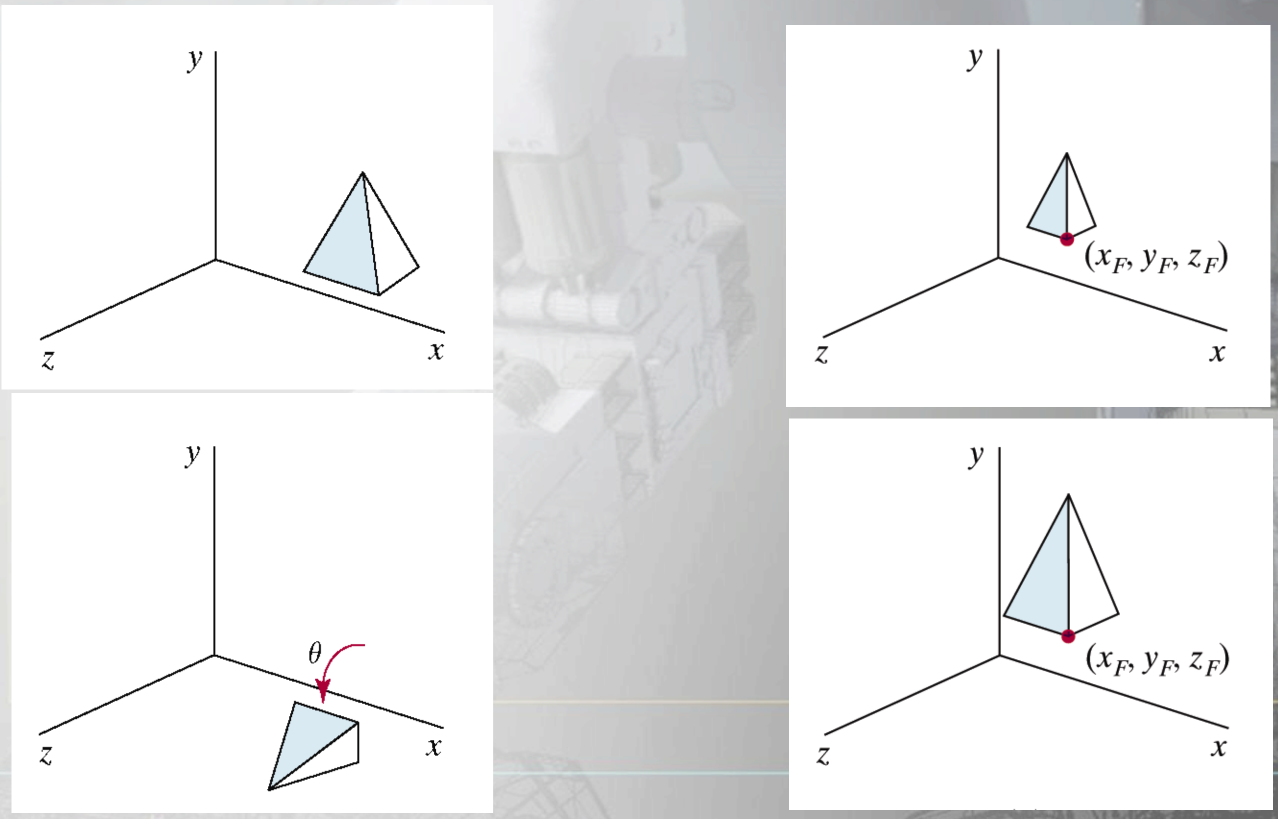
Computer Graphics 3: 2D Transformations

Contents

- In today's lecture we'll cover the following:
    i. Why transformations
    ii. Transformations
        - Translation
        - Scaling
        - Rotation
    iii. Homogeneous coordinates
    iv. Matrix multiplications
    v. Combining transformations

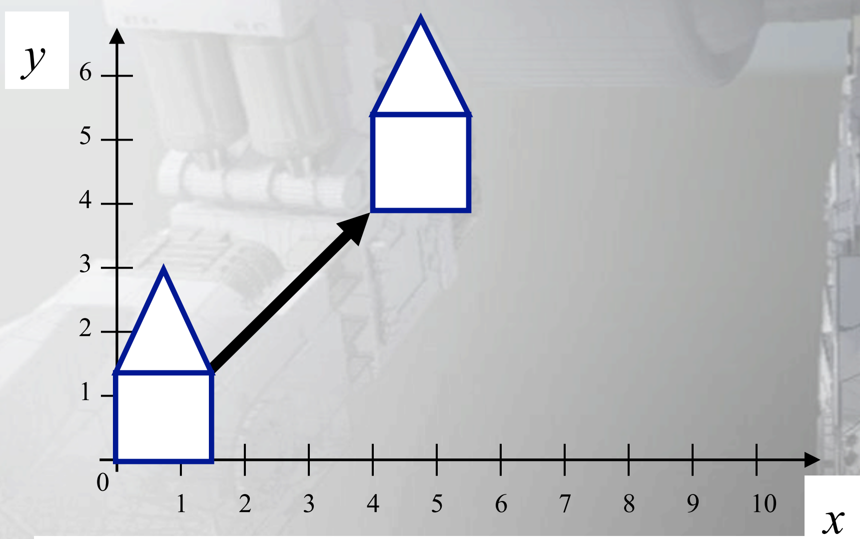
Why Transformations?

