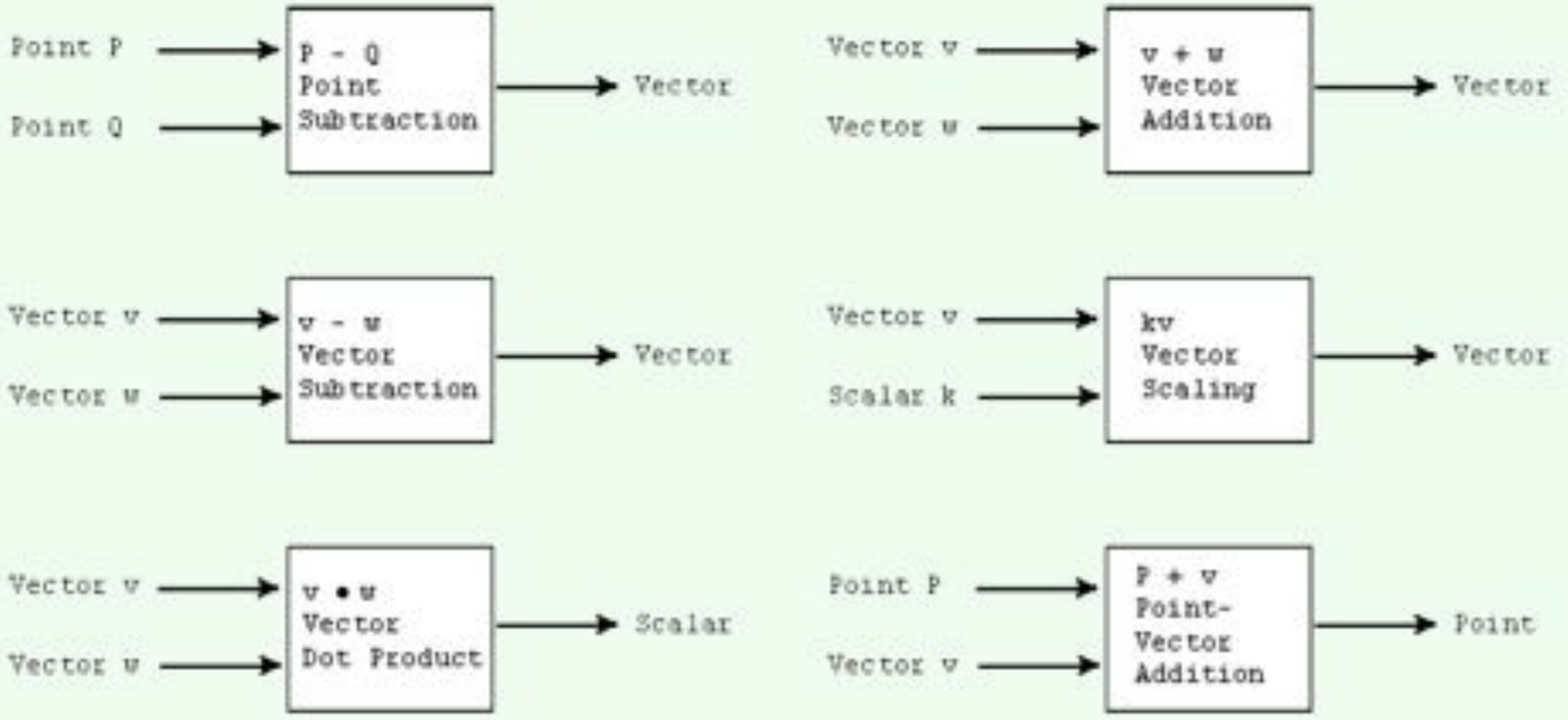
Computer Graphics 3:
Maths vector operations cont’d

