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Mixed Bundling of Two Independently Valued Goods

Hemant K. Bhargava
Graduate School of Management, University of California, Davis, Davis, California 95616, hemantb@ucdavis.edu

This paper develops analytical results and insights for the mixed bundling problem of pricing a product line consisting of two component goods (with valuations distributed uniformly and independently in $[0, a_1]$ and $[0, a_2]$, respectively, with $a_1 \leq a_2$) and a bundle of the two goods. This setting has previously been considered analytically intractable. By deriving component good prices as exact algebraic functions of the optimal bundle price, I reduce the multiproduct pricing problem to a univariate nonlinear optimization problem in the bundle price. When the two component goods are sufficiently asymmetric in valuations, the optimal solution is partial mixed bundling (one component good is not sold separately), for which I derive exact conditions and optimal prices. An exact analytical solution is also given for mixed bundling of two information goods (component goods with zero marginal costs). For the general case, I derive a closed-form approximation of the optimal bundle price: $p_B = 0.5724w_1 + 0.5695w_2 + 0.3516a_1 + 0.4889a_2 + 0.0054(a_2/a_1)w_1 - 0.0201(a_2/a_1)w_2$. The approximation is highly accurate (the resulting profit is within a percent of the exact optimal profit), efficient (instant solutions for specific problem instances), and useful in other applications where mixed bundling is a subproblem. I demonstrate this by applying the solution to the mixed bundle problem in a vertical channel where a retailer must bundle and price component goods from multiple independent manufacturers.

Key words: bundling; price discrimination; pricing; marketing; information systems
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1. Introduction

Mixed bundling is a product line strategy where the firm sells individual goods and, typically at a discount, a bundle of all or a subset of these goods. Examples include bundles of office productivity software, telecommunications services (Internet, TV, phone), camera kits, vacation packages, etc. Although the individual goods have stand-alone consumption value, it is common to refer to them as "components." This paper develops analytical results and insights for the classical mixed bundling problem of pricing a product line consisting of two component goods, which have independent demand, and a bundle of the two goods for which consumers' valuations are the sum of their component valuations. After reducing the pricing problem to a univariate nonlinear optimization problem, I develop tight bounds on the optimal solution and a highly accurate closed-form expression for optimal prices.

Product bundling is extensively studied in the marketing, economics, and information systems literatures and widely practiced in industry. The economic intuition behind bundling, exhibited in the earliest work on bundling (Stigler 1963, Adams and Yellen 1976, Schmalensee 1984, McAfee et al. 1989), is that consumer valuations for the bundle are dispersed in a narrower range, relative to mean, than valuations for individual component goods, enabling the seller to extract more surplus.[1] Because bundling is most attractive when unit marginal costs are very low, bundling has also been widely studied for information goods (Bakos and Brynjolfsson 1999, 2000; Altinkemer and Bandyopadhyay 2000; Wu et al. 2008). Firms love mixed bundling because it empowers them with another tool for segmenting customers: the bundle targets consumers who have moderate to high valuations for both components, whereas the component goods can be sold to consumers who have very high valuations for one product but low for the other.

Hanson and Martin (1990) developed a mixed-integer programming model and algorithmic procedure for the mixed bundle pricing problem. Not only is the procedure computationally expensive, but it can only solve instances of the problem. Solving the

[1] This is most easily seen when consumer valuations for one product are independent of their valuations for the other. The gains from bundling are higher when valuations are negatively correlated or are superadditive (products are complements); conversely, bundling is less attractive when valuations are positive correlated or are subadditive (products are substitutes) (Venkatesh and Mahajan 2009).

general pricing problem under mixed bundling is challenging, in part because of consumer self-selection among the items being offered (the power set of component goods). Unlike the case of vertically differentiated products, where the single-crossing property facilitates computation of optimal market shares and optimal prices (Maskin and Riley 1984), mixed bundling poses a combinatorial problem because the component products cannot be ordered in terms of consumer valuations. Yet it is important to analyze because, as demonstrated by McAfee et al. (1989, for bundles of two components), mixed bundling is superior to both component and pure bundle sales when the demand distributions for the component goods are independent and almost always so for joint distributions. The discussion in the rest of this paper is with respect to this basic modeling construct of two products with independent demand distributions.

Because of the complexity of the mixed bundling optimization problem, analytical results are relatively sparse. Even for the barebones model consisting of two components with independent demand, the literature offers no analytical solution or tight approximation for the optimal mixed bundle problem (even though the *superiority* of mixed bundling can be established analytically; e.g., McAfee et al. 1989, Prasad et al. 2010).[2] Eckalbar (2010) provides an elegant analytical expression for optimal mixed bundling under zero marginal costs but only proves the superiority of mixed bundling under positive costs. Some authors have developed asymptotic results when a very large number of goods is being bundled, but without explicit form for optimal prices (Bakos and Brynjolfsson 1999, Ibragimov and Walden 2010). The complexity of mixed bundling has spurred alternative designs, a form of innovation by necessity. Hitt and Chen (2005) propose that firms could practice customized bundling, pricing a bundle by the number of component goods in it and letting each customer choose which goods to include in the bundle. This approach simplifies problem complexity relative to mixed bundling and works well when the goods have low marginal costs and consumers are budget constrained. Banciu et al. (2010) propose a tractable model of mixed bundling using two component goods that are vertically differentiated, thereby imposing a vertical order on all the elements of the mixed bundle (the bundle and component goods). However, this assumption (that all consumers have identical ranking for the component goods) is highly restrictive for bundling in general. Several other studies of bundling simply do not consider mixed bundling or disallow it by assumption (e.g., Bakos and Brynjolfsson 2000, Geng et al. 2005, Fang and Norman 2006).

The majority of research on mixed bundling employs examples and numerical computations to identify parameter conditions under which the mixed bundle strategy dominates pure components or pure bundle selling (see, e.g., Stigler 1963, Adams and Yellen 1976, Schmalensee 1984, Venkatesh and Kamakura 2003). These regions of dominance are usually displayed graphically on the basis of extensive numerical computation. The reliance on computational approaches for the mixed bundling problem has some drawbacks. First, it does not yield analytical or explicit characterization of the prices. This reduces transparency of the model and results. Second, occasionally, it leads to faulty results. For instance, Venkatesh and Kamakura (2003) find that with two independently valued and symmetric goods, a pure components selling strategy does as well as mixed bundling when marginal costs are high (result 4(c) and Figure 3(b), applied at $\Theta = 0, a = 1$). In an outstanding and comprehensive survey of bundling literature, Venkatesh and Mahajan (2009) repeat this assertion (guideline G4(a)). This claim is not true. Mixed bundling is always and *strictly* better than pure bundling (as long as positive profits are possible for each product). This indeed is the McAfee et al. (1989) result. More generally, the lack of analytical and accurate solutions for mixed bundling inhibits the extension of the analysis to other settings such as when the products (components and bundle) are distributed in a vertical distribution channel rather than sold directly by the manufacturer.

This paper develops multiple analytical solutions and approximations for mixed bundling of two independently valued goods. The model structure, described in §2, has two component goods (having unit marginal costs w_1, w_2) for which demands are independent, with consumer valuations distributed uniformly in $[0, a_1]$ and $[0, a_2]$. Section 3 specifies optimal prices for component goods as an algebraic function of the bundle price, thereby reducing the three-variable optimization problem (optimal prices of the two component goods and the bundle) to a single variable problem (bundle price). It also identifies the level of valuation asymmetry between the two components that leads to partial mixed bundling (i.e., no separate sales for one of the goods). Notably, such a solution features the lower-valuation good and the bundle and only occurs when this good has zero marginal cost. For mixed bundling, although the univariate optimization problem is highly nonlinear, it can be solved efficiently and exactly using global

[2] Venkatesh and Kamakura (2003) cite Wilson (1993) to argue that there are no closed-form solutions for this problem. Armstrong (1996) notes that the profit function is piecewise cubic for the symmetric case and implies lack of an analytical solution. Prasad et al. (2010) also note the intractability of finding an explicit form for the optimal solution.

nonlinear solvers. The reduction facilitates development of a closed-form and near-optimal expression for the bundle price, $p_B = 0.5724w_1 + 0.5695w_2 + 0.3516a_1 + 0.4889a_2 + 0.0054(a_2/a_1)w_1 - 0.0201(a_2/a_1)w_2$. For the special case of zero marginal costs (which applies well to information goods), §4 provides a complete analytical specification of the optimal bundle and component prices. For the case of symmetric valuations (but positive, unequal costs), §5 analytically derives tight lower and upper bounds for the optimal solution (the bundle price). The midpoint function of these bounds produces an extremely accurate approximation of the exact optimal solution, $p_B = (11/12)(1 + (w_1 + w_2)/2)$. All approximations are highly accurate, producing a profit that is within a percent of the exact optimal profit. These closed-form results are used in §6 to examine properties of bundling and in §7 to examine the economics of mixed bundling in a vertical distribution channel.

