Intermediate complex analysis

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Abstract

This article consists of some notes taken by the author while studying **Intermediate complex analysis**.

References: [1], [2], [3], [4], [5], [6].

References

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Contents

1 Preliminaries

1.1 Preliminaries in complex analysis

1.1.1 Cauchy integral formula

Theorem 1.1 (Cauchy Integral Formula 1^{st} form). *Let f be a holomorphic function defined on a neighborhood of $\overline{\Delta(a;r)}$. For any $z \in \Delta(a;r)$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta(a;r)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

More generally, if $\Omega \subset \mathbb{C}$ is a domain with piecewise C^1 boundary, then

Corollary 1.1 (Generalized Cauchy Integral Formula).

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Proof. Let $\Omega_\varepsilon = \Omega - \delta(z,\varepsilon)$ where ε is sufficiently small such that $\overline{\delta(z,\varepsilon)} \cap \partial\Omega = \emptyset$. Since $f(\zeta)/(\zeta - z)$ is holomorphic as a function of ζ on Ω_ε. Using Stoke's Theorem and the above theorem yields the result. □

1.1.2 Residue theorem

Let Ω be defined as above and f be a meromorphic function on $\overline{\Omega}$ without pole on $\partial\Omega$. It automatically implies that f has only finite number of poles. Then, we have

Definition 1.1 (principal part). *Principal part means the sum of all negative–power terms in the Laurent expansion of f at the point a.*

If we take the ***Laurent series*** *of f at point a:*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \qquad (0 < |z-a| < R),$$

then the principal part of f at a is

$$\sum_{n=1}^{m} \frac{c_{-n}}{(z-a)^n}.$$

(If a is a pole, the negative powers stop at some finite order m; if a is an essential singularity, there are infinitely many negative–power terms.)

Theorem 1.2 (Residue Theorem)**.**

$$\int_{\partial\Omega} f(z)dz = 2\pi i \sum_j \mathrm{Res}(f; a_j)$$

where a_j's are poles of meromorphic function f.

Proof. For f meromorphic, let $\{a_j\}$ be the finite set of poles of f in Ω. Consider the function

$$g = f - \sum_j P_j(f)$$

where $P_j(f)$ is the principal part of f at a_j. Then, apply the generalized Cauchy integral formula yields the result. □

We have a generalization for the residue theorem. Take $\overline{\Omega} \subset \Omega'$, consider a discrete set of points $E \subset \Omega$ such that $E \cap \partial\Omega = \emptyset$. If $f \in \mathcal{O}(\Omega' - E)$, then the Residue theorem still holds for f. The proof follows the same argument as above while the principal parts

$$P_j(f) = \sum_{n=-\infty}^{-1} c_n^{(j)}(z-a_j)^n$$

1.1.3 Winding number

Definition 1.2 (winding number)**.** *Let $\Gamma : [a, b] \to \mathbb{C}$ be a closed piecewise C^1 curve. For $z \notin \mathrm{Im}(\Gamma)$. Define*

$$n(\Gamma, z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z}$$

$n(\Gamma, z)$ is called the winding number of Γ at z.

Remark 1.1. *$n(\Gamma, z)$ is an integer and hence is constant on each connected component of $\mathbb{C} - \mathrm{Im}(\Gamma)$ by continuity.*

1.1.4 Cauchy theorem with winding numbers.

Using the winding number, we can generalize the Cauchy integral formula by allowing the domain Ω be multi-connected open subset of $\mathbb{C}$.

Theorem 1.3 (Homotopy form of Cauchy integral theorem)**.** *Let γ be a closed piecewise C^1 curve. Assume that γ is homotopic to a constant in Ω. Then, for any $z \in \Omega - \mathrm{Im}(\gamma)$,*

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

We say that a closed continuous curve γ is **homotopic** to Γ if and only if $\exists F : [0,1] \times [0,1] \to \Omega$ such that

$$\begin{cases} F(0,s) = \gamma(s) \\ F(1,s) = \Gamma(s) \end{cases}$$

For our purpose, taking γ, Γ to be piecewise C^1. We may assume that $\gamma_t(s) = F(t,s)$ is piecewise C^1 in s.

If $\Gamma(s) = p \in \Omega$, then γ is said to be **homotopic to a constant**. In this case, one may assume that $F(t,0) = p, \ \forall t \in [0,1]$ which means that the end point of each curve for each time t is fixed at p.