Revision of vector operations

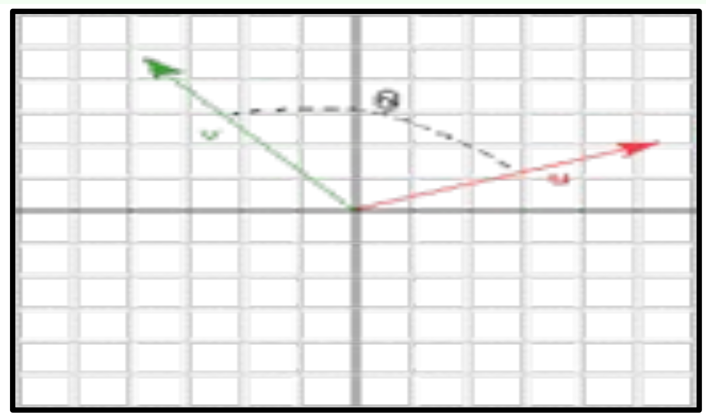


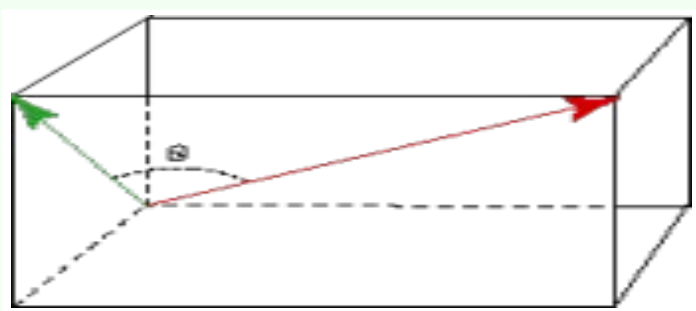
The Dot Product

For geometrical vectors in two or three dimensional space, the **dot product** of two vectors u and v is:

$$u \cdot v = |u|\,|v| \cos \theta$$

θ is the angle between the two vectors.





The dot product is indicated by the dot, " · " between the two vectors. Don't write two vectors next to each other like this: **uv** when you want the dot product. Always put a dot between them: **u · v** .

In 2D the two vectors lie in a plane (of course) and the angle between them is easy to visualize.

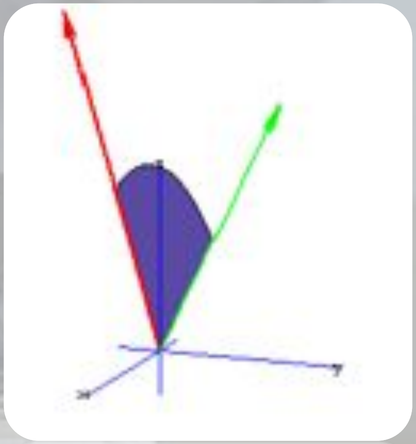
In 3D two vectors also lie in a plane embedded within the 3D space, except when the two vectors are co-linear (when they both point in the same direction).

When two vectors are co-linear, the angle between them is zero and so:

$$u \cdot v = |u|\,|v| \cos 0 = |u|\,|v|\, 1 = |u|\,|v|$$

Vector dot product

- Meaning in Euclidean geometry
  1. If $A(x_1, y_{1, \ldots})$, $B(x_2, y_{2, \ldots})$ are vectors
  2. theta is the angle, in radians, between A and B
  3. Dot Product (A, B) = A . B = = |A|*|B|*cos(theta)



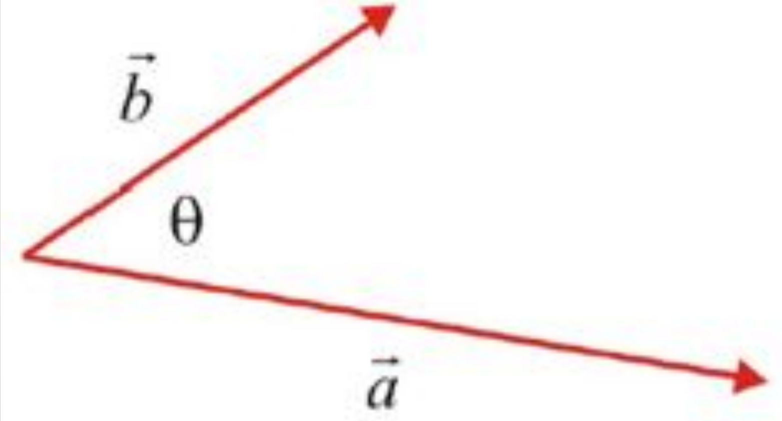
- Applies to all dimensions (1D, 2D, 3D, 4D, … nD)

7.3 Dot Product of two Geometric Vectors

A Definition

The *Dot Product* of two *geometric* vectors $\vec{a}$ and $\vec{b}$ with an angle $\theta = \angle(\vec{a}, \vec{b})$ between them (when positioned tail to tail) is a *scalar* defined by:

$$\vec{a} \cdot \vec{b} = \| \vec{a} \| \| \vec{b} \| \cos\theta$$



Note. By convention $0° \le \theta \le 180°$.