- In graphics, once we have an object described, transformations are used to move that object, scale it and rotate it



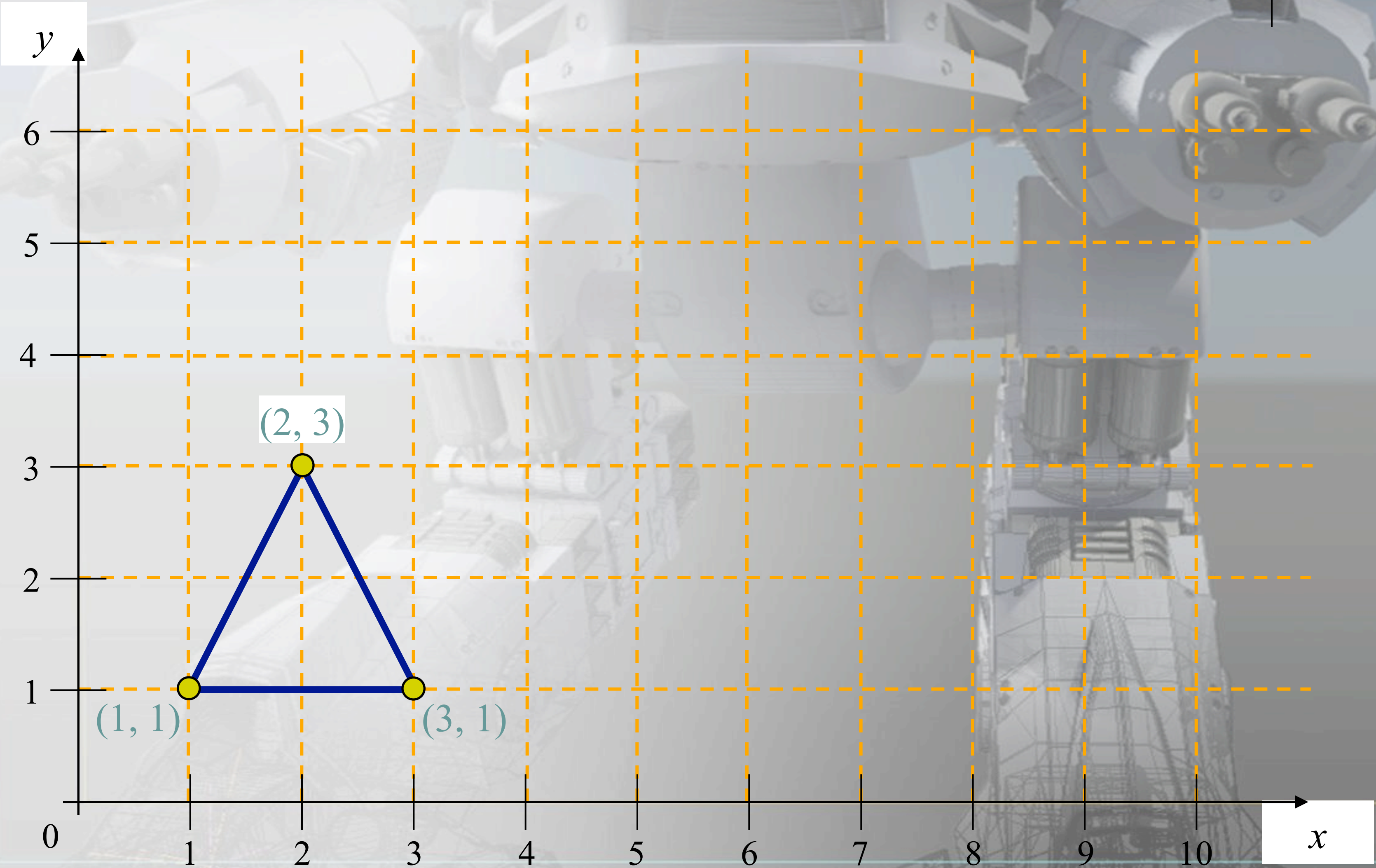
Translation

- Simply moves an object from one position to another
- $$x_{new} = x_{old} + dx \qquad\qquad y_{new} = y_{old} + dy$$



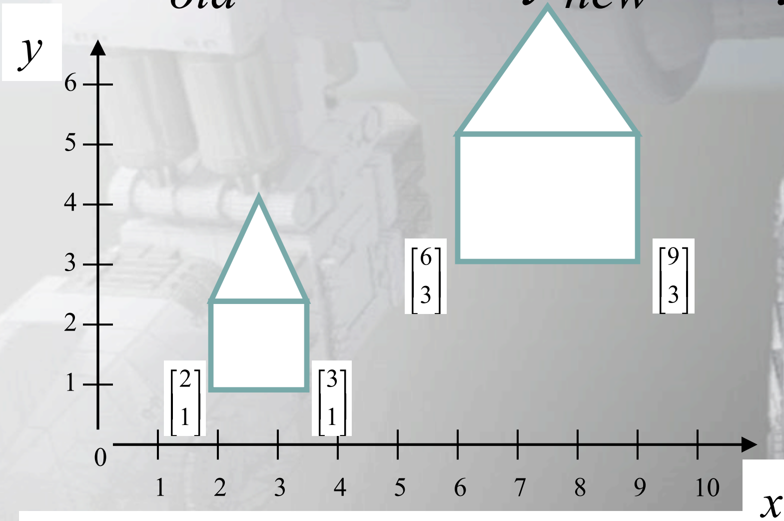
Note: House shifts position relative to origin

