

# Krótkie wprowadzenie do metody FDTD (cz. 2(2D))

# Równania Maxwella

$$\bullet \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} (\nabla \times \mathbf{H} - \mathbf{J})$$

$$\bullet \frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E}$$

$$\mathbf{J} = \mathbf{J}_{source} + \sigma \mathbf{E}$$

Mikroskopowe  
prawo Ohma

Źródło prądu

$$\bullet \frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - J_x - \sigma E_x \right)$$

$$\bullet \frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} - J_y - \sigma E_y \right)$$

$$\bullet \frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_z - \sigma E_z \right)$$

$$\bullet \frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right)$$

$$\bullet \frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right)$$

$$\bullet \frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right)$$

# Równania Maxwella – ograniczenie do 2D

Zakładamy niezmienniczość ze względu na translację wzdłuż osi  $z$  (pole EM nie zmienia się w kierunku  $z$ )

$$\begin{aligned} \bullet \frac{\partial E_x}{\partial t} &= \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial y} - \cancel{\frac{\partial H_y}{\partial z}} - J_x - \sigma E_x \right) & \bullet \frac{\partial H_x}{\partial t} &= \frac{1}{\mu} \left( \cancel{\frac{\partial E_y}{\partial z}} - \frac{\partial E_z}{\partial y} \right) \\ \bullet \frac{\partial E_y}{\partial t} &= \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial x} - \cancel{\frac{\partial H_x}{\partial z}} - J_y - \sigma E_y \right) & \bullet \frac{\partial H_y}{\partial t} &= \frac{1}{\mu} \left( \frac{\partial E_x}{\partial z} - \cancel{\frac{\partial E_z}{\partial x}} \right) \\ \bullet \frac{\partial E_z}{\partial t} &= \frac{1}{\varepsilon} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_z - \sigma E_z \right) & \bullet \frac{\partial H_z}{\partial t} &= \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) \end{aligned}$$

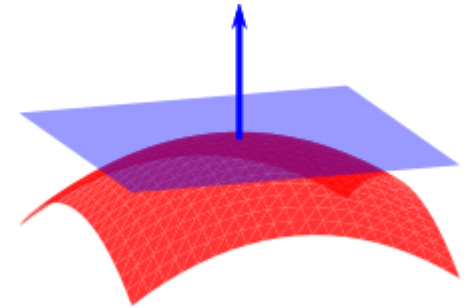
# Równania Maxwella – ograniczenie do 2D

Zakładamy niezmienniczość ze względu na translację wzdłuż osi  $z$

Odpowiednie grupowania daje dwa układy równań całkowicie **niezależne** i opisujące różne zjawiska.

Na przykład weźmy mały fragment powierzchni Ziemi:

- założymy niezmienne pole EM wraz z wysokością nad powierzchnią Ziemi
- używamy jednego z układów równań – niebieskiego lub zielonego
- jeśli założymy prąd płynący wzdłuż osi  $z$  to używamy zielonego

