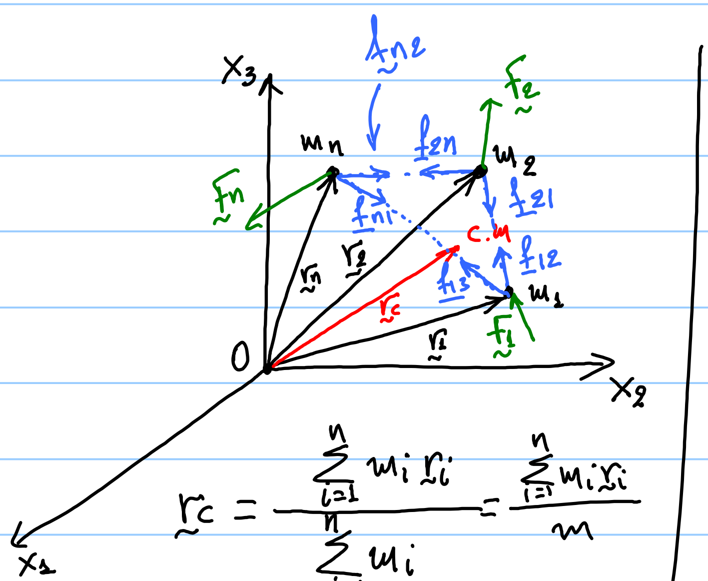
CHAPTER 8: Dynamics of a Rigid Body

Reminder. Chapter 4: System of particles

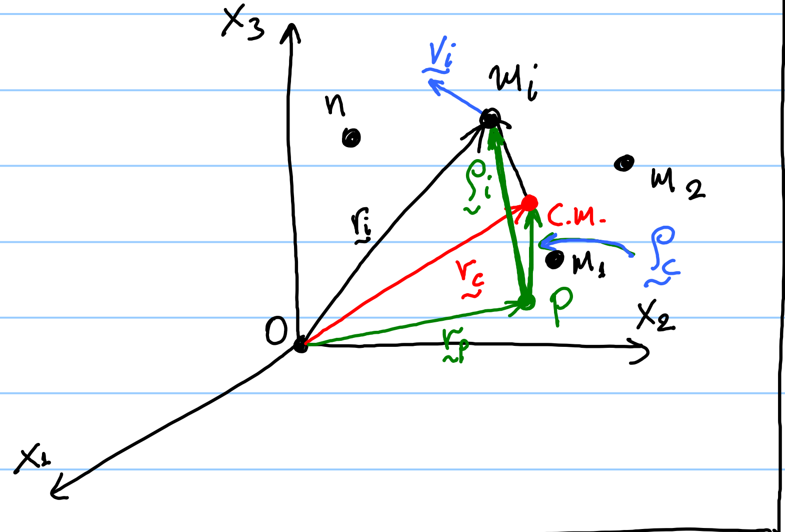


$$\underset{\sim}{r}_c = \frac{\sum_{i=1}^{n} m_i \underset{\sim}{r}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \underset{\sim}{r}_i}{m}$$

Translational motion:

$$\boxed{\sum_{i=1}^{n} \underset{\sim}{F}_i = \underset{\sim}{F} = m\ddot{\underset{\sim}{r}}_c}$$

where $\underset{\sim}{r}_c$ is the location of the c.m.



Equation of rotational motion

(i) Fixed reference point O: $\boxed{\underset{\sim}{M}_O = \dot{\underset{\sim}{H}}_O}$ (1)

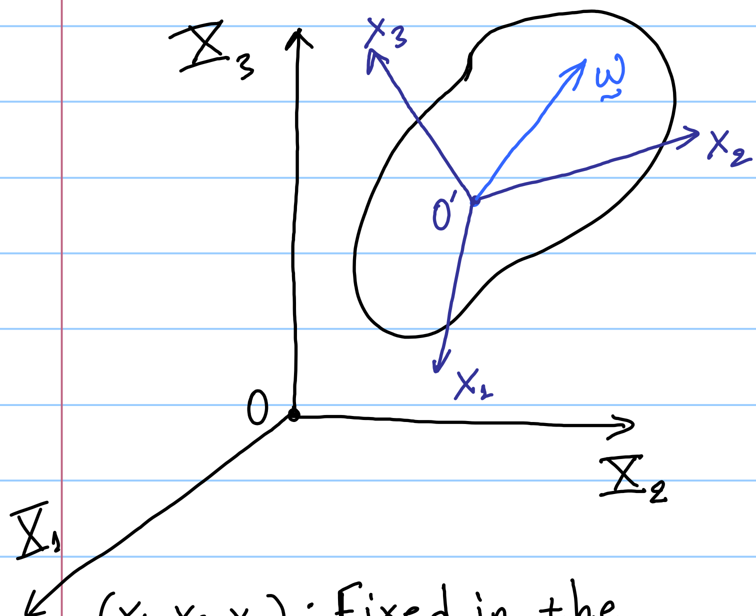
$$\underset{\sim}{M}_O = \sum_{i=1}^{n} \underset{\sim}{r}_i \times \underset{\sim}{F}_i \, , \quad \underset{\sim}{H}_i = \sum \underset{\sim}{r}_i \times \underset{\sim}{P}_i$$

ii) Reference point at the c.m.: $\boxed{\underset{\sim}{M}_c = \dot{H}_c}$ (2)

iii) Arbitrary reference point P: $\boxed{\underset{\sim}{M}_P - \underset{\sim}{\rho}_c \times m \ddot{\underset{\sim}{r}}_P = \dot{\underset{\sim}{H}}_P}$ (3)

Note: If P is fixed or $\dot{\underset{\sim}{r}}_P = \text{const.}$, or if $P \equiv$ c.m., or if $\underset{\sim}{\rho}_c \,//\, \ddot{\underset{\sim}{r}}_P \Rightarrow \underset{\sim}{\rho}_c \times m \ddot{\underset{\sim}{r}}_P = 0$

