Insurance: Mathematics and Economics 69 (2016) 194–201

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Nonparametric estimation of operational value-at-risk (OpVaR)☆

Ainura Tursunalieva [a], Param Silvapulle [b,*]

[a] Department of Statistics, Data Science and Epidemiology, School of Health Sciences, Swinburne University of Technology, Vic 3122, Australia

[b] Department of Econometrics and Business Statistics, Monash Business School, Monash University, Vic 3145, Australia

HIGHLIGHTS

- Nonparametric methods to estimate the 99.9% operational value at risk (OpVaR).
- Nonparametric methods to assess the uncertainty associated with the OpVaR estimate.
- Our method leads to efficient and effective operational risk management during crises.
- We propose algorithms for practitioners in risk management to use our new methods.

ARTICLE INFO

Article history:
Received February 2016
Received in revised form
April 2016
Accepted 18 May 2016
Available online 25 May 2016

JEL classification:
C13
C14
C16
G32

Keywords:
Loss severity distribution
Risk analysis
Operational risk
Simulation
Empirical likelihood confidence band

ABSTRACT

This paper introduces nonparametric methods for estimating 99.9% operational value-at-risk (OpVaR) and its confidence interval (CI), and demonstrates their applications to US business losses. An attractive feature of these new methods is that there is no need to estimate either the entire heavy-tailed loss distribution or the tail region of the distribution. Furthermore, we provide algorithms that facilitate applied researchers and practitioners in risk management area to implement the sophisticated empirical likelihood ratio (ELR) based methodologies to construct the CI of the true underlying 99.9% OpVaR. In a simulation study, we find that the weighted ELR (WELR) CI estimator is more reliable than the ELR CI estimator. The empirical results show that the nonparametric OpVaR estimates are consistently larger than those of other comparable methods, which provide adequate regulatory capitals, particularly during crises. The findings have implications for regulators, and effective and efficient risk financing.

1. Introduction

The last two decades or so witnessed that globalisation of international financial markets and the developments of e-commerce, automated technology, and the like, increased the risk of business operations and encouraged fraud. As large financial losses have regularly surfaced, producing devastating economic and financial impacts on businesses, investors and consumers, the estimation of *operational risk* has been receiving significant attention from the academics, practitioners and regulators (Moosa and Li, 2013). The Basel II Accord (Basel Committee on Banking Supervision, 2006) on capital adequacy implemented in 2007 requires internationally active banks and other businesses to develop statistical models of operational risk and use them to determine the capital to be held against operational risk. Thus, the estimation of operational risk and the associated regulatory capital has become prominent in the risk management area. The pertinent question is how to quantify the operational risk exposure from the distribution of operational losses over certain period. A predominant tool used for this purpose is value-at-risk (VaR) which is also called downside or tail risk. In the operational risk context, this measure is known as operational value-at-risk, denoted by OpVaR. The primary objective of this paper is to propose a nonparametric methodology for estimating the 99.9% OpVaR and its confidence intervals (CIs). First, we introduce nonparametric methods developed by Hall and Yao (2003) and Peng and Qi (2006) for high quantiles[1] and their CIs,

☆ We wish to thank the reviewer for constructive comments which were helpful to improve the presentation of the paper. We wish to thank Eric Renault, David Veredas, Paul Embrechts, Nader Tajvidi and Harry Scheule for their useful comments on the methodologies employed in this article.

* Corresponding author.
E-mail addresses: atursunalieva@swin.edu.au (A. Tursunalieva), param.silvapulle@monash.edu (P. Silvapulle).

[1] We use high quantile & OpVaR interchangeably throughout this paper.

http://dx.doi.org/10.1016/j.insmatheco.2016.05.010
0167-6687/

which are the recent advances in statistics, based on which we then propose a methodology to apply to US business operational losses categorised as four business lines and three event type losses.

Following the recent global financial crisis, Basel Committee on Banking Supervision (2011) has reported that the regulatory capital requirements that the banks provided under advanced measurement approach (AMA) were not sufficient enough to withstand the losses during the crisis, due to inadequate modelling of large tail losses. However, for the purpose of calculating the required capital level, within AMA, the 99.9% credit-VaR is primarily used. As said below in this section, should a widely used parametric extreme value distribution (EVD) be used to model credit losses, then the credit-VaR could be vastly underestimated, so as the regulatory capital, particularly during crises. Moreover, we note at the outset that despite our study demonstrating the application of the nonparametric methods to operational losses to estimate the 99.9% OpVaR, these methods can also be employed to estimate 99.9% credit-VaR and market-VaR, which in turn are used to compute adequate capitals to cover credit losses and market losses respectively.

As will be seen in the next section, the nonparametric methods that we study in this paper would use all the tail losses in calculating the 99.9% OpVaR. Additionally, we employ a simple method introduced by Tursunalieva and Silvapulle (2014) for efficiently and consistently estimating the optimum threshold loss, and the losses over and above this threshold loss constitute the tail losses. In our empirical study, we find that the nonparametric OpVaR estimates of various operational losses are invariably larger than those obtained by the methods such as empirical distribution approach and improved peak-over-threshold (POT) method. The larger the OpVaR the larger the regulatory capital which would result in efficient and effective risk management that is mostly needed during financial and economic crises.

Furthermore, to assess the uncertainties associated with the OpVaR estimate, we construct three sets of CI estimates for the true OpVaR, the empirical likelihood ratio (ELR) method, the weighted ELR (WELR) method and the normal approximation (NA) method. The WELR is a desirable method in this context because it allocates zero weights to losses in the non-tail region and positive and non-decreasing weights to losses in the tail region — the most important region in estimating tail risks. In a simulation study, we assess the properties of the three interval estimators in terms of coverage probability and width (Appendix C). Moreover, we provide a step-by-step approach and algorithms that facilitate applied researchers and practitioners in the risk management area to implement this sophisticated nonparametric methodology for the estimation of 99.9% OpVaR and its CIs. Thus, our work contributes significantly to the literature on the estimation of operational risk.

