ASSIGNMENT

CS21BTECH11020 (Harsh Goyal)

CONTENTS

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1

1.1. Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the file:

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    prob1.1/exrand.c
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    prob1.1/coeffs.h
```

and compile and execute the C program using

```
$ gcc exrand.c -lm -Wall -g
$ ./a.out
```

1.2. Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

Solution: The following code plots Fig. 1.2

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    prob1.2/cdf_plot.py
```

It is executed with

```
$ python3 cdf_plot.py
```

$$F_U(x) = Pr(U \leq x) \tag{1.2.1}$$

Graph of CDF is as follow:

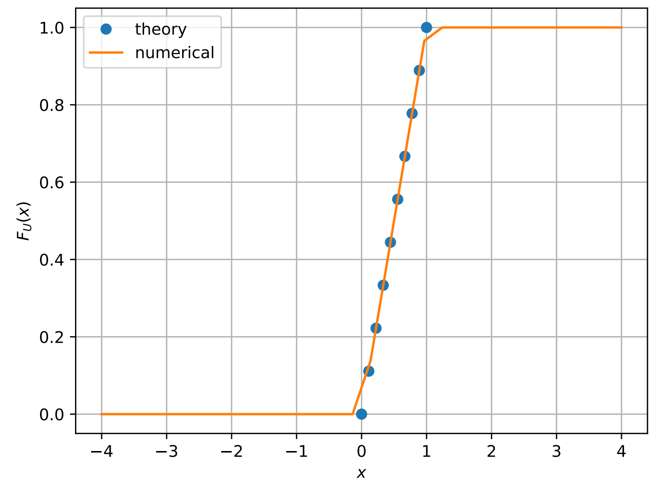


Fig. 1.2.1. CDF of U

1.3. Find a theoretical expression for $F_U(x)$.

Solution: Since We have,

$$P_U(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & otherwise \end{cases} \tag{1.3.1}$$

on integrating for CDF we get,

$$F_U(x) = \int_{-\infty}^{x} P_U(t)dt \tag{1.3.2}$$

$$F_U(x) = \begin{cases} \int_{-\infty}^{x} 0dx & x \in (-\infty, 0) \\ \int_{0}^{x} 1dx & x \in (0,1) \\ \int_{0}^{1} 1dx & x \in (1,\infty) \end{cases} \tag{1.3.3}$$

$$F_U(x) = \begin{cases} 0 & x \in (-\infty, 0) \\ x & x \in (0,1) \\ 1 & x \in (1,\infty) \end{cases} \tag{1.3.4}$$

1.4. Write a C program to find the mean and variance of U.

Solution: download C program

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    prob1.4/mycode1.c
```

and compiled and executed with

```
$ gcc mycode1.c -lm -Wall -g
$ ./a.out
```

$$E[U] = 0.500007 \tag{1.4.1}$$
$$\text{Var}[U] = 0.083301 \tag{1.4.2}$$

1.5. Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.5.1}$$

Solution: we have

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.5.2}$$
$$= \int_0^1 x dx \tag{1.5.3}$$
$$= 0.5 \tag{1.5.4}$$

From (1.4.1), we have,

$$E[U] = 0.500007 \approx 0.5 \tag{1.5.5}$$

Similarly,

$$\text{Var}[U] = E[U^2] - (E[U])^2 \tag{1.5.6}$$
$$= \int_{-\infty}^{\infty} x^2 dF_U(x) - 0.25 \tag{1.5.7}$$
$$= \int_0^1 x^2 dx - 0.25 \tag{1.5.8}$$
$$= 0.3333... - 0.25 = 0.083333... \tag{1.5.9}$$

From (1.4.2),we get

$$\text{Var}[U] = 0.083301 \approx 0.08333.. \tag{1.5.10}$$

Hence Verified.

2 Central Limit Theorem

2.1. Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution: Download the file

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    prob2.1/mycode2.c
```

Use coeffs.h from the prob1.1
And run the code as:

```
$ gcc mycode2.c -lm -Wall -g
$ ./a.out
```

2.2. Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Formula Used to calculate $F_X(x)$ is:

$$F(x) = 1 - Q(x) \tag{2.2.1}$$
$$= 1 - \frac{1}{2} erfc(\frac{x}{\sqrt{2}}) \tag{2.2.2}$$

where,

$$erfc(x) = 1 - erf(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \tag{2.2.3}$$