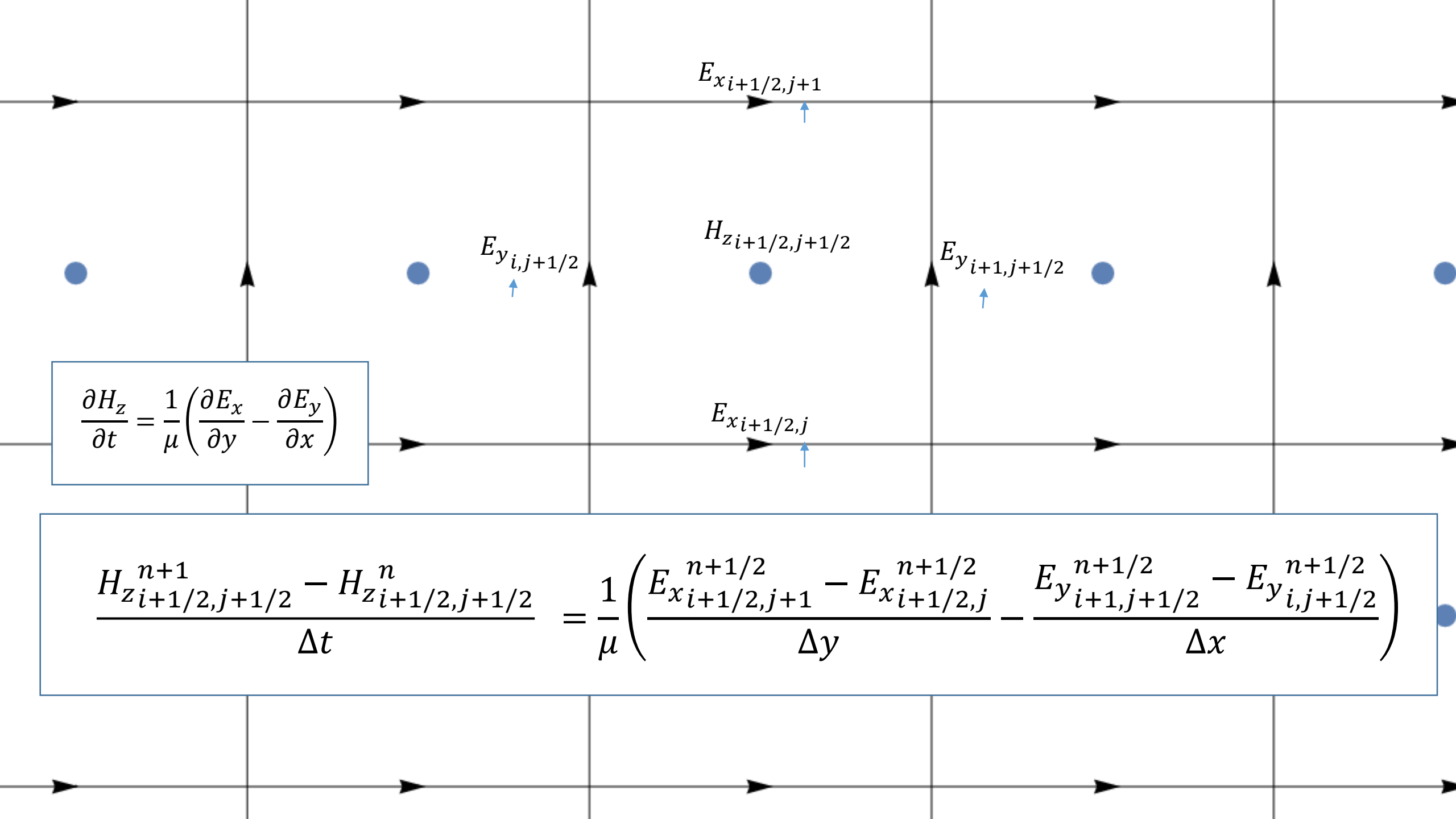


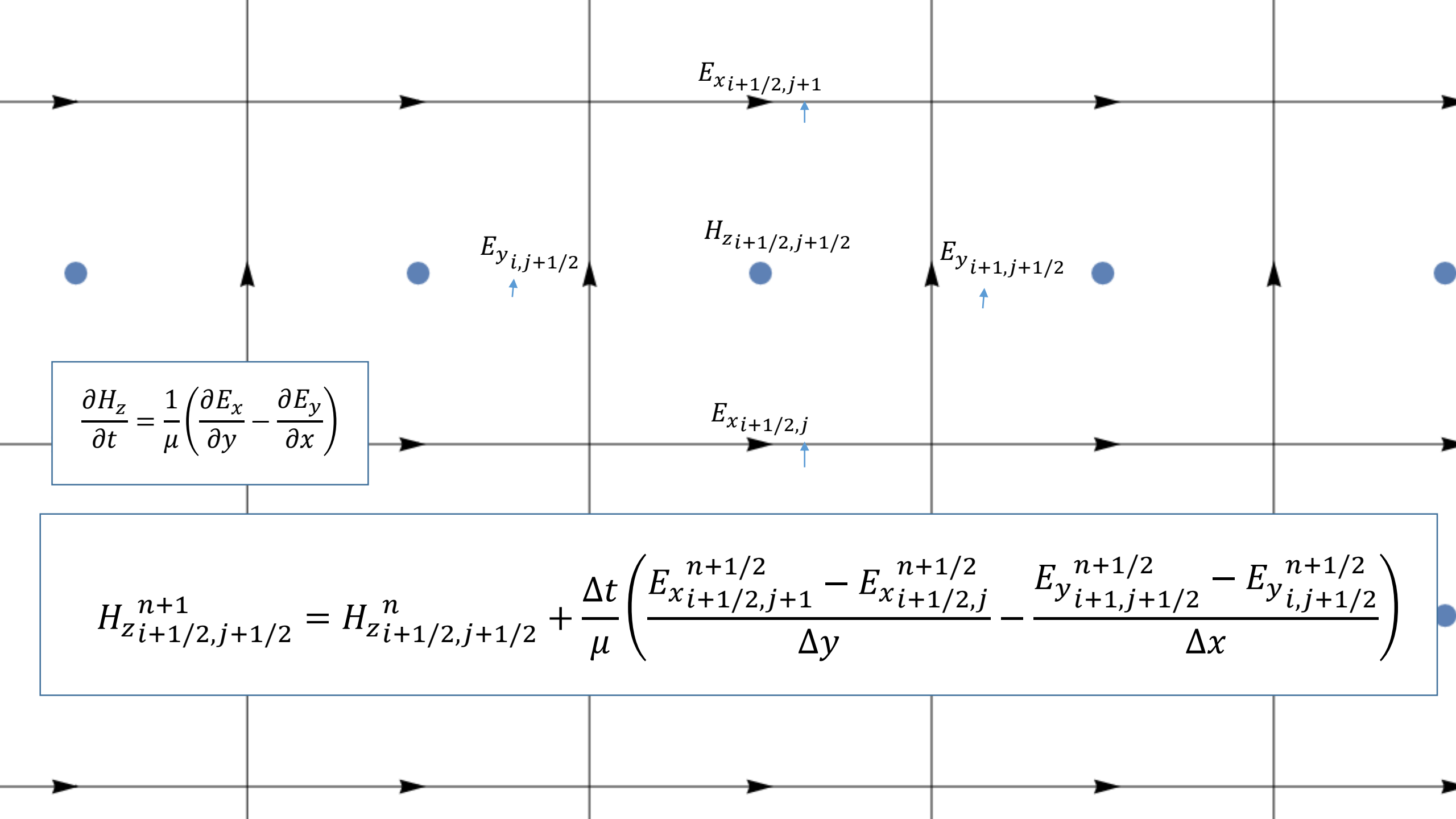
$$\begin{aligned} \bullet \frac{\partial E_x}{\partial t} &= \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial y} - \cancel{\frac{\partial H_y}{\partial z}} - J_x - \sigma E_x \right) \\ \bullet \frac{\partial E_y}{\partial t} &= \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial x} - \cancel{\frac{\partial H_x}{\partial z}} - J_y - \sigma E_y \right) \end{aligned}$$

$$\bullet \frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_z - \sigma E_z \right)$$

$$\begin{aligned} \bullet \frac{\partial H_x}{\partial t} &= \frac{1}{\mu} \left( \cancel{\frac{\partial E_y}{\partial z}} - \frac{\partial E_z}{\partial y} \right) \\ \bullet \frac{\partial H_y}{\partial t} &= \frac{1}{\mu} \left( \cancel{\frac{\partial E_x}{\partial z}} - \frac{\partial E_z}{\partial x} \right) \end{aligned}$$

