ASSIGNMENT

CS21BTECH11020 (Harsh Goyal)

CONTENTS

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1

1.1. Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the file:

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/1.1.c
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/header/coeffs.h
```

and compile and execute the C program using

```
$ gcc 1.1.c -lm -Wall -g
$ ./a.out
```

1.2. Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

Solution: The following code plots Fig. 1.2

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/1.2.py
```

It is executed with

```
$ python3 1.2.py
```

$$F_U(x) = Pr(U \leq x) \tag{1.2.1}$$

Graph of CDF is as follow:

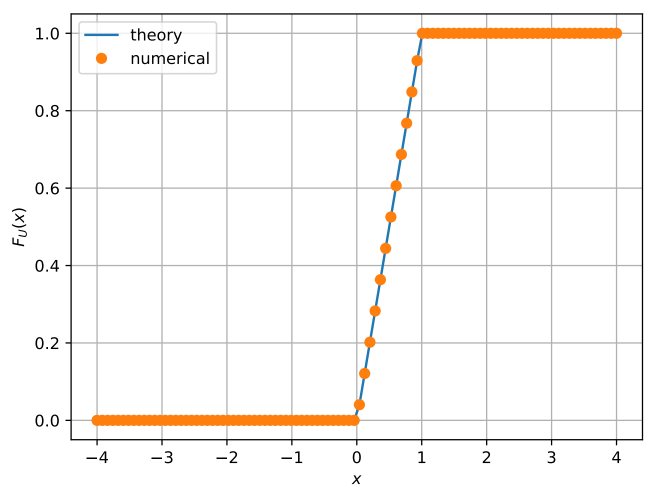

Fig. 1.2.1. CDF of U

1.3. Find a theoretical expression for $F_U(x)$.

Solution: Since We have,

$$P_U(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & otherwise \end{cases} \tag{1.3.1}$$

on integrating for CDF we get,

$$F_U(x) = \int_{-\infty}^{x} P_U(t)dt \tag{1.3.2}$$

$$F_U(x) = \begin{cases} \int_{-\infty}^{x} 0dx & x \in (-\infty, 0) \\ \int_{0}^{x} 1dx & x \in (0,1) \\ \int_{0}^{1} 1dx & x \in (1, \infty) \end{cases} \tag{1.3.3}$$

$$F_U(x) = \begin{cases} 0 & x \in (-\infty, 0) \\ x & x \in (0,1) \\ 1 & x \in (1,\infty) \end{cases} \tag{1.3.4}$$

1.4. Write a C program to find the mean and variance of U.

Solution: download C program

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/1.4.c
```

and compiled and executed with

```
$ gcc 1.4.c -lm -Wall -g
$ ./a.out
```

$$E[U] = 0.500007 \tag{1.4.1}$$
$$\text{Var}[U] = 0.083301 \tag{1.4.2}$$

1.5. Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.5.1}$$

Solution: we have

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.5.2}$$
$$= \int_0^1 x dx \tag{1.5.3}$$
$$= 0.5 \tag{1.5.4}$$

From (1.4.1), we have,

$$E[U] = 0.500007 \approx 0.5 \tag{1.5.5}$$

Similarly,

$$\text{Var}[U] = E[U^2] - (E[U])^2 \tag{1.5.6}$$
$$= \int_{-\infty}^{\infty} x^2 dF_U(x) - 0.25 \tag{1.5.7}$$
$$= \int_0^1 x^2 dx - 0.25 \tag{1.5.8}$$
$$= 0.3333... - 0.25 = 0.083333... \tag{1.5.9}$$

From (1.4.2),we get

$$\text{Var}[U] = 0.083301 \approx 0.08333.. \tag{1.5.10}$$

Hence Verified.

2 Central Limit Theorem

2.1. Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution: Download the file

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/2.1.c
```

Use coeffs.h from the prob1.1
And run the code as:

```
$ gcc 2.1.c -lm -Wall -g
$ ./a.out
```

2.2. Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Formula Used to calculate $F_X(x)$ is:

$$F(x) = 1 - Q(x) \tag{2.2.1}$$
$$= 1 - \frac{1}{2} erfc(\frac{x}{\sqrt{2}}) \tag{2.2.2}$$

where,

$$erfc(x) = 1 - erf(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \tag{2.2.3}$$

Using $mpmath.erfc()$ function to calculate $erfc()$ in python code. The required python file can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/
    ass_manual/code/2.2.py
```

and executed using

```
$ python3 2.2.py
```

Graph is as follow:

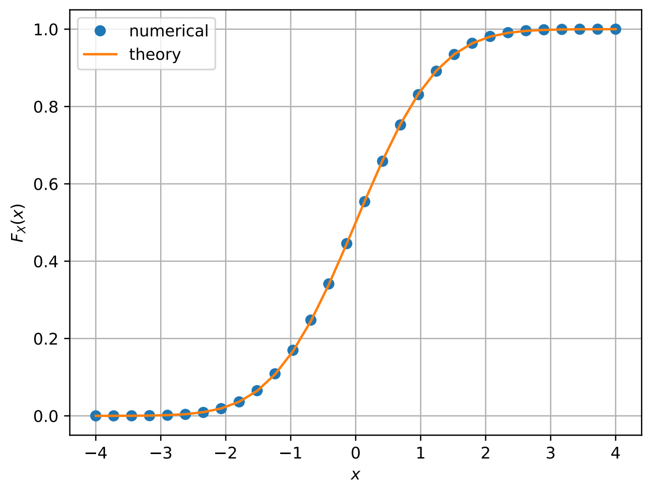

Fig. 2.2.1. CDF of X

CDF has properties:

- CDF is non-decreasing
- $\lim_{x\leftarrow-\infty} F_X(x) = 0$
- $\lim_{x\leftarrow\infty} F_X(x) = 1$
- It is right continous

2.3. Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$P_X(x) = \frac{d}{dx} F_X(x) \tag{2.3.1}$$

What properties does the PDF have?

Solution: The required python file can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/
    ass_manual/code/2.3.py
```

and executed using

```
$ python3 2.3.py
```

Graph is as follow:

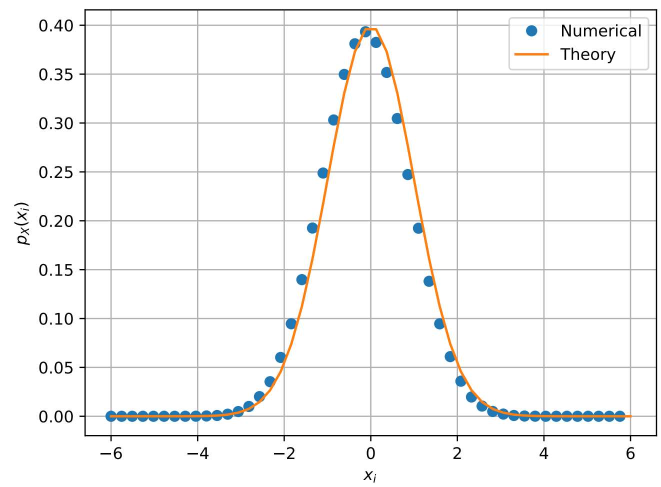

Fig. 2.3.1. PDF of X

PDF has properties:

- $\int_{-\infty}^{\infty} P_X(x)dx = 1$
- $\forall x \in \mathbb{R} \quad P_X(x) \leq 0$
- $\forall a < b \quad a, b \in \mathbb{R}$ $Pr(a < x < b) = Pr(a \leq x \leq b) = \int_a^b P_X(x)dx$

2.4. Find the mean and variance of X by writing a C program.

Solution:
The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/2.4.c
```

and compiled and executed with the following commands

```
$ gcc 2.4.c −lm −Wall −g
$ ./a.out
```

On running, we get

$$E[X] = 0.000326 \tag{2.4.1}$$
$$\text{Var}[X] = 1.000907 \tag{2.4.2}$$

2.5. Given that

$$P_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right), -\infty < x < \infty \tag{2.5.1}$$

Find Mean and Varaince theoretically.
Solution: we have,

$$E[X] = \int_{-\infty}^{\infty} x P_X(x) dx \tag{2.5.2}$$
$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{2.5.3}$$
$$= \frac{1}{\sqrt{2\pi}} (-e^{\frac{-x^2}{2}}) \Big|_{x=-\infty}^{x=\infty} \tag{2.5.4}$$
$$= 0 \tag{2.5.5}$$

Now, Knowing the fact $\int_{-\infty}^{\infty} P_x(x) = 1$
Using Integration by Parts, we get,

$$\text{Var}[X] = E[X^2] - (E[X])^2 \tag{2.5.6}$$
$$= \int_{-\infty}^{\infty} x^2 P_x(x) dx \tag{2.5.7}$$
$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{2.5.8}$$
$$= x \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{2.5.9}$$
$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{2.5.10}$$
$$= x.[-\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}] \Big|_{x=-\infty}^{x=\infty} \tag{2.5.11}$$
$$+ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{2.5.12}$$
$$= 0 + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{2.5.13}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{2.5.14}$$
$$= \int_{-\infty}^{\infty} P_X(x) dx \tag{2.5.15}$$
$$= 1 \tag{2.5.16}$$

3 From Uniform to other

3.1. Generate samples of

$$V = -2\ln(1-U) \tag{3.1.1}$$

and plot its CDF.
Solution:
Download the C code to create the distribution.

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/header/coeffs.h
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/3.1.c
```

and can be executed with

```
$ gcc 3.1.c −lm −Wall −g
$ ./a.out
```

The relevant python code is at

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/
    ass_manual/code/3.1.py
```

and can be executed with

```
$ python3 3.1.py
```

CDF Graph is as follow

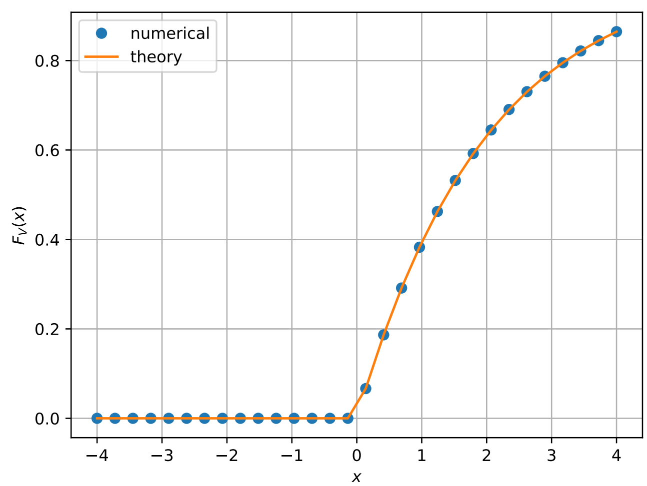

Fig. 3.1.1. CDF of V

3.2. Find a theoretical expression for $F_V(x)$.
Solution: Since $V = f(U) = -2\ln(1-U)$ is a monotonically incresing function in $(0,\infty)$. Therefore, It's Inverse exists:
$U = f^{-1}(V) = 1 - e^{-v/2}$
Hence By monotonicity of f(U), we get

$$F_V(x) = Pr(V < x) \tag{3.2.1}$$
$$= Pr(-2\ln(1-U) < x) \tag{3.2.2}$$
$$= Pr(U < 1 - e^{\frac{-x}{2}}) \tag{3.2.3}$$
$$= F_U(1 - e^{\frac{-x}{2}}) \tag{3.2.4}$$

Therfore,

$$F_V(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ 1 - e^{\frac{-x}{2}} & x \in (0, \infty) \end{cases} \tag{3.2.5}$$

4 Triangular Distribution

4.1. Generate

$$T = U_1 + U_2 \tag{4.1.1}$$

Solution: Download the files:

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/4.1.c
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/header/coeffs.h
```

and compile and execute the C program using

```
$ gcc 4.1.c -lm -Wall -g
$ ./a.out
```

4.2. Find the CDF of T.
Solution: The following code plots 4.2

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/4.2.py
```

and execute with

```
$ python3 4.2.py
```

Experimental graph of CDF is as follow:

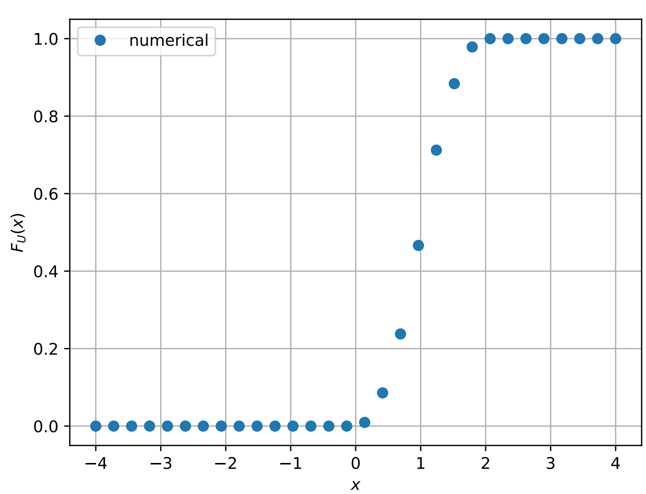

Fig. 4.2.1. Experimental CDF of T

4.3. Find the PDF of T.
Solution: The following code plots 4.3

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/4.3.py
```

and execute with

```
$ python3 4.3.py
```

Experimental graph of PDF is as follow:

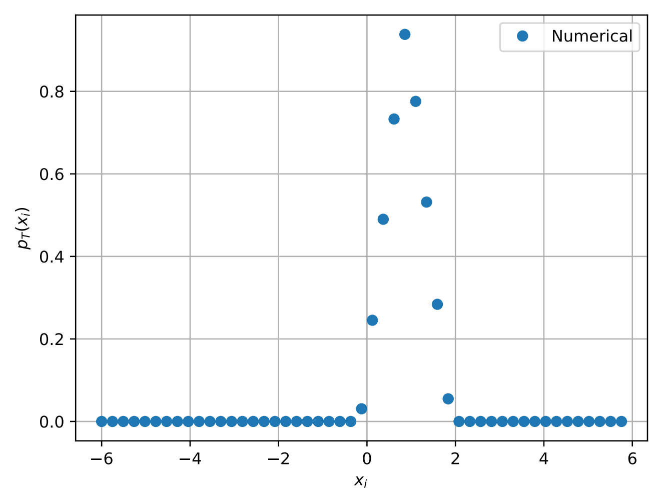

Fig. 4.3.1. Experimental PDF of T

4.4. Find the theoretical expressions for the PDF and CDF of T.
Solution: we have

$$T = U_1 + U_2 \tag{4.4.1}$$

we know,

$$p_U(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & otherwise \end{cases} \tag{4.4.2}$$

By convolution, we have

$$\begin{aligned} p_T(x) &= p_U(u) * p_U(u) && (4.4.3) \\ &= \int_{-\infty}^{\infty} p_U(\tau) p_U(x-\tau) d\tau && (4.4.4) \\ & && (4.4.5) \end{aligned}$$

Since $p_U(\tau)$ is 0 when $x < -\infty$ and $x > 1$
Therefore,

$$\int_{-\infty}^{\infty} p_U(\tau)p_U(x-\tau)d\tau = \int_0^1 p_U(\tau)p_U(x-\tau)d\tau \tag{4.4.6}$$

$$= \int_0^1 p_U(x-\tau)d\tau \tag{4.4.7}$$

Now When $0 < x < 1$

$$\int_0^1 p_U(x-\tau)d\tau = \int_0^x p_U(x-\tau)d\tau \tag{4.4.8}$$

$$= \int_0^x 1d\tau \tag{4.4.9}$$

$$= x \tag{4.4.10}$$

Now, When $1 < x < 2$

$$\int_0^1 p_U(x-\tau)d\tau = \int_{1-x}^1 p_U(x-\tau)d\tau \tag{4.4.11}$$

$$= \int_{1-x}^1 1d\tau \tag{4.4.12}$$

$$= 2 - x \tag{4.4.13}$$

Therefore,

$$p_T(x) = \begin{cases} x & x \in (0,1] \\ 2-x & x \in (1,2) \end{cases} \tag{4.4.14}$$

we know,

$$F_T(x) = \int_{-\infty}^x P_T(t)dt \tag{4.4.15}$$

Therefore,

$$F_T(x) = \begin{cases} 0 & x \in (-\infty, 0) \\ x^2/2 & x \in (0,1] \\ -x^2/2 + 2x - 1 & x \in (1,2) \\ 1 & x \in [2,\infty) \end{cases} \tag{4.4.16}$$

4.5. Verify your results through a plot.
Solution: The following code plots 4.5.1

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/4.5.py
```

and execute with

```
$ python3 4.5.py
```

The follwing code plots 4.5

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    code/4.6.py
```

and execute with

```
$ python3 4.6.py
```

Graph of CDF is as follow:

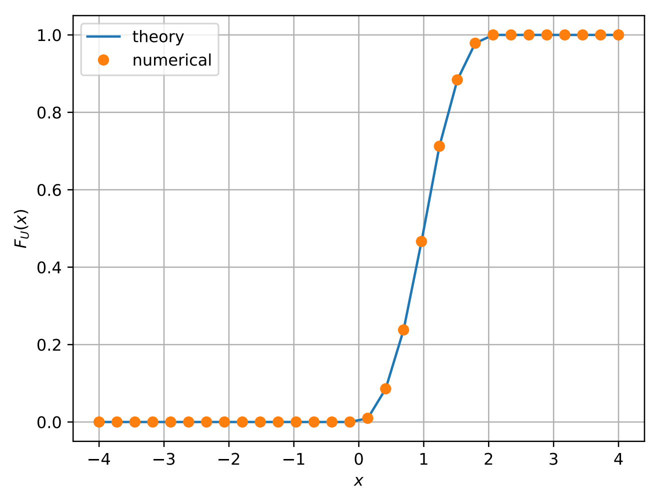

Fig. 4.5.1. CDF of T

Graph of PDF is as follow:

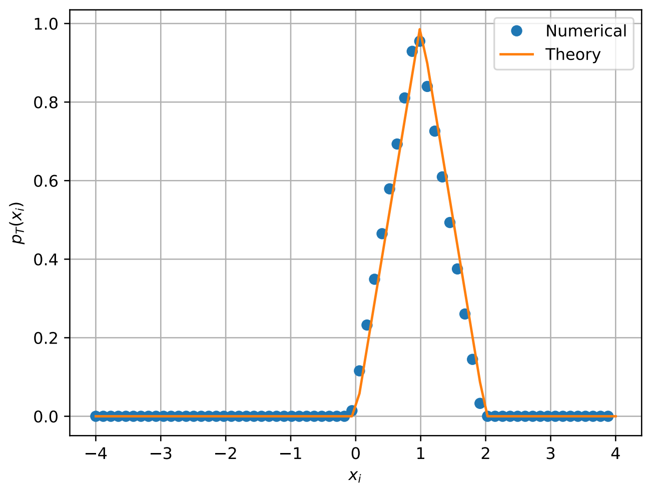

Fig. 4.5.2. PDF of T

5 Maximum Likelihood

5.1. Generate equiprobable $X \in -1, 1$.
Solution: Download the file

```
$ wget https://raw.githubusercontent.com//
    galaxion-tech/AI110/master/ass_manual/
    code/5.1.c
```

Use the coeffs.h downloaded in problem 1.1
Execute the code as follow:

```
$ gcc ./5.1.c -Wall -g -lm
$ ./a.out
```

5.2. Generate

$$Y = AX + N \tag{5.2.1}$$

where $A = 5$ dB, and $N \sim N(0, 1)$.
Solution: Download the file

```
$ wget https://raw.githubusercontent.com//
    galaxion-tech/AI110/master/ass_manual/
    code/5.2.c
```

Use the coeffs.h downloaded in problem 1.1
Execute the code as follow:

```
$ gcc ./5.2.c -Wall -g -lm
$ ./a.out
```

5.3. Plot Y using a scatter plot.
Solution: Download the Python code

```
$ wget https://raw.githubusercontent.com//
    galaxion-tech/AI110/master/ass_manual/
    code/5.3.py
```

```
$ python3 ./5.3.py
```

Noise Produced as follow:

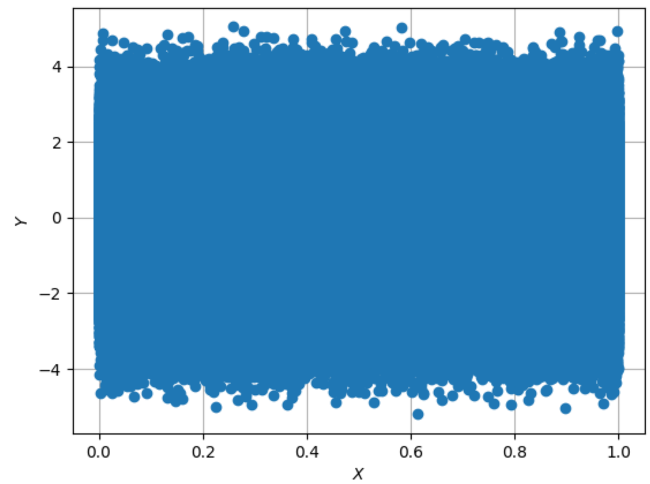

Fig. 5.3.1. Noise of Y

5.4. Guess how to estimate X from Y
Solution: we have

$$Y = AX + N \tag{5.4.1}$$

Estimating X from Y as follows:

$$\hat{X} = sgn(Y) \tag{5.4.2}$$

where $sgn(y)$ is defined as

$$sgn(y) = \begin{cases} -1 & y \in (-\infty, 0) \\ -1 & y \in [0, \infty) \end{cases} \tag{5.4.3}$$

