Assignment 9a - The Discrete-Time Fourier Transform

5.3. Determine the Fourier transform for $-\pi \leq w < \pi$ in the case of each of the following periodic signals:

a) $sin(\frac{\pi}{3}n + \frac{\pi}{4})$

Answers:

Let $x[n] = sin(\pi/3 + \pi/4)$ and $\Omega = \pi/3$. Since $\Omega = 2\pi/N$, we have fundamental period N=6. We can express the signal as:

$$x[n] = (1/2j)e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - (1/2j)e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})} = (1/2)e^{\frac{j\pi}{4}}e^{\frac{j2\pi}{6}} - (1/2j)e^{-\frac{j\pi}{4}}e^{-\frac{j2\pi}{6}}$$

Therefore, we can get the non-zero Fourier coefficients:

$a_1 = (1/2j)e^{\frac{j\pi}{4}}$ and $a_{-1} = -(1/2j)e^{\frac{j\pi}{4}}$.

Therefore, in the interval $-\pi \leq \omega \leq \pi$, we have:

$$X(e^{j\omega}) = 2\pi a_1 \delta(\omega - \frac{2\pi}{6}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{6})$$

$$X(e^{j\omega}) = (\pi/j)e^{j\pi/4}\delta(\omega - 2\pi/6) - e^{-j\pi/4}\delta(\omega + 2\pi/6)$$

b) $2 + cos(\frac{\pi}{6}n + \frac{\pi}{8})$

Answers:

Let $x[n] = 2 + cos(frac\pi 6n + \frac{\pi}{8})$ and $\Omega = \pi/6$, the fundamental perriod N=12.

$$x[n] = (1/2j)e^{j(\frac{\pi}{6}n+\frac{\pi}{8})} - (1/2j)e^{-j(\frac{\pi}{6}n+\frac{\pi}{8})} = (1/2j)e^{j(\frac{\pi}{8}n+\frac{\pi}{12})} - (1/2j)e^{-j(\frac{\pi}{8}n+\frac{\pi}{12})}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_2[n]$ in the range $-5 \leq k \leq 6$ as

$$a_0 = 2, a_1 = (1/2)e^{\frac{j\pi}{8}}, a_{-1} = (1/2)e^{-\frac{j\pi}{8}}.$$

Therefore, in the range $-\pi \leq \omega \leq \pi$, we obtain

$$X(e^{j\omega}) = 2\pi a_0 \delta(\omega) + 2\pi a_1 \delta(\omega - \frac{2\pi}{12}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{12})$$

$$X(e^{j\omega}) = 4\pi\delta(\omega) + \pi e^{\frac{j\pi}{8}}\delta(\omega - \frac{\pi}{6}) + e^{-\frac{j\pi}{8}}\delta(\omega + \frac{\pi}{6})$$

5.4. Use the Fourier transform synthesis equation (5.8) to determine the inverse Fourierb transforms of:

a) $X_1(e^{jw}) = \sum_{k=-\infty}^{\infty}[2\pi\delta(w-2\pi k) + \pi\delta(w-\frac{\pi}{2}-2\pi k) + \pi\delta(w+\frac{\pi}{2}-2\pi k)]$

Answers:

Using the Fourier transform synthese equation 5.8,

$$x_1[n] = (1/2\pi)\int_{-\pi}^{\pi} X_1(e^{j\omega})e^{j\omega n}d\omega$$

$$x_1[n] = (1/2\pi)\int_{-\pi}^{\pi} 2\pi\delta(\omega) + \pi\delta(\omega-\frac{\pi}{2}) + \pi\delta(\omega+\frac{\pi}{2})e^{j\omega n}d\omega$$

$$x_1[n] = e^{j0} + (1/2)e^{j(\pi/2)n} + (1/2)e^{-j(\pi/2)n}$$

$$x_1[n] = 1 + cos(\pi n/2)$$

b) $X_2(e^{jw}) = \begin{cases} 2j, & 0 < w \leq \pi, \\ -2j, & -\pi < w \leq 0 \end{cases}$

Answers:

