Homework 07

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1 Warmup Problems

4. Express $\frac{1}{1}+\frac{1}{3}+\cdots+\frac{1}{2n+1}$ in terms of harmonic numbers.

Solution. The following derivations (Eqn. (1)) can come to the final conclusion $H_{2n+1}-\frac{1}{2}H_n$.

$$\begin{aligned}\frac{1}{1}+\frac{1}{3}+\cdots+\frac{1}{2n+1} &= \left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{2n+1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)\\ &= \sum_{k=1}^{2n+1}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{2k} \\ &= H_{2n+1}-\frac{1}{2}H_n\end{aligned} \tag{1}$$

□

5. Explain how to get the recurrence (Formula (6.75) in textbook) from the definition of $U_n(x,y)$ in Formula (6.74) in textbook, and solve the recurrence.

Solution. The derivations of the recurrence are as follows.

$$\begin{aligned}U_n(x,y) &= \sum_{k\ge 1}\binom{n}{k}\frac{(-1)^{k-1}}{k}(x+ky)^n\\ &= \sum_{k\ge 1}\binom{n}{k}\frac{(-1)^{k-1}}{k}(x+ky)^{n-1}(x+ky)\\ &= x\sum_{k\ge 1}\binom{n}{k}\frac{(-1)^{k-1}}{k}(x+ky)^{n-1}+y\sum_{k\ge 1}\binom{n}{k}(-1)^{k-1}(x+ky)^{n-1}\\ &= x\sum_{k\ge 1}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)\frac{(-1)^{k-1}}{k}(x+ky)^{n-1}+y\sum_{k\ge 1}\binom{n}{k}(-1)^{k-1}(x+ky)^{n-1}\\ &= xU_{n-1}(x,y)+\frac{x}{n}\sum_{k\ge 1}\binom{n}{k}(-1)^{k-1}(x+ky)^{n-1}+y\sum_{k\ge 1}\binom{n}{k}(-1)^{k-1}(x+ky)^{n-1}\end{aligned}$$

If we let $S_n(x,y)$ be $\sum_{k\ge 1}\binom{n}{k}(-1)^{k-1}(x+ky)^{n-1}$, then according to the derivations above, we have the following formula (Eqn. (2))

$$U_n(x,y) = xU_{n-1}(x,y)+\frac{x}{n}S_n(x,y)+yS_n(x,y) \tag{2}$$

Now let us evaluate the value of S_n.

$$\begin{aligned}S_n(x,y) &= x^{n-1}+\sum_{k}\binom{n}{k}(-1)^{k-1}(x+ky)^{n-1}\\ &= x^{n-1}+\sum_{k}\binom{n}{k}(-1)^{k-1}\left(\sum_{i}\binom{n-1}{i}k^i x^{n-1-i}y^i\right)\\ &= x^{n-1}+\sum_{i}\binom{n-1}{i}x^{n-1-i}y^i\left(\sum_{k}\binom{n}{k}(-1)^{k-1}k^i\right)\end{aligned}$$

To continue the derivation, we must use an important formula (5.40) in textbook, which is displayed as follows.

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k)$$

Here we denote $f_i(x)$ be x^i, then according to the previous formula, we know that

$$\Delta^n f_i(0) = \sum_k \binom{n}{k} (-1)^{n-k} k^i = -\sum_k \binom{n}{k} (-1)^{k-1} k^i$$

It's easy to notice that for all $0 \le i < n$, we have $\Delta^n f_i(x) = 0$. Thus, we can continue our derivations of $S_n(x, y)$.

$$\begin{aligned}
S_n(x,y) &= x^{n-1} + \sum_i \binom{n-1}{i} x^{n-1-i} y^i \left(\sum_k \binom{n}{k} (-1)^{k-1} k^i \right) \\
&= x^{n-1} - \sum_i \binom{n-1}{i} x^{n-1-i} y^i \Delta^n f_i(0) \\
&= x^{n-1} - \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i} y^i \Delta^n f_i(0) \\
&= x^{n-1} - \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i} y^i \cdot 0 \\
&= x^{n-1}
\end{aligned}$$

We plug our result $S_n(x, y) = x^{n-1}$ in Equation (2), then we can get the recurrence (6.75) in textbook.