Proof. Take homotopy $F : [0,1] \times [0,1] \to \Omega$,

$$\begin{cases} F(0,s) = \gamma(s), \quad \gamma(0) = p = \gamma(1) \\ F(1,s) = p \end{cases}$$

Consider

$$h(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

which is a holomorphic function in ζ defined on $\Omega - \{z\}$. Then, $\mathrm{Res}(h, z) = f(z)$ and the principal part is $f(z)/(\zeta - z)$. Define

$$\begin{aligned} g(\zeta) &= h(\zeta) - \text{principal part at } z \\ &= \frac{f(\zeta)}{\zeta - z} - \frac{f(z)}{\zeta - z} \\ &= \frac{f(\zeta) - f(z)}{\zeta - z} \end{aligned}$$

which has a removable singularity at z. Therefore,

$$\int_{\gamma_t} g(\zeta) \, d\zeta = \int_{\gamma_1} g(\zeta) \, d\zeta = 0.$$

$\square$

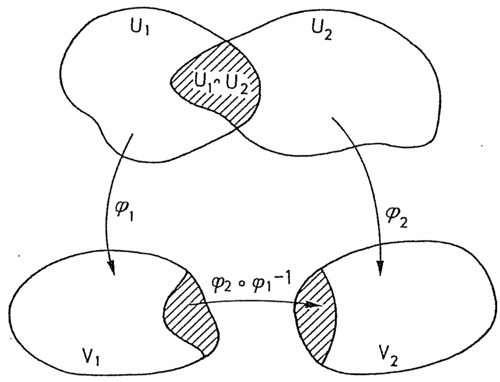

Figure 1: complex manifold

1.2 Preliminaries in Riemann surface

1.2.1 Definition of Riemann surface

In this part we define Riemann surfaces, holomorphic and meromorphic functions on them and also holomorphic maps between Riemann surfaces.

Riemann surfaces are two-dimensional manifolds together with an additional structure which we are about to define. As is well known, an

n-dimensional manifold is a **Hausdorff topological space** X such that every point $a \in X$ has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 1.3. *Let X be a two-dimensional manifold. A complex chart on X is a homeomorphism $\varphi : U \to V$ of an open subset $U \subset X$ onto an open subset $V \subset \mathbb{C}$. Two complex charts $\varphi_i : U_i \to V_i, i = 1, 2$ are said to be holomorphically compatible if the map*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is biholomorphic.

A **complex atlas** on X is a system $\mathcal{A} = \{\varphi_i : U_i \to V_i,\ i \in I\}$ of charts which are holomorphically compatible and which cover X, i.e., $\bigcup_{i \in I} U_i = X$.

Two complex atlases $\mathcal{A}$ and $\mathcal{A}'$ on X are called **analytically equivalent** if every chart of $\mathcal{A}$ is holomorphically compatible with every chart of $\mathcal{A}'$.

Remark 1.2.

1. *If $\varphi : U \to V$ is a complex chart, U_1 is open in U and $V_1 := \varphi(U_1)$, then $\varphi|_{U_1} : U_1 \to V_1$ is a chart which is holomorphically compatible with $\varphi : U \to V$.*

2. *Since the composition of biholomorphic mappings is again biholomorphic, one easily sees that the notion of analytic equivalence of complex atlases is an equivalence relation.*

Definition 1.4 (complex structure)**.** *By a complex structure on a two-dimensional manifold X we mean an equivalence class of analytically equivalent atlases on X.*

Thus a complex structure on X can be given by the choice of a complex atlas. Every complex structure Σ on X contains a unique maximal atlas $\mathcal{A}^*$. If $\mathcal{A}$ is an arbitrary atlas in Σ, then $\mathcal{A}^*$ consists of all complex charts on X which are holomorphically compatible with every chart of $\mathcal{A}$.

Definition 1.5 (Riemann surface)**.** *A Riemann surface is a pair (X, Σ), where X is a connected two-dimensional manifold and Σ is a complex structure on X.*

One usually writes X instead of (X, Σ) whenever it is clear which complex structure Σ is meant. Sometimes one also writes $(X, \mathcal{A})$ where $\mathcal{A}$ is a representative of Σ.

If X is a Riemann surface, then by a chart on X we always mean a complex chart belonging to the maximal atlas of the complex structure on X.