Translation Example



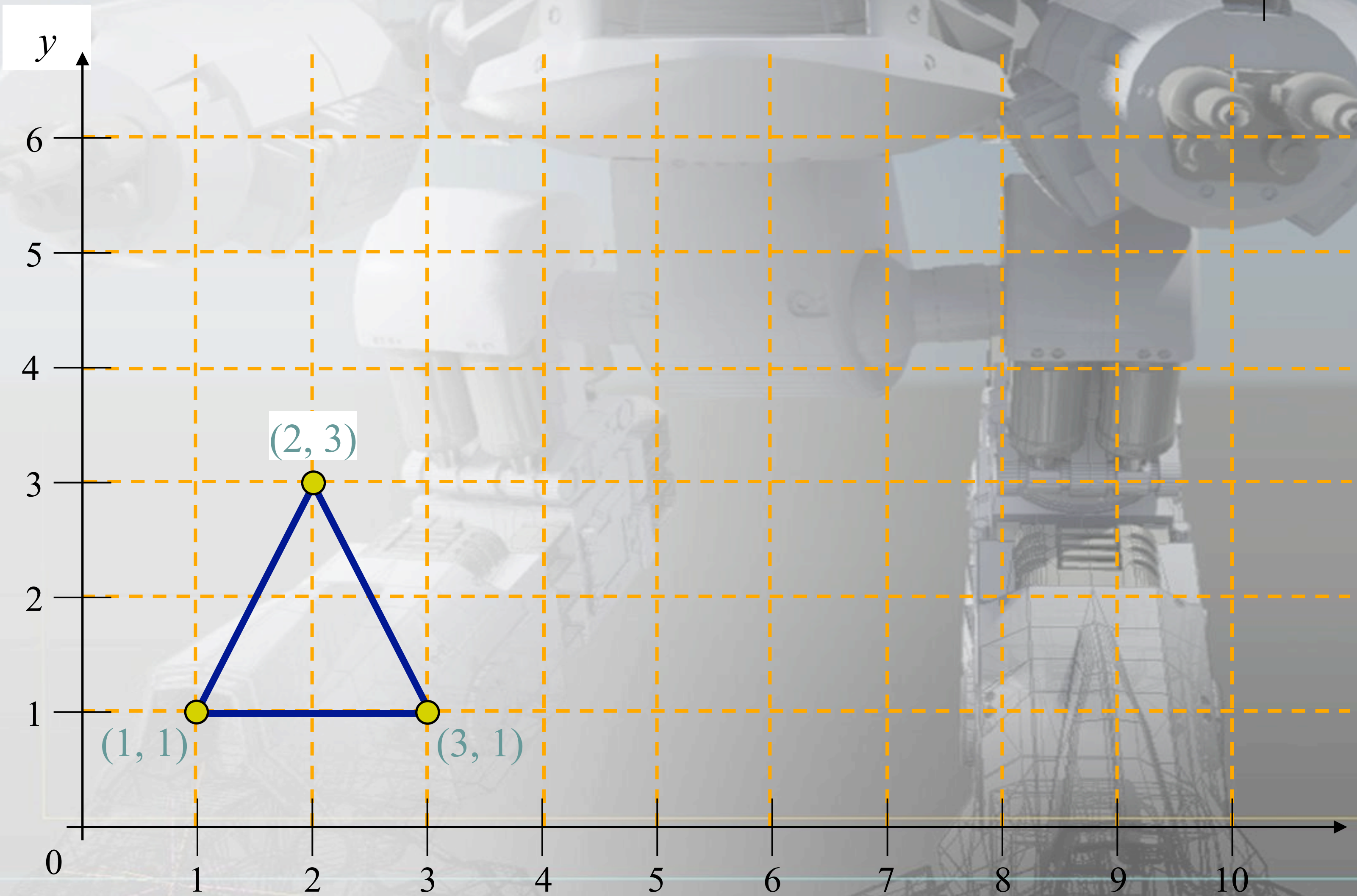
Scaling

- Scalar multiplies all coordinates
- **WATCH OUT:** Objects grow and move!
- $x_{new} = Sx \times x_{old}$ $y_{new} = Sy \times y_{old}$



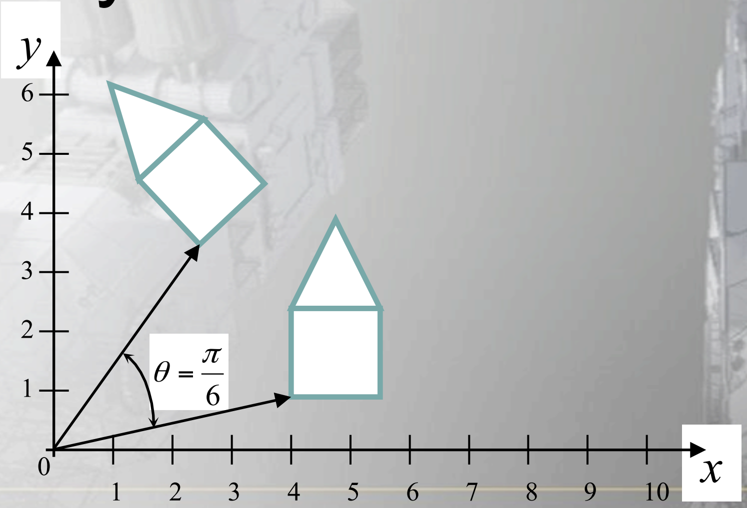
Note: House shifts position relative to origin

