Assignment 8a - Main Textbook Problems

4.3 Determine the Fourier transform of each of the following periodic signals:

(a) $sin(2\pi t + \frac{\pi}{4})$

$$sin(2\pi t + \frac{\pi}{4}) = \frac{e^{j(2\pi t + \pi/4)} - e^{-j(2\pi t + \pi/4)}}{2j}$$

$$sin(2\pi t + \frac{\pi}{4}) <-> \frac{1}{2j} e^{j\pi/4(2\pi\delta(\omega - 2\pi)} - \frac{1}{2j} e^{-j\pi/4(2\pi\delta(\omega + 2\pi))}$$

$$sin(2\pi t + \frac{\pi}{4}) = j\pi((\frac{1-j}{\sqrt{2}})\delta(\omega + 2\pi) - (\frac{1+5}{\sqrt{2}})\delta(\omega - 2\pi))$$

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$$x_1(t) = \frac{1}{2j}(e^{j(2\pi t + \pi/4)} - e^{-j(2\pi t + \pi/4)}) = \frac{1}{2j}(e^{j\pi/4} e^{j2\pi t}) - \frac{1}{2j} e^{-j\pi/4} e^{-j2\pi t}$$

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(b) $1 + cos(6\pi t + \frac{\pi}{8})$

$$1 + cos(6\pi t + \frac{\pi}{8}) = \frac{1 + e^{j(6\pi t + \pi 8)} + e^{-j(6\pi t + \pi 8)}}{2}$$

$$1 + cos(6\pi t + \frac{\pi}{8}) <-> 2\pi\delta(\omega) + \frac{1}{2} e^{j\pi/8(2\pi\delta(\omega - 6\pi))} + \frac{1}{2} e^{-j\pi/8(2\pi\delta(\omega + 6\pi))}$$

$$= 2\pi\delta(\omega) + \pi[e^{j\pi/8\delta(\omega - 6\pi)} + e^{-j\pi/8 delta(\omega + 6\pi)}]$$

4.6 Given that x(t) has the Fourier transform $X(j\omega)$, express the Fourier transform of the signals listed below in terms of $X(j\omega)$. You may find useful the Fourier transform properties listed in Table 4.1

(a) $x_1(t) = x(1-t) + x(-1-t)$

$$x_1(t) = x(1-t) + x(-1-t)$$

$$x_1(j\omega) = e^{-2\pi f} x(j\omega) + e^{-2\pi f} x(-j\omega)$$

$$F[\delta(t-a)] = e^{-2\pi f a} G(f)$$

$F[(\delta(-t))] = G(-f)$

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$e^{-j\omega t}X(-j\omega) + e^{j\omega t}X(-j\omega)$

$2X(-j\omega)cos\omega$

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(b) $x_2(t) = x(3t-6)$

$x_2(t) = x(3t-6)$

$x_2(j\omega) = \frac{1}{3}e^{-2\pi f}x(\frac{j\omega}{3})$

$x_2(j\omega) = \frac{1}{3}e^{-12\pi f}x(\frac{j\omega}{3})$

(c) $x_3(t) = \frac{d^2}{dt^2}x(t-1)$

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$\frac{dx(t)}{dt} <-> j\omega X(j\omega)$

$\frac{d^2x(t)}{dt^t} <-> -\omega^2 X j\omega)$

$x_3(t) = \frac{d^2x(t-1)}{dt^2} <-> -\omega X(j\omega)e^{-j\omega t}$

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4.8 Consider the signal

$$x(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ t + \frac{1}{2}, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 1, & t > \frac{1}{2} \end{cases}$$

(a) Use the differentiation and integration properties in Table 4.1 and the Fourier transform pair for the rectangular pulse in Table 4.2 to find a closed-form expression for $X(j\omega)$.

