The Proof of Disjoint Path Matroid

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Disjoint Path Matroid

The definition of the Disjoint Path Matroid is as follows.

Definition 1 (Disjoint Path Matroid)

Let $G = (V, E)$ be an arbitrary directed graph, and let s be a fixed vertex of G. A subset $I \subseteq V$ is independent if and only if there are edge-disjoint paths from s to each vertex in I. Let $\mathbf{C}$ be the collection of independent subsets of V, then $M = (V, \mathbf{C})$ is a Disjoint Path Matroid.

And we want to prove that M is indeed a matroid.

Hereditary

Lemma 1 (Hereditary)

The collection $\mathbf{C}$ is hereditary.

Proof.

$\forall B \in \mathbf{C}$, B is independent, that is, there are edge-disjoint paths from s to each vertex in B. $\forall A \subseteq B$, then there are edge-disjoint paths from s to each vertex in A, since the vertex in A must be in B. Hence, $A \in \mathbf{C}$. Therefore, the collection $\mathbf{C}$ is hereditary. $\square$

Overlapped Path

Assume that

- All lemmas and definitions are under the condition of a given graph $G = (V, E)$;
- A path P consists of the edges in path, so P is a subset of E;
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Before we prove the exchange property of M, I would like to propose the following useful definitions and lemmas.

Definition 2 (Overlapped Path)

Two paths P_1, P_2 is overlapped if and only if there exists an edge $e \in E$ satisfying $e \in P_1$ and $e \in P_2$ (i.e., $P_1 \cap P_2 \neq \varnothing$).

Construction Lemma

Lemma 2 (Construction Lemma)

Suppose $A \in \mathbf{C}, B \in \mathbf{C}$, *and* $m = |A| < |B| = n$. *According to the definition of* $\mathbf{C}$, *there exist* m *non-overlapped paths with the destinations of every vertex in* A, *say* $\mathbf{P} = \{P_1, P_2, ..., P_m\}$. *Similarly, there also exist* n *non-overlapped paths with the destinations of every vertex in* B, *say* $\mathbf{Q} = \{Q_1, Q_2, ..., Q_n\}$. *Then we are able to construct a non-overlapped path set* $\mathbf{R} = \{R_1, R_2, ..., R_n\}$ *satisfying*

$$A \subseteq \{dst(R_i) \mid i = 1, 2, ..., n\}$$

Proof of Construction Lemma

- Initially, we set $\mathbf{R}_0 = \mathbf{Q}$. Then we prove that after c steps ($c \in \mathbb{R}$), $\mathbf{R}_c$ meets the requirement $A \subseteq \{dst(R) \mid R \in \mathbf{R}_c\}$.

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- In every step, we will maintain the property that each path in $\mathbf{R}_i$ is not overlapped with other paths in $\mathbf{R}_i$.
- First, we give the following process of every step, and try to prove that if the process ends, then it will give the correct answer to the problem.
- In i-th step, we will check that whether $\mathbf{R}_i$ meets the requirement $A \subseteq \{dst(R) \mid R \in \mathbf{R}_i\}$ first.
- If the answer is true, then let $c = i$, we prove that $\mathbf{R}_c$ meets the requirement above.
- If the answer is false, then there exists $P \in \mathbf{P}$ satisfying $dst(P) \notin \{dst(R) \mid R \in \mathbf{R}_i\}$. We will discuss this case in next slide.

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  - Case 1 (P is not overlapped with any paths in $\mathbf{R}_i$):
    Randomly choose a path in $\mathbf{R}_i$ called R', which is not in $\mathbf{P}$ (there must exist because paths are distinct in $\mathbf{P}$ and $|\mathbf{P}| < |\mathbf{R}_i|$). Then $\mathbf{R}_{i+1} = \mathbf{R}_i - \{R'\} + \{P\}$ will maintain the property above, since every two paths in $\mathbf{R}_i$ do not have overlapped edge and P is not overlapped with any paths in $\mathbf{R}_i$.

