SDE

Stochastic Calculus

Ito's Calculus

Ito's Formula

M_t: Semi Martingale

$$f(t, M_t) - f(0, x_0)$$

$$= \int_0^t \frac{\partial f}{\partial s} ds + \int_0^t \sum_i \frac{\partial f}{\partial x_j} dM^j + \frac{1}{2} \int_0^t \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} d[M^i, M^j]_t$$

For any f sqaure integrable:
$f : [0, t] \to \mathbb{R}$, $\int_0^t f^2 dx < \infty$
then

$$\int_0^t f(t) dW_t = \xi_t \sim N\left(0, \int_0^t f^2 ds\right)$$

Ito's Isometry

Integration By parts

SDE

Girsanov's theorem

Novikov's condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \theta_s^2 ds\right)\right] < \infty$$

Girsanov's kernel:

$$L_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\theta_s^2 ds\right)$$

1. W_t: Brownian motion under $\mathcal{F}_t$
2. θ_t: $\mathcal{F}_T$ adapted process satisfying Novikov's condition
3. L_t: Girasnov's kernel
4. $\mathbb{Q}[A] := \int_A L_t d\mathbb{P}$
5. $\tilde{W}_t = W_t + \int_0^t \theta_s ds$: new Brownian motion under $\mathbb{Q}$

Results

1. Change of measure

$$\mathbb{E}^{\mathbb{Q}}[h_t] = \mathbb{E}^{\mathbb{P}}[h_t L_t], \forall h_t\ \mathcal{F}_t\text{measurable}$$

Conditional change of measure $t > s$

$$\mathbb{E}^{\mathbb{Q}}[h_t|\mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}[h_t \frac{L_t}{L_s}|\mathcal{F}_s] = \frac{1}{L_s}\mathbb{E}^{\mathbb{P}}[h_t L_t|\mathcal{F}_s]$$

2. Equivalence of $\mathbb{Q}$ and $\mathbb{P}$
3. Example:
$B_t = \sigma W_t + \mu t$, let $\theta = \mu/\sigma$,
4. Exponential Martingale and its power
θ: adapted process
$R(t) = \exp(v_t)$ where

$$v_t = \int_0^t \theta_s dW_s + \int_0^t \theta_s^2 ds$$

Then

$$R_t^m = \exp(m v_t) = \widetilde{R}_t \cdot \exp\left(\frac{1}{2}\int_0^t (m^2 - m)\theta_s ds\right)$$

where

$$\widetilde{R}_t = \exp\left(\int_0^t m\theta_s dW_s + \int_0^t (m\theta)_s^2 ds\right)$$

$$\mathbb{E}[R_t^m]$$
$$= \mathbb{E}\left[\widetilde{R}_t \cdot \exp\left(\frac{1}{2}\int_0^t (m^2 - m)\theta_s ds\right)\right]$$
$$\le \exp\left(\frac{m^2 - m}{2} t \|\theta\|_\infty\right)\mathbb{E}[\widetilde{R}_0]$$

Existence and uniqueness of solutions

$$\mathrm{d}Z_t = b(t, Z_t)\mathrm{d}t + \sigma(t, Z_t)\mathrm{d}W_t$$

b: drift, output is $\mathbb{R}$
σ: diffusion/volatility, matrix

Lipchitz continuity
$f : \mathbb{R}^d \to \mathbb{R}$ is Lipchitz continuous if there's a constant K such that for any $x, y \in \mathbb{R}^d$,

$$|f(x) - f(y)| \le K\|x - y\|_2$$

Linear growth condition

$$|b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \le K|x - y\|_2$$

$$|b(t,x)|^2 + \|\sigma(t,x)\|_2^2 \le K(1 + \|x\|^2)$$

Here $\|\sigma\|_2^2 = \sum_{ij}\sigma_{ij}^2$

THEOREM
If σ and b satisfy
1. Lipchitz continuity
2. Linear growth condition
Then exists a unique process X_t adapted to filtration of W that solves the equation

