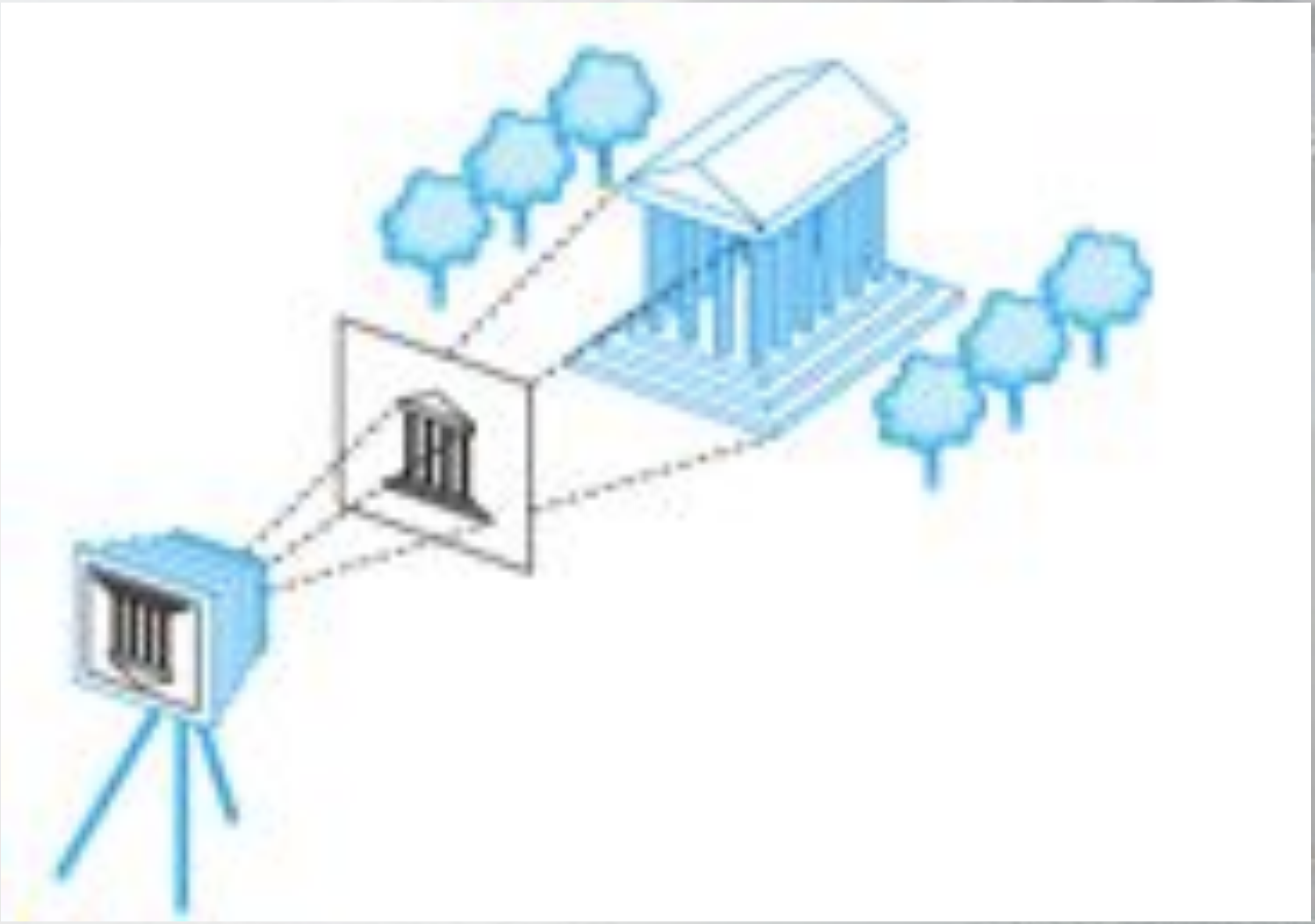
Computer Graphics 2: Maths Preliminaries

Introduction

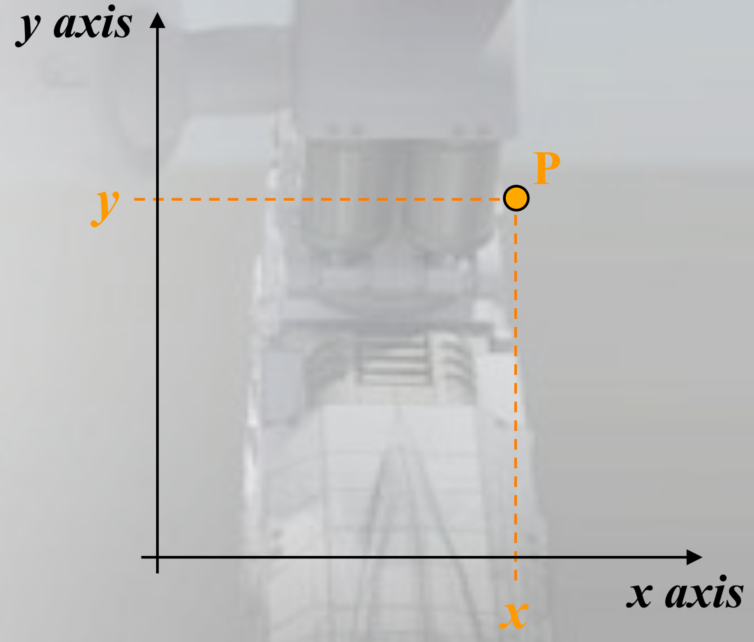
- Computer graphics is all about maths!
- None of the maths is hard, but we need to understand it well in order to be able to understand certain techniques
- Today we'll look at the following:
    i. Coordinate reference frames
    ii. Points & lines
    iii. Vectors
    iv. Matrices