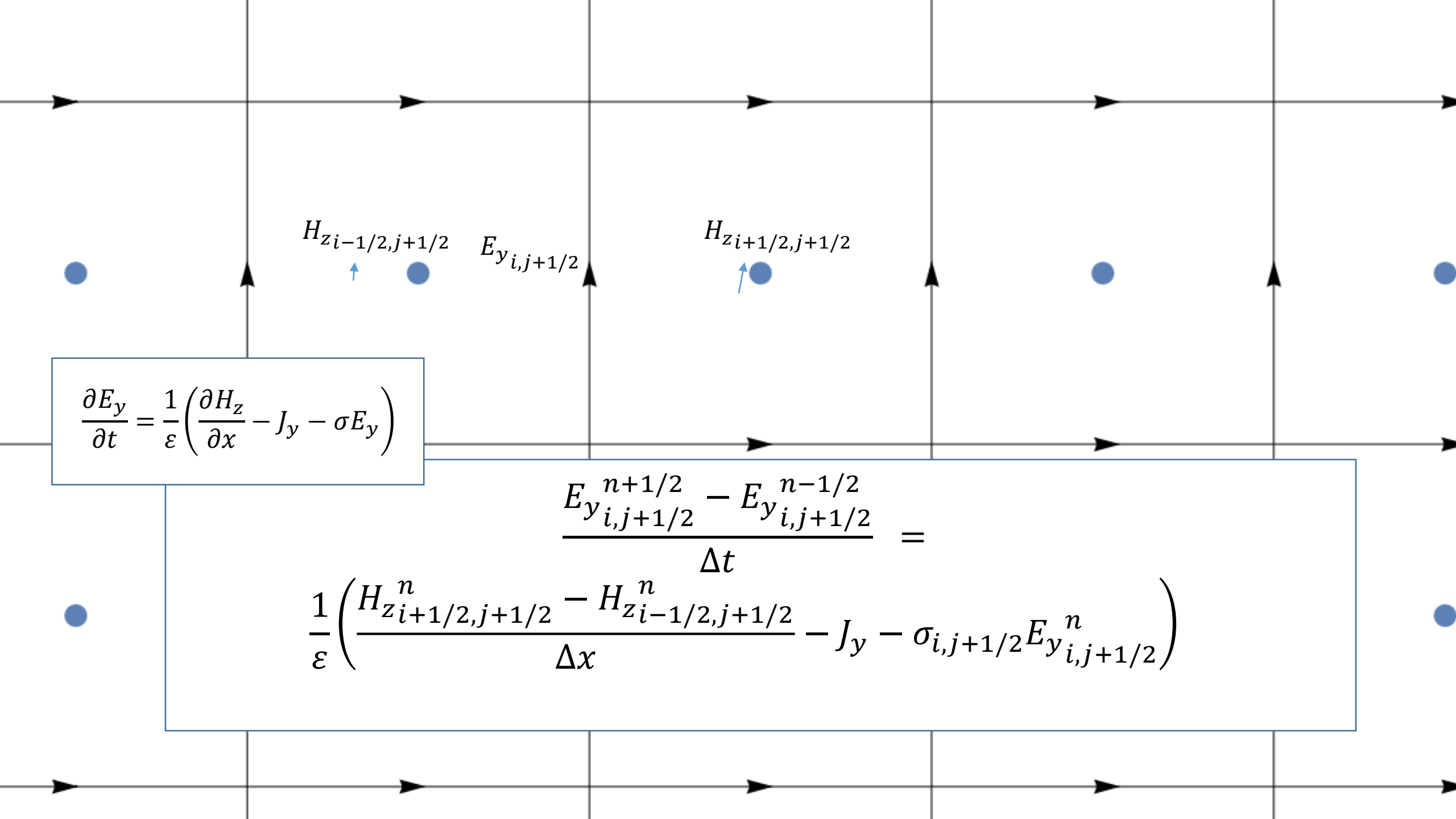
$$\bullet \frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right)$$





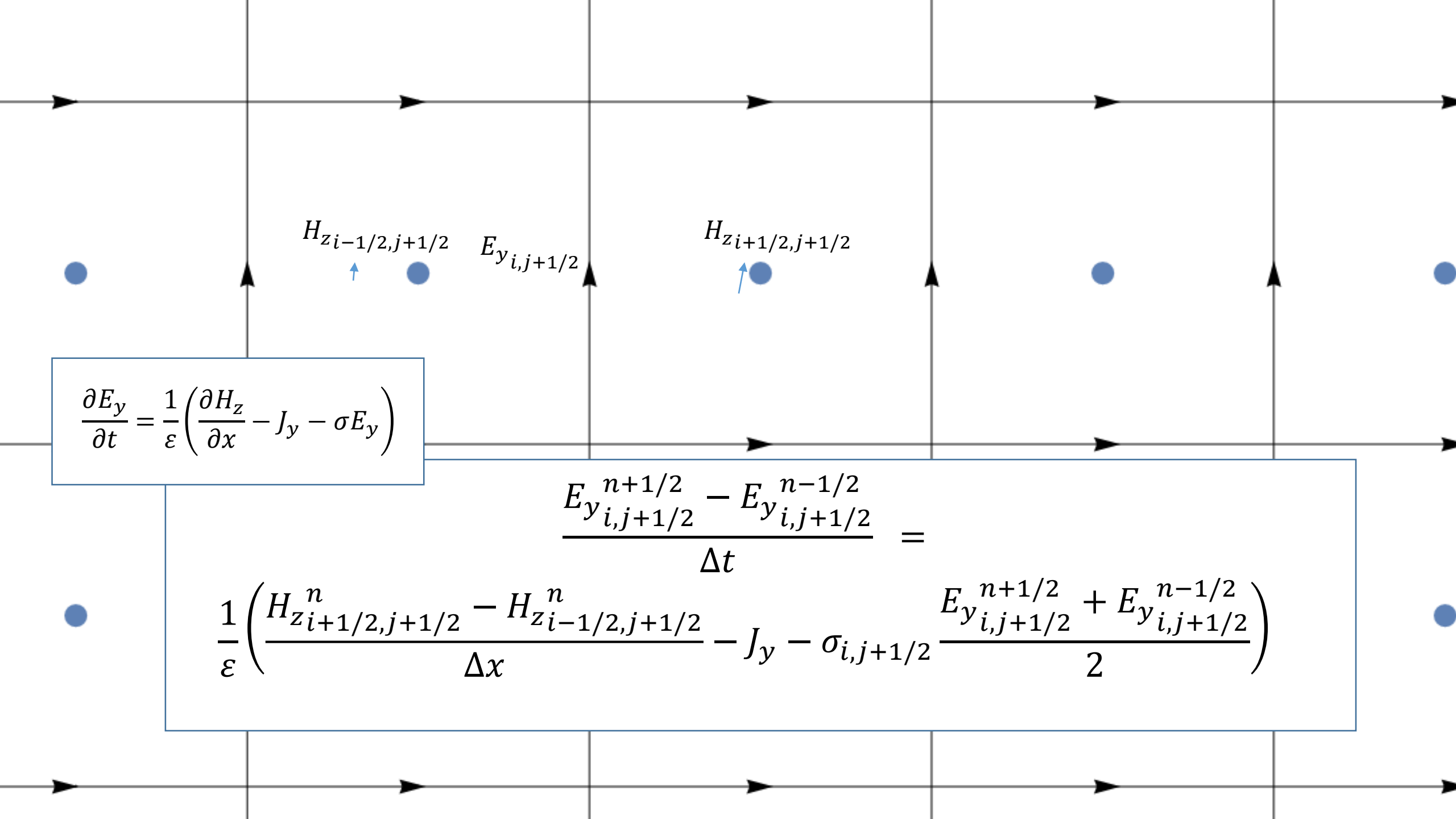
$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right)$$

$$H_{zi+1/2,j+1/2}^{n+1} = H_{zi+1/2,j+1/2}^n + \frac{\Delta t}{\mu} \left( \frac{E_{xi+1/2,j+1}^{n+1/2} - E_{xi+1/2,j}^{n+1/2}}{\Delta y} - \frac{E_{y_{i+1,j+1/2}}^{n+1/2} - E_{y_{i,j+1/2}}^{n+1/2}}{\Delta x} \right)$$



$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial x} - J_y - \sigma E_y \right)$$