In the OpVaR literature, several studies used parametric distributions such as log-normal, Weibull, gamma, Gumbel, exponential, Pareto, Burr, log-logistic for modelling loss distributions and found them unsuitable for modelling loss data; see, for example, Moscadelli (2004), de Fontnouvelle et al. (2005), Jobst (2007), among others. Moreover, a number of studies used semi-parametric methods, known as the POT method (Chavez-Demoulin et al., 2006), where a nonparametric distribution is fitted to small to medium operational losses and an extreme value distribution (EVD) fitted to large losses. Recently, Tursunalieva and Silvapulle (2014) have proposed an improved peaks over thresholds (POT) method and applied to US business losses.[2] The most commonly used EVDs for modelling large losses include Pareto, generalised Pareto distribution (GPD) and the g-and-h distribution, among others. Although these EVDs have been introduced specifically for modelling high quantiles of heavy-tailed distributions, Dutta and Perry (2007), Jobst (2007), Embrechts et al. (1997), and Tursunalieva and Silvapulle (2014), among others, show that these EVDs are not suitable for capturing very large losses and they do not provide consistent OpVaR estimates.

The remainder of the article is organised as follows. Section 2 briefly discusses the methodologies introduced in this paper. Section 3 describes the operational loss data. In a step-by-step approach, the proposed methodology is applied to an event type losses, followed by the discussion of the overall results. Section 4 concludes this paper. Appendices A and B provide respectively, algorithms to implement ELR CI and WELR CI estimation methods. The simulation results are reported in Appendix C. Appendix D provides the relevant graphs and tables for the empirical study.[3]

2. Methodologies

In this section, we introduce nonparametric methods for estimating the 99.9% OpVaR of a heavy-tailed distribution, as well as three methods for constructing CI estimates for the underlying true OpVaR.[4]

2.1. Estimation of a high quantile of a heavy-tailed distribution

To explain the methodology, let the observations $X_1, X_2, \ldots, X_n$ be *i.i.d.* random variables with a heavy-tailed distribution function F, a simple form of which is defined as:

$$1 - F(x) = cx^{-\gamma} \quad \text{for } x > T, \tag{1}$$

where T is a threshold loss in our context. Let $X_{n,1} \leq X_{n,2} \leq \cdots \leq X_{n,n}$ denote the order statistics of $X_1, \ldots, X_n$. Let $T = X_{n,n-k}$, where $X_{n,n-k}$ is the optimum threshold loss and k is tail length representing the count of tail losses. Clearly, the distribution F is defined for the heavy tail losses. It is crucial to estimate the optimum threshold loss reliably, because the tail losses are defined as the losses that are over and above this threshold loss. These losses are used in the estimation of both OpVaR and its CIs. In this paper, the optimum threshold loss is estimated efficiently and consistently by a simple approach proposed by Tursunalieva and Silvapulle (2014). Now, we define, $\delta_i = \mathbf{I}(X_i > X_{n,n-k})$, where $I(.)$ is an indicator function. Then, the likelihood for the censored data $\{\delta_i, max(X_i, X_{n,n-k})\}$ is:

$$L(\gamma, c) = \prod_{i=1}^{n} (c\gamma X_i^{-\gamma-1})^{\delta_i} (1 - cX_{n,n-k}^{-\gamma})^{1-\delta_i}. \tag{2}$$

As seen below, this likelihood is used to estimate the OpVaR, x_p, and its CIs by ELR and WELR methods.

Estimation of $100(1 - p_n)\,\%$ *quantile* x_{p_n}

For a given small value of $0 < p_n < 1$, and p_n is close to zero, say 0.001, the true $100(1 - p_n)\%$ quantile x_{p_n} for the distribution F is defined as $x_{p_n} = (1 - F)^{-}(p_n)$, where $(\cdot)^{-}$ denotes the inverse function of $(\cdot)$. To estimate this quantile, maximise the log likelihood defined in (2) and obtain the maximum likelihood estimates (MLE) of c and γ as: $\hat{c}_n = \frac{k}{n} X_{n,n-k}^{\hat{\gamma}_n}$ and $\hat{\gamma}_k$. The latter is the MLE of the well-known tail index. Eq. (1) yields the natural estimator of the quantile x_p as:

$$\hat{x}_{p_n} = \left(\frac{p_n}{\hat{c}_n}\right)^{-1/\hat{\gamma}_n}. \tag{3}$$

[2] Tursunalieva and Silvapulle (2014) discuss the literature on the POT method as well as the operational risk in general.

[3] Matlab code was used in all calculations. The code will be available from the authors upon request.

[4] See Peng and Qi (2006) and Hall and Yao (2003) for mathematical derivations of the high quantile estimator and the ELR based CI estimators.

That is, $\hat{x}_{p_n}$ is the required $100(1-p_n)$% OpVaR. In what follows, we will discuss briefly NA, ELR and WELR methods for constructing CI estimates for a high quantile of a heavy-tailed distribution.

2.2. Normal approximation method

For a given small value of p_n, the $100(1-p_n)$% NA CI estimator[5] for x_p is defined as follows:

$$I_{x_p}^n = \left[\hat{x}_p \exp\left(-z_\alpha \log\left(\frac{k}{np_n}\right)\left(\hat{\gamma}_n\sqrt{k}\right)\right),\right.$$
$$\left.\hat{x}_p \exp\left(z_\alpha \log\left(\frac{k}{np_n}\right)\left(\hat{\gamma}_n\sqrt{k}\right)\right)\right], \tag{4}$$

where $P(|N(0,1)| \leq z_\alpha) = \alpha$, k is the optimum tail length and $\hat{x}_p$ is the quantile estimator.