8-1 General equations of motion



(x_1, x_2, x_3): Fixed in the rigid body and rotates with it

Body-axis coordinate system

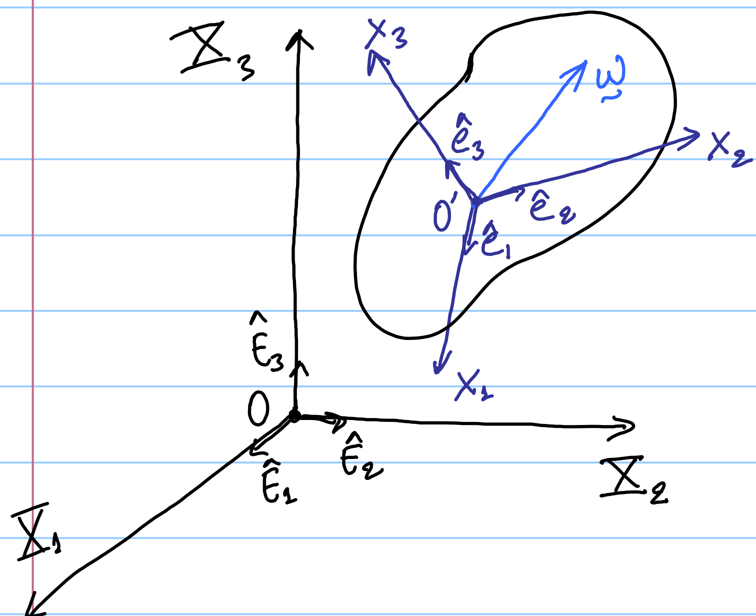
i) Translational motion: $\boxed{\underset{\sim}{F} = m\ddot{\underset{\sim}{r}}_c}$ (4)

ii) Rotational motion

If O' is a fixed point or the c.m.: $\boxed{\underset{\sim}{M}_{O'} = \dot{\underset{\sim}{H}}_{O'}}$ (5)

If O' is some other reference point then we have to use eqn. (3)

In almost all cases that we will discuss: $\underset{\sim}{M} = \underset{\sim}{M}_{O'}$



(x_1, x_2, x_3): Body-axis c.s.

$\underset{\sim}{\omega}$: Absolute angular velocity w.r.t. X_i

Rotational motion:

$$\boxed{\underset{\sim}{M} = \dot{\underset{\sim}{H}} = \underset{\approx}{I} \cdot \dot{\underset{\sim}{\omega}} + \underset{\sim}{\omega} \times \underset{\sim}{H} = \underset{\approx}{I} \cdot \dot{\underset{\sim}{\omega}} + \underset{\sim}{\omega} \times \underset{\approx}{I} \cdot \underset{\sim}{\omega}} \quad (6a)$$

$$\boxed{M_i = I_{ij}\,\dot{\omega}_j + \varepsilon_{ijk}\,\omega_j H_k} \quad (6b) \qquad \text{or}$$

$$M_1 = I_{11}\dot{\omega}_1 + I_{12}\dot{\omega}_2 + I_{13}\dot{\omega}_3 + \omega_2 H_3 - \omega_3 H_2 \quad (6c)$$

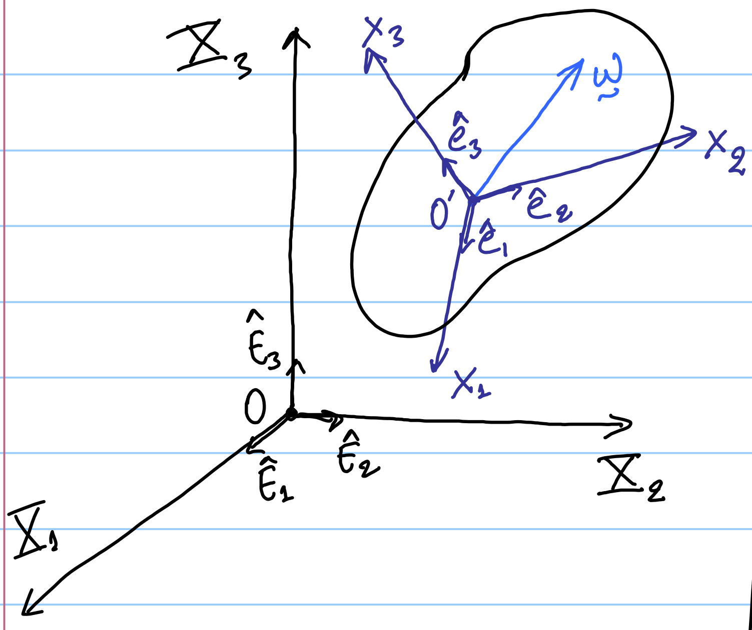
(Direct proof in chapter 7) or

$$\left.\begin{aligned} \dot{\underset{\sim}{H}} &= (\dot{H})_r + \underset{\sim}{\omega} \times \underset{\sim}{H} \\ (\dot{\underset{\sim}{H}}_r) &= (\dot{H}_i)_r \hat{e}_i = (\dot{\overline{I_{ij}\,\omega_j}})_r \hat{e}_i = I_{ij}\,\dot{\omega}_j\,\hat{e}_i \end{aligned}\right\} \Rightarrow$$

$$\boxed{\dot{\underset{\sim}{H}} = \underset{\approx}{I} \cdot \dot{\underset{\sim}{\omega}} + \underset{\sim}{\omega} \times \underset{\sim}{H}} \quad (6a)$$

$\dot{\underset{\sim}{H}}$: absolute rate of change

Euler's equations of Motion



(x_1, x_2, x_3): Body-axis c.s.

