6.4 EXERCISES

In Exercises 1–6, the given set is a basis for a subspace W. Use the Gram–Schmidt process to produce an orthogonal basis for W.

1. $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$

2. $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$

3. $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

4. $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$

5. $\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix}$

6. $\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix}$

7. Find an orthonormal basis of the subspace spanned by the vectors in Exercise 3.

8. Find an orthonormal basis of the subspace spanned by the vectors in Exercise 4.

Find an orthogonal basis for the column space of each matrix in Exercises 9–12.

9. $\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$ 10. $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$ 12. $\begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$

In Exercises 13 and 14, the columns of Q were obtained by applying the Gram–Schmidt process to the columns of A. Find an upper triangular matrix R such that $A = QR$. Check your work.

13. $A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$, $Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$

14. $A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$, $Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}$

15. Find a QR factorization of the matrix in Exercise 11.

16. Find a QR factorization of the matrix in Exercise 12.

In Exercises 17 and 18, all vectors and subspaces are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

17. a. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W, then multiplying $\mathbf{v}_3$ by a scalar c gives a new orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, c\mathbf{v}_3\}$.

b. The Gram–Schmidt process produces from a linearly independent set $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ an orthogonal set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ with the property that for each k, the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ span the same subspace as that spanned by $\mathbf{x}_1, \dots, \mathbf{x}_k$.

c. If $A = QR$, where Q has orthonormal columns, then $R = Q^T A$.

18. a. If $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent, and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.

b. If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x} - \text{proj}_W\, \mathbf{x}$ is not zero.

c. In a QR factorization, say $A = QR$ (when A has linearly independent columns), the columns of Q form an orthonormal basis for the column space of A.

19. Suppose $A = QR$, where Q is $m \times n$ and R is $n \times n$. Show that if the columns of A are linearly independent, then R must be invertible. [*Hint:* Study the equation $R\mathbf{x} = \mathbf{0}$ and use the fact that $A = QR$.]

20. Suppose $A = QR$, where R is an invertible matrix. Show that A and Q have the same column space. [*Hint:* Given $\mathbf{y}$ in Col A, show that $\mathbf{y} = Q\mathbf{x}$ for some $\mathbf{x}$. Also, given $\mathbf{y}$ in Col Q, show that $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$.]

21. Given $A = QR$ as in Theorem 12, describe how to find an orthogonal $m \times m$ (square) matrix Q_1 and an invertible $n \times n$ upper triangular matrix R such that

$$A = Q_1 \begin{bmatrix} R \\ 0 \end{bmatrix}$$

The MATLAB `qr` command supplies this "full" QR factorization when rank $A = n$.

22. Let $\mathbf{u}_1, \dots, \mathbf{u}_p$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\mathbf{x}) = \text{proj}_W\, \mathbf{x}$. Show that T is a linear transformation.

23. Suppose $A = QR$ is a QR factorization of an $m \times n$ matrix A (with linearly independent columns). Partition A as $[A_1 \quad A_2]$, where A_1 has p columns. Show how to obtain a QR factorization of A_1, and explain why your factorization has the appropriate properties.

24. [M] Use the Gram–Schmidt process as in Example 2 to produce an orthogonal basis for the column space of

$$A = \begin{bmatrix} -10 & 13 & 7 & -11 \\ 2 & 1 & -5 & 3 \\ -6 & 3 & 13 & -3 \\ 16 & -16 & -2 & 5 \\ 2 & 1 & -5 & -7 \end{bmatrix}$$

25. [M] Use the method in this section to produce a QR factorization of the matrix in Exercise 24.

26. [M] For a matrix program, the Gram–Schmidt process works better with orthonormal vectors. Starting with $\mathbf{x}_1, \dots, \mathbf{x}_p$ as in Theorem 11, let $A = [\,\mathbf{x}_1 \ \cdots \ x_p\,]$. Suppose Q is an $n \times k$ matrix whose columns form an orthonormal basis for the subspace W_k spanned by the first k columns of A. Then for $\mathbf{x}$ in $\mathbb{R}^n$, $QQ^T\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto W_k (Theorem 10 in Section 6.3). If $\mathbf{x}_{k+1}$ is the next column of A, then equation (2) in the proof of Theorem 11 becomes

$$\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - Q(Q^T\mathbf{x}_{k+1})$$

(The parentheses above reduce the number of arithmetic operations.) Let $\mathbf{u}_{k+1} = \mathbf{v}_{k+1}/\|\mathbf{v}_{k+1}\|$. The new Q for the next step is $[\,Q \quad \mathbf{u}_{k+1}\,]$. Use this procedure to compute the QR factorization of the matrix in Exercise 24. Write the keystrokes or commands you use.

WEB

SOLUTION TO PRACTICE PROBLEMS

1. Let $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - 0\mathbf{v}_1 = \mathbf{x}_2$. So $\{\mathbf{x}_1, \mathbf{x}_2\}$ is already orthogonal. All that is needed is to normalize the vectors. Let

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Instead of normalizing $\mathbf{v}_2$ directly, normalize $\mathbf{v}_2' = 3\mathbf{v}_2$ instead:

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2'\|}\mathbf{v}_2' = \frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

Then $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for W.

2. Since the columns of A are linearly dependent, there is a nontrivial vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. But then $QR\mathbf{x} = \mathbf{0}$. Applying Theorem 7 from Section 6.2 results in $\|R\mathbf{x}\| = \|QR\mathbf{x}\| = \|\mathbf{0}\| = 0$. But $\|R\mathbf{x}\| = 0$ implies $R\mathbf{x} = \mathbf{0}$, by Theorem 1 from Section 6.1. Thus there is a nontrivial vector $\mathbf{x}$ such that $R\mathbf{x} = \mathbf{0}$ and hence, by the Invertible Matrix Theorem, R cannot be invertible.