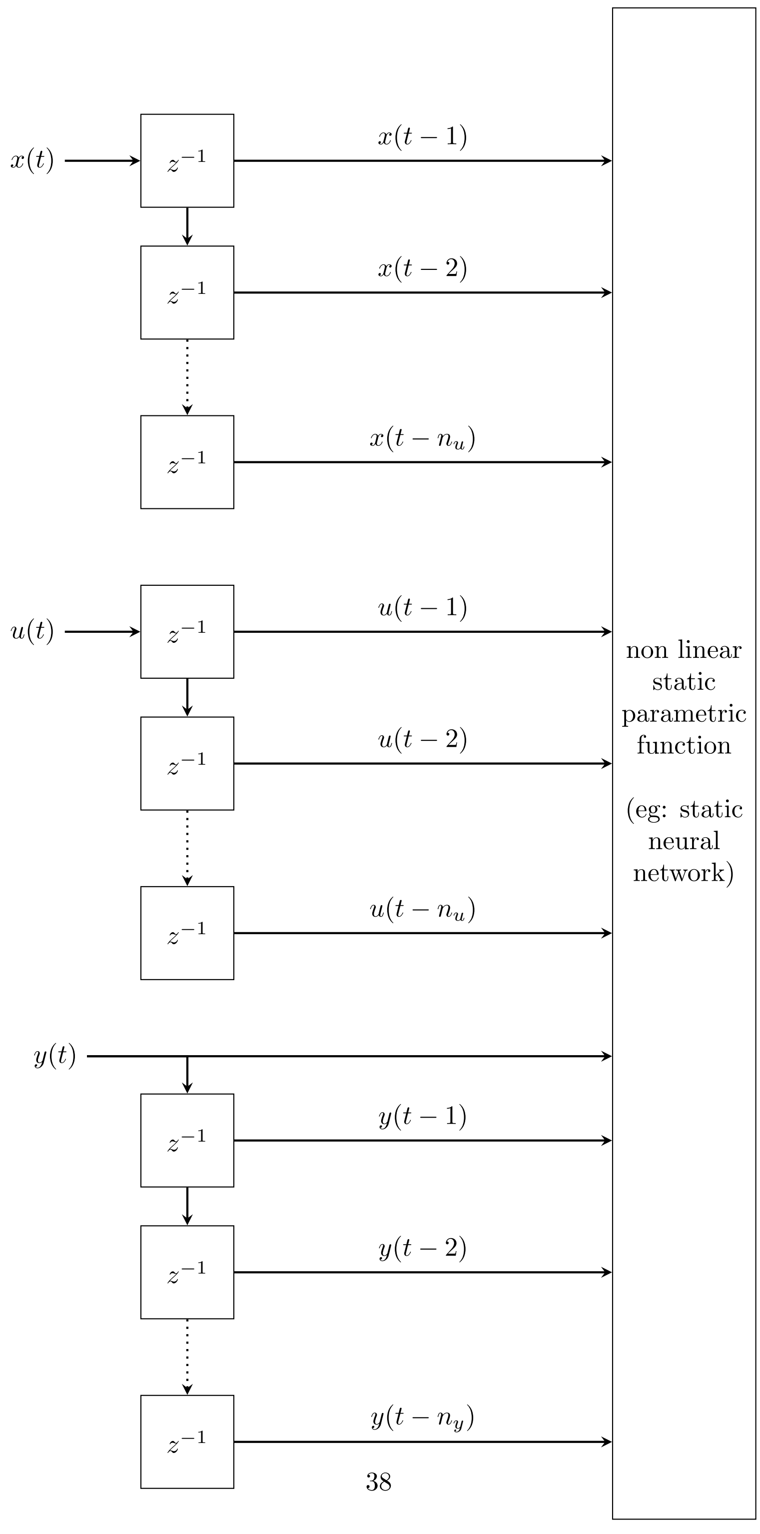
6.2.2 FIR architecture

split the system into a static non-linear system and linear dynamics



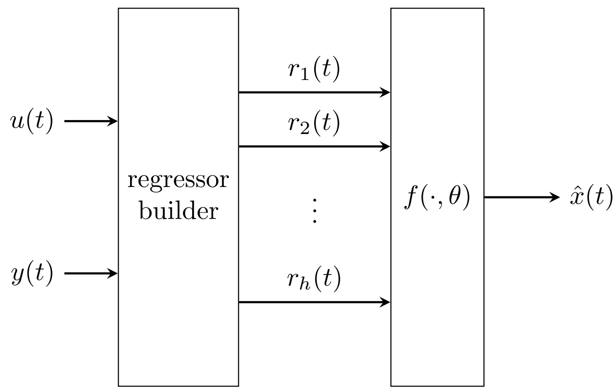
$u(t)$
z^{-1}
$u(t-1)$
z^{-1}
$u(t-2)$
z^{-1}
$u(t-n_u)$
$y(t)$
z^{-1}
$y(t-1)$
z^{-1}
$y(t-2)$
z^{-1}
$y(t-n_y)$
non linear static parametric function
(eg: static neural network)

6.2.3 IRR scheme



6.2.4 Physical regressors

Using the physical knowledge of the system, provide a set of regressors (ie: smaller and more meaningful set of signals) elaborated by a static non-linear system



Chapter 7

Grey-box System Identification

7.1 Using Kalman Filter

7.1.1 Problem definition

- We have a model:

$$S : \begin{cases} x(t+1) = f(x(t), u(t), \theta) + v_1(t) \\ y(t) = h(x(t), \theta) + v_2(t) \end{cases}$$

- f and h are functions (linear or not) depending on some unknown parameter θ carrying physical meaning (mass, resistance,...)
- We want to estimate $\hat{\theta}$
- This is archieved by managing the unknown parameters as extended states

7.1.2 State extension

$$S : \begin{cases} x(t+1) = f(x(t), u(t), \theta(t)) + v_1(t) \\ \theta(t+1) = \theta(t) + v_\theta(t) \\ y(t) = h(x(t), \theta(t)) + v_2(t) \end{cases}$$

And the extended state vector is $x_E = \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix}$
The noise in the equation of θ is added to prevent the KF form getting stuck on the initial conditions.

7.1.3 Design choice

The choice of the covariance matrix of $v_\theta \sim WN(0, V_\theta)$