2.3. Empirical likelihood ratio method

To explain the ELR method developed by Peng and Qi (2006) for constructing CI for the high quantile, consider the set of three constraints: (i) $\gamma > 0$, (ii) $c > 0$, and (iii) $\gamma \log x_p + \log(p_n(c)^{-1}) = 0$. Note that the constraint (iii) defines the quantile. For small p_n, such as 0.001, x_p corresponds to 99.9% quantile of a loss distribution. To construct ELR CI, first, estimate the log likelihood (2) subject to constraints (i) and (ii), and let the maximum likelihood be $l_1 = max_{\gamma>0,c>0} \log L(\gamma_n, c_n) = \log L(\hat{\gamma}_n, \hat{c}_n)$.

Now, maximise $\log L(\gamma, c)$ subject to all three constraints (i), (ii) and (iii) defined above. Denote the maximum log likelihood function by $l_2(x_p) = \log L(\bar{\gamma}, \bar{c})$. It can then be shown that

$$l_2(x_p) = \log L(\bar{\gamma}_n(\lambda), \bar{c}_n(\lambda)), \tag{5}$$

where

$$\bar{\gamma}_n(\lambda) = k\left(\sum_{i=1}^{k}(\log X_{n,n-i+1} - \log X_{n,n-k})\right.$$
$$\left. + \lambda \log X_{n,n-k} - \lambda \log x_p\right)^{-1} \tag{6}$$

and

$$\bar{c}_n(\lambda) = X_{n,n-k}^{\bar{\gamma}(\lambda)} \frac{k-\lambda}{n-\lambda}, \tag{7}$$

and λ satisfies two conditions:

$$\bar{\gamma}_n(\lambda) \log x_p + \log\left(\frac{p_n}{c_n(\lambda)}\right) = 0, \tag{8}$$

$$\bar{\gamma}_n(\lambda) > 0 \quad \text{and} \quad \lambda < k. \tag{9}$$

Therefore, the log likelihood ratio multiplied by −2 takes the form:

$$l(x_p) = -2(l_2(x_p) - l_1). \tag{10}$$

At the true quantile x_p, $l\left(x_p\right)$ has a χ^2 distribution with one degree of freedom, $l(x_p) \xrightarrow{d} \chi^2_{(1)}$, and the $100(1-\alpha)$% confidence interval estimator for x_p is $I_\alpha^{elr} = \left\{x_p : l\left(x_p\right) \leq u_\alpha\right\}$, where u_α is the α nominal level critical value of a $\chi^2_{(1)}$ distribution. Clearly, the ELR CI cannot be directly estimated and the profile likelihood function $l(x_p)$ against x_p is used to estimate the $100(1-\alpha)$% CI. In the Appendix A, we present an algorithm for implementing this method.

[5] The delta method (Caroll and Ruppert, 1991) can also be used to construct approximate confidence interval for x_p. However, it works well only in large samples.

2.4. Weighted empirical likelihood ratio method

To employ the WELR method for constructing the confidence interval of x_p, developed by Hall and Yao (2003) and extended to a high quantile by Peng and Qi (2006), consider the weights $q = (q_1, \ldots, q_n)$, such that $q_i \geq 0$, and $\sum_{i=1}^n q_i = 1$, and solve:

$$(\hat{\gamma}(q), \hat{c}(q))$$
$$= argmax_{(\gamma,c)} \sum_{i=1}^{n} \log((c\gamma X_i^{-\gamma-1})^{\delta_i}(1 - cX_{n,n-k}^{-\gamma})^{(1-\delta_i)}), \tag{11}$$

Solving Eq. (11) results in

$$\tilde{\gamma}_n(q) = \frac{\sum_{i=1}^{n} q_i\delta_i}{\sum_{i=1}^{n} q_i\delta_i(\log X_i - \log X_{n,n-k})}, \tag{12}$$

and

$$\tilde{c}_n(q) = X_{n,n-k}^{\tilde{\gamma}_k(q)} \sum_{i=1}^{n} q_i\delta_i. \tag{13}$$

Now, define a function $D_p(q)$ which is the distance between q and a uniform distribution $q_i = n^{-1}$ as follows

$$D_p(q) = \sum_{i=1}^{n} q_i \log(nq_i), \tag{14}$$

where the weights $q_i (i = 1, \ldots, n)$ are calculated subject to constraints: $q_i \geq 0$, $\sum_{i=1}^n q_i = 1$ and

$$\tilde{\gamma}_n(q) \log \frac{x_p}{X_{n,n-k}} = \log \frac{\sum_{i=1}^{n} q_i\delta_i}{p_n}. \tag{15}$$

This minimisation procedure results in solving $(2n)^{-1}L(x_p) = min_q D_p(q)$, which gives the following two measures:

$$A_1(\lambda_1) = 1 - \frac{n-k}{n}e^{-1-\lambda_1} \tag{16}$$

and

$$A_2(\lambda_1) = A_{(\lambda_1)} \frac{\log(x_p/X_{n,n-k})}{\log(A_1(\lambda_1)/(p_n))}. \tag{17}$$

By the standard method of Lagrange multipliers, we have $q_i = q_i(\lambda_1, \lambda_2) =$

$$\begin{cases} n^{-1}\exp(-1-\lambda_1), & \text{if } \delta_i = 0 \\ n^{-1}\exp\{-1-\lambda_1 + \lambda_2(\varphi A_2^{-1}(\lambda_1) - A_1^{-1}(\lambda_1) \\ \quad -A_1(\lambda_1)\phi\varphi A_2^{-2}(\lambda_1))\} & \text{if } \delta_i = 1, \end{cases} \tag{18}$$

where $\varphi = \log(\hat{x}_p X_{n,n-k}^{-1})$ and $\phi = \log(X_j X_{n,n-k}^{-1})$ for $j = n-k, \ldots, n$, and both λ_1 and λ_2 satisfy

$$\sum_{i=1}^{n} q_i = 1, \quad \text{and} \quad \tilde{\gamma}_n(q) \log \frac{x_p}{X_{n,n-k}} = \log \frac{\sum_{i=1}^{n} q_i\delta_i}{p_n}. \tag{19}$$