$$U_n(x,y) = xU_{n-1}(x,y) + \frac{x^n}{n} + yx^{n-1} \tag{3}$$

Solving the recurrence is quite simple and the derivations are as follows (Eqn. (4)).

$$\begin{aligned}
U_n(x,y) &= xU_{n-1}(x,y) + \frac{x^n}{n} + yx^{n-1} \\
&= x\left(xU_{n-2}(x,y) + \frac{x^{n-1}}{n-1} + yx^{n-2} \right) + \frac{x^n}{n} + yx^{n-1} \\
&= \cdots \\
&= x^n U_0(x,y) + nx^{n-1}y + x^n \sum_{i=1}^{n} \frac{1}{i} \\
&= x^n H_n + nx^{n-1}y
\end{aligned} \tag{4}$$

Therefore, we come to the same conclusion as the conclusion in textbook. □

9. About how many square kilometers are in 8 square miles?

Solution. We have talked about unit transform from miles to kilometers in textbook, and we find out the coefficient is approximately ϕ in coincidence. Thus, the coefficient of unit transform from square miles to square kilometers should be ϕ^2. Therefore, we just need to use Fibonacci radix system to represent the number of square miles, and left-shift all the digit twice then we can get the approximate number of square kilometers. Since $8 = F_6$, so the answer should be approximately $F_8 = 21$ square kilometers. Actually, the more accurate result is 20.72, which is quite close to our approximation. □

10. What is the continued fraction representation of ϕ?

Solution. Here is an interesting property of ϕ.

$$\phi = \frac{1+\sqrt{5}}{2} = 1 + \frac{\sqrt{5}-1}{2} = 1 + \frac{2}{\sqrt{5}+1} = 1 + \frac{1}{\phi}$$

Therefore, we can use this property infinitely to get the continued representation of ϕ.

$$\phi = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$$

Therefore, all the coefficients a_i in the continued fraction representation of ϕ is 1, that is, $a_0 = 1, a_1 = 1, a_2 = 1, \cdots$. What's more, according to textbook, the Stern-Brocot representation is $RLRLRLRL\cdots$. □

2 Basic Problems

14. Prove the power identity for Eulerian numbers (Formula (6.37) in textbook).

Proof. Let us prove the power identity for Eulerian numbers by induction on n. When $n = 1$, it is easy to verify that

$$1 = x^0 = \left\langle {0 \atop 0} \right\rangle \binom{x}{0} = \sum_k \left\langle {0 \atop k} \right\rangle \binom{x+k}{0}$$

Now suppose the conclusion holds for $n = m$ and we will prove it still holds for $n = m + 1$.

$$\begin{aligned}
\sum_k \left\langle {m+1 \atop k} \right\rangle \binom{x+k}{m+1} &= \sum_k \left((k+1) \left\langle {m \atop k} \right\rangle + (m+1-k) \left\langle {m \atop k-1} \right\rangle \right) \binom{x+k}{m+1} \\
&= \sum_k (k+1) \left\langle {m \atop k} \right\rangle \binom{x+k}{m+1} + \sum_k (m+1-k) \left\langle {m \atop k-1} \right\rangle \binom{x+k}{m+1} \\
&= \sum_k (k+1) \left\langle {m \atop k} \right\rangle \binom{x+k}{m+1} + \sum_k (m-k) \left\langle {m \atop k} \right\rangle \binom{x+k+1}{m+1} \\
&= \sum_k \left\langle {m \atop k} \right\rangle \left((k+1) \binom{x+k}{m+1} + (m-k) \binom{x+k+1}{m+1} \right) \\
&= \sum_k \left\langle {m \atop k} \right\rangle \left(\frac{(x+k)!(k+1)}{(m+1)!(x+k-m-1)!} + \frac{(x+k+1)!(m-k)}{(m+1)!(x+k-m)!} \right)
\end{aligned}$$

Let us focus on the formula in brackets, which is

$$\frac{(x+k)!(k+1)}{(m+1)!(x+k-m-1)!} + \frac{(x+k+1)!(m-k)}{(m+1)!(x+k-m)!} \triangleq S(k)$$

Simplify the formula above as follows.

