Assignment 8

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May 31, 2022

LaTeX

Outline

Problem Statement

Papoulis Ch-6 Ex 6.34

$\mathbf{x}$ and $\mathbf{y}$ are independent and identically distributed normal random variables with zero mean and variance σ^2. Define

$$\mathbf{u} = \frac{\mathbf{x}^2 - \mathbf{y}^2}{\sqrt{\mathbf{x}^2 + \mathbf{y}^2}} \quad \mathbf{v} = \frac{2\mathbf{x}\mathbf{y}}{\sqrt{\mathbf{x}^2 + \mathbf{y}^2}} \tag{1}$$

(a) Find the joint p.d.f $f_{\mathbf{uv}}(u, v)$ of the random variables $\mathbf{u}$ and $\mathbf{v}$.
(b) Show that $\mathbf{u}$ and $\mathbf{v}$ are independent normal random variables.
(c) Show that $\frac{[(\mathbf{x}-\mathbf{y})^2 - 2\mathbf{y}^2]}{\sqrt{\mathbf{x}^2+\mathbf{y}^2}}$ is also a normal random variables.

Thus nonlinear function of normal random variables can lead to normal random variables!(This result is due to Shepp.)

Solution

Joint Density

Let $g(x, y)$ and $h(x, y)$ be two continuous and differentiable function such that

$$g(x, y)y = z \quad h(x, y) = w \tag{2}$$

For a given point (z,w), (2) can have many solutions. Let us say $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$ represent these multiple solutions such that

$$g(x_i, y_i) = z \quad h(x_i, y_i) = w \tag{3}$$

Finally,

$$f_{zw}(z, w) = \sum_i \frac{1}{|J(x_i, y_i)|} f_{xy}(x_i, y_i) \tag{4}$$

where the determinant $J(x_i, y_i)$ represents the Jacobian of orignal transformation given by:

Solution

$$J(x_i, y_i) = \begin{vmatrix} \frac{\delta g}{\delta x} & \frac{\delta g}{\delta y} \\ \frac{\delta h}{\delta x} & \frac{\delta h}{\delta y} \end{vmatrix}_{x=x_i, y=y_i} \tag{5}$$

Joint Density Function

If x and y are zero mean independent random variables, then

$$f_{x,y}(x, y) = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}} \tag{6}$$

Let $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ where θ vary in the interval $(-\pi, \pi)$.

$$f_{r,\theta}(r, \theta) = rf_{xy}(x, y) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} = f_r(r)f_\theta(\theta) \quad 0 < r < \infty \quad |\theta| < \pi \tag{7}$$

Note: r and θ are independent random variables

Solution

(a) Let,

$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}(y/x) \tag{8}$$

From (7), we have r and θ as independent random variables. In term of r and θ we get, $x = r\cos\theta$ and $y = r\sin\theta$ and hence we obtain

$$u = \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = r\cos 2\theta = g(r,\theta) \tag{9}$$

$$v = \frac{2xy}{\sqrt{x^2 - y^2}} = r\sin 2\theta = h(r,\theta) \tag{10}$$

This gives Jacobian $J(r,\theta)$ (independent of θ) as

$$J(r,\theta) = \begin{vmatrix} \frac{\delta g}{\delta r} & \frac{\delta g}{\delta \theta} \\ \frac{\delta h}{\delta r} & \frac{\delta h}{\delta \theta} \end{vmatrix} = \begin{vmatrix} \cos 2\theta & -2r\sin 2\theta \\ \sin 2\theta & 2r\cos 2\theta \end{vmatrix} = 2r = 2\sqrt{u^2 + v^2} \tag{11}$$

Solution

Since x and y are independent and identically distributed normal random variables. Therefor There will be two soution $(x_1, y_1)$and (x_2, y_2) or (r_1, θ_1) and (r_2, θ_2) of the equation $(u, v) = (g(r, \theta), h(r, \theta))$ for some u and v

And, Since x and y are i.i.d random variables, there p.d.f are same at these two solution, Therefore

$$r_1 = r_2 \quad 2\theta_2 = \pi + 2\theta_1 \implies f_{r\theta}(r_1, \theta_1) = f_{r\theta}(r_2, \theta_2) \tag{12}$$

Now, By (4) and (7), we get

$$f_{uv}(u, v) = \frac{f_{r\theta}(r_1, \theta_1)}{J(r_1, \theta_1)} + \frac{f_{r\theta}(r_2, \theta_2)}{J(r_2, \theta_2)} f = \frac{2}{J(r, \theta)} f_{r\theta}(r_1, \theta_1) \tag{13}$$

$$= \frac{2}{2r} \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} = \frac{1}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} \tag{14}$$

Soltuion

(b) From equation (14), we obtain

$$f_{uv}(u, v) = \frac{1}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} \tag{15}$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-u^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-v^2/2\sigma^2} \tag{16}$$
$$= f_u(u) f_v(v) \tag{17}$$

Thus, u and v are independent normal random variables.

Solution

(c) Let z be a random varaible such that

$$z = \frac{[(x-y)^2 - 2y^2]}{\sqrt{x^2+y^2}} = \frac{[(x^2-y^2) - 2yx]}{\sqrt{x^2+y^2}} \tag{18}$$

$$= u - v \sim N(0, 2\sigma^2) \tag{19}$$

Thus, z is a normal random variable.