

Poisson Processes

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Ago-Dic
2020

Introducción

In many applications of stochastic processes, the random variable can be a continuous function of the time t .

Interpreting the term population in the broad sense, we might be interested typically in the probability that the population size is, say, n at time t . We shall represent this probability usually by $p_n(t)$.

Introducción

- For the Geiger 1 counter application it will represent the probability that n particles have been recorded up to time t .
- The arrival of telephone calls, at an office, it could represent the number of calls logged up to time t .
- Births and deaths can occur at any time, in a population.

The Poisson process

Let $N(t)$ be a time-varying random variable representing the population size at time t . Consider the probability of population size n at time t given by

$$p_n(t) = \mathbf{P}[N(t) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad (t \geq 0)$$

for $n = 0, 1, 2, \dots$ (remember $0! = 1$). It is assumed that $N(t)$ can take the integer values $n = 0, 1, 2, \dots$

The Poisson process

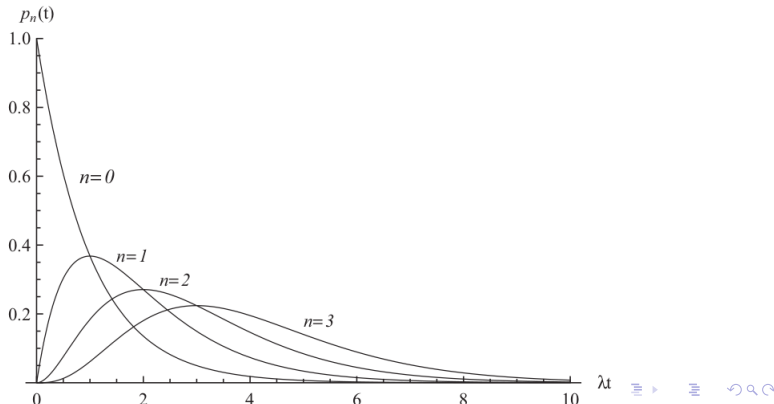
We can confirm that the previous equation is a probability distribution by observing that

$$\sum_{n=0}^{\infty} p_n(t) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = e^{-\lambda t} e^{\lambda t} = 1,$$

Note that $p_0(0) = 1$, that is, the initial population size is 1.

The Poisson process

In fact, $p_n(t)$ is a Poisson probability (mass) function with parameter or intensity λ . For this reason any application for which Eqn holds is known as a Poisson process.



The Poisson process

Since, for $n \geq 1$

$$\frac{dp_n(t)}{dt} = \frac{(n - \lambda t)}{n!} \lambda^n t^{n-1} e^{-\lambda t},$$

The maximum values of the probabilities for fixed n occur at time $t = n/\lambda$, where $dp_n(t)/dt = 0$. For $n = 0$.

$$\frac{dp_0(t)}{dt} = -\lambda e^{-\lambda t}$$

The Poisson process

The mean $\mu(t)$ of the Poisson distribution is given by

$$\begin{aligned}\mu(t) = \mathbf{E}[N(t)] &= \sum_{n=0}^{\infty} n p_n(t) = \sum_{n=1}^{\infty} n \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\ &= e^{-\lambda t} \lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} = e^{-\lambda t} \lambda t e^{\lambda t} = \lambda t.\end{aligned}$$

Note that the mean value increases linearly with time at rate λ

The Poisson process

Expressing the variance in terms of means, the variance is given by

$$\mathbf{V}[N(t)] = \mathbf{E}[N(t)^2] - [\mathbf{E}[N(t)]]^2 = \mathbf{E}[N(t)^2] - (\lambda t)^2$$

Also

$$\begin{aligned} \mathbf{E}[N(t)^2] &= \sum_{n=1}^{\infty} \frac{n^2 (\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{n (\lambda t)^n}{(n-1)!} \\ &= e^{-\lambda t} t \frac{d}{dt} \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{(n-1)!} = e^{-\lambda t} t \frac{d}{dt} [\lambda t e^{\lambda t}] = \lambda t + (\lambda t)^2. \end{aligned}$$

$$\mathbf{V}[N(t)] = \lambda t + (\lambda t)^2 - (\lambda t)^2 = \lambda t.$$

Note that the Poisson distribution has the property that its mean is the same as its variance.

Probabilidades de transición

From

$$p_n(t) = \mathbf{P}[N(t) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad (t \geq 0)$$

$$\frac{dp_n(t)}{dt} = \frac{(n - \lambda t)}{n!} \lambda^n t^{n-1} e^{-\lambda t}, \quad \frac{dp_0(t)}{dt} = -\lambda e^{-\lambda t}$$

given

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t),$$

$$\begin{aligned} \frac{dp_n(t)}{dt} &= \frac{(n - \lambda t)}{n!} \lambda^n t^{n-1} e^{-\lambda t} = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} - \frac{\lambda^{n+1}}{n!} t^n e^{-\lambda t} \\ &= \lambda[p_{n-1}(t) - p_n(t)], \quad (n \geq 1). \end{aligned}$$

Probabilidades de transición

These are differential-difference equations for the sequence of probabilities $p_n(t)$. From the definition of differentiation, the derivatives are obtained by the limiting process

$$\frac{dp_n(t)}{dt} = \lim_{\delta t \rightarrow 0} \frac{p_n(t + \delta t) - p_n(t)}{\delta t},$$

so that approximately, for small $\delta t > 0$,

$$\frac{dp_n(t)}{dt} \approx \frac{p_n(t + \delta t) - p_n(t)}{\delta t}$$

Probabilidad estacionaria o absoluta

we can replace the equations by

$$\frac{p_0(t + \delta t) - p_0(t)}{\delta t} \approx -\lambda p_0(t),$$

$$\frac{p_n(t + \delta t) - p_n(t)}{\delta t} \approx \lambda[p_{n-1}(t) - p_n(t)],$$

so that

$$\left. \begin{aligned} p_0(t + \delta t) &\approx (1 - \lambda\delta t)p_0(t) \\ p_n(t + \delta t) &\approx p_{n-1}(t)\lambda\delta t + p_n(t)(1 - \lambda\delta t), \quad (n \geq 1) \end{aligned} \right\}$$

Probabilidad estacionaria o absoluta

We can interpret the equations as follows. For the Geiger counter, we can infer from these formulas that the probability that a particle is recorded in the short time interval δt is $\lambda \delta t$, and that the probability that two or more particles are recorded is negligible, and consequently that no recording takes place with probability $(1 - \lambda \delta t)$. The only way in which the outcome reading $n(\geq 1)$ can occur at time $t + \delta t$ is that either one particle was recorded in the interval δt when n particles were recorded at time t , or that nothing occurred with probability $(1 - \lambda \delta)$ when $n - 1$ particles were recorded at time t .