7.3 Dot Product of two Geometric Vectors

Ex 1. If $\|\vec{u}\| = 4$, $\|\vec{v}\| = 6$, and $\theta = \angle(\vec{u}, \vec{v}) = 120^\circ$, find $\vec{u} \cdot \vec{v}$.

7.3 Dot Product of two Geometric Vectors

Ex 1. If $||\vec{u}||=4$, $||\vec{v}||=6$, and $\theta=\angle(\vec{u},\vec{v})=120^\circ$, find $\vec{u}\cdot\vec{v}$.

$$\vec{u}\cdot\vec{v}=||\vec{u}||\,||\vec{v}||\cos\theta=(4)(6)\cos 120^\circ=-12$$

$$\therefore\ \vec{u}\cdot\vec{v}=-12$$

7.3 Dot Product of two Geometric Vectors

Ex 2. Find the angle between two unit vectors with a dot product equal to $1/\sqrt{2}$.

7.3 Dot Product of two Geometric Vectors

Ex 2. Find the angle between two unit vectors with a dot product equal to $1/\sqrt{2}$.

$$||\vec{u}||=1\,,\ ||\vec{v}||=1\,,\ \vec{u}\cdot\vec{v}=1/\sqrt{2}$$

$$\vec{u}\cdot\vec{v}=||\vec{u}||\,||\vec{v}||\cos\theta \Rightarrow \cos\theta=\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}=\frac{1/\sqrt{2}}{(1)(1)}=\frac{1}{\sqrt{2}}$$

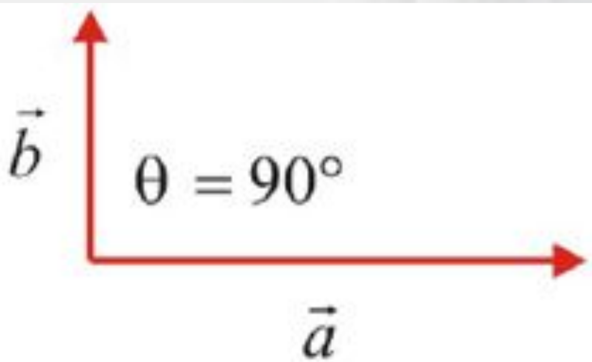
$$\therefore \theta=\cos^{-1}\frac{1}{\sqrt{2}}=45^{\circ}$$

7.3 Dot Product of two Geometric Vectors

B Properties of Dot Product

1. $\vec{a} \cdot \vec{b}$ is a scalar (a real number).