Remark 1.3. *Locally a Riemann surface X is nothing but an open set in the complex plane. For, if $\varphi : U \to V \subset \mathbb{C}$ is a chart on X, then φ maps the open set $U \subset X$ bijectively onto V. However, any given point of X is contained in many different charts and no one of these is distinguished from the others. For this reason we may only carry over to Riemann surfaces those notions from complex analysis in the plane which remain invariant under biholomorphic mappings, i.e., those notions which do not depend on the choice of a particular chart.*

1.2.2 Examples of Riemann Surfaces

Example 1.1 (Complex Plane $\mathbb{C}$)**.** *Its complex structure is defined by the atlas whose only chart is the identity map $\mathbb{C} \to \mathbb{C}$.*

Example 1.2 (Domains)**.** *Suppose X is a Riemann surface and $Y \subset X$ is a domain, i.e., a connected open subset. Then Y has a natural complex structure which makes it a Riemann surface. Namely, one takes as its atlas all those complex charts $\varphi : U \to V$ on X, where $U \subset Y$. In particular, every domain $Y \subset \mathbb{C}$ is a Riemann surface.*

Example 1.3 (Riemann sphere $\mathbb{P}^1$)**.** *Let $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$, where ∞ is a symbol not contained in $\mathbb{C}$. Introduce the following topology on $\mathbb{P}^1$. The open sets are the usual open sets $U \subset \mathbb{C}$ together with sets of the form $V \cup \{\infty\}$, where $V \subset \mathbb{C}$ is the complement of a compact set $K \subset \mathbb{C}$. With this topology $\mathbb{P}^1$ is a compact Hausdorff topological space, homeomorphic to the 2-sphere S^2. Set*

$$U_1 := \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}, \qquad U_2 := \mathbb{P}^1 \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}.$$

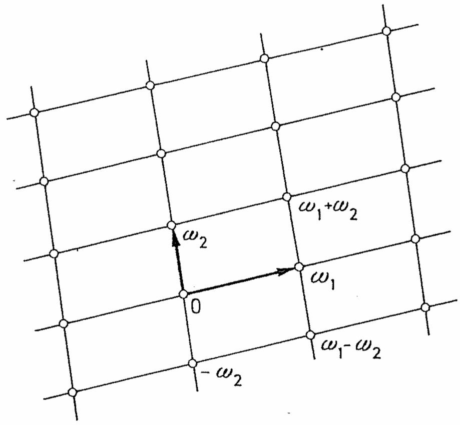

Figure 2: Tori

Define maps $\varphi_i : U_i \to \mathbb{C},\ i = 1, 2,$ *as follows.* φ_1 *is the identity map and*

$$\varphi_2(z) := \begin{cases} 1/z, & \text{for } z \in \mathbb{C}^*, \\ 0, & \text{for } z = \infty. \end{cases}$$

Clearly these maps are homeomorphisms and thus $\mathbb{P}^1$ *is a two-dimensional manifold. Since* U_1 *and* U_2 *are connected and have non-empty intersection,* $\mathbb{P}^1$ *is also connected.*

The complex structure on $\mathbb{P}^1$ *is now defined by the atlas consisting of the charts* $\varphi_i : U_i \to \mathbb{C}, i = 1, 2$*. We must show that the two charts are holomorphically compatible. But* $\varphi_1(U_1 \cap U_2) = \varphi_2(U_1 \cap U_2) = \mathbb{C}^*$ *and*

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^*, \quad z \mapsto 1/z,$$

is biholomorphic.

Remark 1.4. *The notation* $\mathbb{P}^1$ *comes from the fact that one may consider* $\mathbb{P}^1$ *as the 1-dimensional projective space over the field of complex numbers.*

Example 1.4 (Tori)**.** *Suppose* $\omega_1, \omega_2 \in \mathbb{C}$ *are linearly independent over* $\mathbb{R}$*. Define*

$$\Gamma := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{\, n\omega_1 + m\omega_2 : n, m \in \mathbb{Z} \,\}.$$

Γ *is called the* ***lattice*** *spanned by* ω_1 *and* ω_2 *(Fig. 2). Two complex numbers* $z, z' \in \mathbb{C}$ *are called equivalent mod* Γ *if* $z - z' \in \Gamma$*. The set of all equivalence classes is denoted by* $\mathbb{C}/\Gamma$*. Let* $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ *be the canonical projection, i.e., the map which associates to each point* $z \in \mathbb{C}$ *its equivalence class mod* Γ*.*

Introduce the following topology (the quotient topology) on $\mathbb{C}/\Gamma$*. A subset* $U \subset \mathbb{C}/\Gamma$ *is open precisely if* $\pi^{-1}(U) \subset \mathbb{C}$ *is open. With this topology* $\mathbb{C}/\Gamma$ *is a Hausdorff topological* *space and the quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is continuous. Since $\mathbb{C}$ is connected, $\mathbb{C}/\Gamma$ is also connected. As well $\mathbb{C}/\Gamma$ is compact, for it is covered by the image under π of the compact parallelogram*

$$P := \{\lambda\omega_1 + \mu\omega_2 : \lambda, \mu \in [0,1]\}.$$

The map π is open, i.e., the image of every open set $V \subset \mathbb{C}$ is open. To see this one has to show that $\widehat{V} := \pi^{-1}(\pi(V))$ is open. But

$$\widehat{V} = \bigcup_{\omega\in\Gamma} (\omega + V).$$

Since every set $\omega + V$ is open, so is $\widehat{V}$.

