Homework 05

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1. What is 11^4? Why is this number easy to compute for a person who knows binomial coefficients?

   Solution. Actually, $11^4 = 14641$, which is exactly the binomial coefficients arranged in order. This is because of the following equation (Equation (1)).

$$11^n = (10+1)^n = \sum_{k=0}^{n} \binom{n}{k} 10^k \tag{1}$$

   where, $\binom{n}{k}$ is the binomial coefficients. Therefore, we can compute 11^n easily by combining the binomial coefficients together in order.

   What's more, we have $(11)_r^4 = (14641)_r$, where r is the radix number satisfying $r \geq 7$, which is a more general conclusion. □

2. For which value(s) of k is $\binom{n}{k}$ a maximum, when n is a given positive integer? Prove your answer.

   Conclusion. *The maximum occurs when $k = \lfloor n/2 \rfloor$ and $k = \lceil n/2 \rceil$.*

   Proof. Considering two consecutive binomial coefficients $\binom{n}{k}$ and $\binom{n}{k+1}$, we have Equation (2).

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n-k}{k+1} \tag{2}$$

   Therefore,

   - When $k < \lceil \frac{n}{2} \rceil$, we have $\frac{n-k}{k+1} \geq 1$, which means $\binom{n}{k+1} \geq \binom{n}{k}$.
   - When $k \geq \lfloor \frac{n}{2} \rfloor$, we have $\frac{n-k}{k+1} \leq 1$, which means $\binom{n}{k+1} \leq \binom{n}{k}$.

   Thus,

   - If n is an odd number, then the maximum occurs when $k = \lfloor n/2 \rfloor$ and $k = \lceil n/2 \rceil$;
   - If n is an even number, then the maximum occurs when $k = \lfloor n/2 \rfloor = \lceil n/2 \rceil = n/2$.

   In conclusion, the maximum occurs when $k = \lfloor n/2 \rfloor$ and $k = \lceil n/2 \rceil$. □

3. Prove the hexagon property.

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}$$

   Proof. We can prove the property by simply unfold the notations of binomial coefficients, as Equation (3) shown.

$$\begin{aligned}
\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{n!}{(k+1)!(n-k-1)!} \cdot \frac{(n+1)!}{k!(n-k+1)!} \\
&= \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{(n+1)!}{(k+1)!(n-k)!} \cdot \frac{n!}{(k-1)!(n-k+1)!} \\
&= \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}
\end{aligned} \tag{3}$$

   □

4. Evaluate $\binom{-1}{k}$ by negating (actually un-negating) its upper index.

Solution. With formula (5.14) in textbook, we can derive Equation (4).

$$\binom{-1}{k} = (-1)^k \binom{k-(-1)-1}{k} = (-1)^k \binom{k}{k} = (-1)^k \tag{4}$$

$\square$

5. Let p be prime. Show that $\binom{p}{k} \bmod p = 0$ for $0 < k < p$. What does this imply about the binomial coefficients $\binom{p-1}{k}$?

Solution. We can derive that $p \mid \binom{p}{k}$ as follows.

$$\binom{p}{k} = \frac{p!}{(p-k)!k!} = p \cdot \frac{(p-1)!}{(p-k)!k!} \quad \Longrightarrow \quad p \mid \binom{p}{k}$$

The result indicates that $\binom{p}{k} \bmod p = 0$ for $0 < k < p$.
Suppose $\binom{p-1}{k} \equiv f(k) \pmod p$. Because of the facts that $\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}$ and $\binom{p}{k} \bmod p = 0$, we can derive the following equation (Equation (5)).

$$f(k) + f(k-1) \equiv 0 \pmod p \tag{5}$$

Therefore, we have $f(k) \equiv f(k-2) \pmod p$ for $k \geq 2$.
Then we are going to determine the values of $f(0)$ and $f(1)$. With the facts that $\binom{p-1}{0} = 1$ and $\binom{p-1}{1} = p-1$, we can set $f(0) = 1$ and $f(1) = -1$ since $-1 \equiv p-1 \pmod p$. Therefore, we can derive the general formula $f(k) = (-1)^k$. The property of binomial coefficients $\binom{p-1}{k}$ is as follows (Equation (6)).

$$\binom{p-1}{k} \equiv (-1)^k \pmod p \tag{6}$$

$\square$

6. Fix up the text's derivation in Problem 6, Section 5.2, by correctly applying symmetry.

Solution. We must add a restriction $k \geq 0$ when we use symmetry. Then we fix a part of the derivation as follows (Equation (7)).

$$\begin{aligned}
\frac{1}{n+1}\sum_k \binom{n+k}{k}\binom{n+1}{k+1}(-1)^k &= \frac{1}{n+1}\sum_{k\geq 0} \binom{n+k}{(n+k)-k}\binom{n+1}{k+1}(-1)^k \\
&= \frac{1}{n+1}\sum_{k\geq 0} \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k \\
&= \frac{1}{n+1}\left(-\binom{n-1}{n}\binom{n+1}{0}(-1)^{-1} + \sum_k \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k\right) \\
&= [n=0] + \frac{1}{n+1}(-1)^n\binom{n-1}{-1} \\
&= [n=0]
\end{aligned} \tag{7}$$

