Homework 08

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1 Warmup Problems

1. An eccentric collector of $2 \times n$ domino tilings pay \$4 for each vertical domino and \$1 for each horizontal domino. How many tilings are worth exactly \$$m$ by this criterion? For example, when $m = 6$ there are three solutions as follows.

Solution. Suppose there are T_m tilings worth exactly \$$m$ by this criterion. And let us focus on some small cases first.

- When $m = 0$, we know $T_0 = 1$, since there is a null tiling worth exact \$0 by this criterion.
- When $m = 1$, we know $T_1 = 0$, since there is no tiling worth exact \$1 by this criterion.
- When $m = 2$, we know $T_2 = 1$, since there is only one tiling worth exact \$2 by this criterion, which uses two horizontal dominos to form a 2×2 rectangle. Therefore, we have the conclusion that $T_2 = T_0$.
- When $m = 3$, we know $T_3 = 0$, since there is no tiling worth exact \$3 by this criterion.

Now let us see what will happen when $m \geq 4$.

- If we choose a horizontal domino, then we must choose another horizontal domino to form a 2×2 small rectangle. Therefore, we need \$2 in total, which leaves us T_{m-2} ways to arrange the rest of our money by this criterion.
- If we choose a vertical domino, then we it can form a 2×1 small rectangle by itself. Therefore, we need \$4 in total, which leaves us T_{m-4} ways to arrange the rest of our money by this criterion.

Therefore, we have the following recurrence (Eqn. (1)).

$$T_m = T_{m-2} + T_{m-4}, \quad (m \geq 4) \tag{1}$$

Suppose the generating function of T_m is $G(z)$, which is displayed as follows.

$$G(z) = \sum_{m \geq 0} T_m z^m$$

According to the recurrence of T_m, we can get an equation of $G(z)$ as follows.

$$\begin{aligned}
G(z) &= \sum_{m \geq 0} T_m z^m \\
&= T_0 + T_2 z^2 + \sum_{m \geq 4} (T_{m-2} + T_{m-4}) z^m \\
&= 1 + T_0 z^2 + T_1 z^3 + \sum_{m \geq 2} T_m z^{m+2} + \sum_{m \geq 0} T_m z^{m+4} \\
&= 1 + z^2 \sum_{m \geq 0} T_m z^m + z^4 \sum_{m \geq 0} T_m z^m \\
&= 1 + z^2 G(z) + z^4 G(z)
\end{aligned}$$

Solve the simple equation, we get the closed form of $G(z)$ (Eqn. (2)).

$$G(z) = \frac{1}{1 - z^2 - z^4} \tag{2}$$

It's a bit familiar! We know that the generating function for Fibonacci Numbers series $\{0, 1, 1, 2, 3, 5, 8, \cdots\}$ is

$$F(z) = \frac{z}{1 - z - z^2}$$

Then there is a simple relation between $F(z)$ and $G(z)$, which is

$$G(z) = \frac{1}{z^2} F(z^2)$$

Let us write the relation above in a summation form, which can shows the relation between T_m and Fibonacci Numbers F_m more explicitly.

$$\begin{aligned}
\sum_{m \geq 0} T_m z^m &= \frac{1}{z^2} \sum_{m \geq 0} F_m (z^2)^m \\
&= \sum_{m \geq 2} F_m z^{2m-2} \\
&= \sum_{m \geq 0} [m \text{ is even}] F_{\lfloor \frac{m}{2} \rfloor + 1} z^m
\end{aligned}$$

Therefore, we know the value of T_m by referring to the corresponding term of the right-hand side. The answer of the question is as follows (Eqn. (3)).

$$T_m = \begin{cases} 0 & (m \text{ is odd}) \\ F_{\frac{m}{2}+1} & (m \text{ is even}) \end{cases} \tag{3}$$

□

2. Give the generating function and the exponential generating function for the sequence

$$\{2, 5, 13, 35, \cdots\} = \{2^n + 3^n\}$$

in closed form.