The complex structure on $\mathbb{C}/\Gamma$ is defined in the following way. Let $V \subset \mathbb{C}$ be an open set such that no two points in V are equivalent under Γ. Then $U := \pi(V)$ is open and $\pi|_V : V \to U$ is a homeomorphism. Its inverse $\varphi : U \to V$ is a complex chart on $\mathbb{C}/\Gamma$. Let $\mathcal{U}$ be the set of all charts obtained in this fashion. We have to show that any two charts $\varphi_i : U_i \to V_i$, $i = 1, 2$, belonging to $\mathcal{U}$ are holomorphically compatible. Consider the map

$$\psi := \varphi_2 \circ \varphi_1^{-1} : \ \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2).$$

For every $z \in \varphi_1(U_1 \cap U_2)$ one has $\pi(\psi(z)) = \varphi_1^{-1}(z) = \pi(z)$ and thus $\psi(z) - z \in \Gamma$. Since Γ is discrete and ψ is continuous, this implies that $\psi(z) - z$ is constant on every connected component of $\varphi_1(U_1 \cap U_2)$. Thus ψ is holomorphic. Similarly ψ^{-1} is also holomorphic.

Now let $\mathbb{C}/\Gamma$ have the complex structure defined by the complex atlas $\mathcal{U}$.

Remark 1.5. *Let $S^1 = \{z \in \mathbb{C} : \ |z| = 1\}$ be the unit circle. The map which associates to the point of $\mathbb{C}/\Gamma$ represented by $\lambda\omega_1 + \mu\omega_2$, $(\lambda, \mu \in \mathbb{R})$, the point*

$$\left(e^{2\pi i\lambda},\, e^{2\pi i\mu}\right) \in S^1 \times S^1,$$

is a homeomorphism of $\mathbb{C}/\Gamma$ onto the torus $S^1 \times S^1$.

Definition 1.6 (holomorphic function and sructure sheaf on Riemann surface)**.** *Let X be a Riemann surface and $Y \subset X$ an open subset. A function $f : Y \to \mathbb{C}$ is called* ***holomorphic****, if for every chart $\psi : U \to V$ on X the function*

$$f \circ \psi^{-1} : \ \psi(U \cap Y) \longrightarrow \mathbb{C}$$

is holomorphic in the usual sense on the open set $\psi(U \cap Y) \subset \mathbb{C}$. The set of all functions holomorphic on Y will be denoted by $\mathcal{O}(Y)$.

Remark 1.6. *1. The sum and product of holomorphic functions are again holomorphic. Also constant functions are holomorphic. Thus $\mathcal{O}(Y)$ is a $\mathbb{C}$-algebra.*

2. *Of course the condition in the definition does not have to be verified for all charts in a maximal atlas on X, just for any family of charts covering Y. Then it is automatically fulfilled for all other charts.*

3. *Every chart $\psi : U \to V$ on X is, in particular, a complex-valued function on U. Trivially it is holomorphic. One also calls ψ a local coordinate or a uniformizing parameter and (U, ψ) a* coordinate neighborhood *of any point $a \in U$. In this context one generally uses the letter z instead of ψ.*

Theorem 1.4 (Riemann's Removable Singularities Theorem)**.** *Let U be an open subset of a Riemann surface and let $a \in U$. Suppose the function $f \in \mathcal{O}(U \setminus \{a\})$ is bounded in some neighborhood of a. Then f can be extended uniquely to a function $\widetilde{f} \in \mathcal{O}(U)$.*

This follows directly from Riemann's Removable Singularities Theorem in the complex plane.

We now define holomorphic mappings between Riemann surfaces.