$$\begin{aligned}
S(k) &= \frac{(x+k)!}{(x+k-m)!m!} \left(\frac{(k+1)(x+k-m)}{m+1} + \frac{(x+k+1)(m-k)}{m+1} \right) \\
&= \binom{x+k}{m} \frac{x+mx}{m+1} \\
&= \binom{x+k}{m} x
\end{aligned}$$

Plug the result of S in the derivations above, we know that

$$\begin{aligned}\sum_k \left\langle {m+1 \atop k} \right\rangle \binom{x+k}{m+1} &= \sum_k \left\langle {m \atop k} \right\rangle \cdot S(k) \\ &= \sum_k \left\langle {m \atop k} \right\rangle \binom{x+k}{m} x \\ &= x \sum_k \left\langle {m \atop k} \right\rangle \binom{x+k}{m} \\ &= x \cdot x^m \\ &= x^{m+1}\end{aligned}$$

Therefore, the conclusion is still holds for $n = m+1$. Hence, the conclusion is correct. □

15. Prove the Eulerian identity (Formula (6.39) in textbook) by taking the m-th difference of Formula (6.37) in textbook.

Proof. Notice that $\Delta\left(\binom{x+k}{n}\right) = \binom{x+k+1}{n} - \binom{x+k}{n} = \binom{x+k}{n-1}$, so $\Delta^m\left(\binom{x+k}{n}\right) = \binom{x+k}{n-m}$. Take the m-th difference of Formula (6.37) in textbook, we have:

$$\begin{aligned} &\Delta^m x^n = \sum_k \left\langle {n \atop k} \right\rangle \Delta^m \binom{x+k}{n} \\ \overset{(5.40)}{\Longrightarrow}\quad &\sum_k \binom{m}{k} (-1)^{m-k} (x+k)^n = \sum_k \left\langle {n \atop k} \right\rangle \binom{x+k}{n-m} \\ \overset{x=0}{\Longrightarrow}\quad &\sum_k \binom{m}{k} (-1)^{m-k} k^n = \sum_k \left\langle {n \atop k} \right\rangle \binom{k}{n-m} \\ \overset{(6.19)}{\Longrightarrow}\quad &m! \left\{ {n \atop m} \right\} = \sum_k \left\langle {n \atop k} \right\rangle \binom{k}{n-m} \end{aligned}$$

Thus, the formula is proved. □

16. What is the general solution of the double recurrence

$$\begin{aligned} &A_{n,0} = a_n[n \geq 0]; \qquad A_{0,k} = 0, \quad \text{if } k > 0; \\ &A_{n,k} = kA_{n-1,k} + A_{n-1,k-1}, \quad \text{integers } k, n; \end{aligned}$$

when k and n range over the set of all integers?

Conclusion. *Our conclusion about $A_{n,k}$ is as follows (Eqn. (5)).*

$$A_{n,k} = \sum_{j \geq 0} \left\{ {n-j \atop k} \right\} a_j \tag{5}$$

Proof. We will prove our conclusion by induction to n.

- **(Induction Basis)** When $n = 0$, our conclusion is $A_{0,k} = \sum_{j\geq 0} \left\{ {0-j \atop k} \right\} a_j = a_0[k=0]$, which satisfies the condition that $A_{0,k} = 0$ $(k > 0)$ and $A_{0,0} = a_0$.

- **(Induction Step)** Assume the conclusion holds for $n = m - 1$, and we will prove that the conclusion still holds for $n = m$. Thus we can have the following derivations.

$$\begin{aligned}
A_{m,k} &= kA_{m-1,k} + A_{m-1,k-1} \\
&= k\sum_{j\geq 0} \left\{ {m-1-j \atop k} \right\} a_j + \sum_{j\geq 0} \left\{ {m-1-j \atop k-1} \right\} a_j \\
&= \sum_{j\geq 0} a_j \left(k\left\{ {m-1-j \atop k} \right\} + \left\{ {m-1-j \atop k-1} \right\} \right) \\
&= \sum_{j\geq 0} \left\{ {m-j \atop k} \right\} a_j
\end{aligned}$$

  The derivations above tell us the conclusion still holds for $n = m$.

Therefore, the conclusion holds for all integer n and k.

We also want to emphasize that the summation is always finite for a given n, which can be easily verified by connecting $A_{n,k}$ with the Stirling's Numbers for subsets $\left\{ {n \atop k} \right\}$. $\square$