$\underset{\sim}{\omega}$: Absolute angular velocity w.r.t. X_i

If x_1, x_2, x_3 are the principal axes w.r.t. O' and O': fixed point or c.m. then

$$H_i = \left(\underline{\underline{I}} \cdot \underset{\sim}{\omega}\right)_i = \underline{\underline{I_{ij}}}\, \delta_{ij}\, \omega_j \,, \quad H_1 = I_{11}\, \omega_1 \,, \ldots \qquad (7)$$

$$(6b) \Rightarrow \boxed{M_i = \underline{\underline{I_{ij}}}\, \delta_{ij}\, \dot{\omega}_j + \varepsilon_{ijk}\, \omega_j\, \underline{\underline{I_{kl}}}\, \delta_{kl}\, \omega_l} \qquad (8a)$$

$$\Rightarrow M_1 = I_{11}\, \dot{\omega}_1 + \varepsilon_{123}\, \omega_2\, I_{33}\, \omega_3 + \varepsilon_{132}\, \omega_3\, I_{22}\, \omega_2$$

$$\Rightarrow \boxed{\begin{cases} M_1 = I_{11}\, \dot{\omega}_1 + \left(I_{33} - I_{22}\right) \omega_2\, \omega_3 \\ M_2 = I_{22}\, \dot{\omega}_2 + \left(I_{11} - I_{33}\right) \omega_1\, \omega_3 \\ M_3 = I_{33}\, \dot{\omega}_3 + \left(I_{22} - I_{11}\right) \omega_1\, \omega_2 \end{cases}} \qquad (8b)$$

Body-axis translational equations

$\underset{\sim}{F} = m \ddot{\underset{\sim}{r}}_c \Longleftrightarrow \underset{\sim}{F} = m \dot{\underset{\sim}{v}}_c$, $\underset{\sim}{F}$: total external force

$\underset{\sim}{v}_c$: absolute velocity of the c.m.,

$\dot{\underset{\sim}{v}}_c$ referred to the body-axis coordinate system

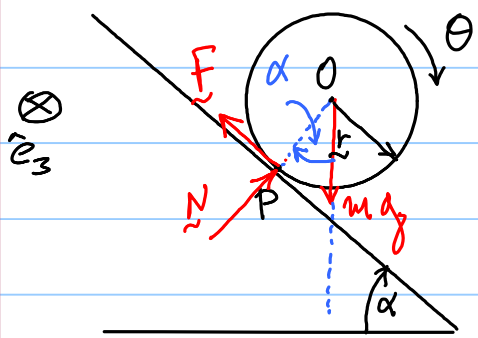
$$\dot{\underset{\sim}{v}}_c = (\dot{\underset{\sim}{v}}_c)_r + \underset{\sim}{\omega} \times \underset{\sim}{v}_c \ , \ (\dot{\underset{\sim}{v}})_r = \dot{v}_i \hat{e}_i + \underset{\sim}{\omega} \times \underset{\sim}{v} \Rightarrow F_i/m = \dot{v}_i + \varepsilon_{ijk}\, \omega_j v_k \quad (9)$$

e.g. $F_1 = m \dot{v}_{c1} + m (\omega_2 v_{c3} - \omega_3 v_{c2})$

$\underset{\sim}{v}_c, \dot{\underset{\sim}{v}}_c$: are the absolute velocity and acceleration of the c.m. referred to the body-axis c.s.

Examples

1. Rigid body motion in a plane (2-D) (Example 7-5)



(a)

Uniform cylinder rolls without slipping

$\ddot{\theta} = ?$

(A) First method

(O ≡ C.M. the reference point)

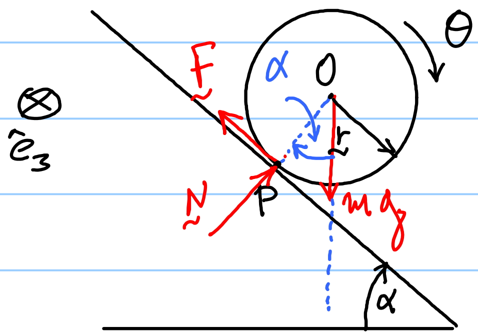
(i) Translation : $mg\sin\alpha - F = m\dot{v} = mr\ddot{\theta}$ (1)

(ii) Rotation: $\underset{\sim}{M} = \dot{\underset{\sim}{H}}$ (2)

$\underline{H} = \underline{\underline{I}} \cdot \underset{\sim}{\omega}$, $\underset{\sim}{\omega} = \omega\, \hat{e}_3 = \dot{\theta}\, \hat{e}_3$, $I_{ij} = 0$ for $i \neq j$

$\underline{H} = I_{33}\dot{\theta}\, \hat{e}_3$ (3)

(2), (3) $\Longrightarrow$ $M = I_{33}\ddot{\theta}$ (4) , $I_{33} = \frac{1}{2} m r^2$ (5)



(a)

Uniform cylinder rolls without slipping

$\ddot{\theta} = ?$

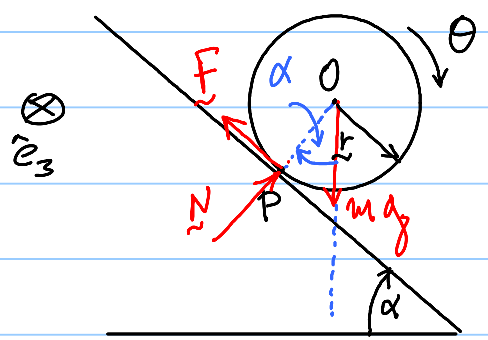
From the free-body diagram: $M = Fr \qquad (6)$