2. Model

There are two products $i = 1, 2$ with constant unit marginal costs w_1, w_2, respectively. The firm must choose whether to sell only components (the *pure components* case), only the bundle (*pure bundling*), or the bundle and one or more component good(s) (*mixed bundling*). Under mixed bundling, there are two possibilities, (i) positive separate sales for both component goods and the bundle and (ii) *partial mixed bundling* with separate sales for only one component good and the bundle. The latter is also referred to in the literature as tie-in sales (Burstein 1960). Consumers are heterogeneous in their valuations for the two goods, and these valuations follow the cumulative distribution functions F_1 and F_2, respectively (with uniform density f_1 and f_2) over the support $[0, a_1]$ and $[0, a_2]$.[3] Without loss of generality, designate the higher valuation product as good 2, i.e., suppose that $a_2 \geq a_1$. I assume that f_1 and f_2 are independent and that each consumer's valuation of the bundle, z, is the sum of her valuations for individual products. Therefore, consumer valuations for the bundle have an isosceles trapezoidal distribution in the interval $[0, a_1 + a_2]$ (with the shorter parallel side ranging from a_1 to $a_2 - a_1$ or from $a_2 - a_1$ to a_2 depending on whether $a_2 - a_1$ is $>$ or $< a_1$), with density g given by

$$
\begin{aligned}
g(z) &= (f_1 * f_2)(z) = \int_0^z f_1(z-y) f_2(y)\, dy \\
&= \int_0^{a_2} f_1(z-y) \frac{1}{a_2}\, dy \\
&\qquad\qquad \text{because } f_2 = 1/a_2 \text{ only in } [0, a_2] \\
&= \int_{z-a_2}^{z} f_1(x) \frac{1}{a_2}\, dx \quad \text{transform } z-y \text{ to } x \\
&= \begin{cases}
\displaystyle\int_0^z \frac{1}{a_1 a_2}\, dz = \frac{z}{a_1 a_2} & \text{if } z \in [0, a_1], \\
\displaystyle\int_0^{a_1} \frac{1}{a_1 a_2}\, dz = \frac{a_1}{a_2} & \text{if } z \in [a_1, a_2], \\
\displaystyle\int_{z-a_1}^{a_1} \frac{1}{a_1 a_2}\, dz = \frac{a_1 + a_2 - z}{a_1 a_2} & \\
& \text{if } z \in [a_2, a_1 + a_2], \\
0 & \text{elsewhere.}
\end{cases}
\end{aligned}
\tag{1}
$$

Denote the prices under mixed bundling by p_1, p_2, and p_B. The sales levels for each product are determined after accounting for consumer self-selection constraints. Each consumer picks the product—bundle or component(s)—that offers her the best net surplus. This might lead to positive sales for only components, only the bundle, or both. From McAfee et al. (1989), mixed bundling is strictly optimal relative to pure components or pure bundling. Hence, the optimal prices yield positive sales for at least one component and the bundle. The optimal solution satisfies the properties, $w_i < p_i \leq \min\{a_i, p_B\}$, $w_1 + w_2 < p_B$ and $p_B < p_1 + p_2$. Subject to these constraints, the prices imply the following sales levels for each product,

$$Q_1 = (a_1 - p_1)(p_B - p_1), \tag{2a}$$

$$Q_2 = (a_2 - p_2)(p_B - p_2), \tag{2b}$$

$$Q_B = (a_2 - p_B + p_1)(a_1 - p_B + p_2) - \frac{(p_1 + p_2 - p_B)^2}{2}. \tag{2c}$$

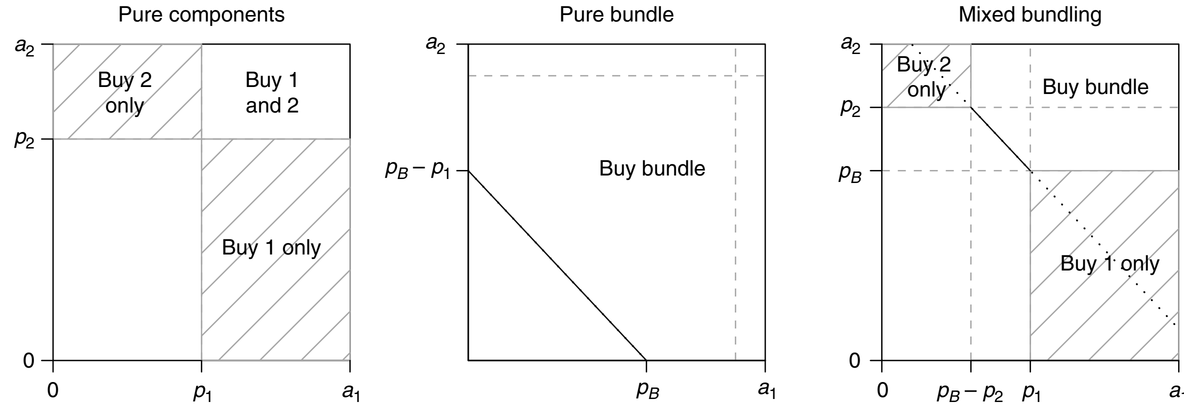
The sales equations are intuitively easy to understand for the case of symmetric valuations ($a_1 = a_2$; illustrated in Figure 1). Fortunately, the same equations remain valid—conditional on the properties stated above—even under asymmetric valuations. However, partial mixed bundling now becomes possible (e.g., when $p_B \leq p_2$).

The firm's mixed bundle pricing profit maximization problem is

$$
\begin{aligned}
\underset{p_1, p_2, p_B}{\text{maximize}}\ \Pi &= (p_1 - w_1) Q_1 + (p_2 - w_2) Q_2 \\
&\quad + (p_B - w_1 - w_2) Q_B \\
&= (p_1 - w_1)(a_1 - p_1)(p_B - p_1) \\
&\quad + (p_2 - w_2)(a_2 - p_2)(p_B - p_2) \\
&\quad + (p_B - w_1 - w_2) \\
&\quad \cdot \Big((a_2 - p_B + p_1)(a_1 - p_B + p_2) \\
&\qquad - \frac{(p_1 + p_2 - p_B)^2}{2} \Big)
\end{aligned}
\tag{3}
$$

[3] I assume a uniform distribution on this interval because a specific distribution is needed to develop specific analytical forms for the mixed bundling problem. The uniform distribution (linear demand) is widely used in analytical work on pricing and bundling.

Figure 1 Product Sales Under Pure Components, Pure Bundle, and Mixed Bundling



$$\text{s.t. } w_1 < p_1 < \min\{a_1, p_B\}, \quad w_2 < p_2 < \min\{a_2, p_B\},$$
$$w_1 + w_2 < p_B < p_1 + p_2.$$

I solve the unconstrained maximization problem and then, among the multiple candidate optima, pick the optimal solution by a process of eliminating other candidate solutions that fail to satisfy the constraints. The three first-order conditions for the unconstrained pricing problem are

$$(p_B - p_1)(2a_1 + w_1 - 3p_1) - (a_1 - p_1)w_2 = 0, \tag{4a}$$
$$(p_B - p_2)(2a_2 + w_2 - 3p_2) - (a_2 - p_2)w_1 = 0, \tag{4b}$$
$$\left(\frac{3(p_B^2 - p_1^2 - p_2^2)}{2} + a_1a_2 + p_1w_1 + p_2w_2 - p_B(w_1 + w_2) + a_1(2p_1 + w_2 - 2p_B) + a_2(2p_2 + w_1 - 2p_B)\right) = 0. \tag{4c}$$

3. Analysis

This section analyzes the general two-product mixed bundle pricing problem posed in §2. The analysis process consists of two parts. Section 3.1 reduces the three-variable optimization problem to one in a single variable p_B by specifying the other two variables p_1 and p_2 as unique polynomial functions of p_B. The boundary case where the mixed bundle strategy reduces to partial mixed bundling (i.e., where one component good has no separate sales) is covered in §3.2. Finally, §3.3 develops a closed-form near-optimal solution for p_B.

3.1. Relation Between Component and Bundle Prices

Consider the first two quadratic equations in Equation (4). Each equation generates two possible solutions for p_1 and p_2, respectively, in terms of p_B. Combining these would yield four alternative specifications for p_1, p_2 as functions of p_B. However, the following result simplifies the analysis by reducing Equations 4(a) and 4(b) to a single solution that uniquely specifies p_1, p_2 in terms of the sole remaining variable, p_B.

LEMMA 1 (COMPONENT PRICES). *The component prices p_1 and p_2 are uniquely specified as a function of the bundle price p_B,*

$$p_1 = \frac{1}{6}\left(3p_B + (2a_1 + w_1 - w_2) - \sqrt{(3p_B - 2a_1 + (w_2 - w_1))^2 + 12w_2(a_1 - p_B)}\right), \tag{5a}$$

$$p_2 = \frac{1}{6}\left(3p_B + (2a_2 + w_2 - w_1) - \sqrt{(3p_B - 2a_2 + (w_1 - w_2))^2 + 12w_1(a_2 - p_B)}\right). \tag{5b}$$

Lemma 1 motivates the following notational convention. Because square root terms feature frequently in the analysis, and because Lemma 1 isolates which root gives the correct solution, it simplifies the exposition to write a square root term as representing a positive value, so that the negative root of a term x is then written as $-\sqrt{x}$. When p_j exceeds p_B in the above equation, then good j is never purchased separately, and the mixed bundling strategy reduces to the special case of *partial mixed bundling* where the firm only sells product i and the bundle. In this case, the equation overstates the component good j's price; however, the overstatement is immaterial because it still leads to zero separate sales for good j. Section 3.2 examines the partial mixed bundling case, which is useful in its own right but also as a stepping stone to the general solution.

3.2. Partial vs. Full Mixed Bundling

Intuitively, when the component goods have highly asymmetric valuations, the firm's product line under mixed bundling turns lopsided. It has one item (good 1) with relatively low valuations and two (good 2 and the bundle) with much higher and very similar valuations. Thus, two of the products have very little differentiation, which increases the problem of cannibalization within the product line, making it attractive to drop one component good and employ a partial bundling approach. Which component should be dropped and under what conditions? Continuing with the solution property defined in Lemma 1, I demonstrate that the higher-valuation good is dropped from separate sale, and only when the lower valuation good has zero marginal cost.

Proposition 1 (Partial vs. Full Mixed Bundling). *When the good with lower range of valuations has positive marginal cost* ($w_1 > 0$), *then it is always optimal to have separate sales of both component goods* (*in addition to the bundle*). *When* $w_1 = 0$ *and the goods are substantially asymmetric, i.e.,* a_1 *is below a threshold* A_1*, specifically,*

$$a_1 \leq A_1 = \frac{1}{48a_2}\left(28a_2^2 - 12a_2w_2 - w_2^2 - (2a_2 + w_2)\sqrt{4a_2^2 + 20a_2w_2 + w_2^2}\right), \quad (6)$$

then partial mixed bundling is optimal (*with positive sales for the bundle and good* 1), *and then the bundle price is* $p_B < (2a_2 + w_2)/3 < a_2$.

Proposition 1 identifies when a partial mixed bundling strategy might be useful and also identifies the design of the firm's product line under partial bundling. Equation (6) can be simplified for the special case of zero marginal costs, yielding the condition for a partial mixed bundling solution as $a_1 \leq (1/2)a_2$. To illustrate the proposition further, normalize a_2 to 1 and vary w_2 from 0 to 0.5 (in intervals of 0.05). The corresponding thresholds for partial mixed bundling for these values of w_2 are $a_1 \leq (0.5, 0.475, 0.45, 0.427, 0.403, 0.378, 0.354, 0.33, 0.306, 0.28, 0.257) * a_2$, which suggests that Equation (6) can be tightly approximated and simplified to $a_1 \leq (0.5 - 0.486w_2)a_2$.