- Assume $v_1 \perp v_\theta$ and $v_2 \perp v_\theta$:

$$V_\theta = \begin{bmatrix} \lambda_{1\theta}^2 & & & \\ & \lambda_{2\theta}^2 & & \\ & & \ddots & \\ & & & \lambda_{n_\theta\theta}^2 \end{bmatrix}_{n_\theta \times n_\theta}$$

- Usually, it is assumed that $\lambda_{i\theta} = \lambda_{j\theta} \quad \forall i \forall j$
- Assume that $v_\theta(t)$ is a set of <u>independent</u> WN all with the same variance λ_θ^2
- Bigger values of λ_θ^2 lead to a quicker convergence, but less stable (stronger obscillations around the steady-state)
- The selection of λ_θ^2 is leaded by application-specific constraints

7.1.4 Applicability

In theory, this trick can work with any number of sensors, states, and parameters. In practice it works well only on a limited number of parameters (~3 sensors, 5 states, 2 parameters)

7.2 Simulation Error Method

$$u(t) \longrightarrow \boxed{\begin{array}{c}\text{Model with}\\ \text{some unknown}\\ \text{parameters}\end{array}} \longrightarrow y(t)$$

7.2.1 Dataset

from an experiment, collect:

$$\{\overline{u}(1), \overline{u}(2), \dots, \overline{u}(N)\}$$
$$\{\overline{y}(1), \overline{y}(2), \dots, \overline{y}(N)\}$$

7.2.2 Model

$$y(t) = \mathcal{M}(u(t), \overline{\theta}, \theta)$$

$\overline{\theta}$ is the set of known parameters, θ the set of unknown parameters

7.2.3 Performance index

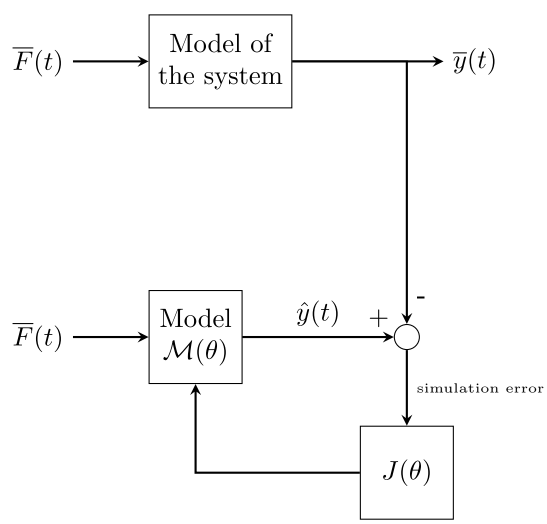
$$J_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}\left(\overline{y}(t) - \mathcal{M}(\overline{u}(t), \overline{\theta}, \theta)\right)^2$$