$(4\text{–}6) \Rightarrow \quad F = \frac{1}{2} m r \ddot{\theta} \qquad (7)$

$(1),(7) \Rightarrow \quad mg\sin\alpha - \frac{1}{2} m r \ddot{\theta} = m r \ddot{\theta}$

$$\Rightarrow \boxed{\ddot{\theta} = 2g\sin\alpha / 3r}$$

(B) Second method (P: fixed point in the inclined plane)



(a)

Uniform cylinder rolls without slipping

$\ddot{\theta} = ?$

Since P is fixed → 1 d.o.f. ⇒ 1 equation

$$\underset{\sim}{M}_P = \dot{\underset{\sim}{H}}_P \quad (1), \quad \underset{\sim}{H}_P = \underline{\underline{I}}_P \cdot \underline{\omega} \Rightarrow H_P = I_P \dot{\theta} \quad (2)$$

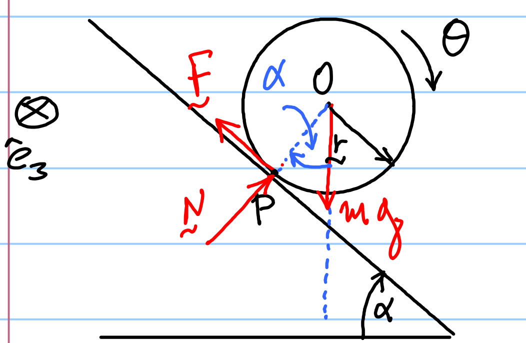
$$I_P = I_O + mr^2 = \tfrac{1}{2} mr^2 + mr^2 = \tfrac{3}{2} mr^2 \quad (3)$$

$$M_P = mg\, r \sin\alpha \quad (4)$$

$$(1-4) \Rightarrow mgr\sin\alpha = \tfrac{3}{2} mr^2 \ddot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} = 2g\sin\alpha / 3r}$$

(c) Third method: Reference point P fixed in the cylinder



(a)

Uniform cylinder rolls without slipping

$\ddot{\theta} = ?$

Because P is moving and it is not the c.m., we have to use the equation: $\underset{\sim}{M}_P - \underset{\sim}{\rho}_c \times m\ddot{\underset{\sim}{r}}_P = \dot{\underset{\sim}{H}}_P$ (1)

$\underset{\sim}{\rho}_c$ is the position of the c.m. O w.r.t. to P. In this specific case: $\underset{\sim}{\rho}_c \parallel \ddot{r}_P \Rightarrow \underset{\sim}{\rho}_c \times m\ddot{\underset{\sim}{r}}_P = 0$ (2)

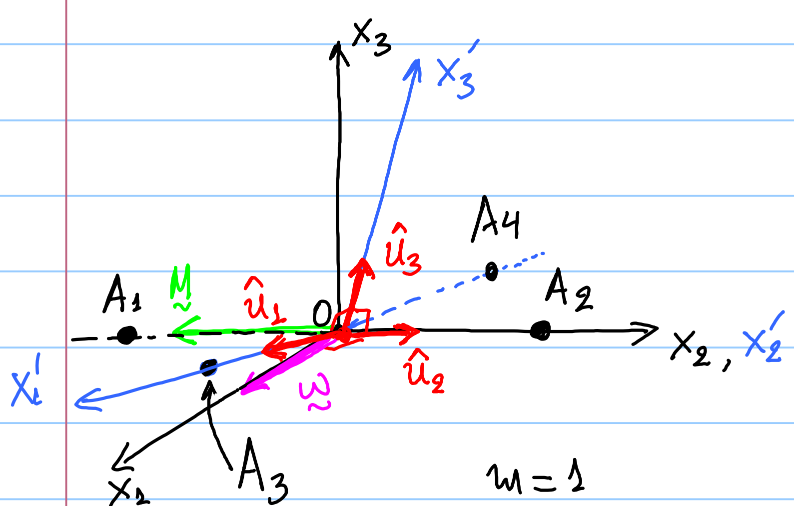
If the c.m. was not at O then $\underset{\sim}{\rho}_c \times m\ddot{\underset{\sim}{r}}_P \neq 0$

$\dot{H}_P = I_P\dot{\theta}$, $M_P = mgr\sin\alpha$

$\Rightarrow \boxed{\ddot{\theta} = 2g\sin\alpha/3r}$

2. 3-D examples

Example 8-1



$m = 1$

$A_1 = (0, -\sqrt{50}, 0)$, $A_2 = (0, \sqrt{50}, 0)$

$A_3 = (10, 0, 5)$, $A_4 = (-10, 0, -5)$

$$[I] = \begin{pmatrix} 150 & 0 & -100 \\ 0 & 250 & 0 \\ -100 & 0 & 300 \end{pmatrix} \text{ kg}\cdot\text{m}^2$$

$$I_{11} = \sum_{i=1}^{4} m_i \left(x_{i2}^2 + x_{i3}^2\right) = 50 + 50 + 25 + 25 = 150$$

$$I_{13} = -\sum_{i=1}^{4} m_i\, x_{i1}\, x_{i3} = -0\cdot 0 - 0\cdot 0 - 10\cdot 5 - 10\cdot 5 = -100$$

Eigenvectors - Principal directions

$$\hat{u}_1 = \begin{pmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{pmatrix}, \quad \hat{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{u}_3 = \begin{pmatrix} -1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{pmatrix}$$

Consider a rigid body with inertia matrix w.r.t. the c.m.

$$[I] = \begin{pmatrix} 150 & 0 & -100 \\ 0 & 250 & 0 \\ -100 & 0 & 300 \end{pmatrix} \text{kg}\cdot\text{m}^2 \quad \text{and} \quad \underset{\sim}{\omega} = (10 \quad 0 \quad 0)^T \text{ rad/s} ;$$

What is $\underset{\sim}{M}$

General expression: $M_i = I_{ij}\,\dot{\omega}_j + \varepsilon_{ijk}\,\omega_j H_k$