Orstein-Uhlenbeck process

Filtering

- Filtering framework
  - Signal process
  - Observation process
  - Filtering problem
- Tools
  - Girsanov's theorem
  - Novikov's condition
  - Zakai equation
  - Kushner-Stranovich equation
  - Kallianpur-striebel's formula
  - Kalman-Bucy
- Innovation process

Martingale

Classification

Semi-martingale
$X_t = M_t + V_t$
M_t: Martingale
V_t: finite variation process

Martingale:
1. Adapted
2. Integrable, i.e. finite L^1 expectation.
3. $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$

- Sub $\sim$: $\mathbb{E} \nearrow$
- Super $\sim$: $\mathbb{E} \searrow$

Local martingale
$\exists$ sequence of non-decreasing stopping times T_n, such that $M_t^{T_n}$ are all (continuous) martingales
Informally, this means that if we continue observing, it always become a martingale at some point.

- Mar $\Longrightarrow$ local
converse not true
- NONEXAMNINABLE
1. Non-negative local mar is super mar

Continuous process

A process is continuous(right/left) if its paths are continuous almost surely.
i.e. the map $t \mapsto X_t(\omega)$ satisfy

$$\mathbb{P}(\omega : X_t(\omega) : \mathbb{R}_+ \to \mathbb{R} \text{ is cont.}) = 1$$

Cadlag or RCLL: right continuous with left limit

Properties

1. $Y_t = g(X_t)$, g convex, X_t martingale.
If Y_t integrable, then Y_t is sub-martingale.
2. Doob's Maximal ineq
3. A sub martingale of constant expectation is martingale

Quadratic Variation

Doob-Meyer:
a. Sub-martingale=Martingale+Increasing process
b. M_t martingale $\Longrightarrow$ M_t^2 sub martingale
c. $M_t^2 = \widetilde{M}_t + [M]_t$
d. $\widetilde{M}_t$: martingale, $[M]_t$: increasing process, called Q.V.

Cross variation process

$$[X,Y]_t = \frac{1}{4}([X+Y]_t - [X-Y]_t)$$

- Proof
$(X+Y)^2 - [X+Y], (X-Y)^2 - [X-Y]$ are martinales
- $XY - [X,Y]_t$ is a martingale
- $[\cdot,\cdot]_t$ defines "an inner product"
  - Orthogonality
  $[X,Y]_t = 0\ a.e.$
  $\Longleftrightarrow$ strongly orthogonal
  $\Longleftrightarrow$ both are martingale
  - Linearity
  1. $[aX + bY, Z] = a[X,Z] + b[Y,Z]$
  2. $[X,Y] = [Y,X]$
  - Cauchy-Schwartz
  $|[X,Y]|^2 \le [X][Y]$

p-variation

- Notes on Total variation
- Brownian motion has finite 2 variation,
$0\ x < 2 - variation, \infty\ x > 2 - varaition$

Brownian Motion

- Markov+Gaussian+Martingale
- Standard BM
1. Continuous a.s., $W_0 = 0$
2. Independent increments
3. $W_t - W_s \sim \mathcal{N}(0, t-s)$
- $\boxed{[W]_t = t}$
- Levy's characterisation
  - M is a Brownan motion if:
  1. M is an adapted martingale
  2. $M_0 = 0$
  3. $[M]_t = tI$
- Results
1. $\int_0^t B_s dB_s = \frac{B_t^2 - t}{2} \Longrightarrow [W]_t = t$
2. Martingale representation theorem
Any Square integrable martingale M_t can be uniquely decomposed w.r.t. $\{\phi^j\}_{j=1}^d$

$$M_t = M_0 + \sum_{j=1}^d \int_0^t \phi_t^j dW_t^j$$

Poisson Process with parameter λ

$$N_t - N_s \sim Poi(\lambda t - \lambda s)$$

$$Pr_\lambda(X = k) = \frac{\lambda^k r^{-\lambda}}{k!}$$

Compensated Poisson Process M_t
$M_t = N_t - \lambda t$
Is a martingale!