Homework 09

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1 Warmup Problems

4. The general expansion theorem for rational functions $P(z)/Q(z)$ is not completely general, because it restricts the degree of P to be less than the degree of Q. What happens if P has a larger degree than this?

Solution. We can divide $P(z)$ by $Q(z)$, and get a quotient polynomial $T(z)$ and a remainder polynomial $R(z)$. The degree of remainder polynomial $R(z)$ is less than the degree of $Q(z)$, therefore we can perform general expansion theorem to rational functions $R(z)/Q(z)$. After adding $T(z)$ to the result, we expand $P(z)/Q(z)$ successfully.

To state more formally, first we define $P(z)$ as $\sum_{i=0}^{n} p_i z^i$ and $Q(z)$ as $\sum_{i=0}^{m} q_i z^i$, and according to the problem descriptions, we have $n > m$. Therefore, we can perform polynomial division to get the quotient $T(z)$ and remainder $R(z)$, that is,

$$\frac{P(z)}{Q(z)} = T(z) + \frac{R(z)}{Q(z)}$$

Then, according to the polynomial division, we know that $T(z)$ is a polynomial whose degree is $n - m$, say

$$T(z) = \sum_{i=0}^{n-m} t_i z^i$$

And the degree of $R(z)$ is less than m, so it is also less than the degree of $Q(z)$. Thus, we perform the general expansion theorem to $R(z)/Q(z)$, and assume that we get the following expansion result.

$$\frac{R(z)}{Q(z)} = \sum_i r'_i z^i$$

Therefore, we can derive the following formula (Eqn. (1)).

$$\begin{aligned}
\frac{P(z)}{Q(z)} &= T(z) + \frac{R(z)}{Q(z)} \\
&= \sum_{i=0}^{n-m} t_i z^i + \sum_i r'_i z^i \\
&= \sum_i ([i \le n-m] \cdot t_i + r'_i) z^i
\end{aligned} \tag{1}$$

Hence, we expand $P(z)/Q(z)$ successfully. □

5. Find a generating function $S(z)$ such that

$$[z^n]S(z) = \sum_k \binom{r}{k} \binom{r}{n-2k}$$

Solution. The given formula tells us an important property of $S(z)$, which is shown as follows.

$$\begin{aligned}
S(z) &= \sum_n z^n \cdot \sum_k \binom{r}{k}\binom{r}{n-2k} \\
&= \sum_n \sum_k \binom{r}{k} z^{2k} \cdot \binom{r}{n-2k} z^{n-2k} \\
&= \sum_n \sum_k [k \text{ is even}] \binom{r}{\frac{k}{2}} z^k \cdot \binom{r}{n-k} z^{n-k} \\
&= \left(\sum_n [n \text{ is even}] \binom{r}{\frac{n}{2}} z^n\right) \cdot \left(\sum_n \binom{r}{n} z^n\right) \\
&\triangleq F(z) \cdot G(z)
\end{aligned}$$

Hence, $S(z)$ is the convolution of two generating functions $F(z)$ and $G(z)$, where

$$\begin{aligned}
F(z) &= \sum_n [n \text{ is even}] \binom{r}{\frac{n}{2}} z^n = \sum_n \binom{r}{n} z^{2n} = (1+z^2)^r \\
G(z) &= \sum_n \binom{r}{n} z^n = (1+z)^r
\end{aligned}$$

Therefore, we can derive the closed form of $S(z)$ (Eqn. (2)).

$$S(z) = F(z) \cdot G(z) = (1+z^2)^r \cdot (1+z)^r = (1+z+z^2+z^3)^r \tag{2}$$

□

2 Basic Problems

10. Set $r = s = -\frac{1}{2}$ in identity (7.62) in textbook and then remove all occurrences of $\frac{1}{2}$ by using tricks like (5.36) in textbook. What amazing identity do you deduce?