The scene

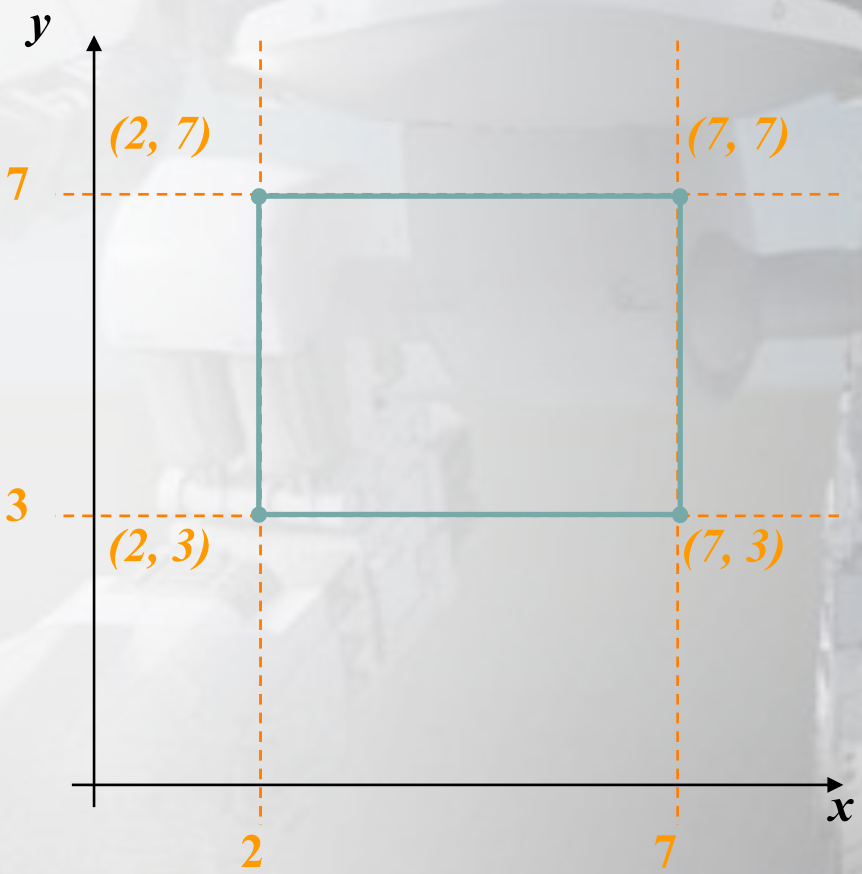


Coordinate Reference Frames – 2D

- When setting up a scene in computer graphics we define the scene using simple geometry
- For 2D scenes we use simple two dimensional Cartesian coordinates
- All objects are defined using simple coordinate pairs

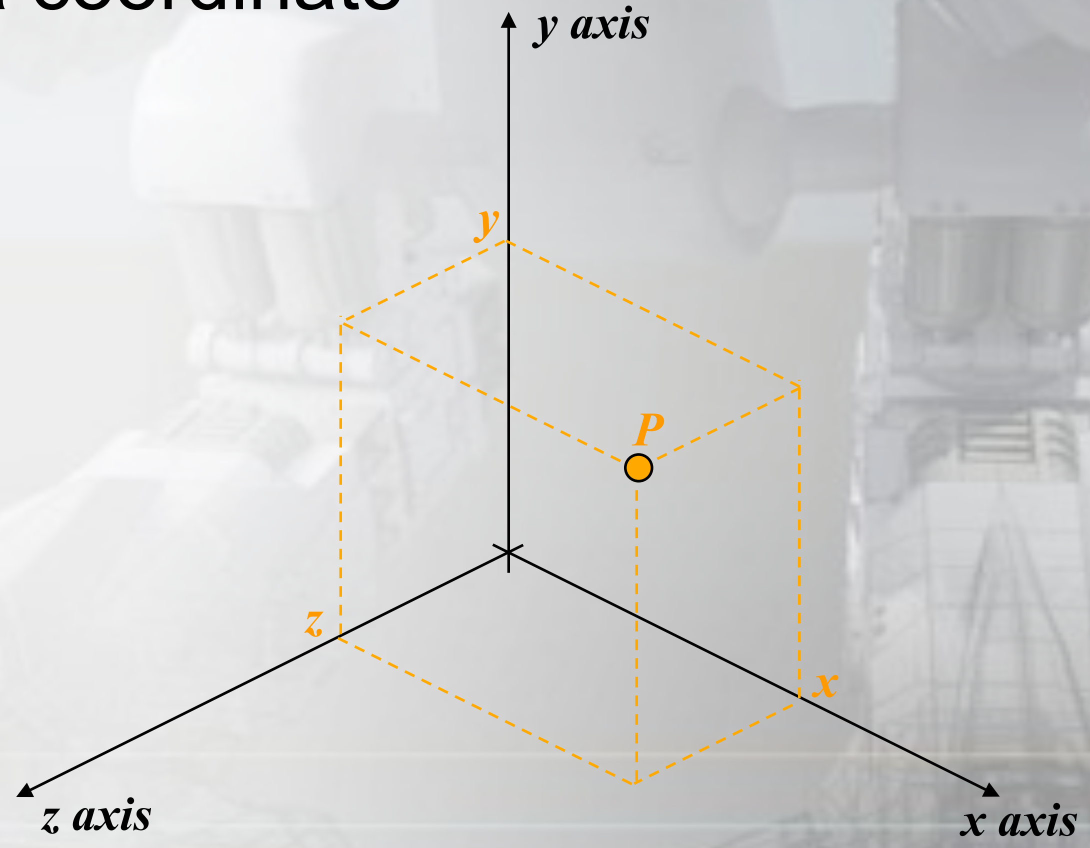


Coordinate Reference Frames – 2D (cont…)