Using the Fourier transform synthese equation 5.8,

$$x_2[n] = (1/2\pi)\int_{-\pi}^{\pi} X_2(e^{j\omega})e^{j\omega n}d\omega$$

$$x_2[n] = -(1/2\pi)\int_{-\pi}^{0} 2je^{j\omega n}d\omega + (1/2\pi)\int_{0}^{\pi} 2je^{j\omega n}d\omega$$

$$x_2[n] = (j/\pi) - \frac{1-e^{-jn\pi}}{jn} + \frac{e^{jn\pi}-1}{jn}$$

$$x_2[n] = -(4/(n\pi))sin^2(n\pi/2)$$

5.6. Given that $x[n]$ has Fourier transform $X(e^{jw})$, express the Fourier transforms of the following signals in terms of $X(e^{jw})$. You may use the Fourier transform properties listed in Table 5.1.

a) $x_1[n] = x[1-n] + x[-1-n]$

Given $x[n] <-> X(e^{j\omega})$, by time reversal property, we have:

$x[-n] <-> X(e^{-j\omega})$

Apply time shifting property, we have

$x[-n+1] <-> e^{-j\omega n}X(e^{-j\omega})$ and $x[-n-1] <-> e^{j\omega n}X(e^{-j\omega})$

Therefore,

$x_1[n] = x[-n+1] + x[-n-1] <-> e^{-j\omega n}X(e^{-j\omega}) + e^{j\omega n}X(e^{-j\omega})$

$x_1[n] = x[-n+1] + x[-n-1] <-> 2X(e^{-j\omega})cos\omega$

Answers:

b) $x_2[n] = \frac{x*[-n] + x[n]}{2}$

Using time reversal property (Sec 5.3.6), we have

$x[-n] <-> X(e^{-j\omega})$

Using the conjugation property on this, we have

$x*[-n] <-> X*(e^{j\omega})$

Therefore,

$x_2[n] = (1/2)(x*[-n] + x[n]) <-> (1/2)(X(e^{j\omega}) + X*(e^{j\omega}))$

$x_2[n] = (1/2)(x*[-n] + x[n]) <-> ReX(e^{j\omega})$

Answers:

c) $x_3[n] = (n-1)^2 x[n]$

Answers:

Using the differentiation in frequency property (Sec 5.3.8), we have

$nx[n] <-> j\frac{dX(e^{j\omega})}{d\omega}$

Using the same property a second time,

$n^2x[n] <-> \frac{d^2X(e^{j\omega})}{d^2\omega}$

Therefore,

$$x_3[n] = n^2 x[n] - 2nx[n] + 1 <-> \frac{d^2X(e^{j\omega})}{d\omega^2} - 2j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

5.28. The signals $x[n]$ and $g[n]$ are known to have Fourier transforms $X(e^{jw})$ and $G(e^{jw})$, respectively. Furthermore, $X(e^{jw})$ and $G(e^{jw})$ are related as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})G(e^{j(w-\theta)})d\theta \;=\; 1 + e^{-jw} \qquad \text{(P5.28-1)}$$

a) If $x[n] = (-1)^n$ determine a sequence $g[n]$ such that its Fourier transform $G(e^{jw})$ satisfies eq. (P5.28-1). Are there other possible solutions for $g[n]$?

Answers:

If $x[n] = (-1)^n$, g[n] only have non-zero value when n=0 or n=1, such that, g[0] =1 and g[1] = -1.

Therefore, $g[n] = \delta[n] - \delta[n-1]$.

b) Repeat the previous part for $x[n] = (\frac{1}{2})^n \; u[n]$

Answers:

If $x[n] = (\frac{1}{2})^n \; u[n]$ has tp be chosen such that g[n] = { 1, n=0; 2, n=1; 0, n> 1; any value, otherwise

Therefore there are many possible choices for g[n]

5.31. An LTI systemS with impulse response $h[n]$ and frequency response $H(e^{jw})$ is known to have the property that, when $-\pi \leq w_0 \leq \pi$,

$cosw_0 n \rightarrow w_0 \; cosw_0 n.$

a) Determine $H(e^{jw})$.

Answers:

From the given information, it is clear that when the input to the system is a comp[lex exponential; oif frequency ω_0, the output is a complex exponential of the same frequency but scaled by the | ω_0 |. Therefore, the frequency response of the system is $H(e^{j\omega}) = |\omega|$, for $0 \leq |\omega| \leq \pi$.

b) Determine $h[n]$.

Answers:

Take the inverse Fourier transform of the frequency response, we obtain

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{0} -\omega e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{0}^{\pi} \omega e^{j\omega n}d\omega$$

$$h[n] = \frac{1}{\pi}\int_{0}^{\pi} \omega cos(\omega n)d\omega$$

$$h[n] = \frac{1}{2\pi}\frac{cos(n\pi) - 1}{n^2}$$