ASSIGNMENT

CS21BTECH11020 (Harsh Goyal)

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1

1.1. Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the file:

$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/1.1.c
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/header/coeffs.h

and compile and execute the C program using

$ gcc 1.1.c −lm −Wall −g
$ ./a.out

1.2. Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

Solution: The following code plots Fig. 1.2

$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/1.2.py

It is executed with

$ python3 1.2.py

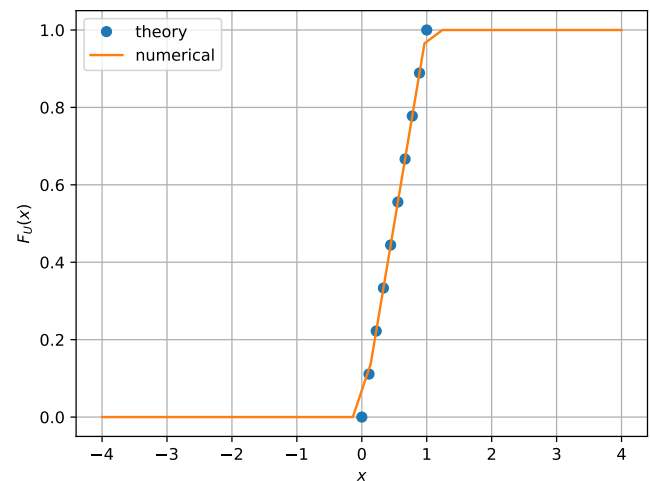$$F_U(x) = Pr(U \le x) \qquad (1.2.1)$$

Graph of CDF is as follow:

Fig. 1.2.1. CDF of U

1.3. Find a theoretical expression for $F_U(x)$.

Solution: Since We have,

$$P_U(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & otherwise \end{cases} \qquad (1.3.1)$$

on integrating for CDF we get,

$$F_U(x) = \int_{-\infty}^{x} P_U(t)dt \qquad (1.3.2)$$

$$F_U(x) = \begin{cases} \int_{-\infty}^{x} 0 dx & x \in (-\infty, 0) \\ \int_{0}^{x} 1 dx & x \in (0,1) \\ \int_{0}^{1} 1 dx & x \in (1,\infty) \end{cases} \qquad (1.3.3)$$

$$F_U(x) = \begin{cases} 0 & x \in (-\infty, 0) \\ x & x \in (0,1) \\ 1 & x \in (1,\infty) \end{cases} \qquad (1.3.4)$$

1.4. Write a C program to find the mean and variance of U.

Solution: download C program

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/1.4.c
```

and compiled and executed with

```
$ gcc 1.4.c −lm −Wall −g
$ ./a.out
```

$$E[U] = 0.500007 \qquad (1.4.1)$$
$$\text{Var}[U] = 0.083301 \qquad (1.4.2)$$

1.5. Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \qquad (1.5.1)$$

Solution: we have

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \qquad (1.5.2)$$
$$= \int_0^1 x dx \qquad (1.5.3)$$
$$= 0.5 \qquad (1.5.4)$$

From (1.4.1), we have,

$$E[U] = 0.500007 \approx 0.5 \qquad (1.5.5)$$

Similarly,

$$\text{Var}[U] = E[U^2] - (E[U])^2 \qquad (1.5.6)$$
$$= \int_{-\infty}^{\infty} x^2 dF_U(x) - 0.25 \qquad (1.5.7)$$
$$= \int_0^1 x^2 dx - 0.25 \qquad (1.5.8)$$
$$= 0.3333... - 0.25 = 0.083333... \qquad (1.5.9)$$

From (1.4.2),we get

$$\text{Var}[U] = 0.083301 \approx 0.08333.. \qquad (1.5.10)$$

Hence Verified.

2 CENTRAL LIMIT THEOREM

2.1. Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \qquad (2.1.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution: Download the file

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/2.1.c
```

Use coeffs.h from the prob1.1
And run the code as:

```
$ gcc 2.1.c −lm −Wall −g
$ ./a.out
```

2.2. Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Formula Used to calculate $F_X(x)$ is:

$$F(x) = 1 - Q(x) \qquad (2.2.1)$$
$$= 1 - \frac{1}{2} erfc(\frac{x}{\sqrt{2}}) \qquad (2.2.2)$$

where,

$$erfc(x) = 1 - erf(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \qquad (2.2.3)$$

Using $mpmath.erfc()$ function to calculate $erfc()$ in python code. The required python file can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/
    ass_manual/code/2.2.py
```

and executed using

