EE447 Mobile Internet HW 4
Heterogeneous Graphs

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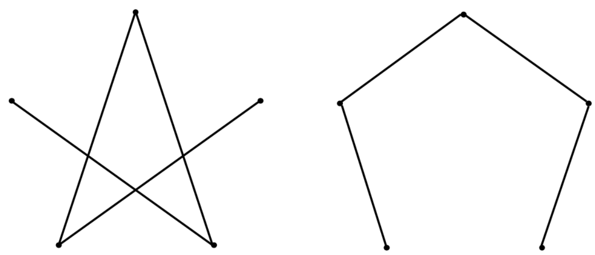
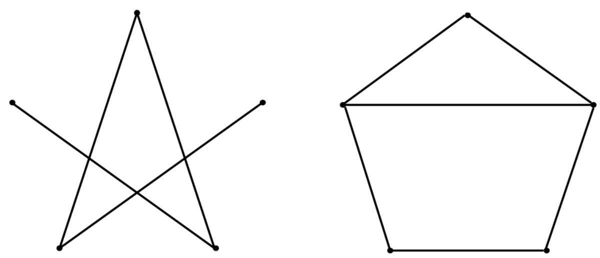
Problem[1].

Given 5 identical nodes, how many different networks can be built? Use symmetry and complementarity to enumerate as few network structures as possible.

Symmetry: As shown in the figure below, two diagrams remain isomorphic under different edge relations, and only one of them should be included in the enumeration of "networks that are not isomorphic to each other".



Complementarity: As shown in the figure below, the two diagrams overlap to form a complete diagram, so the former structure can be directly derived from the latter structure given in the enumeration.



Solution. First we consider complementarity. Since there is totally $5 \times (5-1) \times \frac{1}{2} = 10$ edges in a graph which consists of 5 identical nodes, we just need to enumerate the number of edges from 0 to 5 to eliminate complementarity, which is based on the following lemmas.

Lemma 1. *For a graph consists of n nodes and m edges, there is a corresponding graph consists of n nodes and $\left(\frac{n(n-1)}{2} - m\right)$ edges as its complementary graph; and vice versa.*

Lemma 2. *For two heterogeneous graphs which consists of n nodes and m edges each, their complementary graphs are also heterogeneous; and vice versa.*

[1]Translated by DeepL Translator: `https://www.deepl.com/translator`

From Lemma 1 and Lemma 2 we can draw the conclusion that we can enumerate m from 0 to $\left\lceil \frac{n(n-1)}{4} \right\rceil$ to count the heterogeneous graphs.

Then, let us consider that the number of heterogeneous graphs which consist of n nodes and m edges, where $0 \le m \le \left\lceil \frac{n(n-1)}{4} \right\rceil$. In the settings of this problem, we just need to count the number of heterogeneous graphs of $n = 5$ and $0 \le m \le 5$. For graphs of n nodes and m edges, we can enumerate the degree of nodes to prevent omissions. For example, when $n = 5$ and $m = 4$, the sum of node degrees will be $2m = 8$, therefore we can enumerate the multi-set of node degrees such as $[4, 1, 1, 1, 1]$, $[3, 2, 2, 1, 0]$, $[3, 2, 1, 1, 1]$, $[2, 2, 2, 1, 1]$ and $[2, 2, 2, 2, 0]$, and the corresponding graphs are listed as Fig. 1 shown.

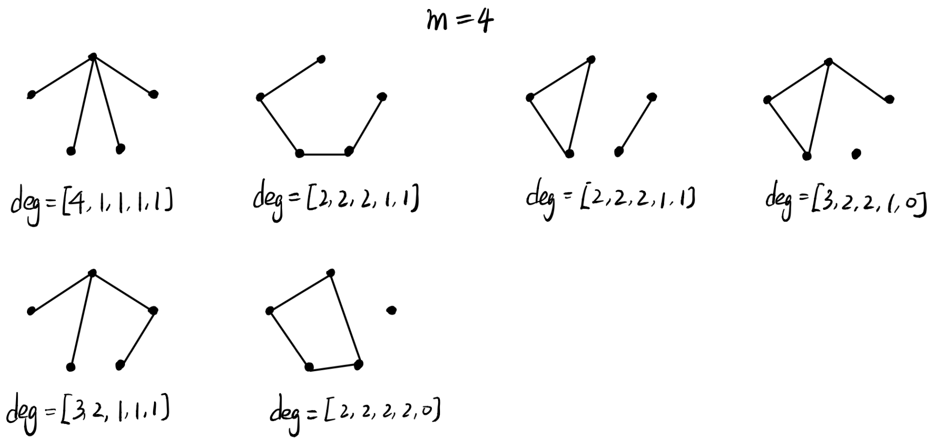


Figure 1: The heterogeneous graphs when $n = 5, m = 4$.

Similarly, we can list the heterogeneous graphs when $m = 0, 1, 2, 3$ and $m = 5$, as shown in Fig. 2 and Fig. 3.