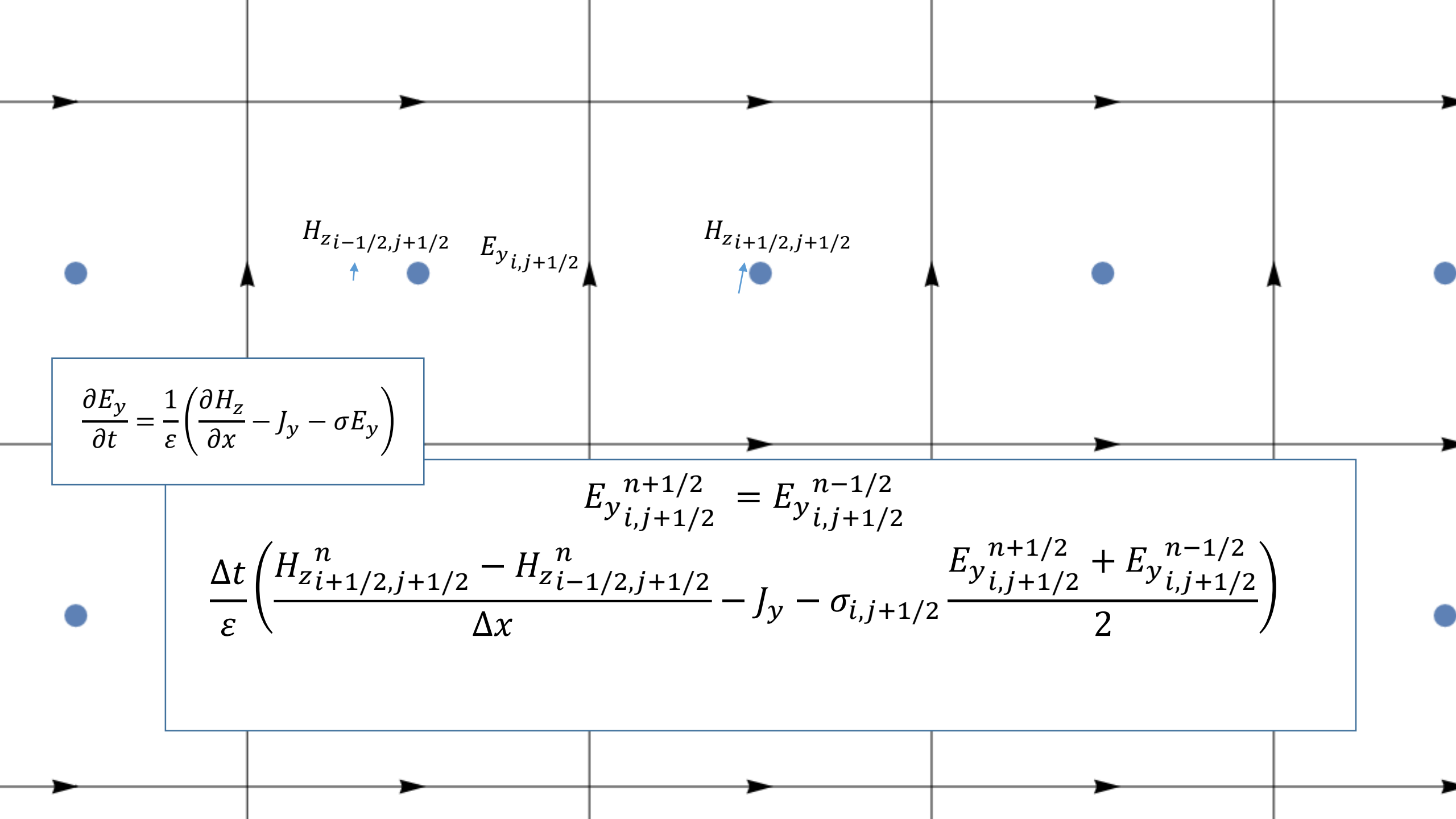
$$\frac{E_{y,i,j+1/2}^{n+1/2} - E_{y,i,j+1/2}^{n-1/2}}{\Delta t} = \frac{1}{\varepsilon} \left( \frac{H_{zi+1/2,j+1/2}^n - H_{zi-1/2,j+1/2}^n}{\Delta x} - J_y - \sigma_{i,j+1/2} E_{y,i,j+1/2}^n \right)$$



$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial x} - J_y - \sigma E_y \right)$$

$$\frac{E_{y,i,j+1/2}^{n+1/2} - E_{y,i,j+1/2}^{n-1/2}}{\Delta t} = \frac{1}{\varepsilon} \left( \frac{H_{zi+1/2,j+1/2}^n - H_{zi-1/2,j+1/2}^n}{\Delta x} - J_y - \sigma_{i,j+1/2} \frac{E_{y,i,j+1/2}^{n+1/2} + E_{y,i,j+1/2}^{n-1/2}}{2} \right)$$