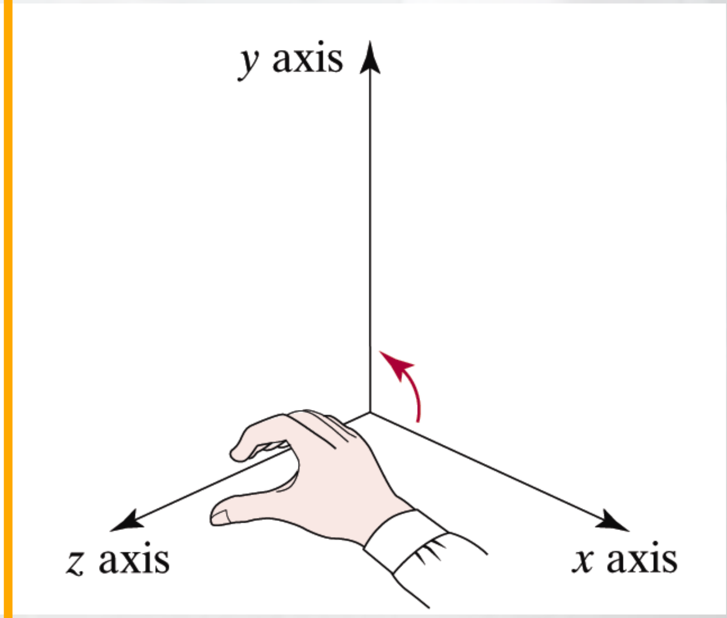
Coordinate Reference Frames – 3D

- For three dimensional scenes we simply add an extra coordinate

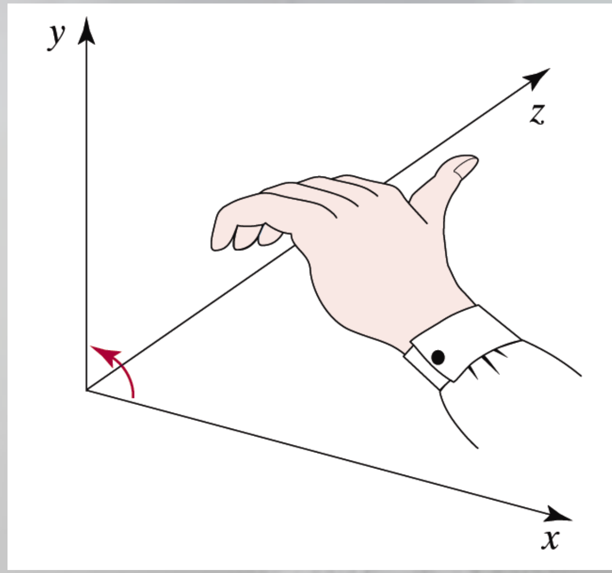


Left Handed Or Right Handed?

- There are two different ways in which we can do 3D coordinates – *left handed* or *right handed*



Right-Hand
Reference System



Left-Hand
Reference System

Left Handed Or Right Handed?

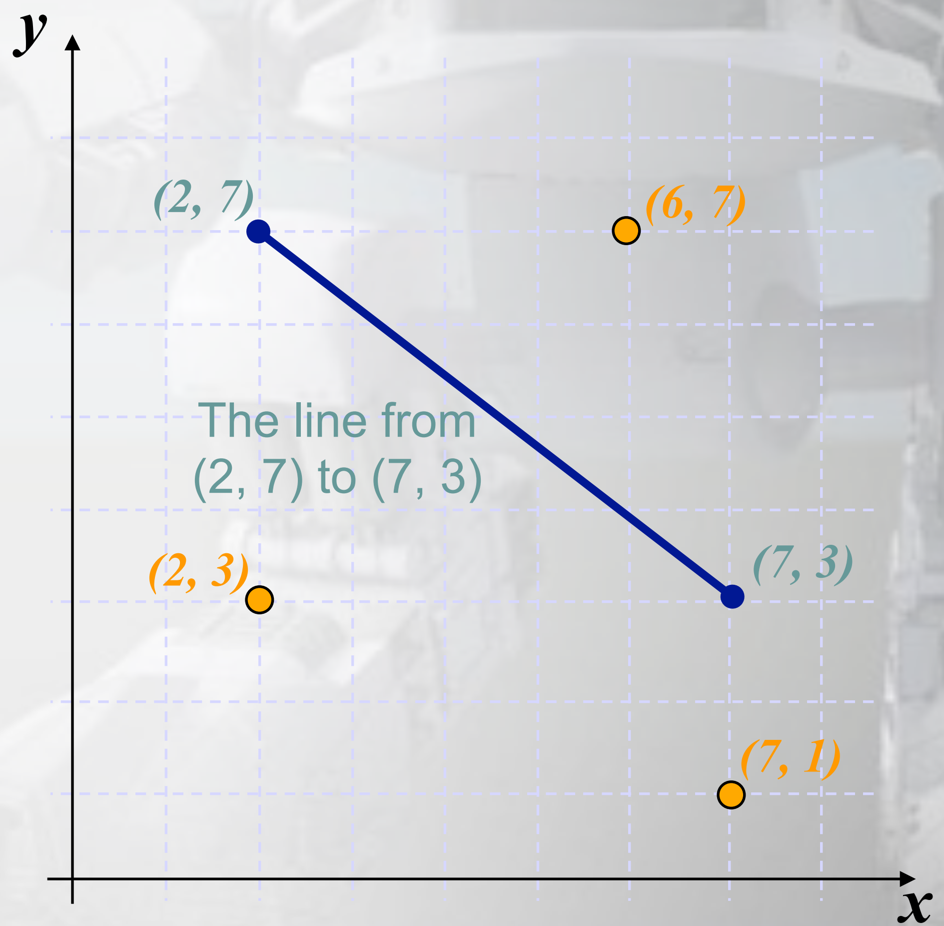
- There are two different ways in which we can do 3D coordinates – *left handed* or *right handed*