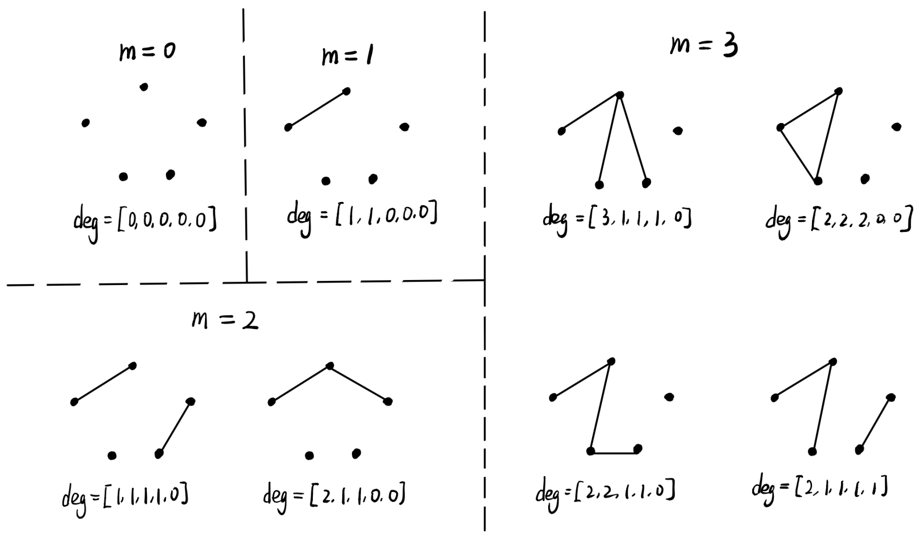


Figure 2: The heterogeneous graphs when $n = 5, m = 0, 1, 2, 3$.

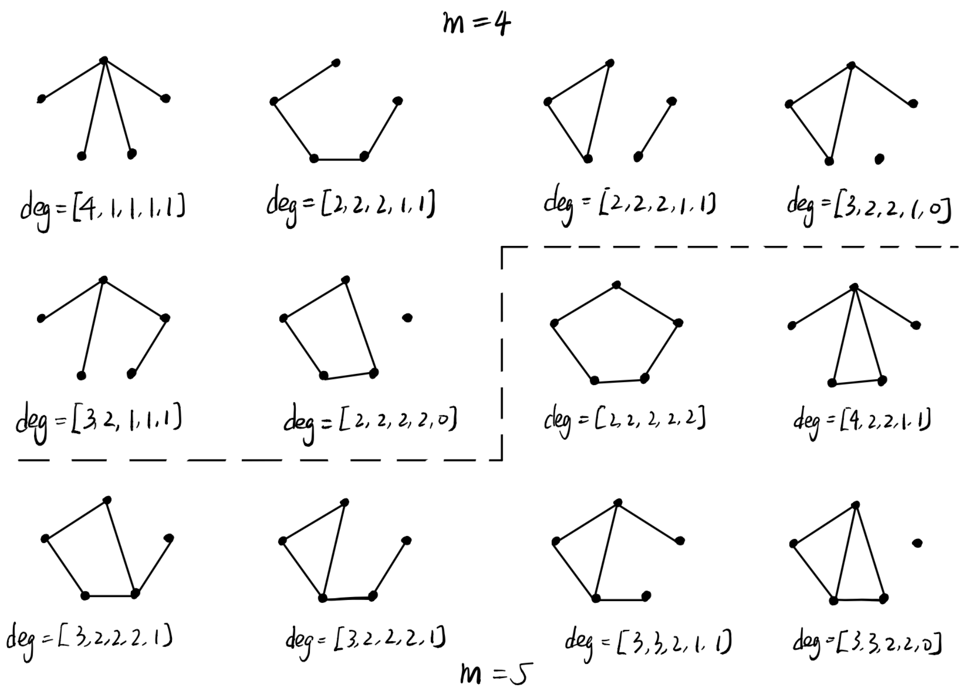


Figure 3: The heterogeneous graphs when $n = 5, m = 4, 5$.

Notice that for $m = 5$, since $2m = 10 = \frac{n(n-1)}{2}$, the complementarity are inside the graph set of $m = 5$. Suppose for n nodes and m edges, there are $N(n, m)$ heterogeneous graphs. Therefore Lemma 1 and Lemma 2 states that,

$$N(n, m) = N\left(n, \frac{n(n-1)}{2} - m\right)$$

Combining the previous lemmas with the result in Fig. 2 and Fig. 3, we can know that

- $N(5, 0) = 1$;
- $N(5, 1) = 1$;
- $N(5, 2) = 2$;
- $N(5, 3) = 4$;
- $N(5, 4) = 6$;
- $N(5, 5) = 6$;
- $N(5, 6) = N(5, 4) = 6$;
- $N(5, 7) = N(5, 3) = 4$;
- $N(5, 8) = N(5, 2) = 2$;

- $N(5,9) = N(5,1) = 1$;
- $N(5,10) = N(5,0) = 1$;

Therefore, for $n = 5$, there are totally

$$\sum_{m=0}^{\frac{5\cdot(5-1)}{2}} N(5,m) = 34$$

heterogeneous graphs, that is, given 5 identical nodes, totally 34 different networks can be built. We can double-check the result using OEIS[2], an online encyclopedia of integer sequences website. As shown in Fig. 4, we can see that our result is correct.

A000088 Number of graphs on n unlabeled nodes. 212
(Formerly M1253 N0479)

1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864, 165091172592, 50502031367952, 29054155657235488, 31426485969804308768, 64001015704527557894928, 245935864153532932683719776, 17875777251456117005478781908 48, 24637809253125004524383007491432768 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS Euler transform of the sequence A001349.
Also, number of equivalence classes of sign patterns of totally nonzero symmetric n X n matrices.

Figure 4: The result using OEIS

□

[2] http://oeis.org/