On Levi-Civita pseudotensor

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In this paper we present properties of Levi-Civita pseudotensor. Let us firstly denote n as dimention of space. We define covariant Levi-Civita pseudorensor as follows

$$\varepsilon^{i_1 \dots i_n} = \begin{cases} 1 & \text{if } (i_1,\ \dots, i_n) \text{ is an even permutation} \\ -1 & \text{if } (i_1,\ \dots, i_n) \text{ is an odd permutation} \\ 0 & \text{if } i_p = i_q \text{ for some } p,\ q \end{cases} \tag{1}$$

and consistently contravaraint

$$\varepsilon_{i_1 \dots i_n} = g_{i_1 j_1} \dots g_{i_n j_n} \varepsilon^{j_1 \dots j_n}, \tag{2}$$

where g_{ij} is metric tensor ($g^{ik} g_{kj} = \delta^i_j$) and $\sqrt{-g} = 1$. One can easly see that we can, using equality (2), define determinant of matrix M

$$\det(M) \varepsilon_{i_1 \dots i_n} = M_{i_1 j_1} \dots M_{i_n j_n} \varepsilon^{j_1 \dots j_n}. \tag{3}$$

It is worth mentioning that in curved space unit antisimetric pseudotensor $E^{i_1 \dots i_n}$ will be definied as follows

$$E^{i_1 \dots i_n} = \frac{1}{\sqrt{-g}} e^{i_1 \dots i_n} \tag{4}$$

$$E_{i_1 \dots i_n} = \sqrt{-g} e_{i_1 \dots i_n} \tag{5}$$

Now we will present usefull properties of Levi-Civita psedotensor.

$$\varepsilon^{i_1 \dots i_n} \varepsilon_{i_1 \dots i_n} = n! \tag{6}$$

$$\varepsilon^{i_1 i_2 \dots i_n} \varepsilon_{j_1 i_2 \dots i_n} = (n-1)!\ \delta^{i_1}_{j_1} \tag{7}$$

2 dimensions

$$(\varepsilon^{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8}$$

$$(\varepsilon_{ij}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

$$\varepsilon^{ij}\varepsilon_{pq} = \delta^i_p\delta^j_q - \delta^i_q\delta^j_p \tag{10}$$

$$\varepsilon^{ij}\varepsilon_{jk} = \delta^i_k \tag{11}$$

$$\varepsilon^{ij}\varepsilon_{ij} = 2 \tag{12}$$

3 dimensions

$$\varepsilon^{ijk}\varepsilon_{pqr} = \delta^i_p\delta^j_q\delta^k_r + \delta^i_q\delta^j_r\delta^k_p + \delta^i_r\delta^j_p\delta^k_q - \delta^i_p\delta^j_r\delta^k_q - \delta^i_r\delta^j_q\delta^k_p - \delta^i_q\delta^j_p\delta^k_r \tag{13}$$

$$\varepsilon^{ijk}\varepsilon_{ipq} = \delta^j_p\delta^k_q - \delta^j_q\delta^k_p \tag{14}$$

$$\varepsilon^{ipq}\varepsilon_{jpq} = 2\delta^i_j \tag{15}$$

$$\varepsilon^{ijk}\varepsilon_{ijk} = 6 \tag{16}$$