According to Theorem 1 of Peng and Qi (2006), there exists a solution $(\tilde{\lambda}_1(x_p), \tilde{\lambda}_2(x_p))$, subject to the following inequalities:

$$-\log\left(1 + \frac{\sqrt{k}\sqrt{\omega}}{n-k}\right) \leq 1 + \lambda_1 \leq -\log\left(1 - \frac{\sqrt{k}\sqrt{\omega}}{n-k}\right), \tag{20}$$

and

$$|\lambda_2| \leq k^{-1/4}\frac{k}{n\omega}, \tag{21}$$

where $\omega = \log(k(np_n)^{-1})$ and $L(x_{p,0}) \xrightarrow{d} \chi^2_{(1)}$, with $(\lambda_1, \lambda_2) = (\tilde{\lambda}_1(x_p), \tilde{\lambda}_2(x_p))$ in the expression of $L(x_p)$. Therefore, the $100(1-\alpha)\%$ WELR CI estimator for x_p is $I^{welr}_\alpha = \{x_p : l(x_p) \le u_\alpha\}$, where u_α is the α-level critical value of a $\chi^2_{(1)}$ distribution. In Appendix B, we present an algorithm for constructing the WELR CI interval estimates for the 99.9% OpVaR.

We conduct a limited simulation study to assess and compare the properties of ELR CI, WELR CI and NA CI estimates in terms of the coverage probability and the width. The simulation results reported in Appendix C show that NA CIs tend to have low coverage probabilities even in large samples and are very wide. While both ELR and WELR CIs have correct coverage probabilities, the latter has the narrowest width among three CI estimators.

3. Data and empirical results

The methodology discussed in the previous section is now applied to operational (nominal and real) losses in excess of \$1M incurred by some large banks and businesses in the United States. The nominal operational losses that used in our study are available in FitchRisk database,[6] and cover the period May 1984 to May 2007, consisting 24 years of losses. The real losses were obtained by dividing the nominal losses by the January 1997 based consumer price indices. Moreover, the losses are pooled across the various sectors and categorised according to the Basel classification of business lines and event types. Thus, there are four business line and three event-type losses included in this study. Tursunalieva and Silvapulle (2014) provide a description of this loss data set and a summary statistics. In what follows, first, we will illustrate the application of the methodology to *Employment PWS* nominal losses in a step-by-step approach, followed by a discussion of the overall results.

3.1. An application to Employment PWS losses

The 99.9% OpVaR of *Employment PWS* nominal losses is estimated and the 95% CIs are constructed for the true OpVaR. The application of the methodology explained in Section 2 to *Employment PWS* losses includes the following steps:

1. The estimate of the tail index $\gamma_k = 1.3$; the optimal tail length $k = 21$; and the optimum threshold loss is \$70.4 M. This means there are 21 tail losses that exceed the threshold loss. These quantities are obtained by an approach proposed in Section 2.3 in Tursunalieva and Silvapulle (2014).
2. Estimate of the 99.9% OpVaR = 2475 M, by using the formula (3).
3. The estimate of the 95% NA CI for the true 99.9% OpVaR is [\$538M \$11,040M], by using the formula (4).
4. Estimate the ELR $l(x_p)$ defined in (23) (Appendix A) as a function of x_p. Use the algorithm provided in Appendix A to plot the profile log likelihood function in Fig. 3 against a possible range of values for the quantile x_p. As shown in Fig. 3, use the $\chi^2_{(1)} = 3.84$ (5% critical value), and deduce the 95% ELR CI estimate [\$1,615M \$4,567M] for the true OpVaR.
5. To estimate the WELR CI, first, estimate the weights, which are defined in (19), for the likelihood function. It is noteworthy that the larger the losses in the tail the higher the weights allocated to them. The weights allocated to tail losses are plotted in Fig. 2. Use the algorithm provided in Appendix B to plot the profile likelihood of the WELR as a function of x_p in Fig. 3. As shown in Fig. 3, the WELR CI estimate is [\$1,583M \$3,876M]. Clearly, it is much narrower than the ELR CI estimate for the OpVaR.

The results in Table 5 indicate that, in comparison, for the *Employment PWS* real operational losses: (i) the estimate of the tail index is 1.2 which is smaller than for the nominal losses, and it indicates that the severity distribution of real losses is heavier than that for nominal losses; and (ii) the 99.9% OpVaR is \$2,462M for real losses and it is \$2,433M for the nominal losses. The results in Table 5 indicate that 95% WELR CI estimate is [\$1,618M \$3,874M], which is narrower than that for nominal losses. In what follows, the overall empirical results are discussed.

3.2. Discussion of empirical results

Table 5 presents the optimum threshold loss in column 2, the number of tail losses in column 3, which are used to compute the tail index (column 4), and 99.9% OpVaR (column 6). The estimates of the tail index are all close to 1.0, which indicate that all seven loss distributions have very heavy right tails. The three sets of 95% CI estimates for each OpVaR are reported in Table 6. The NA CI estimates are very wide, with huge upper bounds. The ELR CI estimates and WELR CI estimates are much narrower than those of the NA CIs. According to the results of the simulation study presented in Appendix C, the WELR method provides the most reliable confidence interval estimates in terms of widths and coverage probabilities. The WELR CI estimate has the full coverage probability, the least non-right coverage ratio and the narrowest width in comparison to its counterparts; see Tables 2–4 in the Appendix C for details. The overall empirical results in Tables 5 and 6 indicate that WELR CIs are the narrowest intervals, which are consistent with the simulation results.

On the analysis of real operational losses, in general, the estimated quantities are smaller than those for the nominal losses, although the difference is marginal in some cases. However, for the *Commercial Banking* business line real losses, the tail index estimate is much smaller than that for nominal losses, indicating heavier loss severity distribution for real losses. As a result, the OpVaRs for the *Commercial Banking* is much larger for real losses than for the corresponding nominal losses.