7.2.4 Optimization

$$\hat{\theta}_N = \arg\min_{\theta} J_N(\theta)$$

7.2.5 Limitations

- Usually J_N has no analytic expression
- Computing the value of J_N requires an entire simulation from $t = 1$ to $t = N$
- Usually J_N is non-quadratic and non-convex $\rightarrow$ iterative and randomizad optimization must be used
- Computationally demanding



Chapter 8

Minimum Variance Control

The goal is to design a feedback system

- Control design is the ain motivation of system identification and software sensing
- MVC is based on prediction theory

8.1 Setup the problem

Consider a generic ARMAX model:

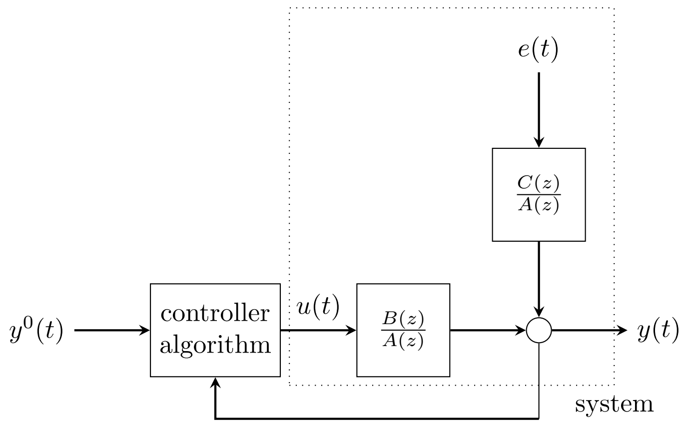
$$y(t) = \frac{B(z)}{A(z)}u(t-k) + \frac{C(z)}{A(z)}e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

$$\begin{aligned} B(z) &= b_0 + b_1 z^{-1} + \cdots + b_p z^{-p} \\ A(z) &= 1 + a_1 z^{-1} + \cdots + a_m z^{-m} \\ C(z) &= 1 + c_1 z^{-1} + \cdots + c_n z^{-n} \end{aligned}$$

Assumptions

- $\frac{C(z)}{A(z)}$ is in canonical form
- $b_0 \neq 0 \rightarrow$ k is the actual delay
- $\frac{B(z)}{A(z)}$ is minimum phase $\iff$ all the roots of $B(z)$ are strictly inside the unit circle
- $y^0(t) \perp e(t)$
- $y^0(t)$ is known only up to the present time ($y^0(t)$ is totally unpredictable)

We want to determine the optimal tracking of the desired output behaviour:



Formally, MVC tries to minimize the following performance index:

$$J = E\left[(y(t) - y^0(t))^2\right)$$

Which is the variance of the tracking error

8.1.1 Simplified problem 1

$$S : y(t) = ay(t-1) + b_0u(t-1) + b_1u(t-2) \qquad y(t) = \frac{b_0 + b_1z^{-1}}{1 - az^{-1}}u(t-1)$$

Assuming:

- $y^0(t) = \overline{y}^0$
- no noise
- $b_0 \neq 0$
- Root of numerator inside the unit circle

$$\begin{aligned} J &= \left(y(t) - y^0(y)\right)^2 \\ &= \left(y(t) - \overline{y}^0\right)^2 \\ &= \Big(ay(t-1) + b_0u(t-1) + b_1u(t-2) - \overline{y^0}\Big)^2 \\ &= \Big(ay(t) + b_0u(t) + b_1u(t-1) - \overline{y^0}\Big)^2 \end{aligned}$$

$$\frac{\delta J}{\delta u(t)} = 2\left(ay(t) + b_0u(t) + b_1\underbrace{u(t-1)}_{\substack{\text{number}\\\text{not variable}}} - \overline{y^0}\right)(b_0)$$

$$\frac{\delta J}{\delta u(t)} = 0 \implies ay(t) + b_0u(t) + b_1u(t-1) - \overline{y^0} = 0 \implies u(t) = (\overline{y}^0 - ay(t))\frac{1}{b_0 + b_1z^{-1}}$$

