Plasma Spray Coating Research for Pratt and Whitney by Max Plomer

Summary: The following is a preliminary research of the Plasma Spray Coating process; we will review basic heat transfer material and then build a simple analytical/numerical estimate of the location of a particle when it has reached melting temperature and then when it has fully melted. There are 3 main problems that are combined into the solution:
(1) Lumped capacitance model is used for low Biot number spheres being heated to melting temperature
(2) Once particle reaches melting temperature this becomes a Stefan Problem where we calculate the particle's solid fraction using it's latent heat of melting
(3) Linear and quadratic drag coefficients are used to calculate the location of the particle when melted.

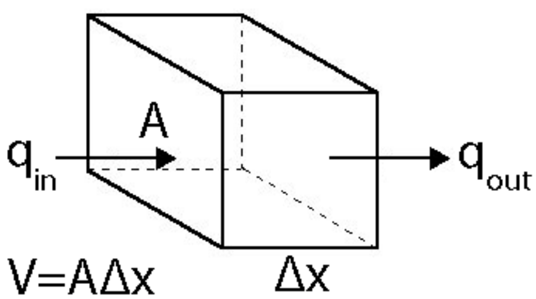
Sources for the following material is:
1. MIT material processing lecture notes part of the MIT Open Courseware initiative
2. Classical Mechanics. John R. Taylor
3. Fundamentals of Heat and Mass Transfer. Incropera, DeWitt, Bergman and Lavine
4. "Scaling Analysis and Prediction of Thermal Aspects of the Plasma Spraying Process Using a Discrete Particle Approach" Jinho Lee and Theodore L. Bergman

Fourier's Law

$$\vec{q} = -k\vec{\nabla}T$$

$$\vec{q} = heat\ flux\ \left[\frac{J}{m^2\ s} = \frac{W}{m^2}\right] \qquad k = thermal\ conductivity\ \left[\frac{W}{m\ K}\right]$$

Heat Balance in Small Element



heat in – heat out + heat generation (chemical reaction) = heat accumulation

$$Aq_{in} - Aq_{out} = V\frac{\partial H}{dt} \qquad H = enthalpy\ per\ unit\ volume$$

$$[q]_x - [q]_{x+\Delta x} = \Delta x\frac{\partial H}{dt}$$

$$\frac{\left(\left[-k\frac{\partial T}{dt}\right]_x-\left[-k\frac{\partial T}{dt}\right]_{x+\Delta x}\right)}{\Delta x}=\frac{\partial H}{dt}$$

Limit as $\Delta x \to 0$

$$\frac{\partial}{dx}\left(k\frac{\partial T}{dx}\right)=\frac{\partial H}{dt}$$

If k is constant

$$k\frac{\partial^2 T}{dx^2}=\frac{\partial H}{dt}$$

From definition of heat capacity at constant pressure

$$\rho c_p=\frac{\partial H}{\partial T}$$

Divide by ∂t

$$\frac{\partial H}{\partial t}=\rho c_p\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{dx^2}$$

Heat Conduction Equation

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{dx^2}$$

$$\alpha = thermal\ diffusivity=\frac{k\left[\frac{W}{m*K}\right]}{\rho\left[\frac{kg}{m^3}\right]\ c_p\left[\frac{J}{K*kg}\right]}=\left[\frac{m^2}{s}\right]$$

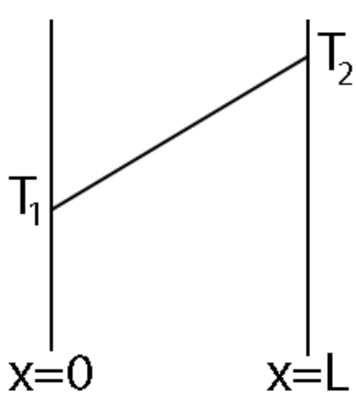
Assumes $k \neq f(x)$

$$\rho c_p\frac{\partial T}{\partial t}=\vec{\nabla}\cdot k\vec{\nabla}T \quad went\ to \quad \frac{\partial T}{\partial t}=\alpha\vec{\nabla}^2 T$$

