

Assignment 9

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Outline

1 Problem Statement

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Papoulis Chapter 7 Exercise 7.29

Using the central limit theorem. show that for large n :

$$\frac{c^n}{(n-1)!} x^{n-1} e^{-cx} \approx \frac{c}{\sqrt{2\pi n}} e^{-(cx-n)^2/2n} \quad (1)$$

Solution

Erlang Density Function

For $N = 0, 1, 2, 3, \dots$ and constant $a > 0$

$$f_X(x) = \frac{a^N}{(N-1)!} a^{N-1} e^{-ax} \quad (2)$$

$$\bar{X} = \frac{N}{a} \quad \sigma^2 = \frac{N}{a^2} \quad (3)$$

$$\Phi_X(\omega) = \left(\frac{a}{a - j\omega} \right)^N \quad (4)$$

Convolution theorem of Fourier transform

It states that Convolution of two function in real space is the same as the product of their respective Fourier transforms in Fourier space.

$$f(r) * g(r) \Leftrightarrow F(k)G(k) \quad (5)$$

Solution

Characteristics function and Normality

The characteristic function of a random vector is by definition the function:

$$\Phi(\Omega) = E(e^{j\Omega\mathbf{X}'}) = E(e^{j(\omega_1x_1+\omega_2x_2+\dots+\omega_nx_n)}) = \Phi(j\Omega) \quad (6)$$

where,

$$\mathbf{X} = [x_1, \dots, x_n] \quad \Omega = [\omega_1, \dots, \omega_n] \quad (7)$$

Let $z = x_1 + x_2 + \dots + x_n$ be a random variable

Now, If the random variables x_i are independent with respective densities $f_i(x_i)$, then

$$E(e^{j(\omega_1x_1+\omega_2x_2+\dots+\omega_nx_n)}) = E(e^{j\omega_1x_1}) \dots E(e^{j\omega_nx_n}) \quad (8)$$

Solution

Hence,

$$\Phi_z(\omega) = E(e^{j(\omega_1 x_1 + \omega_2 x_2 + \dots + \omega_n x_n)}) = \Phi_1(\omega) \dots \Phi_n(\omega) \quad (9)$$

where $\Phi_i(\omega)$ is the characteristic function of x_i

On Applying the convolution theorem for Fourier transform, we obtain

$$f_z(z) = f_1(z) * \dots * f_n(z) \quad (10)$$

Central Limit Theorem

Given n independent random variable x_i , we form their sum

$$x = x_1 + \dots + x_n \quad (11)$$

This is a random variable with mean $\eta = \eta_1 + \dots + \eta_n$ and variance $\sigma^2 = \sigma_1^2 + \dots + \sigma_n^2$

Solution

The Central Limit Theorem (CLT) states that under certain general conditions, the distribution $F(x)$ of x approaches a normal distribution with the same mean and variance as n increases.

$$F(x) \simeq G\left(\frac{x - \eta}{\sigma}\right) \quad (12)$$

Futhermore, if the random variables x_i are of continous type, the density $f(x)$ of x approaches a normal density.

$$f(x) \simeq \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\eta)^2/2\sigma^2} \quad (13)$$

Solution

Let x_1, \dots, x_n be some i.i.d random variables with exponential density function $\lambda e^{-\lambda x}$ and characteristic function $\Phi_{x_i}(\omega)$. Consider a random variable $X = x_1 + \dots + x_n$.

$$\Phi_{x_i}(\omega) = \frac{a}{a - j\omega} \quad (14)$$

Since from equation (9), we have,

$$\Phi_X(\omega) = \Phi_{x_1}(\omega) \dots \Phi_{x_n}(\omega) = \left(\frac{a}{a - j\omega} \right)^n \quad (15)$$

On applying Convolution Theorem, we obtain

$$f_X(x) = \frac{a^n}{(n-1)!} a^{n-1} e^{-ax} \quad \textbf{(Erlang Density Function)} \quad (16)$$

Solution

On Applying Central Limit Theorem on random variable X (For some Large Value of n). we get

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\eta)^2/2\sigma^2} \quad (17)$$

So On Comparing equation (16) and (17), we conclude:
Erlang Density Function approaches Normal Density Curve with same mean and same variance at large value of n .

$$\frac{c^n}{(n-1)!} c^{n-1} e^{-cx} \approx \frac{1}{\sqrt{2\pi \frac{n}{c^2}}} e^{(x-\frac{n}{c})^2/2 \frac{n}{c^2}} = \frac{c}{\sqrt{2\pi n}} e^{-(cx-n)^2/2n} \quad (18)$$

Hence Shown.