Homework 02

CS499-Mathematical Foundations of Computer Science, Jie Li, Spring 2020.

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1. What does the following notation mean?

$$\sum_{k=4}^{0} q_k$$

Solution. The meaning of the formula depends on the specific situation.

- It can have the same meaning as $\sum_{4 \le k \le 0} q_k$, so the value of this formula is 0.
- It can have the same meaning as $\sum_{k \le 0} q_k - \sum_{k \le 4} q_k$. As a result, the value of this formula is $-q_1 - q_2 - q_3 - q_4$.

□

2. Simplify the expression $x \cdot ([x > 0] - [x < 0])$.

Solution. Let $f(x)$ be $x \cdot ([x > 0] - [x < 0])$. Then we are able to derive the following equation (Equation (1)).

$$f(x) = \left\{ \begin{array}{ll} x \cdot (1 - 0) = x & (x > 0) \\ 0 \cdot (0 - 0) = 0 & (x = 0) \\ x \cdot (0 - 1) = -x & (x < 0) \end{array} \right\} = |x| \tag{1}$$

Thus, $x \cdot ([x > 0] - [x < 0])$ can be simplified to $|x|$. □

3. Demonstrate your understanding of Σ-notation by writing out the following sums in full.

$$\sum_{0 \le k \le 5} a_k \qquad \text{and} \qquad \sum_{0 \le k^2 \le 5} a_{k^2}$$

Solution. The sums can be written as follows (Equation (2)).

$$\begin{aligned} \sum_{0 \le k \le 5} a_k &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 \\ \sum_{0 \le k^2 \le 5} a_{k^2} = \sum_{-2 \le k \le 2} a_{k^2} &= a_0 + 2a_1 + 2a_4 \end{aligned} \tag{2}$$

□

4. Express the following triple sum as a three-fold summation (with three Σ's).

$$\sum_{1 \le i < j < k \le 4} a_{ijk}$$

(a) summing first on k, then j, then i;

(b) summing first on i, then j, then k.

Also write your triple sums out in full without the Σ-notation, using parentheses to show what is being added together first.

Solution. The sums can be written as follows (Equation (3) and Equation (4)).

(a)

$$\sum_{1\le i<j<k\le 4} a_{ijk} = \sum_{i=1}^{2}\sum_{j=i+1}^{3}\sum_{k=j+1}^{4} a_{ijk} = ((a_{123}+a_{124})+a_{134})+a_{234} \tag{3}$$

(b)

$$\sum_{1\le i<j<k\le 4} a_{ijk} = \sum_{k=3}^{4}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} a_{ijk} = a_{123}+(a_{124}+(a_{134}+a_{234})) \tag{4}$$

□

5. What's wrong with the following derivation?

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k} = \sum_{k=1}^{n} n = n^2$$

Solution. We can't change the variable j's name to k, because the formula already has a variable k in the inner summation. Therefore, we can only derive the following result (Equation (5)).

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k} \tag{5}$$

□

6. What is the value of $\sum_k [1 \le j \le k \le n]$, as a function of j and n?

Solution. We define $f(j,n)$ as the summation above. Thus, we can derive the following answer (Equation (6)).

$$f(j,n) = \sum_k [1\le j\le k\le n] = \begin{cases} 0 & (j<1 \text{ or } \ j>n) \\ \displaystyle\sum_{j\le k\le n} 1 = n-j+1 & (1\le j\le n) \end{cases} \tag{6}$$

□

7. Let $\nabla f(x) = f(x) - f(x-1)$. What is $\nabla(x^{\overline{m}})$?

Solution. $\nabla(x^{\overline{m}})$ can be calculated as follows (Equation (7)).

$$\begin{aligned} \nabla(x^{\overline{m}}) &= x^{\overline{m}} - (x-1)^{\overline{m}} \\ &= \prod_{i=0}^{m-1}(x+i) - \prod_{i=0}^{m-1}(x-1+i) \\ &= (x+m-1-(x-1))\prod_{i=0}^{m-2}(x+i) \\ &= mx^{\overline{m-1}} \end{aligned} \tag{7}$$

□

8. What is the value of $0^{\underline{m}}$, where m is a given integer?

   Solution.

$$
0^{\underline{m}} = \begin{cases} \prod\limits_{i=1-m}^{0} i = 0 & (m > 0) \\ 1 & (m = 0) \\ \prod\limits_{i=1}^{-m} \dfrac{1}{i} = \dfrac{1}{(-m)!} & (m < 0) \end{cases} \qquad (m \in \mathbb{N}) \tag{8}
$$

   We assume $0! = 1$, then we can simplify Equation (8) to the following equation (Equation (9))

$$
0^{\underline{m}} = \begin{cases} 0 & (m > 0) \\ \dfrac{1}{(-m)!} & (m \leq 0) \end{cases} \qquad (m \in \mathbb{N}) \tag{9}
$$

   □

9. What is the law of exponents for rising factorial powers, analogous to (2.52) in the textbook? Use this to define $x^{\overline{-n}}$.

   Solution. The law of exponents for rising factorial powers is as follows.

$$
x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}} \qquad (m, n \in \mathbb{Z}) \tag{10}
$$

   Thus, we can define $x^{\overline{-n}}$ by applying $m = -n$ and $n = n$ in Equation (10), and we can derive the following formulas.

$$
\begin{aligned} & x^{\overline{-n}}(x-n)^{\overline{n}} = x^{\overline{-n+n}} = x^{\overline{0}} = 1 \\ \Longrightarrow \quad & x^{\overline{-n}} = \frac{1}{(x-n)^{\overline{n}}} = \frac{1}{(x-1)^{\underline{n}}} \qquad (n \in \mathbb{N}_+) \end{aligned} \tag{11}
$$

   With Equation (11), we define $x^{\overline{-n}}$ as $\frac{1}{(x-1)^{\underline{n}}}$ when $n \in \mathbb{N}_+$. □

10. The text derives the following formula for the difference of a product:

$$
\Delta(uv) = u\Delta v + Ev\Delta u \tag{12}
$$

   How can this formula be correct, when the left-hand side is symmetric with respect to u and v but the right-hand side is not?

   Solution. The derivation of Equation (12) is on the textbook, so it is obviously correct. Actually, on one hand, we have the Equation (12); on the other hand, we can have the following equation (Equation (13)).

$$
\Delta(uv) = v\Delta u + Eu\Delta v \tag{13}
$$

   The two right-hand sides of Equation (12) and Equation (13) are symmetric.

   What's more, if u and v are differentiable functions, then we can substitute the notation Δ with the notation D by taking the limit as the delta number of independent variable goes to 0. As a result, Ev equals to v under D. So the equation becomes $\mathrm{D}(uv) = u\mathrm{D}u + v\mathrm{D}u$, which is symmetric on both sides. □