Steady State 1D Conduction

$$\vec{\nabla}^2 T=0 \quad Laplace\ Equation$$

$$1D \quad \frac{\partial^2 T}{dx^2}=0 \quad (a\ line)$$



$$\int \left[\frac{d}{dx}\left(\frac{dT}{dx}\right) = 0\right] dx$$

$$\int \left[\frac{dT}{dx} = A\right] dx$$

$$T = Ax + B$$

At x=0, T=T_1

$$T_1 = B$$

At x=L, T=T_2

$$T_2 = AL + T_1 \qquad A = \frac{T_2 - T_1}{L}$$

$$T = \left(\frac{T_2 - T_1}{L}\right) x + T_1$$

$$\frac{T - T_1}{T_2 - T_1} = \frac{x}{L}$$

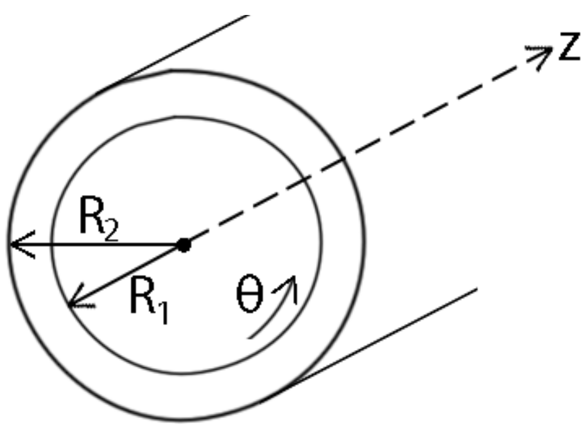
Dimensionless variables T* and x*

$$T^* = \frac{T - T_1}{T_2 - T_1} \qquad x^* = \frac{x}{L} \qquad T^* = x^*$$

Constant heat flux

$$q = -k\frac{dT}{dx} \qquad \frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

Steady State in Cylinder



$$\vec{\nabla}^2 T = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

T constant in z and θ directions

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

$$\int \left[\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0\right] dr$$

$$r\frac{dT}{dr} = A$$

$$\int \left[\frac{dT}{dr} = \frac{A}{r}\right] dr$$

Note: $\int dx/x = \ln(x) + C$

$$T = Aln(r) + B$$

At r=R_1, T=T_1

$$T_1 = Aln(R_1) + B \quad eq1$$

At r=R_2, T=T_2

$$T_2 = Aln(R_2) + B \quad eq1$$

From eq1

$$B = T_1 - Aln(R_1)$$

From eq2

$$B = T_2 - Aln(R_2)$$

Substitute eq2 → eq1

$$T_1 - Aln(R_1) = T_2 - Aln(R_2)$$

$$T_1 - T_2 = \; Aln(R_1) - Aln(R_2) = Aln\left(\frac{R_1}{R_2}\right)$$

$$A = \frac{T_1 - T_2}{ln\left(\frac{R_1}{R_2}\right)} \qquad\qquad B = T_1 - \frac{T_1 - T_2}{ln\left(\frac{R_1}{R_2}\right)} ln(R_1)$$

$$T = \frac{T_1 - T_2}{ln\left(\frac{R_1}{R_2}\right)} ln(r) + T_1 - \frac{T_1 - T_2}{ln\left(\frac{R_1}{R_2}\right)} ln(R_1)$$

Subtract T_1, divide by (T_1-T_2)

$$\frac{T - T_1}{T_1 - T_2} = \frac{\ln\left(\frac{r}{R_1}\right)}{\ln\left(\frac{R_1}{R_2}\right)}$$

