

Midterm Exam 1

April 7, 2023

Name: _____ Student No.: _____ Seat No.: _____

1. (20%) Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} n^{-\sin^2}$. (b) $\sum_{n=2}^{\infty} n \tan \frac{1}{n}$. (c) $\sum_{n=3}^{\infty} n e^{-n^2}$. (d) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$.

2. (10%) Given $f(x) = e^{x^3}$. Find $f^{(4n)}(0)$.

3. (10%) Find the limit of the sequence $\left\{ \sqrt{7}, \sqrt{7\sqrt{7}}, \sqrt{7\sqrt{7\sqrt{7}}}, \dots \right\}$.

4. (10%) Find the interval of convergence for $\sum_{n=1}^{\infty} \left[\ln \left(1 + \frac{5}{\sqrt{n}} \right) \right] x^n$.

5. (20%) Given two positive sequences $\{a_n\}$ and $\{b_n\}$, i.e., $a_n, b_n > 0$ for any $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} a_n$ is convergent.

(a) Prove or disprove that if $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(b) Prove or disprove that $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent.

(c) Prove or disprove that if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ is convergent, then $\sum_{n=1}^{\infty} b_n$ is convergent.

(d) Prove or disprove that $\sum_{n=1}^{\infty} \ln(1 + a_n)$ is convergent.

6. (10%) Find the first three nonzero terms of the Maclaurin series of $g(x) = \ln \frac{2+x}{2-x}$.

7. (20%) Assume that $\sum_{n=1}^{\infty} a_n x^n$ is convergent at $x = 2$ and divergent at $x = 3$.

(a) Prove or disprove that $\lim_{n \rightarrow \infty} a_n = 0$.

(b) Prove or disprove that $\sum_{n=1}^{\infty} a_n 2^{-n}$ is convergent.

(c) Prove or disprove that $\sum_{n=1}^{\infty} a_n 4^n$ is convergent.

(d) Prove or disprove that $\sum_{n=1}^{\infty} a_n n x^{n-1}$ is convergent at $x = 4$.