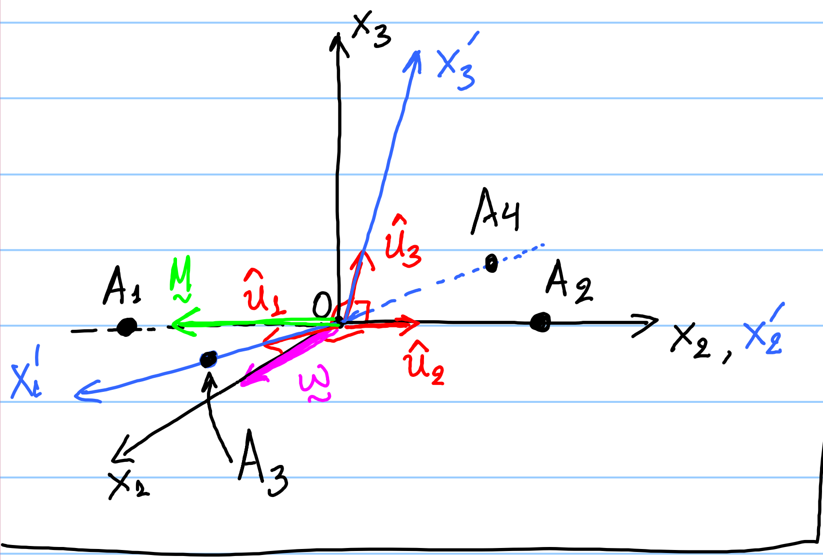
$\dot{\omega}_j = 0 \quad , j = 1,2,3 \quad ; \quad \underset{\sim}{H} = \underline{\underline{I}} \cdot \underline{\omega} \Rightarrow H_i = I_{ij}\,\omega_j \Rightarrow \begin{cases} H_1 = I_{11}\,\omega_1 = 1500 \text{ kg m}^2/\text{s} \\ H_2 = 0 \\ H_3 = I_{31}\,\omega_1 = -1000 \text{ kg m}^2/\text{s} \end{cases}$

$\Rightarrow \quad M_1 = \omega_2 H_3 - \omega_3 H_2 = 0 , \quad M_2 = \cancel{\omega_3 H_1} - \omega_1 H_3 = -10^4 \text{ Nm} \quad , \quad M_3 = 0$

$\underset{\sim}{M} = (0 \quad -10^4 \quad 0)^T$, absolute value of $\underset{\sim}{M}$ w.r.t. the Body-fixed coord. Syst.

(13)

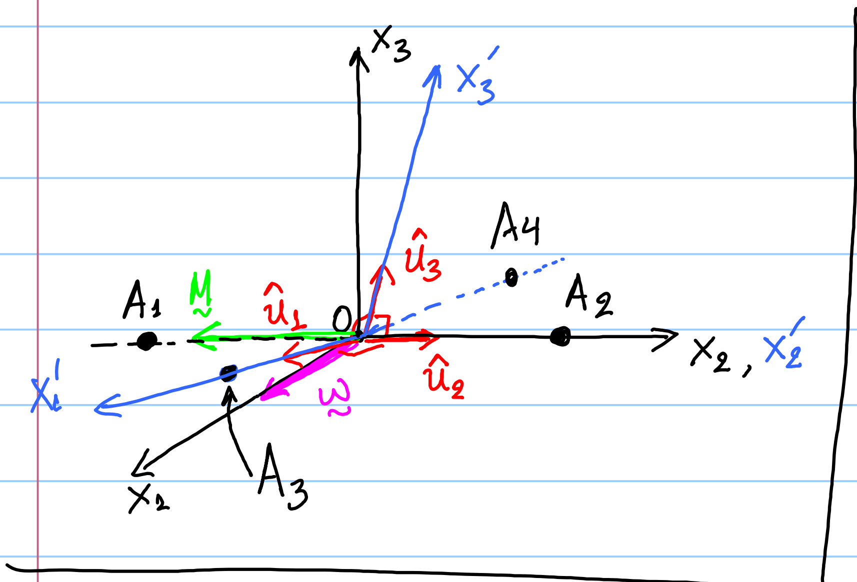
$\underset{\sim}{M}$ = const. ⟹ it rotates with the body



The external torque balances the inertia couple due to the centrifugal force of the particles A_3 and A_4 on the x_1' axis which rotates about the x_1 axis.

If we analyze the problem w.r.t. the principal axes

$$[I'] = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 250 & 0 \\ 0 & 0 & 350 \end{pmatrix} \text{kg.m}^2$$



$$\omega_i' = \hat{u}_i \cdot \underset{\sim}{\omega} \quad \text{or} \quad \underset{\sim}{\omega}' = \begin{pmatrix} u_1^T \\ u_2^T \\ u_3^T \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

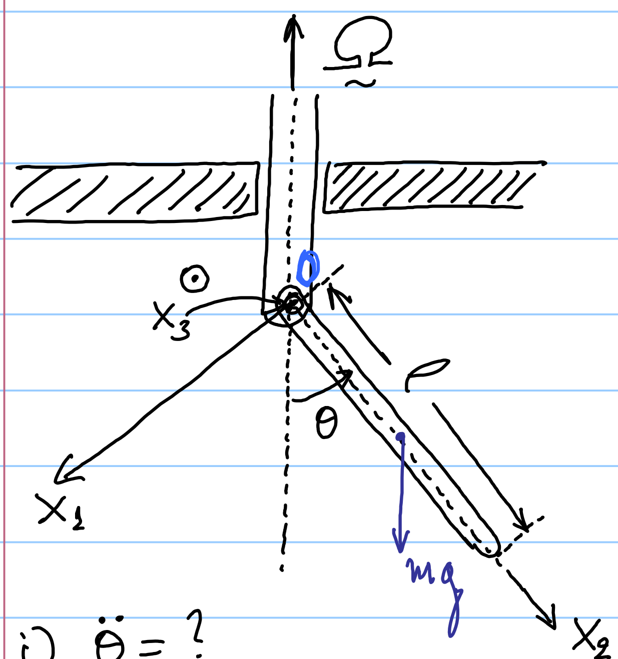
$$\Rightarrow \begin{pmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4\sqrt{5} \\ 0 \\ -2\sqrt{5} \end{pmatrix}$$