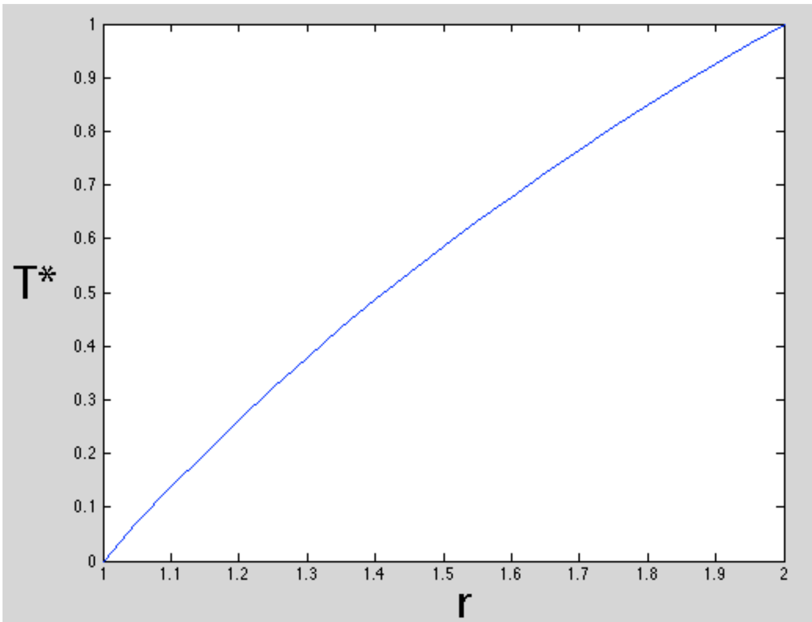
Dimensionless variable T*

$$T^* = \frac{T - T_1}{T_2 - T_1} = \frac{\ln\left(\frac{r}{R_1}\right)}{\ln\left(\frac{R_2}{R_1}\right)}$$

Constant total heat flux, but flux not constant

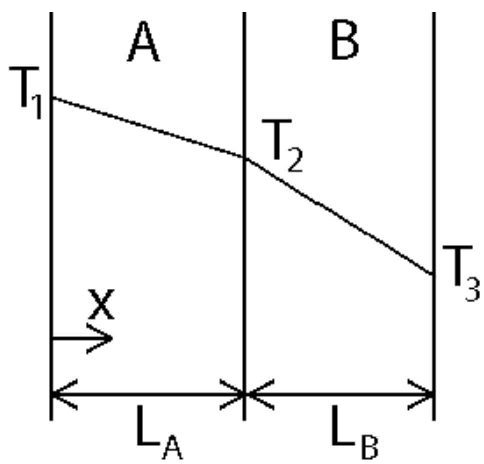
$$q = -k\frac{dT}{dr} \neq constant$$

$$A = 2\pi r \qquad qA = constant$$



At r=1 $T=T_1$
At r=2 $T=T_2$

Steady State in Composite Wall



T_2 is unknown

$$\frac{\partial^2 T}{dx^2} = 0 \quad in\ material\ A\ and\ B$$

At $x=L_A$, $T=T_2$, $q_{in}=q_{out}$

$$q_{in} = -k_A\left[\frac{dT}{dx}\right]_{L_A-} = -k_B\left[\frac{dT}{dx}\right]_{L_A+} = q_{out}$$

$$k_A \frac{T_1 - T_2}{L_A} = k_B \frac{T_2 - T_3}{L_B}$$

$$T_1 \frac{k_A}{L_A} - T_2 \frac{k_A}{L_A} = T_2 \frac{k_B}{L_B} - T_3 \frac{k_B}{L_B}$$

$$T_2 = \frac{T_1 \frac{k_A}{L_A} + T_3 \frac{k_B}{L_B}}{\frac{k_A}{L_A} + \frac{k_B}{L_B}}$$

Useful conclusions

$$\frac{\Delta T_A}{\Delta T_B} = \frac{\frac{L_A}{k_A}}{\frac{L_B}{k_B}}$$

$$q = -k \frac{\Delta T}{L} \quad \rightarrow \quad \Delta T \propto \frac{L}{k}$$

Thermal circuit

$$R_{cond,A} = \frac{L_A}{k_A} \qquad R_{cond,A} = \frac{L_B}{k_B} \qquad q = \frac{T_1 - T_3}{\frac{L_A}{k_A} + \frac{L_B}{k_B}}$$

Note: since $q = -k \frac{\partial T}{dx}$, we can use the q from the thermal circuit to go directly to the temperature gradient

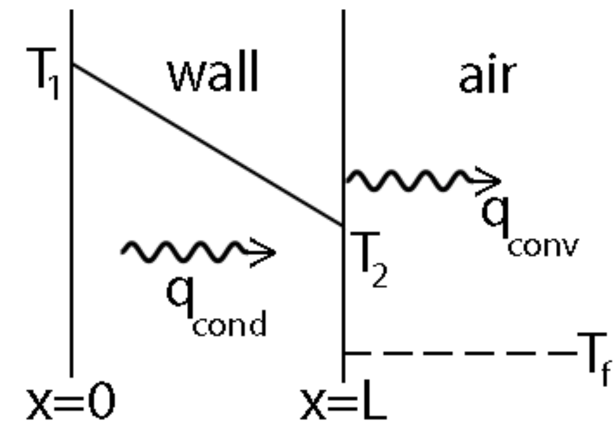
$$\frac{\partial T}{dx} = \frac{-q}{k}$$

Comparison of Steel vs. Mullite

$$k_{Steel} = 30 \frac{W}{m\ K} \qquad k_{Mullite} = 3 \frac{W}{m\ K}$$

$$if\ L_{Steel} = L_{Mullite}, \quad \frac{\Delta T_{Mullite}}{\Delta T_{Steel}} = 10$$

Steady State Convection on Wall



T_2 is unknown
At x=L

$$q = h(T_2 - T_f)$$

$$h = convection\ heat\ transfer\ coefficient\ \left[\frac{W}{m^2\ K}\right]$$

$$q = heat\ flux\ \left[\frac{W}{m^2}\right]$$

$$\frac{\partial^2 T}{dx^2} = 0 \quad in\ wall \qquad T = T_1 + \frac{T_2 - T_1}{L} x$$

At x=L, q_{cond}=q_{conv}

$$-k\frac{dT}{dx} = h(T_2 - T_f) \qquad \frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

$$-\frac{k}{L}T_2 + \frac{k}{L}T_1 = hT_2 - hT_f \qquad \frac{k}{L}T_1 + hT_f = T_2\left(h + \frac{k}{L}\right)$$

$$T_2 = \frac{\frac{k}{L}T_1 + hT_f}{h + \frac{k}{L}}$$

$$T = T_1 + \frac{x}{L}\left[\frac{\frac{k}{L}T_1 + hT_f}{h + \frac{k}{L}} - T_1\right]$$

$$T - T_1 = \frac{x}{L}\left[\frac{h(T_f - T_1)}{h + \frac{k}{L}}\right]$$

$$\frac{T-T_1}{T_f-T_1}=\frac{x}{L}\left[\frac{1}{1+\frac{k}{hL}}\right]$$

Dimensionless variables T* and x*

$$T^*=\frac{T-T_1}{T_f-T_1} \qquad x^*=\frac{x}{L} \qquad T^*=x^*\left[\frac{1}{1+\frac{k}{hL}}\right]$$

Thermal circuit

$$R_{cond}=\frac{L}{k} \qquad R_{conv}=\frac{1}{h} \qquad q=\frac{T_1-T_f}{\frac{L}{k}+\frac{1}{h}}$$

$$Biot\ Number=B_i=\frac{hL}{k}=ratio\ of\ conductive\ resistence\ to\ convective$$

