Polynomials with infinite solutions

Rahul Chhabra

St. Joseph's College, Prayagraj, Uttar Pradesh, India

Abstract

A solution to any polynomial $P(x)$ is a value of x that satisfies $P(x) = 0$. A polynomial of degree 1 forms an equation called a "linear" equation. A linear equation can be expressed as $ax + b = 0$. Its solution is then :

$$x = \frac{-b}{a}$$

For polynomials of degree 2, things go a little complicated :

$$P(x) = ax^2 + bx + c$$

Nonetheless, a quadratic equation can be solved rather easily with the help of the quadratic formula :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubics and quartics each have 3 and 4 roots respectively. Overall, any polynomial of degree n has exactly n solutions or roots. But can there be a polynomial equation which has infinite solutions? As it turns out - yes.

Keywords: Polynomial, solution, degree of a polynomial, linear, quadratic, quadratic formula, roots, cubics, quartics

1. The null polynomial

In general, any polynomial can be expressed as :

$$P(x) = a_0 + a_1x + a_2x^2 + ... + a_ix^i$$

The null polynomial is a polynomial that returns 0 for any value of x. $P(x) = 0$ It can be understood better as a polynomial with every coefficient equal to 0:

$$P(x) = 0 + 0x + 0x^2 + ... + 0x^i$$

It will return 0 no matter what. Since it returns 0 for any given value of x, it has infinite solutions.[**?**]

2. A polynomial of degree infinity

Most of the time when we consider polynomials, we consider the degree to be finite. However, there are instances when a polynomial can be thought of as having an infinite degree. Like a power series. One prime example is of a Taylor series.[**?**]

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Taylor series is defined to be infinite. It is called a Maclaurin series when $a = 0$.

3. Roots of a degree infinity polynomial

By the fundamental theorem of algebra, we know that a degree n polynomial's equation has n roots. Therefore, a polynomial of degree ∞ has ∞ solutions.

Given a polynomial $P(x)$, let us assume that it has a solution set S. Let us also assume that the solution set is finite, i.e., there are a finite number of solutions for $P(x)$. Now, let us take n to be the largest root in S. Since it is a root,

$$P(n) = 0$$

$$a_0 + a_1 n + a_2 n^2 + ... = 0 \tag{1}$$

Let us also take any other arbitrary m. Now, let us see if $m + n$ is a root of $P(x)$.

$$\begin{aligned} a_0 + a_1(n+m) + a_2(n+m)^2 + ... &= 0 \\ a_0 + a_1 n + a_1 m + a_2(n^2 + 2mn + m^2) + ... &= 0 \\ a_0 + a_1 n + a_1 m + a_2 n^2 + a_2 m^2 + 2a_2 mn + ... &= 0 \end{aligned}$$

We see that simplifying produces $a_i n^i + a_i m^i$ along with all the intermediate terms of a bionomial expansion of the form $(a \pm b)^n$. Let the sum of all these intermediate terms be T_{m+n}.

Now,

$$a_1 n + a_1 m + a_2 n^2 + a_2 m^2 + ... + T_{m+n} = 0$$
$$(a_1 n + a_2 n^2 + ...) + (a_1 m + a_2 m^2 + ...) + T_{m+n} = 0$$

But from 1,

$$(a_1 m + a_2 m^2 + ...) + T_{m+n} = 0 \tag{2}$$

Since it is possible for the expression 2,

$$P(m + n) = 0$$

But n is by definition, the largest root, then, by contradiction, the solution set S has to be infinite. Therefore, in this case,there are infinite solutions to the polynomial $P(x)$.

But what if there is no value of m that satisfies $T_{m+n} = 0$? Then, in that case, the solution set can be thought of as not having as many elements as the degree of the polynomial. Just like how

$$x^2 + 1 - 2x = 0$$

Has one solution $x = 1$. The solution set for this quadratic would be $\{1\}$. Here, there are two roots - but they are both equal. The case when m does not exist would also be similar. In other words, the infinity of the solution set (∞_S) would be smaller than or equal to the infinity of the degree of the polynomial (∞_P), for any given infinite polynomial. Or,

$$\infty_S \leqslant \infty_P \tag{3}$$

4. Such equations in action

Here are a few examples :

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

This is the Taylor series for the $sin(x)$ function. It is how your calculator get the value when you feed x to it.

A note on other fields

A field is a set on which the binary operations $+$, $-$, $\times$ and $\div$ are defined. Besides integral fields, there are other fields where polynomials will behave differently and a polynomial with finite terms and of a finite degree can also have infinite solutions.

Bibliography