Scaling Example

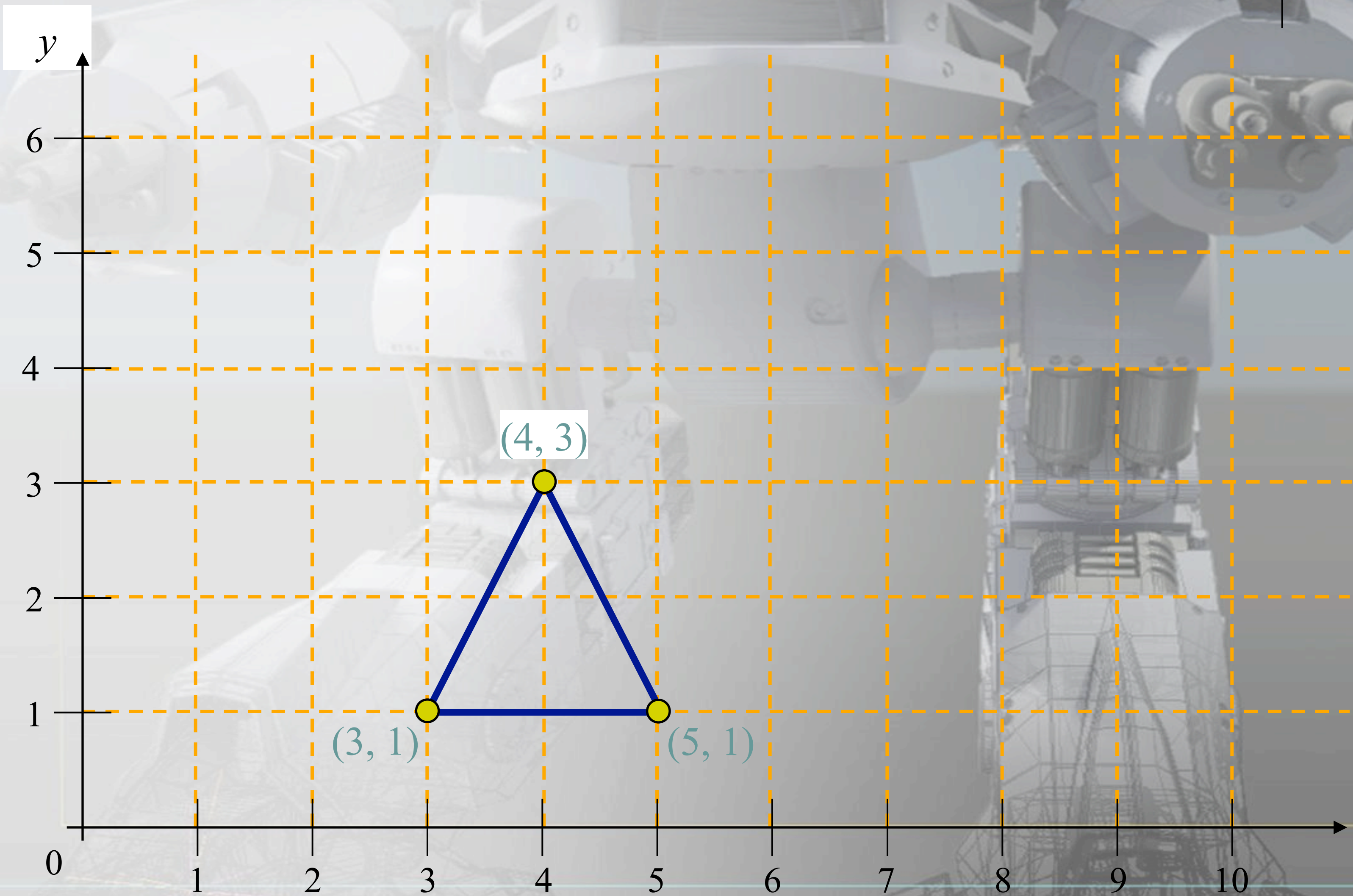


Rotation

- Rotates all coordinates by a specified angle
- $$x_{new} = x_{old} \times \cos\theta - y_{old} \times \sin\theta$$
- $$y_{new} = x_{old} \times \sin\theta + y_{old} \times \cos\theta$$
- Points are always rotated about the origin



Rotation Example



Homogeneous Coordinates

- A point (x, y) can be re-written in **homogeneous coordinates** as (x_h, y_h, h)
- The **homogeneous parameter** h is a non-zero value such that:

$$x = \frac{x_h}{h} \qquad y = \frac{y_h}{h}$$

- We can then write any point (x, y) as (hx, hy, h)
- We can conveniently choose $h = 1$ so that (x, y) becomes $(x, y, 1)$

Why Homogeneous Coordinates?

- Mathematicians commonly use homogeneous coordinates as they allow scaling factors to be removed from equations
- We will see in a moment that all of the transformations we discussed previously can be represented as 3*3 matrices
- Using homogeneous coordinates allows us use matrix multiplication to calculate transformations – extremely efficient!