2. If $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$ (because $\theta = 90^\circ$ and $\cos 90^\circ = 0$).

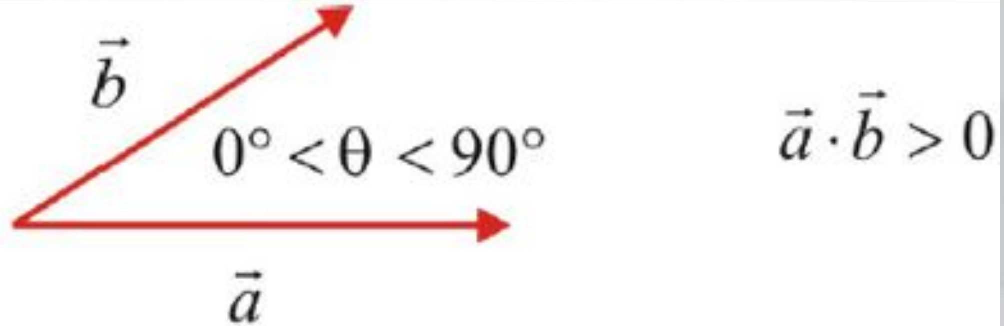


$$\vec{a} \cdot \vec{b} = 0$$

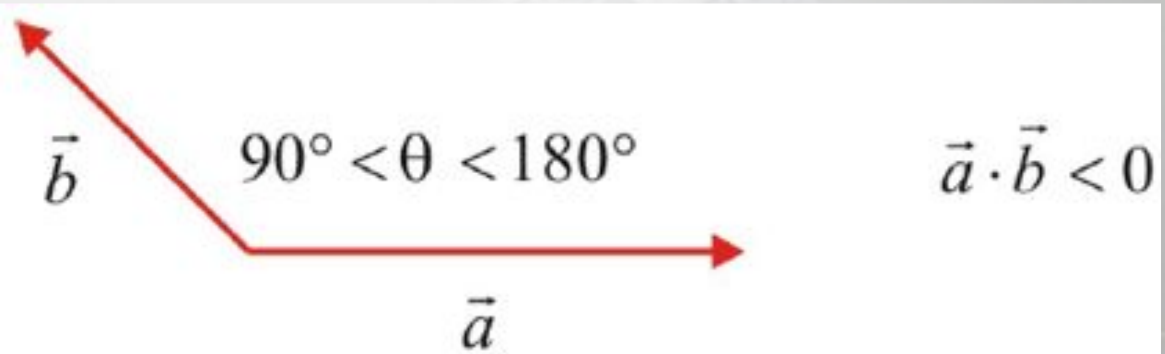
3. If $\vec{a} \cdot \vec{b} = 0$ then $\|\vec{a}\| = 0$ or $\|\vec{b}\| = 0$ or $\vec{a} \perp \vec{b}$.

7.3 Dot Product of two Geometric Vectors

4. If $0° < \theta < 90°$ then $\cos\theta > 0$ and $\vec{a} \cdot \vec{b} > 0$.



5. If $90° < \theta < 180°$ then $\cos\theta < 0$ and $\vec{a} \cdot \vec{b} < 0$.



7.3 Dot Product of two Geometric Vectors

6. If $\vec{a} \uparrow\uparrow \vec{b}$ then $\theta = 0°$, $\cos\theta = 1$, and $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\|$

7. If $\vec{a} \uparrow\downarrow \vec{b}$ then $\theta = 180°$, $\cos\theta = -1$, and $\vec{a} \cdot \vec{b} = -\|\vec{a}\| \|\vec{b}\|$

8. $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

9. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutative property)

10. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ (distributative property)

11. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$

12. $\vec{a} \cdot \vec{0} = 0$

Note. $\vec{0}$ is the zero vector and 0 is the number zero.

7.3 Dot Product of two Geometric Vectors

Ex 3. Use the dot product to prove each relation:

a) $||\vec{a}+\vec{b}||^2 = ||\vec{a}||^2 + ||\vec{b}||^2 + 2\vec{a}\cdot\vec{b}$

7.3 Dot Product of two Geometric Vectors

Ex 3. Use the dot product to prove each relation:

a) $\|\vec{a}+\vec{b}\|^2=\|\vec{a}\|^2+\|\vec{b}\|^2+2\vec{a}\cdot\vec{b}$

$$LS=\|\vec{a}+\vec{b}\|^2=(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$=\|\vec{a}\|^2+\|\vec{b}\|^2+2\vec{a}\cdot\vec{b}=RS$$

7.3 Dot Product of two Geometric Vectors

Ex 3. Use the dot product to prove each relation:

b) $\|\vec{a}-\vec{b}\|^2=\|\vec{a}\|^2+\|\vec{b}\|^2-2\vec{a}\cdot\vec{b}$

7.3 Dot Product of two Geometric Vectors

Ex 3. Use the dot product to prove each relation:

b) $\|\vec{a}-\vec{b}\|^2=\|\vec{a}\|^2+\|\vec{b}\|^2-2\vec{a}\cdot\vec{b}$

$$LS=\|\vec{a}-\vec{b}\|^2=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$=\|\vec{a}\|^2+\|\vec{b}\|^2-2\vec{a}\cdot\vec{b}=RS$$

7.3 Dot Product of two Geometric Vectors

Ex 3. Use the dot product to prove each relation:

c) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \|\vec{a}\|^2 - \|\vec{b}\|^2$

7.3 Dot Product of two Geometric Vectors

Ex 3. Use the dot product to prove each relation:

c) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \|\vec{a}\|^2 - \|\vec{b}\|^2$

$$LS = (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b}$$

$$= \|\vec{a}\|^2 - \|\vec{b}\|^2 = RS$$

Commutative

The dot product is **commutative** The order of operands does not make any difference.