For comparison purpose, we have also reported 99.9% OpVaRs for the seven business lines and event type losses (columns 2, 3, and 4 in Table 6), by employing empirical loss distribution approach as well as the improved POT method[7] of Tursunalieva and Silvapulle (2014). A noteworthy point is that the nonparametric OpVaR estimates are consistently larger than those obtained by the empirical loss distribution approach and improved POT method.

4. Conclusion

This paper proposes a nonparametric methodology for estimating 99.9% OpVaR and demonstrates their applications to US business operational losses categorised as four business lines and three event types. We also construct three sets of confidence interval estimates, ELR CI, WELR CI and NA CI estimates for the underlying true OpVaR. We use these interval estimates to assess the levels of uncertainties associated with the OpVaR estimates. We conduct a simulation study to assess the finite sample properties of the interval estimates. We find that the WELR CI estimate has the full coverage and the narrowest widths in comparison to its counterparts. This method has such favourable properties because it allocates nearly zero weights to central losses, and large and positive weights to tail losses — most crucial losses in this context. We provide a step-by-step approach and algorithms that facilitate applied

6 We thank Professor Imad Moosa for kindly providing the US operational loss data set.

7 The last two methods were studied by Tursunalieva and Silvapulle (2014) and the OpVaRs were estimated for the same operational losses as used in this paper.

researchers and practitioners in the risk management area to implement these sophisticated nonparametric methodology for the estimation of 99.9% OpVaR and its CIs. Furthermore, the nonparametric methodology studied in this paper has much wider applications than that illustrated here. For example, they can be applied to market losses and credit losses to estimate the 99.9% VaR and the 99.9% credit-VaR and the respective regulatory capitals.

The 99.9% OpVaR is widely estimated for regulatory purpose since the introduction of Basel II and III accords. However, Alexander et al. (2003) suggest that this 99.9% level can be lower for some financial institutions. The nonparametric estimation methods that we study in this paper can easily be employed for precision levels lower than the 99.9%. Furthermore, McConnell (2006) is concerned that the use of the 99.9th percentile is "an unrealistic level of precision" that would introduce moral hazard, thus encouraging managers to claim that "…risk has been fully mitigated" rather than addressing the serious issues underlying large loss events in particular. Our empirical and simulation results show that the impression that "…risk has been fully mitigated" may be misleading because the true 99.9% OpVaR can be vastly underestimated. Moreover, as the main focus our study is on accurately quantifying 99.9% OpVaR and its confidence interval, addressing any issues related to "moral hazard" due to large operational losses is beyond the scope of this paper.

In the empirical study, we find that our nonparametric OpVaR estimates are invariably larger than those obtained by comparable methods, such as empirical loss distribution approach and an improved POT method. The larger the 99.9% OpVaR the larger the regulatory capitals, which would provide adequate cover for operational risks, particularly during crises. The WELR CI estimates are asymmetric and very wide indicating a large degree of uncertainty in the OpVaR estimates. Following the recent global financial crisis, Basel Committee on Banking Supervision (2011) has reported that the regulatory capital requirements that the banks provided under AMA were not sufficient enough to withstand credit losses during the crisis, due to inadequate modelling of large tail losses. Clearly, the nonparametric methodology studied in this paper for estimating the regulatory capital appear to be a better risk financing tool than the existing methods. The findings have implications for regulators, and effective and efficient risk financing.

Appendix A. An algorithm for the estimation of ELR CI for a high quantile

In what follows, we provide an algorithm which simplifies the estimation of interval estimates for a high quantile by the ELR method. It involves the following steps:

1. Find the tail index estimate $\hat{\beta}_0^{wls}$ and the tail length k using the methods described in Section 2.1, and the corresponding threshold loss $X_{n,n-k}$. Then, find the $100(1-p_n)\%$ quantile estimate $\hat{x}_p$ using Eq. (3).
2. Find the range for λ from:

$$|\lambda| < \frac{k}{\left(\hat{\gamma}_k \log\left(\hat{x}_p/X_{n,n-k}\right)\right)^{-2}}. \tag{22}$$

   Denote this range by $\lambda_l < \lambda < \lambda_u$, with λ_l and λ_u being the lower and upper limits of λ. If $\lambda_u \geq k$, then set the range to $\lambda_l < \lambda < k$.
3. Choose λ_i in this range for $i = 1, \ldots, N$, where N is large, say 5,000.
4. For each value of λ_i, compute $\bar{\gamma}_k(\lambda_i)$ and $\bar{c}_k(\lambda_i)$ using Eqs. (6) and (7) respectively. Discard values of $\bar{\gamma}_k(\lambda_i)$ and $\bar{c}_k(\lambda_i)$ which do not satisfy the positivity constraints given in (9), for $i = 1, \ldots, N$.
5. Using Eq. (3), compute $\bar{x}_{p_i}$ for each pair of $(\bar{\gamma}_k(\lambda_i), \bar{c}_k(\lambda_i))$, for $i = 1, \ldots, N$.
6. For $i = 1, \ldots, N$, define the empirical log-likelihood ratio function (ELLF) as follows:

$$l(\bar{x}_{pi}) = -2k\left(\log(\psi) - (\psi - 1) - 2k\log\left(1 - \frac{\lambda_i}{n}\right) + 2n\log\left(1 - \frac{\lambda_i}{n}\right)\right), \tag{23}$$

   where $\psi = \bar{\gamma}_k(\lambda_i)/\hat{\gamma}_k$.
7. Plot the profile likelihood function $l(\bar{x}_{p_i})$ against $\bar{x}_{p_i}$, for $i = 1, \ldots, N$. Since the likelihood function has an asymptotic $\chi^2_{(1)}$ distribution with one degree of freedom, the lower and upper bounds of the ELR confidence interval estimator are obtained by minimising the distance between the 5% critical value of the $\chi^2_{(1)}$ distribution and the profile likelihood function $l(\bar{x}_p)$. The values of $\bar{x}_p$ that correspond to these minimum distances do constitute the 95% ELR confidence interval estimator for the true 99.9% x_p. The details of its empirical application are provided in Section 3.1.