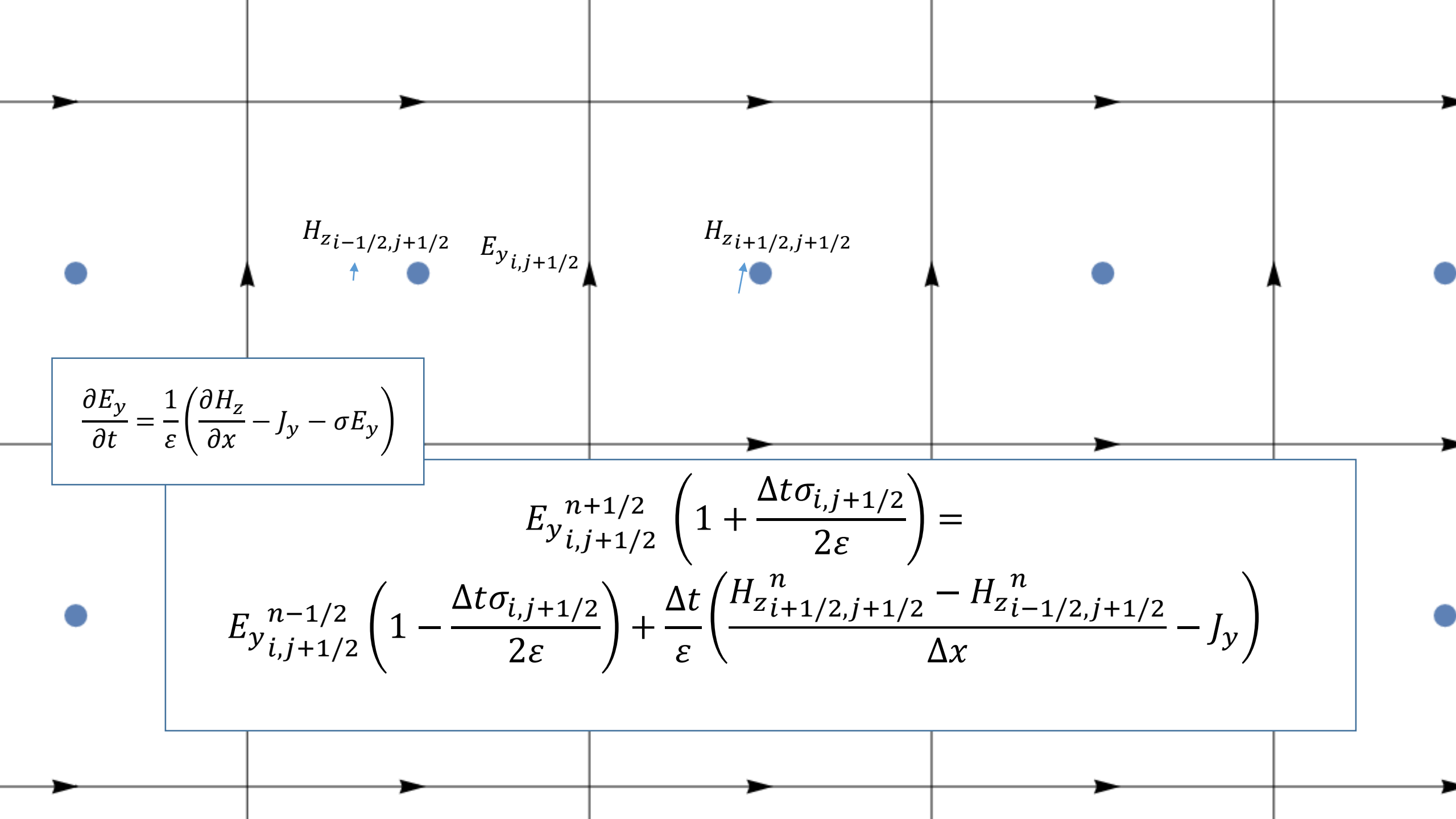




$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial x} - J_y - \sigma E_y \right)$$

$$E_y^{n+1/2} = E_y^{n-1/2}$$

$$\frac{\Delta t}{\varepsilon} \left( \frac{H_{zi+1/2,j+1/2}^n - H_{zi-1/2,j+1/2}^n}{\Delta x} - J_y - \sigma_{i,j+1/2} \frac{E_y^{n+1/2} + E_y^{n-1/2}}{2} \right)$$



$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial x} - J_y - \sigma E_y \right)$$

$$E_{y i,j+1/2}^{n+1/2} \left( 1 + \frac{\Delta t \sigma_{i,j+1/2}}{2\varepsilon} \right) = E_{y i,j+1/2}^{n-1/2} \left( 1 - \frac{\Delta t \sigma_{i,j+1/2}}{2\varepsilon} \right) + \frac{\Delta t}{\varepsilon} \left( \frac{H_{zi+1/2,j+1/2}^n - H_{zi-1/2,j+1/2}^n}{\Delta x} - J_y \right)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_z}{\partial x} - J_y - \sigma E_y \right)$$

$$E_{xi+1/2,j}^{n+1/2} \left( 1 + \frac{\Delta t \sigma_{i,j+1/2}}{2\varepsilon} \right) =$$

$$E_{xi+1/2,j}^{n-1/2} \left( 1 - \frac{\Delta t \sigma_{i,j+1/2}}{2\varepsilon} \right)$$

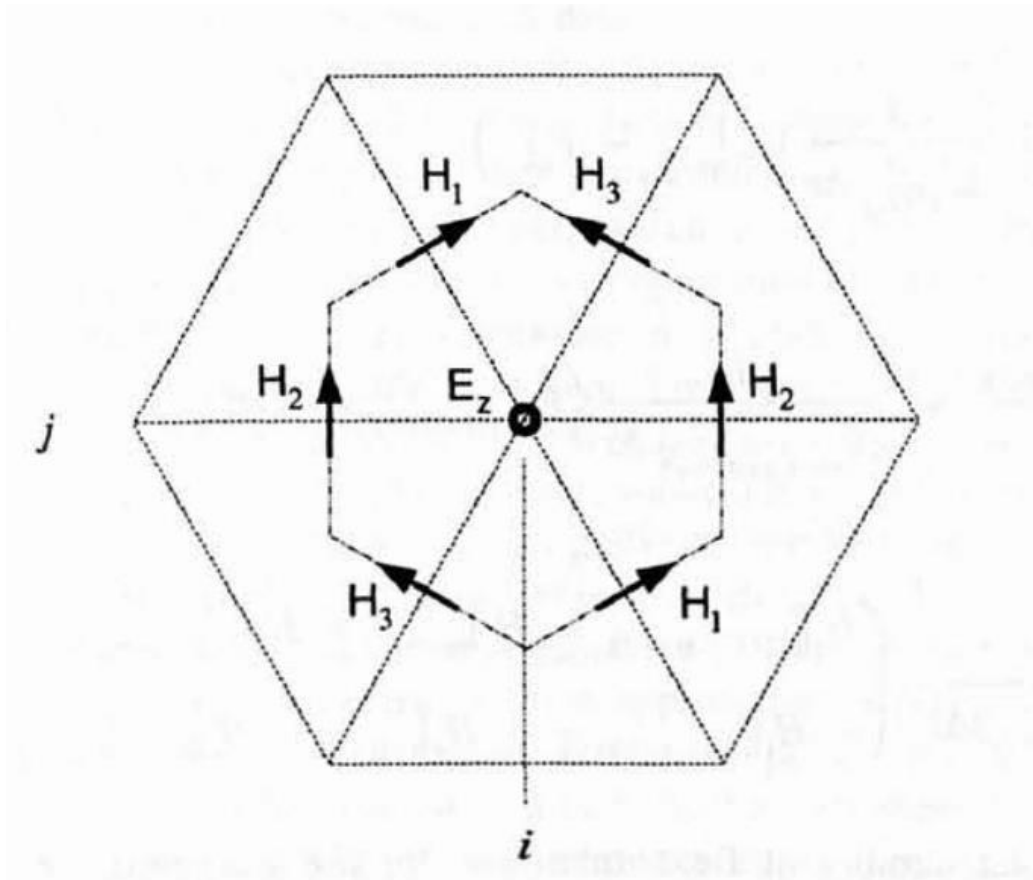
$$+ \frac{\Delta t}{\varepsilon} \left( \frac{H_{zi+1/2,j+1/2}^n - H_{zi+1/2,j-1/2}^n}{\Delta y} - J_x \right)$$