5.5. Find

$$P_{e|0} = Pr(\hat{X} = -1|X = 1) \tag{5.5.1}$$

and

$$P_{e|1} = Pr(\hat{X} = 1|X = -1) \tag{5.5.2}$$

Solution: Download the code

```
$ wget https://raw.githubusercontent.com//
    galaxion-tech/AI110/master/ass_manual/
    code/5.5.c
```

Execute it

```
$ gcc 5.5.c −lm −Wall −g
$ ./a.out
```

Now Download the python code

```
$ wget https://raw.githubusercontent.com//
    galaxion−tech/AI110/master/ass_manual/
    code/5.5.py
```

Execute it

```
$ python3 5.5.py
```

On executing, we get

$$P_{e|0} = 0.3100037999240015 \tag{5.5.3}$$
$$P_{e|1} = 0.3106582131642633 \tag{5.5.4}$$

5.6. Find P_e assuming that X has equiprobable symbols.
Solution: we get

$$P_e = \frac{P_{e|0} + P_{e|1}}{2} \tag{5.6.1}$$
$$= \frac{0.3106582131642633 + 0.3106582131642633}{2} \tag{5.6.2}$$
$$= 0.31033100654413237 \tag{5.6.3}$$

5.7. Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: Download the file

```
$ wget https://raw.githubusercontent.com//
    galaxion−tech/AI110/master/ass_manual/
    code/5.7.c
```

Use the coeffs.h downloaded in problem 1.1
Now compile it

```
$ gcc ./5.7.c −o 5.7 −lm −Wall −g
```

Now Download the Python code

```
$ wget https://raw.githubusercontent.com//
    galaxion−tech/AI110/master/ass_manual/
    code/5.7.py
```

and Execute It

```
$ python3 5.7.py
```

On executing, we have

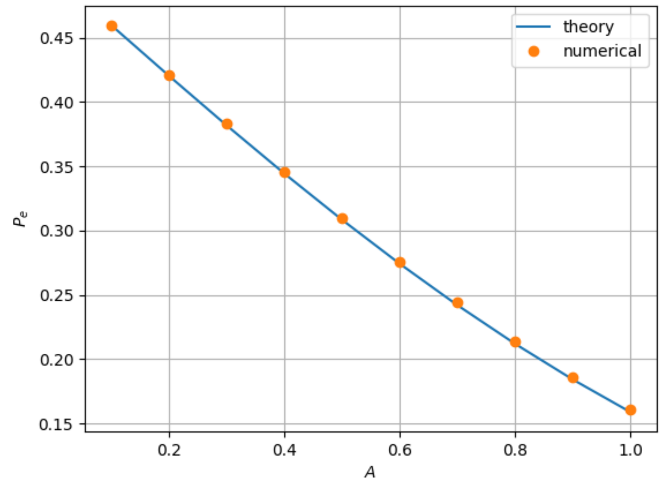

Fig. 5.7.1.

Theoretical Explanation:
For equiprobable X, we have

$$P_e = P_{e|0} = P_{e|1} \tag{5.7.1}$$

Now,

$$P_e = P_{e|0} \tag{5.7.2}$$
$$= Pr(\hat{X} = -1|X = 1) \tag{5.7.3}$$
$$= Pr(sgn(Y) = -1|X = 1) \tag{5.7.4}$$
$$= Pr(Y < 0|X = 1) \tag{5.7.5}$$
$$= Pr(AX + N < 0|X = 1) \tag{5.7.6}$$
$$= Pr(A + N < 0) \tag{5.7.7}$$
$$= Pr(N < -A) \tag{5.7.8}$$
$$= 1 - Pr(N < A) \tag{5.7.9}$$
$$= 1 - F_N(A) \tag{5.7.10}$$
$$= Q_N(A) \tag{5.7.11}$$

Hence Shown

5.8. Now consider a threshold δ while estimating X from Y. Find the value of δ that minimize the theoretical P_e.
Solution: Let estimation of X from Y have an threshold δ

$$\hat{X} = \begin{cases} 1 & Y > \delta \\ -1 & Y < \delta \end{cases} \tag{5.8.1}$$

Since we know,

$$P_{e|0} = Pr(\hat{X} = -1|X = 1) \quad (5.8.2)$$
$$= Pr(Y < \delta|X = 1) \quad (5.8.3)$$
$$= Pr(A + N < \delta) \quad (5.8.4)$$
$$= Pr(N < \delta - A) \quad (5.8.5)$$
$$= F_N(\delta - A) \quad (5.8.6)$$
$$P_{e|1} = Pr(\hat{X} = 1|X = -1) \quad (5.8.7)$$
$$= Pr(Y > \delta|X = -1) \quad (5.8.8)$$
$$= Pr(-A + N > \delta) \quad (5.8.9)$$
$$= Pr(N > \delta + A) \quad (5.8.10)$$
$$= Q_N(\delta + A) \quad (5.8.11)$$

Therefore,

$$P_e = \frac{P_{e|0} + P_{e|1}}{2} \quad (5.8.12)$$
$$= \frac{F_N(\delta - A) + Q_N(\delta + A)}{2} \quad (5.8.13)$$

For minimizing, differnciating w.r.t δ

$$\frac{dP_e}{d\delta} = \frac{d}{d\delta}\left(\frac{F_N(\delta - A) + Q_N(\delta + A)}{2}\right) \quad (5.8.14)$$
$$= \frac{P_N(\delta - A) - P_N(\delta + A)}{2} \quad (5.8.15)$$
$$= \frac{1}{2\sqrt{2\pi}}\left[e^{\frac{-(\delta-A)^2}{2}} - e^{\frac{-(\delta+A)^2}{2}}\right] \quad (5.8.16)$$
$$= 0 \quad (5.8.17)$$

Only Possible solution for δ are

$$\delta = 0, \pm\infty \quad (5.8.18)$$

Since for minima

$$\frac{d^2P_e}{d\delta^2} > 0 \quad (5.8.19)$$

On calculating double derivative

$$\frac{d^2P_e}{d\delta^2} = \frac{1}{2\sqrt{2\pi}}[(A - \delta)e^{\frac{-(\delta-A)^2}{2}} \quad (5.8.20)$$
$$+ (A + \delta)e^{\frac{-(\delta+A)^2}{2}}] \quad (5.8.21)$$

when $\delta = 0$, we get

$$\frac{d^2P_e}{d\delta^2} = \frac{1}{2\sqrt{2\pi}}[(A)e^{\frac{-(A)^2}{2}} + (A)e^{\frac{-(A)^2}{2}}] \quad (5.8.22)$$
$$> 0 \quad (5.8.23)$$

Hence $\delta = 0$ is the threshold on which P_e minimized.

5.9. Repeat the above exercise when

$$p_X(0) = p \quad (5.9.1)$$

Solution: Now we have

$$p_X(x) = \begin{cases} p & x = 1 \\ 1 - p & x = -1 \end{cases} \quad (5.9.2)$$

Since from (5.7.1) and (5.7.11) we have

$$P_{e|0} = P_{e|1} = Q_N(A) \quad (5.9.3)$$

Therfore,

$$P_e = (p)P_{e|0} + (1 - p)P_{e|1} \quad (5.9.4)$$
$$= (p)Q_N(A) + (1 - p)Q_N(A) \quad (5.9.5)$$
$$= (p + 1 - p)Q_N(A) \quad (5.9.6)$$
$$= Q_N(A) \quad (5.9.7)$$

Hence P_e is independent of p when threshold $\delta = 0$

Now Consider a threshold δ

From (5.8.6) and (5.8.11), we have

$$P_e = (p)p_{e|0} + (1 - p)p_{e|1} \quad (5.9.8)$$
$$= (p)F_N(\delta - A) + (1 - p)Q_N(\delta + A) \quad (5.9.9)$$

On differnciating, we have

$$\frac{dP_e}{d\delta} = (p)P_N(\delta - A) - (1 - p)P_N(\delta + A) \quad (5.9.10)$$
$$= \frac{1}{\sqrt{2\pi}}\left[(p)e^{\frac{-(\delta-A)^2}{2}} - (1 - p)e^{\frac{-(\delta+A)^2}{2}}\right] \quad (5.9.11)$$
$$= 0 \quad (5.9.12)$$

On calculating

$$\left[(p)e^{\frac{-(\delta-A)^2}{2}} - (1 - p)e^{\frac{-(\delta+A)^2}{2}}\right] = 0 \quad (5.9.13)$$
$$e^{\frac{(\delta+A)^2}{2} - \frac{(\delta-A)^2}{2}} = \frac{1}{p} - 1 \quad (5.9.14)$$
$$e^{2\delta A} = \frac{1}{p} - 1 \quad (5.9.15)$$

Since e^x is monotonic.

$$\delta = \frac{1}{2A}\ln\left(\frac{1}{p}-1\right) \tag{5.9.16}$$

On double differnciating P_e at $\delta = \frac{1}{2A}\ln(\frac{1}{p}-1)$ for minima, we get

$$\frac{d^2P_e}{d\delta^2} = \frac{1}{2\sqrt{2\pi}}[(p)(A-\delta)e^{\frac{-(\delta-A)^2}{2}} \tag{5.9.17}$$
$$+ (1-p)(A+\delta)e^{\frac{-(\delta+A)^2}{2}}] \tag{5.9.18}$$
$$> 0 \tag{5.9.19}$$

Now,

$$p(A-\delta)e^{\frac{-(\delta-A)^2}{2}} > -(1-p)(\delta+A)e^{\frac{-(\delta+A)^2}{2}} \tag{5.9.20}$$
$$p(A-\delta)e^{2\delta\pi} > -(1-p)(A+\delta) \tag{5.9.21}$$
$$(A-\delta)(1-p) > (p-1)(A+\delta) \tag{5.9.22}$$
$$A - Ap > Ap - A \tag{5.9.23}$$
$$2A > 2Ap \quad \text{Since} A \neq 0 \tag{5.9.24}$$
$$p < 1 \tag{5.9.25}$$

Hence Always true
Therefore, $\delta = \frac{1}{2A}\ln(\frac{1}{p}-1)$ is the threshold where P_e minimized given $p_X(0) = 1$

5.10. Repeat the above exercise using the MAP criterion.
Solution:

6 Gaussian To Other

6.1. Let $X_1 \sim N(0,1)$ and $X_2 \sim N(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1.1}$$

Solution: Downlaod the C code to generate the distribution

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1!10/master/ass_manual/
    code/6.1.c
```

Use the coeffs.h downloaded in problem 1.1
Now execute it to get distribution in v.dat file