Definition 1.7 (holomorphic map)**.** *Suppose X and Y are Riemann surfaces. A continuous mapping $f : X \to Y$ is called* ***holomorphic****, if for every pair of charts $\psi_1 : U_1 \to V_1$ on X and $\psi_2 : U_2 \to V_2$ on Y with $f(U_1) \subset U_2$, the mapping*

$$\psi_2 \circ f \circ \psi_1^{-1} : V_1 \longrightarrow V_2$$

is holomorphic in the usual sense. A mapping $f : X \to Y$ is called ***biholomorphic*** *if it is bijective and both $f : X \to Y$ and $f^{-1} : Y \to X$ are holomorphic. Two Riemann surfaces X and Y are called* ***isomorphic*** *if there exists a biholomorphic mapping $f : X \to Y$.*

Remark 1.7. 1. *In the special case $Y = \mathbb{C}$, holomorphic mappings $f : X \to \mathbb{C}$ are clearly the same as holomorphic functions.*

2. *If X, Y and Z are Riemann surfaces and $f : X \to Y$ and $g : Y \to Z$ are holomorphic mappings, then the composition $g \circ f : X \to Z$ is also holomorphic.*

3. *A continuous mapping $f : X \to Y$ between two Riemann surfaces is holomorphic precisely if for every open set $V \subset Y$ and every holomorphic function $\psi \in \mathcal{O}(V)$, the "****pull–back****" function $\psi \circ f : f^{-1}(V) \to \mathbb{C}$ is contained in $\mathcal{O}(f^{-1}(V))$.*

   In this way a holomorphic mapping $f : X \to Y$ induces a mapping

$$f^* : \mathcal{O}(V) \longrightarrow \mathcal{O}\big(f^{-1}(V)\big), \quad f^*(\psi) = \psi \circ f.$$

   One can easily check that f^* is a ring homomorphism. If $g : Y \to Z$ is another holomorphic mapping, W is open in Z, $V := g^{-1}(W)$ and $U := f^{-1}(V)$, then

$(g \circ f)^* : \mathcal{O}(W) \to \mathcal{O}(U)$ is the composition of the mappings $g^* : \mathcal{O}(W) \to \mathcal{O}(V)$ and $f^* : \mathcal{O}(V) \to \mathcal{O}(U)$, i.e.,

$$(g \circ f)^* = f^* \circ g^*.$$

Theorem 1.5 (Identity Theorem)**.** *Suppose X and Y are Riemann surfaces and $f_1, f_2 : X \to Y$ are two holomorphic mappings which coincide on a set $A \subset X$ having a limit point $a \in X$. Then f_1 and f_2 are identically equal.*

Proof. Let G be the set of all points $x \in X$ having an open neighborhood W such that $f_1|_W = f_2|_W$. By definition G is open. We claim that G is also closed. For, suppose b is a boundary point of G. Then $f_1(b) = f_2(b)$ since f_1 and f_2 are continuous. Choose charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y with $b \in U$ and $f_i(U) \subset U'$. We may also assume that U is connected. The mappings

$$g_i := \psi \circ f_i \circ \varphi^{-1} : V \to V' \subset \mathbb{C}$$

are holomorphic. Since $U \cap G \neq \varnothing$, the Identity Theorem for holomorphic functions on domains in $\mathbb{C}$ implies g_1 and g_2 are identically equal. Thus $f_1|_U = f_2|_U$. Hence $b \in G$ and thus G is closed. Now since X **is connected** either $G = \varnothing$ or $G = X$. But the first case is excluded since $a \in G$ (using the Identity Theorem in the plane again). Hence f_1 and f_2 coincide on all of X. ☐

Definition 1.8 (meromorphic functions in Riemann surface)**.** *Let X be a Riemann surface and Y be an open subset of X. By a* ***meromorphic function*** *on Y we mean a holomorphic function $f : Y' \to \mathbb{C}$, where $Y' \subset Y$ is an open subset, such that the following hold:*

1. *$Y \setminus Y'$ contains only isolated points.*
2. *For every point $p \in Y \setminus Y'$ one has*

$$\lim_{x \to p} |f(x)| = \infty.$$

The points of $Y \setminus Y'$ are called the ***poles*** *of f. The set of all meromorphic functions on Y is denoted by $\mathcal{M}(Y)$.*

Remark 1.8. 1. *Let (U, z) be a coordinate neighborhood of a pole p of f with $z(p) = 0$. Then f may be expanded in a Laurent series*

$$f = \sum_{v=-k}^{\infty} c_v z^v$$

in a neighborhood of p.