$H_{zi+1/2,j+1/2}$

$E_{yi+1/2,j}$

$H_{zi+1/2,j-1/2}$

# A co dla siatek skomplikowanych?

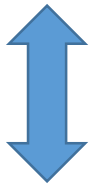


źr: Allen Taflove, Susan C. Hagness, *Computational electrodynamics – FDTD method*

# Równania Maxwella

$$\bullet \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}$$

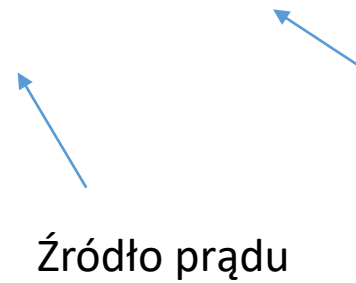
$$\bullet \mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}$$



$$\bullet \varepsilon_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{s} = \int_C \mathbf{H} \cdot d\mathbf{l} - I$$

$$\bullet \mu_0 \frac{\partial}{\partial t} \int_S \mathbf{H} \cdot d\mathbf{s} = -\int_C \mathbf{E} \cdot d\mathbf{l}$$

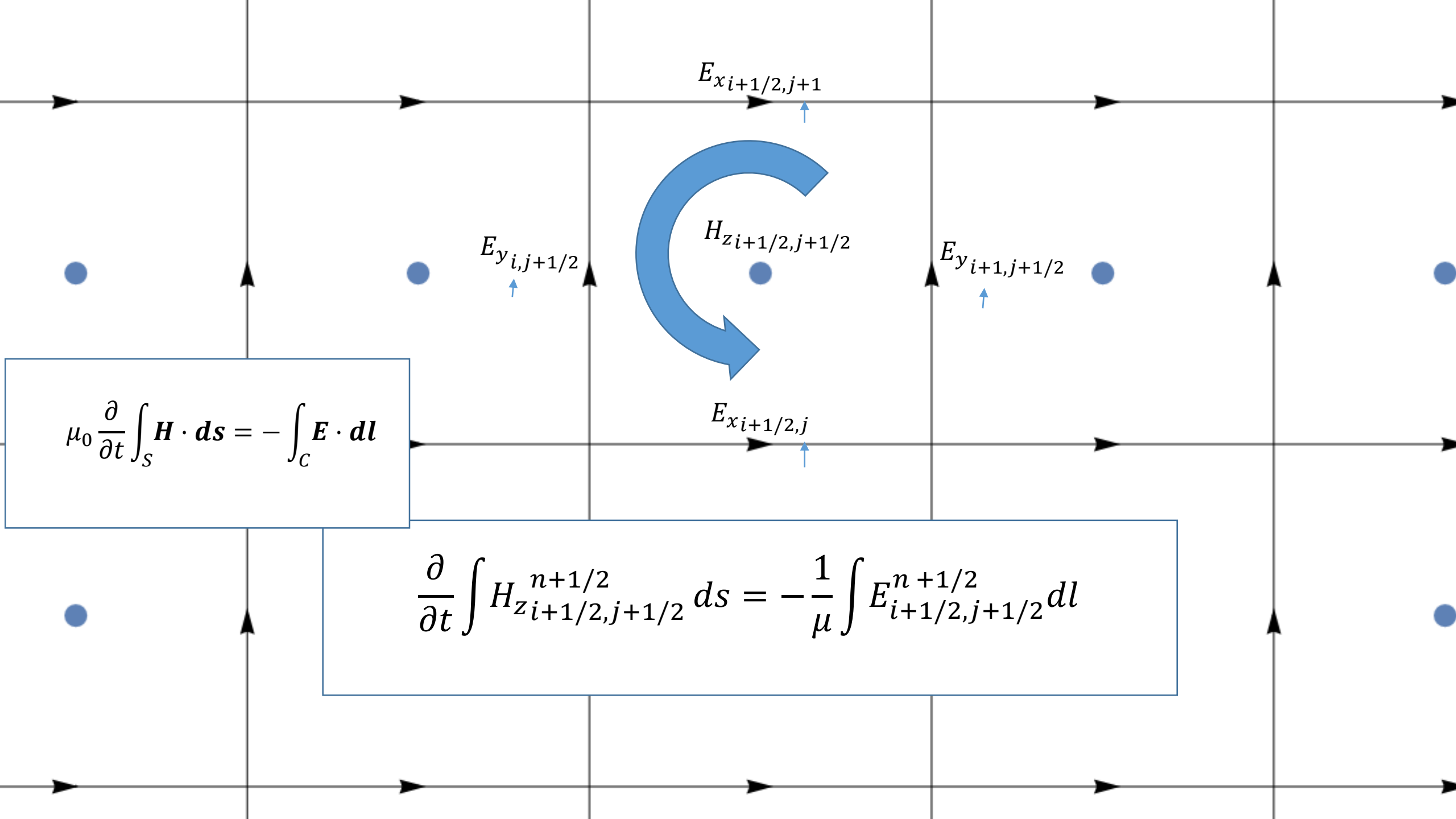
$$\mathbf{J} = \mathbf{J}_{source} + \sigma \mathbf{E}$$



Źródło prądu

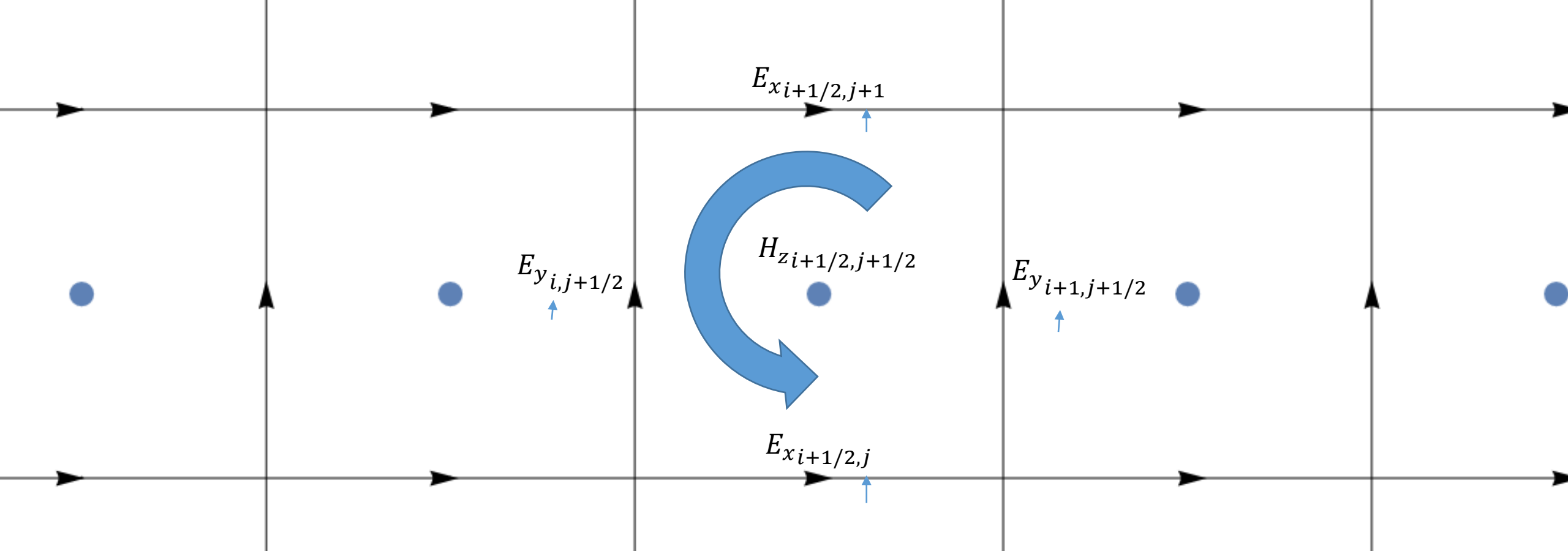
Mikroskopowe  
prawo Ohma

$$I = \int_S \mathbf{J} \cdot d\mathbf{s}$$

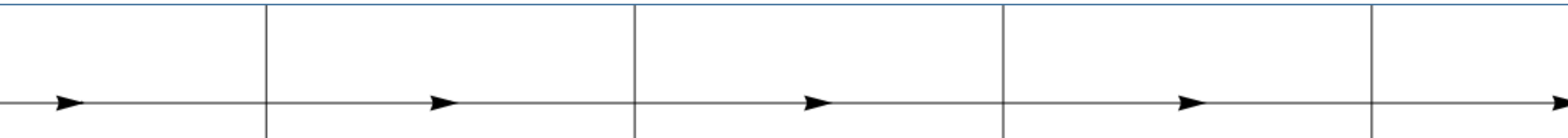


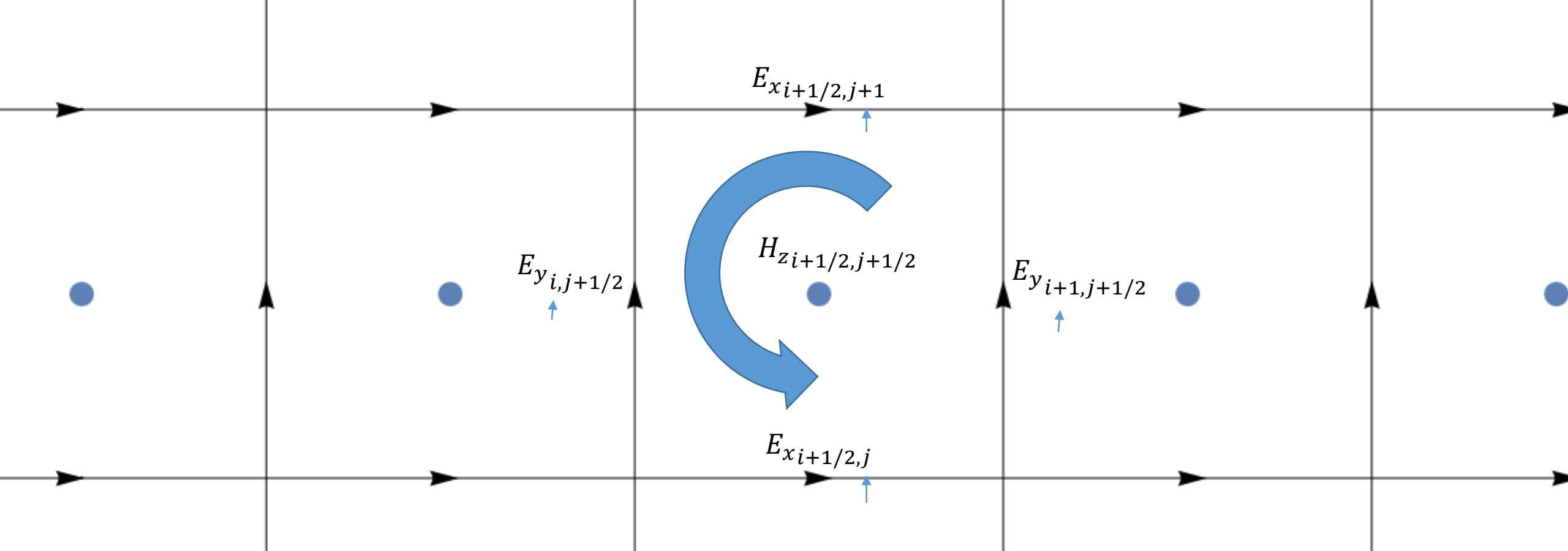
$$\mu_0 \frac{\partial}{\partial t} \int_S \mathbf{H} \cdot d\mathbf{s} = - \int_C \mathbf{E} \cdot d\mathbf{l}$$

$$\frac{\partial}{\partial t} \int H_{zi+1/2,j+1/2}^{n+1/2} ds = - \frac{1}{\mu} \int E_{i+1/2,j+1/2}^{n+1/2} dl$$

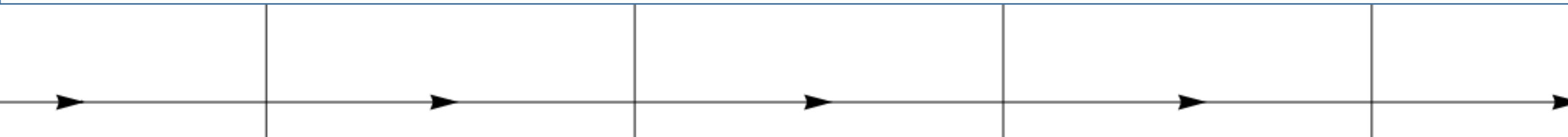


$$\frac{\partial}{\partial t} \Delta x \Delta y H_{z i+1/2, j+1/2}^{n+1/2} = \frac{1}{\mu} \left( \Delta x \left( E_{x i+1/2, j+1}^{n+1/2} - E_{x i+1/2, j}^{n+1/2} \right) + \Delta y \left( E_{y i+1, j+1/2}^{n+1/2} - E_{y i, j+1/2}^{n+1/2} \right) \right)$$