Then the answer $[n=0]$ is the correct answer according to textbook. $\square$

7. Is the following formula (Equation (8)) true also when $k < 0$?

$$r^{\underline{k}}\left(r-\frac{1}{2}\right)^{\underline{k}} = \frac{(2r)^{\underline{2k}}}{2^{2k}} \tag{8}$$

Solution. In Chapter 2, we talk about the properties of falling power, including extending its exponent to all the integers. Therefore, we have the following formula (Equation (9)).

$$a^{\underline{-m}} = \frac{1}{(a+1)(a+2)\cdots(a+m)} = \prod_{i=1}^{m}\frac{1}{a+i} \tag{9}$$

Now let's find out what happens if we have $k < 0$ in Equation (8). We assume that $k = -m$ and therefore m is a positive integer, that is, $m \in \mathbb{Z}^{+}$. We can make the following derivations (Equation (10)).

$$\begin{aligned}
r^{\underline{k}}\left(r-\frac{1}{2}\right)^{\underline{k}} &= r^{\underline{-m}}\left(r-\frac{1}{2}\right)^{\underline{-m}} \\
&= \left(\prod_{i=1}^{m}\frac{1}{r+i}\right)\cdot\left(\prod_{i=1}^{m}\frac{1}{r-\frac{1}{2}+i}\right) \\
&= \prod_{i=1}^{2m}\frac{2}{2r+i} \\
&= 2^{2m}\cdot(2r)^{\underline{-2m}} \\
&= \frac{(2r)^{\underline{2k}}}{2^{2k}}
\end{aligned} \tag{10}$$

Therefore, the formula still holds for $k < 0$.

What's more, with the similar methods, we know that the formula of ascending power also holds, that is,

$$r^{\overline{k}}\left(r-\frac{1}{2}\right)^{\overline{k}} = \frac{(2r)^{\overline{2k}}}{2^{2k}} \qquad (k \in \mathbb{Z})$$

□

8. Evaluate

$$\sum_{k}\binom{n}{k}(-1)^{k}\left(1-\frac{k}{n}\right)^{n}$$

What is the approximate value of this sum, when n is very large? Hint: The sum is $\Delta^{n} f(0)$ for some function f.

Solution. We define a function $f(\cdot)$ as follows (Equation (11)).

$$f(x) = \left(\frac{x}{n}-1\right)^{n} \tag{11}$$

Therefore, according to formula (5.40) in the textbook, we can make the following derivation.

$$\begin{aligned}
\Delta^{n} f(0) &= \sum_{k}\binom{n}{k}(-1)^{n-k} f(k) \\
&= \sum_{k}\binom{n}{k}(-1)^{n+k}\left(\frac{k}{n}-1\right)^{n} \\
&= \sum_{k}\binom{n}{k}(-1)^{k}\left(1-\frac{k}{n}\right)^{n}
\end{aligned}$$

The derivation shows that $\Delta^n f(0)$ equals to the formula in the problem. Then we make the following derivations (Equation (12)).

$$\begin{aligned}
\Delta^n f(0) &= \Delta^n f(x)\big|_{x=0} \\
&= \Delta^n \left(\frac{x}{n} - 1\right)^n\bigg|_{x=0} \\
&= \Delta^n \left(\sum_k \binom{n}{k} (-1)^{n-k} \left(\frac{x}{n}\right)^k\right)\Bigg|_{x=0} \\
&= \Delta^n \left(\frac{x}{n}\right)^n\bigg|_{x=0} \\
&= \frac{n!}{n^n}
\end{aligned} \tag{12}$$

Therefore, we know that $\Delta^n f(0) = \frac{n!}{n^n}$, which indicates that the original formula in the problem also equals to $\frac{n!}{n^n}$. When n is very large, the Stirling Approximation tells us that:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Therefore, we know the following formula (Equation (13)).

$$\begin{aligned}
\sum_k \binom{n}{k} (-1)^k \left(1 - \frac{k}{n}\right)^n &= \Delta^n f(0) \\
&= \frac{n!}{n^n} \\
&\approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot n^n \\
&= \sqrt{2\pi n} e^{-n}
\end{aligned} \tag{13}$$

In summary, when n is very large, the approximation value of the formula in the problem statement is $\sqrt{2\pi n} e^{-n}$. $\square$

9. Show that the generalized exponentials of (5.58) in the textbook obey the law

$$\varepsilon_t(z) = \varepsilon(tz)^{\frac{1}{t}} \qquad (t \neq 0)$$

where $\varepsilon(z)$ is an abbreviation for $\varepsilon_1(z)$.

Solution. Apply $r = t$ in formula (5.60) in the textbook, then we have the following equation.

$$\varepsilon_t(z)^t = \sum_{k \geq 0} t \frac{(tk+t)^{k-1}}{k!} z^k \tag{14}$$

According to the definition of $\varepsilon_t(z)$ in formula (5.58) in the textbook and Equation (14), we can make the following derivations (Equation (15)).

$$\begin{aligned}
\varepsilon(tz) &= \sum_{k \geq 0} (k+1)^{k-1} \frac{(tz)^k}{k!} \\
&= \sum_{k \geq 0} (tk+t)^{k-1} \frac{tz^k}{k!} \\
&= \varepsilon_t(z)^t
\end{aligned} \tag{15}$$

Therefore, $\varepsilon_t(z) = \varepsilon(tz)^{\frac{1}{t}}$, which completes the proof. $\square$