Homogeneous Translation

- The translation of a point by (dx, dy) can be written in matrix form as:

$$\begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

- Representing the point as a homogeneous column vector we perform the calculation as:

$$\begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1*x + 0*y + dx*1 \\ 0*x + 1*y + dy*1 \\ 0*x + 0*y + 1*1 \end{bmatrix} = \begin{bmatrix} x + dx \\ y + dy \\ 1 \end{bmatrix}$$

Remember Matrix Multiplication

- Recall how matrix multiplication takes place:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a*x+b*y+c*z \\ d*x+e*y+f*z \\ g*x+h*y+i*z \end{bmatrix}$$

Homogenous Coordinates

- To make operations easier, 2-D points are written as homogenous coordinate column vectors

Translation: $$\begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + dx \\ y + dy \\ 1 \end{bmatrix} : v' = T(dx, dy)v$$

Scaling: $$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x \times x \\ s_y \times y \\ 1 \end{bmatrix} : v' = S(s_x, s_y)v$$

Homogenous Coordinates (cont…)

Rotation:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta \times x - \sin\theta \times y \\ \sin\theta \times x + \cos\theta \times y \\ 1 \end{bmatrix} : v' = R(\theta)v$$

Inverse Transformations

- Transformations can easily be reversed using inverse transformations

$$T^{-1} = \begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

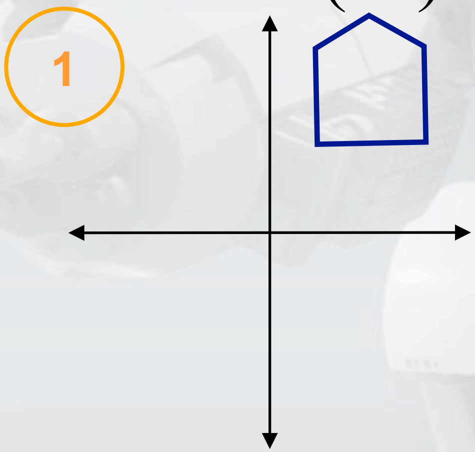
$$R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combining Transformations

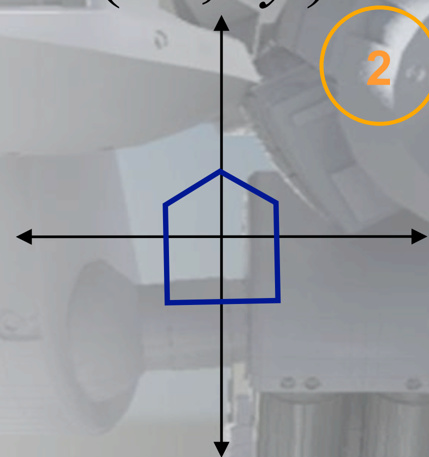
- A number of transformations can be combined into one matrix to make things easy
  i. Allowed by the fact that we use homogenous coordinates
- Imagine rotating a polygon around a point other than the origin
  i. Transform to centre point to origin
  ii. Rotate around origin
  iii. Transform back to centre point

Combining Transformations (cont…)

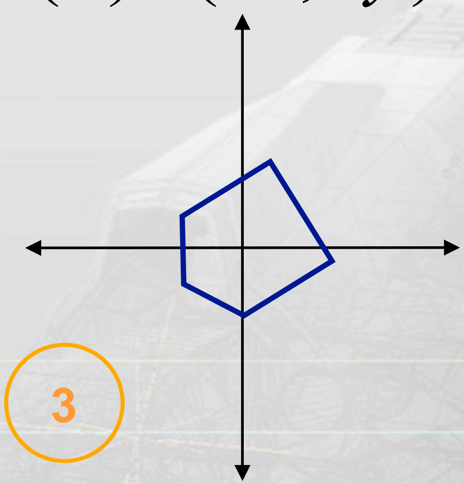
$House\ (H)$



$T(dx, dy)H$



$R(\theta)T(dx, dy)H$



$T(-dx, -dy)R(\theta)T(dx, dy)H$

Combining Transformations (cont…)

- The three transformation matrices are combined as follows

$$\begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$v' = T(-dx, -dy)R(\theta)T(dx, dy)v$$