Using $mpmath.erfc()$ function to calculate $erfc()$ in python code. The required python file can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/
    ass_manual/prob2.2/mycode3.py
```

and executed using

```
$ python3 mycode3.py
```

Graph is as follow:

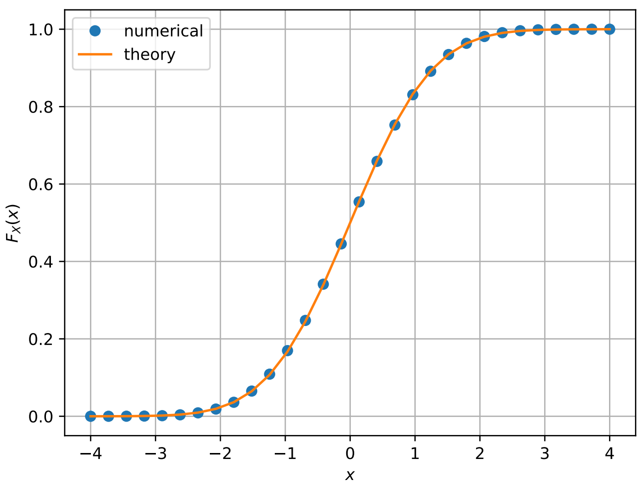


Fig. 2.2.1. CDF of X

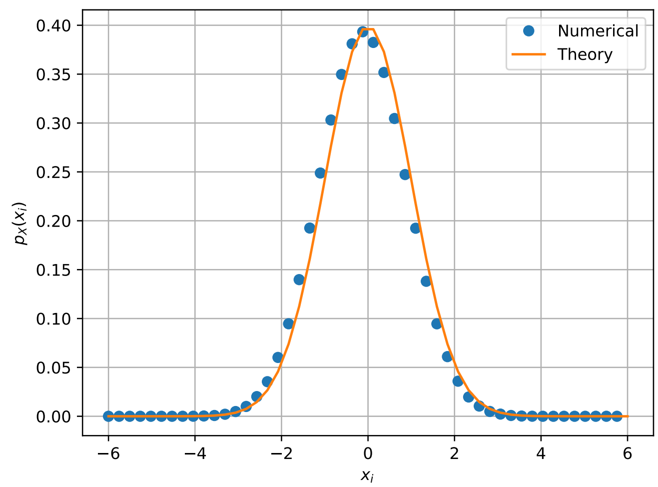


Fig. 2.3.1. PDF of X

CDF has properties:

- CDF is non-decreasing
- $\lim_{x \leftarrow -\infty} F_X(x) = 0$
- $\lim_{x \leftarrow \infty} F_X(x) = 1$
- It is right continous

2.3. Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$P_X(x) = \frac{d}{dx} F_X(x) \tag{2.3.1}$$

What properties does the PDF have?

Solution: The required python file can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/
    ass_manual/prob2.3/pdf_plot.py
```

and executed using

```
$ python3 pdf_plot.py
```

Graph is as follow:

PDF has properties:

- $\int_{-\infty}^{\infty} P_X(x)dx = 1$
- $\forall x \in \mathbb{R} \quad P_X(x) \leq 0$
- $\forall a < b \quad a, b \in \mathbb{R}$ $Pr(a < x < b) = Pr(a \leq x \leq b) = \int_a^b P_X(x)dx$

2.4. Find the mean and variance of X by writing a C program.

Solution:
The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    prob2.4/mycode4.c
```

and compiled and executed with the following commands

```
$ gcc mycode4.c -lm -Wall -g
$ ./a.out
```

On running, we get

$$E[X] = 0.000326 \tag{2.4.1}$$
$$\text{Var}[X] = 1.000907 \tag{2.4.2}$$

2.5. Given that

$$P_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right), -\infty < x < \infty \tag{2.5.1}$$

Find Mean and Varaince theoretically.
Solution: we have,

$$E[X] = \int_{-\infty}^{\infty} xP_X(x)dx \tag{2.5.2}$$
$$= \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx \tag{2.5.3}$$
$$= \frac{1}{\sqrt{2\pi}}(-e^{\frac{-x^2}{2}})\Big|_{x=-\infty}^{x=\infty} \tag{2.5.4}$$
$$= 0 \tag{2.5.5}$$

Now, Knowing the fact $\int_{-\infty}^{\infty} P_x(x) = 1$
Using Integration by Parts, we get,

$$\text{Var}[X] = E[X^2] - (E[X])^2 \tag{2.5.6}$$
$$= \int_{-\infty}^{\infty} x^2P_x(x)dx \tag{2.5.7}$$
$$= \int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx \tag{2.5.8}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx \tag{2.5.9}$$
$$= \int_{-\infty}^{\infty} P_X(x)dx \tag{2.5.10}$$
$$= 1 \tag{2.5.11}$$

3 From Uniform to other

3.1. Generate samples of

$$V = -2\ln(1-U) \tag{3.1.1}$$

and plot its CDF.
Solution:
Download the C code to create the distribution.

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/ass_manual/
    prob3.1/mycode6.c
```

