Poisson Processes

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Introducción

In many applications of stochastic processes, the random variable can be a continuous function of the time t.
Interpreting the term population in the broad sense, we might be interested typically in the probability that the population size is, say, n at time t. We shall represent this probability usually by $p_n(t)$.

Introducción

- For the Geiger 1 counter application it will represent the probability that n particles have been recorded up to time t.
- The arrival of telephone calls, at an office, it could represent the number of calls logged up to time t.
- Births and deaths can occur at any time, in a population.

The Poisson process

Let $N(t)$ be a time-varying random variable representing the population size at time t. Consider the probability of population size n at time t given by

$$p_n(t) = \mathbf{P}[N(t) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \qquad (t \geq 0)$$

for $n = 0, 1, 2, \ldots$ (remember $0! = 1$). It is assumed that $N(t)$ can take the integer values $n = 0, 1, 2, \ldots$

The Poisson process

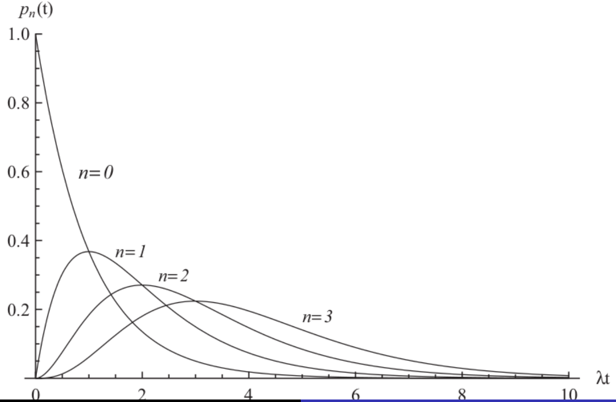
We can confirm that the previuos equation is a probability distribution by observing that

$$\sum_{n=0}^{\infty} p_n(t) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = e^{-\lambda t} e^{\lambda t} = 1,$$

Note that $p_0(0) = 1$, that is, the initial population size is 1.

The Poisson process

In fact, $p_n(t)$ is a Poisson probability (mass) function with parameter or intensity λ. For this reason any application for which Eqn holds is known as a Poisson process.



The Poisson process

Since, for $n \geq 1$

$$\frac{dp_n(t)}{dt} = \frac{(n - \lambda t)}{n!} \lambda^n t^{n-1} e^{-\lambda t},$$

The maximum values of the probabilities for fixed n occur at time $t = n/\lambda$, where $dp_n(t)/dt = 0$. For $n = 0$.

$$\frac{dp_0(t)}{dt} = -\lambda e^{-\lambda t}$$

The Poisson process

The mean $\mu(t)$ of the Poisson distribution is given by

$$\begin{aligned}\boldsymbol{\mu}(t) = \mathbf{E}[N(t)] &= \sum_{n=0}^{\infty} n p_n(t) = \sum_{n=1}^{\infty} n \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\ &= e^{-\lambda t} \lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} = e^{-\lambda t} \lambda t e^{\lambda t} = \lambda t.\end{aligned}$$

Note that the mean value increases linearly with time at rate λ

The Poisson process

Expressing the variance in terms of means, the variance is given by

$$\mathbf{V}[N(t)] = \mathbf{E}[N(t)^2] - [\mathbf{E}[N(t)]]^2 = \mathbf{E}[N(t)^2] - (\lambda t)^2$$

Also

$$\begin{aligned} \mathbf{E}[N(t)^2] &= \sum_{n=1}^{\infty} \frac{n^2(\lambda t)^n}{n!} e^{-\lambda t} = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{n(\lambda t)^n}{(n-1)!} \\ &= e^{-\lambda t} t \frac{d}{dt} \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{(n-1)!} = e^{-\lambda t} t \frac{d}{dt} [\lambda t e^{\lambda t}] = \lambda t + (\lambda t)^2. \end{aligned}$$

$$\mathbf{V}[N(t)] = \lambda t + (\lambda t)^2 - (\lambda t)^2 = \lambda t.$$

Note that the Poisson distribution has the property that its mean is the same as its variance.

The Poisson process

From

$$p_n(t) = \mathbf{P}[N(t) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \qquad (t \geq 0)$$

$$\frac{dp_n(t)}{dt} = \frac{(n - \lambda t)}{n!} \lambda^n t^{n-1} e^{-\lambda t}, \qquad \frac{dp_0(t)}{dt} = -\lambda e^{-\lambda t}$$

given

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t),$$

$$\begin{aligned} \frac{dp_n(t)}{dt} &= \frac{(n - \lambda t)}{n!} \lambda^n t^{n-1} e^{-\lambda t} = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} - \frac{\lambda^{n+1}}{n!} t^n e^{-\lambda t} \\ &= \lambda[p_{n-1}(t) - p_n(t)], \qquad (n \geq 1). \end{aligned}$$

The Poisson process

These are differential-difference equations for the sequence of probabilities $p_n(t)$. From the definition of differentiation, the derivatives are obtained by the limiting process

$$\frac{dp_n(t)}{dt} = \lim_{\delta t \to 0} \frac{p_n(t+\delta t) - p_n(t)}{\delta t},$$

so that approximately, for small $\delta t > 0$,

$$\frac{dp_n(t)}{dt} \approx \frac{p_n(t+\delta t) - p_n(t)}{\delta t}$$

The Poisson process

we can replace the equations by

$$\frac{p_0(t+\delta t)-p_0(t)}{\delta t} \approx -\lambda p_0(t),$$

$$\frac{p_n(t+\delta t)-p_n(t)}{\delta t} \approx \lambda[p_{n-1}(t)-p_n(t)],$$

so that

$$\left.\begin{aligned} p_0(t+\delta t) &\approx (1-\lambda\delta t)p_0(t) \\ p_n(t+\delta t) &\approx p_{n-1}(t)\lambda\delta t + p_n(t)(1-\lambda\delta t), \qquad (n \geq 1) \end{aligned}\right\}$$

The Poisson process

We can interpret the equations as follows. For the Geiger counter, we can infer from these formulas that the probability that a particle is recorded in the short time interval δt is $\lambda\delta t$, and that the probability that two or more particles are recorded is negligible, and consequently that no recording takes place with probability $(1 - \lambda\delta t)$. The only way in which the outcome reading $n(\geq 1)$ can occur at time $t + \delta t$ is that either one particle was recorded in the interval δt when n particles were recorded at time t, or that nothing occurred with probability $(1 - \lambda\delta)$ when $n - 1$ particles were recorded at time t.

Partition theorem approach

For the Geiger counter application (a similar argument can be adapted for the call-logging problem, etc.) we assume that the probability that one particle is recorded in the short time interval δt is

$$\lambda \delta t + o(\delta t)$$

The term $o(\delta t)$ described in words as 'little o δt' means that the remainder or error is of lower order than δt, that is

$$\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0.)$$

Partition theorem approach

The probability of two or more hits is assumed to be $o(\delta t)$, that is, negligible as $\delta t \to 0$, and the probability of no hits is $1 - \lambda \delta t + o(\delta t)$. We now apply the partition theorem on the possible outcomes. The case $n = 0$ is special since reading zero can only occur through no event occurring. Thus

$$p_0(t + \delta t) = [1 - \lambda \delta t + o(\delta t)] p_0(t),$$

$$p_n(t + \delta t) = p_{n-1}(t)(\lambda \delta t + o(\delta t)) + p_n(t)(1 - \lambda \delta t + o(\delta t)) + o(\delta t), \ (n \geq 1)$$

Partition theorem approach

$$p_0(t+\delta t) = [1 - \lambda \delta t + o(\delta t)] p_0(t),$$
$$p_n(t+\delta t) = p_{n-1}(t)(\lambda \delta t + o(\delta t)) + p_n(t)(1 - \lambda \delta t + o(\delta t)) + o(\delta t), \ (n \geq 1)$$

Dividing through by δt and re-organizing the equations, we find that

$$\frac{p_0(t+\delta t) - p_0(t)}{\delta t} = -\lambda p_0(t) + o(1),$$
$$\frac{p_n(t+\delta t) - p_n(t)}{\delta t} = \lambda [p_{n-1}(t) - p_n(t)] + o(1).$$

Partition theorem approach

$$\frac{p_0(t+\delta t) - p_0(t)}{\delta t} = -\lambda p_0(t) + o(1),$$

$$\frac{p_n(t+\delta t) - p_n(t)}{\delta t} = \lambda[p_{n-1}(t) - p_n(t)] + o(1).$$

All terms $o(1)$ tend to zero as $\delta t \to 0$. Not surprisingly, we have recovered firts Eqns

Iterative method

Suppose that our model of the Geiger counter is based on Eqn

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t),$$

and that we wish to solve the equations to recover $p_n(t)$, which we assume to be unknown for the purposes of this exercise. Equation is an ordinary differential equation for one unknown function $p0(t)$.

Iterative method

It is of first-order, and it can be easily verified that its general solution is

$$p_0(t) = C_0 e^{-\lambda t},$$

where C_0 is a constant. We need to specify initial conditions for the problem. Assume that the instrumentation of the Geiger counter is set to zero initially. Thus we have the certain event for which $p_0(0) = 1$, and consequently $p_n(0) = 0$, $(n \geq 1)$

Iterative method

the probability of any reading other than zero is zero at time $t = 0$. Hence $C_0 = 1$ and

$$p_0(t) = e^{-\lambda t}.$$

Now put $n = 1$ in

dt

$$\frac{dp_n(t)}{dt} = \lambda[p_{n-1}(t) - p_n(t)], \qquad (n \geq 1).$$

so that

$$\frac{dp_1(t)}{dt} = \lambda p_0(t) - \lambda p_1(t),$$

Iterative method

after substituting for $p_0(t)$. This first order differential equation for $p_1(t)$ is of integrating factor type with integrating factor

$$e^{\int \lambda dt} = e^{\lambda t},$$

in which case it can be rewritten as the following separable differential equation:

$$\frac{d}{dt}\left(e^{\lambda t} p_1(t)\right) = \lambda e^{\lambda t} e^{-\lambda t} = \lambda.$$

Hence, integration with respect to t results in

$$e^{\lambda t} p_1(t) = \int \lambda dt = \lambda t + C_1.$$

Iterative method

Hence, integration with respect to t results in

$$e^{\lambda t}p_1(t) = \int \lambda dt = \lambda t + C_1.$$

Thus

$$p_1(t) = \lambda t e^{-\lambda t} + C_1 e^{-\lambda t} = \lambda t e^{-\lambda t}$$

since $p_1(0) = 0$.

Iterative method

We now repeat the process by putting $n = 2$ in Eqn and substituting in the $p_1(t)$ which has just been found. The result is the equation

$$\frac{dp_2(t)}{dt} + \lambda p_2(t) = \lambda p_1(t) = \lambda t e^{-\lambda t}.$$

This is a further first-order integrating-factor differential equation which can be solved using the same method. The result is, using the initial condition $p_2(0) = 0$,

$$p_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!}.$$

Iterative method

The method can be repeated for $n = 3, 4, \ldots$, and the results imply that

$$p_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!},$$

which is the probability given by Eqn PDF as we would expect. The result in Eqn can be justified rigorously by constructing a proof by induction.