2. *$\mathcal{M}(Y)$ has the natural structure of a $\mathbb{C}$-algebra. First of all the sum and the product of two meromorphic functions $f, g \in \mathcal{M}(Y)$ are holomorphic functions at those points where both f and g are holomorphic. Then one holomorphically extends, using Riemann's Removable Singularities Theorem, $f+g$ (resp. fg) across any singularities which are removable.*

Example 1.5. *Suppose $n \geq 1$ and let*

$$F(z) = z^n + c_1 z^{n-1} + \cdots + c_n, \qquad c_k \in \mathbb{C},$$

be a polynomial. Then F defines a holomorphic mapping $F : \mathbb{C} \to \mathbb{C}$. If one thinks of $\mathbb{C}$ as a subset of $\mathbb{P}^1$, then $\lim_{z\to\infty} |F(z)| = \infty$. Thus $F \in \mathcal{M}(\mathbb{P}^1)$.

We now interpret meromorphic functions as holomorphic mappings into the Riemann sphere.

Theorem 1.6. *Suppose X is a Riemann surface and $f \in \mathcal{M}(X)$. For each pole p of f, define $f(p) := \infty$. Then $f : X \to \mathbb{P}^1$ is a holomorphic mapping. Conversely, if $f : X \to \mathbb{P}^1$ is a holomorphic mapping, then f is either identically equal to ∞ or else $f^{-1}(\infty)$ consists of isolated points and $f : X \setminus f^{-1}(\infty) \to \mathbb{C}$ is a meromorphic function on X.*

From now on we will identify a meromorphic function $f \in \mathcal{M}(X)$ with the corresponding holomorphic mapping $f : X \to \mathbb{P}^1$.

Proof.

1. Let $f \in \mathcal{M}(X)$ and let P be the set of poles of f. Then f induces a mapping $f : X \to \mathbb{P}^1$ which is clearly continuous. Suppose $\varphi : U \to V$ and $\psi : U' \to V'$ are charts on X and $\mathbb{P}^1$ respectively with $f(U) \subset U'$. We have to show that

$$g := \psi \circ f \circ \varphi^{-1} : V \longrightarrow V'$$

is holomorphic. Since f is holomorphic on $X \setminus P$, it follows that g is holomorphic on $V \setminus \varphi(P)$. Hence by Riemann's Removable Singularities Theorem, g is holomorphic on all of V.

2. The converse follows from the Identity Theorem.

□

Remark 1.9. *The Identity Theorem also holds for meromorphic functions on a Riemann surface. Thus any function $f \in \mathcal{M}(X)$ which is not identically zero has only isolated zeros. This implies that $\mathcal{M}(X)$ is a field.*

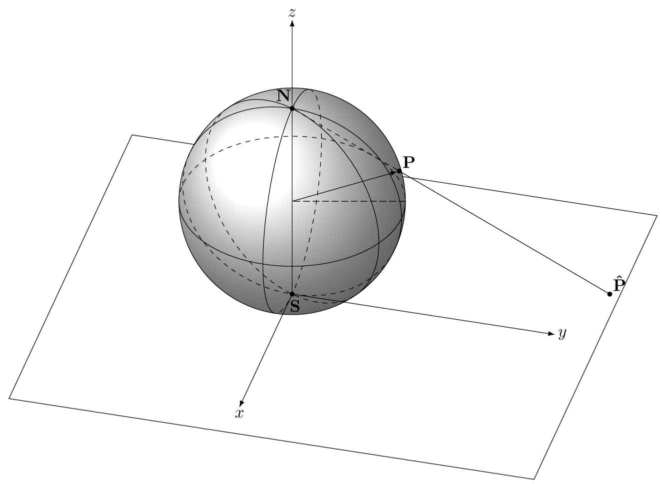

Figure 3: Riemann sphere

2 Mittag-Leffler problem, Weierstrass problem and the field of meromorphic functions on Riemann sphere $\mathbb{P}^1$

2.1 Riemann sphere $\mathbb{P}^1$

Consider the Euclidean unit sphere in $\mathbb{R}^3$, the set $\{(X,Y,Z) \mid X^2+Y^2+Z^2=1\}$. Define the 2 charts:

$$V_1 = \{(X,Y,Z) \mid X^2+Y^2+Z^2=1, Z > -\frac{3}{5}\}, \quad \phi_1 : (X,Y,Z) \mapsto \frac{X+iY}{1+Z}.$$

$$V_2 = \{(X,Y,Z) \mid X^2+Y^2+Z^2=1, Z < \frac{3}{5}\}, \quad \phi_2 : (X,Y,Z) \mapsto \frac{X-iY}{1-Z}.$$