Bi<<1

-Slow convection, no temperature gradients in solid

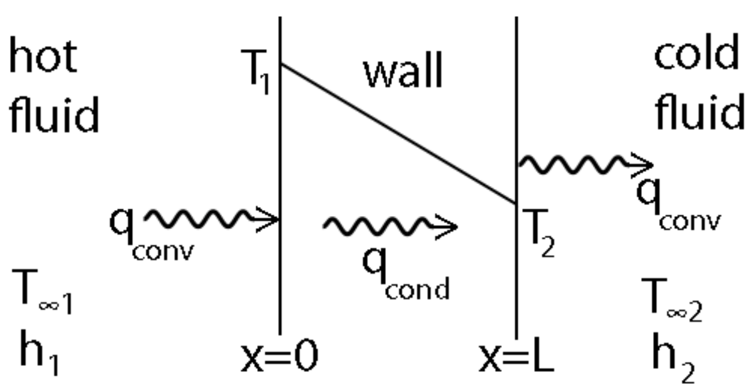
Bi≈1

-Conduction and convection equally important, transient solution

Bi>>1

-T_2=T_f, rapid convection

1D Plane Wall Conduction with Convection on Both Sides



T_1 and T_2 are unknown, assuming k is constant otherwise $\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$

$$\frac{\partial^2 T}{dx^2}=0 \quad in\ wall \qquad T=Ax+B$$

At x=0, q_{cond} = q_{conv} At x=L, q_{cond}=q_{conv}

$$-k\left[\frac{dT}{dx}\right]_{x=0+}=h_1(T_{\infty 1}-T_1) \qquad -k\left[\frac{dT}{dx}\right]_{x=L-}=h_2(T_2-T_{\infty 2})$$

$$\frac{dT}{dx} = A \qquad T_1 = B \qquad T_2 = AL + B$$

$$eq1: \quad -kA = h_1(T_{\infty 1} - B)$$

$$eq2: \quad -kA = h_2(AL + B - T_{\infty 2})$$

$$from\ eq1: \quad B = \frac{kA}{h_1} + T_{\infty 1} \quad substitute\ into\ eq2$$

$$-kA = h_2\left(AL + \frac{kA}{h_1} + T_{\infty 1} - T_{\infty 2}\right)$$

Solve for A

$$-kA = Ah_2\left(L + \frac{k}{h_1}\right) + h_2(T_{\infty 1} - T_{\infty 2})$$

$$A\left(-k - h_2\left(L + \frac{k}{h_1}\right)\right) = h_2(T_{\infty 1} - T_{\infty 2})$$

$$A = \frac{h_2(T_{\infty 1} - T_{\infty 2})}{-k - h_2\left(L + \frac{k}{h_1}\right)} = \frac{-(T_{\infty 1} - T_{\infty 2})}{k\left(\frac{1}{h_1} + \frac{1}{h_2} + \frac{L}{k}\right)}$$

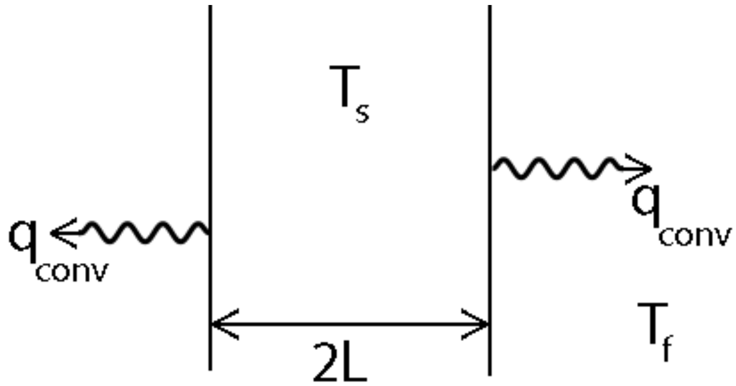
$$B = \frac{kA}{h_1} + T_{\infty 1}$$

$$q = -k\frac{dT}{dx} = -kA$$

Thermal circuit

$$R_{cond} = \frac{L}{k} \qquad R_{conv1} = \frac{1}{h_1} \qquad R_{conv2} = \frac{1}{h_2} \qquad q = \frac{T_{\infty 1} - T_{\infty 2}}{\frac{L}{k} + \frac{1}{h_1} + \frac{1}{h_2}}$$