Solution. Identity (7.62) in textbook is displayed as follows.

$$\sum_k \binom{r+k}{k}\binom{s+n-k}{n-k}(H_{r+k} - H_r) = \binom{r+s+n+1}{n}(H_{r+s+n+1} - H_{r+s+1})$$

Plug $r = s = -\frac{1}{2}$ in the identity above, we can make some derivations as follows.

$$\begin{aligned}
& \sum_k \binom{-\frac{1}{2}+k}{k}\binom{-\frac{1}{2}+n-k}{n-k}\left(H_{-\frac{1}{2}+k} - H_{-\frac{1}{2}}\right) = \binom{n}{n}(H_n - H_0) \\
\overset{(5.36)}{\Longrightarrow}\quad & \sum_k \frac{1}{2^{2k}}\binom{2k}{k} \cdot \frac{1}{2^{2(n-k)}}\binom{2(n-k)}{n-k} \cdot \left(H_{-\frac{1}{2}+k} - H_{-\frac{1}{2}}\right) = H_n \\
\Longrightarrow\quad & 4^n H_n = \sum_k \binom{2k}{k}\binom{2n-2k}{n-k} \cdot \left(\frac{1}{k-\frac{1}{2}} + \frac{1}{k-\frac{3}{2}} + \cdots + \frac{1}{\frac{1}{2}}\right) \\
\Longrightarrow\quad & 4^n H_n = \sum_k \binom{2k}{k}\binom{2n-2k}{n-k} \cdot 2\left(\frac{1}{2k-1} + \frac{1}{2k-3} + \cdots + \frac{1}{1}\right) \\
\overset{Ex.(6.4)}{\Longrightarrow}\quad & 4^n H_n = \sum_k \binom{2k}{k}\binom{2n-2k}{n-k} \cdot (2H_{2k} - H_k)
\end{aligned}$$

In the end, we deduce the following identity (Eqn. (3)).

$$4^n H_n = \sum_k \binom{2k}{k}\binom{2n-2k}{n-k} \cdot (2H_{2k} - H_k) \tag{3}$$

$\square$

11. This problem, whose three parts are independent, gives practice in the manipulation of generating functions. We assuume that $A(z) = \sum_n a_n z^n$, $B(z) = \sum_n b_n z^n$, $C(z) = \sum_n c_n z^n$, and that the coefficients are zero for negative n.

    a. If $c_n = \sum_{j+2k \le n} a_j b_k$, express C in terms of A and B.

    b. If $nb_n = \sum_{k=0}^{n} a_k \cdot \frac{2^k}{(n-k)!}$, express A in terms of B.

    c. If r is a real number and if $a_n = \sum_{k=0}^{n} \binom{r+k}{k} b_{n-k}$, express A in terms of B; then use your formula to find coefficient $f_k(r)$ such that $b_n = \sum_{k=0}^{n} f_k(r) a_{n-k}$.

Solution. Here are the solutions to the sub-problems.

a. We can derive $C(z)$ as follows.

$$\begin{aligned}
C(z) &= \sum_n c_n z^n \\
&= \sum_n z^n \cdot \sum_{j+2k \le n} a_j b_k \\
&= \sum_n z^n \cdot \sum_{n'=0}^{n} \left(\sum_{j+2k=n'} a_j b_k \right) \\
&= \sum_n \sum_{n'=0}^{n} \left(z^{n'} \sum_{j+2k=n'} a_j b_k \right) \cdot \left(z^{n-n'} \cdot 1 \right) \\
&= \left(\sum_n z^n \sum_{j+2k=n} a_j b_k \right) \cdot \left(\sum_n z^n \cdot 1 \right) \\
&= \left(\sum_n \sum_j \left(a_j z^j \right) \cdot \left([n-j \text{ is even}] \cdot b_{\frac{n-j}{2}} z^{n-j} \right) \right) \cdot \frac{1}{1-z} \\
&= \left(\sum_n a_n z^n \right) \cdot \left(\sum_n [n \text{ is even}] \cdot b_{\frac{n}{2}} z^n \right) \cdot \frac{1}{1-z} \\
&= A(z) \cdot \left(\sum_n \cdot b_n z^{2n} \right) \cdot \frac{1}{1-z} \\
&= \frac{A(z)B(z^2)}{1-z}
\end{aligned}$$

