# COMPLEX NUMBERS

# Complex Numbers

## B.1 REPRESENTATION OF COMPLEX NUMBERS

The complex number $z$ can be expressed in several ways.

Cartesian or rectangular form:

$$z = a + jb \tag{B.1}$$

where $j = \sqrt{-1}$ and $a$ and $b$ are real numbers referring to the real part and the imaginary part of $z$. $a$ and $b$ are expressed as

$$a = \mathrm{Re}\{z\} \quad \text{and} \quad b = \mathrm{Im}\{z\} \tag{B.2}$$

where Re denotes the 'real part of' and Im denotes the 'imaginary part of'.

Polar form:

$$z = r\,e^{j\theta} \tag{B.3}$$

where $r > 0$ is the *magnitude* of $z$ and $\theta$ is the *angle* or *phase* of $z$. These quantities are often written as

$$r = |z| \quad \text{and} \quad \theta = \angle z \tag{B.4}$$

Figure B.1 is the graphical representation of $z$. Using Euler's formula,

$$\begin{aligned} z &= r\,e^{j\theta} \\ a + jb &= r[\cos(\theta) + j\sin(\theta)] \end{aligned} \tag{B.5}$$

Thus, the relationships between the Cartesian and polar representations of $z$ are

$$a = r\cos(\theta) \quad \text{and} \quad b = r\sin(\theta) \tag{B.6}$$

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{b}{a}\right) \tag{B.7}$$

## B.2 ADDITION, MULTIPLICATION, AND DIVISION

If $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, then

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \tag{B.8}$$

$$z_1 z_2 = (a_1a_2 - b_1b_2) + j(a_1b_2 + b_1a_2) \tag{B.9}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} \\ &= \frac{(a_1a_2 + b_1b_2) + j(-a_1b_2 + b_1a_2)}{a_2^2 + b_2^2} \end{aligned} \tag{B.10}$$

If $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then

$$z_1 z_2 = (r_1 r_2)\, e^{j(\theta_1 + \theta_2)} \tag{B.11}$$

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{j(\theta_1 - \theta_2)} \tag{B.12}$$

## B.3 COMPLEX CONJUGATE

The complex conjugate of $z$ is denoted by $z^*$ and is given by

$$z^* = a - jb = r\, e^{-j\theta} \tag{B.13}$$

Some useful relationships are as follows:

| | |
|---|---|
| 1. $zz^* = r^2$ | 5. $(z_1 + z_2)^* = z_1^* + z_2^*$ |
| 2. $\dfrac{z}{z^*} = e^{j2\theta}$ | 6. $(z_1 z_2)^* = z_1^* z_2^*$ |
| 3. $z + z^* = 2\mathrm{Re}\{z\}$ | 7. $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$ |
| 4. $z - z^* = j2\mathrm{Im}\{z\}$ | |

## B.4 POWERS AND ROOTS OF COMPLEX NUMBERS

The $n$th power of complex number $z = r\, e^{j\theta}$ is

$$z^n = r^n\, e^{jn\theta} = r^n[\cos(n\theta) + j\sin(n\theta)] \tag{B.14}$$

from which we have DeMoivre's relation

$$[\cos(\theta) + j\sin(\theta)]^n = \cos(n\theta) + j\sin(n\theta) \tag{B.15}$$

The $n$th root of a complex number $z$ is the number $w$ such that

$$w^n = z = r\, e^{j\theta} \tag{B.16}$$

Thus, to find the $n$th root of a complex number $z$, we must solve

$$w^n - r\, e^{j\theta} = 0$$

which is an equation of degree $n$ and has $n$ roots. These roots are given by

$$w_k = r^{1/n}\, e^{j[\theta + 2(k-1)\pi]/n}, \qquad k = 1, 2, \ldots, n \tag{B.17}$$