Low Biot Number Solutions



Global heat balance, $T = f(t)$

$$Ah(T - T_f) = -\rho c_p \frac{dT}{dt} V$$

Separation of variable

$$\int \frac{-Ah}{\rho c_p V} dt = \int \frac{1}{T - T_f} dT$$

Note: Integrate RHS using u-substitution,

$$u = T - T_f, \qquad du = dT, \qquad \int \frac{1}{T - T_f} dT = \ln(u) + C$$

$$\frac{-Ah}{\rho c_p V} t + C = \ln(T - T_f)$$

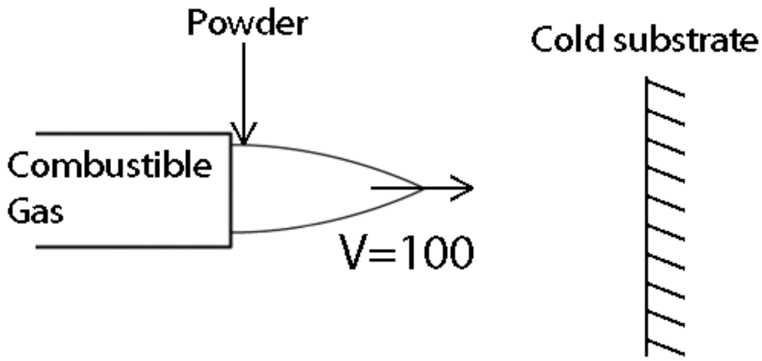
At t=0, T=Ts

$$C = \ln(T_s - T_f)$$

$$\frac{-Ah}{\rho c_p V} t = \ln(T - T_f) - \ln(T_s - T_f) = \ln\left(\frac{T - T_f}{T_s - T_f}\right)$$

$$\frac{T - T_f}{T_s - T_f} = \exp\left(\frac{-Ah}{\rho c_p V} t\right)$$

Thermal Spray Coatings / Plasma Spray



Spray is quasi-steady state because it is assumed that neither the plasma torch nor substrate moves with respect to the other, and that the particle is heated isothermally. This assumption is justified with Bi << 1 .

oxyacetylene torch: T = 3000K
powder: Ni alloy MAR-M200 R = 2 – 50 μm $(\mu m = 10^{-6} m)$

Problem: need to find time that spherical particle reaches melting temperature T_m

At r=0, symmetry $\frac{\partial T}{\partial r} = 0$
At r=R convection into ambient air, $q = h(T - T_f)$ $T_f = 3000K$ $h \approx 500 \frac{W}{m^2\, K}$

Governing equations

$$Bi = \frac{hL_c}{k}$$

L_c is the character length defined as volume/area ratio, R/3 for sphere, L for wall of thickness 2L, R/2 for cylinder

$$h = 500\ \frac{W}{m^2\ K} \qquad R = 50 * 10^{-6} m \qquad k = 16 \frac{W}{m\ K}$$

$$Bi = \frac{h\ R}{k\ 3} = 5.2 * 10^{-4} \ll 1 \quad \textit{Can use lumped capacitance model}$$

$$\frac{T - T_f}{T_0 - T_f} = \exp\left(\frac{-Ah}{\rho c_p V} t\right)$$

$$\frac{V}{A} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{R}{3}$$

Time to reach melting temperature

$$t_1 = \ln\left(\frac{T_m - T_f}{T_0 - T_f}\right)\frac{-\rho c_p R}{h\ 3} = 0.1035\ s$$

$$\mathrm{T_f} = 3000\mathrm{K} \qquad T_0 = 300K \qquad h = 500\ \frac{W}{m^2\ K}$$

$$\rho = 8500\frac{kg}{m^3} \qquad c_p = 500\frac{J}{kg\ K} \qquad T_{m,Ni} = 1700K$$

Once particle gets to melting temperature, still need to transfer energy to melt it