We will mostly use the right-handed system

Points & Lines

- Points:
  i. A point in two dimensional space is given as an ordered pair (x, y)
  ii. In three dimensions a point is given as an ordered triple (x, y, z)
- Lines:
  i. A line is defined using a start point and an end-point
     - In 2d: (x_{start}, y_{start}) to (x_{end}, y_{end})
     - In 3d: $(x_{start}, y_{start}, z_{start})$ to $(x_{end}, y_{end}, z_{end})$

Points & Lines (cont…)



The Equation of A Line

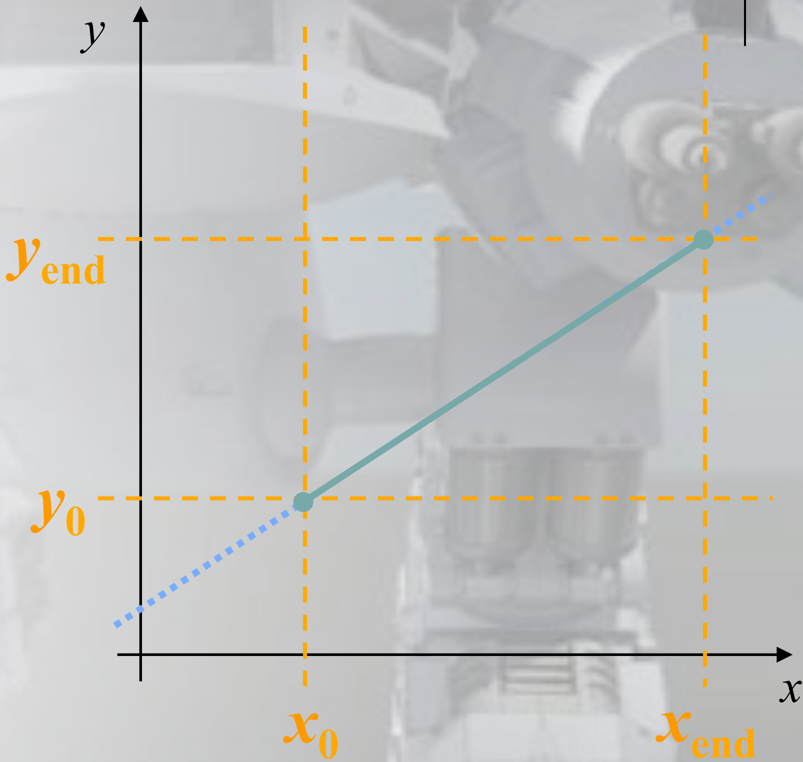
- The *slope-intercept equation of a line* is:

$$y = m \cdot x + b$$

- where:

$$m = \frac{y_{end} - y_0}{x_{end} - x_0}$$

$$b = y_0 - m \cdot x_0$$



- The equation of the line gives us the corresponding y point for every x point

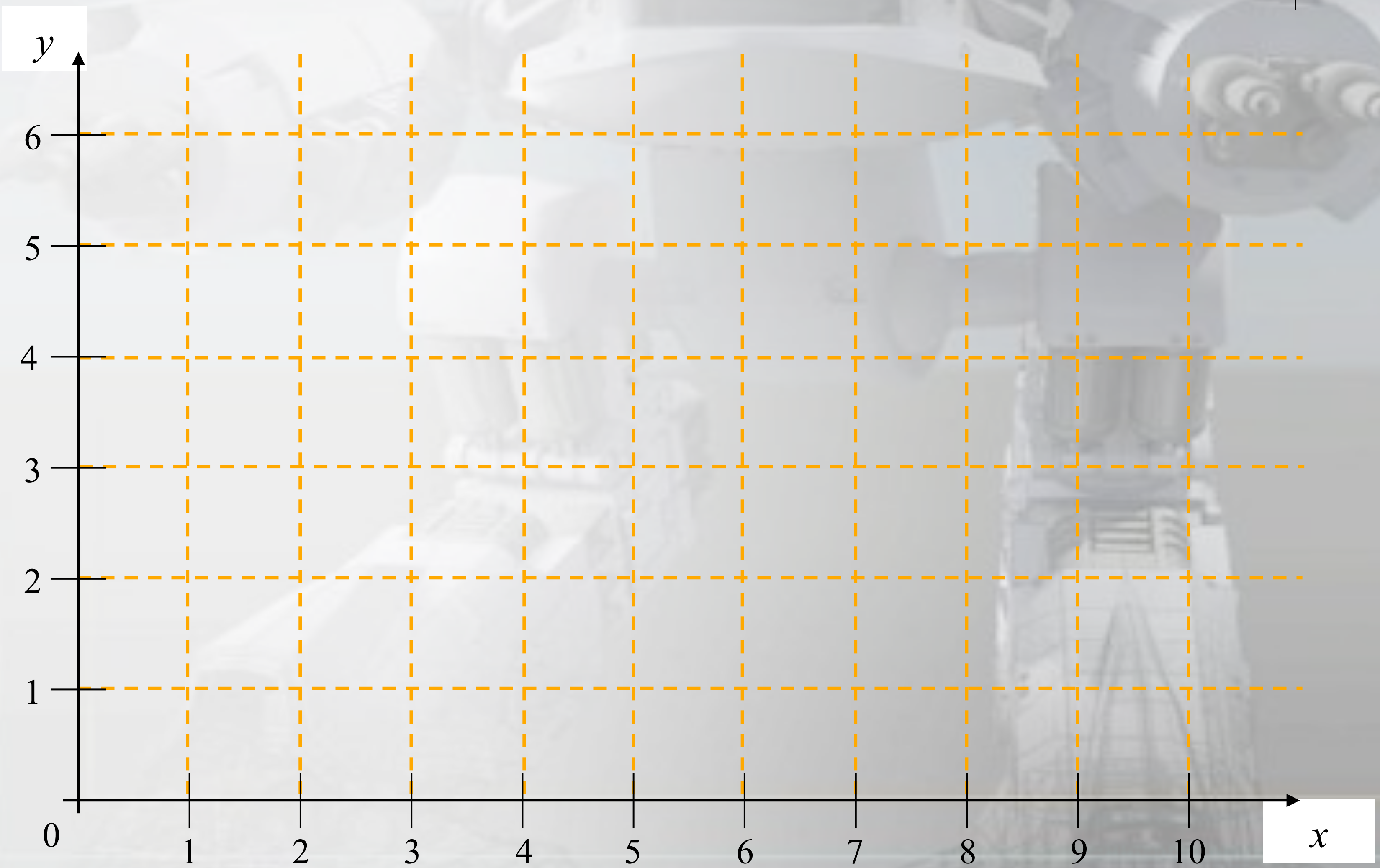
A Simple Example

- Let's draw a portion of the line given by the equation:

$$y = \frac{3}{5}x + \frac{4}{5}$$

- Just work out the y coordinate for each x coordinate

A Simple Example (cont…)



A Simple Example (cont…)

For each x value just work out the y value:

$$y(2) = \frac{3}{5} \cdot 2 + \frac{4}{5} = 2$$

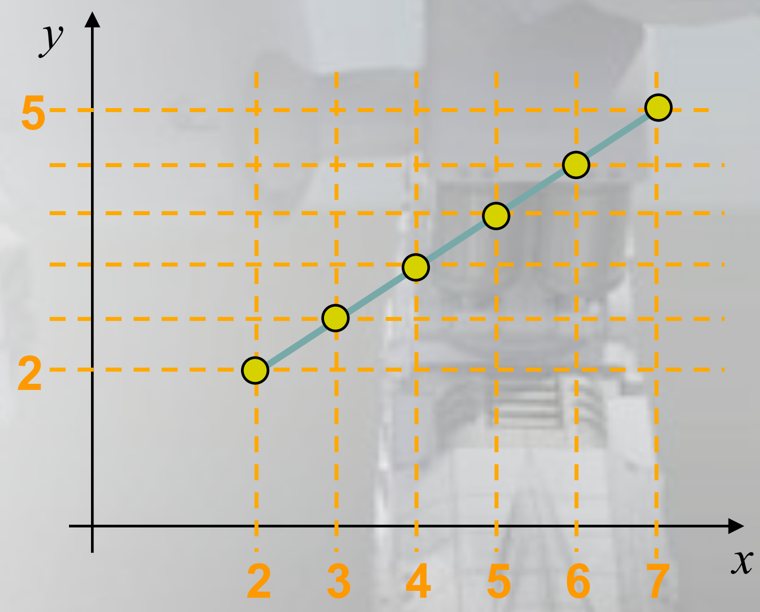
$$y(3) = \frac{3}{5} \cdot 3 + \frac{4}{5} = 2\frac{3}{5}$$