Proposition 1 reveals some noteworthy points about the partial mixed bundling strategy. First, it is the higher-valuation good that is not offered separately. This design intuitively is appealing because offering the higher quality good and the bundle would increase cannibalization. An example is selling music albums and single songs. If the primary attraction of the album was a highly popular single, then selling the single separately would likely kill sales of the album.[4] Second, the partial bundling strategy does not occur except when the lower-valuation good has zero costs. This suggests that real-world observations of partial mixed bundling (or tie-in sales) are more often likely on account of strategic reasons rather than demand management and price discrimination (one such scenario is discussed by Prasad et al. 2010 under the title of "mixed bundling-1"). This is particularly likely when the bundling firm procures the good from an upstream manufacturer and hence does not have zero cost. Finally, the last part of the result ($p_B < a_2$ in the partial bundling case) simplifies the formulation and analysis of the partial bundling strategy by eliminating one of the two branches that would otherwise have had to be considered (as shown in Figure 3, the demand formulation is different for $p_B < a_2$ and $p_B > a_2$, but now the latter can be dropped).

3.3. General Case: Asymmetric Valuations, Positive Costs

Lemma 1 specified optimal prices for component goods as a function of the bundle price. From Proposition 1 partial mixed bundling may be optimal when $w_1 = 0$, but the specification of p_1 and p_2 remains valid (except that p_2 becomes irrelevant when it exceeds p_B). Lemma 1 has two important implications. Making appropriate substitutions from Lemma 1 into Equation (3), the new univariate optimization problem is

$$\begin{aligned}\underset{p_B}{\text{maximize}}\, \Pi &= 6a_1((9p_B - w_1)w_2 - (3p_B - w_1)^2)\\ &\quad + 6a_2((9p_B - w_2)w_1 - (3p_B - w_2)^2)\\ &\quad + 12(a_1^2 + a_2^2)(3p_B - w_1 - w_2)\\ &\quad + 108a_1a_2(p_B - w_1 - w_2)\\ &\quad - 8a_1^3 - 8a_2^3 + 12(a_1w_2^2 + a_2w_1^2 - a_2^2w_1 - a_1^2w_2)\\ &\quad - ((3p_B - 2a_1 + w_2 - w_1)^2 + 12w(a_1 - p_B))^{3/2}\\ &\quad - ((3p_B - 2a_2 + w_1 - w_2)^2 + 12w(a_2 - p_B))^{3/2} \quad (7)\end{aligned}$$

after neglecting a multiplier 1/108 because it does not affect the optimization, and subject to the original constraints. This problem has the optimality condition

$$\begin{aligned}&2[a_1(6p_B - 3w_2 - 2w_1) + a_2(6p_B - 3w_1 - 2w_2) - 2a_1^2 - 2a_2^2 - 6a_1a_2]\\ &\quad = \Big((3p_B - 2a_1 - w_1 - w_2)\cdot\sqrt{(3p_B - 2a_1 + w_2 - w_1)^2 + 12w_2(a_1 - p_B)}\\ &\qquad + (3p_B - 2a_2 - w_1 - w_2)\cdot\sqrt{(3p_B - 2a_2 + w_1 - w_2)^2 + 12w_1(a_2 - p_B)}\Big). \quad (8)\end{aligned}$$

[4] I appreciate an anonymous reviewer's suggestion to include this example.

Hence, I have reduced the mixed bundle pricing problem to this final step of determining the optimal bundle price as the correct root of this one equation (subject to the constraints stated in Equation (3)) or, alternately, solving the univariate optimization in Equation (7). Because the right-hand side (RHS) of Equation (8) is positive, the left-hand side (LHS) must also be positive, implying that

$$p_B \geq \frac{1}{3}(a_1+a_2)+\frac{1}{3}(w_1+w_2) + \frac{1}{6}\frac{a_1(a_2-w_2)+a_2(a_1-w_1)}{a_1+a_2}, \tag{9}$$

and establishing a lower bound for the optimal bundle price.

Despite being highly nonlinear, the univariate optimization problem can be solved efficiently, either as an optimization problem or by finding the roots of Equation (8). Moreover, Lemma 5 and Equation (9) make this computational process highly efficient by pruning the search space. I computed optimal solutions for about 25,000 problem instances and cross-checked them using multiple alternative solution techniques: (i) the nonlinear optimizer BARON (the Branch-And-Reduce Optimization Navigator, http://archimedes.cheme.cmu.edu/?q=baron, a computational system for facilitating the solution of nonconvex optimization problems to global optimality); (ii) the optimization modeling software LINGO (http://www.lindo.com/, which uses a combination of solvers for nonlinear problems); (iii) a nonlinear optimizer in the open-source computing environment *R* (the function "optimize," which employs golden section search and successive parabolic interpolation); and (iv) brute force numerical computation.

Proposition 2 (Optimal Solution for Mixed Bundling Problem). *The optimal bundle price is* $p_B = 0.5724w_1 + 0.5695w_2 + 0.3516a_1 + 0.4889a_2 + 0.0054(a_2/a_1)w_1 - 0.0201(a_2/a_1)w_2$, *with the optimal component prices given by Equation* (5).

Proposition 2 has the usual advantages of a closed-form solution in that it enables comparative statics and sensitivity of optimal prices to the various parameters. But it is also highly accurate. The excellent statistical fit (adjusted $R^2 = 0.9999$) vindicates the linear relationship (with interaction terms) assumed in the approximation. Other functional forms could be used in this process; however, the linear form already produces high fit and is also consistent with intuition about bundle pricing as well as the analytical formulæ available for other special cases. The profit value from the regression-based estimate of the optimal price can be compared with the exact value obtained by solving the pricing problem or Equation (8). Table 1 provides this comparison for a small set of problem instances. Figure 2 displays the results for the entire computation, revealing that the regression-based solution achieves more than 99% of the profit potential over the parameter space of interest (see Figure 2). The worst-case error (of 1%) occurs when w_2 is quite high relative to a_2; however, that region is one where bundling is not too attractive to begin with. For the case of symmetric valuations, a tighter approximation can be obtained using a subset of the results corresponding to $a_1 = a_2 = 1$, and this is quite similar to the bounds-driven approximation presented in Corollary 4 (provided in §5).

Table 1 **Comparison of Actual and Approximate Solutions (with a_1 Normalized to 1)**

Parameters			Actual solution				Approximation result			
w_1	w_2	a_2	p_B	p_1	p_2	Profit	p_B	p_1	p_2	Profit
0.2	0.2	1	1.08	0.68	0.68	0.341	1.07	0.678	0.678	0.3409
0.2	0.2	1.5	1.31	0.7	0.963	0.706	1.31	0.701	0.963	0.7058
0.2	0.2	2	1.54	0.71	1.23	1.2	1.55	0.710	1.238	1.1977
0.4	0.5	1	1.36	0.728	0.783	0.158	1.35	0.726	0.782	0.1581
0.4	0.5	1.5	1.58	0.75	1.05	0.401	1.59	0.750	1.055	0.4015
0.4	0.5	2	1.82	0.762	1.32	0.772	1.83	0.763	1.321	0.7716
0.6	0.8	1	1.65	0.805	0.909	0.0507	1.63	0.803	0.908	0.0505
0.6	0.8	1.5	1.86	0.821	1.17	0.187	1.86	0.821	1.172	0.1871
0.6	0.8	2	2.1	0.831	1.43	0.45	2.10	0.831	1.431	0.4499

Corollary 1. *When* $a_1 = a_2$ (*and normalized to* 1), *the approximately optimal bundle price expression is* $p_B = 0.86032 + 0.55929(w_1 + w_2)$, *and it achieves more than* 99% *of the optimal profit across* w_1, w_2. *For* $w_1 = w_2 = w$, *an alternative expression is* $p_B = 0.8598 + 1.1155w$.

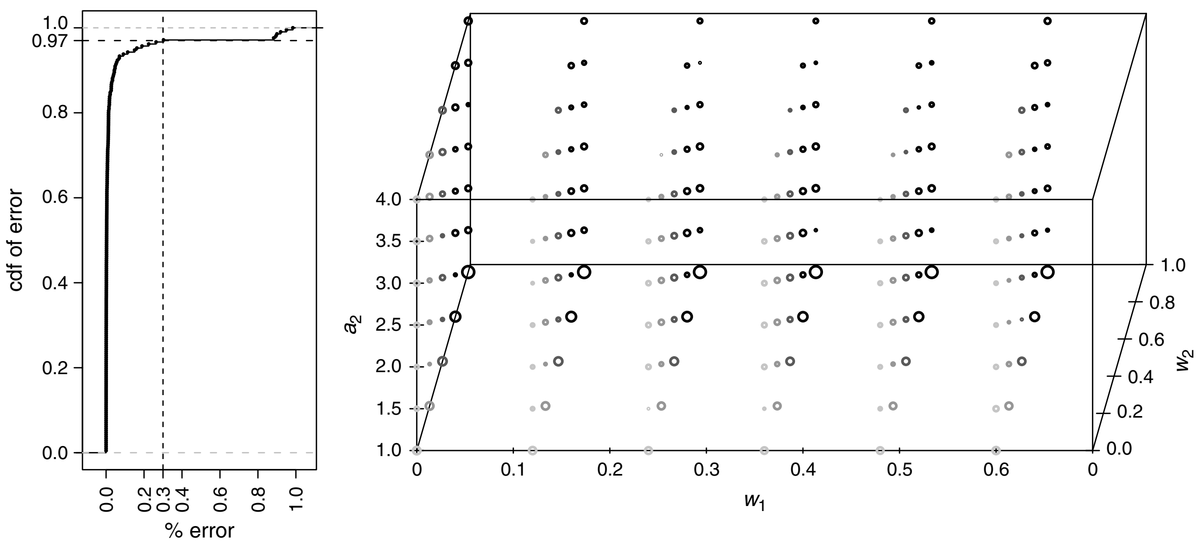
4. Asymmetric Valuations, Zero Costs (Information Goods)

Producers of information goods or services that directly sell to consumers face very low incremental costs for selling additional units. For example, many firms employ their own websites to distribute software drivers or mobile phone apps or to offer online transactional services. For such applications, it is reasonable to assume that marginal costs of production and distribution are negligible; i.e., $w_1 = 0 = w_2$. Many service goods, perishable products, and products with sunk costs of production might also fit this pattern. Applying Proposition 1 and Lemma 1 to the case of zero marginal costs yields the following result.