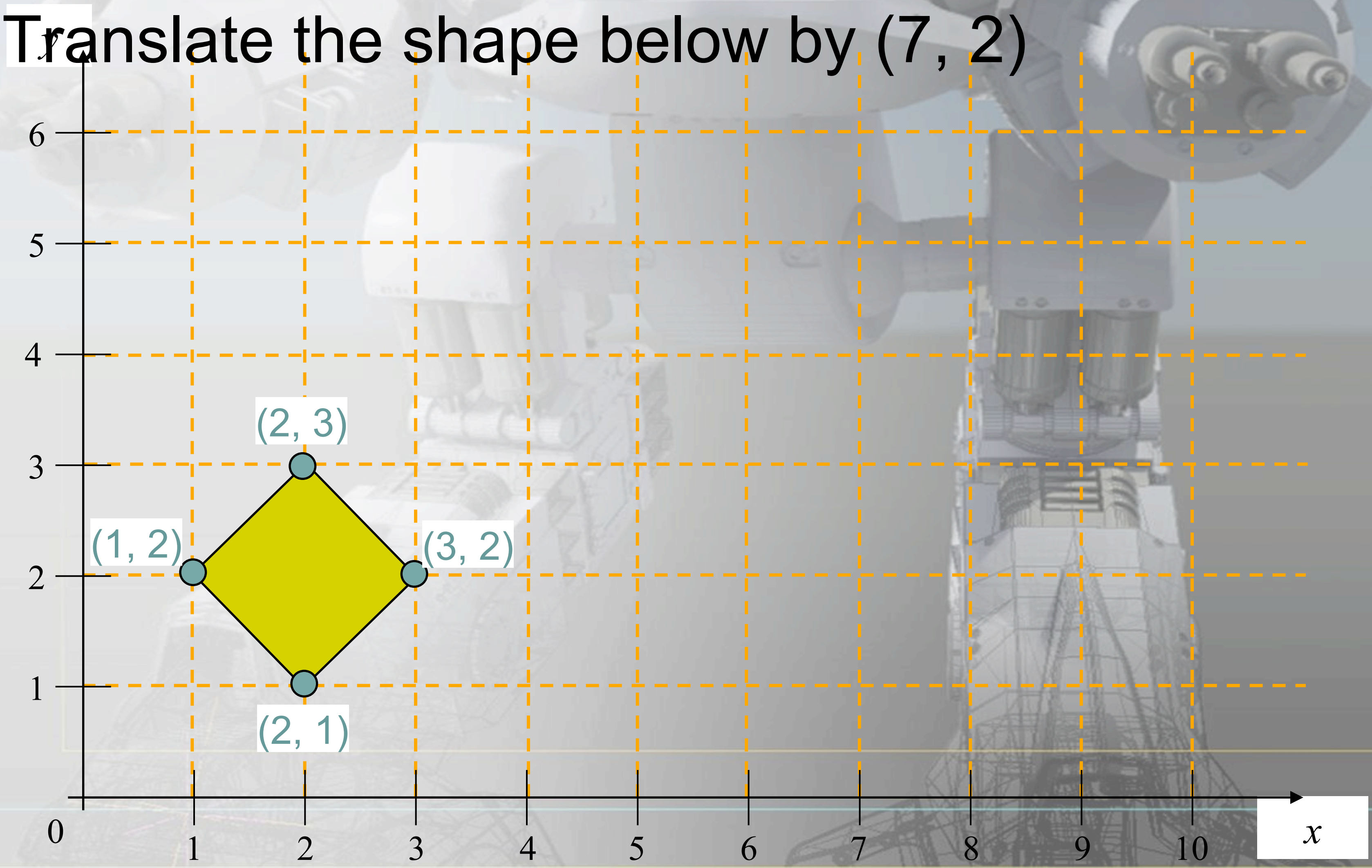
REMEMBER: Matrix multiplication is not commutative so order matters

Summary

- In this lecture we have taken a look at:
    i. 2D Transformations
        - Translation
        - Scaling
        - Rotation
    ii. Homogeneous coordinates
    iii. Matrix multiplications
    iv. Combining transformations
- Next time we'll start to look at how we take these abstract shapes etc and get them on-screen