8.1.2 Simplified problem 2

$$S: y(t) = ay(t-1) + b_0 u(t-1) + b_1 u(t-2) + e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

Reference variable $y^0(t)$

Performance index $E\left[(y(t) - y^0(t))^2\right]$

Trick rewrite $y(t)$ as

$$y(t) = \hat{y}(t|t-1) + \epsilon(t)$$

$$k = 1 \implies \epsilon(t) = e(t) \implies y(t) = \hat{y}(t|t-1) + e(t)$$

$$\begin{aligned} J &= E\left[\left(\hat{y}(t|t-1) + e(t) - y^0(t)\right)^2\right] \\ &= E\left[\left((\hat{y}(t|t-1) - y^0(t)) + e(t)\right)^2\right] \\ &= E\left[\left(\hat{y}(t|t-1) - y^0(t)\right)^2\right] + E\left[e(t)^2\right] + \cancel{2E\left[e(t)(\hat{y}(t|t-1) - y^0(t))\right]} \end{aligned}$$

$$\arg\min_{y^0(t)} E\left[\left(\hat{y}(t|t-1) - y^0(t)\right)^2\right] + E\left[e(t)^2\right] = \arg\min_{y^0(t)} E\left[\left(\hat{y}(t|t-1) - y^0(t)\right)^2\right]$$

The best result is when $\hat{y}(t|t-1) = y^0(t)$. Writing the system in terms of transfer functions we get:

$$S: y(t) = \frac{b_0 + b_1 z^{-1}}{1 - az^{-1}} u(t-1) + \frac{1}{1 - az^{-1}} e(t)$$

Applying the general 1-step predictor for $ARMAX$:

$$\hat{y}(t|t-1) = \frac{b_0 + b_1 z^{-1}}{1} u(t-1) + \frac{1 - 1 + az^{-1}}{1} y(t) = (b_0 b_1 z^{-1}) u(t-1) + ay(t-1)$$

Imposing the optimality condition we get:

$$\begin{aligned} b_0 u(t-1) + b_1 u(t-2) + ay(t-1) &= y^0(t) \\ b_0 u(t) + b_1 u(t-1) + ay(t) &= y^0(t+1) \\ u(t) &= \left(y^0(t+1) - ay(t)\right) \frac{1}{b_0 b_1 z^{-1}} \end{aligned}$$

Since we cannot predict the future we must approximate:

$$u(t) \approx \left(y^0(t) - ay(t)\right) \frac{1}{b_0 + b_1 z^{-1}}$$

8.1.3 General solution

$$S : y(t) = \frac{B(z)}{A(z)}u(t-k) + \frac{C(z)}{A(z)}e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

Assumptions

- $b_0 \neq 0$
- $B(z)$ has all roots inside the unit circle
- $\frac{C(z)}{A(z)}$ is in canonical form
- $y^0(t) \perp e(t)$
- $y^0(t)$ is unpredictable

Trick rewrite $y(t) = \hat{y}(t|t-k) + \epsilon(t)$

$$\begin{aligned}
J &= E\left[\left(\hat{y}(t|t-k) + \epsilon(t) - y^0(t)\right)^2\right] \\
&= E\left[\left((\hat{y}(t|t-k) - y^0(t)) + \epsilon(t)\right)^2\right] \\
&= E\left[\left(\hat{y}(t|t-k) - y^0(t)\right)^2\right] + E\left[\epsilon(t)^2\right] + \cancel{2E\left[\epsilon(t)\left(\hat{y}(t|t-k) - y^0(t)\right)\right]} \\
&= E\left[\left(\hat{y}(t|t-k) - y^0(t)\right)^2\right] + \textbf{constant}
\end{aligned}$$

Chapter 9

Recursive Identification

9.1 Least square

$$\hat{\theta}_N = \arg\min_{\theta} \left\{ J_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} (y(t) - \hat{y}(t|t-1,\theta))^2 \right\}$$

We want to find the predictor model $\hat{y}(t|t-1,\theta)$

$$y(t) = \phi(t)^T\theta + e(t)$$

Since $e(t)$ is unpredictable, the best possible 1-step predictor is $\hat{y}(t|t-1,\theta) = \phi(t)^T\theta$. Substituting in J:

$$J_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} \left(y(t) - \phi(t)^T\theta\right)^2$$