Therefore, we can express $C(z)$ in terms of $A(z)$ and $B(z)$ as follows (Eqn. (4)).

$$C(z) = \frac{A(z)B(z^2)}{1-z} \tag{4}$$

b. First we multiply the both sides of the equation by z^n, and we add a summation on n to both sides, then we can get the following formula.

$$\sum_n nb_n z^n = \sum_n z^n \sum_{k=0}^{n} a_k \cdot \frac{2^k}{(n-k)!}$$

The left side of the formula above can be rewritten as follows using the differentiate formula.

$$\sum_n nb_n z^n = z \sum_n b_n \cdot (nz_{n-1}) = z \sum_n b_n (z_n)' = zB'(z)$$

Therefore, we can continue our derivations as follows.

$$\begin{aligned}
zB'(z) &= \sum_n z^n \sum_{k=0}^{n} a_k \cdot \frac{2^k}{(n-k)!} \\
&= \sum_n \sum_{k=0}^{n} \left(a_k 2^k \cdot z^k\right) \cdot \left(\frac{1}{(n-k)!} \cdot z^{n-k}\right) \\
&= \left(\sum_n a_n 2^n \cdot z^n\right) \cdot \left(\sum_n \frac{1}{n!} \cdot z^n\right) \\
&= \left(\sum_n a_n (2z)^n\right) \cdot e^z \\
&= e^z A(2z)
\end{aligned}$$

Therefore, we can express $A(z)$ in terms of $B(z)$ as follows (Eqn. (5)).

$$A(z) = \frac{z}{2} e^{-\frac{z}{2}} B'\left(\frac{z}{2}\right) \tag{5}$$

c. We can derive $A(z)$ as follows.

$$\begin{aligned}
A(z) &= \sum_n a_n z^n \\
&= \sum_n z^n \cdot \sum_{k=0}^{n} \binom{r+k}{k} b_{n-k} \\
&= \sum_n \sum_{k=0}^{n} \left(\binom{r+k}{k} \cdot z^k\right) \cdot \left(b_{n-k} \cdot z^{n-k}\right) \\
&= \left(\sum_n \binom{r+n}{n} z^n\right) \cdot \left(\sum_n b_n z^n\right) \\
&= \frac{B(z)}{(1-z)^{r+1}}
\end{aligned}$$

Therefore, we can express $A(z)$ in terms of $B(z)$ as follows (Eqn. (6)).

$$A(z) = \frac{B(z)}{(1-z)^{r+1}} \tag{6}$$

Hence, we can also express $B(z)$ in terms of $A(z)$ as follows.

$$B(z) = (1-z)^{r+1} A(z)$$

We can unfold the generating function into summation forms, then we can make the following derivations.

$$
\begin{aligned}
\sum_n b_n z^n &= (1-z)^{r+1} \cdot \left(\sum_n a_n z^n\right) \\
&= \left(\sum_n \binom{r+1}{n} (-1)^n z^n\right) \cdot \left(\sum_n a_n z^n\right) \\
&= \sum_n \sum_k \left(\binom{r+1}{k} (-1)^k \cdot z^k\right) \cdot \left(a_{n-k} \cdot z^{n-k}\right) \\
&= \sum_n z^n \cdot \sum_k (-1)^k \binom{r+1}{k} a_{n-k}
\end{aligned}
$$

Therefore, the corresponding coefficient must be the same, that is,

$$b_n = \sum_k (-1)^k \binom{r+1}{k} a_{n-k}$$

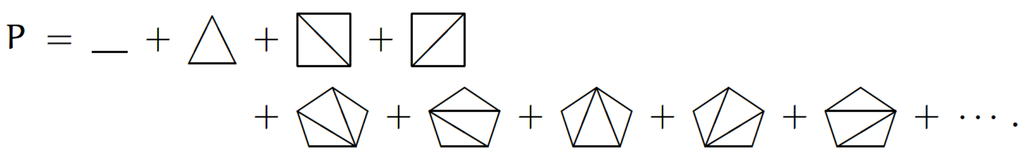
Hence, we can get the coefficient $f_k(r)$ as follows (Eqn. (7)).

$$f_k(r) = (-1)^k \binom{r+1}{k} \tag{7}$$

$\square$

3 Homework Exercises

22. **(Bonus Problem)** Let P be the sum of all ways to "triangulate" polygons.