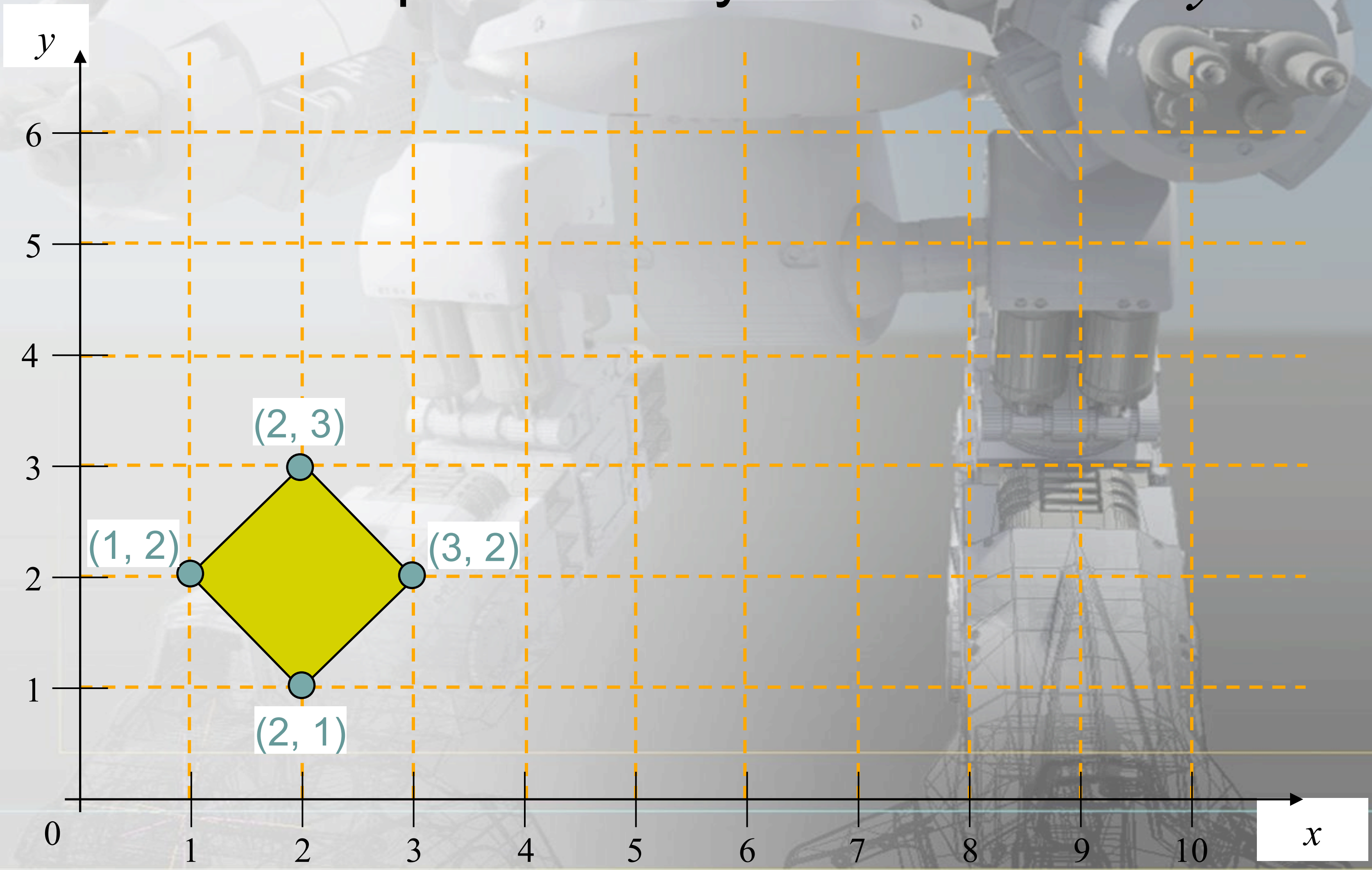
Exercises 1

- Translate the shape below by (7, 2)