Analytically we can find the minimum:

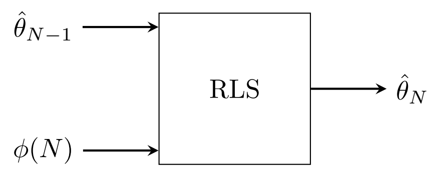
$$\frac{\delta J_N(\theta)}{\delta\theta} = 0$$

$$\hat{\theta}_N = S(N)^{-1} \sum_{t=1}^{N} \phi(t)y(t)$$

$$S(N) = \sum_{t=1}^{N} \phi(t)\phi(t)^T$$

But the procedure must be repeated at each t step

9.1.1 Recursive Least Square



9.1.2 First form

$$
\begin{aligned}
\hat{\theta}_N &= S(N)^{-1}\sum_{t=1}^{N}\phi(t)y(t) \\
S(N)\hat{\theta}_N &= \sum_{t=1}^{N}\phi(t)y(t) \\
S(N-1)\hat{\theta}_{N-1} &= \sum_{t=1}^{N-1}\phi(t)y(t) \\
\sum_{t=1}^{N}\phi(t)y(t) &= \sum_{t=1}^{N-1}\phi(t)y(t) + \phi(N)y(N) \\
\sum_{t=1}^{N}\phi(t)y(t) &= S(N-1)\hat{\theta}_{N-1} + \phi(N)y(N) \\
&\boxed{\hat{\theta}_N = \hat{\theta}_{N-1} + S(N)^{-1}\phi(N)\left[y(N) - \phi(N)^T\hat{\theta}_{N-1}\right]} \\
S(N) &= S(N-1) + \phi(N)\phi(N)^T
\end{aligned}
$$

However, this way $S(N) \to \infty$, so the domputing unit saturates.

9.1.3 Second form

Normalize wrt. N

$$
\begin{aligned}
S(N) &= S(N-1) + \phi(N)\phi(N)^T \\
R(N) &= \frac{1}{N}S(N) \\
R(N) &= \frac{N-1}{N}R(N-1) + \frac{1}{N}\phi(N)\phi(N)^T \\
K(N) &= \frac{1}{N}R(N)^{-1}\phi(N) \\
\epsilon(N) &= y(N) - \phi(N)^T\hat{\theta}_{N-1} \\
&\boxed{\hat{\theta}_N = \hat{\theta}_{N-1} + K(N)\epsilon(N)}
\end{aligned}
$$

This however reqires matrix inversion at each iteration, which is expensive

9.1.4 Third form

It can be proven that the following works

$$\epsilon(N) = y(N) - \phi(N)^T \hat{\theta}_{N-1}$$
$$\boxed{V(N) = V(N-1) - \frac{V(N-1)\phi(N)\phi(N)^T V(N-1)}{1 + \phi(N)^T V(N-1)\phi(N)}}$$
$$\boxed{K(N) = V(N)\phi(N)}$$
$$\hat{\theta}_N = \hat{\theta}_{N-1} + K(N)\epsilon(N)$$

This does not require matrix inversion

9.1.5 Forgetting factor

If theta contains some parameter changing over time, the objective function should "forget" old data. The new objective function becomes:

$$J_N(\theta) = \frac{1}{N} \rho^{N-t} \left(\hat{y}(t|t-1, \theta)\right)^2$$

Where $\rho \in [0, 1]$ is the forgetting factor (the smaller, the more forgetful). The new equations become:

$$\epsilon(N) = y(N) - \phi(N)^T \hat{\theta}_{N-1}$$
$$\boxed{S(N) = \rho S(N-1) + \phi(N)\phi(N)^T}$$
$$K(N) = S(N)^{-1}\phi(N)$$
$$\hat{\theta}_N = \hat{\theta}_{N-1} + K(N)\epsilon(N)$$

Part III

Cheatsheet

Appendix A

MIDA 1

A.1 Probability Recall

Cross-Variance $Var[v, u] = E[(v - E[v])(u - E[u])]$

Variance Matrix $\begin{vmatrix} Var[v_1] & . & . & Var[v_1, v_k] \\ . & . & & . \\ . & & . & . \\ Var[v_k, v_1] & . & . & Var[v_k] \end{vmatrix}$