Let U_1 (resp. U_2), the image of ϕ_1 (resp. ϕ_2), be the open disc $D(0,2) \subset \mathbb{C}$. The image by ϕ_1 (resp. ϕ_2) of $V_1 \cap V_2$ is the annulus $\frac{1}{2} < |z| < 2$ in U_1 (resp. U_2). On this annulus, the transition map

$$\phi_2 \circ \phi_1^{-1} = \psi : z \mapsto \frac{1}{z}$$

is analytic, bijective, and its inverse is analytic. This defines the **Riemann sphere**, which is compact (the 2 charts are bounded discs $D(0,2)$), connected and simply connected Riemann surface. The map ϕ_1 (resp. ϕ_2) is called the **stereographic projection** from the south (resp. north) pole of the sphere to the Euclidean plane $Z=0$ in $\mathbb{R}^3$, identified with $\mathbb{C}$.

Another definition of the Riemann sphere is the **complex projective plane** $\mathbb{CP}^1$:

$$\mathbb{CP}^1 = \frac{\{(z_1,z_2)\in\mathbb{C}\times\mathbb{C} \mid (z_1,z_2)\neq(0,0)\}}{(z_1,z_2)\sim(\lambda z_1,\lambda z_2),\ \forall\,\lambda\in\mathbb{C}^*}.$$

It has also an atlas of 2 charts,

$$V_1 = \{[(z_1,z_2)] \mid z_2\neq 0\}, \quad \phi_1:\ [(z_1,z_2)]\mapsto z_1/z_2,$$

$$V_2 = \{[(z_1,z_2)] \mid z_1\neq 0\}, \quad \phi_2:\ [(z_1,z_2)]\mapsto z_2/z_1,$$

with transition map $z\mapsto 1/z$ (everything is well defined on the quotient by equivalent relation $\sim$). Thus $\mathbb{CP}^1$ is analytically isomorphic to the Riemann sphere previously defined.

Another definition of the Riemann sphere is from an abstract atlas of 2 charts

$$U_1 = D(0,R_1)\subset\mathbb{C} \quad \text{and} \quad U_2 = D(0,R_2)\subset\mathbb{C}$$

whose radii satisfy $R_1R_2>1$. The 2 discs are glued by the analytic transition map

$$\psi:\ z\mapsto\frac{1}{z}$$

from the annulus $\frac{1}{R_2}<|z|<R_1$ in U_1 to the annulus $\frac{1}{R_1}<|z|<R_2$ in U_2.

In other words, consider the following subset of $\mathbb{C}\times\{1,2\}$

$$\frac{\{(z,i)\in\mathbb{C}\times\{1,2\} \mid z\in U_i\}}{\sim}.$$

quotiented by the equivalence relation

$$(z,i)\sim(\tilde z,j) \iff \big(i=j \text{ and } z=\tilde z\big) \ \text{ or } \ \big(i+j=3 \text{ and } z\tilde z=1\big).$$

This Riemann surface is analytically isomorphic to the Riemann sphere previously defined.

Notice that one can choose R_1 very large, and R_2 very small, and even consider a projective limit $R_1\to\infty$ and $R_2\to 0$, in other words glue the whole $U_1=\mathbb{C}$ to the single point $U_2=\{0\}$. Notice that the point $z'=0$ in U_2 should correspond to the point $z=1/z'=\infty$ in $U_1=\mathbb{C}$. In this projective limit, by adding a single point to $\mathbb{C}$, we turn it into a compact Riemann surface $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. The topology of $\overline{\mathbb{C}}$ is generated by the open sets of $\mathbb{C}$, as well as all the sets

$$V_R = \{\infty\}\cup\{\, z\in\mathbb{C} \mid |z|>R\,\} \qquad \text{for all } R\geq 0.$$

These open sets form a basis of neighborhoods of ∞. With this topology, $\overline{\mathbb{C}}$ is compact.

This justifies that the Riemann sphere is called a compactification of $\mathbb{C}$:

$$\mathbb{CP}^1 = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

2.2 $\mathbf{Aut}(\mathbb{P}^1)$ and fractional linear transformations on $\mathbb{P}^1$

Definition 2.1 (automorphism group: $\mathrm{Aut}(X)$)**.** *Let X be a Riemann surface,*

$$\mathrm{Aut}(X) = \{\varphi : X \to X \mid \varphi \text{ is biholomorphic}\},$$

where "biholomorphic" means holomorphic, bijective, and with holomorphic inverse.

Remark 2.1. *Conformal equivalence and biholomorphism for any complex dimensions are the same.*

Definition 2.2 (Conformal equivalence)**.** *Two Riemann surfaces X, Y are **conformally equivalent** (or **biholomorphically equivalent**) if there is a biholomorphism $\varphi : X \to Y$.*

Definition 2.3 (Fractional linear (Möbius) transformation)**.** *Let $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, we call*

$$\varphi(z) = \frac{az + b}{cz + d},$$

a fractional linear (Möbius) transformation.

