Homework 12

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1 Chapter 5: Binomial Coefficients

3. Prove the hexagon property.

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}$$

Proof. We can prove the property by simply unfolding the notations of binomial coefficients, as Equation (1) shown.

$$\begin{aligned}\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!}\cdot\frac{n!}{(k+1)!(n-k-1)!}\cdot\frac{(n+1)!}{k!(n-k+1)!}\\ &= \frac{(n-1)!}{k!(n-k-1)!}\cdot\frac{(n+1)!}{(k+1)!(n-k)!}\cdot\frac{n!}{(k-1)!(n-k+1)!}\\ &= \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}\end{aligned} \tag{1}$$

□

13. Find relations between the superfactorial function $P_n = \prod_{k=1}^n k!$, the hyperfactorial function $Q_n = \prod_{k=1}^n k^k$, and the product $R_n = \prod_{k=0}^n \binom{n}{k}$.

Solution. Let us first simplify R_n as follows.

$$R_n = \prod_{k=0}^n \binom{n}{k} = \prod_{k=0}^n \frac{n!}{k!(n-k)!} = \frac{(n!)^{n+1}}{\left(\prod_{k=0}^n k!\right)^2} = \frac{(n!)^{n+1}}{P_n^2}$$

Actually this result can be simplified further as follows.

$$R_n = \frac{(n!)^{n+1}}{P_n^2} = \frac{\prod_{k=1}^n k^{n+1}}{\left(\prod_{k=1}^n k!\right)\cdot P_n} = \frac{\prod_{k=1}^n k^{n+1}}{\left(\prod_{k=1}^n k^{n-k+1}\right)\cdot P_n} = \frac{\prod_{k=1}^n k^k}{P_n} = \frac{Q_n}{P_n}$$

Therefore, we derive an amazing relation among three functions, that is, $R_n = Q_n/P_n$. In our derivation process, another important relation between P_n and Q_n is that $P_nQ_n = (n!)^{n+1}$. □

2 Chapter 6: Special Numbers

11. What is $\sum_k (-1)^k \left[{n \atop m}\right]$, the row sum of Stirling's cycle-number triangle with alternating signs, when n is a nonnegative integer?

Solution. According to the general formula (6.11) in textbook, we can make the following derivations (Eqn. (2)).

$$\sum_k \left[{n \atop m}\right](-1)^k = (-1)^{\overline{n}} = \begin{cases} 1 & (n=0) \\ -1 & (n=1) \\ 0 & (n>1)\end{cases} = [n=0]-[n=1] \tag{2}$$

□

12. Prove that Stirling numbers have an inversion law analogous to Formula (5.48) in textbook:

$$g(n) = \sum_k \left\{ {n \atop k} \right\} (-1)^k f(k) \quad \Longleftrightarrow \quad f(n) = \sum_k \left[{n \atop k} \right] (-1)^k g(k)$$

Proof. We can split the conclusion to two sub-conclusions and prove them respectively.

- **($\Longrightarrow$)** If we have $g(n) = \sum_k \left\{ {n \atop k} \right\} (-1)^k f(k)$, then $f(n) = \sum_k \left[{n \atop k} \right] (-1)^k g(k)$ is correct. According to the similar form of the formula (6.31) in textbook, we can make the following derivations.

$$\begin{aligned} f(n) &= \sum_k \left[{n \atop k} \right] (-1)^k g(k) \\ &= \sum_k \left[{n \atop k} \right] (-1)^k \left(\sum_m \left\{ {k \atop m} \right\} (-1)^m f(m) \right) \\ &= \sum_m \sum_k \left[{n \atop k} \right] \left\{ {k \atop m} \right\} (-1)^{k+m} f(m) \\ &= \sum_m (-1)^{n+m} f(m) \sum_k \left[{n \atop k} \right] \left\{ {k \atop m} \right\} (-1)^{n-k} \\ &= \sum_m (-1)^{n+m} f(m) [m = n] \\ &= f(n) \end{aligned}$$

  Therefore, the conclusion's correctness is proved.

- **($\Longleftarrow$)** If we have $f(n) = \sum_k \left[{n \atop k} \right] (-1)^k g(k)$, then $g(n) = \sum_k \left\{ {n \atop k} \right\} (-1)^k f(k)$ is correct. The proof process is similar to the previous one and we can draw the similar conclusion that the conclusion is correct.

Therefore, the conclusion is correct. □

16. What is the general solution of the double recurrence

$$\begin{aligned} A_{n,0} &= a_n [n \geq 0]; \qquad A_{0,k} = 0, \quad \text{if } k > 0; \\ A_{n,k} &= k A_{n-1,k} + A_{n-1,k-1}, \quad \text{integers } k, n; \end{aligned}$$

when k and n range over the set of all integers?

Conclusion. *Our conclusion about $A_{n,k}$ is as follows (Eqn. (3)).*

$$A_{n,k} = \sum_{j \geq 0} \left\{ {n - j \atop k} \right\} a_j \tag{3}$$

Proof. We will prove our conclusion by induction to n.

- **(Induction Basis)** When $n = 0$, our conclusion is $A_{0,k} = \sum_{j \geq 0} \left\{ {0-j \atop k} \right\} a_j = a_0 [k = 0]$, which satisfies the condition that $A_{0,k} = 0$ $(k > 0)$ and $A_{0,0} = a_0$.

- **(Induction Step)** Assume the conclusion holds for $n = m - 1$, and we will prove that the conclusion still holds for $n = m$. Thus we can have the following derivations.

$$\begin{aligned} A_{m,k} &= kA_{m-1,k} + A_{m-1,k-1} \\ &= k\sum_{j\geq 0} \left\{ {m-1-j \atop k} \right\} a_j + \sum_{j\geq 0} \left\{ {m-1-j \atop k-1} \right\} a_j \\ &= \sum_{j\geq 0} a_j \left(k\left\{ {m-1-j \atop k} \right\} + \left\{ {m-1-j \atop k-1} \right\} \right) \\ &= \sum_{j\geq 0} \left\{ {m-j \atop k} \right\} a_j \end{aligned}$$

  The derivations above tell us the conclusion still holds for $n = m$.

Therefore, the conclusion holds for all integer n and k.

We also want to emphasize that the summation is always finite for a given n, which can be easily verified by connecting $A_{n,k}$ with the Stirling's Numbers for subsets $\left\{ {n \atop k} \right\}$. $\square$