Covariance coefficient $\delta[i, j] = \frac{Var[i,j]}{\sqrt{Var[i]}\sqrt{Var[j]}}$

Stationary process

- m constant
- λ^2 constant
- covariance $\gamma(\tau)$ depends only on time difference
- $|\gamma(\tau)| \leq \gamma(0) \quad \forall \tau$

White noise $\eta(t) \sim WN(m, \lambda^2)$

- Stationary process
- $\gamma(\tau) = 0 \quad \forall \tau \neq 0$
- $v(t) = \alpha\eta(t) + \beta \implies v(t) \sim WN(\beta, \alpha^2\lambda^2)$

Canonical representation

- Monic
- Same degree
- Coprime
- Poles and zeros in unit disk

A.2 Spectral analysis

Spectrum

- $\Gamma(\omega) = \gamma(0) + 2cos(\omega)\gamma(1) + 2cos(2\omega)\gamma(2) + ...$
- Periodic $T = 2\pi$
- Even
- $\Gamma_\eta(\omega) = \gamma_\eta(0) = \lambda^2$

Complex spectrum

- $\Phi(z) = \sum_{\tau=-\infty}^{+\infty} \omega(\tau) z^{-\tau}$
- $\Gamma(\omega) = \Phi(e^{j\omega})$

Fundamental theorem of spectral analysis

- $\Gamma_{\text{out}}(\omega) = |W(e^{j\omega})|^2 \cdot \Gamma_{\text{in}}(\omega)$
- $\Phi_{\text{out}}(z) = W(z)W(z^{-1}) \cdot \Phi_{\text{in}}(z)$

A.3 Moving Average MA(n)

- $W(z) = \frac{c_0 z^n + c_1 z^{n-1} + ... + c_n}{z_n}$
- $m = 0$
- $\gamma(\tau) = \begin{cases} (c_0 c_\tau + c_1 c_{1+\tau} + ... + c_{n-\tau} c_\tau)\lambda^2 & |\tau| \leq n \\ 0 & \text{otherwise} \end{cases}$

A.3.1 MA(∞)

- $\gamma(0) = (c_0^2 + c_1^2 + ... + c_k^2 + ...)\lambda^2$
- $\gamma(0)$ must converge to a finite value

A.4 Auto Regressive AR(n)

- $m = 0$
- $W(z) = \frac{z^n}{z^n - a_1 z_{n-1} - \ldots - a_n}$
- Covariance calculated by its definition

A.5 Known predictors

Appendix B

MIDA 2

AR(1) $\hat{v}(t|t-r) = a^r v(t-r)$

MA(1) $\hat{v}(t|t-1) = v(t-1) - c\hat{v}(t-1|t-2)$

MA(n) $\hat{v}(t|t-\mathbf{k}) = 0 \quad \forall k > n$

ARMA(n_a, n_b) $\hat{v}(t|t-1) = \frac{C(z)-A(z)}{C(z)} v(t)$

ARMAX(n_a, n_b) $\hat{y}(t|t-1) = \frac{C(z)-A(z)}{C(z)} y(t) + \frac{B(z)}{C(z)} u(t-1)$

B.1 State-space representation

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) & \text{state equations} \\ y(t) = Hx(t) + Du(t) & \text{output equations} \end{cases}$$

B.2 BB representations shift

B.2.1 State space to Transfer function

$$W(z) = H(zI - F)^{-1}G$$

B.2.2 Transfer Function to State Space

$$W(z) = \frac{b_0 z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1}}{z^n + a_0 z^{n-1} + \cdots + a_n}$$

$$\Longrightarrow$$

$$F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & \dots & \dots & -a_1 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_0 \end{bmatrix} \quad D = 0$$

B.2.3 Transfer function to Impulse response

∞ long division of $W(z)$

B.2.4 Impulse response to Tranfer function

Z-transform

$$\mathcal{Z} = \sum_{t=0}^{\infty} s(t) z^{-t}$$

$$W(z) = \mathcal{Z}(\omega(t)) = \sum_{t=0}^{\infty} \omega(t) z^{-1}$$

B.2.5 State space to Impulse response

$$\omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ HF^{t-1}G & \text{if } t > 0 \end{cases}$$

B.3 Controllability and Observability

B.3.1 Fully observable system

$\iff$ O is full rank:

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \qquad rank(O) = n$$

B.3.2 Fully controllable system

Fully controllable $\iff$ R is full rank:

$$R = \begin{bmatrix} G & FG & \dots & F^{n-1}G \end{bmatrix} \qquad rank(R) = n$$

B.3.3 Hankel Matrix

Starting from $\omega(1), \omega(2), \ldots, \omega(N)$ where $N \geq 2n-1$, we can build the Hankel Matrix of order n:

$$H_n = \begin{bmatrix} \omega(1) & \omega(2) & \omega(3) & \ldots & \omega(n) \\ \omega(2) & \omega(3) & \omega(4) & \ldots & \omega(n+1) \\ \omega(3) & \omega(4) & \omega(5) & \ldots & \omega(n+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega(n) & \omega(n+1) & \omega(n+2) & \ldots & \omega(2n-1) \end{bmatrix}$$

$$H_n = \begin{bmatrix} HG & HFG & HF^2G & \ldots & HF^{n-1}G \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ HF^{n-1}G & \ldots & \ldots & \ldots & HF^{2n-2}G \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \cdot \begin{bmatrix} G & FG & \ldots & F^{n-1}G \end{bmatrix} = O \cdot R$$

Appendix C

System Identification

Impulse experiment Measure $y(t)$ under the input $u(t) = \text{impulse}(0)(0)$

C.1 F, G, H from noise-free IR

1. $$H_1 = \begin{bmatrix} \omega(1) \end{bmatrix} \quad H_2 = \begin{bmatrix} \omega(1) & \omega(2) \\ \omega(2) & \omega(3) \end{bmatrix} \quad H_3 = \dots \quad \dots \quad H_n = \dots$$

   If $rank(H_n) = rank(H_{n+1})$, then n is the order of the system

2. Factorize H_{n+1} as $H_{n+1} = O_{n+1[(n+1)\times n]} \cdot R_{n+1[n\times(n+1)]}$

   $$O_{n+1} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^n \end{bmatrix} \qquad R_{n+1} = \begin{bmatrix} G & FG & \dots & F^nG \end{bmatrix}$$

3. $H = O[0]$, $G = R[0]$

4. Define

   $$O_1 = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \qquad O_2 = \begin{bmatrix} HF \\ \vdots \\ HF^n \end{bmatrix}$$

5. $F = O_1^{-1} O_2$

C.2 Obtain F, G, H from a noisy IR (TODO: not well understood)

The measurement is of $\hat{\omega}(t) = \omega(t) + \eta(t)$. To identify the process:

1. Build the Hankel matrix using all the N data:

$$\hat{H}_{q\times d} = \begin{bmatrix} \hat{\omega}(1) & \hat{\omega}(2) & \dots & \hat{\omega}(d) \\ \hat{\omega}(2) & \hat{\omega}(3) & \dots & \hat{\omega}(d+1) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\omega}(q) & \hat{\omega}(q+1) & \dots & \hat{\omega}(q+d+1=N) \end{bmatrix}$$

2. Calculate the Singular Value Decomposition of $\hat{H}_{q\times d}$:

$$\hat{H}_{q\times d} = \hat{U}_{q\times q} \cdot \hat{S}_{q\times d} \cdot \hat{V}^T_{d\times d}$$

$\hat{U}$ and $\hat{V}$ are unitary matrices: they are invertible and their inverses are equal to their transpose.

$$\hat{S} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{bmatrix}$$

3. Plot the singular values (σ_i)and cut-off the three matrices:

   - Ideally, after a certain n (the order of the IR) there would be a jump dividing the signal (before) from the noise (after)
   - In reality no clear distinction exists, but it's possible to identify an interval of possible values of n. A tradeoff between complexity, precision and oferfitting takes place

4. Split $\hat{U}, \hat{S}, \hat{V}^T$ obtaining $U_{q\times n}$, $S_{n\times n}$, $V^T_{n\times d}$ and then recreate $H_{qd} = USV^T$