Solution. Suppose the generating function for the sequence is $F(z)$ and the exponential generating function for the sequence is $G(z)$. It's easy to derive the closed form of generating function $F(z)$ as follows.

$$\begin{aligned}
F(z) &= \sum_{n \geq 0} (2^n + 3^n) z^n \\
&= \sum_{n \geq 0} (2z)^n + \sum_{n \geq 0} (3z)^n \\
&= \frac{1}{1 - 2z} + \frac{1}{1 - 3z}
\end{aligned}$$

The derivations of exponential generating function $G(z)$'s closed form are as follows.

$$\begin{aligned}
G(z) &= \sum_{n \geq 0} (2^n + 3^n) \frac{z^n}{n!} \\
&= \sum_{n \geq 0} \frac{(2z)^n}{n!} + \sum_{n \geq 0} \frac{(3z)^n}{n!} \\
&= e^{2z} + e^{3z}
\end{aligned}$$

Therefore, the generating function $F(z)$ and exponential generating function $G(z)$ is as follows (Eqn. (4)).

$$F(z) = \frac{1}{1-2z} + \frac{1}{1-3z}, \quad G(z) = e^{2z} + e^{3z} \tag{4}$$

□

3. What is $\sum_{n\geq 0} \frac{H_n}{10^n}$?

Solution. The convergence radius of the formula above is 1. The generating function $H(z)$ of Harmonic Numbers H_n is as follows, according to Formula (7.43) in textbook.

$$H(z) = \sum_{n\geq 0} H_n z^n = \frac{1}{1-z} \ln \frac{1}{1-z}$$

Since $\frac{1}{10}$ is within the convergence radius, we can set $z = \frac{1}{10}$ in the formula above, and we can simplify the given formula.

$$\begin{aligned}\sum_{n\geq 0} \frac{H_n}{10^n} &= \sum_{n\geq 0} H_n \left(\frac{1}{10}\right)^n \\ &= \frac{1}{1-\frac{1}{10}} \ln \frac{1}{1-\frac{1}{10}} \\ &= \frac{10}{9} \ln \frac{10}{9}.\end{aligned}$$

Therefore, the value of the given formula is $\frac{10}{9} \ln \frac{10}{9}$. □

2 Basic Problems

7. Solve the recurrence

$$\begin{aligned}&g_0 = 1 \\ &g_n = g_{n-1} + 2g_{n-2} + \cdots + ng_0, \quad \text{for } n > 0\end{aligned}$$

Solution. Suppose the generating function of $\{g_0, g_1, g_2, \cdots\}$ is $G(z)$, which means

$$G(z) = \sum_{n\geq 0} g_n z^n$$

According to the textbook, the the generating function $F(z)$ for sequence $\{0, 1, 2, 3, 4, \cdots\}$ is

$$F(z) = \frac{z}{(1-z)^2}$$

It's easy to find out that the recurrence has a form of convolution between $\{g_n\}$ and $\{0, 1, 2, 3, 4, \cdots\}$. Therefore, we can derive a equation of $G(z)$ as follows.

$$\begin{aligned}G(z) &= \sum_{n\geq 0} g_n z^n \\ &= g_0 + \sum_{n\geq 1} \left(\sum_{k=0}^{n} k g_{n-k}\right) \\ &= g_0 + \sum_{n\geq 0} \left(\sum_{k=0}^{n} k g_{n-k}\right) \\ &= 1 + F(z) \cdot G(z)\end{aligned}$$

Solve the equation, we can get the closed form of $G(z)$ as follows (Eqn. (5)).

$$G(z) = \frac{1}{1-F(z)} = \frac{1}{1-\frac{z}{(1-z)^2}} = \frac{z^2-2z+1}{z^2-3z+1} = 1 + \frac{z}{z^2-3z+1} \tag{5}$$

This closed form is very similar to Formula (7.24) in textbook, which is displayed as follows.

$$\sum_{n\geq 0} F_{2n} z^n = \frac{z}{1-3z+z^2}$$

Therefore, the generating function of $G(z)$ can be written as follows.

$$G(z) = 1 + \sum_{n\geq 0} F_{2n} z^n$$

Hence, we have the following conclusion (Eqn. (6)).

$$g_n = F_{2n} + [n=0] \tag{6}$$

□

8. What is the value of the following formula?

$$[z^n]\frac{(\ln(1-z))^2}{(1-z)^{m+1}}$$

Solution. Table 335 in the textbook tells us the following property.

$$\sum_{n\geq 0}\binom{c+n-1}{n} z^n = \frac{1}{(1-z)^c}$$

Set $x = c-1$ in the formula above, and differentiate twice with respect to x, we can make the following derivations.

$$\begin{aligned}
& \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sum_{n\geq 0}\binom{x+n}{n} z^n = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{1}{(1-z)^{x+1}} \\
\Longrightarrow\ & \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sum_{n\geq 0}\frac{(x+n)^{\underline{n}}}{n!} z^n = \frac{(\ln(1-z))^2}{(1-z)^{x+1}} \\
\Longrightarrow\ & \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n\geq 0}\frac{(x+n)^{\underline{n}}}{n!}(H_{x+n}-H_x) z^n = \frac{(\ln(1-z))^2}{(1-z)^{x+1}} \\
\Longrightarrow\ & \sum_{n\geq 0}\frac{(x+n)^{\underline{n}}}{n!}((H_{x+n}-H_x)^2 - (H_{x+n}^{(2)} - H_x^{(2)})) z^n = \frac{(\ln(1-z))^2}{(1-z)^{x+1}} \\
\Longrightarrow\ & \sum_{n\geq 0}\binom{x+n}{n}((H_{x+n}-H_x)^2 - (H_{x+n}^{(2)} - H_x^{(2)})) z^n = \frac{(\ln(1-z))^2}{(1-z)^{x+1}}
\end{aligned}$$

where, $H_x^{(2)}$ stands for

$$\sum_{1\leq k\leq x}\frac{1}{k^2}$$

Set $x = m$, then the value of the formula is as follows (Eqn. (7))

$$[z^n]\frac{(\ln(1-z))^2}{(1-z)^{m+1}} = \binom{m+n}{n}((H_{m+n}-H_m)^2 - (H_{m+n}^{(2)} - H_m^{(2)})) \tag{7}$$

□

9. Use the result of the previous exercise to evaluate $\sum_{k=0}^{n} H_k H_{n-k}$.

Solution. According to (7.43) in textbook, we know that the generating function $H(z)$ for Harmonic Numbers H_n is as follows.

$$H(z) = \sum_{n\geq 0} H_n z^n = \frac{1}{1-z} \ln \frac{1}{1-z}$$

Suppose the generating function for the number series $\{\sum_{k=0}^{n} H_k H_{n-k}\}$ is $H^*(z)$, then it's obvious that $H^*(z) = H^2(z)$ since $\sum_{k=0}^{n} H_k H_{n-k}$ is a convolution form. Hence,

$$H^*(z) = \frac{(\ln(1-z))^2}{(1-z)^2}$$

Therefore, using the result of the last exercise, we can get the value of the given formula as follows.

$$\begin{aligned}
\sum_{k=0}^{n} H_k H_{n-k} &= [z^n]H^*(z) \\
&= [z^n]\frac{(\ln(1-z))^2}{(1-z)^2} \\
&= \binom{1+n}{n}((H_{1+n} - H_1)^2 - (H_{1+n}^{(2)} - H_1^{(2)})) \\
&= (n+1)\left(\left(H_n + \frac{1}{n+1} - 1\right)^2 - H_n^{(2)} - \frac{1}{(n+1)^2} + 1\right) \\
&= (n+1)\left((H_n - 1)^2 + 2\frac{H_n - 1}{n+1} - H_n^{(2)} + 1\right) \\
&= (n+1)\left(H_n^2 - 2H_n + 2\frac{H_n - 1}{n+1} - H_n^{(2)} + 2\right) \\
&= (n+1)(H_n^2 - H_n^{(2)}) - 2(H_n - 1)(n+1) + 2(H_n - 1) \\
&= (n+1)(H_n^2 - H_n^{(2)}) - 2n(H_n - 1)
\end{aligned}$$