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  - Case 2 (P is overlapped with some paths in $\mathbf{R}_i$):
    Suppose the last overlapped edge in P is e, and it is the overlapped edge between P and R', where $R' \in \mathbf{R}_i$. Then we construct a new path P' by mainly copying path R', except that after edge e it follows path P. Therefore, the destination of P' is the same as the destination of P, and P' do not overlapped with any path in $\mathbf{R}_i - \{R'\}$. It's because the front part of P' follows R', which is not overlapped with any other paths in $\mathbf{R}_i$, and the last part of P' follows path P after edge e, which is not overlapped with any paths since e is the last overlapped edge in P. Hence $\mathbf{R}_{i+1} = \mathbf{R}_i - \{R'\} + \{P'\}$ will maintain the property above.

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- Second, we prove that the process will end in finite steps.
- Notice that when the first case happens, we substitutes an element in $\mathbf{R}$ with an element in $\mathbf{P}$ and keeps the elements in $\mathbf{R}$ distinct. And the second case won't change these elements in $\mathbf{P}$. Therefore, the first case happens at most $|\mathbf{P}| = m$ times.

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- In summary, we have proved that if the process ends successfully, it will give us the correct answer to the question.
- Second, we prove that the process will end in finite steps.
- Notice that when the first case happens, we substitutes an element in $\mathbf{R}$ with an element in $\mathbf{P}$ and keeps the elements in $\mathbf{R}$ distinct. And the second case won't change these elements in $\mathbf{P}$. Therefore, the first case happens at most $|\mathbf{P}| = m$ times.
- Now let us consider the second case. Let E_i be the **multi-set** (i.e. set that allows duplicate elements) of edges that have been operated after i-th second-case step. At first, we stipulate that $E_0 = \varnothing$. In i-th second-case step $(i \in \mathbb{N}^+)$, we set e as a new operated edge, that is, $E_i = E_{i-1} + \{e\}$. Then it is obvious that $|E_i|$ is strictly increasing.

Proof of Construction Lemma (Cont.)

- Notice that an edge can not be operated twice, that is, the multi-set E_i is actually a set. In the second case, we set edge e as operated edge and we know that $e \in \cup\mathbf{P}$ because of overlapping. After the setting, edge e cannot be set as operated edge again because:
  - Edge e cannot be overlapped again by another path in $\mathbf{P} - \{P\}$, and for path P we won't operate the e again because $dst(R') \in A$;
  - If an edge e' in path R' before edge e is operated in future, then e will maintain the property that $e \in \cup\mathbf{P}$ and $e \in \cup\mathbf{Q}$, which will ensure that it won't be operated again.

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- Since the process starts with an empty set $E_0 = \varnothing$ and only involves edges in a finite set $\mathbf{E} \triangleq (\cup\mathbf{P}) \cup (\cup\mathbf{Q})$, the second case happens at most $|\mathbf{E}|$ times.
- Therefore, the whole process will end in at most $(|\mathbf{E}| + m)$ steps. Thus, the lemma is proved. QED.

Exchange Property

Lemma 3 (Exchange Property)

The independent system $M = (V, \mathbf{C})$ satisfies the exchange property.

Proof.

$\forall A \in \mathbf{C}, \forall B \in \mathbf{C}$ and $|A| < |B|$, with the Construction Lemma, we can construct a path set $\mathbf{R}$ satisfying

$$A \subseteq \{dst(R) \mid R \in \mathbf{R}\}$$

Thus, it's easy to choose $(|A| + 1)$ paths in $\mathbf{R}$ and keep the property above, and we suppose these paths form path set $\mathbf{R}'$. Then, we construct the set C as follows.

$$C = \{dst(R) \mid R \in \mathbf{R}'\}$$

It's easy to see that $A \subseteq C$ and $C \in \mathbf{C}$. Thus the exchange property is proved. □

Final: Matroid

Theorem 4

Disjoint Path Matroid M is indeed a matroid.

Proof.

According to the Lemma Hereditary and Lemma Exchange Property, the theorem is proved. $\square$