During heating:

$$Ah(T - T_f) = -\rho c_p \frac{dT}{dt}V$$

During melting:

$$Ah(T_m - T_f) = \rho V h_{sf}\frac{df_s}{dt}$$

$$f_s = particle's\ solid\ fraction \qquad h_{sf} = particle's\ latent\ heat\ of\ melting$$

$$h_{sf,Ni} = 297 * 10^3\frac{J}{kg}$$

$$Ah(T_m - T_f) = \rho V h_{sf}\frac{df_s}{dt}$$

Separation of variable

$$\int \frac{Ah(T_m - T_f)}{\rho V h_{sf}}dt = \int df_s$$

$$f_s = \frac{Ah(T_m - T_f)}{\rho V h_{sf}}t + C$$

At t=0, f_s=1, C=1

$$\frac{V}{A} = \frac{R}{3}$$

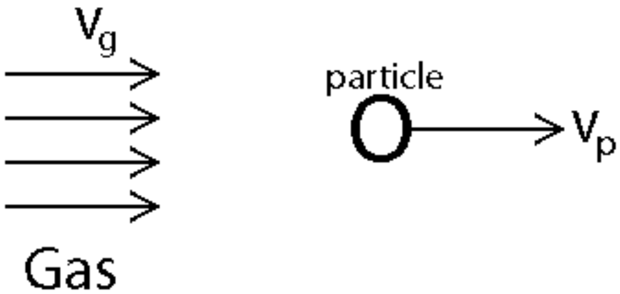
Want time when f_s=0

$$t_2 = \frac{-\rho V h_{sf}}{Ah(T_m - T_f)} = \frac{-\rho R h_{sf}}{3h(T_m - T_f)} = 0.0647\ s$$

Add time to reach T_m and time to melt to get total time

$$t_{tot} = 0.1683\ s$$

Calculating Acceleration and Velocity of Particle Using Air Resistance



For speeds not approaching the speed of sound

$$f_{drag} = f_{lin} + f_{quad} = bv + cv^2$$

For spherical projectile b and c have the form

$$b = \beta D \qquad c = \gamma D^2$$

For air at standard temperature (0°C) and pressure (1 atm), which is inaccurate but they were the values that were available

$$\beta = 1.6 * 10^{-4} \frac{N\,s}{m^2} \qquad \gamma = 0.25\ \frac{N\,s^2}{m^4}$$

We can compare f_{lin} and f_{quad} to see if one can be neglected, using v=100m/s and $D = 100 * 10^{-6} m$

$$\frac{f_{quad}}{f_{lin}} = \frac{\gamma D^2 v^2}{\beta D v} = \frac{\gamma D v}{\beta} = 15.625$$

Quadratic is dominant at v=100m/s but it's dominance diminishes as the particle gets up to speed, for the sake of completeness we will solve using both linear and quadratic drag.

From F=ma

$$\frac{dv}{dt} = \frac{b}{m}\left(v_g - v\right) + \frac{c}{m}\left(v_g - v\right) * abs\left(v_g - v\right)$$

$$m = V\rho = \frac{4}{3}\pi r^3 \rho \qquad v_g = 100\frac{m}{s} \qquad \gamma = 0.25\ \frac{N\,s^2}{m^4} \qquad c = \gamma D^2$$

$$\beta = 1.6 * 10^{-4} \frac{N\,s}{m^2} \qquad b = \beta D$$

$$D = 100 * 10^{-6} m \qquad R = 50 * 10^{-6} m \qquad \rho = 8500 \frac{kg}{m^3}$$

I chose to solve this numerically using Matlab

At $t = 0.1683\ s$ distance traveled is 13.1060 m

Code used:

```
function velocity_using_quadandlinear_drag
roe=8500;
R=50e-6;
D=2*R;
gamma=0.25;
beta=1.6e-4;
c=gamma*D^2;
b=beta*D;
V=4/3*pi*R^3;
m=roe*V;
vg=100;

%initial conditions
v0=0;

%time span 0 to 10 sec
tspan = [0 .1683];

[t ,v] = ode15s(@quaddrag, tspan, v0);

%calculate distance
distance=trapz(t,v)

    function dvdt = quaddrag(t, v)%time, mass faction
        dvdt = b/m*(vg-v)+c/m*(vg-v)*abs(vg-v);
    end
end
```