```
$ gcc 6.1.c −lm −Wall −g
$ ./a.out
```

Now Download the python code

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1!10/master/ass_manual/
    code/6.1_cdf.py
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1!10/master/ass_manual/
    code/6.1_pdf.py
```

execute them

```
$ python3 6.1_cdf.py
$ python3 6.1_pdf.py
```

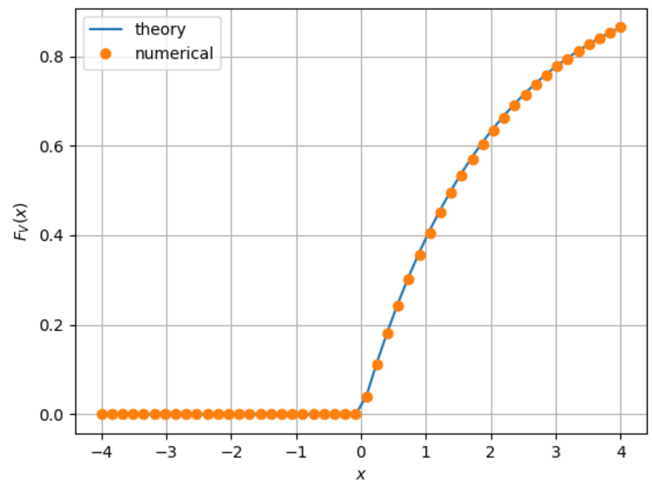

Fig. 6.1.1. CDF of V

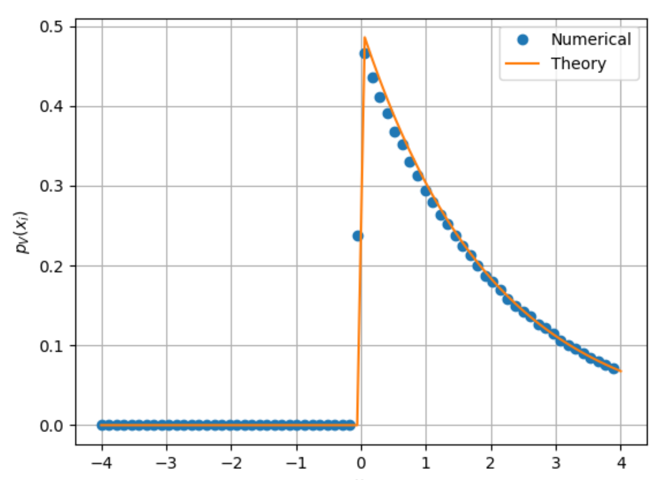

Fig. 6.1.2. PDF of V

Theoritical Explanation:

$$V = X_1^2 + X_2^2 \tag{6.1.2}$$

where X_1 and X_2 are i.i.d normal random variable
Consider two random variable R and Θ such that $X_1 = R\sin\Theta$ and $X_2 = R\cos\Theta$,
Using transformation, we have

$$f_{R,\Theta}(r,\theta) = \|J\| f_{X_1,X_2}(x_1,x_2) \tag{6.1.3}$$

where J is Jacobian matrix

$$J = \begin{pmatrix} \frac{\delta x_1}{\delta r} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta r} & \frac{\delta x_2}{\delta \theta} \end{pmatrix} \tag{6.1.4}$$

$$= \begin{pmatrix} \sin\theta & r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix} \tag{6.1.5}$$

$$= -r(\cos^2\theta + \sin^2\theta) \tag{6.1.6}$$

$$\|J\| = r \tag{6.1.7}$$

Now, since X_1 and X_2 are independent,

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1).f_{X_2}(x_2) \tag{6.1.8}$$

$$= \left(\frac{1}{\sqrt{2\pi}} e^{\frac{-x_1^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{\frac{-x_2^2}{2}}\right) \tag{6.1.9}$$

$$= \left(\frac{1}{2\pi} e^{\frac{-(x_1^2+x_2^2)}{2}}\right) \tag{6.1.10}$$

Now Since $X_1^2 + X_2^2 = R^2$, using (6.1.3) we get

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi} e^{\frac{-r^2}{2}} \tag{6.1.11}$$

Now we know R and Θ are independent, Therefore

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta)d\theta \tag{6.1.12}$$

$$= \int_0^{2\pi} \frac{r}{2\pi} e^{\frac{-r^2}{2}} d\theta \tag{6.1.13}$$

$$= re^{\frac{-r^2}{2}} \tag{6.1.14}$$

Now Since $V = X_1^2 + X_2^2 = R^2$ where $R \geq 0$, we have CDF of V as follow

$$F_V(v) = Pr(V < v) \tag{6.1.15}$$

$$= Pr(R^2 < v) \tag{6.1.16}$$

$$= Pr(R < \sqrt{v}) \quad \text{Since } R \geq 0 \tag{6.1.17}$$

$$= F_R(\sqrt{v}) \tag{6.1.18}$$

where $v \geq 0$
Since when $v < 0$,From (6.1.18) we have

$$F_V(v) = 0 \tag{6.1.19}$$

Therefore,

$$f_V(v) = 0 \tag{6.1.20}$$

when $v \geq 0$,
On differnciating both side w.r.t v, we get,

$$F_V(v) = F_R(\sqrt{v}) \tag{6.1.21}$$

$$\frac{dF_V(v)}{dv} = \frac{dF_R(\sqrt{v})}{dv} \tag{6.1.22}$$

$$f_V(v) = f_R(\sqrt{v})\frac{1}{2\sqrt{v}} \tag{6.1.23}$$

$$= \sqrt{v}e^{\frac{-v}{2}}\frac{1}{2\sqrt{v}} \tag{6.1.24}$$

$$= \frac{1}{2}e^{\frac{-v}{2}} \tag{6.1.25}$$

Now, On Integrating $f_V(v)$ for $v \geq 0$, we get

$$F_V(v) = \int_{-\infty}^{v} f_V(v)dv \tag{6.1.26}$$

$$= \int_{-\infty}^{v} \frac{1}{2}e^{\frac{-v}{2}}dv \tag{6.1.27}$$

$$= 1 - e^{\frac{-v}{2}} \tag{6.1.28}$$

Hence we have

$$f_V(v) = (\frac{1}{2}e^{\frac{-v}{2}})u(v) \tag{6.1.29}$$

$$F_V(v) = (1 - e^{\frac{-v}{2}})u(v) \tag{6.1.30}$$

where $u(v)$ is a unit step function
It is Chi-square Distribution.