Corollary 2. *When* $w_1 = w_2 = 0$, *then partial bundling is optimal if and only if* $a_2 \geq 2a_1$, *and then* $p_1 < 2a_1/3$ *while* $p_B < 2a_2/3$. *Component good prices under full bundling are* $p_i = 2a_i/3$.

Corollary 2 implies that $3p_B > 2a_i$ when $a_2 < 2a_1$ (full bundling), whereas otherwise $p_B < 2a_2$ under partial mixed bundling (but $p_B > 2a_1$ because $p_1 = 2a_1/3$).

Figure 2 Accuracy of Approximate Solution for the General Case



Notes. The left panel summarizes the error performance across the entire parameter space as a cumulative distribution. It reveals that about 97% of the problem instances solved have an error below 0.3%, and all have error below 1%. The right panel describes the error level at various points in the three-dimensional parameter space (w_1, w_2, a_2). Bubble size represents the extent of loss from the exact optimal profit for various parameter conditions (missing dots represent negligible error). Color shading is used to help depth perception: lighter gray bubbles represent w_2 values closer to 0, black bubbles closer to 1. Inaccuracy is greater toward higher w_2, and the worst-case error (0.9838)% occurs when w_2 is almost the same as a_2.

Combining all this with Equation (4c), the optimal bundle price is obtained by solving the equation

$$\begin{aligned}4(3p_B(a_1+a_2)-a_1^2-a_2^2-3a_1a_2)\\=(3p_B-2a_1)^2\pm(3p_B-2a_2)\sqrt{(3p_B-2a_2)^2},\end{aligned}\quad(10)$$

where $\sqrt{(3p_B-2a_2)^2}$ yields $+(3p_B-2a_2)$ or $-(3p_B-2a_2)$ under full versus partial bundling, respectively.

Proposition 3. *Under zero marginal costs, the optimal bundle price is*

$$p_B=\begin{cases}\frac{1}{3}(2a_1+2a_2-\sqrt{2a_1a_2}) & \text{if } a_2\in[1,2a_1],\\ \frac{1}{6}(2a_1+3a_2) & \text{if } a_2\geq 2a_1,\end{cases}\quad(11)$$

while the component good prices are $p_1=(2a_1)/3$ *and* $p_2=(2a_2)/3$.

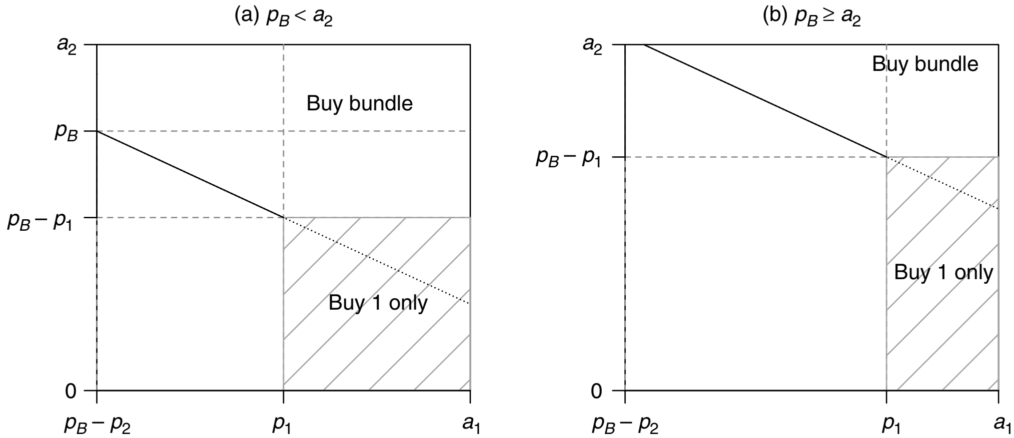
Corollary 3 (Zero Costs and Symmetric Valuations). *When* $w_1=w_2=0$ *and* $a_1=a=a_2$, *then the optimal bundle price is* $p_B=0.8619a$, *while the optimal component prices are* $p_1=p_2=(2/3)a$.

Equation (11) in Proposition 3 matches Eckalbar's (2010) result for mixed bundling under zero costs. The derivation above is more concise and transparent because of the reduction provided by Lemma 1. As with single product pricing under a linear demand function, optimal price is a linear function of the range of valuations for the single good and increases less than linearly for the bundle. When the two component goods are substantially asymmetric in distribution of valuations and good 2 is not sold separately ($a_2\geq 2a_1$; see Corollary 2), Equation (11) is not valid (p_2 exceeds p_B), but this is irrelevant because product 2 is not purchased separately anyway. Dropping this p_2 value from the lemma, the partial mixed bundle solution is fully defined as

$$\begin{aligned}&\{p_1^*=\tfrac{2}{3}a_1,\ p_B^*=\tfrac{1}{6}(2a_1+3a_2)\};\\&\Pi^*=\tfrac{1}{108}a_1(27a_2^2+36a_1a_2-4a_1^2).\end{aligned}\quad(12)$$

It is worth examining how the partial bundling approach might extend to the case where the firm has $N>2$ component goods. Here, suppose that goods $1,\ldots,N-1$ are of similar valuation, whereas good N has substantially high valuations in aggregate. For instance, the market for TV channels may exhibit several channels with relatively similar distribution of valuations, whereas valuations for HBO might cover a range with higher values. Now there are two possibilities. If the combined valuations for the first $N-1$ goods (which should be relatively homogeneous across the consumers because of demand smoothing) were comparable to valuations for the Nth good, then the firm should sell the Nth good, a package of the first $N-1$ goods, and the entire bundle. If, however, even the combined valuations for the first $N-1$ goods were low compared with the Nth good, then asymmetry between these products would drive the firm to sell the sub-bundle of $N-1$ goods and a super-bundle that includes all goods but not separate the Nth good. This strategy is often used in the back end of the TV channel supply chain, where

Figure 3 **Product Sales Under Partial Mixed Bundling**



content networks prebundle their marquee show with several other less popular shows (i.e., the buyer cannot purchase just the highly desired marquee show).

5. Symmetric Valuations, Positive Costs

From the previous section we see that mixed bundling is most relevant when products are reasonably symmetric because when the two products become very different in valuations, then it is no longer optimal to sell both components separately. Moreover, because aggregate valuations for the bundle (goods 1 and 2 combined) far exceed the valuations for the lower-valuation product (1), the dominant share of the firm's profit is determined by the bundle alone, and the selling strategy essentially resembles pure bundling. Hence, in this section I focus on the case where the two products are symmetric in valuations; i.e., $a_1 = a_2 = a$. Without further loss of generality, set $a = 1$; results for $a \neq 1$ can be obtained by multiplying all results in this section with a. Using Equation (4c), the optimality condition for p_B is

$$\begin{aligned}
&2\left(12p_B - 10\left(1 + \frac{w_1 + w_2}{2}\right)\right) \\
&\quad = \left(3p_B - 2\left(1 + \frac{w_1 + w_2}{2}\right)\right) \\
&\qquad \cdot \left(\sqrt{(3p_B - 2 + (w_2 - w_1))^2 + 12w_2(1 - p_B)}\right. \\
&\qquad\quad \left. + \sqrt{(3p_B - 2 + (w_1 - w_2))^2 + 12w_1(1 - p_B)}\right). \quad (13)
\end{aligned}$$

The optimality condition Equation (13) is essentially an eighth-order polynomial in p_B. However, the following reduction procedure can be employed both to reduce the order of the condition and also to pick the global optimal solution (or develop a close analytical approximation to it) among the multiple roots of Equation (13). The process is easiest to demonstrate for symmetric costs ($w_1 = w_2$), as explained below. The general result for asymmetric costs is stated in Proposition 4, with its proof in the appendix.

For symmetric costs, the optimality condition for p_B is

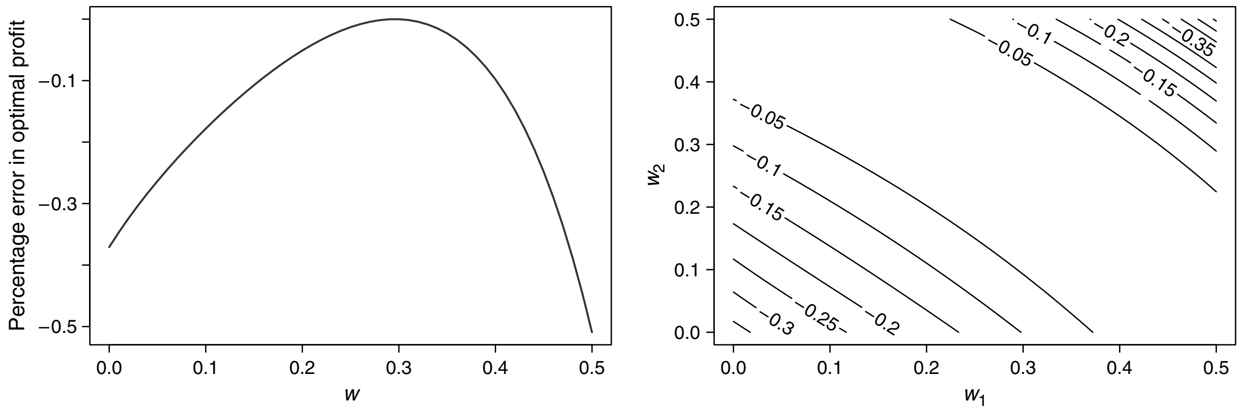
$$\begin{aligned}
&12p_B - 10(1 + w) \\
&\quad = (3p_B - 2(1 + w))\sqrt{(3p_B - 2)^2 + 12w(1 - p_B)}. \quad (14)
\end{aligned}$$

Recall that only the positive sign in front of the square root in Equation (13) is valid.[5] Squaring both sides of the equation, it may be seen that it is a quartic (fourth power) polynomial in f_B. From the Abel–Ruffini theorem, this is the highest order polynomial for which general algebraic solutions exist (i.e., an analytical statement of the solution in terms of parameters of the equation) and can be computed mechanically with only the basic arithmetic operations (Auckly 2007, Faucette 1996). Among the multiple solutions for Equation (14), the correct solution is easily obtained by elimination because each of the other solutions violates one of the feasibility constraints.