Remark 2.2. *If $c = d = 0$ then $ad - bc = 0$, so that case is not allowed.*

We extend φ from $\mathbb{C}$ to the sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by the following natural conventions:

- (Regular points) If $z_0 \in \mathbb{C}$ and $cz_0 + d \neq 0$, set

$$\varphi(z_0) = \frac{az_0 + b}{cz_0 + d} \in \mathbb{C}.$$

- (Poles) If $z_0 \in \mathbb{C}$ and $cz_0 + d = 0$ (equivalently $z_0 = -\dfrac{d}{c}$ with $c \neq 0$), then $az_0 + b = -\frac{ad}{c} + b = \frac{-ad+bc}{c}$ is constant, set

$$\lim_{z \to z_0} \varphi(z) = \lim_{z \to z_0} \left| \frac{az + b}{cz + d} \right| = \infty,$$

  so we define $\varphi(z_0) = \infty \in \mathbb{P}^1$

- (At infinity) For $z_0 = \infty$, we have

$$\varphi(\infty) = \lim_{z \to \infty} \varphi(z) = \lim_{z \to \infty} \frac{az + b}{cz + d} = \lim_{z \to \infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} \longrightarrow \begin{cases} \dfrac{a}{c}, & c \neq 0, \\ \infty, & c = 0. \end{cases}$$

The set of all the fractional linear transformations on $\mathbb{P}^1$ forms a group, with the multiplication operation given by composition.

Proposition 2.1 (FractLin($\mathbb{P}^1$))**.** *The group of all the fractional linear transformations on $\mathbb{P}^1$:*

$$\mathrm{FractLin}(\mathbb{P}^1) := \left\{ \frac{az+b}{cz+d} \,:\, ad-bc \neq 0 \right\}.$$

Proof.

- (Closure) For $\varphi(z) = \dfrac{az+b}{cz+d}$ and $\psi(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$,

$$(\varphi\circ\psi)(z) = \frac{(a\alpha+b\gamma)z+(a\beta+b\delta)}{(c\alpha+d\gamma)z+(c\beta+d\delta)},$$

  which is again fractional linear. Its determinant equals $(ad-bc)(\alpha\delta-\beta\gamma) \neq 0$.

- (Associativity) Follows from associativity of function composition, equivalently, matrix multiplication is associative:

$$(\varphi\circ\psi)\circ\theta = \varphi\circ(\psi\circ\theta) \quad\Longleftrightarrow\quad (AB)C = A(BC).$$

- (Identity)

$$\mathrm{id}(z) = z = \frac{1\cdot z+0}{0\cdot z+1} \quad\Longleftrightarrow\quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

  lies in the set.

- (Inverse) If $\varphi(z) = \dfrac{az+b}{cz+d}$, then

$$\varphi^{-1}(z) = \frac{dz-b}{-cz+a},$$

  corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, hence $\varphi^{-1} \in \mathrm{FractLin}(\mathbb{P}^1)$.

$\square$

The correspondence between the composition of linear transformations and matrix multiplication: take

$$\varphi(w) = \frac{aw+b}{cw+d}, \qquad \psi(z) = \frac{\alpha z+\beta}{\gamma z+\delta}, \qquad ad-bc\neq 0, \quad \alpha\delta-\beta\gamma\neq 0.$$

Compute

$$(\varphi\circ\psi)(z) = \frac{a\frac{\alpha z+\beta}{\gamma z+\delta}+b}{c\frac{\alpha z+\beta}{\gamma z+\delta}+d} = \frac{(a\alpha+b\gamma)z+(a\beta+b\delta)}{(c\alpha+d\gamma)z+(c\beta+d\delta)}.$$

Hence the composition is again fractional linear: closure holds.

Associate

$$\varphi \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \psi \longleftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and

$$(\varphi \circ \psi)(z) = \frac{Az + B}{Cz + D}.$$

Determinants multiply:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

so the composite is nondegenerate, because assumption $ad \neq cd$ means $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Finelly, under composition, $\mathrm{FractLin}(\mathbb{P}^1)$ is a group. Via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \mapsto \frac{az + b}{cz + d} \right),$$

this group is isomorphic to

$$\mathrm{PGL}(2, \mathbb{C}) = \mathrm{GL}(2, \mathbb{C})/\mathbb{C}^*$$