6.2. If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.2.1}$$

find α.

Solution: Since From (6.1.30), we have

$$F_V(x) = \begin{cases} 1 - e^{\frac{-x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.2.2}$$

Since e^{-x} is monotonic, On comparing (6.2.2) with (6.2.1), we get

$$\alpha = \frac{1}{2} \tag{6.2.3}$$

6.3. Plot the CDF and PDF of

$$A = \sqrt{V} \tag{6.3.1}$$

Solution: Download the C code to generate distribution A in aa.dat

```
$ wget http://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/6.3.c
```

execute it to get distribution

```
$ gcc 6.3.c −lm −Wall −g
$ ./a.out
```

Now Downlaod the Python code

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/6.3_cdf.py
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/6.3_pdf.py
```

Execute it

```
$ python3 6.3_cdf.py
$ python3 6.3_pdf.py
```

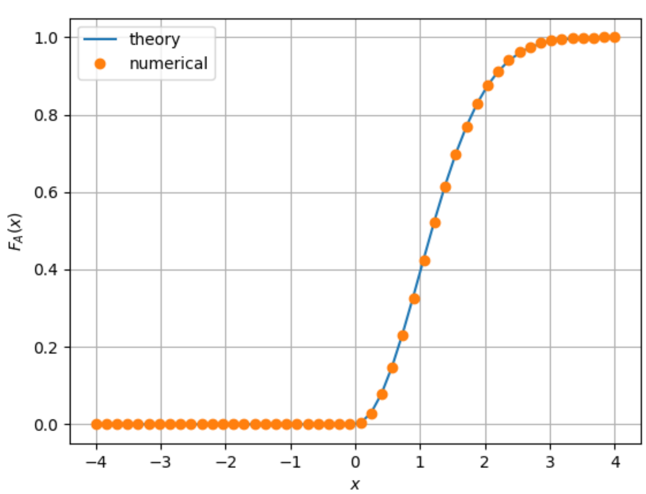

Fig. 6.3.1. CDF of A

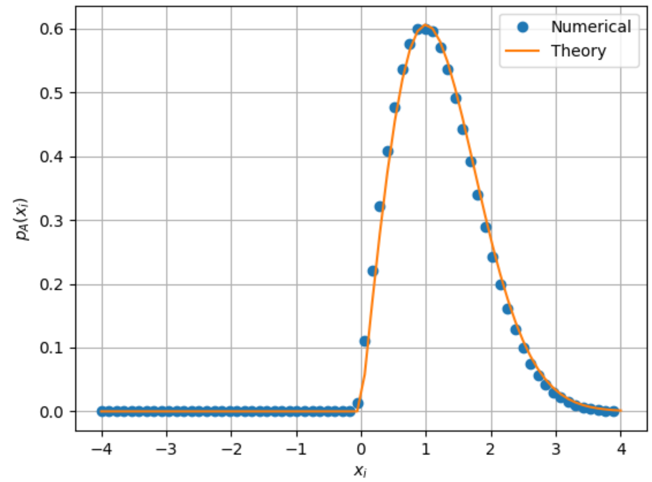

Fig. 6.3.2. PDF of A

Theoritical Explanation:
Since we have $V = R^2$, Hence

$$A = \sqrt{V} = R \tag{6.3.2}$$

Therefore,From (6.1.14), PDf of A is as follow

$$f_A(a) = f_R(a) \tag{6.3.3}$$

$$= ae^{\frac{-a^2}{2}} \tag{6.3.4}$$

where $a \geq 0$
For $a < 0$, we have

$$f_A(a) = 0 \tag{6.3.5}$$

$$F_A(a) = 0 \tag{6.3.6}$$

On Integrating (6.3.4) for $a \geq 0$, we have

$$F_A(a) = \int_{-infty}^{a} f_A(x)dx \tag{6.3.7}$$

$$= \int_{-\infty}^{a} ae^{\frac{-a^2}{2}} \tag{6.3.8}$$

$$= \int_{-\infty}^{0} ae^{\frac{-a^2}{2}} + \int_{0}^{a} ae^{\frac{-a^2}{2}} \tag{6.3.9}$$

$$= 0 + \int_{0}^{a} ae^{\frac{-a^2}{2}} \tag{6.3.10}$$

$$= 1 - e^{\frac{-a^2}{2}} \tag{6.3.11}$$

Therefore, Finally we have

$$f_A(a) = ae^{\frac{-a^2}{2}} u(a) \tag{6.3.12}$$

$$F_A(a) = (1 - e^{\frac{-a^2}{2}})u(a) \tag{6.3.13}$$

where $u(a)$ is unit step function
It is Rayleigh Distribution

7 Conditional Probability

7.1. Plot

$$P_e = Pr(\hat{X} = -1|X = 1) \tag{7.1.1}$$

for

$$Y = AX + N \tag{7.1.2}$$

where A is Raleigh with $E[A^2] = \gamma, N \sim N(0,1), X \in \{-1,1\}$ for $0 \le \gamma \le 10$ dB.

7.2. Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3. For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx \tag{7.3.1}$$

Find $P_e = E[P_e(N)]$.

7.4. Plot P_e in problem (7.1.1) and (7.3.1) on the same graph w.r.t γ. Comment.