```
$ python3 2.2.py
```
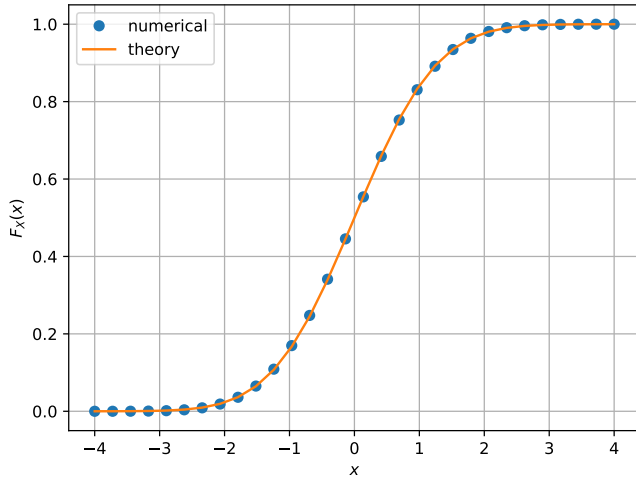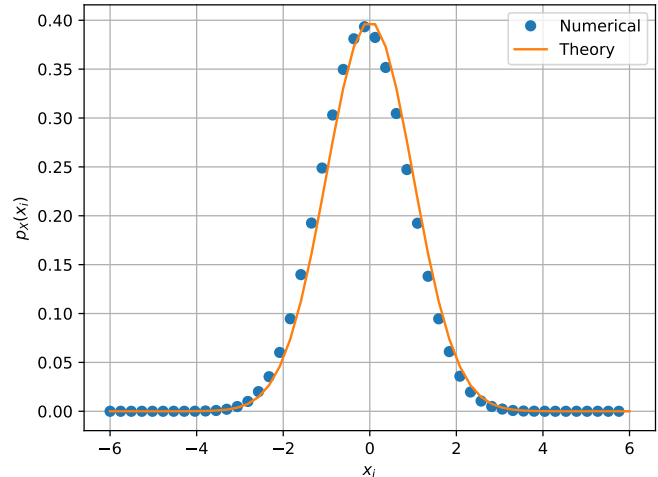
Graph is as follow:

Fig. 2.2.1. CDF of X

Fig. 2.3.1. PDF of X

CDF has properties:
- CDF is non-decreasing
- $\lim_{x \leftarrow -\infty} F_X(x) = 0$
- $\lim_{x \leftarrow \infty} F_X(x) = 1$
- It is right continous

2.3. Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$P_X(x) = \frac{d}{dx} F_X(x) \qquad (2.3.1)$$

What properties does the PDF have?

Solution: The required python file can be downloaded using

$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/
    ass_manual/code/2.3.py

and executed using

$ python3 2.3.py

Graph is as follow:

PDF has properties:
- $\int_{-\infty}^{\infty} P_X(x)dx = 1$
- $\forall x \in \mathbb{R} \quad P_X(x) \leq 0$
- $\forall a < b \quad a, b \in \mathbb{R}$
  $Pr(a < x < b) = Pr(a \leq x \leq b) = \int_a^b P_X(x)dx$

2.4. Find the mean and variance of X by writing a C program.

Solution:
The C program can be downloaded using

$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/2.4.c

and compiled and executed with the following commands

$ gcc 2.4.c −lm −Wall −g
$ ./a.out

On running, we get

$$E[X] = 0.000326 \qquad (2.4.1)$$
$$\text{Var}[X] = 1.000907 \qquad (2.4.2)$$

2.5. Given that

$$P_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right), -\infty < x < \infty \qquad (2.5.1)$$

Find Mean and Varaince theoretically.
Solution: we have,

$$E[X] = \int_{-\infty}^{\infty} x P_X(x) dx \qquad (2.5.2)$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \qquad (2.5.3)$$

$$= \frac{1}{\sqrt{2\pi}} (-e^{\frac{-x^2}{2}}) \Big|_{x=-\infty}^{x=\infty} \qquad (2.5.4)$$

$$= 0 \qquad (2.5.5)$$

Now, Knowing the fact $\int_{-\infty}^{\infty} P_x(x) = 1$
Using Integration by Parts, we get,

$$\text{Var}[X] = E[X^2] - (E[X])^2 \qquad (2.5.6)$$

$$= \int_{-\infty}^{\infty} x^2 P_x(x) dx \qquad (2.5.7)$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \qquad (2.5.8)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \qquad (2.5.9)$$

$$= \int_{-\infty}^{\infty} P_X(x) dx \qquad (2.5.10)$$

$$= 1 \qquad (2.5.11)$$

3 FROM UNIFORM TO OTHER

3.1. Generate samples of

$$V = -2\ln(1 - U) \qquad (3.1.1)$$

and plot its CDF.
Solution:
Download the C code to create the distribution.

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/header/coeffs.h
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/3.1.c
```

and can be executed with

```
$ gcc 3.1.c −lm −Wall −g
$ ./a.out
```

The relevant python code is at

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/
    ass_manual/code/3.1.py
```

and can be executed with