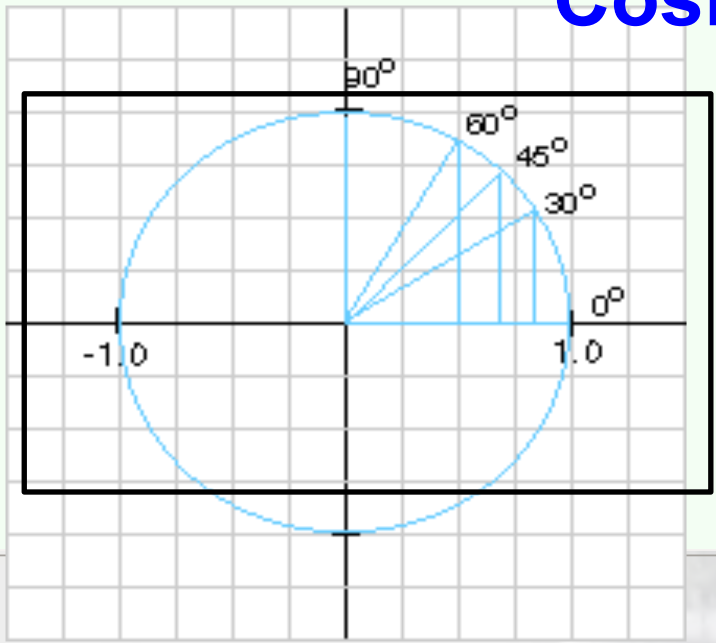
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$ Another

property is:

$$\mathbf{0} \cdot \mathbf{0} = 0.$$

This means that the dot product of the *zero vector* with itself results in the *scalar value* of zero. There are two different kinds of zero in the equation. Remember that operands of the dot product are two vectors, and the output is a scalar (a real number).

Cosine of 90°



You may be somewhat fuzzy about how the cosine function behaves. Rather than memorize abstract stuff, I prefer to visualize the unit circle with its radius projected onto the x-axis.

From the picture, the cosine of 0° = 1.0, the cosine of 30° = 0.866, the cosine of 45° = 0.707, the cosine of 60° = 0.500, the cosine of 90° = 0.0.

Recall that

$u \cdot v = |u||v| \cos$

If u and v are orthogonal, then $u \cdot v = |u||v| \cos 90° = |u||v| 0.0 = 0.0$

Dot Product of Orthogonal Vectors

This fact is of fundamental importance. It works for vectors of all dimensions: The dot product of orthogonal vectors is zero.

"Orthogonal" means "oriented at 90° to each other". To keep things consistent, the zero vector is regarded as orthogonal to all other vectors since $\mathbf{0} \cdot \mathbf{v} = 0.0$ for all vectors $\mathbf{v}$.

Pure Orthogonality Detector

When two vectors are orthogonal (to each other) then their dot product is zero, regardless of their lengths. The dot product "detects" orthogonality no matter what the lengths.

Now look at the dot product of a vector with itself: $v \cdot v \quad =$

$$|v|\,|v| \cos 0^\circ \quad = \quad |v|\,|v|\, 1.0 \quad = \quad |v|^2$$

The dot product of a vector with itself yields the square of its length, or:

$$|\, v \,| \;=\; \sqrt{(v \cdot v)}$$

Used in this fashion, the dot product is a *pure length detector*. Since the two properties of a vector are length and orientation, you might suspect that the dot product is useful.

Vector Cross Product

Definition, Features, Application

Vector cross product

- Cross product
  i. Operates on vectors with up to 3 dimensions
  ii. Forms a determinant of a matrix of the vectors
  iii. Result – depends on the dimension
    - In 2D – a scalar number (1D)
    - In 3D – a vector (3D)
    - Not defined for 1D and dimensions higher than 3

The vector cross product takes two vector operands to produce a vector result. The result, like all geometric vectors, has two properties: length and orientation.

2D Vector cross product

- 2D Cross product
  1. Take the vectors $U(x_1, y_1)$ and $V(x_2, y_2)$
  2. Multiply their coordinates across and subtract:
     - $U(x_1, y_1) \ x \ V(x_2, y_2) = (x_1 * y_2) - (x_2 * y_1)$
  3. Result
     - A scalar number

2D Vector cross product

- Scalar meaning in Euclidean geometry
  i. If $U(x_1, y_1)$ and $V(x_2, y_2)$ are 2D vectors
  ii. theta is the angle between *U* and *V*
  iii. Cross Product (*U*, *V*) = *U* x *V* =
  = |*U*| * |*V*| * sin(theta)