I exploit the reduction developed above to generate very tight bounds on the optimal bundle price. Again, I demonstrate the process for symmetric costs ($w_1 = w_2$) and provide the general result in Proposition 4. The first step is to obtain a lower bound. Because the square root term in Equation (14) is positive, the equation's validity requires that the other two terms ($12p_B - 10(1 + w)$ and $3p_B - 2(1 + w)$) either be both positive or both negative. Negativity is ruled out because it implies $p_B < (2/3)(1 + w)$, which is infeasible (because $p_B > p_i$ and $p_i = 2/3$ at

[5] The invalidity of the negative sign can be further confirmed: It would require that $10(1+w) - 12p_B$ and $3p_B - 2(1+w)$ have opposite signs. Both combinations fail.

Figure 4 **Percentage Difference Between Exact Optimal Profit and the Profit Approximated Using the Bounds on Bundle Price**



Notes. The left panel is for the symmetric case, whereas the right panel displays a contour plot of the percentage difference across a range of values for w_1 and w_2. The approximation is least accurate when w_1 and w_2 are far apart, but even then the percentage difference is below 0.4%.

$w = 0$). Positivity requires that $p_B > (5/6)(1+w)$, yielding a lower bound on the optimal price. Second, to obtain an upper bound, combine Equation (14) and Equation (17) using the common (square root) term. Then,

$$6p = 3p_B + 2 - \frac{12p_B - 10(1+w)}{3p_B - 2(1+w)}. \tag{15}$$

Now, applying the constraint $p_1 + p_2 > p_B$ (i.e., for the symmetric case, $2p > p_B$),

$$\begin{aligned} &3p_B(3p_B - 2(1+w)) \\ &\quad < (3p_B+2)(3p_B - 2(1+w)) - (12p_B - 10(1+w)) \\ &\quad \equiv 6p_B < 6(1+w). \end{aligned} \tag{16}$$

Now I have established both lower and upper bounds for the optimal bundle price. The general case of nonsymmetric costs has a remarkably similar solution: simply replace w from the symmetric result with the average cost, $(w_1 + w_2)/2$.

Proposition 4 (Optimal Bundle Price: Bounds and Approximation). *The optimal bundle price p_B lies in $[(5/6)(1 + (w_1 + w_2)/2), (1 + (w_1 + w_2)/2)]$. An approximation for the optimal bundle price is the function representing the average of the lower and upper bounds, specifically, $p_B^* \approx (11/12)(1 + (w_1 + w_2)/2)$.*

Corollary 4 (Symmetric Valuations and Equal Costs). *If $w_1 = w_2 = w$,*

$$p_i = \tfrac{1}{6}\left(3p_B + 2 - \sqrt{(3p_B - 2)^2 + 12w(1-p_B)}\right). \tag{17}$$

The optimal bundle price p_B is in the interval $[(5/6)(1+w), (1+w)]$ and is the unique constraint-satisfying solution of the quartic equation

$$\begin{aligned} &[(3p_B - 2)^2 + 12w(1-p_B)](3p_B - 2(1+w))^2 \\ &\quad = (10 - 12p_B + 10w)^2. \end{aligned} \tag{18}$$

The bounds established above are useful in two ways. First, when an exact numerical solution is needed, they provide a simple procedure for selecting the optimal bundle price among the multiple roots of Equation (13). Second, if an explicit analytical statement of optimal price is needed, the bounds easily yield one as stated in Proposition 4. Moreover, this approximation is highly accurate, which can be verified by solving Equation (13) directly, and is demonstrated in Figure 4. The maximum percentage error over the entire solution space ($w \in [0, 2/3]$), relative to the exact optimal solution, is no more than (1/2)%.

6. Properties of Optimal Mixed Bundling

One advantage of developing analytical solutions for the mixed bundling problem is their amenability to comparative statics and other procedures for gaining additional insight into the nature of the problem. I provide two examples below. The first demonstrates a commonly understood insight about the bundle price—and sensitivity of price to costs—relative to prices under the pure component and pure bundle strategies. The second demonstrates a surprising aspect of the optimal mixed bundle prices: prices for the individual components *fall* as marginal costs *increase* near zero. I develop these results using the symmetric valuations solution of §5 and Corollary 1, which facilitates comparison with previous work (Venkatesh and Kamakura 2003, Eckalbar 2010, Bhargava 2012). For comparison, the optimal prices under pure components are $p_i = (1 + w_1)/2$ and for pure bundling,

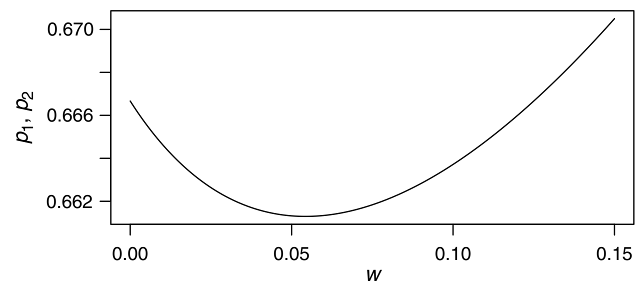
$$p_B = \begin{cases} \tfrac{1}{3}\left(w_1 + w_2 + \sqrt{6 + (w_1 + w_2)^2}\right) & \text{if } w_1 + w_2 < \tfrac{1}{2}, \\ \tfrac{2}{3}(1 + w_1 + w_2) & \text{otherwise.} \end{cases} \tag{19}$$

Proposition 5. *The bundle price under mixed bundling offers a discount relative to the sum of component prices under a pure components selling strategy but is higher than the bundle price under the pure bundling strategy. Conversely, the component prices under the mixed bundling strategy are higher than those under pure component sales.*

The results are consistent with existing bundling literature and economic intuition. The firm offers the bundle at a discount relative to the pure components case to incentivize bundle sales, and this in turn enables it to maintain higher prices for the component goods. Compared with pure bundling, the presence of the component goods in the bundle leads to a higher bundle price. We can also examine the nature of variation in the bundle price as marginal costs change. For ease of comparison with the other pricing strategies, I explain the argument for the symmetric case, with per-component unit cost w, and in terms of the prorated per-component bundle price. For pure components (where optimal price is $(1+w)/2$), component prices increase at a rate of 1/2 per unit of increase in cost. Under the pure bundling strategy, bundle price increases at a faster rate, 2/3 per unit of increase in component prices (the rate is lower for $w < 0.25$). For mixed bundling, applying Corollary 1, the rate of change in bundle price is 0.5593, slower than for the pure bundle but faster than for pure components. This is intuitive because an increase in costs makes bundling less attractive; hence, bundle price increases faster than if the firm were selling components only and slower than the pure bundle case, which experiences a greater penalty from cost increases. Figure 5 displays the prices under the three strategies.

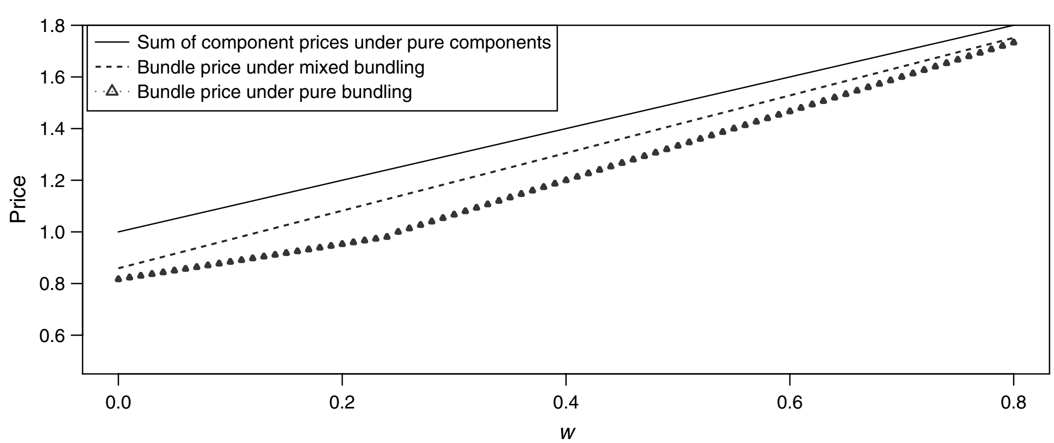
Next, consider the effect of marginal costs of component goods on component prices. Intuitively, a cost increase should lead to higher prices for component products (and, as explained above, for the bundle). But an examination of the analytical form of the optimal component prices (applied to the symmetric case, Equation (17)) indicates a counterintuitive result. At $w=0$, the term $12w(1-p_B)$ vanishes, so that $p_1=p_2=2/3$, and note that $p_B<1$. Hence, as w increases at zero, the term $12w(1-p_B)$ is positive and the term in the square root increases, which implies that component prices will be lower than 2/3. Component prices fall as costs increase, until $w=0.0542$ (this is where the function for p_i attains its minimum), returning back to 2/3 at $w=1/8$ with $p_B=1$ (where $12w(1-p_B)$ again equals 0). This surprising result can be confirmed using the exact computational solutions, and the effect is visualized in Figure 6. It happens because an increase in marginal costs makes bundling less attractive, implying a shift in sales from the bundle to components (and also an overall reduction in sales due to the increase in costs). At w near 0—i.e., at a bundle price less than 1—the bundle demand itself is highly elastic. An increase in bundle price produces a disproportionate loss in sales; therefore, the firm finds it optimal to implement the shift (in part) by lowering the component prices instead. I formalize the finding in the following result.

Figure 6 **Component Prices Drop as w Increases in $[0, 0.0542)$, Returning to the $w=0$ Level at $w=1/8$**



Proposition 6. *Under mixed bundling* (*when $w_1 = w_2 = w$*), *the optimal component prices fall as marginal costs increase, when $w \in [0, 0.0542)$.*

Figure 5 **Bundle Price (Under Mixed Bundling, Dashed Line) as a Function of Marginal Costs, Compared Against the Pure Bundle Price and the Sum of Component Prices Under the Pure Components Strategy**



7. Application: Bundling in a Distribution Channel

Although the bundling literature, including in information goods, has modeled a problem where a single firm bundles its own products, bundling often occurs in a distribution channel where a downstream firm (retailer) adopts a bundling strategy involving products made by separate and multiple upstream firms (manufacturers). This sort of practice is common for information goods such as online services (e.g., Expedia sells bundles of air travel, hotel, and car rental) and media and entertainment products (e.g., Netflix and Comcast sell bundles of entertainment content products made by multiple studios and programming networks). Bhargava (2012) examined the economics of this form of bundling by comparing a component-selling strategy to pure bundling. This problem is analyzed as a two-stage game where in stage 2 the retailer chooses prices (and selling strategy) and in stage 1 the manufacturers choose prices for their component goods. The manufacturers' profit is a function of their prices and sales quantities, which in turn depend on the retailer's market prices. Hence, solution of the stage 1 pricing game requires a closed-form expression for the retailer's prices for component goods and/or the bundle.