Appendix B. An algorithm for the estimation of WELR CI for a high quantile

To simplify the application of this method, we provide the following steps:

1. This step is the same as Step 1 of Appendix A.
2. Using Eqs. (20) and (21), compute the upper and lower bounds for λ_1 and λ_2 respectively. Choose the values of λ_{1i} and λ_{2i}, for $i = 1, \ldots, N$ (for a large value of N, say 5000), within their respective ranges.
3. Compute $A_1(\lambda_{1i})$ and $A_2(\lambda_{1i})$ for $i = 1, \ldots, N$ using Eqs. (12) and (13) respectively.
4. Using Eq. (19), calculate the weights q_i.
5. Compute $\tilde{\gamma}_k(q)$ and $\tilde{c}_k(q)$ using Eqs. (12) and (13) respectively.
6. Using Eq. (3), compute $\tilde{x}_{p_i}$ for each pair of $\tilde{\gamma}_k(\lambda_i)$ and $\tilde{c}_k(\lambda_i)$, for $i = 1, \ldots, N$.
7. Calculate the likelihood function $L(\tilde{x}_{p_i})$ as follows:

$$L(\tilde{x}_{p_i}) = \tilde{\gamma}_k^2(q_i)k\left[\frac{\log(\tilde{x}_{pi}/x_p)}{\log(k/np_n)}\right]^2. \tag{24}$$

8. Plot $L(\tilde{x}_{p_i})$ against $\tilde{x}_{p_i}$ for $i = 1, \ldots, N$. Following the step 7 for the WELR confidence interval, the lower and upper bounds of the 95% WELR confidence interval estimator for the true 99.9% x_p can be calculated.

Appendix C. Simulation study

We conduct a limited simulation study to assess the finite sample properties of the three CI estimators for a high quantile of a heavy-tailed distribution. The three methods used for interval estimation include NA, ELR and WELR methods. Both ELR and WELR are non-parametric methods and have been developed specifically for estimating confidence intervals for a high quantile of a heavy-tailed distribution. The NA method is the standard method included as the benchmark for a comparison purpose.

The design of the simulation study

The random samples are generated from two heavy-tailed distributions for a range of sample sizes and tail indices. First, 10,000 random samples of size n are generated from the following two distributions:

1. Fréchet(γ_k): $F(x) = \exp\left\{-x^{-\gamma_k}\right\}$,
2. Burr($\gamma_k, \gamma_k + \beta$): $F(x) = 1 - (1 - x^{\beta-\gamma_k})^{\frac{-\gamma_k}{\beta-\gamma_k}}$ for $x > 0$, where $\beta = \gamma_k$.

Table 1
True 99.9% quantiles of the distributions.

Distribution	Tail index	
	1.5	1.2
Fréchet (γ_k) *distribution*	369.8	379.1
Burr $(\gamma_k, \gamma_k + \beta)$ *distribution*	133.8	149.5

Table 2
Coverage probabilities of 95% CIs for a high quantile.

Methods	*NA*[a]		*ELR*[b]		*WELR*[c]	
γ_k	1.5	1.2	1.5	1.2	1.5	1.2
n	*Fréchet* (γ_k) *distribution*					
500	90.1	89.2	63.7	62.4	70.3	69.1
300	89.9	88.1	57.7	53.1	60.1	59.1
200	88.1	87.7	47.1	46.2	54.9	53.1
	Burr $(\gamma_k, \gamma_k + \beta)$ *distribution*					
500	90.1	88.8	64.3	61.7	70.6	68.6
300	89.3	88.7	54.7	52.0	62.3	59.8
200	87.8	87.2	47.4	45.7	55.8	53.6

[a] Normal approximation method.
[b] Empirical likelihood ratio method.
[c] Weighted empirical likelihood ratio method.

Table 3
Estimates of the widths of the 95% CIs for a high quantile.

Methods	*NA*		*ELR*		*WELR*	
γ_k	1.5	1.2	1.5	1.2	1.5	1.2
n	*Fréchet* (γ_k) *distribution*					
500	561	2392	434	2103	397	1898
300	1039	4681	725	2907	663	2624
200	1773	8784	1124	5168	979	4667
	Burr $(\gamma_k, \gamma_k + \beta)$ *distribution*					
500	471	3229	363	2695	325	2455
300	822	5877	679	4306	585	3916
200	1396	10,148	905	7502	822	6717

Table 4
Right non-coverage probabilities of 95% CIs for a high quantile.

Methods	*NA*[a]		*ELR*[b]		*WELR*[c]	
γ_k	1.5	1.2	1.5	1.2	1.5	1.2
n	*Fréchet* (γ_k) *distribution*					
500	12.3	34.2	4.3	3.7	4.0	3.0
300	15.9	38.1	3.4	4.7	3.2	4.1
200	15.8	39.7	4.0	5.2	3.5	4.6
	Burr $(\gamma_k, \gamma_k + \beta)$ *distribution*					
500	20.8	48.2	5.00	4.8	4.6	4.7
300	22.3	58.8	5.9	5.2	5.3	4.9
200	22.8	59.0	4.4	4.8	4.1	4.3

[a] Normal approximation method.
[b] Empirical likelihood ratio method.
[c] Weighted empirical likelihood ratio method.

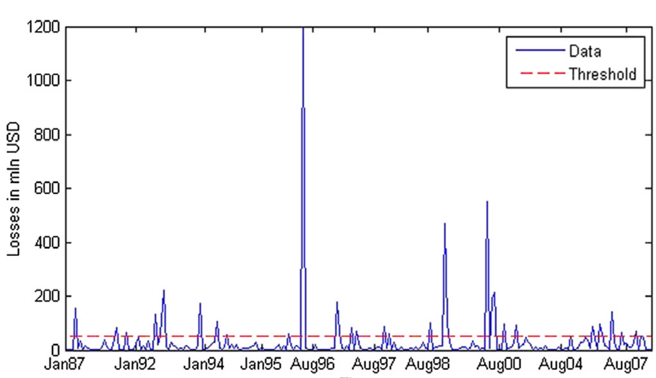

Fig. 1. Time series plot of *Employment PWS losses*.