and can be executed with

```
$ gcc mycode6.c -lm -Wall -g
$ ./a.out
```

The relevant python code is at

```
$ wget https://raw.githubusercontent.com/
    galaxion-tech/AI1110/master/
    ass_manual/prob3.1/mycode5.py
```

and can be executed with

```
$ python3 mycode5.py
```

CDF Graph is as follow

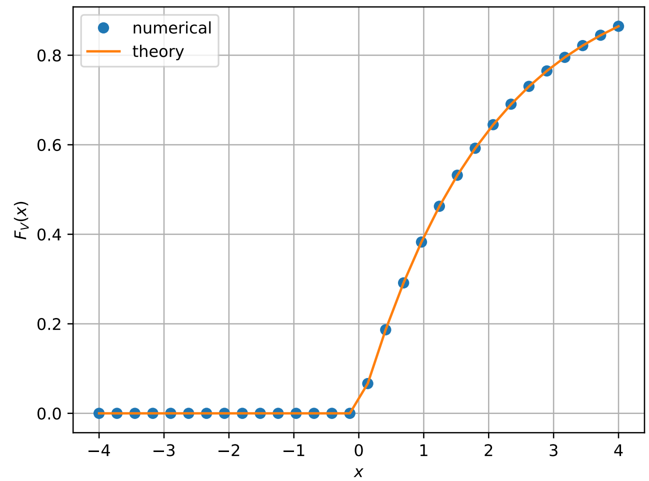


Fig. 3.1.1. CDF of V

3.2. Find a theoretical expression for $F_V(x)$.
Solution: Since $V = f(U) = -2\ln(1-U)$ is a monotonically incresing function in $(0,\infty)$. Therefore, It's Inverse exists:
$U = f^{-1}(V) = 1 - e^{-v/2}$
Hence By monotonicity of f(U), we get

$$F_V(x) = Pr(V < x) \tag{3.2.1}$$
$$= Pr(-2\ln(1-U) < x) \tag{3.2.2}$$
$$= Pr(U < 1 - e^{\frac{-x}{2}}) \tag{3.2.3}$$
$$= F_U(1 - e^{\frac{-x}{2}}) \tag{3.2.4}$$

Therfore,

$$F_V(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ 1 - e^{\frac{-x}{2}} & x \in (0,\infty) \end{cases} \tag{3.2.5}$$

4 Triangular Distribution

4.1. Generate

$$T = U_1 + U_2 \tag{4.1.1}$$

Solution: Download the files:

```
$ wget
$ wget
```

and compile and execute the C program using

```
$ gcc 4.1.c -lm -Wall -g
$ ./a.out
```

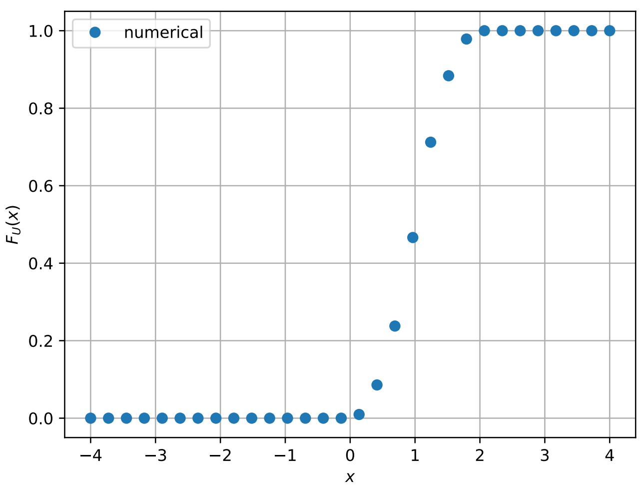
4.2. Find the CDF of T. **Solution:** The following code plots 4.2

```
$
```

and execute with

```
$
```

Experimental graph of CDF is as follow:



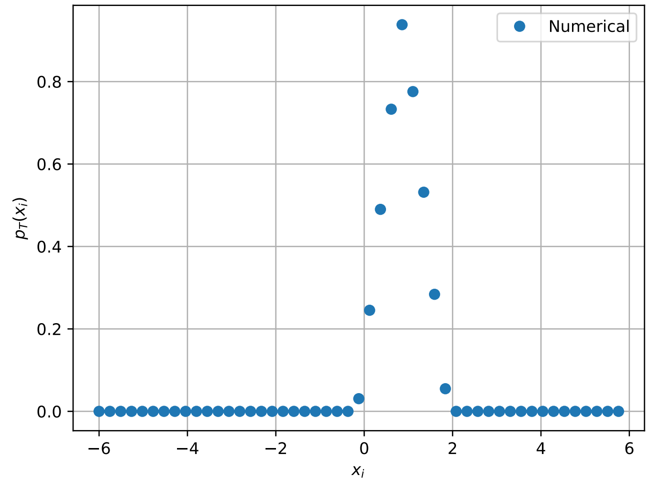
4.3. Find the PDF of T. **Solution:** The following code plots 4.3

```
$
```

and execute with

```
$
```

Experimental graph of PDF is as follow:



4.4. Find the theoretical expressions for the PDF and CDF of T. **Solution:** we have

$$T = U_1 + U_2 \tag{4.4.1}$$

we know,

$$p_U(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & otherwise \end{cases} \tag{4.4.2}$$

By convolution, we have

$$p_T(x) = p_U(u) * p_U(u) \tag{4.4.3}$$

$$= \int_{-\infty}^{\infty} p_U(\tau) p_U(x-\tau) d\tau \tag{4.4.4}$$

$$\tag{4.4.5}$$

Since $p_U(\tau)$ is 0 when $x < -\infty$ and $x > 1$ Therefore,

$$\int_{-\infty}^{\infty} p_U(\tau) p_U(x-\tau) d\tau = \int_0^1 p_U(\tau) p_U(x-\tau) d\tau \tag{4.4.6}$$

$$= \int_0^1 p_U(x-\tau) d\tau \tag{4.4.7}$$

Now When $0 < x < 1$

$$\int_0^1 p_U(x-\tau) d\tau = \int_0^x p_U(x-\tau) d\tau \tag{4.4.8}$$

$$= \int_0^x 1 d\tau \tag{4.4.9}$$

$$= x \tag{4.4.10}$$

Now, When $1 < x < 2$

$$\int_0^1 p_U(x-\tau)d\tau = \int_{1-x}^1 p_U(x-\tau)d\tau \tag{4.4.11}$$

$$= \int_{1-x}^1 1d\tau \tag{4.4.12}$$

$$= 2 - x \tag{4.4.13}$$

Therefore,

$$p_T(x) = \begin{cases} x & x \in (0,1] \\ 2-x & x \in (1,2) \end{cases} \tag{4.4.14}$$

4.5. Verify your results through a plot. **Solution:** The following code plots 4.5

```
$
```

and execute with

```
$
```

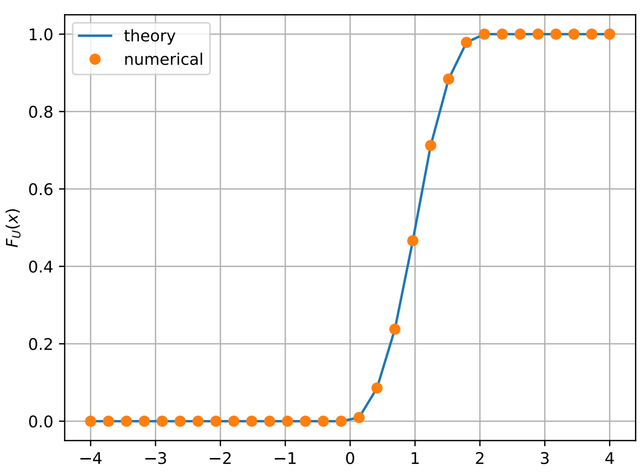
The follwing code plots 4.5

```
$
```

and execute with

```
$
```

Graph of CDF is as follow:



Graph of PDF is as follow: