17. a. Let A_1 consist of the r pivot columns in A. The columns of A_1 are linearly independent. So A_1 is an $m \times r$ with rank r.

b. By the Rank Theorem applied to A_1, the dimension of Row A is r, so A_1 has r linearly independent rows. Use them to form A_2. Then A_2 is $r \times r$ with linearly independent rows. By the Invertible Matrix Theorem, A_2 is invertible.

19. $$[\,B \quad AB \quad A^2B\,] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix has rank 3, so the pair (A, B) is controllable.

21. [M] rank $[\,B \quad AB \quad A^2B \quad A^3B\,] = 3$. The pair (A, B) is not controllable.

Chapter 5

Section 5.1, page 289

1. Yes **3.** No **5.** Yes, $\lambda = 0$ **7.** Yes, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

9. $\lambda = 1$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\lambda = 5$: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **11.** $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

13. $\lambda = 1$: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $\lambda = 2$: $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$; $\lambda = 3$: $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

15. $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ **17.** $0, 2, -1$

19. 0. Justify your answer.

21. See the *Study Guide*, after you have written your answers.

23. *Hint:* Use Theorem 2.

25. *Hint:* Use the equation $A\mathbf{x} = \lambda\mathbf{x}$ to find an equation involving A^{-1}.

27. *Hint:* For any λ, $(A - \lambda I)^T = A^T - \lambda I$. By a theorem (which one?), $A^T - \lambda I$ is invertible if and only if $A - \lambda I$ is invertible.

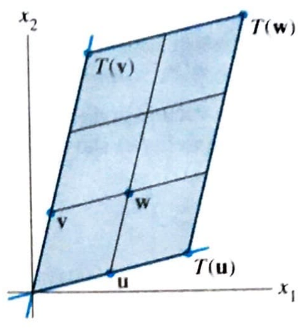
29. Let $\mathbf{v}$ be the vector in $\mathbb{R}^n$ whose entries are all 1's. Then $A\mathbf{v} = s\mathbf{v}$.

31. *Hint:* If A is the standard matrix of T, look for a nonzero vector $\mathbf{v}$ (a point in the plane) such that $A\mathbf{v} = \mathbf{v}$.

33. a. $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}$

b.
$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}) \\ &= c_1\lambda^k A\mathbf{u} + c_2\mu^k A\mathbf{v} && \text{Linearity} \\ &= c_1\lambda^k \lambda\mathbf{u} + c_2\mu^k \mu\mathbf{v} && \mathbf{u} \text{ and } \mathbf{v} \text{ are eigenvectors.} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

35.



37. [M] $\lambda = 3$: $\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$; $\lambda = 13$: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. You can speed up your calculations with the program `nulbasis` discussed in the *Study Guide*.

39. [M] $\lambda = -2$: $\begin{bmatrix} -2 \\ 7 \\ -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -5 \\ 0 \\ 5 \end{bmatrix}$;

$\lambda = 5$: $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Section 5.2, page 297

1. $\lambda^2 - 4\lambda - 45$; $9, -5$ **3.** $\lambda^2 - 2\lambda - 1$; $1 \pm \sqrt{2}$

5. $\lambda^2 - 6\lambda + 9$; 3 **7.** $\lambda^2 - 9\lambda + 32$; no real eigenvalues

9. $-\lambda^3 + 4\lambda^2 - 9\lambda - 6$ **11.** $-\lambda^3 + 9\lambda^2 - 26\lambda + 24$

13. $-\lambda^3 + 18\lambda^2 - 95\lambda + 150$ **15.** $4, 3, 3, 1$

17. $3, 3, 1, 1, 0$

19. *Hint:* The equation given holds for all λ.

21. The *Study Guide* has hints.

23. *Hint:* Find an invertible matrix P so that $RQ = P^{-1}AP$.

25. a. $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = .3$

b. $\mathbf{x}_0 = \mathbf{v}_1 - \frac{1}{14}\mathbf{v}_2$

c. $\mathbf{x}_1 = \mathbf{v}_1 - \frac{1}{14}(.3)\mathbf{v}_2$, $\mathbf{x}_2 = \mathbf{v}_1 - \frac{1}{14}(.3)^2\mathbf{v}_2$, and $\mathbf{x}_k = \mathbf{v}_1 - \frac{1}{14}(.3)^k\mathbf{v}_2$. As $k \to \infty$, $(.3)^k \to 0$ and $\mathbf{x}_k \to \mathbf{v}_1$.

27. a. $A\mathbf{v}_1 = \mathbf{v}_1$, $A\mathbf{v}_2 = .5\mathbf{v}_2$, $A\mathbf{v}_3 = .2\mathbf{v}_3$. (This also shows that the eigenvalues of A are 1, .5, and .2.)

b. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent because the eigenvectors correspond to distinct eigenvalues (Theorem 2). Since there are 3 vectors in the set, the set is a basis for $\mathbb{R}^3$. So there exist (unique) constants such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Then

$$\mathbf{w}^T\mathbf{x}_0 = c_1\mathbf{w}^T\mathbf{v}_1 + c_2\mathbf{w}^T\mathbf{v}_2 + c_3\mathbf{w}^T\mathbf{v}_3 \qquad (*)$$

Since $\mathbf{x}_0$ and $\mathbf{v}_1$ are probability vectors and since the entries in $\mathbf{v}_2$ and in $\mathbf{v}_3$ each sum to 0, $(*)$ shows that $1 = c_1$.

c. By (b),

$$\mathbf{x}_0 = \mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Using (a),

$$\begin{aligned} \mathbf{x}_k &= A^k\mathbf{x}_0 = A^k\mathbf{v}_1 + c_2A^k\mathbf{v}_2 + c_3A^k\mathbf{v}_3 \\ &= \mathbf{v}_1 + c_2(.5)^k\mathbf{v}_2 + c_3(.2)^k\mathbf{v}_3 \\ &\to \mathbf{v}_1 \text{ as } k \to \infty \end{aligned}$$

29. [M] Report your results and conclusions. You can avoid tedious calculations if you use the program `gauss` discussed in the *Study Guide*.

Section 5.3, page 304

1. $\begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$ **3.** $\begin{bmatrix} a^k & 0 \\ 3(a^k - b^k) & b^k \end{bmatrix}$

5. $\lambda = 5$: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\lambda = 1$: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

When an answer involves a diagonalization, $A = PDP^{-1}$, the factors P and D are not unique, so your answer may differ from that given here.

7. $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **9.** Not diagonalizable

11. $P = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. $P = \begin{bmatrix} -1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

15. $P = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

17. Not diagonalizable

19. $P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

21. See the *Study Guide*. **23.** Yes. (Explain why.)

25. No, A must be diagonalizable. (Explain why.)

27. *Hint:* Write $A = PDP^{-1}$. Since A is invertible, 0 is not an eigenvalue of A, so D has nonzero entries on its diagonal.

29. One answer is $P_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, whose columns are eigenvectors corresponding to the eigenvalues in D_1.

31. *Hint:* Construct a suitable 2×2 triangular matrix.

33. [M] $P = \begin{bmatrix} 2 & 2 & 1 & 6 \\ 1 & -1 & 1 & -3 \\ -1 & -7 & 1 & 0 \\ 2 & 2 & 0 & 4 \end{bmatrix}$,

$D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

35. [M] $P = \begin{bmatrix} 6 & 3 & 2 & 4 & 3 \\ -1 & -1 & -1 & -3 & -1 \\ -3 & -3 & -4 & -2 & -4 \\ 3 & 0 & -1 & 5 & 0 \\ 0 & 3 & 4 & 0 & 5 \end{bmatrix}$,

$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Section 5.4, page 311

1. $\begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}$

3. a. $T(\mathbf{e}_1) = -\mathbf{b}_2 + \mathbf{b}_3$, $T(\mathbf{e}_2) = -\mathbf{b}_1 - \mathbf{b}_3$, $T(\mathbf{e}_3) = \mathbf{b}_1 - \mathbf{b}_2$

b. $[T(\mathbf{e}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $[T(\mathbf{e}_2)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$,

$[T(\mathbf{e}_3)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

5. a. $10 - 3t + 4t^2 + t^3$

b. For any $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_2$ and any scalar c,

$$\begin{aligned} T[\mathbf{p}(t) + \mathbf{q}(t)] &= (t+5)[\mathbf{p}(t) + \mathbf{q}(t)] \\ &= (t+5)\mathbf{p}(t) + (t+5)\mathbf{q}(t) \\ &= T[\mathbf{p}(t)] + T[\mathbf{q}(t)] \\ T[c \cdot \mathbf{p}(t)] &= (t+5)[c \cdot \mathbf{p}(t)] = c \cdot (t+5)\mathbf{p}(t) \\ &= c \cdot T[\mathbf{p}(t)] \end{aligned}$$

c. $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

9. a. $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

b. *Hint:* Compute $T(\mathbf{p} + \mathbf{q})$ and $T(c \cdot \mathbf{p})$ for arbitrary $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_2$ and an arbitrary scalar c.

c. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ 13. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

15. $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

17. a. $A\mathbf{b}_1 = 2\mathbf{b}_1$, so $\mathbf{b}_1$ is an eigenvector of A. However, A has only one eigenvalue, $\lambda = 2$, and the eigenspace is only one-dimensional, so A is not diagonalizable.

b. $\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

19. By definition, if A is similar to B, there exists an invertible matrix P such that $P^{-1}AP = B$. (See Section 5.2.) Then B is invertible because it is the product of invertible matrices. To show that A^{-1} is similar to B^{-1}, use the equation $P^{-1}AP = B$. See the *Study Guide*.

21. *Hint:* Review Practice Problem 2.

23. *Hint:* Compute $B(P^{-1}\mathbf{x})$.

25. *Hint:* Write $A = PBP^{-1} = (PB)P^{-1}$, and use the trace property.

27. For each j, $I(\mathbf{b}_j) = \mathbf{b}_j$. Since the standard coordinate vector of any vector in $\mathbb{R}^n$ is just the vector itself, $[I(\mathbf{b}_j)]_{\mathcal{E}} = \mathbf{b}_j$. Thus the matrix for I relative to $\mathcal{B}$ and the standard basis $\mathcal{E}$ is simply $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$. This matrix is precisely the *change-of-coordinates* matrix $P_{\mathcal{B}}$ defined in Section 4.4.

29. The $\mathcal{B}$-matrix for the identity transformation is I_n, because the $\mathcal{B}$-coordinate vector of the jth basis vector $\mathbf{b}_j$ is the jth column of I_n.

31. [M] $\begin{bmatrix} -7 & -2 & -6 \\ 0 & -4 & -6 \\ 0 & 0 & -1 \end{bmatrix}$

Section 5.5, page 318

1. $\lambda = 2 + i, \begin{bmatrix} -1 + i \\ 1 \end{bmatrix}$; $\lambda = 2 - i, \begin{bmatrix} -1 - i \\ 1 \end{bmatrix}$

3. $\lambda = 2 + 3i, \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix}$; $\lambda = 2 - 3i, \begin{bmatrix} 1 + 3i \\ 2 \end{bmatrix}$

5. $\lambda = 2 + 2i, \begin{bmatrix} 1 \\ 2 + 2i \end{bmatrix}$; $\lambda = 2 - 2i, \begin{bmatrix} 1 \\ 2 - 2i \end{bmatrix}$

7. $\lambda = \sqrt{3} \pm i, \varphi = \pi/6$ radian, $r = 2$

9. $\lambda = -\sqrt{3}/2 \pm (1/2)i, \varphi = -5\pi/6$ radians, $r = 1$

11. $\lambda = .1 \pm .1i, \varphi = -\pi/4$ radian, $r = \sqrt{2}/10$

In Exercises 13–20, other answers are possible. Any P that makes $P^{-1}AP$ equal to the given C or to C^T is a satisfactory answer. First find P; then compute $P^{-1}AP$.

13. $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

15. $P = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$

17. $P = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}, C = \begin{bmatrix} -.6 & -.8 \\ .8 & -.6 \end{bmatrix}$

19. $P = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}, C = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$

21. $\mathbf{y} = \begin{bmatrix} 2 \\ -1 + 2i \end{bmatrix} = \frac{-1 + 2i}{5} \begin{bmatrix} -2 - 4i \\ 5 \end{bmatrix}$

23. (a) Properties of conjugates and the fact that $\overline{\mathbf{x}}^T = \overline{\mathbf{x}^T}$; (b) $\overline{A\mathbf{x}} = A\overline{\mathbf{x}}$ and A is real; (c) because $\mathbf{x}^T A \overline{\mathbf{x}}$ is a scalar and hence may be viewed as a 1×1 matrix; (d) properties of transposes; (e) $A^T = A$, definition of q

25. *Hint:* First write $\mathbf{x} = \operatorname{Re} \mathbf{x} + i(\operatorname{Im} \mathbf{x})$.

27. [M] $P = \begin{bmatrix} 1 & -1 & -2 & 0 \\ -4 & 0 & 0 & 2 \\ 0 & 0 & -3 & -1 \\ 2 & 0 & 4 & 0 \end{bmatrix}$,

$C = \begin{bmatrix} .2 & -.5 & 0 & 0 \\ .5 & .2 & 0 & 0 \\ 0 & 0 & .3 & -.1 \\ 0 & 0 & .1 & .3 \end{bmatrix}$

Other choices are possible, but C must equal $P^{-1}AP$.

Section 5.6, page 327

1. a. *Hint:* Find c_1, c_2 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Use this representation and the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A to compute $\mathbf{x}_1 = \begin{bmatrix} 49/3 \\ 41/3 \end{bmatrix}$.

b. In general, $\mathbf{x}_k = 5(3)^k\mathbf{v}_1 - 4(\frac{1}{3})^k\mathbf{v}_2$ for $k \geq 0$.

3. When $p = .2$, the eigenvalues of A are .9 and .7, and

$$\mathbf{x}_k = c_1(.9)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(.7)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \to \mathbf{0} \quad \text{as } k \to \infty$$

The higher predation rate cuts down the owls' food supply, and eventually both predator and prey populations perish.

5. If $p = .325$, the eigenvalues are 1.05 and .55. Since $1.05 > 1$, both populations will grow at 5% per year. An eigenvector for 1.05 is (6, 13), so eventually there will be approximately 6 spotted owls to every 13 (thousand) flying squirrels.

7. a. The origin is a saddle point because A has one eigenvalue larger than 1 and one smaller than 1 (in absolute value).