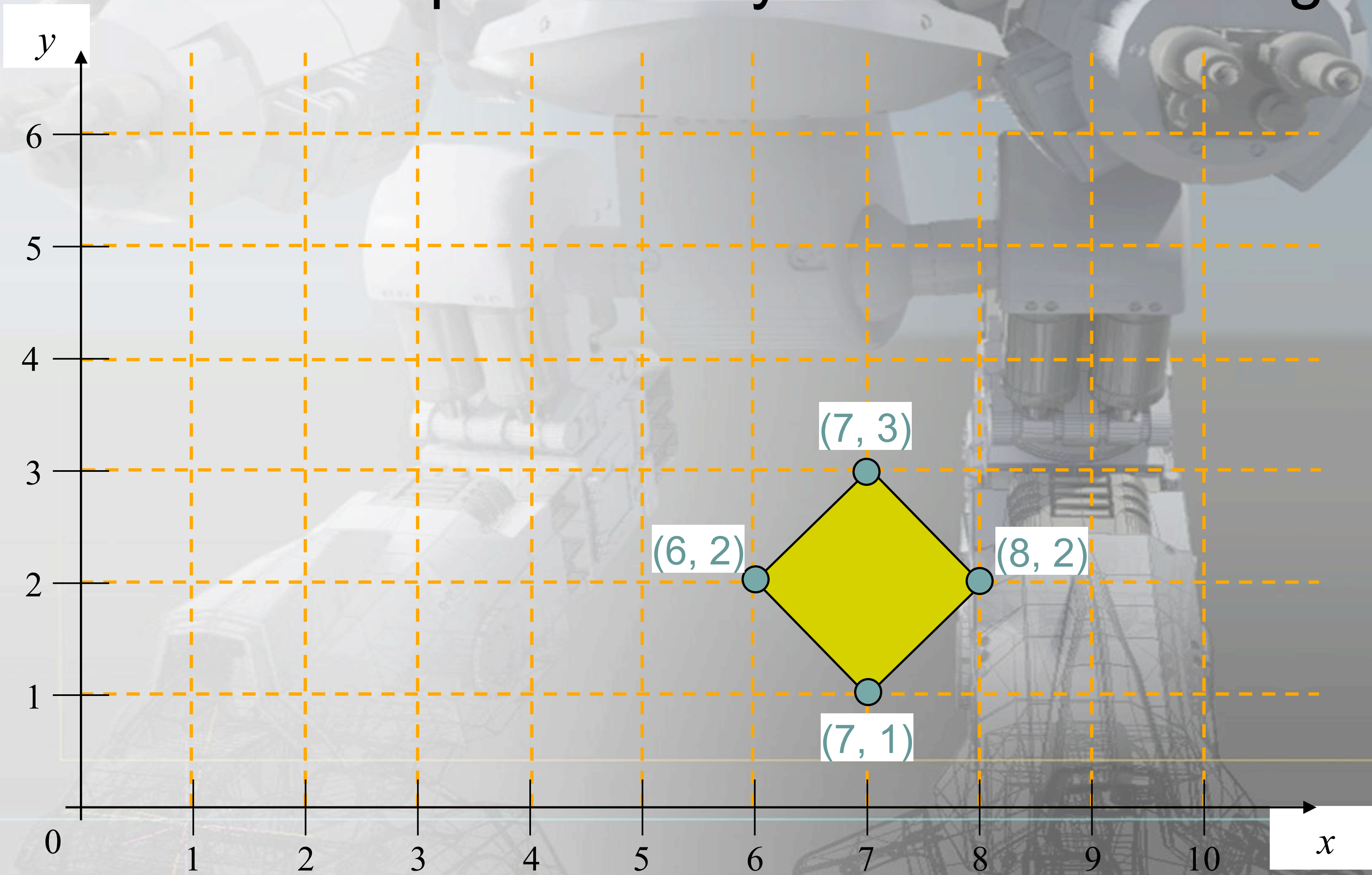
Exercises 2

- Scale the shape below by 3 in x and 2 in y



Exercises 3

- Rotate the shape below by 30° about the origin



Exercise 4

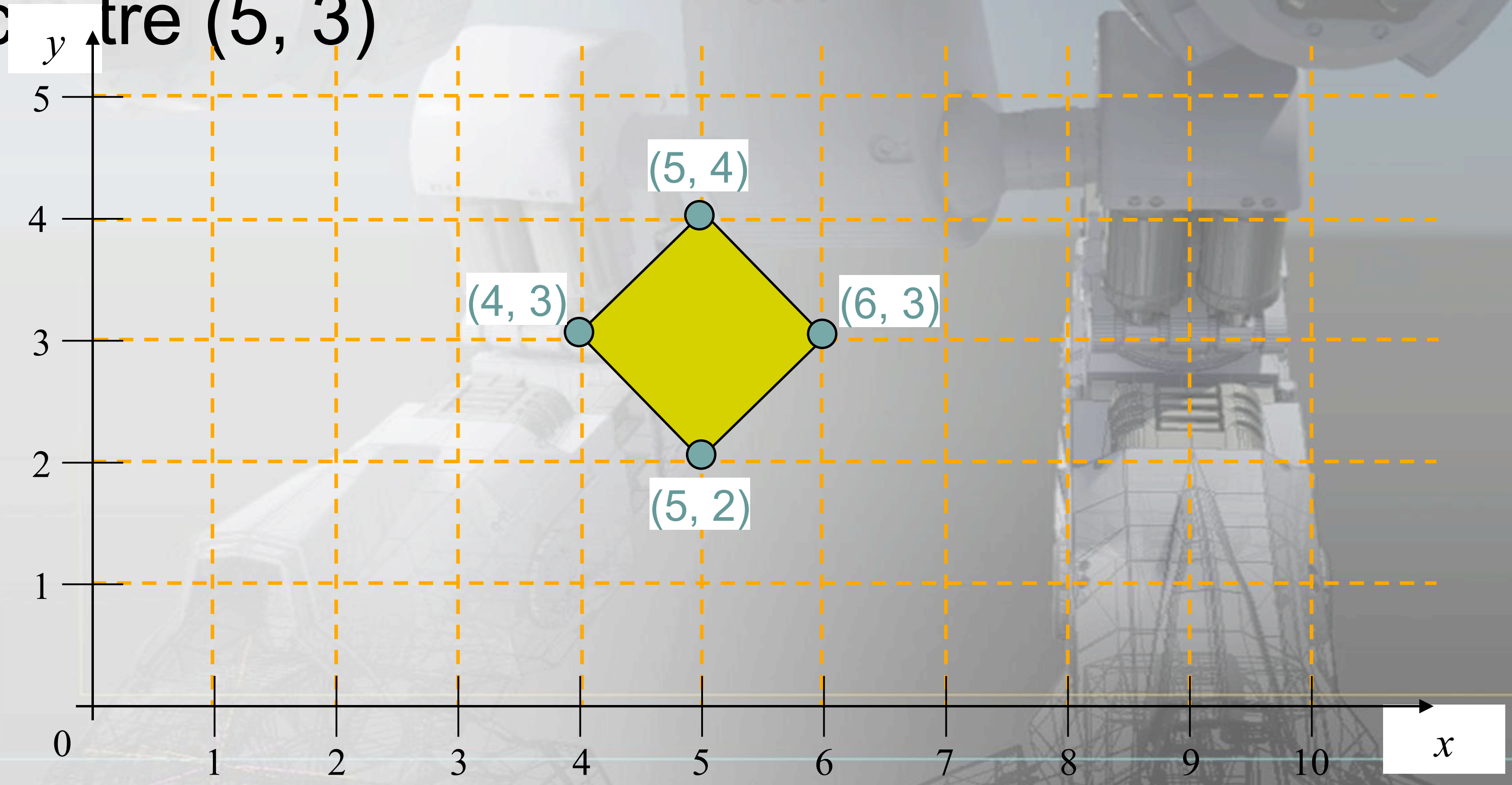
- Write out the homogeneous matrices for the previous three transformations

Translation | Scaling | Rotation

$$\begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix} \quad \begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix} \quad \begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix}$$

Exercises 5

- Using matrix multiplication calculate the rotation of the shape below by 45° about its c[illegible]tre (5, 3)



Equations

- Translation:
- $x_{new} = x_{old} + dx \qquad y_{new} = y_{old} + dy$

- Scaling:
- $x_{new} = Sx \times x_{old} \qquad y_{new} = Sy \times y_{old}$
- Rotation
- $x_{new} = x_{old} \times \cos\theta - y_{old} \times \sin\theta$
- $y_{new} = x_{old} \times \sin\theta + y_{old} \times \cos\theta$