$$y(t) = \frac{dx(t)}{dt} = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0, & t > \frac{1}{2} \end{cases}$$

$$= \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}$$

$$Y(j\omega) = \frac{2sin(\omega\frac{1}{2})}{\omega}$$

$$Y(j\omega) = \frac{2sin(\frac{\omega}{2})}{\omega}$$

$$x(t) = \int_{-\infty}^{t} y(t)dt$$

$$X(j\omega) = \frac{1}{j\omega}Y(j\omega) + \pi Y(0)\delta(\omega)$$

$$\left(\frac{2sin\frac{\omega}{2}}{\omega}\right) => Y(j\omega)$$

$$X(j\omega) = \frac{1}{j\omega}\left(\frac{2sin\frac{\omega}{2}}{\omega}\right) + \pi\left(\frac{2sin\frac{0}{2}}{0}\right)\delta(\omega)$$

$$X(j\omega) = \frac{1}{j\omega}\left(\frac{2sin\frac{\omega}{2}}{\omega}\right) + \pi\left(\frac{2sin\frac{0}{2}}{\frac{0}{2}}\right)\delta(\omega)$$

L'Hospital Rule

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1$$

$$\frac{sin\frac{0}{2}}{\frac{0}{2}} = 1$$

$$X(j\omega) = \frac{1}{j\omega}\left(\frac{2sin\frac{(\frac{\omega}{2})}{\omega}}{\omega}\right) + \pi(1)\delta(\omega)$$

$$X(j\omega) = \left(\frac{2sin\frac{(\frac{\omega}{2})}{\omega}}{j\omega^2}\right) + \pi\delta(\omega)$$

The closed form expression for $X(j\omega)$ is $\left(\frac{2sin\frac{(\frac{\omega}{2})}{\omega}}{j\omega^2}\right) + \pi\delta(\omega)$

(b) What is the Fourier transform of $g(t) = x(t) - \frac{1}{2}$?

$$g(t) = x(t) - \frac{1}{2}$$

$$G(j\omega) = X(j\omega) - \frac{1}{2}(2\pi\delta(\omega))$$

$$G(j\omega) = X(j\omega) - (\pi\delta(\omega))$$

$$\left(\frac{2sin(\frac{\omega}{2})}{j\omega^2}\right) + \pi\delta(\omega) => Y(j\omega)$$

$$G(j\omega) = \frac{2sin(\frac{\omega}{2})}{j\omega^2} + \pi\delta(\omega) - \pi\delta(\omega)$$

$$G(j\omega) = \frac{2sin(\frac{\omega}{2})}{j\omega^2}$$

The Fourier transform of g(t) is $\frac{2sin(\frac{\omega}{2})}{j\omega^2}$

4.27 Consider the signals

$x(t) = u(t-1) - 2u(t-2) + u(t-3)$

and

$x^*(t) = \Sigma_{k=-\infty}^{\infty} x(t-kT)$

where ***T > 0*** . Let a_k denote the Fourier series coefficients of $x^*(t),\ and\ let\ X(j\omega)$ denote the Fouries transform of $x(t)$.

(a) Determine a closed-form expression for $X(j\omega)$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} [u(t-1) - 2u(t-2) + u(t-3)]e^{-j\omega t}dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} u(t-1)e^{-j\omega t}dt - 2\int_{-\infty}^{\infty} u(t-2)e^{-j\omega t}dt + \int_{-\infty}^{\infty} u(t-3)e^{-j\omega t}dt$$

$$X(j\omega) = \int_{1}^{\infty} e^{-j\omega t}dt - 2\int_{2}^{\infty} e^{-j\omega t}dt + \int_{3}^{\infty} e^{-j\omega t}dt$$

$$X(j\omega) = \frac{e^{-j\omega t}}{-j\omega}\int_{1}^{\infty} - \frac{2e^{-j\omega t}}{-j\omega}\int_{2}^{\infty} + \frac{e^{-j\omega t}}{-j\omega}\int_{3}^{\infty}$$

$$X(j\omega) = \frac{e^{-j\omega}}{-j\omega} - \frac{-2e^{-2j\omega}}{-j\omega} + \frac{-e^{-3j\omega}}{-j\omega}$$