Past literature on bundling has not provided a closed-form expression for the mixed bundle selling strategy, and the analysis in Bhargava (2012) is limited to pure bundling. With the apparatus developed in the present paper, it becomes possible to analyze mixed bundling in this distribution setting where the retailer combines component goods from multiple separate manufacturers. I demonstrate this using the case of symmetric valuations ($a_1 = a_2 = 1$), which facilitates comparison with the base case of bundling by a single firm. Let c_1 and c_2 represent the per-unit costs faced by manufacturers 1 and 2, respectively. The manufacturers' profit functions are

$$\pi_1 = (w_1 - c_1)(Q_1 + Q_B), \tag{20a}$$

$$\pi_2 = (w_2 - c_2)(Q_2 + Q_B), \tag{20b}$$

where Q_1, Q_2, and Q_B are functions of the retailer's prices p_1, p_2, and p_B, respectively, as stated in Equation (2), and the retailer's prices are obtained from Lemma 1 and Proposition 2 for the general case of asymmetric valuations (or the other results for the special cases). Manufacturer i sells $\hat{Q}_i = Q_i + Q_B$ units. After substituting all these, and with algebraic rearrangement and simplification, the sales of manufacturer 1 can be expressed as

$$\begin{aligned}\hat{Q}_1 = {} & (0.6164 - 0.5772w_1 - 0.3550w_2) \\ & + 0.0161 f(w_1, w_2) - 0.0394 f(w_2, w_1) \\ & + 0.0188(w_1 + 3.9503w_2) \\ & \times (f(w_1, w_2) + f(w_2, w_1)))\end{aligned} \tag{21}$$

$$\begin{aligned}f(w_1, w_2) = {} & (0.3374 + 0.7876(w_1 + w_2) + 4w_2 \\ & - 3.0809w_1w_2 + 0.4595(w_1^2 + w_2^2))^{1/2}.\end{aligned}$$

The expression for $\hat{Q}_2$ is symmetric. The manufacturers' equilibrium prices w_1^*, w_2^* are obtained by solving a simultaneous game where manufacturers seek to maximize individual profit $\Pi_i(w_1, w_2) = (w_i - c_i)\hat{Q}_i$.

$$\begin{aligned} w_1^* &= \arg\max_{w_1}(\Pi_1(w_1, w_2) = (w_1 - c_1)\hat{Q}_1), \\ w_2^* &= \arg\max_{w_2}(\Pi_2(w_1, w_2) = (w_2 - c_2)\hat{Q}_2)\end{aligned} \tag{22}$$

The nonlinearity of the $\hat{Q}_i$ terms and the expanded profit expressions prevent a direct analytical solution to Equation (22). However, a near-exact linear approximation for the sales quantities is

$$\begin{aligned}\hat{Q}_1 &\approx 0.5539 - 0.5189w_1 - 0.0554w_2, \\ \hat{Q}_2 &\approx 0.5539 - 0.5189w_2 - 0.0554w_1.\end{aligned} \tag{23}$$

With this approximation, the manufacturers' profit functions are quadratic and the pricing game of Equation (22) yields best-response functions that are linear in the two variables. The system of two simultaneous equations therefore has a unique equilibrium solution summarized below. More details regarding this argument are provided in the proof.

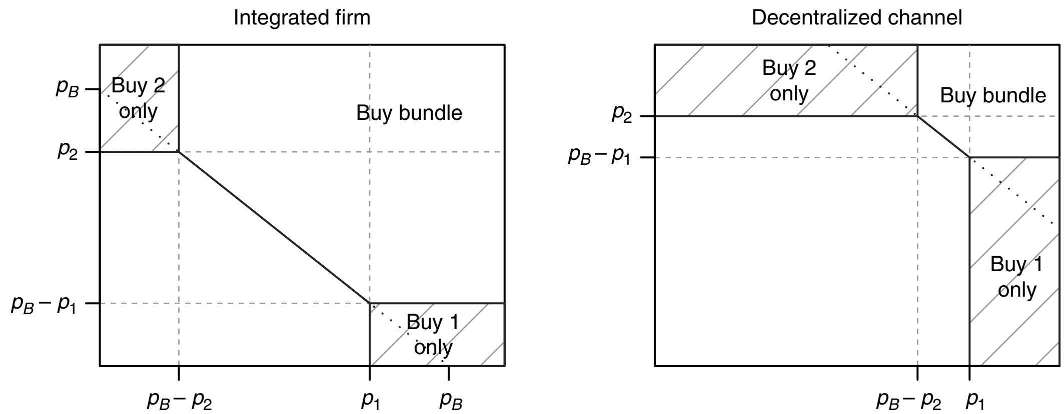
Lemma 2 (Mixed Bundling in Vertical Channel). *The equilibrium solution in the decentralized vertical channel is the p_1^*, p_2^*, p_B^* given in Lemma* 1 *and Proposition* 2, *with w_1^*, w_2^* given by*

$$\begin{aligned} w_1^* &= 0.5067 + 0.5014c_1 - 0.0268c_2, \\ w_2^* &= 0.5067 + 0.5014c_2 - 0.0268c_1.\end{aligned}$$

Corollary 5. *Each manufacturer's best-response price $\Omega_j(w_i)$ is a decreasing function of the other manufacturer's price, specifically, $\Omega_j(w_i) = 0.5337 + 0.5c_j - 0.0534w_i$.*

The equilibrium solution outlined in Lemma 2 helps identify how the channel structure impacts the attractiveness of bundling. One metric is the extent to which mixed bundling increases profit in the vertical channel (over a component-selling strategy) versus for the base case of a single firm. The latter is solved in Venkatesh and Kamakura (2003) and Table 1 in Bhargava (2012), and mixed bundling raises the firm's profit 9.84% from 0.5 (under component selling). For $c_1 = c_2 = 0$, manufacturer prices are $w_1^* = w_2^* = 0.5067$, and plugging this into Corollary 1 yields $p_B^* = 1.4271$, which in turn (using Lemma 1) yields $p_1^* = p_2^* = 0.7778$. Each manufacturer earns 0.1317, the

Figure 7 **Optimal Mixed Bundle Solution for $c_1 = c_2 = 0$**



Note. Price levels and proportion of sales quantities for component goods and bundle.

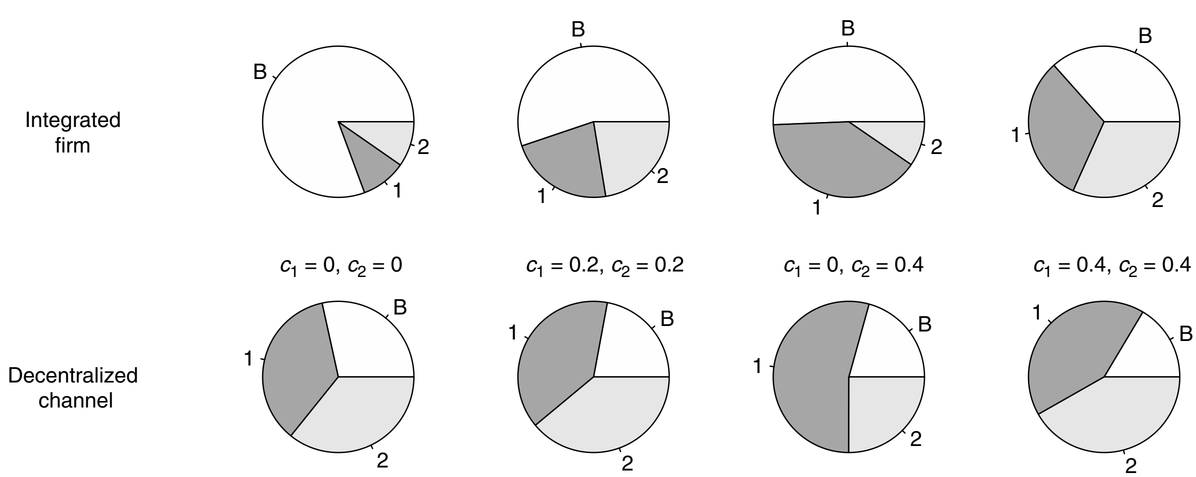
retailer's profit is 0.1248, and the combined firms' profit is 0.3882. Compared with component selling, mixed bundling in the vertical channel delivers only a modest 3.5% increase in industry profits from 0.375 to 0.3882, as against a 9.84% gain for the integrated firm. The results hold true in general for positive marginal costs. For instance, if the costs were $c_1 = c_2 = 0.2$, bundling increases industry profit by only 3.248% under the channel structure, compared with the 6.57% gain for the integrated firm. Hence, the profitability metric indicates that the channel structure negatively impacts the attractiveness of bundling.

Another metric for evaluating the attractiveness of bundling is the fraction of bundle sales $Q_B/(Q_1 + Q_2 + Q_B)$ in equilibrium, which captures the extent to which bundling is part of the selling strategy. The sales levels for the two component products and the bundle (at $c_1 = c_2 = 0$) are 0.1443, 0.1443, and 0.1147, respectively. The bundle price in the vertical channel is significantly higher than for the single firm (1.4271 versus 0.8619), whereas component prices are only minimally higher (0.5067 versus 0.5), so the retailer primarily sells component goods rather than the bundle. This is vividly illustrated in Figure 7, which demonstrates that the share of the bundle as a fraction of overall sales drops substantially from 80.6589% under the integrated-firm structure to 28.4442%. The same property also holds in the more general case of positive marginal costs and is depicted in Figure 8.