$$\begin{cases} M_1' = I_{11}' \dot{\omega}_1' + (I_{33}' - I_{22}')\, \omega_2' \omega_3' \\ M_2' = I_{22}' \dot{\omega}_2' + (I_{11}' - I_{33}')\, \omega_1' \omega_3' \\ M_3' = I_{33}' \dot{\omega}_3' + (I_{22}' - I_{11}')\, \omega_1' \omega_2' \end{cases}$$

$\dot{\omega}_i' = 0$,

$M_1' = 0$, $\quad M_2' = (100 - 350)\, 4\sqrt{5}\,(-2\sqrt{5}) = 10^4$ N·m , $\quad M_3' = 0 \quad$ (x_2 axis $\equiv$ x_2' axis)

Example 8-2



i) $\ddot{\theta} = ?$

ii) Stability for $\theta \ll 1$

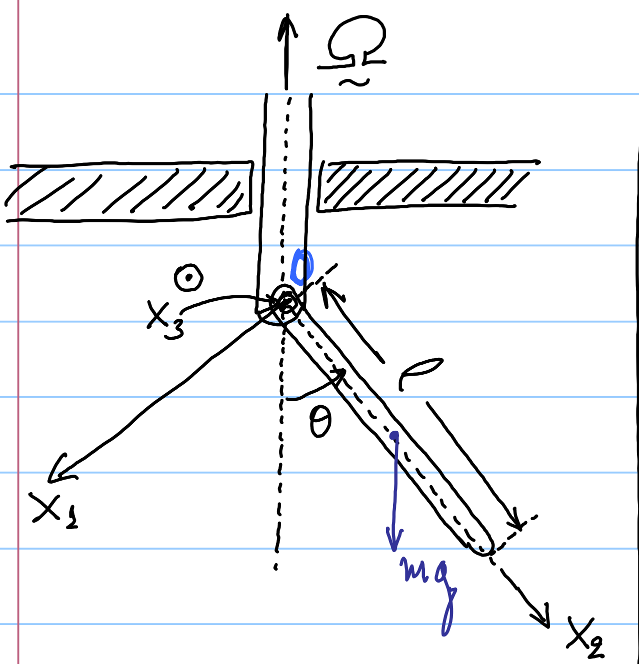
We consider the equations of motion w.r.t. the fixed point O.

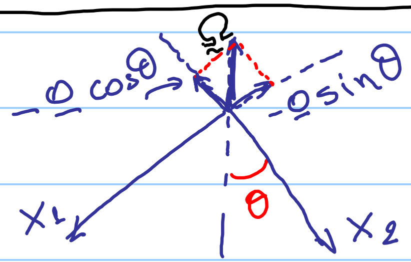
$$I_{11} = I_{33} = 1/3\, m \ell^2, \quad I_{22} = 0, \quad I_{ij} = 0 \text{ for } i \neq j \qquad (1)$$

For the chosen axes $[I]$ is diagonal and the eqns of motion are the Euler's eqns.

$$\boxed{\begin{cases} M_1 = I_{11}\dot{\omega}_1 + (I_{33} - I_{22})\,\omega_2\omega_3 \\ M_2 = I_{22}\dot{\omega}_2 + (I_{11} - I_{33})\,\omega_1\omega_3 \\ M_3 = I_{33}\dot{\omega}_3 + (I_{22} - I_{11})\,\omega_1\omega_2 \end{cases}}$$

(2a), (2b), (2c)





We will compute the coordinates of the absolute angular velocity of the rod w.r.t. (x_1, x_2, x_3) C.S.

$$\omega_1 = -\Omega\sin\theta\,, \quad \omega_2 = -\Omega\cos\theta\,, \quad \omega_3 = \dot{\theta} \qquad (3)$$

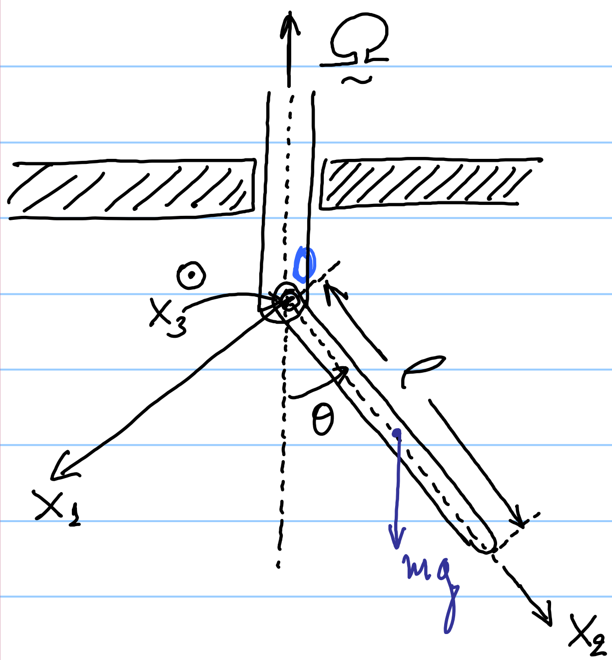
Applied moment: $M_3 = -\,mg\,\ell/2\,\sin\theta \qquad (4)$