```
$ python3 3.1.py
```
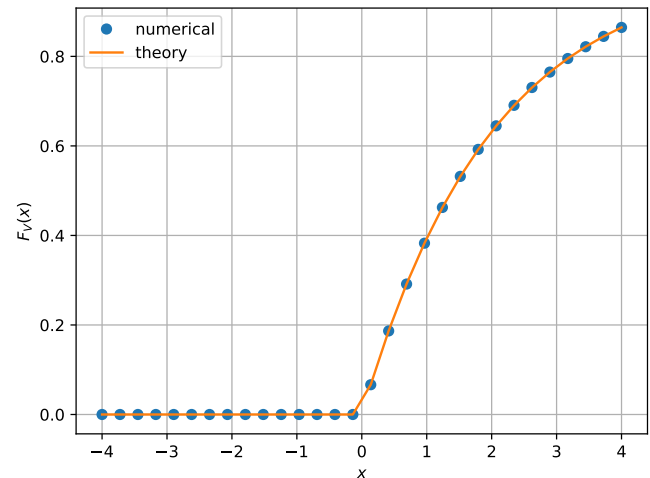
CDF Graph is as follow

Fig. 3.1.1. CDF of V

3.2. Find a theoretical expression for $F_V(x)$.
Solution: Since $V = f(U) = -2\ln(1 - U)$ is a monotonically incresing function in $(0, \infty)$. Therefore, It's Inverse exists:
$U = f^{-1}(V) = 1 - e^{-v/2}$
Hence By monotonicity of f(U), we get

$$F_V(x) = Pr(V < x) \qquad (3.2.1)$$

$$= Pr(-2\ln(1 - U) < x) \qquad (3.2.2)$$

$$= Pr(U < 1 - e^{\frac{-x}{2}}) \qquad (3.2.3)$$

$$= F_U(1 - e^{\frac{-x}{2}}) \qquad (3.2.4)$$

Therfore,

$$F_V(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ 1 - e^{\frac{-x}{2}} & x \in (0, \infty) \end{cases} \qquad (3.2.5)$$

4 TRIANGULAR DISTRIBUTION

4.1. Generate

$$T = U_1 + U_2 \qquad (4.1.1)$$

Solution: Download the files:

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/4.1.c
```

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/header/coeffs.h
```

and compile and execute the C program using

```
$ gcc 4.1.c −lm −Wall −g
$ ./a.out
```

4.2. Find the CDF of T.
Solution: The following code plots 4.2

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/4.2.py
```

and execute with

```
$ python3 4.2.py
```

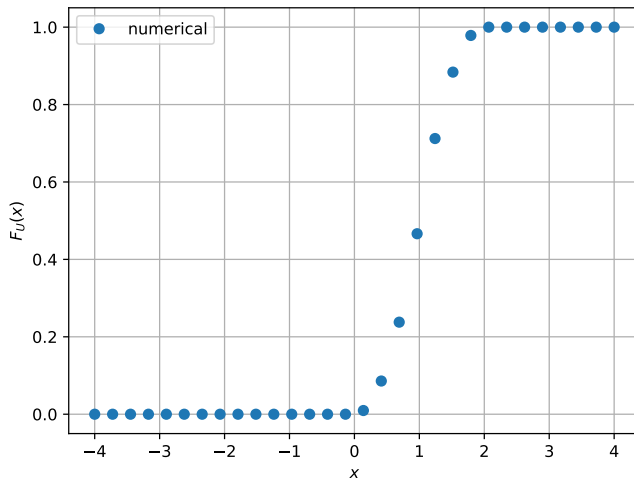Experimental graph of CDF is as follow:

Fig. 4.2.1. Experimental CDF of T

4.3. Find the PDF of T.
Solution: The following code plots 4.3

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/4.3.py
```

and execute with