Comparisons along both these metrics shed further light on the results in Bhargava (2012), which were derived by comparing the extreme case of pure bundling and pure components. That comparison produced an extreme finding, that channel conflicts between the firms would prevent bundling from emerging as a stable equilibrium outcome. The above comparisons reveal a more nuanced finding, summarized below.

Proposition 7 (Mixed Bundling in the Vertical Channel). *The decentralized vertical channel structure weakens the benefits from bundling. Mixed bundling in the*

Figure 8 **Fraction of Sales of Bundle (B) vs. Component Goods 1 and 2, for Integrated Firm vs. Decentralized Channel, at Several Levels of c_1, c_2**



Note. In each case, the bundled good's share of overall sales is substantially lower in the channel structure than for the integrated firm.

decentralized vertical channel performs more like component selling than bundling. It features a lower fraction of bundle sales and leads to a smaller percentage increase in profit when a retailer bundles component goods from independent manufacturers, relative to when a firm bundles its own products.

8. Conclusion

This paper develops analytical solutions for the mixed bundling pricing problem with two independently valued component goods. The development is notable because this two-good problem is pervasive in the bundling literature, and several authors have consigned the mixed bundling case to the bin of analytical intractability (e.g., Schmalensee 1984 under a bivariate normal distribution and Venkatesh and Kamakura 2003 and others with a bivariate uniform). Many researchers have disallowed mixed bundling and focused on pure bundling (even though mixed bundling is known to be better, usually), whereas others resorted to numerical routines that solve specific instances of the pricing problem computationally. This paper reduces the mixed bundle pricing problem to a univariate nonlinear optimization problem in the bundle price by deriving optimal component prices as algebraic functions of the bundle price. The global maximum of this univariate problem can be identified systematically because the other solutions can be eliminated as infeasible. The reduction also facilitates development of closed-form near-optimal solutions. For goods with zero marginal costs, I develop a complete analytical specification of the optimal bundle and component prices and also specify conditions and optimal design for partial mixed bundling. Under symmetric valuations, I derive a near-exact closed-form specification of the optimal solution.

It is important to acknowledge the limitation that these results were derived under a particular set of assumptions: only two products with independent and additive valuations. However, what is important here is the achievement of analytical tractability of the mixed bundling problem for any formulation at all. I am optimistic that this small step of progress will catalyze researchers to prove similar results for other, and perhaps more general, models of product bundling. The use of analytically derived approximations is another significant aspect of this paper. It demonstrates that, occasionally, modelers would be well-guided to jettison the quest for an exact analytical solution—which may be a fairly complex polynomial or perhaps only an implicit characterization—in favor of an approximation that relinquishes a tiny amount of accuracy in return for a huge gain in elegance and simplicity.

The results of this paper can benefit managers and researchers in various ways. The closed-form approximations of optimal prices can be employed in practical applications of bundle pricing. These analytical expressions can also be employed directly, or with suitable variations, for example, by researchers who study "bundling plus" (e.g., bundling of products that exhibit network externalities). When the mixed bundle pricing problem is subordinate to a broader question (e.g., in two-stage problems where bundling is the second stage, as discussed in §7), such approximations can be very useful in deriving analytical solutions for the main problem. In the broadest terms, this paper contributes to research in bundling and pricing by opening up a category of research problems previously considered analytically intractable.

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Appendix

Proof of Lemma 1. The two first-order conditions (Equation (4)) yield the two simultaneous equations symmetric in i, j:

$$6p_i = \left(3p_B + 2a_i + (w_i - w_j) \pm \sqrt{(3p_B - 2a_i + (w_j - w_i))^2 + 12w_j(a_i - p_B)}\right). \quad (24)$$

I show that minus in front of the square root gives the correct p_1, p_2. The only time the positive root gives the correct value is when it is zero, but then the negative root gives the same result as well. The proof employs contradiction, that the positive square root yields infeasible p_i.

(a) $p_B < a_i$: First, consider the effect of dropping the positive term $12w_j(a_i - p_B)$ from the square root term. Then, (i) when $2a_i + w_i - w_j \geq 3p_B$, the RHS trivially exceeds $6p_B$ (after dropping the positive square root term entirely); the only exception is when RHS exactly equals p_B, which occurs when $2a_i + w_i - w_j = 3p_B$ and $a_i = p_B$ or $w_j = 0$, but then the negative sign will give exactly the same result as well. (ii) When $3p_B > 2a_i + w_i - w_j$, the RHS exceeds $(2a_i + w_i - w_j) + 3p_B + 3p_B - (2a_i + w_i - w_j) = 6p_B$. This implies $p_i \geq p_B$, which is not feasible.

(b) $p_B > a_i$: Then it also follows, because $p_B > w_1 + w_2$, that $3p_B > 2a_i + w_1 + w_2$. Rewrite the square root term as $\sqrt{(3p_B - (2a_i + w_1 + w_2))^2 + 4w_j(a_i - w_i)}$. Hence, (with the positive square root) the RHS exceeds $3p_B + 2a_i + (w_i - w_j) + \sqrt{4w_j(a_i - w_i)}$. When $w_i + w_j > a_i$ (i.e., $w_j > (a_i - w_i)$), then the RHS exceeds $3p_B + 2a_i + (w_i - w_j) + 2(a_i - w_i) = 3p_B + 4a_i - (w_i + w_j)$. Because $p_B > w_i + w_j$, this term strictly exceeds $6a_i$, which is a contradiction to $p_i < a_i$. For the last case where $w_i + w_j < a_i$, we again show that the RHS exceeds $6a_i$. For this we just need to show that the square root term exceeds

$a_i + w_j - w_i$. Equivalently, $(3(p_B - a_i) + (a_i + w_j - w_i))^2 - 12w_j(p_B - a_i)$ must exceed $(a_i + w_j - w_i)^2$. This reduces to $3(p_B - a_i)^2 > 6(p_B - a_i)(w_i + w_j - a_i)$, which is trivially satisfied (RHS negative when $w_i + w_j < a_i$).

Both $p_i > p_B$ and $p_i > a_i$ are infeasible; hence, the positive square root can be eliminated, leaving behind a unique specification of p_1, p_2 in terms of p_B. □

Proof of Proposition 1. First, suppose $w_i > 0$. From Lemma 1, $6p_j$ is of the form $(3p_B + (2a_j + w_j - w_i)) - \sqrt{x}$; hence, $6p_j < (3p_B + (2a_j + w_j - w_i))$. (1) If $3p_B > 2a_j + w_j - w_i$, then trivially $p_j < p_B$, and the partial bundling boundary solution is never achieved. (2) If $3p_B < 2a_j + w_j - w_i$—which implies $p_B < a_j$ because $p_B > w_j - w_i$—then (because $12w_i(a_j - p_B) \geq 0$ inside the $\sqrt{x}$ term), we get $6p_j < (3p_B + (2a_j + w_j - w_i)) - ((2a_j + w_j - w_i) - p_B) = 6p_B$, so that again the constraint is never binding. Finally, (3) suppose $3p_B = 2a_j + w_j - w_i$, then trivially $6p_j < 6p_B$ because $\sqrt{x}$ is nonzero. Combining all these, when both goods have positive marginal costs, the constraint $p_i \leq p_B$ is never binding for either. Hence, a necessary condition for partial mixed bundling is that $w_i = 0$ for at least one of the goods.

Now, suppose $w_i = 0$. I examine what the partial mixed bundling solution would look like if it were optimal. With $w_1 = 0$, Lemma 1 yields $p_j = 3p_B - (2a_j + w_j) - |3p_B - (2a_j + w_j)|$. Therefore, if $3p_B = (2a_j + w_j)$, then $p_j = p_B = (2a_j + w_j)/3$, and this is the point where the constraint first becomes binding. When $3p_B < 2a_j + w_j$, then also we get $p_j = p_B < (2a_j + w_j)/3$. (As shown above, partial bundling will not occur when $3p_B > 2a_j + w_j$.) To summarize, when $w_i = 0$, then $p_j = p_B$ so that the firm only has separate sales of good i and the bundle, and the bundle price does not exceed $(2a_j + w_j)/3$, which also ensures $p_B < a_j$.

Next, I examine under what conditions this partial mixed bundling strategy would be optimal. To do this, I compute $\partial\Pi/\partial p_B$ under the conditions $w_i = 0$, p_i as given in Lemma 1, and $p_j = p_B = (2a_j + w_j)/3$. Setting this term to zero because we want this to be a stationary point and solving, yields that the goods must be substantially asymmetric, i.e., that a_i should be below a threshold A_i, specifically,

$$a_i < A_i = \frac{1}{48a_j}\left(28a_j^2 - 12a_j w_j - w_j^2 - (2a_j + w_j)\sqrt{4a_j^2 + 20a_j w_j + w_j^2}\right).$$

Finally, because by convention we have good 2 as the higher valuation good, the above condition can occur only with good j as 2 and good i as 1. □

Proof of Proposition 2. I obtained optimal solutions for about 25,000 problem instances uniformly distributed in a wide parameter space of interest. The parameter space is $(w_1, w_2 \in ([0, a_1 = 1] \times [0, a_2])$ combined with varying a_2 in $[1, A]$; $A = 4$. Without loss of generality, the parameter a_1 was normalized to 1. This is equivalent to dividing all the true values of the remaining parameters by a_1 so that the optimal solution that is computed truly represents p_B/a_1. The marginal costs w_1, w_2 range within $[0, a_1]$ and $[0, a_2]$, respectively. Varying a_2 in $[1, 4]$ covers a sufficient region of interest because bundling ceases to be important once the products become too asymmetric in valuations. I cross-checked the results by computing them using the four alternative methods mentioned earlier. Using this library of exact solutions, I estimated a closed-form specification of the optimal solution using multiple linear regression, estimating p_B as a function of the w_1, w_2, a_1, a_2. This yields $p_B = 0.3516 + 0.5724w_1 + 0.5695w_2 + 0.4889a_2 + 0.0054w_1a_2 - 0.0201w_2a_2$. Finally, rescaling each term (i.e., replace p_B, w_1, w_2, a_2 in the estimated equation with itself divided by a_1) yields the solution stated in Proposition 2. □

