Исследовать ф-цию на условной экстремум

1. $U = 3 - 8x + 6y$, $x^2 + y^2 = 36$

$L(\lambda_1, x, y) = 3 - 8x + 6y + \lambda_1 (x^2 + y^2 - 36)$

$$\begin{cases} L'_x = -8 + \lambda_1 \cdot 2x = 0 \\ L'_y = 6 + \lambda_1 \cdot 2y = 0 \\ L'_{\lambda_1} = x^2 + y^2 - 36 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{8}{2\lambda_1} \\ y = -\frac{6}{2\lambda_1} \\ \frac{64}{4\lambda_1^2} + \frac{36}{4\lambda_1^2} = 36 \end{cases} \Rightarrow$$

$$\begin{cases} x = \frac{8}{2\lambda_1} \\ y = -\frac{6}{2\lambda_1} \\ \lambda_1^2 = \frac{25}{36} \end{cases} \Rightarrow \left(\frac{5}{6}, \frac{24}{5}, -\frac{18}{5}\right), \left(-\frac{5}{6}, -\frac{24}{5}, \frac{18}{5}\right)$$

$L''_{xx} = 2\lambda_1 \quad L''_{yy} = 2\lambda_1 \quad L''_{\lambda_1\lambda_1} = 0$

$L''_{xy} = 0 \quad L''_{x\lambda_1} = 2x \quad L''_{y\lambda_1} = 2y$

$$\begin{vmatrix} 0 & 2x & 2y \\ 2x & 2\lambda_1 & 0 \\ 2y & 0 & 2\lambda_1 \end{vmatrix} = -2x \cdot 2x \cdot 2\lambda_1 - 4y^2 \cdot 2\lambda_1 =$$

$= -8x^2\lambda_1 - 8y^2\lambda_1 = -8\lambda_1 (x^2 + y^2) \Rightarrow$

$\Rightarrow \left(\frac{5}{6}, \frac{24}{5}, -\frac{18}{5}\right)$ - мин, $\left(-\frac{5}{6}, -\frac{24}{5}, \frac{18}{5}\right)$ - max

2. $U = 2x^2 + 12xy + 32y^2 + 15$, $x^2 + 16y^2 = 64$

в ф-ции U $2x^2 + 32y^2$ можем заменить на 128:

$U = 12xy + 143$

$L(\lambda_1, x, y) = 12xy + 143 + \lambda_1(x^2 + 16y^2 - 64)$

$$\begin{cases} L'_x = 12y + \lambda_1 \cdot 2x = 0 \\ L'_y = 12x + \lambda_1 \cdot 32y = 0 \\ L'_{\lambda_1} = x^2 + 16y^2 - 36 = 0 \end{cases}$$

① $y = -\frac{\lambda_1 \cdot x}{6}$

② $12x - \lambda_1 \cdot 32\frac{\lambda_1 x}{6} = 0 \Rightarrow \lambda^2 = \frac{9}{4} \Rightarrow \lambda_{1,2} = \pm\frac{3}{2}$

Подставим λ_1 в систему (1-е и 2-е уравнения станут равнозначными):

$$\begin{cases} 12y + 3x = 0 \\ x^2 + 16y^2 = 0 \end{cases} \Rightarrow \begin{cases} x = -4y \\ y^2 = \frac{9}{8} \end{cases}$$

$y_{1,2} = \pm\frac{3}{2\sqrt{2}}$, $x_{1,2} = \mp 3\sqrt{2}$

$$\begin{cases} 12y - 3x = 0 \\ x^2 + 16y^2 = 0 \end{cases} \Rightarrow \begin{cases} x = 4y \\ y^2 = \frac{9}{8} \end{cases}$$

$y_{3,4} = \pm\frac{3}{2\sqrt{2}}$, $x_{3,4} = \pm 3\sqrt{2}$

Точки экстремума:

$M_0\left(\frac{3}{2}, -3\sqrt{2}, \frac{3}{2\sqrt{2}}\right)$, $M_1\left(\frac{3}{2}, 3\sqrt{2}, -\frac{3}{2\sqrt{2}}\right)$

$M_2\left(-\frac{3}{2}, 3\sqrt{2}, \frac{3}{2\sqrt{2}}\right)$, $M_3\left(-\frac{3}{2}, -3\sqrt{2}, -\frac{3}{2\sqrt{2}}\right)$

$L''_{xx} = 2\lambda_1 \quad L''_{yy} = 32\lambda_1 \quad L''_{\lambda_1\lambda_1} = 0$

$L''_{xy} = 12 \quad L''_{x\lambda_1} = 2x \quad L''_{y\lambda_1} = 32y$

$$\begin{vmatrix} 0 & 2x & 32y \\ 2x & 2\lambda_1 & 12 \\ 32y & 12 & 32\lambda_1 \end{vmatrix} = -2x(2x \cdot 32\lambda_1 - 384y) + 32y(24x - 64\lambda_1 y) =$$

$= -128x^2\lambda_1 + 768xy + 768xy - 2048y^2\lambda_1$

разделим определитель на 128:

$-x^2\lambda_1 + 12xy - 16y^2\lambda_1 = 12xy - 16y^2\lambda_1 - x^2\lambda_1$

M_0: $12\left(-\frac{9}{2}\right) - 16 \cdot \frac{9}{2}\frac{3}{2} - 18 \cdot \frac{3}{2} < 0$ - min

M_1: $12\left(-\frac{9}{2}\right) - 16\frac{9}{8} \cdot \frac{3}{2} - 18\frac{3}{2} < 0$ - min

M_2: $12 \cdot \frac{9}{2} + 16\frac{9}{8} \cdot \frac{3}{2} + 18\frac{3}{2} > 0$ - max

M_3: $12 \cdot \frac{9}{2} + 16 \cdot \frac{9}{8} \cdot \frac{3}{2} + 18 \cdot \frac{3}{2} > 0$ - max

3. Найти производную функции по направлению в точку:

$U = x^2 + y^2 + z^2$, $\vec{c}(-9, 8, -12)$, $M(2; -12; 9)$

$|\vec{c}| = \sqrt{81 + 64 + 144} = \sqrt{289} = 17$

$\vec{c_0} = \frac{\vec{c}}{|\vec{c_0}|} = \left(-\frac{9}{17}; \frac{8}{17}; -\frac{12}{17}\right)$

$U'_x = 2x$, $U'_y = 2y$, $U'_z = 2z$

$\text{grad } U|_{(2;-12;9)} = (16, -24, 18)$

$U'_{\vec{c}}|_{(2;-12;9)} = -\frac{16 \cdot 9}{17} - \frac{8 \cdot 24}{17} - \frac{18 \cdot 12}{17} = -\frac{462}{17}$

4. $U = e^{x^2+y^2+z^2}$, $\vec{d} = (4, -13, -16)$, $L(-16, 4, -13)$

$|\vec{d}| = \sqrt{16 + 169 + 256} = 21$

$U'_x = 2x e^{x^2+y^2+z^2}$, $U'_y = 2y e^{x^2+y^2+z^2}$, $U'_z = 2z e^{x^2+y^2+z^2}$

$\text{grad } U|_{(-16,4,-13)} = (-32e^{441}, 4e^{441}, -26e^{441})$

$U'_{\vec{d}}|_{(-16,4,-13)} = \frac{-32 \cdot 4e^{441} - 13 \cdot 4e^{441} + 16 \cdot 26 e^{441}}{21} =$

$= \frac{236 \cdot e^{441}}{21}$