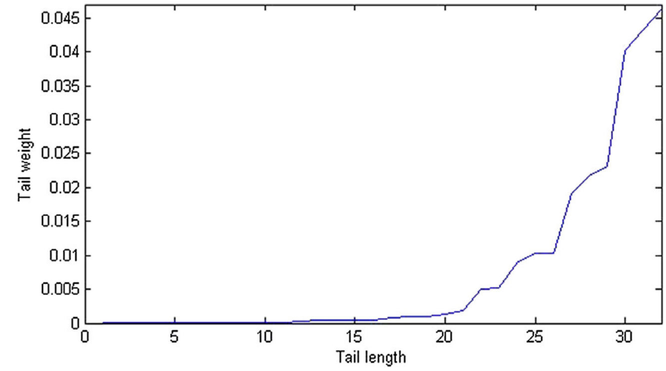

Fig. 2. The tail weights for the WELR method for the *Employment PWS* losses, computed using Eq. (19).

Table 5
Estimates of the threshold loss, the tail index, CI and OpVaR.

Loss type	Threshold	k[a]	$\hat{\gamma}_k$[b]	CI for $\hat{\gamma}_k$[c]	OpVaR[d]
Operational losses: Basel business line					
Retail Brokerage	36.0	23	1.2	[0.5 1.3]	2092
Trading and Sales	64.0	29	1.1	[0.5 1.3]	6913
Commercial Banking	149.0	13	1.2	[0.4 1.4]	5791
Retail Banking	99.0	18	1.4	[0.6 1.8]	3040
Operational losses: Basel event type					
Internal Fraud	94.0	80	1.2	[0.9 1.5]	7032
External Fraud	26.0	22	1.2	[0.8 1.8]	2196
Employment PWS	70.4	21	1.3	[1.1 1.9]	2433
Operational losses: Basel business line (in real terms)					
Retail Brokerage	32.8	22	1.2	[0.5 1.3]	1796
Trading and Sales	63.9	29	1.2	[0.5 1.3]	6150
Commercial Banking	156.6	13	1.1	[0.3 1.3]	7346
Retail Banking	87.9	18	1.4	[0.6 1.6]	2302
Operational losses: Basel event type (in real terms)					
Internal Fraud	102.0	71	1.2	[1.0 1.6]	5598
External Fraud	23.9	22	1.2	[0.8 1.7]	1479
Employment PWS	64.2	23	1.2	[1.1 1.9]	2462

[a] Number of tail losses.
[b] The tail index.
[c] Bootstrap CI estimate for γ_k.
[d] The 99.9% operational value-at-risk.

The following sample sizes and tail indices are used for both distributions:

1. $n = \{200, 300, 500\}$.
   The tail length was selected and fixed for each sample size as the following:
   (a) for $n = 200$, the tail length $k = 16$
   (b) for $n = 300$, the tail length $k = 20$
   (c) for $n = 500$, the tail length $k = 26$.
2. $\gamma_k = \{1.2, 1.5\}$. To ensure that the simulated samples have the correct tail index, the tail index is computed for each of the simulated samples for a given value of k. Then, the simulated samples are filtered out. The results are reported for $p_n = 0.001$.

The coverage probability is estimated as the proportion of CI estimates that cover the true quantile. The true quantile is obtained by taking the 99.9th percentile of one million observations for both distributions and for each tail index. The CI width is estimated as the sum of the widths of the CI estimates that cover the true quantile divided by the total number of CIs which cover the true quantile. To assess how well the CI estimates capture the positive skewness of the underlying distribution, the right non-coverage rate is also estimated. It shows the proportion of intervals that lie to the left of the true quantile. Using the outcomes of this simulation experiment, recommendations are made as to the most

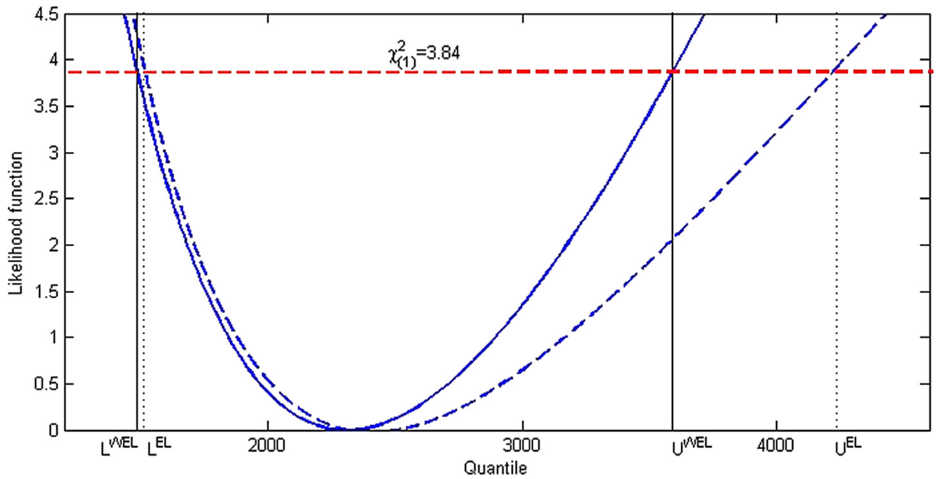

Fig. 3. The 95% ELR CI estimate is [$1,615M, $4,567M] (solid line) and the 95% WELR CI estimate is [$1,583M, $3,876M] (broken line).

Table 6
Estimates of OpVaR and CI estimates.

