1. We want to find $(\alpha, \beta) \in \mathbb{N}^2$ s.t. $g^{\alpha+x} \cdot h^{-(y+\beta)} = g^x \cdot h^{-y} \mod p$.

Let's denote by α' and β' the order of g and h in $\mathbb{Z}_p^*$, respectively. It means that $g^{\alpha'} = 1 \mod p$ and $h^{\beta'} = 1 \mod p$. We have, for any $(x,y) \in \mathbb{N}^2$,

$$g^{(\alpha'+x)} \cdot h^{-(\beta'+y)} = g^{\alpha'} \cdot g^x \cdot h^{-\beta'} \cdot h^{-y} = (g^{\alpha'} \cdot h^{-\beta'})\, g^x \cdot h^{-y} \underset{g^{\alpha'}=1 \bmod p}{=} (h^{\beta'})^{-1} \cdot g^x \cdot h^{-y} \underset{h^{\beta'}=1 \bmod p}{=} (1)^{-1} \cdot g^x \cdot h^{-y} \mod p.$$

So, the fct. is periodic with period (α', β').

Given a period (α, β), you can compute the discrete logarithm. We know that

$g^{x+\alpha} \cdot h^{-(y+\beta)} = g^x \cdot h^{-y} \mod p$. And this happens $\iff g^\alpha \cdot h^{-\beta} = 1 \mod p$

$\iff g^\alpha = h^\beta \mod p$

$\iff g^\alpha = g^{a\beta} \mod p$

$\iff a = \alpha \cdot \beta^{-1} \mod q$ where q is the order of g.