    - |*U*| and |*V*| denote the length of *U* and *V*
    - Applies to 2D and 3D

2D Vector cross product

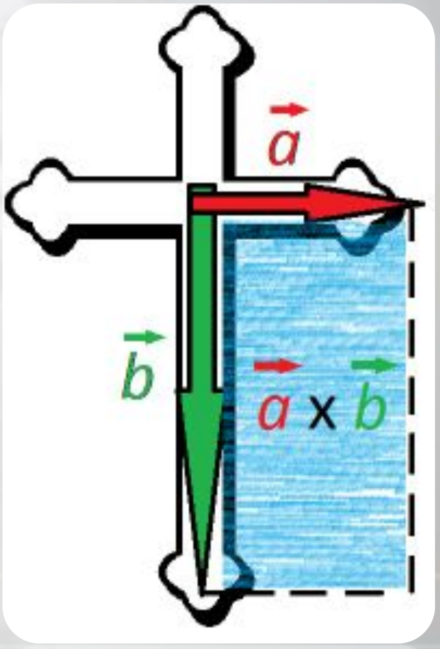
- Scalar meaning in Euclidean geometry (2)
  i. For every two 2D vectors *U* and *V*
     - *U* x *V* = the oriented face of the parallelogram, defined by *U* and *V*

  ii. For every three 2D points *A, B* and *C*
     - If *U* x *V* = 0, then *A*, *B* and *C* are collinear
     - If *U* x *V* > 0, then *A*, *B* and *C* constitute a 'left turn'
     - If *U* x *V* < 0, then *A*, *B* and *C* constitute a 'right turn'

2D Vector cross product

- Applications
  i. Graham scan (2D convex hull)
  ii. Easy polygon area computation
     - Cross product divided by two equals oriented (signed) triangle area
  iii. 2D orientation
     - ‘left’ and ‘right’ turns

2D Cross Product Computation



3D Vector cross product

- If u and v are vectors in three dimensional space (only), then u × v is a three dimensional vector,
- where:
- Length:
- |u × v | = | u | | v | sin , where is the angle between u and v.
- Orientation:
- u × v is perpendicular to both u and v. The choice (out of two) orientations
- perpendicular to u and v is made by the right hand rule.

3D Vector cross product

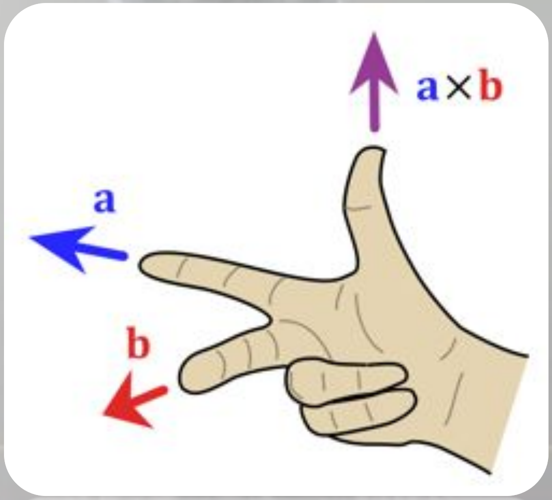
- 3D Cross product
  i. Take two 3D vectors *U*(x_1, y_1, z_1) and *V*(x_2, y_2, z_2)
  ii. Calculate the following 3 coordinates
     - $x_3 = y_1 * z_2 – y_2 * z_1$
     - $y_3 = z_1 * x_2 – z_2 * x_1$
     - $z_3 = x_1 * y_2 – x_2 * y_1$
  iii. Result
     - A 3D vector with coordinates (x_3, y_3, z_3)

3D Vector cross product

- Meaning in Euclidean geometry
  i. The magnitude
     - Always positive (length of the vector)
     - Has the unsigned properties of the 2D dot product
  ii. The vector
     - Perpendicular to the initial vectors *U* and *V*
       - Normal to the plane defined by *U* and *V*
     - Direction determined by the right-hand rule

3D Vector cross product

- The right-hand rule
  1. Index finger points in direction of first vector (*a*)
  2. Middle finger points in direction of second vector (*b*)
  3. Thumb points up in direction of the result of *a* x *b*



3D Vector cross product

- Unpredictable results occur with
  1. Cross product of two collinear vectors
  2. Cross product with a zero-vector
- Applications
  1. Calculating normals to surfaces
  2. Calculating torque (physics)