

PRACTICE PROBLEM

Find the characteristic equation and eigenvalues of $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

5.2 EXERCISES

Find the characteristic polynomial and the eigenvalues of the matrices in Exercises 1–8.

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 3 & -4 \\ 4 & 8 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix, using either a cofactor expansion or the special formula for 3×3 determinants described

prior to Exercises 15–18 in Section 3.1. [Note: Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations, because the variable λ is involved.]

$$\begin{array}{ll} 9. \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix} & 10. \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \\ 11. \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix} & 12. \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \\ 13. \begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix} & 14. \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix} \end{array}$$

For the matrices in Exercises 15–17, list the eigenvalues, repeated according to their multiplicities.

$$\begin{array}{ll} 15. \begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix} & 16. \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix} \\ 17. \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix} & \end{array}$$

18. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ . Find h in the matrix A below such that the eigenspace for $\lambda = 5$ is two-dimensional:

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

19. Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \dots, \lambda_n$, repeated according to multiplicities, so that $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$.

Explain why $\det A$ is the product of the n eigenvalues of A . (This result is true for any square matrix when complex eigenvalues are considered.)

20. Use a property of determinants to show that A and A^T have the same characteristic polynomial.

In Exercises 21 and 22, A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

21. a. The determinant of A is the product of the diagonal entries in A .
b. An elementary row operation on A does not change the determinant.
c. $(\det A)(\det B) = \det AB$
d. If $\lambda + 5$ is a factor of the characteristic polynomial of A , then 5 is an eigenvalue of A .

22. a. If A is 3×3 , with columns \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_3 , then $\det A$ equals the volume of the parallelepiped determined by \mathbf{a}_1 , \mathbf{a}_2 and \mathbf{a}_3 .
b. $\det A^T = (-1) \det A$.
c. The multiplicity of a root r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A .
d. A row replacement operation on A does not change the eigenvalues.

A widely used method for estimating eigenvalues of a general matrix A is the *QR algorithm*. Under suitable conditions, this algorithm produces a sequence of matrices, all similar to A , that become almost upper triangular, with diagonal entries that approach the eigenvalues of A . The main idea is to factor A (or another matrix similar to A) in the form $A = Q_1 R_1$, where $Q_1^T = Q_1^{-1}$ and R_1 is upper triangular. The factors are interchanged to form $A_1 = R_1 Q_1$, which is again factored as $A_1 = Q_2 R_2$; then to form $A_2 = R_2 Q_2$, and so on. The similarity of A , A_1, \dots follows from the more general result in Exercise 23.

23. Show that if $A = QR$ with Q invertible, then A is similar to $A_1 = RQ$.
24. Show that if A and B are similar, then $\det A = \det B$.
25. Let $A = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. [Note: A is the stochastic matrix studied in Example 5 of Section 4.9.]

- a. Find a basis for \mathbb{R}^2 consisting of \mathbf{v}_1 and another eigenvector \mathbf{v}_2 of A .
b. Verify that \mathbf{x}_0 may be written in the form $\mathbf{x}_0 = \mathbf{v}_1 + c\mathbf{v}_2$.
c. For $k = 1, 2, \dots$, define $\mathbf{x}_k = A^k \mathbf{x}_0$. Compute \mathbf{x}_1 and \mathbf{x}_2 , and write a formula for \mathbf{x}_k . Then show that $\mathbf{x}_k \rightarrow \mathbf{v}_1$ as k increases.

26. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Use formula (1) for a determinant (given before Example 2) to show that $\det A = ad - bc$. Consider two cases: $a \neq 0$ and $a = 0$.

27. Let $A = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$,
 $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

- a. Show that $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 are eigenvectors of A . [Note: A is the stochastic matrix studied in Example 3 of Section 4.9.]
b. Let \mathbf{x}_0 be any vector in \mathbb{R}^3 with nonnegative entries whose sum is 1. (In Section 4.9, \mathbf{x}_0 was called a probability vector.) Explain why there are constants c_1, c_2 , and c_3 such that $\mathbf{x}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3$. Compute $\mathbf{w}^T \mathbf{x}_0$, and deduce that $c_1 = 1$.
c. For $k = 1, 2, \dots$, define $\mathbf{x}_k = A^k \mathbf{x}_0$, with \mathbf{x}_0 as in part (b). Show that $\mathbf{x}_k \rightarrow \mathbf{v}_1$ as k increases.

28. [M] Construct a random integer-valued 4×4 matrix A , and verify that A and A^T have the same characteristic polynomial (the same eigenvalues with the same multiplicities). Do A and A^T have the same eigenvectors? Make the same analysis of a 5×5 matrix. Report the matrices and your conclusions.

29. [M] Construct a random integer-valued 4×4 matrix A .

- Reduce A to echelon form U with no row scaling, and use U in formula (1) (before Example 2) to compute $\det A$. (If A happens to be singular, start over with a new random matrix.)
- Compute the eigenvalues of A and the product of these eigenvalues (as accurately as possible).

- List the matrix A , and, to four decimal places, list the pivots in U and the eigenvalues of A . Compute $\det A$ with your matrix program, and compare it with the products you found in (a) and (b).

30. [M] Let $A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & a & 25 \end{bmatrix}$. For each value of a in

the set $\{32, 31.9, 31.8, 32.1, 32.2\}$, compute the characteristic polynomial of A and the eigenvalues. In each case, create a graph of the characteristic polynomial $p(t) = \det(A - tI)$ for $0 \leq t \leq 3$. If possible, construct all graphs on one coordinate system. Describe how the graphs reveal the changes in the eigenvalues as a changes.

SOLUTION TO PRACTICE PROBLEM

The characteristic equation is

$$\begin{aligned} 0 &= \det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & -4 \\ 4 & 2 - \lambda \end{bmatrix} \\ &= (1 - \lambda)(2 - \lambda) - (-4)(4) = \lambda^2 - 3\lambda + 18 \end{aligned}$$

From the quadratic formula,

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(18)}}{2} = \frac{3 \pm \sqrt{-63}}{2}$$

It is clear that the characteristic equation has no real solutions, so A has no real eigenvalues. The matrix A is acting on the real vector space \mathbb{R}^2 , and there is no nonzero vector \mathbf{v} in \mathbb{R}^2 such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ .