PARTIAL FRACTION EXPANSION

Partial Fraction Expansion

A rational function $X(s)$ is ratio of two polynomials $N(s)$ and $D(s)$:

$$X(s) = \frac{N(s)}{D(s)} = k\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

where m is the highest power of numerator polynomial $N(s)$ and n is the highest power of denominator polynomial $D(s)$. Depending upon the highest power, rational function can be of two types:

1. If $m > n$, rational function $X(s)$ is called improper rational function.
2. If $m < n$, $X(s)$ is proper rational function.

For improper rational function, $X(s)$ is first separated by long division such that

$$X(s) = Q(s) + \frac{R(s)}{D(s)}$$

where $R(s)/D(s)$ will be a proper rational function. Then partial fraction is continued on the line of proper rational function.

For proper rational function, $X(s)$ may have three different types of denominator roots (poles):

1. All the poles are simple.
2. Poles are complex conjugate and simple.
3. Multiple poles at same point.

Case I (All poles are simple) Suppose that the poles $p_1,\ p_2,\ \ldots,p_n$ are all different (distinct), then, using the partial fraction expansion, we may write

$$X(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_n}{s-p_n} \tag{C.1}$$

The problem is to determine the coefficients A_1, A_2, A_n. We can determine the coefficients A_1, A_2, A_n by multiplying both sides of Eq. (C.1) by each of the terms $(s-p_k)$, $k = 1, 2, \cdots, n$, and by evaluating the resulting expressions at the corresponding pole positions, $p_1, p_2, \cdots, p_n$. Thus, we have, in general,

$$(s-p_k)X(s) = \frac{(s-p_k)A_1}{s-p_1} + \cdots + A_k + \cdots + \frac{(s-p_k)A_n}{s-p_n} \tag{C.2}$$

Consequently, with $s = p_k$, Eq. (C.2) yields the kth coefficient as

$$A_k = (s-p_k)X(s)\bigg|_{s=p_k} \qquad k = 1, 2, \cdots, n \tag{C.3}$$

The expansion given in Eq. (C.1) and the formula given in Eq. (C.3) hold for both real and complex poles. The only constraint is that all poles be distinct.

Case II (Multiple order poles) If $X(s)$ has a pole of multiplicity r, i.e., it contains in its denominator the factor $(s-p_k)^r$, then the expansion given in Eq. (C.1) is no longer true. If a pole p_k is repeated r times, then there are r terms in the partial fraction expansion associated with that pole. The partial fraction expansion must contain the terms

$$\frac{A_{1k}}{s-p_k} + \frac{A_{2k}}{(s-p_k)^2} + \cdots + \frac{A_{rk}}{(s-p_k)^r} = \sum_{i=1}^{r} \frac{A_{ik}}{(s-p_k)^i}$$

The coefficients A_{ik} are computed from the equation

$$A_{ik} = \frac{1}{(r-i)!}\left[\frac{d^{r-i}}{ds^{r-i}}\Big((s-p_k)^r X(s)\Big)\right]\Bigg|_{s=p_k} \tag{C.4}$$

Example C.1 Obtain partial fraction expansion of

$$X(s) = \frac{s}{(s+1)(s+2)}$$

Solution

Partial fraction expansion of $X(s)$ is given by

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

where, from Eq. (C.3)

$$A_1 = (s+1)\frac{s}{(s+1)(s+2)}\bigg|_{s=-1}$$

$$= \frac{-1}{-1+2} = -1$$

and

$$A_2 = (s+2)\frac{s}{(s+1)(s+2)}\bigg|_{s=-2}$$

$$= \frac{-2}{-2+1} = 2$$

Thus,

$$X(s) = \frac{-1}{s+1} + \frac{2}{s+2}$$

Example C.2 Obtain partial fraction expansion of

$$X(s) = \frac{s+1}{(s+3)(s+2)^2}$$

Solution
Partial fraction expansion of $X(s)$ is given by

$$X(s) = \frac{A_1}{s+3} + \frac{A_{12}}{s+2} + \frac{A_{22}}{(s+2)^2}$$

where from Eq. (C.3)

$$A_1 = (s+3)\frac{s+1}{(s+3)(s+2)^2}\bigg|_{s=-3}$$

$$= \frac{-2}{(-1)^2} = -2$$

$$A_{12} = \frac{1}{1!}\frac{d}{ds}\left[(s+2)^2\frac{s+1}{(s+3)(s+2)^2}\right]\bigg|_{s=-2}$$

$$= \frac{d}{ds}\left[\frac{s+1}{s+3}\right]\bigg|_{s=-2} = 2$$

and

$$A_{22} = \left[(s+2)^2\frac{s+1}{(s+3)(s+2)^2}\right]\bigg|_{s=-2} = -1$$

Thus,

$$X(s) = \frac{-2}{s+3} + \frac{2}{s+2} - \frac{1}{(s+2)^2}$$