5. H and G are estimated as for the unnoisy case. To estimate F we can build O_1 and O_2 as before, but then the system $O_1 \cdot F = O_2$ cannot be solved directly as O_1 is not square. We can instead compute the approximate least-square solution of the system:

$$F = (O_1^T O_1)^{-1} O_1^T O_2$$

Appendix D

Kalman Filter

D.1 Representations

D.1.1 For basic systems

$$
\begin{aligned}
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + K(t)e(t) && \text{state equation}\\
\hat{y}(t|t-1) &= H\hat{x}(t|t-1) && \text{output equation}\\
e(t) &= y(t) - \hat{y}(t|t-1) && \text{prediction error}\\
K(t) &= \left(FP(t)H^T + V_{12}\right)\left(HP(t)H^T + V_2\right)^{-1} && \text{gain of the KF}\\
P(t+1) &= \left(FP(t)F^T + V_1\right)\\
&\quad - \left(FP(t)H^T + V12\right)\left(HP(t)H^T + V_2\right)^{-1}\left(FP(t)H^T + V_{12}\right)^T && \text{difference Riccati equation}\\
\hline
\hat{x}(1|0) &= E\left[x(1)\right] = X_0 && \text{Initial state}\\
P(1) &= var\left[x(1)\right] = P_0 && \text{initial DRE}
\end{aligned}
$$

D.1.2 With input

$$
\begin{aligned}
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + Gu(t) + K(t)e(t) \qquad \text{state equation}\\
\text{other equations} &= \text{unchanged}
\end{aligned}
$$

D.1.3 Multi-step prediction

Knowing $\hat{x}(t+1|T)$ from the basic solution we can derive

$$
\begin{aligned}
\hat{x}(t+k|t) &= F^{k-1}\hat{x}(t+1|t)\\
\hat{y}(t+k|t) &= H\hat{x}(t+k|t)
\end{aligned}
$$

D.1.4 Filter ($\hat{x}(t|t)$)

F invertible

$$\hat{x}(t+1|t) = F\hat{x}(t|t) \qquad \Longrightarrow \qquad \hat{x}(t|t) = F^{-1}\hat{x}(t+1|t)$$

F not invertible   assuming $V_{12} = 0$:

$$\begin{aligned}
\hat{x}(t|t) &= F\hat{x}(t-1|t-1) + Gu(t-1) + K_0(t)e(t) \\
\hat{y}(t|t-1) &= H\hat{x}(t|t-1) \\
e(t) &= y(t) - \hat{y}(t|t-1) \\
K_0(t) &= \left(P(t)H^T\right)\left(HP(t)H^T + V_2\right)^{-1} \\
P(t+1) &= \text{unchanged}
\end{aligned}$$

D.1.5 Time-varying systems

$$S : \begin{cases} x(t+1) = F(t)x(t) + G(t)u(t) + v_1(t) \\ y(t) = H(t)x(t + v_2(t)) \end{cases}$$

K.F. equations are unchanged

D.2 Asymptotic KF

D.2.1 ARE

$$\overline{P} = \left(F\overline{P}F^T + V_1\right) - \left(F\overline{P}H^T + V_{12}\right)\left(H\overline{P}H^T + V_2\right)^{-1}\left(F\overline{P}H^T + V_{12}\right)^T$$

First asymptotic theorem

$V_{12} = 0$ and the system is asymptotically stable (all eigenvalues of F strictly inside the unit circle) $\Longrightarrow$:

- $\exists!$ solution of ARE: $\overline{P} \geq 0$
- DRE converges to $\overline{P} \quad \forall P_0 \geq 0$
- The corresponding $\overline{K}$ will make the KF asymptotically stable

Second asymptotic theorem

$V_{12} = 0$, (F, H) is observable, (F, Γ) is controllable $\Longrightarrow$

- $\exists!$ solution of ARE: $\overline{P} \geq 0$
- DRE converges to $\overline{P} \quad \forall P_0 \geq 0$
- The corresponding $\overline{K}$ will make the KF asymptotically stable

D.3 Non-linear systems

$$F(t) = \left.\frac{\delta f(x(t), u(t))}{\delta x(t)}\right|_{x(t)=\hat{x}(t|t-1)}$$

$$H(t) = \left.\frac{\delta h(x(t))}{\delta x(t)}\right|_{x(t)=\hat{x}(t|t-1)}$$

D.4 Optimizing K

Appendix E

Minimum Variance Control

$$y(t) = ay(t-1)+b_0u(t-1)+b_1u(t-2)+e(t) \implies u(t) \approx \big(y^0(t) - ay(t)\big)\,\frac{1}{b_0 + b_1 z^{-1}}$$