Navier-Stokes Equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \vec{U} = 0 \qquad \text{mass}$$

$$\frac{\partial \rho \vec{U}}{\partial t} + \boldsymbol{\nabla} \cdot \rho \vec{U} \times \vec{U} + \boldsymbol{\nabla} \cdot (\tau - PI) = 0 \qquad \text{momentum}$$

$$\frac{\partial \rho e}{\partial t} + \boldsymbol{\nabla} \cdot (\rho e + P)\vec{U} - \boldsymbol{\nabla} \cdot \tau \vec{U} - \boldsymbol{\nabla} \cdot k \boldsymbol{\nabla} T = 0 \qquad \text{energy}$$

$$\vec{U} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

ρ - density [kg/m^3]

u, v, w - velocity in x, y, z in Cartesian coordinates [m/s]

P - pressure [Pa]

e - internal energy [J/kg]

k - thermal conductivity [$\frac{W}{m \cdot k}$]

T - temperature [K]

μ - dynamic viscosity [$\frac{N \cdot s}{m^2}$]

τ - viscous stress tensor (symmetric)

$$\begin{aligned}
\tau_{xy} &= \tau_{yx} = \mu(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}) \\
\tau_{xz} &= \tau_{zx} = \mu(\tfrac{\partial u}{\partial z} + \tfrac{\partial w}{\partial x}) \\
\tau_{yz} &= \tau_{yz} = \mu(\tfrac{\partial v}{\partial z} + \tfrac{\partial w}{\partial y}) \\
\tau_{xx} &= \tfrac{2}{3}\mu(2\tfrac{\partial u}{\partial x} - \tfrac{\partial v}{\partial y} - \tfrac{\partial w}{\partial z}) \\
\tau_{yy} &= \tfrac{2}{3}\mu(-\tfrac{\partial u}{\partial x} + 2\tfrac{\partial v}{\partial y} - \tfrac{\partial w}{\partial z}) \\
\tau_{zz} &= \tfrac{2}{3}\mu(-\tfrac{\partial u}{\partial x} - \tfrac{\partial v}{\partial y} + 2\tfrac{\partial w}{\partial z})
\end{aligned}$$

Navier-Stokes equations are a set of equations that are solved simultaneously. For a 2D fluids domain, we need to solve 4 equations: Density (ρ), X-momentum (ρu), Y-momentum (ρv), and Energy (ρe). Once these equations are solved, the temperature gradient normal to the wall can be computed using e (Energy). The temperature gradient is not directly calculated from the Navier-Stokes equations. T is computed for a given e through the ideal gas law (second thermodynamics equation below), which is used to calculate the gradient of T. As a reminder, in the challenge problem, q_w is transferred from the fluid to the solid as a Neumann boundary condition, and the new T from the solid is fed back into the fluid domain. To get q_w from T, take the dot product of the gradient of T with the wall normal vector and multiply by thermal conductivity (k):

$$q_w = k\boldsymbol{\nabla} T \cdot \vec{n}$$

Equation of State (closure equations)

Based on the problem assumptions and the physical model

$$
\begin{aligned}
P &= \rho R T && \text{thermodynamics, ideal gas law} \\
e &= \frac{R}{\gamma - 1} T && \text{thermodynamics, ideal gas law} \\
e &= \frac{P}{\gamma - 1} + \frac{1}{2}\sqrt{u^2 + v^2 + w^2} && \text{tie kinetic energy} \\
\mu &= \mu_0 \frac{T_0 + C}{T + C}\left(\frac{T}{T_0}\right)^{\frac{3}{2}} && \text{dynamic viscosity} \\
k &= \frac{\gamma R \mu}{P_r(\gamma - 1)} && \text{heat conduction}
\end{aligned}
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