```
$ python3 4.3.py
```

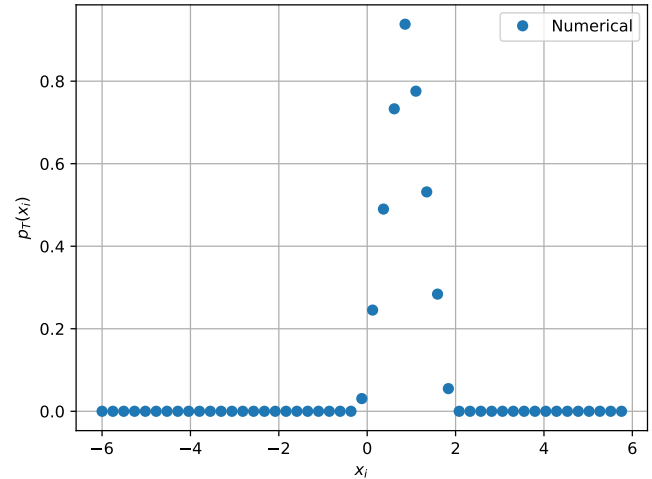Experimental graph of PDF is as follow:

Fig. 4.3.1. Experimental PDF of T

4.4. Find the theoretical expressions for the PDF and CDF of T.
Solution: we have

$$T = U_1 + U_2 \qquad (4.4.1)$$

we know,

$$p_U(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & otherwise \end{cases} \qquad (4.4.2)$$

By convolution, we have

$$p_T(x) = p_U(u) * p_U(u) \qquad (4.4.3)$$

$$= \int_{-\infty}^{\infty} p_U(\tau) p_U(x - \tau) d\tau \qquad (4.4.4)$$

$$(4.4.5)$$

Since $p_U(\tau)$ is 0 when $x < -\infty$ and $x > 1$
Therefore,

$$\int_{-\infty}^{\infty} p_U(\tau) p_U(x - \tau) d\tau = \int_0^1 p_U(\tau) p_U(x - \tau) d\tau \qquad (4.4.6)$$

$$= \int_0^1 p_U(x - \tau) d\tau \qquad (4.4.7)$$

Now When $0 < x < 1$

$$\int_0^1 p_U(x - \tau) d\tau = \int_0^x p_U(x - \tau) d\tau \qquad (4.4.8)$$

$$= \int_0^x 1 d\tau \qquad (4.4.9)$$

$$= x \qquad (4.4.10)$$

Now, When $1 < x < 2$

$$\int_0^1 p_U(x - \tau)d\tau = \int_{1-x}^1 p_U(x - \tau)d\tau \tag{4.4.11}$$

$$= \int_{1-x}^1 1 d\tau \tag{4.4.12}$$

$$= 2 - x \tag{4.4.13}$$

Therefore,

$$p_T(x) = \begin{cases} x & x \in (0, 1] \\ 2 - x & x \in (1, 2) \end{cases} \tag{4.4.14}$$

4.5. Verify your results through a plot.
Solution: The following code plots 4.5.1

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/4.5.py
```

and execute with

```
$ python3 4.5.py
```

The follwing code plots 4.5

```
$ wget https://raw.githubusercontent.com/
    galaxion−tech/AI1110/master/ass_manual/
    code/4.6.py
```

and execute with

```
$ python3 4.6.py
```
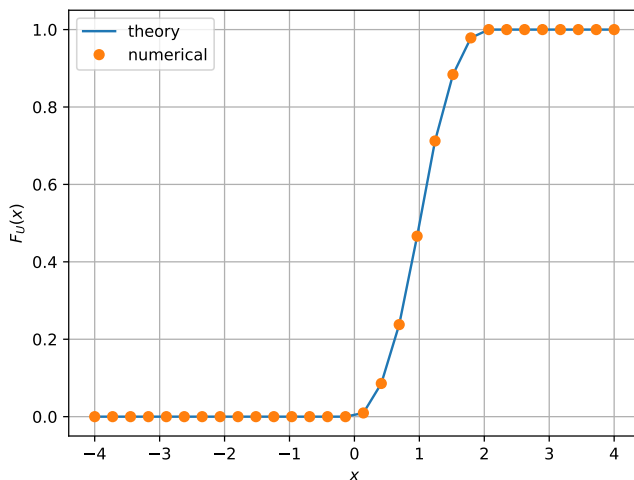
Graph of CDF is as follow:

Graph of PDF is as follow:

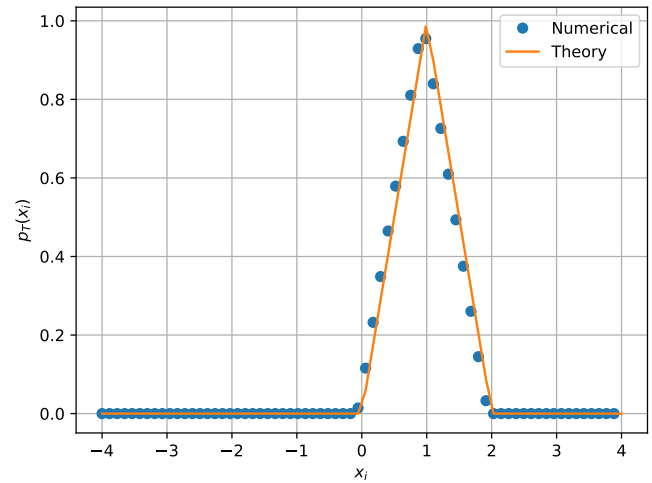Fig. 4.5.2. PDF of T

Fig. 4.5.1. CDF of T