$$y(4) = \frac{3}{5} \cdot 4 + \frac{4}{5} = 3\frac{1}{5}$$

$$y(5) = \frac{3}{5} \cdot 5 + \frac{4}{5} = 3\frac{4}{5}$$

$$y(6) = \frac{3}{5} \cdot 6 + \frac{4}{5} = 4\frac{2}{5}$$

$$y(7) = \frac{3}{5} \cdot 7 + \frac{4}{5} = 5$$

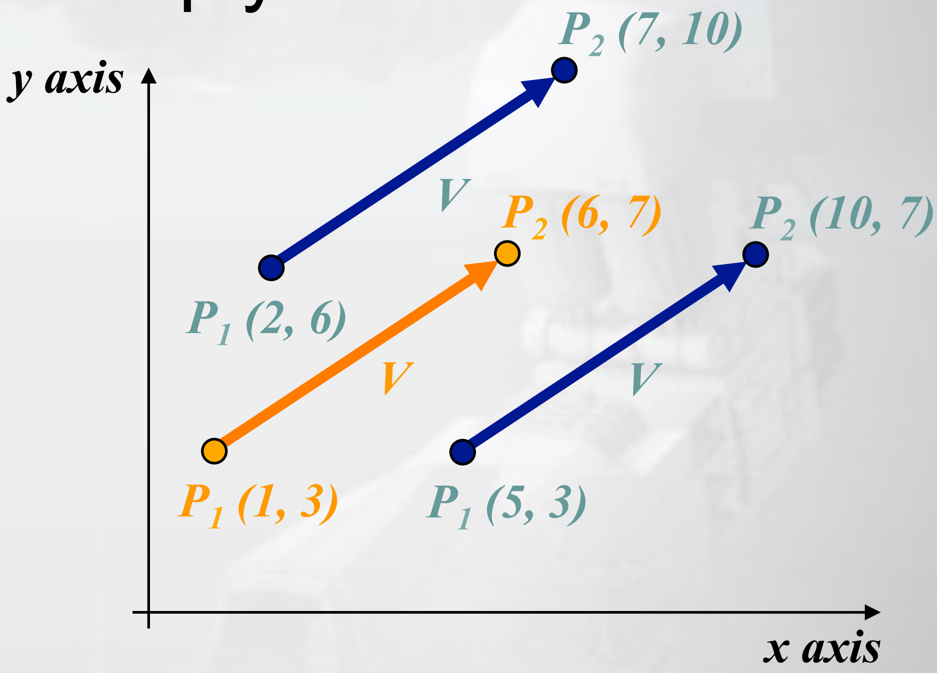


Vectors

- Vectors:
  i. A vector is defined as the difference between two points
  ii. The important thing is that a vector has a direction and a length
- What are vectors for?
  i. A vector shows how to move from one point to another
  ii. Vectors are very important in graphics - especially for transformations

Vectors (2D)

- To determine the vector between two points simply subtract them

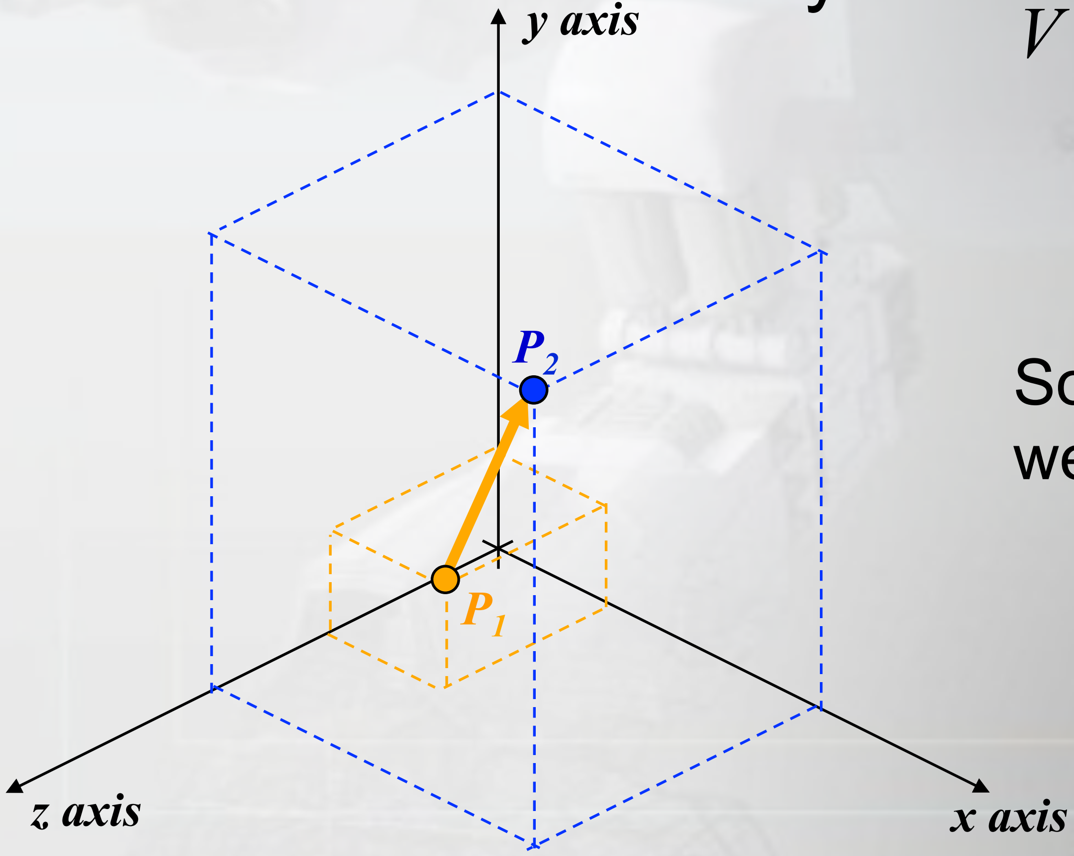


$$V = P_2 - P_1$$
$$= (x_2 - x_1, y_2 - y_1)$$
$$= (6 - 1, 7 - 3)$$
$$= (5, 4)$$

WATCH OUT: Lots of pairs of points share the same vector between them

Vectors (3D)

