Mathematical Relations

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A.1 TRIGONOMETRIC IDENTITIES

$$e^{\pm j\theta} = \cos(\theta) \pm j\sin(\theta)$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} = \sin\left(\theta + \frac{\pi}{2}\right)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \cos\left(\theta - \frac{\pi}{2}\right)$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$$

$$\cos^2(\theta) = \frac{1}{2}[1 + \cos(2\theta)]$$

$$\sin^2(\theta) = \frac{1}{2}[1 - \cos(2\theta)]$$

$$\cos^3(\theta) = \frac{1}{4}[3\cos(\theta) + \cos(3\theta)]$$

$$\sin^3(\theta) = \frac{1}{4}[3\sin(\theta) - \sin(3\theta)]$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a)\tan(b)}$$

$$\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$$

$$\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$$

$$\sin(a)\cos(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$$

$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos(a) - \cos(b) = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$A\cos(\alpha) + B\sin(\beta) = \sqrt(A^2 + B^2)\cos\left(\alpha - \tan^{-1}\frac{B}{A}\right)$$

$$\sinh(a) = \frac{e^a - e^{-a}}{2}$$

$$\cosh(a) = \frac{e^a + e^{-a}}{2}$$

$$\tanh(a) = \frac{\sinh(a)}{\cosh(a)}$$

$$\cosh^2(a) - \sinh^2(a) = 1$$

$$\cosh(a) + \sinh(a) = e^a$$

$$\cosh(a) - \sinh(a) = e^{-a}$$

A.2 POWER SERIES EXPANSION

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \binom{n}{k}x^k + \cdots + x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots (-1)^{n+1}\frac{x^n}{n} + \cdots, \qquad |x| < 1$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

$$a^x = 1 + x\ln(a) + \frac{(x\ln(a))^2}{2!} + \cdots + \frac{(x\ln(a))^n}{n!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

A.3 SUMS OF POWERS OF NATURAL NUMBERS

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{k=1}^{N} k^3 = \frac{N^2(N+1)^2}{4}$$

A.3.1 Series of Exponentials

$$\sum_{n=0}^{N} a^n = \begin{cases} \dfrac{1-a^{N+1}}{1-a}, & a \neq 1 \\ N+1, & a = 1 \end{cases}$$

$$\sum_{n=0}^{N} e^{j2\pi kn/N} = \begin{cases} N, & k = 0, N \\ 0, & 1 \leq k \leq N-1 \end{cases}$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \qquad |a| < 1$$

$$\sum_{n=0}^{\infty} na^n = \frac{1}{(1-a)^2}, \qquad |a| < 1$$

$$\sum_{n=0}^{\infty} n^2 a^n = \frac{a^2+a}{(1-a)^3}, \qquad |a| < 1$$

A.4 DERIVATIVES

$$\frac{d}{dt}[t^n] = nt^{n-1}$$

$$\frac{d}{dt}[\ln(t)] = \frac{1}{t}$$

$$\frac{d}{dt}[e^{at}] = a\,e^{at}$$

$$\frac{d}{dt}[\sin(at)] = a\cos(at)$$

$$\frac{d}{dt}[\cos(at)] = -a\sin(at)$$

$$\frac{d}{dt}[\tan(at)] = a\sec^2(at)$$

$$\frac{d}{dt}[\sin^{-1}(at)] = \frac{a}{\sqrt{1-(at)^2}}$$

$$\frac{d}{dt}[\cos^{-1}(at)] = -\frac{a}{\sqrt{1-(at)^2}}$$

$$\frac{d}{dt}[\tan^{-1}(at)] = \frac{a}{1+(at)^2}$$

$$\frac{d}{dt}[x(t)y(t)] = x(t)\frac{d}{dt}[y(t)] + y(t)\frac{d}{dt}[x(t)]$$

$$\frac{d}{dt}\left[\frac{x(t)}{y(t)}\right] = \frac{\left\{y(t)\frac{d[x(t)]}{dt} - x(t)\frac{d}{dt}[y(t)]\right\}}{y^2(t)}$$

A.5 DEFINITE INTEGRALS

$$\int_0^\infty \left[\frac{a}{a^2+t^2}\right] dt = \frac{\pi}{2}, \qquad a > 0$$

$$\int_0^\infty t^n e^{-at}\, dt = \frac{n!}{a^{n+1}}, \qquad a > 0, n \text{ is a positive integer}$$

$$\int_0^\infty e^{-a^2t^2}\, dt = \frac{\sqrt{\pi}}{2a}, \qquad a > 0$$

$$\int_0^\pi \sin^2(nt)\, dt = \int_0^\pi \cos^2(nt)\, dt = \frac{\pi}{2}, \qquad n \text{ is an integer}$$

$$\int_0^\pi \sin(nt)\sin(mt)\, dt = \int_0^\pi \cos(nt)\cos(mt)\, dt = 0, \qquad n \text{ and } m \text{ are unequal integers}$$

$$\int_0^\pi \sin(nt)\cos(mt)\, dt = \begin{cases} \dfrac{2n}{n^2-m^2}, & n-m \text{ is an odd integer} \\ 0, & n-m \text{ is an even integer} \end{cases}$$

$$\int_0^\infty \operatorname{sinc}(at)\, dt = \frac{1}{2a}, \quad a > 0$$

$$\int_0^\infty \operatorname{sinc}^2(at)\, dt = \frac{1}{2a}, \quad a > 0$$

$$\int_0^\infty e^{-at}\cos(bt)\, dt = \frac{a}{a^2+b^2}, \quad a > 0$$

$$\int_0^\infty e^{-at}\sin(bt)\, dt = \frac{b}{a^2+b^2}, \quad a > 0$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

A.6 INDEFINITE INTEGRALS

$$\int t^n\,dt = \frac{t^{n+1}}{n+1}, \qquad n \neq 1$$

$$\int \frac{1}{t}\,dt = \ln(t)$$

$$\int e^{at}\,dt = \frac{e^{at}}{a}, \qquad a \neq 0$$

$$\int \sin(at)\,dt = -\frac{\cos(at)}{a}$$

$$\int \cos(at)\,dt = \frac{\sin(at)}{a}$$

$$\int \sin^2(at)\,dt = \frac{2at - \sin(2at)}{4a}$$

$$\int \cos^2(at)\,dt = \frac{2at + \sin(2at)}{4a}$$

$$\int t^n \sin(at)\,dt = \frac{1}{a}\left[-t^n\cos(at) + n\int t^{n-1}\cos(at)\,dt\right]$$

$$\int t^n \cos(at)\,dt = \frac{1}{a}\left[t^n\sin(at) - n\int t^{n-1}\sin(at)\,dt\right]$$

A.7 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

$$e^a\,e^b = e^{a+b}$$

$$\frac{e^a}{e^b} = e^{a-b}$$

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

$$\ln(a^b) = b\,\ln(a)$$

$$\log_b(N) = \log_a(N)\,\log_b(a) = \frac{\log_a(N)}{\log_a(b)}$$

A.8 TAYLOR SERIES

$$f(x) = f(a) + \frac{(x-a)}{1!}\dot{f}(a) + \frac{(x-a)^2}{2!}\ddot{f}(a) + \cdots$$