Heat Equation

$$\frac{\partial T}{\partial t} - D_T \nabla^2 T = 0$$

$T(x, y, z, t)$ = temperature (K) at position and time, also represented as u
D_T = thermal diffusivity constant, also represented as α (constant, m^2/s)
$D_T = \frac{k}{c\rho}$
k = thermal conductivity of the material (constant, watts per meter kelvin $\frac{W}{mK}$)
c = specific heat capacity of the material, if gas, at constant pressure (constant, joule per kelvin per kilogram $\frac{J}{kgK}$)
ρ = density of the material (constant, $\frac{kg}{m^3}$)

Derivation

Derivation follows from conduction of heat and conservation of energy
Conduction of heat: the rate of flow of heat energy per unit area through a surface is proportional to the negative temperature gradient across it

$$\mathbf{q} = -k\nabla T$$

1D case, where $T(x, t)$:

$$\mathbf{q} = -k\frac{\partial T}{\partial x}$$

The rate of change in internal heat energy per unit volume in the material ($\frac{\partial Q}{\partial t}$) is proportional to the rate change of its temperature ($\frac{\partial T}{\partial t}$), where $Q = Q(x, t)$ is the internal heat energy per unit volume of the rod

$$\frac{\partial Q}{\partial t} = c\rho\frac{\partial T}{\partial t}$$

Conservation of energy: the rate at which heat accumulates at a given point x is equal to the derivative of the heat flow at that point, negated (in laymen's terms, amount of heat flowing into the rod = amount of heat flowing out of the rod, which means, flow of heat in = amount of heat at that point):

$$\frac{\partial Q}{\partial t} = -\frac{\partial q}{\partial x}$$

So we have the following:

$$
\begin{aligned}
\mathbf{q} &= -k\frac{\partial T}{\partial x} \\
\frac{\partial q}{\partial x} &= -k\frac{\partial^2 T}{\partial x^2} \\
\frac{\partial Q}{\partial t} &= -\frac{\partial q}{\partial x} = c\rho\frac{\partial T}{\partial t} \\
\frac{\partial q}{\partial x} &= -c\rho\frac{\partial T}{\partial t} \\
-c\rho\frac{\partial T}{\partial t} &= -k\frac{\partial^2 T}{\partial x^2} \\
\frac{\partial T}{\partial t} &= \frac{k}{c\rho}\frac{\partial^2 T}{\partial x^2}
\end{aligned}
$$

So if we know the temperature of the metal rod at some particular time and D_T, then we can figure out the temperature of each little piece of the rod for some little timestep (Δt).

$$
\begin{aligned}
\frac{\Delta T}{\Delta t} &= D_T\frac{\partial^2 T}{\partial x^2} \\
\frac{T_i^{new} - T_i^{old}}{\Delta t} &= D_T\frac{\partial^2 T}{\partial x^2} \\
T_i^{new} - T_i^{old} &= D_T\frac{\partial^2 T}{\partial x^2}\Delta t \\
T_i^{new} &= T_i^{old} + D_T\frac{\partial^2 T}{\partial x^2}\Delta t
\end{aligned}
$$

$\Delta t =$ simulation timestep
$T_i =$ temperature at position x_i, where the rod is broken up into $x_1, x_2, ..., x_n$
$D_T =$ thermal diffusivity, a constant