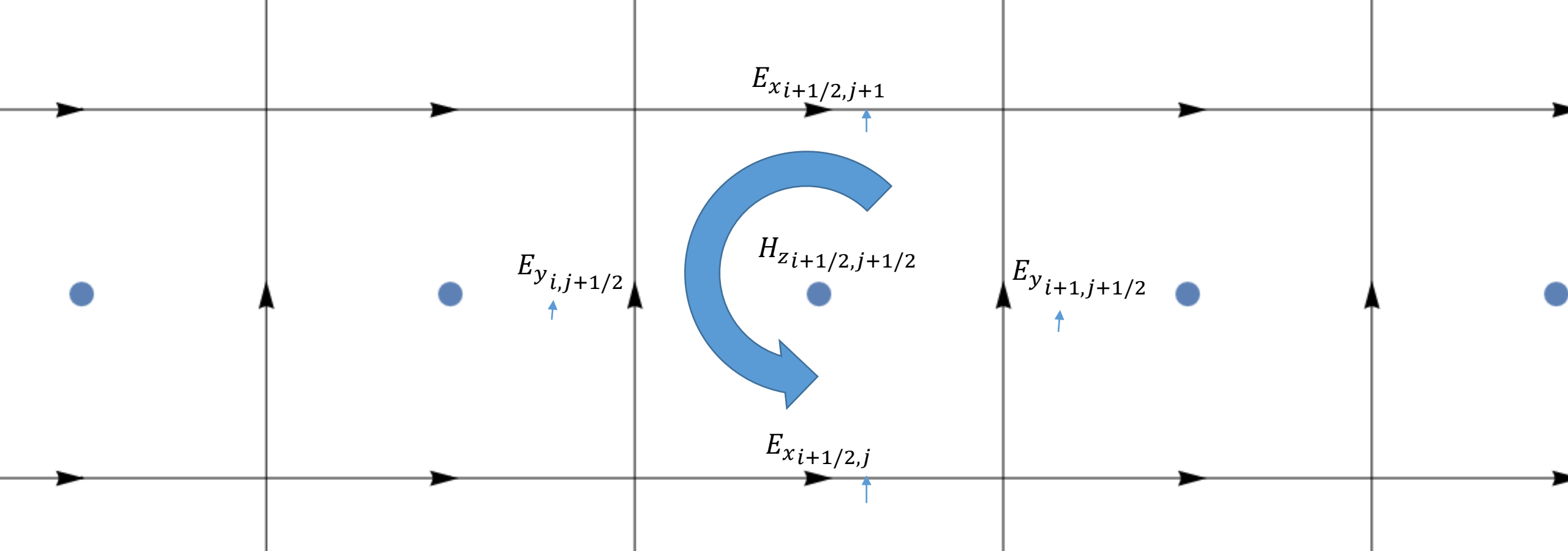




$$\Delta x \Delta y \frac{\partial}{\partial t} H_{zi+1/2,j+1/2}^{n+1/2} = \frac{1}{\mu} \left( \Delta x \left( E_{xi+1/2,j+1}^{n+1/2} - E_{xi+1/2,j}^{n+1/2} \right) + \Delta y \left( E_{yi+1,j+1/2}^{n+1/2} - E_{yi,j+1/2}^{n+1/2} \right) \right)$$

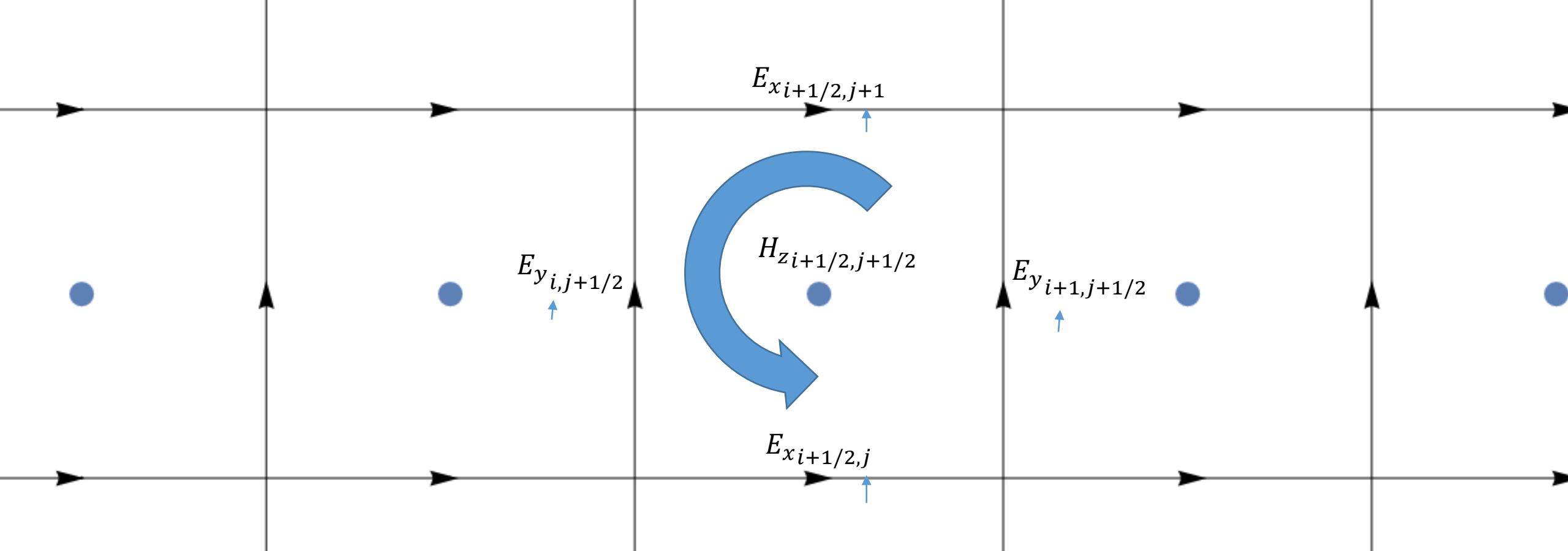






$$\Delta x \Delta y \frac{H_{zi+1/2,j+1/2}^{n+1} - H_{zi+1/2,j+1/2}^n}{\Delta t}$$

$$= \frac{1}{\mu} \left( \Delta x \left( E_{xi+1/2,j+1}^{n+1/2} - E_{xi+1/2,j}^{n+1/2} \right) + \Delta y \left( E_{yi+1,j+1/2}^{n+1/2} - E_{yi,j+1/2}^{n+1/2} \right) \right)$$



$$\frac{H_{zi+1/2,j+1/2}^{n+1} - H_{zi+1/2,j+1/2}^n}{\Delta t} = \frac{1}{\mu} \left( \frac{E_{xi+1/2,j+1}^{n+1/2} - E_{xi+1/2,j}^{n+1/2}}{\Delta y} + \frac{E_{y i+1,j+1/2}^{n+1/2} - E_{y i,j+1/2}^{n+1/2}}{\Delta x} \right)$$