(The first term represents a degenerate polygon with only two vertices, every other term shows a polygon that has been divided into triangles. For example, a pentagon can be triangulated in five ways.) Define a "multiplication" operation $A\Delta B$ on triangulated polygons A and B so that the equation

$$\mathsf{P} = \_\_ + \mathsf{P} \triangle \mathsf{P}$$

is valid. Then replace each triangle by z; what does this tell you about the number of ways to decompose a n-gon into triangles?

Solution. We define the base of each polygon as the line segment at the bottom. If A and B are triangulated polygons, we can define $A \triangle B$ as the result of pasting the base of A to the upper left diagonal of $\triangle$, and pasting the base of B to the upper right diagonal of $\triangle$ (Notice the $\triangle$ here means both an operator and a triangle).

Here are some examples of the operator Δ.

Notice we re-shape the result polygons to make it look better.

Every triangulation can be constructed uniquely in this way, because the base line is part of a unique triangle $\triangle$, and triangulated polygon A and B are at its left and right.

If we replace triangle $\triangle$ by z, then we can get a power series $P(z)$, which can be also regarded as a generating function. The coefficient of z^n in the power series is the number of triangulations with n triangles, which is also the number of ways to decompose $(n+2)$-gon into triangles. And we can get the following equation (Eqn. (8)) according to the previous derivations.

$$P(z) = 1 + zP^2(z) \tag{8}$$

Therefore, we can solve $P(z)$.

$$P(z) = \frac{1 \pm \sqrt{1-4z}}{2z}$$

When $z = 0$, $P(0)$ should be a finite number, therefore we must choose the substraction sign according to the L'Hospital Law, or $P(z)$ will be an infinite number. Therefore,

$$P(z) = \frac{1 - \sqrt{1-4z}}{2z}$$

We can expand $\sqrt{1-4z}$ as follows using the general binomial theorem.

$$\sqrt{1-4z} = \sum_n \binom{\frac{1}{2}}{n} (-4z)^n$$

Hence, we can continue our derivations of $P(z)$.

$$\begin{aligned} P(z) &= \frac{1 - \sqrt{1-4z}}{2z} \\ &= \frac{1 - \sum_n \binom{\frac{1}{2}}{n}(-4z)^n}{2z} \\ &= \frac{1 - \sum_n \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}(-4z)^n}{2z} \\ &= \sum_n \frac{(2n)!}{n!(n+1)!} z^n \\ &= \sum_n \frac{\binom{2n}{n}}{n+1} z^n \end{aligned}$$

Therefore, the number of ways to decompose a n-gon into triangles, which is named p_n here, is as follows (Eqn. (9)).

$$p_n = \frac{\binom{2n}{n}}{n+1} \tag{9}$$

Actually, $\{p_n\}$ is actually the famous **Catalan Numbers**.

Therefore, the number of ways to decompose n-gon into triangles is p_{n-2}, which is

$$\frac{\binom{2n-4}{n-2}}{n-1}$$

$\square$

23. **(Bonus Problem)** In how many ways can a $2 \times 2 \times n$ pillar be built out of $2 \times 1 \times 1$ bricks?

Solution. Let a_n denote the number of ways to build a $2 \times 2 \times n$ pillar using $2 \times 1 \times 1$ bricks. Let b_n denote the number of ways to build a $2 \times 2 \times n$ pillar with a $2 \times 1 \times 1$ notch missing on the top layer.

If n is negative, then we specially define $a_n = 0$ and $b_n = 0$. Notice an important thing is, when we consider about b_n, we take the direction of the missing notch into account, that is, we regard the following four situations of different directions (Fig. 1) as four different ways when counting b_n.

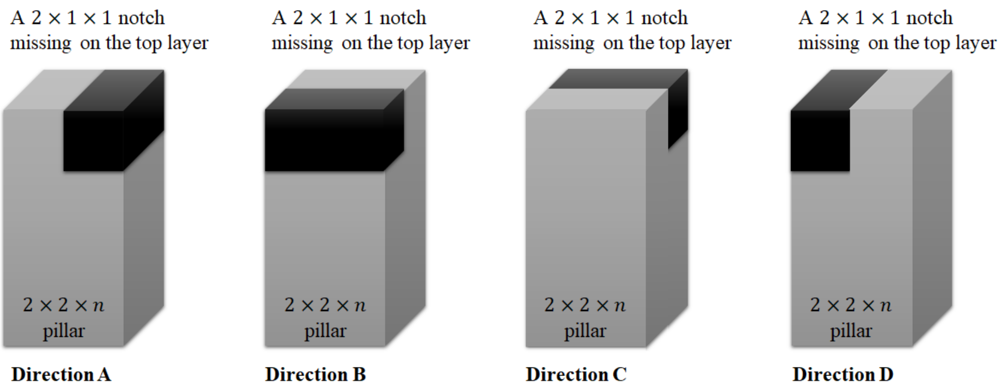


Figure 1: Situations of different directions

Therefore, there are several ways to build a $2 \times 2 \times n$ pillar.

- Add a 2×2 layer based on the $2 \times 2 \times (n-1)$ pillar, and there are two ways to add the top layer (Fig. 2).

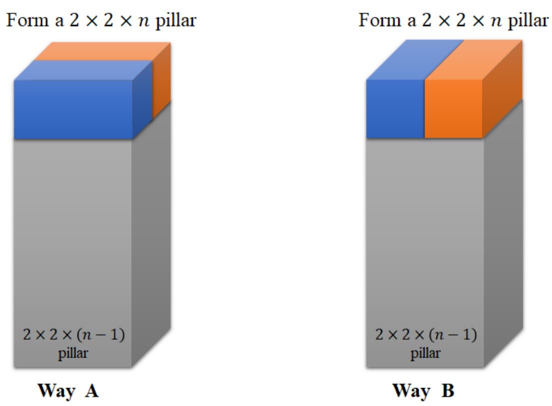


Figure 2: Two ways of adding the top layer

- Add three $2 \times 1 \times 1$ bricks based on the $2 \times 2 \times (n-1)$ pillar with a $2 \times 1 \times 1$ notch missing on the top layer, and there is only one way to add three bricks (Fig. 3).

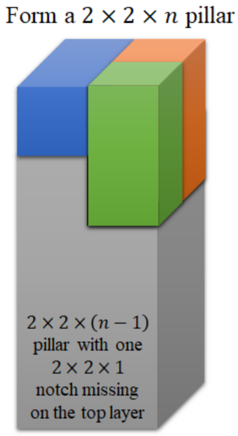


Figure 3: One way of adding three bricks

Notice here we do not allow to add a $2 \times 1 \times 1$ brick directly into the missing notch, because we want to prevent aliases with the previous method.

- Add two 2×2 layers based on the $2 \times 2 \times (n-2)$ pillar using four $2 \times 2 \times 1$ bricks, and there is only one way to add four bricks (Fig. 4).

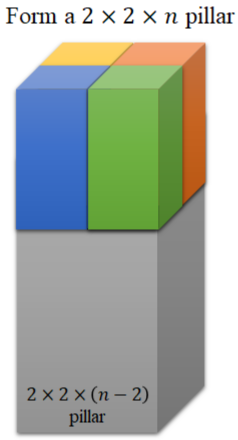


Figure 4: One way of adding four bricks

Notice here we do not allow other method to form two 2×2 layers because we want to prevent aliases with the previous methods.