$(2b) \Rightarrow \quad 0 = 0$ ✗

$$I_{11}\,\dot{\omega}_1 = \tfrac{1}{3} m \ell^2 (-\Omega\cos\theta)\,\dot{\theta}$$

$$(I_{33} - I_{22})\,\omega_2\omega_3 = \tfrac{1}{3} m \ell^2\,\dot{\theta}\,(-\Omega\cos\theta)\,, \qquad M_1 = 0$$

$(2a) \Rightarrow M_1 = I_{11}\,\dot{\omega}_1 + (I_{33} - I_{22})\,\omega_2\omega_3 \Rightarrow 0 = 0$ ✗



(2c) $\Rightarrow M_3 = I_{33}\,\dot{\omega}_3 + (I_{22} - I_{11})\,\omega_1\,\omega_2$

$$\rightsquigarrow\ -mg\,\tfrac{\ell}{2}\sin\theta = \tfrac{1}{3}m\ell^2\,\ddot{\theta} + \left(-\tfrac{1}{3}m\ell^2\right)\underline{\Omega}^2\sin\theta\cos\theta$$

$$\Rightarrow\ \boxed{\ddot{\theta} + \left(\frac{3g}{2\ell} - \underline{\Omega}^2\cos\theta\right)\sin\theta = 0} \qquad (5)$$

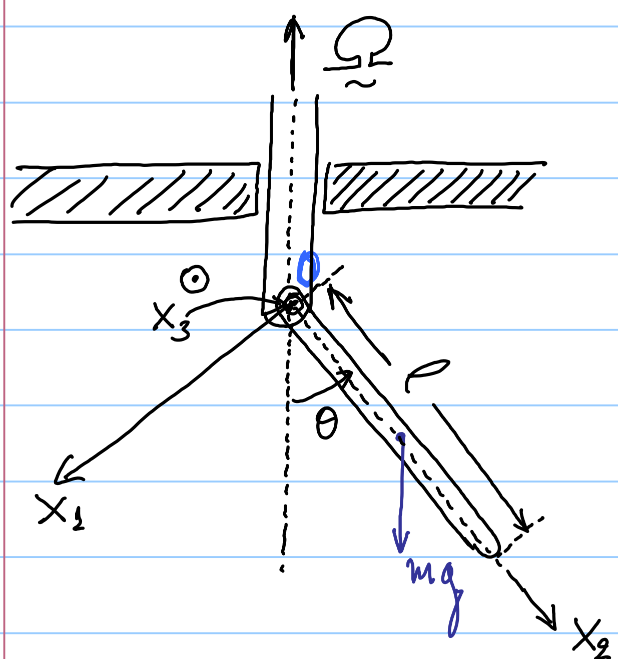
a) if $\underline{\Omega} = 0 \Rightarrow$ simple pendulum with $L = 2\ell/3$

b) for $\theta \ll 1 \Rightarrow \cos\theta \simeq 1,\ \sin\theta \simeq \theta$

$$(5) \Rightarrow \boxed{\ddot{\theta} + \left(\frac{3g}{2\ell} - \underline{\Omega}^2\right)\theta = 0} \qquad (6)$$

Harmonic eqn if $\frac{3g}{2\ell} > \underline{\Omega}^2$

Natural freq. $\omega_n = \sqrt{\frac{3g}{2\ell} - \underline{\Omega}^2}$



$$\ddot{\theta} + \left(\frac{3g}{2\ell} - \Omega^2\right)\theta = 0 \qquad (6)$$

For $\Omega^2 > 3g/2\ell$ unstable near $\theta = 0$
solution shifts from trigonometric to hyperbolic

However, even for $\Omega^2 > \frac{3g}{2\ell}$ a stable solution exists for $\theta \gg 0$.

$$\ddot{\theta} + \left(\frac{3g}{2\ell} - \Omega^2 \cos\theta\right)\sin\theta = 0 \qquad (5)$$

By setting $\ddot{\theta} = 0 \xRightarrow{(5)} \cancel{\theta_0 = 0}\,, \quad \theta_0 = \cos^{-1}\left(\frac{3g}{2\ell\Omega^2}\right) \qquad (7)$

$$\ddot{\theta} + \left(\frac{3g}{2l} - \underline{\Omega}^2\cos\theta\right)\sin\theta = 0, \qquad \theta_0 = \cos^{-1}\left(\frac{3g}{2l\underline{\Omega}^2}\right), \qquad \theta = \theta_0 + \delta\theta$$

$$\ddot{\theta} = \delta\ddot{\theta}, \qquad \cos\theta = \cos(\theta_0 + \delta\theta) = \cos\theta_0 - \sin\theta_0\,\delta\theta$$

$$\sin\theta = \sin(\theta_0 + \delta\theta) = \sin\theta_0 + \cos\theta_0\,\delta\theta$$

$$\delta\ddot{\theta} + \underline{\Omega}^2\left(\cancel{\frac{3g}{2l\underline{\Omega}^2}} - \cancel{\cos\theta_0} + \sin\theta_0\,\delta\theta\right)\left(\sin\theta_0 + \cos\theta_0\,\delta\theta\right) = 0$$

$$\Rightarrow \quad \delta\ddot{\theta} + \underline{\Omega}^2\sin^2\theta_0\,\delta\theta = 0$$

We can investigate the stability about $\theta_0 = \cos^{-1}\left(\frac{3g}{2l\Omega^2}\right)$
by perturbing the differential equation about θ_0, $\theta = \theta_0 + \delta\theta$

$$(5) \Longrightarrow \boxed{\delta\ddot{\theta} + \Omega^2 \sin^2\theta_0\, \delta\theta = 0} \qquad (8)$$