Hence, the result of the given formula is

$$(n+1)(H_n^2 - H_n^{(2)}) - 2n(H_n - 1)$$

□

3 Homework Exercises

21. **(Bonus Problem)** A robber holds up a bank and demands \$500 in tens and twenties. He also demands to know the number of ways in which the cashier can give him the money. Find a generating function $G(z)$ for which this number is $[z^{500}]G(z)$, and a more compact generating function $\check{G}(z)$ for which this number is $[z^{50}]\check{G}(z)$. Determine the required number of ways by (a) using partial fractions; (b) using a method like (7.39).

Solution. The generating function for only using \$10 is $\frac{1}{1-z^{10}}$, and similarly, the generating function for only using \$20 is $\frac{1}{1-z^{20}}$. Therefore, the generating function $G(z)$ for using \$10 and \$20 is as follows (Eqn. (8))

$$G(z) = \frac{1}{1-z^{10}} \cdot \frac{1}{1-z^{20}} \tag{8}$$

According to the definition of $\check{G}(z)$, we know that

$$\check{G}(z^{10}) = G(z) \quad \Longrightarrow \quad \check{G}(z) = \frac{1}{(1-z)(1-z^2)}$$

a. **(Partial Fractions Decomposition)** Suppose $\check{G}(z)$ can be decomposed as follows.

$$\check{G}(z) = \frac{1}{(1-z)(1-z^2)} = \frac{1}{(1-z)^2(1+z)} = \frac{A}{1+z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

where A, B, C are const numbers. Then we can simplify the formula above.

$$\begin{aligned}\check{G}(z) &= \frac{A(1-z)^2 + B(1-z^2) + C(1+z)}{(1-z)^2(1+z)} \\ &= \frac{(A-B)z^2 + (C-2A)z + A + B + C}{(1-z)^2(1+z)}\end{aligned}$$

According to the previous result, the numerator should be 1. Thus,

$$\begin{cases} A - B = 0 \\ C - 2A = 0 \\ A + B + C = 1 \end{cases}$$

Solve the equations then we can get $A = \frac{1}{4}, B = \frac{1}{4}, C = \frac{1}{2}$. Hence we can rewrite $\check{G}(z)$ as follows (Eqn. (9)).

$$\begin{aligned}\check{G}(z) &= \frac{1}{4} \cdot \frac{1}{1+z} + \frac{1}{4} \cdot \frac{1}{1-z} + \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\ &= \frac{1}{4}\sum_{n\geq 0}(-z)^n + \frac{1}{4}\sum_{n\geq 0} z^n + \frac{1}{2}\sum_{n\geq 0}(n+1)z^n \\ &= \sum_{n\geq 0}\frac{1}{4}(2n+3+(-1)^n)z^n\end{aligned} \tag{9}$$

Therefore,

$$[z^n]\check{G}(z) = \frac{1}{4}(2n+3+(-1)^n) \tag{10}$$

Plug $n = 50$ into the Equation (10), and we get $[z^{50}]\check{G}(z) = 26$. So there are 26 ways in which the cashier can give the robber the money.

b. **(Method like (7.39))** We can rewrite $\check{G}(z)$ as follows (Eqn. (11)).

$$\begin{aligned}\check{G}(z) &= \frac{1+z}{(1-z^2)^2} \\ &= (1+z)\sum_{n\geq 0}(n+1)(z^2)^n \\ &= \sum_{n\geq 0}(n+1)z^{2n} + \sum_{n\geq 0}(n+1)z^{2n+1} \\ &= \sum_{n\geq 0}\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)z^n\end{aligned} \tag{11}$$

Therefore,

$$[z^n]\check{G}(z) = \left\lfloor\frac{n}{2}\right\rfloor + 1 \tag{12}$$

Plug $n = 50$ into the Equation (12), and we get $[z^{50}]\check{G}(z) = 26$. So there are 26 ways in which the cashier can give the robber the money.

□