Notice we need to set $a_0 = 1$ because there is one way to form a null pillar. Therefore we can write down the recurrence formula of a_n as follows (Eqn. (10)).

$$a_n = 2a_{n-1} + b_{n-1} + a_{n-2} + [n = 0] \tag{10}$$

Now let us consider about the construction of a $2 \times 2 \times n$ pillar with a $2 \times 1 \times 1$ notch missing on the top layer.

- An simple way is, we directly add a $2 \times 1 \times 1$ brick lying down on the top layer based on the $2 \times 2 \times (n-1)$ pillar, and there are four ways to add the brick because there are four different directions (Fig. 1).

- Another way is, add two $2\times2\times1$ bricks based on the $2\times2\times(n-1)$ pillar with a $2\times1\times1$ notch missing on the top layer. There is only one way to add the brick (Fig. 5).

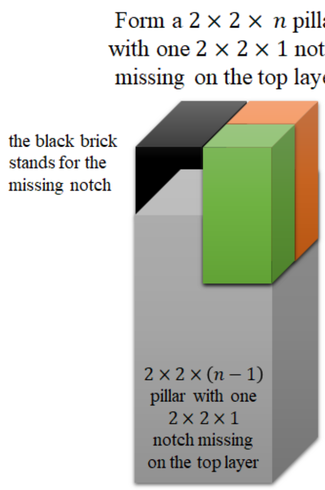


Figure 5: One way of adding two bricks

Notice here we do not allow other method to add two $2\times1\times1$ bricks, because we want to prevent aliases with the previous method.

Therefore we can write down the recurrence formula of b_n as follows (Eqn. (11)).

$$b_n = 4a_{n-1} + b_{n-1} \tag{11}$$

Suppose the generating function of $\{a_n\}$ and $\{b_n\}$ are $A(z)$ and $B(z)$ respectively. According to Equation (10) and Equation (11), we can write the relations between the $A(z)$ and $B(z)$ as follows (Eqn. (12)).

$$\begin{cases} A(z) = 2zA(z) + zB(z) + z^2A(z) + 1 \\ B(z) = 4zA(z) + zB(z) \end{cases} \tag{12}$$

Solve the equation, and we can get the closed form of $A(z)$ as follows (Eqn. (13)).

$$A(z) = \frac{1-z}{1-3z-3z^2+z^3} = \frac{1-z}{(1+z)(1-4z+z^2)} \tag{13}$$

We can use the general partial fraction expansion method to solve the problem. We can rewrite $A(z)$ as follows.

$$A(z) = \frac{1-z}{(1+z)(1-4z+z^2)} = \frac{1}{3}\cdot\frac{1}{1+z} + \frac{2-\sqrt{3}}{6}\cdot\frac{1}{1-(2-\sqrt{3})z} + \frac{2+\sqrt{3}}{6}\cdot\frac{1}{1-(2+\sqrt{3})z}$$

Therefore, we can expand $A(z)$ into summation form as follows.

$$\begin{aligned} A(z) &= \frac{1}{3}\cdot\frac{1}{1+z} + \frac{2-\sqrt{3}}{6}\cdot\frac{1}{1-(2-\sqrt{3})z} + \frac{2+\sqrt{3}}{6}\cdot\frac{1}{1-(2+\sqrt{3})z} \\ &= \frac{1}{3}\sum_n(-z)^n + \frac{2-\sqrt{3}}{6}\sum_n\left((2-\sqrt{3})z\right)^n + \frac{2+\sqrt{3}}{6}\sum_n\left((2+\sqrt{3})z\right)^n \\ &= \sum_n\left(\frac{1}{3}(-1)^n + \frac{1}{6}(2-\sqrt{3})^{n+1} + \frac{1}{6}(2+\sqrt{3})^{n+1}\right)z^n \end{aligned}$$

Therefore, we can get the closed form of a_n as follows (Eqn. (14)), which is the answer to the problem.

$$a_n = \frac{1}{3}(-1)^n + \frac{1}{6}(2-\sqrt{3})^{n+1} + \frac{1}{6}(2+\sqrt{3})^{n+1} \tag{14}$$

□