Proof of Proposition 3. Apply Lemma 1 at $w_1 = w_2 = 0$, which trivially yields $p_1 = 2a_1/3$ and $p_2 = 2a_1/3$ (unless $p_B < 2a_i/3$ with $p_i = p_B$, but then the price $p_i = 2a_i/3$ also produces the same result of zero separate sales). For p_B, when $a_2 > 2a_1$, employ the negative sign for $(3p_B - 2a_2)^2$ in the RHS for Equation (10), which then yields the unique solution $p_B = (1/6)(2a_1 + 3a_2)$. When $a_2 \leq 2a_1$, employ the positive sign, which yields a pair of solutions, $p_B = (1/3)(2a_1 + 2a_2 \pm \sqrt{2a_1a_2})$. Among these, the term with the positive sign fails because then $p_B > p_1 + p_2 = (2/3)(a_1 + a_2)$ and implies no bundle sales. □

Proof of Proposition 4. Differentiating Equation (3) (with $a_1 = a_2 = 1$) with respect to p_B yields

$$(p_1 - w_1)(1 - p_1) + (p_2 - w_1)(1 - p_2) + \left((1 - p_B + p_1)(1 - p_B + p_2) - \frac{(p_1 + p_2 - p_B)^2}{2}\right) + (p_B - p_1 - p_2)(-2 + p_B).$$

Replace p_1, p_2 using Lemma 1 and simplify the equation, which yields the optimality condition for p_B,

$$2\left(10 - 12p_B + 10\frac{w_1 + w_2}{2}\right) = \left(3p_B - 2\left(1 + \frac{w_1 + w_2}{2}\right)\right) \times \left[\pm\sqrt{(3p_B - 2 + w_2 - w_1)^2 + 12w_2(1 - p_B)} + \sqrt{(3p_B - 2 + w_1 - w_2)^2 + 12w_1(1 - p_B)}\right]. \quad (25)$$

Now, the term in the large brackets can be either positive or negative. I eliminate the positive possibility, for it would require that

$$\operatorname{sign}\left[\left(10\left(1 + \frac{w_1 + w_2}{2}\right) - 12p_B\right)\right] = \operatorname{sign}\left[\left(3p_B - 2\left(1 + \frac{w_1 + w_2}{2}\right)\right)\right]$$

for which there are two possibilities, both of which fail. The first, that both signs are negative, would imply that $12p_B > 10(1 + (w_1 + w_2)/2)$ and $12p_B < 8(1 + (w_1 + w_2)/2)$, which is impossible. The second, both positive signs, would imply that $12p_B < 10(1 + (w_1 + w_2)/2)$, but we know that this is false, e.g., at $w_1 = 0, w_2 = 0$. Hence, the term in the large brackets must be negative. We can rewrite the optimality condition for p_B as

$$\begin{aligned}2\left(12p_B-10\left(1+\frac{w_1+w_2}{2}\right)\right)&\\=\left(3p_B-2\left(1+\frac{w_1+w_2}{2}\right)\right)&\\ \times\left|\sqrt{(2-3p_B+w_1-w_2)^2+12w_2(1-p_B)}\right.&\\ \left.+\sqrt{(2-3p_B+w_2-w_1)^2+12w_1(1-p_B)}\right|.&\quad(26)\end{aligned}$$

For the above equation to be valid (given the absolute term), the remaining two terms must either be both positive or both negative. Negativity is ruled out because $(3p_B-2(1+(w_1+w_2)/2))<0$ would force a contradiction at $w_1=0$, $w_2=0$ (where $p_1=p_2=2/3$ and $p_B=0.8619$). Positivity provides us with a lower bound for p_B, $p_B>(5/6)(1+(w_1+w_2)/2)$.

Next, to obtain an upper bound, combine this equation with the two equations (Equations 5(a) and 5(b)) from Lemma 1 using the common square root terms. We get

$$6(p_1+p_2)=2(3p_B+2)-2\left(\frac{12p_B-10(1+(w_1+w_2)/2)}{3p_B-2(1+(w_1+w_2)/2)}\right).$$

Apply the constraint $p_B<p_1+p_2$, then

$$2\left(\frac{12p_B-10(1+(w_1+w_2)/2)}{3p_B-2(1+(w_1+w_2)/2)}\right)<4,$$

which yields the upper bound, $p_B<(1+(w_1+w_2)/2)$. □

Proof of Proposition 5. The sum of component prices under the pure component strategy is $1+(w_1+w_2)/2$. Comparing this with the bundle price under mixed bundling, the bundle offers a discount if and only if $w_1+w_2<0.14/0.06$, which is easily seen to be true. Comparing with the pure bundle price (and first using the solution $(2/3)(1+w_1+w_2)$, which is valid when $w_1+w_2\geq 1/2$), the mixed bundle price is higher if $w_1+w_2<0.19/0.11$, which is true in the region of interest ($w_1, w_2<1/2$). For $w_1+w_2<1/2$, the pure bundle price is bounded from above by $(1/3)(w_1+w_2+2.5)$, and the mixed bundle price exceeds this for all $w_1+w_2>0$.

For component prices, I explain the result for the equal costs case to make the comparison meaningful. Substitute p_B into Equation (17), the component price under mixed bundling can be rewritten as $p=0.7635+0.5598w-0.2446\sqrt{(2.6495-w)(0.0591+w)}$. Let $\delta=p-(1+w)/2$ (the difference in component prices between the two selling strategies). Then it is easily seen that $\delta>0$ at $w=0$ and the equation $\delta=0$ has no solution in $[0,1/2]$ (δ does not reach until $w=0.9472$, a region not of interest). Hence, the component prices under mixed bundling are always higher than under component sales. □

Proof of Lemma 2. I start the stage 1 analysis by computing the sales quantities for each manufacturer. These terms ($\hat{Q}_1=Q_1+Q_B$ and $\hat{Q}_2=Q_2+Q_B$, respectively) are computed by substituting into Equation (2) the retailer's pricing rules given in Lemma 1 and Proposition 2. Algebraic rearrangement and simplification yields Equation (21). Because these $\hat{Q}_i$ terms are highly nonlinear, the two-player game in Equation (22) precludes direct analytical solution. However, the Equation (21) expressions for $\hat{Q}_i$ can be replaced with a highly accurate linear approximation, either (i) a Taylor series approximation of the expressions or (ii) a linear regression estimate using the true values of $\hat{Q}_i$ over a vector of values for w_1 and w_2, respectively. The Taylor expression is computed around ($w_1=0.5$, $w_2=0.5$), which are the equilibrium prices for the special case of zero marginal costs, yielding the functions

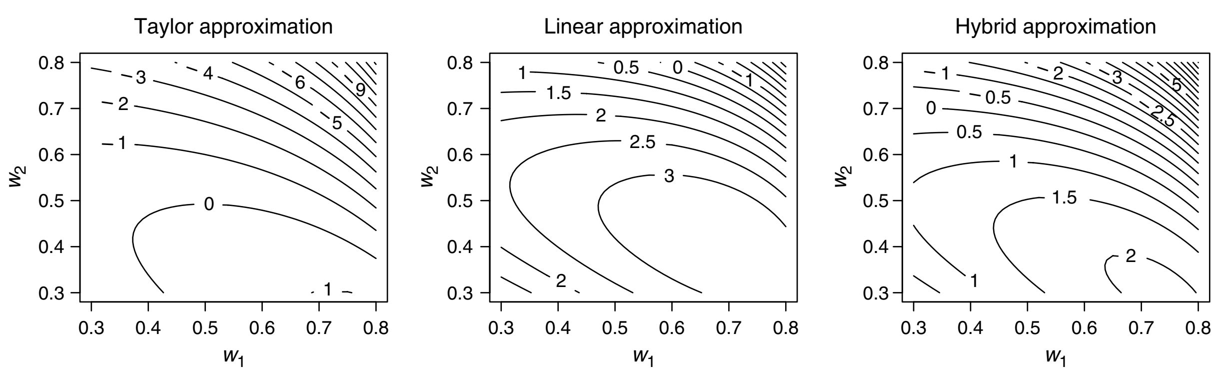
$$\begin{aligned}\hat{Q}_1&\approx 0.5510-0.5143w_1-0.0624w_2,\\ \hat{Q}_2&\approx 0.5510-0.5143w_2-0.0624w_1.\end{aligned}\quad(27)$$

The alternative approximation via linear regression yields

$$\begin{aligned}\hat{Q}_1&\approx 0.5567-0.5234w_1-0.0484w_2,\\ \hat{Q}_2&\approx 0.5567-0.5234w_2-0.0484w_1.\end{aligned}\quad(28)$$

The accuracy of each of these approximations is depicted in Figure A.1, which depicts the percentage deviation from the actual value of $\hat{Q}_1$. Note that the equilibrium prices under positive marginal costs will generally be above 0.5 (which corresponds to zero costs). The Taylor approximation for $\hat{Q}_1$ works best when w_2 is small (below 0.5; a symmetric statement holds for the approximation for $\hat{Q}_2$), and the linear approximation works well for higher values

Figure A.1 Percentage Accuracy of Alternative Approximations of Sales Quantities



Note. Each panel displays a contour map of the percentage gap between the approximation and the exact value for different levels of (w_1, w_2).

of w_2. I employ an hybrid approximation obtained by averaging the two approximation functions. This yields the $\hat{Q}_1, \hat{Q}_2$ functions stated in Equation (23), and the hybrid approximation is fairly robust (within 3% of the exact value throughout the region). The alternative approximations are even closer in terms of the equilibrium values they produce for w_1^*, w_2^* after solving the simultaneous pricing game.

Substituting the linear approximations for $\hat{Q}_1$ and $\hat{Q}_2$ into the manufacturers' profit functions, Equation (22) can be solved analytically to produce the Nash equilibrium prices w_1^*, w_2^*. Manufacturer i's best-response function to j's price and in anticipation of the retailer's pricing rules is

$$\begin{aligned}\Omega_1(w_2) &= 0.5337 + 0.5c_1 - 0.0534w_2,\\ \Omega_2(w_1) &= 0.5337 + 0.5c_2 - 0.0534w_1\end{aligned} \tag{29}$$

Solving the pair of simultaneous equations yields the equilibrium price functions. $\square$

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CORRECTION

In this article, "Mixed Bundling of Two Independently Valued Goods" by Hemant K. Bhargava (first published in *Articles in Advance*, February 15, 2013, *Management Science*, DOI:10.1287/mnsc.1120.1663), Proposition 3 has been corrected to read as follows: *while the component good prices are* $p_1 = (2a_1)/3$ *and* $p_2 = (2a_2)/3$.