$$X(j\omega) = \frac{e^{-j\omega} - 2e^{-2j\omega} - e^{-3j\omega}}{j\omega}$$

$$X(j\omega) = \frac{e^{-j\omega}[1 - e^{-j\omega}] - e^{-2j\omega}[1 - e^{-j\omega}]}{j\omega}$$

$$X(j\omega) = \frac{[e^{-j\omega} - e^{-2j\omega}] - (1 - e^{-j\omega})}{j\omega}$$

$$X(j\omega) = \frac{e^{-j\omega}(1 - e^{-j\omega}) - (1 - e^{-j\omega})}{j\omega}$$

$$X(j\omega) = \frac{2jsin(\frac{\omega}{2})(1 - e^{-j\omega})e^{-3j\frac{\omega}{2}}}{j\omega}$$

$$X(j\omega) = \frac{2sin(\frac{\omega}{2})}{\omega}(1 - e^{-j\omega})e^{-j3\frac{\omega}{2}}$$

(b) Determine an expression for the Fourier coefficients a_k and verify that

$$a_k = \frac{1}{T}X(j\frac{2\pi k}{T}).$$

$$x(t) = \sum_{k->-\infty}^{\infty} x(t - kT)$$

$$a_k = \frac{1}{T}\int_T^{\infty} x(t)e^{-j\frac{2\pi}{T}}ktdt$$

$$a_k = \frac{1}{2}[\int_1^{\infty} e^{\frac{-j2\pi}{T}}ktdt - 2\int_2^{\infty} e^{-j\frac{2\pi}{T}}ktdt + \int_3^{\infty} e^{\frac{-j2\pi}{T}}kxdt]$$

$$a_k = \frac{1}{2}\frac{e^{-j\frac{2\pi}{2}}}{-j2\pi k}\int_1^{\infty} -2\frac{e^{-j\pi kt}}{-j\pi k}\int_2^{\infty} + \frac{e^{-j\pi kt}}{-j\pi k}\int_3^{\infty}$$

$$a_k = \frac{e^{-j\pi k}[1 - e^{-j\pi k}] - e^{-2j\pi k}[1 - e^{-j\pi k}]}{j2\pi k}$$

$$a_k = e^{-j\frac{\pi 3k}{2} - jk\frac{k}{2}}(e^{-j\pi\frac{k}{2}} - e^{-j\pi\frac{k}{2}})(1 - e^{-j\pi k})$$

$$a_k = \frac{sin(\frac{\pi k}{2})(1 - e^{-j\pi k}e^{-j\frac{3\pi k}{2}}}{2k}$$

$$a_k = \frac{1}{T}x(j\frac{2\pi k}{T})$$

$$T = 2$$

4.31

(a) Show that the three LTI system with impulse responses

$h_1(t) = u(t),$

$h_2(t) = -2\delta(t) + 5e^{-t}u(t)$

and

$h_3(t) = 2te^{-t}u(t)$

all have the same response to $x(t) = cos(t)$

$x(t) = cos(t)$

$w = 1$

$h_1(t) = u(t)$

$h_1(s) = \frac{1}{5}$

$h_1(j\omega) = \frac{1}{j\omega}$

$|h_1(jt)| = 1h_1(jt) = -90$

$y_1(t) = |h_1(jt)|cos(t + h_1jt)$

$y_1(t) = cos(t - 90)$

$y_1(t) = sint$

$h_2(t) = -2(t) + 5e^{-2t}u(t)$

$h_2(s) = -2 + \frac{5}{s+2}$

$h_2(s) = \frac{5 - 2s - 4}{s+2}$

$h_2(j\omega) = \frac{1 - j2\omega}{2 + j\omega}$

$h_2(j1) = \frac{1 - j2}{2 + j}$

(b) Find the impulse response of another LTI system with the same response to cos(t).

This problem illustrates the fact that the response to cos(t) cannot used to specify an LTI system uniquely.