Loss type	OpVaR[a]	OpVaR[b]	OpVaR[c]	CI[d]	CI[e]	CI[f]
Operational losses: Basel business line						
Retail Brokerage	2092	1499	2332	[426 12,478]	[1,334 4,113]	[1,242 3,657]
Trading and Sales	6913	4399	7101	[1,317 41,237]	[4,363 13,640]	[4,421 11,241]
Commercial Banking	5791	3599	5543	[807 44,473]	[3,373 13,341]	[3,702 9,369]
Retail Banking	3040	2085	2503	[519 19,444]	[1,872 6,358]	[1,856 5,157]
Operational losses: Basel event type						
Internal Fraud	7032	3399	3671	[2,843 19,148]	[5,518 10,464]	[5,591 9,149]
External Fraud	2196	674	752	[380 16,875]	[1,328 4,622]	[1,157 4,313]
Employment PWS	2433	1199	1116	[538 11,040]	[1,615 4,567]	[1,583 3,876]
Operational losses: Basel business line (in real terms)						
Retail Brokerage	1796	1570	2511	[363 10,911]	[1,141 3,559]	[1,014 3,297]
Trading and Sales	6150	4278	6532	[1,223 35,187]	[3,925 11,929]	[3,980 9,867]
Commercial Banking	7346	3165	5119	[925 62,708]	[4,163 17,661]	[4,759 11,687]
Retail Banking	2302	1860	2148	[401 14,531]	[1,427 4,801]	[1,578 4,325]
Operational losses: Basel event type (in real terms)						
Internal Fraud	5598	2665	3258	[2,289 14,992]	[4,404 8,269]	[4,467 7,251]
External Fraud	1479	528	632	[291 10,010]	[928 2,968]	[707 3,214]
Employment PWS	2462	1225	1096	[558 10,861]	[1,645 4,556]	[1,618 3,874]

[a] The 99.9% operational value-at-risk for non-parametric loss distribution.
[b] The 99.9% operational value-at-risk for empirical loss distribution.
[c] The 99.9% operational value-at-risk for semi-parametric loss distribution.
[d] Normal approximation based confidence interval estimate.
[e] Empirical likelihood ratio confidence interval estimate.
[f] Weighted empirical likelihood ratio confidence interval estimate.

appropriate estimation method for CI for a high quantile of a heavy-tailed distribution. Such a CI interval estimate would indicate the accuracy of the point estimate of the true 99.9% OpVaR.

The simulation results

Table 1 presents the true values of the 99.9% population quantile. It is noticeable that the true (high) quantile increases as the tail index decreases which in turn increases the tail thickness. The values of the true quantile of the Burr distribution are much lower than those of the Fréchet distribution.

Tables 2 and 3 present the coverage probabilities and the widths of the CI estimates. While the NA method does provide the highest coverage probabilities, the average interval width are very wide. Overall, WELR method has the correct coverage probabilities and narrowest intervals. Based on this limited simulation study, we recommended that WELR method be used for estimating CI for the 99.9% quantile of a heavy-tailed distribution. The results of the right non-coverage probabilities are presented in Table 4. Clearly, the NA CI estimates have very high non-coverage probabilities in comparison to both ELR CI and WELR CI estimates. On the other hand, despite being a small difference, the right non-coverage probabilities of the WELR CI estimates are consistently smaller than those of the ELR CI estimates.

Appendix D. Figures and tables

See Figs. 1–3 and Tables 5 and 6.

References

Alexander, C., Anders, U., Blunden, T., Down, V., Hadjiemmanuil, C., Hardin, L., Haubenstock, M., 2003. Operational Risk: Regulation, Analysis, and Management. Prentice Hall.
Basel Committee on Banking Supervision 2006. International convergence of capital measurement and capital standards. Technical report. Bank for International Settlements, Basel, Switzerland.
Basel Committee on Banking Supervision 2011. Basel III: A global regulatory framework for more resilient bank and banking systems. Technical report. Bank for International Settlements, Basel, Switzerland.
Caroll, R., Ruppert, D., 1991. Prediction and tolerance intervals with transformation and/or weighting. Technometrics 33 (2), 197–210.
Chavez-Demoulin, V., Embrechts, P., Neslehova, J., 2006. Quantitative models for operational risk: Extremes, dependence and aggregation. J. Bank. Finance 30, 2635–2658.

de Fontnouvelle, P., Rosengren, E., Jordan, J., 2005. Implications of alternative operational risk modeling techniques. NBER Working Paper 11103, National Bureau of Economic Research. Cambridge, MA. Available at http://www.nber.org/papers/.
Dutta, K., Perry, J., 2007. A tale of tails: An empirical analysis of loss distribution models for estimating operational risk capital. Federal Reserve Bank of Boston, Working Papers No. 06-13.
Embrechts, P., Klüppelberg, C., Mikosh, T., 1997. Modelling Extremal Events for Insurance and Finance. Springer, Berlin, Germany.
Hall, P., Yao, Q., 2003. Data tilting for time series. J. R. Stat. Soc. Ser. B 65, 425–442.
Jobst, A., 2007. It's all in the data — consistent operational risk measurement and regulation. J. Financ. Regul. Compliance 15, 423–449.
McConnell, P., 2006. A perfect storm — why are Some Operational Losses Larger than Others? Unpublished paper. July.
Moosa, I., Li, L., 2013. An operational risk profile: The experience of british firms. Appl. Econ. 45, 2491–2500.
Moscadelli, M., 2004. The modelling of operational risk: experience with the analysis of the data collected by the Basel Committee. Technical report. Banca d'Italia, Roma, Italy.
Peng, L., Qi, Y., 2006. Confidence regions for high quantiles of a heavy-tailed distribution. Ann. Statist. 34, 1964–1986.
Tursunalieva, A., Silvapulle, P., 2014. A semi-parametric approach to estimating the operational risk and expected shortfall. Appl. Econ. 46 (30), 3659–3672.