$\delta\theta$ oscillates about 0 or θ oscillates about θ_0 with the natural frequency

$$\omega_n = |\Omega \sin\theta_0| = \sqrt{\Omega^2(1-\cos^2\theta_0)} = \sqrt{\Omega^2 - \left(\frac{3g}{2l\Omega}\right)^2} \qquad (9)$$

Stability of rotational motion about a principal axis. (Project)

- Free motion of a rigid body rotating about one of its principal axes

$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$: Principal axes at the c.m. fixed in the body

$\underset{\sim}{\omega_0} = (\omega_0 \;\; 0 \;\; 0)^T$; $\omega_{01} = \omega_0 = \text{const.}$, $\omega_{02} = \omega_{03} = 0$; Rotation about x_1 axis

Problem: Analyze the free motion of the rigid body after it is given a small disturbance $(\delta\omega_1 \; \delta\omega_2 \; \delta\omega_3)^T$, $\delta\omega_i << \omega$, $i = 1, 2, 3$

$$M_1 = I_{11}\dot{\omega}_1 + (I_{33} - I_{22})\,\omega_2\omega_3 \qquad (1a)$$

$$M_2 = I_{22}\dot{\omega}_2 + (I_{11} - I_{33})\,\omega_1\omega_3 \qquad (1b) \qquad \text{Euler's eqns.}$$

$$M_3 = I_{33}\dot{\omega}_3 + (I_{22} - I_{11})\,\omega_1\omega_2 \qquad (1c)$$

$$(1a) \Longrightarrow \quad I_{11}\dot{\omega}_1 + (I_{33} - I_{22})\,\underbrace{\cancel{\delta\omega_2\,\delta\omega_3}}_{\simeq 0} = 0 \quad \Rightarrow \dot{\omega}_1 = \text{const} \Rightarrow \omega_1 = \omega_0 \quad (2)$$

$$(1b,c) \Longrightarrow \begin{cases} I_{22}\dot{\omega}_2 + (I_{11} - I_{33})\,\omega_0\,\omega_3 = 0 & (3a) \\ I_{33}\dot{\omega}_3 + (I_{22} - I_{11})\,\omega_0\,\omega_2 = 0 & (3b) \end{cases}$$

where $\omega_2\ ,\ \omega_3 \ll \omega_0$

$$(3a) \Longrightarrow \quad I_{22}\ddot{\omega}_2 + (I_{11} - I_{33})\,\omega_0\,\dot{\omega}_3 = 0$$

$$\xRightarrow{(3b)} \quad I_{22}\ddot{\omega}_2 + (I_{11} - I_{33})\,\omega_0\,\frac{(I_{11} - I_{22})}{I_{33}}\,\omega_0\,\omega_2 = 0$$

$$\ddot{\omega}_2 + \frac{(I_{11}-I_{22})(I_{11}-I_{33})}{I_{22}\, I_{33}}\, \omega_0^2\, \omega_2 = 0 \quad (4a),$$

$$\dot{\omega}_3 = \frac{I_{11}-I_{22}}{I_{33}}\, \omega_0\, \omega_2 \quad (4b), \qquad \dot{\omega}_2 = \frac{I_{33}-I_{11}}{I_{22}}\, \omega_0\, \omega_3 \quad (4c)$$

$$(3) \Rightarrow \begin{pmatrix} \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} = 0 \quad (5a),$$

$$a = -\left[(I_{11}-I_{33})/I_{22}\right]\omega_0$$
$$b = -\left[(I_{22}-I_{11})/I_{33}\right]\omega_0 \quad (5b)$$

Eigenvalues: λ_1, λ_2 ; Eigenvectors: $\hat{u}_1, \hat{u}_2$

solution: $\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = c_1 e^{\lambda_1 t} \hat{u}_1 + c_2 e^{\lambda_2 t} \hat{u}_2$

(ia) $I_{11} = I_{33} \xRightarrow{(3a)} \dot{\omega}_2 = 0 \Rightarrow \omega_2 = \text{const.}$

$(3b) \Rightarrow \dot{\omega}_3 = A = \text{const.} \Rightarrow \omega_3 = At + \omega_{30}$

(ib) similarly if $I_{11} = I_{22} \xRightarrow{(3b)} \omega_3 = \text{const.}$

$I_{22}\dot{\omega}_2 + (I_{11} - I_{33})\,\omega_0\,\omega_3 = 0$ $\quad (3a) \Rightarrow \dot{\omega}_2 = \text{Const.} = B \Rightarrow \omega_2 = Bt + \omega_{20}$

(i) $I_{11} < I_{22}$ and $I_{11} < I_{33}$ or $I_{11} > I_{22}$ and $I_{11} > I_{33}$

(4a) ω_2 is harmonic with $\omega_n = \omega_0 \left[(I_{11} - I_{22})(I_{11} - I_{33}) / I_{22} I_{33} \right]^{1/2}$ (5)

If $\omega_{02} = \omega_{03} = 0 \xRightarrow{(4)} \omega_1 = 10,\ \omega_2 = \omega_3 = 0$

if $\omega_{02} = \delta\omega$ and/or $\omega_{03} = \delta\omega' \Rightarrow \omega_2$ and ω_3 are oscillating but the magnitude of the disturbance <u>does not</u> grow.

$\Rightarrow$ The rotational motion about x_1 is <u>stable</u>.

(ii) If $I_{22} < I_{11} < I_{33}$ or $I_{33} < I_{11} < I_{22}$

ω_2 and ω_3 : exponentially increasing (unstable motion)

Matlab Project 1

Investigate the stability of the rotational motion about the axis x_1 for a rigid body with principal moments of inertia

(i) $I_{11} < I_{22}$ and $I_{11} < I_{33}$, (ii) $I_{11} > I_{22}$ and $I_{11} > I_{33}$

(iii) $I_{22} < I_{11} < I_{33}$, (iv) $I_{33} < I_{11} < I_{22}$