- In three dimensions a vector is calculated in much the same way



$$V = P_2 - P_1$$

$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$= (V_x, V_y, V_z)$$

So for (2, 1, 3) to (7, 10, 5) we get

$$= (7 - 2, 10 - 1, 5 - 3)$$

$$= (5,9,2)$$

Vector Operations

- There are a number of important operations we need to know how to perform with vectors:
  1. Calculation of vector length
  2. Vector addition
  3. Scalar multiplication of vectors
  4. Scalar product
  5. Vector product

Vector Operations: Vector Length

- Vector lengths are easily calculated in two dimensions:

$$|V| = \sqrt{V_x^2 + V_y^2}$$

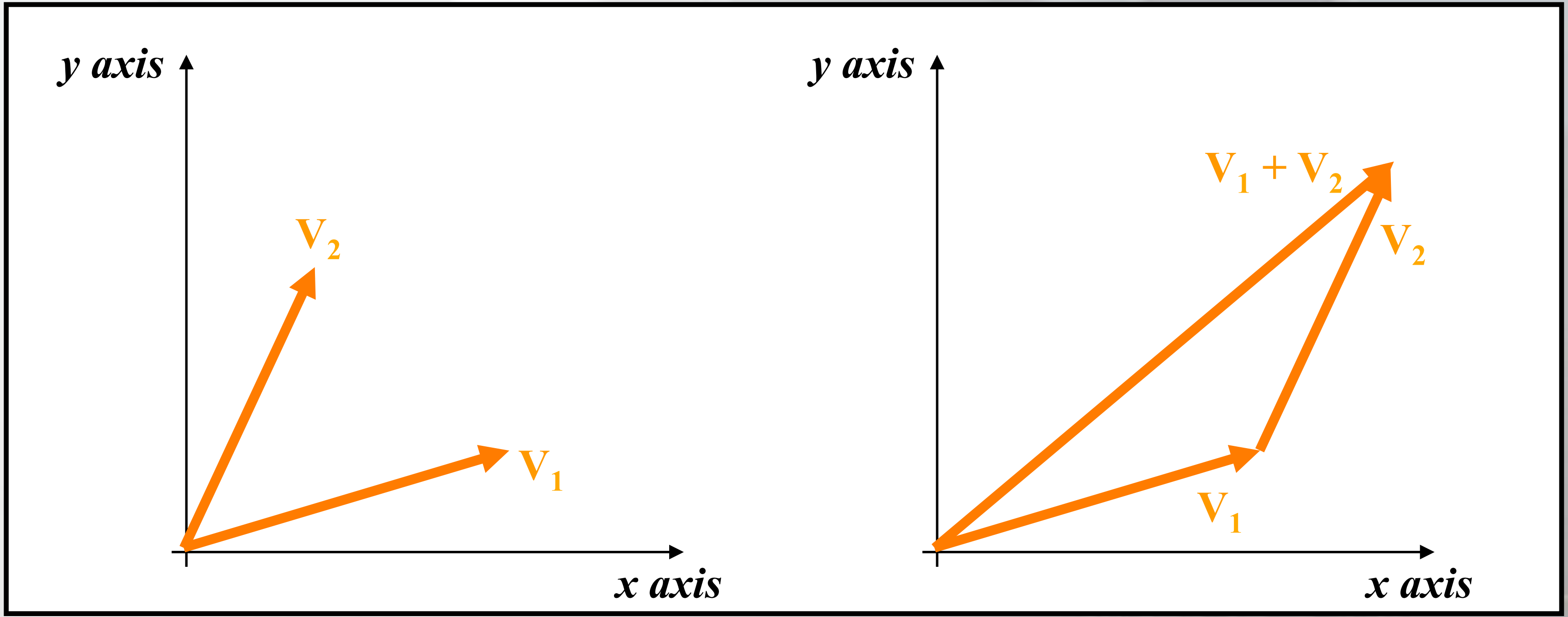
- and in three dimensions:

$$|V| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

Vector Operations: Vector Addition

- The sum of two vectors is calculated by simply adding corresponding components

$$V_1 + V_2 = (V_{1x} + V_{2x}, V_{1y} + V_{2y})$$

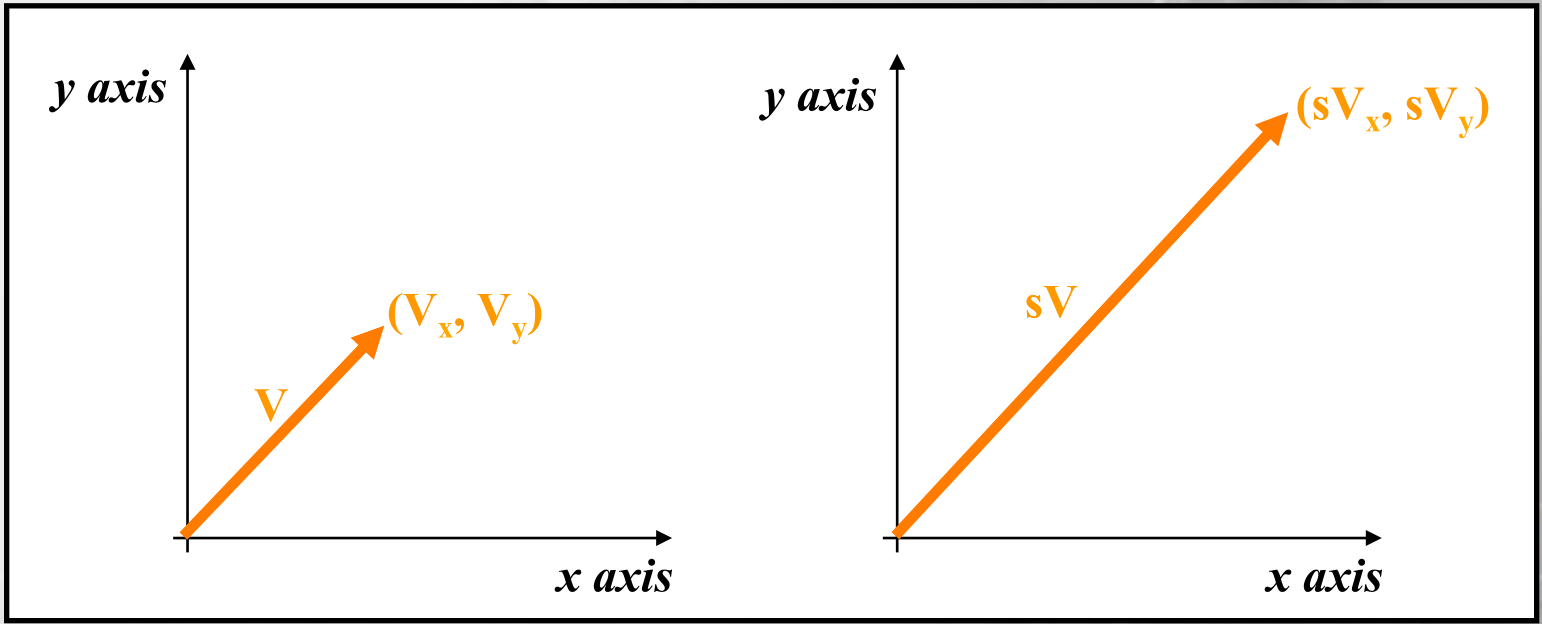


- Performed similarly in three dimensions

Vector Operations: Scalar Multiplication

- Multiplication of a vector by a scalar proceeds by multiplying each of the components of the vector by the scalar

$$sV = (sV_x, sV_y)$$



Other Vector Operations

- There are other important vector operations that we will cover as we come to them
- These include:
  1. Scalar product (dot product)
  2. Vector product (cross product)

Matrices

- A matrix is simply a grid of numbers

$$\begin{bmatrix} 1 & 11 & 13 \\ 10 & 4 & -3 \\ 2 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 4.3 \\ 6.7 \\ 1.2 \end{bmatrix} \quad \begin{bmatrix} 4 & 8 & 15 \\ 16 & 23 & 42 \end{bmatrix}$$

- However, by using matrix operations we can perform a lot of the maths operations required in graphics extremely quickly

Matrix Operations

- The important matrix operations for this course are:
  1. Scalar multiplication
  2. Matrix addition
  3. Matrix multiplication
  4. Matrix transpose
  5. Determinant of a matrix
  6. Matrix inverse

Matrix Operations: Scalar Multiplication

- To multiply the elements of a matrix by a scalar simply multiply each one by the scalar

$$s*\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} s*a & s*b & s*c \\ s*d & s*e & s*f \\ s*g & s*h & s*i \end{bmatrix}$$

- Example:

$$3*\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 18 \\ 24 & 30 & 36 \\ 42 & 48 & 54 \end{bmatrix}$$

Matrix Operations: Addition

- To add two matrices simply add together all corresponding elements

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} a+r & b+s & c+t \\ d+u & e+v & f+w \\ g+x & h+y & i+z \end{bmatrix}$$

- Example:

$$\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \\ 15 & 17 & 19 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ 17 & 21 & 25 \\ 29 & 33 & 37 \end{bmatrix}$$

Both matrices have to be the same size

Matrix Operations: Matrix Multiplication

- We can multiply two matrices **A** and **B** together as long as the number of columns in **A** is equal to the number of rows in **B**
- So, if we have an m by n matrix **A** and a p by q matrix **B** we get the multiplication:
  - **C**=**AB**
- where **C** is a m by q matrix whose elements are calculated as follows:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{ki}$$

Matrix Operations: Matrix Multiplication (cont…)

- Examples:

$$\begin{bmatrix} 0 & -1 \\ 5 & 7 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0*1+(-1)*3 & 0*2+(-1)*4 \\ 5*1+7*3 & 5*2+7*4 \\ -2*1+8*3 & -2*2+8*4 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 26 & 38 \\ 22 & 28 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1*4+2*5+3*6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4*1 & 4*2 & 4*3 \\ 5*1 & 5*2 & 5*3 \\ 6*1 & 6*2 & 6*3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

Matrix Operations: Matrix Multiplication (cont…)

- **Watch Out!** Matrix multiplication is not commutative, so: $AB \neq BA$

Matrix Operations: Transpose

- The transpose of a matrix M, written as M^T is obtained by simply interchanging the rows and columns of the matrix
- For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Other Matrix Operations

- There are some other important matrix operations that we will explain as we need them
- These include:
  1. Determinant of a matrix
  2. Matrix inverse

Summary

- In this lecture we have taken a brief tour through the following:
  1. Basic idea
  2. The mathematics of points, lines and vectors
  3. The mathematics of matrices
- These tools will equip us to deal with the computer graphics techniques that we will begin to look at, starting next time